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YUJI LIU

*Studies on boundary value problems of  
singular fractional differential equations with  
impulse effects* **3**

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## Studies on boundary value problems of singular fractional differential equations with impulse effects\*

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**ABSTRACT.** In this paper, we firstly prove existence and uniqueness of solutions of Cauchy problems for nonlinear fractional differential equations involving the Caputo fractional derivative, the Riemann-Liouville derivative, the Caputo type Hadamard derivative and the Riemann-Liouville type Hadamard fractional derivatives of order  $\alpha \in [n - 1, n)$  respectively by using Picard iterative technique under some suitable assumptions. Meanwhile, we get the iterative approximation solutions of these kind of Cauchy problems. Secondly we obtain exact expression of piecewise continuous solutions of the linear fractional differential equations. These results provide new methods to convert an impulsive fractional differential equation to a fractional integral equation. Thirdly, four classes of boundary value problems for singular fractional differential equations with impulse effects are proposed. Sufficient conditions are given for the existence of solutions of these problems. We allow the nonlinearity  $p(t)f(t, x)$  in fractional differential equations to be singular at  $t = 0, 1$ . Finally, by establishing existence results on solvability of two class of impulsive boundary value problems of fractional differential equations, we make a comparison on impulsive boundary value problems for two kinds of fractional differential equations, one has a single starting point and the other one has multiple starting points. In order to avoid misleading the readers, a mistake in [Impulsive integral boundary value problems of the higher-order fractional differential equation with eigenvalue arguments, *Adv. Differ. Equ.* (2015) 2015: 382] is also corrected.

**KEY WORDS.** higher order singular fractional differential equation, iterative approximation solution, impulsive boundary value problem, Riemann-Liouville fractional derivative, Caputo fractional derivative, Riemann-Liouville type Hadamard fractional derivative, Caputo type Hadamard fractional derivative, fixed point theorem.

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## 1 Introduction

One knows that the fractional derivatives (Riemann-Liouville fractional derivative, Caputo fractional derivative and Hadamard fractional derivative and other type see [44]) are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory.

Fractional order differential equations are generalizations of integer order differential equations. Using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. Fractional differential equations have many applications see Chapter 10 in [71], books [45, 71, 76].

In recent years, there have been many results obtained on the existence and uniqueness of solutions of initial value problems or boundary value problems for nonlinear fractional differential equations, see [17, 19, 55, 65, 69, 70, 72, 83, 102, 109].

Dynamics of many evolutionary processes from various fields such as population dynamics, control theory, physics, biology, and medicine undergo abrupt changes at certain moments of time like earthquake, harvesting, shock, and so forth. These perturbations can be well approximated as instantaneous change of states or impulses. These processes are modeled by impulsive differential equations. In 1960, Milman and Myshkis introduced impulsive differential equations in their paper [63]. Based on their work, several monographs have been published by many authors like Samoilenko and Perestyuk [77], Lakshmikantham et al. [56], Bainov and Simeonov [22, 23], Bainov and Covachev [24], and Benchohra et al. [25].

Fractional differential equation was extended to impulsive fractional differential equations, since Agarwal and Benchohra published the first paper on the topic [3] in 2008. Since then many authors [1, 29, 32, 43, 47, 48, 52–54, 68, 72, 82, 83, 100] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems see [3, 10, 11, 49, 85], impulsive periodic boundary value problems see [18, 81, 93], impulsive initial value problems see [26, 31, 67, 78], two-point, three-point or multi-point impulsive boundary value problems see [9, 36, 84, 92, 106, 111], impulsive boundary value problems on infinite intervals see [105].

In [33], Feckan and Zhou pointed out that the formula of solutions for impulsive fractional differential equations in [2, 8, 14, 21] is incorrect and gave their correct formula. In [90, 92], the authors established a general framework to find the solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of the solutions to a kind of impulsive fractional differential equations respectively. In [88], the authors illustrated their comprehension for the counterexample in [33] and criticized the

viewpoint in [33, 90, 92]. Next, in [34], Feckan et al. expounded for the counterexample in [33] and provided further five explanations in the paper.

In a fractional differential equation, there exist two cases concerning the derivatives: the first case is  $D^\alpha = D_{0+}^\alpha$ , i.e., the fractional derivative has a single start point  $t = 0$ . The other case is  $D^\alpha = D_{t_i^+}^\alpha$ , i.e., the fractional derivative has a multiple start points  $t = t_i (i \in \mathbb{N}[0, m])$ .

There have been many authors concerning the existence and uniqueness of solutions of boundary value problems of impulsive fractional differential equations with multiple start points  $t = t_i (i \in \mathbb{N}[0, m])$ .

Recently, Wang [80] consider the second case in which  $D^\alpha$  has multiple start points, i.e.,  $D^\alpha = D_{t_i^+}^\alpha$ . They studied the existence and uniqueness of solutions of the following initial value problem of the impulsive fractional differential equation

$$\begin{cases} {}^C D_{t_i^+}^\alpha u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u_j, j \in \mathbb{N}[0, n-1], \\ \Delta u^{(j)}(t_i) = I_{ji}(u(t_i)), i \in \mathbb{N}[1, p], j \in [0, n-1], \end{cases} \quad (1.0.1)$$

where  $\alpha \in (n-1, n)$  with  $n$  being a positive integer,  ${}^C D_{t_i^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$ ,  $\mathbb{N}[a, b] = \{a, a+1, \dots, b\}$  with  $a, b$  being integers,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$ ,  $I_{ji} \in C(\mathbb{R}, \mathbb{R}) (i \in \mathbb{N}[1, p], j \in \mathbb{N}[0, n-1])$ ,  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Henderson and Ouahab [41] studied the existence of solutions of the following initial value problems and periodic boundary value problems of impulsive fractional differential equations:

$$\begin{cases} {}^C D_{t_i^+}^\alpha u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u_j, j \in \mathbb{N}[0, 1], \\ u^{(j)}(t_i) = I_{ji}(u(t_i)), i \in \mathbb{N}[1, p], j \in \mathbb{N}[0, 1], \end{cases}$$

and

$$\begin{cases} {}^C D_{t_i^+}^\alpha u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u^{(j)}(b), j \in \mathbb{N}[0, 1], \\ u^{(j)}(t_i) = I_{ji}(u(t_i)), i \in \mathbb{N}[1, p], j \in \mathbb{N}[0, 1], \end{cases}$$

where  $\alpha \in (1, 2]$ ,  $b > 0$ ,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = b$ ,  $f : [0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $I_{ji} : \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions. Readers should also refer [89].

In [104], Zhao and Gong studied existence of positive solutions of the following nonlinear impulsive fractional differential equation with generalized periodic boundary value conditions

$$\begin{cases} {}^C D_{t_i^+}^\alpha u(t) = f(t, u(t)), t \in (0, T] \setminus \{t_1, \dots, t_p\}, \\ \Delta u(t_i) = I_i(u(t_i)), i \in \mathbb{N}[1, p], \\ \Delta u'(t_i) = J_i(u(t_i)), i \in \mathbb{N}[1, p], \\ \alpha u(0) - \beta u(1) = 0, \alpha u'(0) - \beta u'(1) = 0, \end{cases} \quad (1.0.2)$$

where  $q \in (1, 2)$ ,  ${}^C D_{t_i^+}^q$  represents the standard Caputo fractional derivatives of order  $q$ ,  $\alpha > \beta > 0$ ,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$ ,  $\mathbb{N}[a, b] = \{a, a+1, \dots, b\}$  with  $a, b$  being integers,  $I_i, J_i \in C([0, +\infty), [0, +\infty))$  ( $i \in \mathbb{N}[1, p]$ ),  $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous.

Wang, Ahmad and Zhang [87] studied the existence and uniqueness of solutions of the following periodic boundary value problems for nonlinear impulsive fractional differential equation

$$\begin{cases} {}^C D_{t_i^+}^\alpha u(t) = f(t, u(t)), t \in (0, T] \setminus \{t_1, \dots, t_p\}, \\ \Delta u(t_i) = I_i(u(t_i)), i \in \mathbb{N}[1, p], \\ \Delta u'(t_i) = I_i^*(u(t_i)), i \in \mathbb{N}[1, p], \\ u'(0) + (-1)^\theta u(T) = bu(T), u(0) + (-1)^\theta u(T) = 0, \end{cases} \quad (1.0.3)$$

where  $\alpha \in (1, 2)$ ,  ${}^C D_{t_i^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$ ,  $\theta = 1, 2$ ,  $\mathbb{N}[a, b] = \{a, a+1, \dots, b\}$  with  $a, b$  being integers,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$ ,  $I_i, I_i^* \in C(\mathbb{R}, \mathbb{R})$  ( $i \in \mathbb{N}[1, p]$ ),  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

In [9, 10, 99, 107], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^C D_{t_i^+}^\alpha x(t) = w(t)f(t, x(t), x'(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), i \in \mathbb{N}[1, p], \\ \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ ax(0) \pm bx'(0) = g_1(x), cx(1) + dx'(1) = g_2(x), \end{cases} \quad (1.0.4)$$

where  $\alpha \in (1, 2)$ ,  ${}^C D_{t_i^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$ ,  $a, b, c, d \geq 0$  with  $ac + ad + bc \neq 0$ ,  $\mathbb{N}[m, n] = \{m, m+1, \dots, n\}$  with  $m, n$  being integers,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$ ,  $I_i, J_i \in C(\mathbb{R}, \mathbb{R})$  ( $i \in \mathbb{N}[1, p]$ ),  $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous,  $w : [0, 1] \rightarrow [0, +\infty)$  is a continuous function,  $g_1, g_2 : PC(0, 1) \rightarrow \mathbb{R}$  are two continuous functions.

In 2015, Zhou, Liu and Zhang [108] studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^C D_{t_i^+}^\alpha x(t) = \lambda x(t) + f(t, x(t), (Kx)(t), (Hx)(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), i \in \mathbb{N}[1, p], \\ \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ ax(0) - bx'(0) = x_0, cx(1) + dx'(1) = x_1, \end{cases} \quad (1.0.5)$$

where  $\alpha \in (1, 2)$ ,  ${}^C D_{t_i^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$ ,  $a \geq 0, b > 0, c \geq 0, d > 0$  with  $\delta = ac + ad + bc \neq 0$ ,  $\lambda > 0$ ,  $x_0, x_1 \in \mathbb{R}$ ,  $\mathbb{N}[m, n] = \{m, m+1, \dots, n\}$  with  $m, n$  being integers,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$ ,  $I_i, J_i \in C(\mathbb{R}, \mathbb{R})$  ( $i \in \mathbb{N}[1, p]$ ),  $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuous,  $(Hx)(t) = \int_0^1 h(t, s)x(s)ds$  and  $(Kx)(t) = \int_0^t k(t, s)x(s)ds$ .

In [50, 57], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^C D_{t_i^+}^\alpha x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), i \in \mathbb{N}[1, p], \\ \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ ax(0) + bx(1) = g_1(x), ax'(0) + bx'(1) = g_2(x), \end{cases}$$

where  $\alpha \in (1, 2)$ ,  ${}^C D_{t_i^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$ ,  $a, b \in \mathbb{R}$  with  $a \geq b > 0$ ,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$ ,  $I_i, J_i \in C(\mathbb{R}, \mathbb{R}) (i \in \mathbb{N}[1, p])$ ,  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $g_1, g_2 : PC(0, 1] \rightarrow \mathbb{R}$  are two continuous functions.

In [52], Liu and Li investigated the existence and uniqueness of solutions for the following nonlinear impulsive fractional differential equations

$$\begin{cases} {}^C D_{t_i^+}^\alpha u(t) = f(t, u(t), u'(t), u''(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, p], \\ u(0) = \lambda_1 u(T) + \xi_1 \int_0^T q_1(s, u(s), u'(s), u''(s)) ds, \\ u'(0) = \lambda_2 u'(T) + \xi_2 \int_0^T q_2(s, u(s), u'(s), u''(s)) ds, \\ u''(0) = \lambda_3 u''(T) + \xi_3 \int_0^T q_3(s, u(s), u'(s), u''(s)) ds, \\ \Delta u(t_i) = A_i(u(t_i)), i \in \mathbb{N}[1, p], \\ \Delta u'(t_i) = B_i(u(t_i)), i \in \mathbb{N}[1, p], \\ \Delta u''(t_i) = C_i(u(t_i)), i \in \mathbb{N}[1, p], \end{cases} \quad (1.0.6)$$

where  $\alpha \in (2, 3)$ ,  ${}^C D_{t_i^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$ ,  $\mathbb{N}[a, b] = \{a, a+1, \dots, b\}$  with  $a, b$  being integers,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$ ,  $\lambda_i, \xi_i \in \mathbb{R} (i = 1, 2, 3)$  are constants,  $A_i, B_i, C_i \in C(\mathbb{R}, \mathbb{R}) (i \in \mathbb{N}[1, p])$ ,  $f : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuous.

Recently, in [20], to extend the problem for impulsive differential equation  $u''(t) - \lambda u(t) = f(t, u(t))$ ,  $u(0) = u(T) = 0$ ,  $\Delta u'(t_i) = I_i(u(t_i))$ ,  $i = 1, 2, \dots, p$  to impulsive fractional differential equation, the authors studied the existence and the multiplicity of solutions for the Dirichlet's boundary value problem for impulsive fractional order differential equation

$$\begin{cases} {}^C D_{T-}^\alpha ({}^C D_{0+}^\alpha x(t) + a(t)x(t) = \lambda f(t, x(t)), t \in [0, T], t \neq t_i, i \in \mathbb{N}[1, m], \\ \Delta {}^C D_{T-}^{\alpha-1} ({}^C D_{0+}^\alpha x(t_i) = \mu I_i(x(t_i^-)), i \in \mathbb{N}[1, m], x(0) = x(T) = 0, \end{cases} \quad (1.0.7)$$

where  $\alpha \in (1/2, 1]$ ,  $\lambda, \mu > 0$  are constants,  $\mathbb{N}[a, b] = \{a, a+1, \dots, b\}$  with  $a \leq b$ ,  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$ ,  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function,  $I_i : \mathbb{R} \rightarrow \mathbb{R} (i \in \mathbb{N}[1, m])$  are continuous functions,  ${}^C D_{0+}^\alpha$  (or  ${}^C D_{T-}^\alpha$ ) is the standard left (or right) Caputo fractional derivative of order  $\alpha$ ,  $a \in C[0, T]$  and there exist constants  $a_1, a_2 > 0$  such that  $a_1 \leq a(t) \leq a_2$  for all  $t \in [0, T]$ ,  $\Delta x|_{t=t_i} = \lim_{t \rightarrow t_i^+} x(t) - \lim_{t \rightarrow t_i^-} x(t) = x(t_i^+) - x(t_i^-)$  and  $x(t_i^+), x(t_i^-)$  represent

the right and left limits of  $x(t)$  at  $t = t_i$  respectively,  $a, b, x_0$  a constant with  $a + b \neq 0$ . One knows that the boundary condition  $ax(0) + bx(T) = x_0$  becomes  $x(0) - x(T) = \frac{x_0}{a}$  when  $a + b = 0$ , that is so called nonhomogeneous periodic type boundary condition.

For impulsive fractional differential equations whose derivatives have single start points  $t = 0$ , there has been few papers published. In [73], authors presented a new method to converting the impulsive fractional differential equation (with the Caputo fractional derivative) to an equivalent integral equation and established existence and uniqueness results for some boundary value problems of impulsive fractional differential equations involving the Caputo fractional derivatives with single start point. The existence and uniqueness of solutions of the following initial or boundary value problems were discussed in [73]:

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ x(0) = x_0, x'(0) = x_1, \end{cases}$$

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ x(0) + \phi(x) = x_0, x'(0) = x_1, \end{cases}$$

$$\begin{cases} {}^C D_{0+}^\beta x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), i \in \mathbb{N}[1, p], \\ ax(0) + bx(1) = 0, \end{cases}$$

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ ax(0) - bx'(0) = x_0, cx(1) + dx'(1) = x_1, \end{cases}$$

and

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ x(0) - ax(\xi) = x(1) - bx(\eta) = 0, \end{cases}$$

where  $\alpha \in (1, 2], \beta \in (0, 1], D_{0+}^*$  is the Caputo fractional derivative with order  $*$  and single start point  $t = 0$ ,  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $I_i, J_i : \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions,  $a, b, c, d, x_0, x_1 \in \mathbb{R}$  are constants,  $\phi : PC(0, 1] \rightarrow \mathbb{R}$  is a functional.

In [104], authors studied the existence of positive solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), i \in \mathbb{N}[1, p], \\ \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ ax(0) - bx(1) = 0, ax'(0) - bx'(1) = 0, \end{cases}$$

where  $\alpha \in (1, 2)$ ,  ${}^C D_{0+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$ ,  $a > b > 0$ ,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$ ,  $I_i, J_i \in C(\mathbb{R}, \mathbb{R}) (i \in \mathbb{N}[1, p])$ ,  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

We firstly observed that in the above-mentioned work, the authors all require that the fractional derivatives are the Caputo type derivatives, the nonlinear term  $f$  and the impulse functions are continuous. It is easy to see that these conditions are very strongly restrictive and difficult to satisfy in applications. To the author's knowledge, there has been no paper published discussed the existence of solutions of boundary value problems of impulsive fractional differential equations involving other fractional derivatives such as the Riemann-Liouville fractional derivatives, Hadamard fractional derivatives.

Secondly, we known (see Theorem 3.1, Theorem 3.6 in [44]) under the assumption  $f(t, x) \in L(0, 1)$  for any  $x \in \mathbb{R}$  that  $x \in L(0, 1)$  satisfies a.e. the equation

$$D_{0+}^\alpha x(t) = f(t, x(t)), \text{ a.e., } t \in (0, 1),$$

$$D_{0+}^{\alpha-j} x(0) = b_j, \quad j \in \mathbb{N}[1, -[-\alpha]]$$

if and only if  $x$  satisfies

$$x(t) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha-j+1)} t^{\alpha-j} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds, \text{ a.e., } t \in [0, 1].$$

When we consider the following initial value problem

$${}^{RL} D_{0+}^{\frac{3}{2}} x(t) = t^{-\frac{1}{8}}(1-t)^{-\frac{9}{8}} =: f(t), \text{ a.e., } t \in (0, 1), \quad \lim_{t \rightarrow 0^+} t^{\frac{1}{2}} x(t) = {}^{RL} D_{0+}^{\frac{1}{2}} x(0) = 0. \quad (\text{IVP})$$

Let

$$x_0(t) = \int_0^t \frac{(t-s)^{\frac{3}{2}-1}}{\Gamma(3/2)} f(s) ds = \int_0^t \frac{(t-s)^{\frac{3}{2}-1}}{\Gamma(3/2)} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}} ds, \quad t \in (0, 1).$$

Then

$${}^{RL} D_{0+}^{\frac{1}{2}} x_0(t) = \int_0^t (t-s)^{\frac{3}{2}-\frac{1}{2}-1} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}} ds.$$

It is easy to check that

$$\int_0^t (t-s)^{\frac{3}{2}-1} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}} ds \leq \int_0^t (t-s)^{-\frac{5}{8}} s^{-\frac{1}{8}} = t^{\frac{1}{4}} \mathbf{B}(3/8, 7/8), \quad t \in [0, 1],$$

$$\int_0^t (t-s)^{\frac{3}{2}-\frac{1}{2}-1} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}} ds \leq (1-t)^{-\frac{9}{8}} \int_0^t s^{-\frac{1}{8}} = (1-t)^{-\frac{9}{8}} \frac{8t^{\frac{7}{8}}}{7}, \quad t \in [0, 1].$$

So  $\lim_{t \rightarrow 0^+} t^{\frac{1}{2}} x_0(t) = {}^{RL}D_{0^+}^{\frac{1}{2}} x_0(0) = 0$ . Further more, we have

$$\begin{aligned} {}^{RL}D_{0^+}^{\frac{3}{2}} x_0(t) &= \frac{1}{\Gamma(2-3/2)} \left[ \int_0^t (t-s)^{2-3/2-1} \int_0^s \frac{(s-u)^{\frac{3}{2}-1}}{\Gamma(3/2)} f(u) du ds \right]'' \\ &= \frac{1}{\Gamma(1/2)} \left[ \int_0^t \int_u^t (t-s)^{-1/2} \frac{(s-u)^{1/2}}{\Gamma(3/2)} ds f(u) du \right]'' \text{ by interchanging the integral order} \\ &= \frac{1}{\Gamma(1/2)} \left[ \int_0^t (t-u) \int_0^1 (1-w)^{-1/2} \frac{w^{1/2}}{\Gamma(3/2)} dw f(u) du \right]'' \text{ by } \frac{s-u}{t-u} = w \\ &= \left[ \int_0^t (t-u) f(u) du \right]'' = f(t) \text{ by } \mathbf{B}(1/2, 3/2) = \Gamma(1/2)\Gamma(3/2). \end{aligned}$$

It follows that problem (IVP) has a unique solution  $x_0$  in  $C^0[0, 1]$ . However  $f(t)$  is not integral on  $[0, 1]$  ( $f \notin L(0, 1)$ ). Hence it is interesting to investigate the existence and uniqueness of solutions of initial value problems of fractional differential equations under the assumptions the nonlinearity  $f(t, x)$  is not integral for  $x \in \mathbb{R}$ .

In this paper, we firstly establish existence and uniqueness results for initial value problems of fractional differential equations under the assumption that the nonlinearity  $f(t, x)$  is not integral for  $x \in \mathbb{R}$ . Then we obtain piecewise continuous solutions of fractional differential equations. Thirdly, we study the existence of solutions of four classes of impulsive boundary value problems of singular fractional differential equations. The first one is the impulsive mixed type integral boundary value problem as follows:

$$\begin{cases} {}^C D_{0^+}^{\beta} x(t) = p(t)f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \lim_{t \rightarrow 0^+} x(t) = \int_0^1 \phi(s)G(s, x(s))ds, \\ x'(1) = \int_0^1 \psi(s)H(s, x(s))ds, \\ \Delta x(t_i) = I(t_i, x(t_i)), i \in \mathbb{N}[1, m], \\ \Delta x'(t_i) = J(t_i, x(t_i)), i \in \mathbb{N}[1, m], \end{cases} \quad (1.0.8)$$

where

- (a)  $1 < \beta < 2$ ,  ${}^C D_{0^+}^{\beta}$  is the Caputo fractional derivative of order  $\beta$  with starting point 0,
- (b)  $m$  is a positive integer,  $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = 1$ ,  $\mathbb{N}[a, b] = \{a, a + 1, a + 2, \dots, a + n\}$  with  $a, b$  being integers and  $a \leq b$ ,
- (c)  $\phi, \psi : (0, 1) \rightarrow \mathbb{R}$  are measurable functions,
- (d)  $p : (0, 1) \rightarrow \mathbb{R}$  is continuous and there exist numbers  $k > 1 - \beta$  and  $l \in \max\{-\beta, -\beta - k, 0\}$  such that  $|p(t)| \leq t^k(1-t)^l$  for all  $t \in (0, 1)$ ,
- (e)  $f, G, H$  defined on  $(0, 1] \times \mathbb{R}$  are **impulsive I-Carathéodory functions**,  $I, J : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  is a **Discrete I-Carathéodory functions**.

The second one is the impulsive Dirichlet type integral boundary value problem as follows:

$$\begin{cases} {}^{RL}D_{0+}^{\beta}x(t) = q(t)f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \lim_{t \rightarrow 0^+} t^{2-\beta}x(t) = \int_0^1 \phi(s)G(s, x(s))ds, \\ x(1) = \int_0^1 \psi(s)H(s, x(s))ds, \\ \lim_{t \rightarrow t_i^+} (t - t_i)^{2-\beta}x(t) = I(t_i, x(t_i)), i \in \mathbb{N}[1, m], \\ \Delta {}^{RL}D_{0+}^{\beta-1}x(t_i) = J(t_i, x(t_i)), i \in \mathbb{N}[1, m], \end{cases} \quad (1.0.9)$$

where

(f)  $1 < \beta < 2$ ,  ${}^{RL}D_{0+}^{\beta}$  is the Riemann-Liouville fractional derivative of order  $\beta$  with starting point 0,  $n, t_i, \mathbb{N}[a, b]$  satisfies (b),  $\phi, \psi : (0, 1) \rightarrow \mathbb{R}$  satisfy (c),

(g)  $q : (0, 1) \rightarrow \mathbb{R}$  is continuous and there exist numbers  $k > -1$  and  $l \in \max\{-\beta, -\beta - k, 0\}$  such that  $|q(t)| \leq t^k(1 - t)^l$  for all  $t \in (0, 1)$ ,

(h)  $f, G, H$  defined on  $(0, 1] \times \mathbb{R}$  are **impulsive II-Carathéodory functions**,  $I, J : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  are **Discrete II-Carathéodory functions**.

We emphasize that much work on fractional boundary value problems involves either Riemann-Liouville or Caputo type fractional differential equations see [5–7, 11]. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [38], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Recent studies can be seen in [13, 15, 16].

Thirdly we study the following impulsive anti-periodic type integral boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{RLH}D_{1+}^{\beta}x(t) = q(t)f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \lim_{t \rightarrow 1^+} (\log t)^{2-\beta}x(t) + x(e) = \int_0^1 \phi(s)G(s, x(s))ds, \\ \lim_{t \rightarrow 1^+} {}^{RLH}D_{1+}^{\beta-1}x(t) + {}^{RLH}D_{1+}^{\beta-1}x(e) = \int_0^1 \psi(s)H(s, x(s))ds, \\ \lim_{t \rightarrow t_i^+} (\log t - \log t_i)^{2-\beta}x(t) = I(t_i, x(t_i)), i \in \mathbb{N}[1, m], \\ \Delta {}^{RL}D_{1+}^{\beta-1}x(t_i) = J(t_i, x(t_i)), i \in \mathbb{N}[1, m], \end{cases} \quad (1.0.10)$$

where

(i)  $1 < \beta < 2$ ,  ${}^{RLH}D_{1+}^{\beta}$  is the Hadamard fractional derivative of order  $\beta$  with starting point 1,

- (j)  $m$  is a positive integer,  $1 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = e$ ,  $\phi, \psi : (1, e) \rightarrow \mathbb{R}$  are measurable functions,  $q : (1, e) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > -1$  and  $l \in \max\{-\beta, -\beta - k, 0\}$  such that  $|q(t)| \leq (\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ ,
- (k)  $f, G, H$  defined on  $(1, e] \times \mathbb{R}$  are **impulsive III-Carathéodory functions**,  $I, J : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  are **Discrete III-Carathéodory functions**.

Finally we study the following impulsive Sturm-Liouville type integral boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{CH}D_{1+}^{\beta}x(t) = p(t)f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ x(1) - \left(t \frac{d}{dt}\right)x(t)\Big|_{t=1} = \int_0^1 \phi(s)G(s, x(s))ds, \\ x(e) + \left(t \frac{d}{dt}\right)x(t)\Big|_{t=e} = \int_0^1 \psi(s)H(s, x(s))ds, \\ \lim_{t \rightarrow t_i^+} x(t) - x(t_i) = I(t_i, x(t_i)), i \in \mathbb{N}[1, m], \\ \lim_{t \rightarrow t_i^+} \left(t \frac{d}{dt}\right)x(t) - \left(t \frac{d}{dt}\right)x(t)\Big|_{t=t_i} = J(t_i, x(t_i)), i \in \mathbb{N}[1, m], \end{cases} \quad (1.0.11)$$

where

- (l)  $1 < \beta < 2$ ,  ${}^{CH}D_{1+}^{\beta}$  is the Caputo type Hadamard fractional derivative of order  $\beta$  with the starting point 1,  $\left(t \frac{d}{dt}\right)^1 x(t) = tx'(t)$ ,
- (m)  $m, t_i, \mathbb{N}[a, b]$  satisfy (i),  $\phi, \psi : (1, e) \rightarrow \mathbb{R}$  are measurable functions,  $p : (1, e) \rightarrow \mathbb{R}$  is continuous and satisfies  $p : (1, e) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > 1 - \beta$  and  $l \in \max\{-\beta, -\beta - k, 0\}$  such that  $|p(t)| \leq (\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ ,
- (n)  $f, G, H$  defined on  $(1, e] \times \mathbb{R}$  are **impulsive IV-Carathéodory functions**,  $I, J : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  are **Discrete IV-Carathéodory functions**.

A function  $x : (0, 1] \rightarrow \mathbb{R}$  is called a solution of BVP (1.0.9) (or BVP (1.0.8)) if  $x|_{(t_i, t_{i+1}]}$  ( $i = 0, 1, j \in \mathbb{N}[0, m]$ ) is continuous, the limits below exist  $\lim_{t \rightarrow t_i^+} (t - t_i)^{2-\beta}x(t), i \in \mathbb{N}[0, m]$ , (or  $\lim_{t \rightarrow t_i^+} x(t) (i \in \mathbb{N}[0, m])$  and  $x$  satisfies the equations in (1.0.9) (or (1.0.8)).

A function  $x : (1, e] \rightarrow \mathbb{R}$  is called a solution of BVP (1.0.11) (or BVP (1.0.10)) if  $x|_{(t_i, t_{i+1}]}$  ( $i \in \mathbb{N}[0, m]$ ) is continuous, the limits below exist  $\lim_{t \rightarrow t_i^+} \left(\log \frac{t}{t_i}\right)^{2-\beta} x(t), i \in \mathbb{N}[0, m]$ , (or  $\lim_{t \rightarrow t_i^+} x(t) (i \in \mathbb{N}[0, m])$  and  $x$  satisfies equations in (1.0.11) (or (1.0.10)).

To get solutions of a boundary value problem of fractional differential equations, we firstly define a Banach space  $X$ , then we transform the boundary value problem into a integral equation and define a nonlinear operator  $T$  on  $X$  by using the integral equation obtained, finally, we prove that  $T$  has fixed point in  $X$ . The fixed points are just solutions of the

boundary value problem. Three difficulties occur in known papers: one is how to transform the boundary value problem into a integral equation; the other one is how to define and prove a Banach space and the completely continuous property of the nonlinear operator defined; the third one is to choose a suitable fixed point theorem and impose suitable growth conditions on functions to get the fixed points of the operator.

To the best of the authors knowledge, no one has studied the existence of strong weak or weak solutions of **BVP (1.0.i)** ( $i = 8, 9, 10, 11$ ). This paper fills this gap. Another purpose of this paper is to illustrate the similarity and difference of these three kinds of fractional differential equations. We obtain results on the existence of at least one solution for **BVP (1.0.i)** ( $i = 8, 9, 10, 11$ ) respectively. For simplicity we only consider the left-sided operators here. The right-sided operators can be treated similarly. For clarity and brevity, we restrict our attention to BVPs with one impulse, the difference between the theory of one or an arbitrary number of impulses is quite similar.

The remainder of this paper is as follows: in Section 2, we present related definitions. In Section 3 some preliminary results are given (the first purpose is to establish existence and uniqueness of continuous solutions of four classes of initial value problems of non-linear fractional differential equations, the second purpose is to establish explicit expression of continuous solution of four classes of linear fractional differential equations (see Theorem 3.1.2, Theorem 3.2.2, Theorem 3.3.2 and Theorem 3.4.2 in Subsections 3.1, 3.2, 3.3 and 3.4 respectively). In Section 4, the purpose is to get exact expression of piecewise continuous solutions of four classes of linear fractional differential equations. In Section 5, we prove the main results for establishing existence results of solutions of (1.0.i) ( $i=8,9,10,11$ ). In Sections 6, by establishing existence results on solvability of two class of impulsive boundary value problems of fractional differential equations, we make a comparison on two impulsive boundary value problems in which one has a single starting point and the other one has multiple starting points. In order to avoid misleading the readers, a mistake in a published paper is also corrected in this section.

## 2 Related definitions

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [44, 71, 76].

Let the Gamma function, Beta function and the classical Mittag-Leffler special function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx, \quad \mathbf{B}(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx, \quad \mathbf{E}_{\delta, \sigma}(x) = \sum_{\chi=0}^{\infty} \frac{x^{\chi}}{\Gamma(\chi\delta + \sigma)}$$

respectively for  $\alpha > 0, p > 0, q > 0$ . We note that  $\mathbf{E}_{\delta,\delta}(x) > 0$  for all  $x \in \mathbb{R}$  and  $\mathbf{E}_{\delta,\delta}(x)$  is strictly increasing in  $x$ . Then for  $x > 0$  we have  $\mathbf{E}_{\delta,\sigma}(-x) < \mathbf{E}_{\delta,\sigma}(0) = \frac{1}{\Gamma(\sigma)} < \mathbf{E}_{\delta,\sigma}(x)$ .

**Definition 2.1** [44] *Let  $c \in \mathbb{R}$ . The left Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $g : (c, \infty) \rightarrow \mathbb{R}$  is given by*

$$I_{c^+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} g(s) ds,$$

*provided that the right-hand side exists.*

**Definition 2.2** [44] *Let  $c \in \mathbb{R}$ . The left Riemann-Liouville fractional derivative of order  $\alpha > 0$  of a function  $g : (c, +\infty) \rightarrow \mathbb{R}$  is given by*

$${}^{RL}D_{c^+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

*where  $\alpha \in (n-1, n)$ , i.e.,  $n = [\alpha]$ , provided that the right-hand side exists.*

**Definition 2.3** [44] *Let  $c \in \mathbb{R}$ . The left Caputo fractional derivative of order  $\alpha > 0$  of a function  $g : (c, +\infty) \rightarrow \mathbb{R}$  is given by*

$${}^C D_{c^+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

*where  $\alpha \in (n-1, n)$ , i.e.,  $n = [\alpha]$ , provided that the right-hand side exists.*

**Definition 2.4** [44] *Let  $c > 0$ . The left Hadamard fractional integral of order  $\alpha > 0$  of a function  $g : [c, +\infty) \rightarrow \mathbb{R}$  is given by*

$${}^H I_{c^+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s},$$

*provided that the right-hand side exists.*

**Definition 2.5** [44] *Let  $c > 0$ . The left Riemann-Liouville type Hadamard fractional derivative of order  $\alpha > 0$  of a function  $g : [c, +\infty) \rightarrow \mathbb{R}$  is given by*

$${}^{RLH} D_{c^+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_c^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} g(s) \frac{ds}{s},$$

*where  $\alpha \in (n-1, n)$ , i.e.,  $n = [\alpha]$ , provided that the right-hand side exists.*

**Definition 2.6** [42] *Let  $c > 0$ . The left Caputo type Hadamard fractional derivative of order  $\alpha > 0$  of a function  $g : [c, +\infty) \rightarrow \mathbb{R}$  is given by*

$${}^{CH} D_{c^+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s \frac{d}{ds}\right)^n g(s) \frac{ds}{s},$$

*where  $\alpha \in (n-1, n)$ , i.e.,  $n = [\alpha]$ , provided that the right-hand side exists.*

**Definition 2.7** We call  $F : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$  an **impulsive I-Carathéodory function** if it satisfies

- (i)  $t \rightarrow F(t, u)$  is measurable on  $(0, 1)$  for any  $u \in \mathbb{R}$ ,
- (ii)  $u \rightarrow F(t, u)$  are continuous on  $\mathbb{R}$  for all  $t \in (0, 1)$ ,
- (iii) for each  $r > 0$  there exists  $M_r > 0$  such that

$$|F(t, u)| \leq M_r, t \in (0, 1), |u| \leq r.$$

**Definition 2.8** Let  $n - 1 < \alpha < n$  and  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ . We call  $F : \bigcup_{i=0}^m (t_i, t_{i+1}] \times \mathbb{R} \rightarrow \mathbb{R}$  an **impulsive II-Carathéodory function** if it satisfies

- (i)  $t \rightarrow F(t, (t - t_i)^{\alpha-n} u)$  is measurable on  $(t_i, t_{i+1}] (i \in \mathbb{N}[0, m])$  for any  $u \in \mathbb{R}$ ,
- (ii)  $u \rightarrow F(t, (t - t_i)^{\alpha-n} u)$  are continuous on  $\mathbb{R}$  for all  $t \in (t_i, t_{i+1}] (i \in \mathbb{N}[0, m])$ ,
- (iii) for each  $r > 0$  there exists  $M_r > 0$  such that

$$|F(t, (t - t_i)^{\alpha-n} u)| \leq M_r, t \in (t_i, t_{i+1}], |u| \leq r, (i \in \mathbb{N}[0, m]).$$

**Definition 2.9** Let  $n - 1 < \alpha < n$  and  $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$ . We call  $F : \bigcup_{i=0}^m (t_i, t_{i+1}] \times \mathbb{R} \rightarrow \mathbb{R}$  an **impulsive III-Carathéodory function** if it satisfies

- (i)  $t \rightarrow F\left(t, \left(\log \frac{t}{t_i}\right)^{\alpha-n} u\right)$  is measurable on  $(t_i, t_{i+1}] (i \in \mathbb{N}[0, m])$  for any  $u \in \mathbb{R}$ ,
- (ii)  $u \rightarrow F\left(t, \left(\log \frac{t}{t_i}\right)^{\alpha-n} u\right)$  are continuous on  $\mathbb{R}$  for all  $t \in (t_i, t_{i+1}] (i \in \mathbb{N}[0, m])$ ,
- (iii) for each  $r > 0$  there exists  $M_r > 0$  such that

$$\left| F\left(t, \left(\log \frac{t}{t_i}\right)^{\alpha-n} u\right) \right| \leq M_r, t \in (t_i, t_{i+1}], |u| \leq r, (i \in \mathbb{N}[0, m]).$$

**Definition 2.10** We call  $F : (1, e) \times \mathbb{R} \rightarrow \mathbb{R}$  an **impulsive IV-Carathéodory function** if it satisfies

- (i)  $t \rightarrow F(t, u)$  is measurable on  $(1, e)$  for any  $u \in \mathbb{R}$ ,
- (ii)  $u \rightarrow F(t, u)$  are continuous on  $\mathbb{R}$  for all  $t \in (1, e)$ ,
- (iii) for each  $r > 0$  there exists  $M_r > 0$  such that

$$|F(t, u)| \leq M_r, t \in (1, e), |u| \leq r.$$

**Definition 2.11** Let  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ . We call  $I : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  a **discrete I-Carathéodory function** if it satisfies

- (i)  $u \rightarrow I(t_i, u) (i \in \mathbb{N}[1, m])$  are continuous on  $\mathbb{R}$ ,
- (ii) for each  $r > 0$  there exists  $M_r > 0$  such that  $|I(t_i, u)| \leq M_r, |u| \leq r, i \in \mathbb{N}[1, m]$ .

**Definition 2.12** Let  $n - 1 < \alpha < n$  and  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ . We call  $I : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  a **discrete II-Carathéodory function** if it satisfies

- (i)  $u \rightarrow I(t_i, (t_i - t_{i-1})^{\alpha-2}u)$  ( $i \in \mathbb{N}[1, m]$ ) are continuous on  $\mathbb{R}$ ,
- (ii) for each  $r > 0$  there exists  $M_r > 0$  such that

$$|I(t_i, (t_i - t_{i-1})^{\alpha-2}u)| \leq M_r, |u| \leq r, i \in \mathbb{N}[1, m].$$

**Definition 2.13** Let  $n - 1 \leq \alpha < n$  and  $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$ . We call  $I : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  a **discrete III-Carathéodory function** if it satisfies

- (i)  $u \rightarrow I(t_i, (\log t_i - \log t_{i-1})^{\alpha-2}u)$  ( $i \in \mathbb{N}[1, m]$ ) are continuous on  $\mathbb{R}$ ,
- (ii) for each  $r > 0$  there exists  $M_r > 0$  such that

$$|I(t_i, (\log t_i - \log t_{i-1})^{\alpha-2}u)| \leq M_r, |u| \leq r, i \in \mathbb{N}[1, m].$$

**Definition 2.14** Let  $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$ . We call

$I : \{t_i : i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R}$  a **discrete IV-Carathéodory function** if it satisfies

- (i)  $u \rightarrow I(t_i, u)$  ( $i \in \mathbb{N}[1, m]$ ) are continuous on  $\mathbb{R}$ ,
- (ii) for each  $r > 0$  there exists  $M_r > 0$  such that  $|I(t_i, u)| \leq M_r, |u| \leq r, i \in \mathbb{N}[1, m]$ .

**Definition 2.15** [64] Let  $E$  and  $F$  be Banach spaces. A operator  $T : E \rightarrow F$  is called a **completely continuous operator** if  $T$  is continuous and maps any bounded set into relatively compact set.

Suppose that  $n - 1 < \alpha < n$  and  $0 \leq a < b$  are constants. The following Banach spaces are used:

- (i)  $C_0(a, b]$  denote the set of all continuous functions on  $(a, b]$  with the limit  $\lim_{t \rightarrow a^+} x(t)$  existing, and the norm  $\|x\| = \sup_{t \in (a, b]} |x(t)|$ ;
- (ii)  $C_{n-\alpha}(a, b]$  the set of all continuous functions on  $(a, b]$  with the limit  $\lim_{t \rightarrow a^+} (t - a)^{n-\alpha} x(t)$  existing, the norm  $\|x\|_{n-\alpha} = \sup_{t \in (a, b]} (t - a)^{n-\alpha} |x(t)|$ ;
- (iii)  $LC_{n-\alpha}(a, b]$  denote the set of all continuous functions on  $(a, b]$  with the limit  $\lim_{t \rightarrow a^+} (\log \frac{t}{a})^{n-\alpha} x(t)$  existing, and the norm  $\|x\| = \sup_{t \in (a, b]} (\log \frac{t}{a})^{n-\alpha} |x(t)|$ .

Let  $m$  be a positive integer and  $\mathbb{N}[0, m] = \{0, 1, 2, \dots, m\}$ ,  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ . The following Banach spaces are also used in this paper:

- (iv)

$$P_m C_{n-\alpha}(0, 1] = \left\{ x : (0, 1] \rightarrow \mathbb{R} : x|_{(t_i, t_{i+1}]} \in C_{n-\alpha}(t_i, t_{i+1}] : i \in \mathbb{N}[0, m] \right\}$$

with the norm

$$\|x\| = \|x\|_{P_m C_{n-\alpha}} = \max \left\{ \sup_{t \in (t_i, t_{i+1}]} (t - t_i)^{n-\alpha} |x(t)| : i \in \mathbb{N}[0, m] \right\}.$$

(v)

$$P_m C_0(0, 1] = \left\{ x : (0, 1] \rightarrow \mathbb{R} : x|_{(t_i, t_{i+1}]} \in C_0(t_i, t_{i+1}] : i \in \mathbb{N}[0, m] \right\}$$

with the norm

$$\|x\| = \|x\|_{P_m C_0(0,1]} = \max \left\{ \sup_{t \in (t_i, t_{i+1}]} |x(t)| : i \in \mathbb{N}[0, m] \right\}.$$

Let  $m$  be a positive integer and  $\mathbb{N}[0, m] = \{0, 1, 2, \dots, m\}$ ,  $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$ . We also use the Banach spaces

(vi)

$$LP_m C_{n-\alpha}(1, e] = \left\{ x : (1, e] \rightarrow \mathbb{R} : \begin{array}{l} x|_{(t_i, t_{i+1}]} \in C(t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \text{there exist the limits} \\ \lim_{t \rightarrow t_i^+} \left( \log \frac{t}{t_i} \right)^{n-\alpha} x(t), i \in \mathbb{N}[0, m] \end{array} \right\}$$

with the norm

$$\|x\| = \|x\|_{LP_m C_{n-\alpha}} = \max \left\{ \sup_{t \in (t_i, t_{i+1}]} \left( \log \frac{t}{t_i} \right)^{n-\alpha} |x(t)|, i \in \mathbb{N}[0, m] \right\}.$$

(vii)

$$LP_m C_0(1, e] = \left\{ x : (1, e] \rightarrow \mathbb{R} : x|_{(t_i, t_{i+1}]} \in C_0(t_i, t_{i+1}], i \in \mathbb{N}[0, m] \right\}$$

with the norm

$$\|x\| = \|x\|_{LP_m C} = \max \left\{ \sup_{t \in (t_i, t_{i+1}]} |x(t)|, i \in \mathbb{N}[0, m] \right\}.$$

### 3 Some preliminary results

Lakshmikantham et al. [58–61] investigated the basic theory of initial value problems for fractional differential equations involving Riemann-Liouville differential operators of order  $q \in (0, 1)$ . The existence and uniqueness of solutions of the following initial value problems of fractional differential equations were discussed under the assumption that  $f \in C_r[0, 1]$ . We will establish existence and uniqueness results for these problems under more weaker assumptions see (A1)-(A4).

Suppose  $n - 1 < \alpha < n$  and  $\eta_j \in \mathbb{R}$  ( $j \in \mathbb{N}[0, n - 1]$ ). We will consider the following four classes of Cauchy problems for non-linear fractional differential equations:

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = p_1(t)F_1(t, x(t)), & t \in (0, 1), \\ \lim_{t \rightarrow 0^+} x^{(j)}(t) = \eta_j, & j \in \mathbb{N}[0, n - 1], \end{cases} \quad (3.0.1)$$

$$\begin{cases} {}^{RL} D_{0+}^\alpha x(t) = p_2(t)F_2(t, x(t)), & t \in (0, 1), \\ \lim_{t \rightarrow 0^+} t^{n-\alpha} x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}, \\ \lim_{t \rightarrow 0^+} {}^{RL} D_{0+}^{\alpha-j} x(t) = \eta_j, & j \in \mathbb{N}[1, n - 1], \end{cases} \quad (3.0.2)$$

$$\begin{cases} {}^{RLH} D_{0+}^\alpha x(t) = p_3(t)F_3(t, x(t)), & t \in (1, e), \\ \lim_{t \rightarrow 1^+} (\log t)^{n-\alpha} x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}, \\ \lim_{t \rightarrow 1^+} {}^{RLH} D_{1+}^{\alpha-j} x(t) = \eta_j, & j \in \mathbb{N}[1, n - 1], \end{cases} \quad (3.0.3)$$

$$\begin{cases} {}^{CH} D_{0+}^\alpha x(t) = p_4(t)F_4(t, x(t)), & t \in (1, e), \\ \lim_{t \rightarrow 1^+} \left(t \frac{d}{dt}\right)^j x(t) = \eta_j, & j \in \mathbb{N}[0, n - 1]. \end{cases} \quad (3.0.4)$$

where  $\left(t \frac{d}{dt}\right)^j x(t) = t \frac{d\left(t \frac{d}{dt}\right)^{j-1} x(t)}{dt}$  for  $j = 2, 3, \dots$ ,  $p_i$  ( $i = 1, 2, 3, 4$ ) satisfy assumption **(Ai)** below,  $F_i$  ( $i = 1, 2, 3, 4$ ) is ***i*-Carathéodory function** ( $i = \mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}$ ) and satisfies the following assumption **(Hi)** ( $i = 1, 2, 3, 4$ ).

**(A1)** there exists constants  $k > -\alpha + n - 1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -\alpha - k\}$  such that  $|p_1(t)| \leq t^k(1-t)^l$  for all  $t \in (0, 1)$ .

**(A2)** there exists constants  $k > -1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -n - k\}$  such that  $|p_2(t)| \leq t^k(1-t)^l$  for all  $t \in (0, 1)$ .

**(A3)** there exists constants  $k > -1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -n - k\}$  such that  $|p_3(t)| \leq (\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ .

**(A4)** there exists constants  $k > -\alpha + n - 1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -\alpha - k\}$  such that  $|p_4(t)| \leq (\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ .

**(H1)** there exists a constant  $L_1 > 0$  such that  $|F_1(t, x_1) - F_1(t, x_2)| \leq L_1|x_1 - x_2|$  for all  $t \in [0, 1]$  and  $x_1, x_2 \in \mathbb{R}$ .

**(H2)** there exists a constant  $L_2 > 0$  such that  $|F_2(t, t^{\alpha-n}x_1) - F_2(t, t^{\alpha-n}x_2)| \leq L_2|x_1 - x_2|$  for all  $t \in (0, 1]$  and  $x_1, x_2 \in \mathbb{R}$ .

**(H3)** there exists a constant  $L_3 > 0$  such that  $|F_3(t, (\log t)^{\alpha-n}x_1) - F_3(t, (\log t)^{\alpha-n}x_2)| \leq L_3|x_1 - x_2|$  for all  $t \in (1, e]$  and  $x_1, x_2 \in \mathbb{R}$ .

**(H4)** there exists a constant  $L_4 > 0$  such that  $|F_4(t, x_1) - F_4(t, x_2)| \leq L_4|x_1 - x_2|$  for all  $t \in (1, e]$  and  $x_1, x_2 \in \mathbb{R}$ .

### 3.1 Existence and uniqueness of solutions of (3.0.1)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!}, \quad t \in [0, 1],$$

$$\phi_i(t) = \phi_0(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_{i-1}(s)) ds, \quad t \in [0, 1], i = 1, 2, \dots.$$

**Lemma 3.1.1** *Suppose that (A1), (H1) hold. Then  $\phi_i \in C[0, 1]$ .*

**Proof:** One sees  $\phi_0 \in C[0, 1]$ . Since  $F_1$  is a **I-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1)} |F_1(t, \phi_0(t))| \leq \sup_{t \in (0,1), |x| \leq \|\phi_0\|} |F_1(t, x)| < +\infty.$$

We have by using (A1)

$$\begin{aligned} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_0(s)) ds \right| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds M_0 \\ &\leq M_0 \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds = M_0 t^{\alpha+k+l} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw \\ &= M_0 t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \rightarrow 0 \text{ as } t \rightarrow 0^+, \end{aligned}$$

implies  $\phi_1$  is continuous on  $[0, 1]$ . So  $\phi_1 \in C[0, 1]$ . By mathematical induction method, we can prove that  $\phi_i \in C[0, 1]$ .  $\square$

**Lemma 3.1.2** *Suppose that (A1), (H1) hold. Then  $\{\phi_i\}$  is convergent uniformly on  $[0, 1]$ .*

**Proof:** Since  $F_1$  is a **I-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1), |x| \leq \|\phi_0\|} |F_1(t, x)| < +\infty.$$

Then we have for  $t \in [0, 1]$  that

$$|\phi_1(t) - \phi_0(t)| = \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_0(s)) ds \right| \leq M_0 t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.$$

So **(H1)** implies that

$$\begin{aligned}
|\phi_2(t) - \phi_1(t)| &= \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) [F_1(s, \phi_1(s)) - F_1(s, \phi_0(s))] ds \right| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l L_1 |\phi_1(s) - \phi_0(s)| ds \\
&\leq M_0 L_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l s^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} ds \\
&\leq M_0 L_1 \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^{\alpha+2k+l} ds \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \\
&= M_0 L_1 t^{2\alpha+2k+2l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Now suppose that

$$|\phi_j(t) - \phi_{j-1}(t)| \leq M_0 L_1^{j-1} t^{j\alpha+jk+jl} \prod_{i=0}^{j-1} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}.$$

We get

$$\begin{aligned}
|\phi_{j+1}(t) - \phi_j(t)| &= \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) [F_1(s, \phi_j(s)) - F_1(s, \phi_{j-1}(s))] ds \right| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l L_1 |\phi_j(s) - \phi_{j-1}(s)| ds \\
&\leq M_0 L_1^j \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l s^{j\alpha+jk+jl} \prod_{i=0}^{j-1} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)} ds \\
&\leq M_0 L_1^j \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^{j\alpha+(j+1)k+jl} ds \prod_{i=0}^{j-1} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)} \\
&= M_0 L_1^j t^{(j+1)\alpha+(j+1)k+(j+1)l} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)} \prod_{i=0}^{j-1} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)} \\
&= M_0 L_1^j t^{(j+1)\alpha+(j+1)k+(j+1)l} \prod_{i=0}^j \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}.
\end{aligned}$$

From the mathematical induction method, we get for every  $i = 1, 2, \dots$  that

$$\begin{aligned}
|\phi_{i+1}(t) - \phi_i(t)| &\leq M_0 L_1^{i-1} t^{(i+1)\alpha+(i+1)k+(i+1)l} \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)} \\
&\leq M_0 L_1^{i-1} \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}, t \in [0, 1].
\end{aligned}$$

Consider

$$\sum_{i=1}^{+\infty} u_i = \sum_{i=1}^{+\infty} M_0 L_1^{i-1} \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}.$$

One sees for sufficiently large  $n$  that

$$\begin{aligned} \frac{u_{i+1}}{u_i} &= L_1 \frac{\mathbf{B}(\alpha+l, (i+1)\alpha+(i+1)k+(i+1)l)}{\Gamma(\alpha)} = L_1 \int_0^1 (1-x)^{\alpha+l-1} x^{(i+1)\alpha+(i+1)k+(i+1)l} dx \\ &\leq L_1 \int_0^\delta (1-x)^{\alpha+l-1} x^{(i+1)\alpha+(i+1)k+(i+1)l} dx + L_1 \int_\delta^1 (1-x)^{\alpha+l-1} dx \text{ with } \delta \in (0, 1) \\ &\leq L_1 \int_0^\delta (1-x)^{\alpha+l-1} dx \delta^{(i+1)\alpha+(i+1)k+(i+1)l} + \frac{L_1}{\alpha+l} \delta^{\alpha+l} \\ &\leq \frac{L_1}{\alpha+l} \delta^{(i+1)\alpha+(i+1)k+(i+1)l} + \frac{L_1}{\alpha+l} \delta^{\alpha+l}. \end{aligned}$$

For any  $\epsilon > 0$ , it is easy to see that there exists  $\delta \in (0, 1)$  such that  $\frac{L_1}{\alpha+l} \delta^{\alpha+l} < \frac{\epsilon}{2}$ . For this  $\delta$ , there exists an integer  $N > 0$  sufficiently large such that  $\frac{L_1}{\alpha+l} \delta^{(i+1)\alpha+(i+1)k+(i+1)l} < \frac{\epsilon}{2}$  for all  $i > N$ . So  $0 < \frac{u_{i+1}}{u_i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$  for all  $i > N$ . It follows that  $\lim_{i \rightarrow +\infty} \frac{u_{i+1}}{u_i} = 0$ . Then  $\sum_{i=1}^{+\infty} u_i$  is convergent. Hence

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_i(t) - \phi_{i-1}(t)] + \cdots, t \in [0, 1]$$

is uniformly convergent. Then  $\{\phi_i(t)\}$  is convergent uniformly on  $[0, 1]$ .  $\square$

**Lemma 3.1.3** *Suppose that (A1), (H1) hold. Then  $\phi(t) = \lim_{i \rightarrow +\infty} \phi_i(t)$  defined on  $[0, 1]$  is a unique continuous solution of the integral equation*

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_1(s) F_1(s, x(s)) ds, t \in [0, 1]. \quad (3.1.1)$$

**Proof:** By  $\phi(t) = \lim_{i \rightarrow +\infty} \phi_i(t)$  and the uniform convergence, we see  $\phi(t)$  is continuous on  $[0, 1]$ . From

$$\begin{aligned} &\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_i(s)) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_j(s)) ds \right| \\ &\leq L_1 \|\phi_i - \phi_j\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds \\ &\leq L_1 \|\phi_i - \phi_j\| t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \\ &\leq L_1 \|\phi_i - \phi_j\| \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \rightarrow 0 \text{ as } i, j \rightarrow +\infty, \end{aligned}$$

we have

$$\begin{aligned}
\phi(t) &= \lim_{i \rightarrow +\infty} \phi_i(t) = \lim_{i \rightarrow +\infty} \left[ \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_{i-1}(s)) ds \right] \\
&= \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \lim_{i \rightarrow +\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_{i-1}(s)) ds \\
&= \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi(s)) ds.
\end{aligned}$$

Then  $\phi$  is a continuous solution of (3.1.1) defined on  $[0, 1]$ .

Suppose that  $\psi$  defined on  $[0, 1]$  is also a solution of (3.1.1). Since  $F_1$  is a **I-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1), |x| \leq \|\psi\|} |F_1(t, x)| < +\infty.$$

Then

$$\psi(t) = \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \psi(s)) ds, t \in (0, 1].$$

We need to prove that  $\phi(t) \equiv \psi(t)$  on  $[0, 1]$ . One sees that

$$|\psi(t) - \phi_0(t)| = \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \psi(s)) ds \right| \leq M_0 t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$\begin{aligned}
|\psi(t) - \phi_1(t)| &= \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) [F_1(s, \psi(s)) - F_1(s, \phi_0(s))] ds \right| \\
&\leq M_0 L_1 t^{2\alpha+2k+2l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)}.
\end{aligned}$$

By mathematical induction method, we have

$$\begin{aligned}
|\psi(t) - \phi_i(t)| &\leq M_0 L_1^i t^{i\alpha+ik+il} \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)} \\
&\leq M_0 L_1^i \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}, t \in [0, 1].
\end{aligned}$$

Similarly we have

$$\lim_{m \rightarrow +\infty} M_0 L_1^i \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)} = 0.$$

Then  $\lim_{i \rightarrow +\infty} \phi_i(t) = \psi(t)$  uniformly on  $[0, 1]$ . Then  $\phi(t) \equiv \psi(t)$ . Then (3.1.1) has a unique solution  $\phi$ . The proof is completed.  $\square$

**Theorem 3.1.1** *Suppose that (A1), (H1) hold. Then  $x$  is a solution of IVP (3.0.1) if and only if  $x$  is a solution of the integral equation (3.1.1).*

**Proof:** Suppose that  $x$  is a solution of IVP (3.0.1). Then  $\lim_{t \rightarrow 0^+} x(t) = \eta_0$  and  $\|x\| = r < +\infty$ . Since  $F_1$  is a **I-Carathéodory function**, we know

$$M_r = \sup_{t \in (0,1), |x| \leq r} |F_1(t, x)| < +\infty.$$

From (A1), we have for  $t \in [0, 1)$

$$\begin{aligned} \left| \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} p_1(s) F_1(s, x(s)) ds \right| &\leq M_r \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} s^k (1-s)^l ds \\ &\leq M_r (1-t)^l \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} s^k ds = M_r (1-t)^l t^{\alpha+k-n+1} \int_0^1 \frac{(1-w)^{\alpha+l-n}}{\Gamma(\alpha-n+1)} w^k dw \\ &= M_r (1-t)^l t^{\alpha+k-n+1} \frac{\mathbf{B}(\alpha-n+1, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

So  $t \rightarrow \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} p_1(s) F_1(s, x(s)) ds$  is continuous on  $[0, 1)$  by  $k > -\alpha + n - 1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -\alpha - k\}$ . It follows that

$$\lim_{t \rightarrow 0^+} \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} p_1(s) F_1(s, x(s)) ds = 0. \quad (3.1.2)$$

From  ${}^C D_{0+}^\alpha x(t) = p_1(t) F_1(t, x(t))$ , a.e.,  $t \in (0, 1)$ , we have

$$\begin{aligned} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, x(s)) ds &= I_{0+}^\alpha p_1(t) F_1(t, x(t)) = I_{0+}^\alpha {}^C D_{0+}^\alpha x(t) \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left( \frac{1}{\Gamma(n-\alpha)} \int_0^s (s-w)^{-\alpha} x^{(n)}(w) dw \right) ds \text{ interchange the order of integrals} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^t \int_w^t (t-s)^{\alpha-1} (s-w)^{n-\alpha-1} ds x^{(n)}(w) dw \text{ use } \frac{s-w}{t-w} = u \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^t (t-u)^{n-1} \int_0^1 (1-u)^{\alpha-1} u^{n-\alpha-1} du x^{(n)}(w) dw \text{ by } \mathbf{B}(\alpha, n-\alpha) = \frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)} \\ &= \frac{1}{(n-1)!} \int_0^t (t-u)^{n-1} x^{(n)}(w) dw \\ &= \frac{1}{(n-1)!} \left[ (t-u)^{n-1} x^{(n-1)}(w) \Big|_0^t + (n-1) \int_0^t (t-u)^{n-2} x^{(n-1)}(w) dw \right] \\ &= \frac{\eta_{n-1}}{(n-1)!} t^{n-1} + \frac{1}{(n-2)!} \int_0^t (t-u)^{n-2} x^{(n-1)}(w) dw \\ &= \dots \dots \dots \\ &= \sum_{j=1}^{n-1} \frac{\eta_j}{j!} t^j + \int_0^t x'(s) ds = x(t) - \sum_{j=0}^{n-1} \frac{\eta_j}{j!}. \end{aligned}$$

It follows that  $x$  satisfies  $x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} t^j + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, x(s)) ds, t \in [0, 1]$ . Then  $x \in C[0, 1]$  is a solution of (3.1.1).

On the other hand, if  $x$  is a solution of (3.1.1). Then

$$x^{(\nu)}(t) = \sum_{j=\nu}^{n-1} \frac{\eta_j}{(j-\nu)!} t^{j-\nu} + \frac{1}{\Gamma(\alpha-\nu)} \int_0^t (t-s)^{\alpha-\nu-1} p_1(s) F_1(s, x(s)) ds, t \in [0, 1].$$

Together Lemma 3.1.1-Lemma 3.1.3 and (3.1.2), we have  $x \in C[0, 1]$  and  $\lim_{t \rightarrow 0^+} x^{(j)}(t) = \eta_j (j \in \mathbb{N}[0, n-1])$ . Furthermore, we have

$$\begin{aligned} {}^C D_{0^+}^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left( \sum_{j=0}^{n-1} \frac{\eta_j s^j}{j!} + \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} p_1(w) F_1(w, x(w)) dw \right)^{(n)} ds \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left( \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} p_1(w) F_1(w, x(w)) dw \right)^{(n)} ds \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{1}{\Gamma(\alpha-n+1)} \left( \int_0^s (s-w)^{\alpha-n} p_1(w) F_1(w, x(w)) dw \right)' ds \\ &= \frac{1}{\Gamma(\alpha-n+1)} \frac{1}{\Gamma(n+1-\alpha)} \left[ \int_0^t (t-s)^{n-\alpha} \left( \int_0^s (s-w)^{\alpha-n} p_1(w) F_1(w, x(w)) dw \right)' ds \right]' \\ &= \frac{1}{\Gamma(\alpha-n+1)} \frac{1}{\Gamma(n+1-\alpha)} \left[ (t-s)^{n-\alpha} \int_0^s (s-w)^{\alpha-n} p_1(w) F_1(w, x(w)) dw \Big|_0^t \right. \\ &\quad \left. + (n-\alpha) \int_0^t (t-s)^{n-\alpha-1} \int_0^s (s-w)^{\alpha-n} p_1(w) F_1(w, x(w)) dw ds \right]' \\ &= \frac{1}{\Gamma(\alpha-n+1)} \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t (t-s)^{n-\alpha-1} \int_0^s (s-w)^{\alpha-n} p_1(w) F_1(w, x(w)) dw ds \right]' \text{ by (3.1.2)} \\ &= \frac{1}{\Gamma(\alpha-n+1)} \frac{1}{\Gamma(n-\alpha)} \frac{1}{\Gamma(\alpha)} \left[ \int_0^t \int_w^t (t-s)^{n-\alpha-1} (s-w)^{\alpha-n} ds p_1(w) F_1(w, x(w)) dw \right]' \\ &\quad \text{by changing the order of integrals} \\ &= \frac{1}{\Gamma(\alpha-n+1)} \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t \int_0^1 (1-u)^{n-\alpha-1} u^{\alpha-n} du p_1(w) F_1(w, x(w)) dw \right]' \text{ by } \frac{s-w}{t-w} = u \\ &= \left[ \int_0^t p_1(w) F_1(w, x(w)) dw \right]' \text{ by } \mathbf{B}(n-\alpha, \alpha-n+1) = \Gamma(n-\alpha)\Gamma(\alpha-n+1) \\ &= p_1(t) F_1(t, x(t)), a.e., t \in (0, 1). \end{aligned}$$

So  $x \in C[0, 1]$  is a solution of IVP (3.1.1). The proof is completed.  $\square$

**Theorem 3.1.2** *Suppose that (A1), (H1) hold. Then*

(i) IVP (3.0.1) has a unique solution.

(ii) *Suppose that there exists constants  $k > -\alpha + n - 1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -\alpha - k\}$ ,  $M \geq 0$  such that  $|F(t)| \leq Mt^k(1-t)^l$  for all  $t \in (0, 1)$ . Then the following special problem*

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = +F(t), \text{ a.e., } t \in (0, 1), \\ \lim_{t \rightarrow 0^+} x^{(j)}(t) = \eta_j, j \in \mathbb{N}[0, n-1] \end{cases} \quad (3.1.3)$$

has a unique solution

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} t^j + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds, \quad t \in (0, 1]. \quad (3.1.4)$$

**Proof:** (i) From Lemmas 3.1.1-Lemma 3.1.3, Theorem 3.1.1 implies that IVP (3.0.1) has a unique solution.

(ii) From the assumption and (i), we know that (3.1.3) has a unique solution. It is easy to see that

$$x(t) \equiv \sum_{j=0}^{n-1} \frac{\eta_j}{j!} t^j + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds$$

by replacing  $p_1(t)F_1(t, x(t))$  with  $F(t)$ . Then  $x$  is a unique solution of (3.1.3). The proof is completed.  $\square$

### 3.2 Existence and uniqueness of solutions of (3.0.2)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v}, \quad t \in (0, 1],$$

$$\phi_i(t) = \phi_0(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_{i-1}(s)) ds, \quad t \in (0, 1], i = 1, 2, \dots$$

**Lemma 3.2.1** *Suppose that (A2), (H2) hold. Then  $\phi_i \in C_{n-\alpha}[0, 1]$ .*

**Proof:** It is easy to see that  $\phi_0 \in C_{n-\alpha}[0, 1]$ . Since  $F_2$  is a **II-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0, 1]} |F_2(t, \phi_0(t))| = \sup_{t \in (0, 1]} |F_2(t, t^{\alpha-n} t^{n-\alpha} \phi_0(t))| \leq \sup_{t \in (0, 1], |x| \leq \|\phi_0\|} |F_2(t, t^{\alpha-n} x)| \leq +\infty.$$

We have by  $k > -1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -n - k\}$

$$\begin{aligned}
& t^{n-\alpha} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_0(s)) ds \right| \\
& \leq t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l M_0 ds + M_F t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l \\
& \leq M_0 t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds \\
& = M_0 t^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Then  $t \rightarrow t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_0(s)) ds$  is continuous on  $[0, 1]$ . We see  $\phi_1 \in C_{n-\alpha}[0, 1]$ . By mathematical induction method, we can prove that  $\phi_i \in C_{n-\alpha}[0, 1]$ .  $\square$

**Lemma 3.2.2** *Suppose that (A2), (H2) hold. Then  $\{t \rightarrow t^{n-\alpha} \phi_i(t)\}$  is convergent uniformly on  $[0, 1]$ .*

**Proof:** Since  $F_2$  is a **II-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1]} |F_2(t, \phi_0(t))| = \sup_{t \in (0,1]} |F_2(t, t^{\alpha-n} t^{n-\alpha} \phi_0(t))| \leq \sup_{t \in (0,1], |x| \leq \|\phi_0\|} |F_2(t, t^{\alpha-n} x)| \leq +\infty.$$

We have for  $t \in [0, 1]$  similarly to Lemma 3.2.1 that

$$t^{n-\alpha} |\phi_1(t) - \phi_0(t)| = \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_0(s)) ds \right| \leq M_0 t^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.$$

So

$$\begin{aligned}
& t^{n-\alpha} |\phi_2(t) - \phi_1(t)| = \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) [F_2(s, \phi_1(s)) - F_2(s, \phi_0(s))] ds \right| \\
& \leq t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l L_1 |\phi_1(s) - \phi_0(s)| ds \\
& \leq L_2 t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l s^{\alpha-n} [s^{n-\alpha} |\phi_1(s) - \phi_0(s)|] ds \\
& \leq M_0 L_2 t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k s^{\alpha-n} s^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} ds \\
& = M_0 L_2 t^{\alpha+n+2k+2l} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^{\alpha+2k+l} dw \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \\
& = M_0 L_2 t^{\alpha+n+2k+2l} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
t^{n-\alpha}|\phi_3(t) - \phi_2(t)| &= \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) [F_2(s, \phi_2(s)) - F_2(s, \phi_1(s))] ds \right| \\
&\leq t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k+\alpha-n} (1-s)^l L_2 |\phi_2(s) - \phi_1(s)| ds \\
&\leq M_0 L_2^2 t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^{k+\alpha-n+\alpha+n+2k+2l} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} ds \\
&\leq M_0 L_2^2 t^{3n+3k+3l} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^{2\alpha+3k+2l} dw \frac{\mathbf{B}(\alpha+l, n+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \\
&= M_0 L_2^2 t^{2\alpha+n+3k+3l} \frac{\mathbf{B}(\alpha+l, 2\alpha+3k+2l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.
\end{aligned}$$

Similarly by the mathematical induction method, we get for every  $i = 1, 2, \dots$  that

$$\begin{aligned}
&t^{n-\alpha} |\phi_i(t) - \phi_{i-1}(t)| \\
&\leq M_0 L_2^{i-1} t^{(i-1)\alpha+n+ik+il} \frac{\mathbf{B}(\alpha+l, \alpha+k-n+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)} \\
&\leq M_0 L_2^{i-1} \frac{\mathbf{B}(\alpha+l, \alpha+k-n+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}, t \in [0, 1].
\end{aligned}$$

Similarly we can prove that

$$\sum_{i=1}^{+\infty} u_i = \sum_{i=1}^{+\infty} M_0 L_2^{i-1} \frac{\mathbf{B}(\alpha+l, \alpha+k-n+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}$$

is convergent. Hence

$$t^{n-\alpha} \phi_0(t) + t^{n-\alpha} [\phi_1(t) - \phi_0(t)] + t^{n-\alpha} [\phi_2(t) - \phi_1(t)] + \dots + t^{n-\alpha} [\phi_i(t) - \phi_{i-1}(t)] + \dots, t \in [0, 1]$$

is uniformly convergent. Then  $\{t \rightarrow t^{n-\alpha} \phi_i(t)\}$  is convergent uniformly on  $[0, 1]$ .  $\square$

**Lemma 3.2.3** *Suppose that (A2), (H2) hold. Then  $\phi(t) = t^{\alpha-n} \lim_{i \rightarrow +\infty} t^{n-\alpha} \phi_i(t)$  defined on  $(0, 1)$  is a unique continuous solution of the integral equation*

$$x(t) = \sum_{v=1}^n \frac{n_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, x(s)), t \in (0, 1). \quad (3.2.1)$$

**Proof:** By  $\lim_{i \rightarrow +\infty} t^{n-\alpha} \phi_i(t) = t^{n-\alpha} \phi(t)$  and the uniformly convergence, we see  $\phi(t)$  is contin-

uous on  $(0, 1]$ . From

$$\begin{aligned}
& t^{n-\alpha} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_p(s)) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_q(s)) ds \right| \\
& \leq L_2 t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l |\phi_p(s) - \phi_q(s)| ds \\
& \leq L_2 \|\phi_p - \phi_q\| t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^{\alpha+k-n} ds \\
& \leq L_2 \|\phi_p - \phi_q\| t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, \alpha+k-n+1)}{\Gamma(\alpha)} \\
& \leq L_2 \|\phi_p - \phi_q\| \frac{\mathbf{B}(\alpha+l, \alpha+k-n+1)}{\Gamma(\alpha)} \rightarrow 0 \text{ uniformly as } p, q \rightarrow +\infty,
\end{aligned}$$

we know

$$\begin{aligned}
\phi(t) &= t^{\alpha-n} \lim_{i \rightarrow +\infty} t^{n-\alpha} \phi_i(t) \\
&= t^{\alpha-n} \lim_{i \rightarrow +\infty} \left[ t^{n-\alpha} \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_{i-1}(s)) ds \right] \\
&= \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \lim_{i \rightarrow +\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_{i-1}(s)) ds \\
&= \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi(s)) ds.
\end{aligned}$$

Then  $\phi$  is a continuous solution of (3.2.1) defined on  $(0, 1]$ .

Suppose that  $\psi$  defined on  $(0, 1]$  is also a solution of (3.2.1). Since  $F_2$  is a **II-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1]} |F_2(t, \psi(t))| = \sup_{t \in (0,1]} |F_2(t, t^{\alpha-n} t^{n-\alpha} \psi(t))| \leq \sup_{t \in (0,1], |x| \leq \|\psi\|} |F_2(t, t^{\alpha-n} x)| \leq +\infty.$$

Then

$$\psi(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \psi(s)) ds, t \in [0, 1].$$

We need to prove that  $\phi(t) \equiv \psi(t)$  on  $(0, 1]$ . Then

$$t^{n-\alpha} |\psi(t) - \phi(t)| = t^{n-\alpha} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \psi(s)) ds \right| \leq M_0 t^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$\begin{aligned}
t^{n-\alpha} |\psi(t) - \phi_1(t)| &= t^{n-\alpha} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) [F_2(s, \psi(s)) - F_2(s, \phi_0(s))] ds \right| \\
&\leq M_0 L_2 t^{\alpha+n+2k+2l} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.
\end{aligned}$$

By mathematical induction method, we can get

$$\begin{aligned} t^{n-\alpha}|\psi(t) - \phi_{i-1}(t)| &= t^{n-\alpha} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) [F_2(s, \psi(s)) - F_2(s, \phi_{i-2}(s))] ds \right| \\ &\leq M_0 L_2^{i-1} t^{(i-1)\alpha+n+ik+il} \frac{\mathbf{B}(\alpha+l_1, \alpha+k-n+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)} \\ &\leq M_0 L_2^{i-1} \frac{\mathbf{B}(\alpha+l_1, \alpha+k-n+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}, t \in [0, 1]. \end{aligned}$$

Hence

$$t^{n-\alpha}|\psi(t) - \phi_{i-1}(t)| \leq M_0 L_2^{i-1} \frac{\mathbf{B}(\alpha+l_1, \alpha+k-n+1)}{\Gamma(\alpha)} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}, \text{ for all } i = 1, 2, \dots.$$

Similarly we have  $\lim_{i \rightarrow +\infty} t^{n-\alpha}\phi_i(t) = t^{n-\alpha}\psi(t)$  uniformly on  $(0, 1]$ . Then  $\phi(t) \equiv \psi(t)$  on  $(0, 1]$ . Then (3.2.1) has a unique solution  $\phi$ . The proof is completed.  $\square$

**Theorem 3.2.1** *Suppose that (A2), (H2) hold. Then  $x \in C_{n-\alpha}(0, 1]$  is a solution of IVP (3.0.2) if and only if  $x \in C_{n-\alpha}(0, 1]$  is a solution of the integral equation (3.2.1).*

**Proof:** Suppose that  $x \in C_{n-\alpha}(0, 1]$  is a solution of IVP (3.0.2). Then  $t \rightarrow t^{n-\alpha}x(t)$  is continuous on  $(0, 1]$  by defining  $t^{n-\alpha}x(t)|_{t=0} = \lim_{t \rightarrow 0^+} t^{n-\alpha}x(t)$  and  $\|x\| = r < +\infty$ . Since  $F_2$  is a **II-Carathéodory function**, we know

$$M_r = \sup_{t \in (0, 1]} |F_2(t, x(t))| = \sup_{t \in (0, 1]} |F_2(t, t^{\alpha-n} t^{n-\alpha} x(t))| \leq \sup_{t \in (0, 1], |x| \leq r} |F_2(t, t^{\alpha-n} x)| \leq +\infty.$$

By  $\frac{w}{s} = u$ , we get

$$\begin{aligned} \lim_{s \rightarrow 0^+} \int_0^s (s-w)^{n-\alpha-1} x(w) dw &= \lim_{s \rightarrow 0^+} \int_0^s (s-w)^{n-\alpha-1} w^{\alpha-n} w^{n-\alpha} x(w) dw \\ &= \lim_{s \rightarrow 0^+} \xi^{n-\alpha} x(\xi) \int_0^s (s-w)^{n-\alpha-1} w^{\alpha-n} dw \text{ by mean value theorem of integral, } \xi \in (0, s) \\ &= \lim_{s \rightarrow 0^+} \xi^{n-\alpha} x(\xi) \int_0^1 (1-u)^{n-\alpha-1} u^{\alpha-n} du = \frac{\eta_n}{\Gamma(\alpha-n+1)} \mathbf{B}(n-\alpha, \alpha-n+1). \end{aligned} \tag{3.2.2}$$

From (A2), we have similarly to Lemma 3.2.1 that

$$t^{n-\alpha} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, x(s)) ds \right| \leq M_r t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds \leq M_r t^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.$$

So  $t \rightarrow t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, x(s)) ds$  is continuous on  $[0, 1]$  and

$$\lim_{t \rightarrow 0^+} t^{n-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, x(s)) ds = 0. \tag{3.2.3}$$

We have from  ${}^{RL}D_{0+}^\alpha x(t) = p_2(t)F_2(t, x(t))$ , a.e.,  $t \in (0, 1)$  that

$$\begin{aligned}
 & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, x(s)) ds = I_{0+}^\alpha p_2(t) F_2(t, x(t)) = I_{0+}^\alpha {}^{RL}D_{0+}^\alpha x(t) \\
 & = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[ \frac{1}{\Gamma(n-\alpha)} \left( \int_0^s (s-w)^{n-\alpha-1} x(w) dw \right)^{(n)} \right] ds \\
 & = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d \left( \int_0^s \frac{(s-w)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(w) dw \right)^{(n-1)} \\
 & = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d ({}^{RL}D_{0+}^{\alpha-1} x(s)) \\
 & = \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} ({}^{RL}D_{0+}^{\alpha-1} x(s)) \Big|_0^t + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left( \int_0^s (s-w)^{n-\alpha-1} x(w) dw \right)^{(n-1)} ds \\
 & = \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left( \int_0^s (s-w)^{n-\alpha-1} x(w) dw \right)^{(n-1)} ds - \frac{\eta_1}{\Gamma(\alpha)} t^{\alpha-1} \\
 & = \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} \left( \int_0^s (s-w)^{n-\alpha-1} x(w) dw \right)^{(n-2)} ds - \frac{\eta_1}{\Gamma(\alpha)} t^{\alpha-1} - \frac{\eta_2}{\Gamma(\alpha-1)} t^{\alpha-2} \\
 & = \dots \\
 & = \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))} \int_0^t (t-s)^{\alpha-n} \left( \int_0^s (s-w)^{n-\alpha-1} x(w) dw \right)' ds - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
 & = \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-2))} \left[ \int_0^t (t-s)^{\alpha-n+1} \left( \int_0^s (s-w)^{n-\alpha-1} x(w) dw \right)' ds \right]' - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
 & = \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-2))} \left[ (t-s)^{\alpha-n+1} \left( \int_0^s (s-w)^{n-\alpha-1} x(w) dw \right) \Big|_0^t \right. \\
 & \quad \left. + (\alpha-n+1) \int_0^t (t-s)^{\alpha-n} \int_0^s (s-w)^{n-\alpha-1} x(w) dw ds \right]' - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
 & \text{by using (3.1.10)} \\
 & = \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))} \left[ \int_0^t \int_u^t (t-s)^{\alpha-n} (s-w)^{n-\alpha-1} ds x(w) dw \right]' - \frac{\eta_n}{\Gamma(\alpha-n+1)} t^{\alpha-n} - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
 & = \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))} \left[ \int_0^t \int_0^1 (1-w)^{\alpha-n} w^{n-\alpha-1} dw x(w) dw \right]' - \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} \\
 & = x(t) - \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v}.
 \end{aligned}$$

Then

$$x(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, x(s)) ds.$$

Then  $x \in C_{n-\alpha}[0, 1]$  is a solution of (3.2.1).

On the other hand, suppose that  $x \in C_{n-\alpha}[0, 1]$  is a solution of IVP (3.2.1). Then

$${}^{RL}D_{0+}^{\alpha-\nu} x(t) = \sum_{v=1}^{\nu} \frac{\eta_v}{\Gamma(\nu-v+1)} t^{\nu-v} + \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} p_2(s) F_2(s, x(s)) ds.$$

So  $\lim_{t \rightarrow 0^+} t^{n-\alpha} x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}$  and  ${}^{RL}D_{0+}^{\alpha-\nu} x(0) = \eta_\nu, \nu \in \mathbb{N}[1, n-1]$ . Furthermore, we have

$$\begin{aligned} {}^{RL}D_{0+}^{\alpha} x(t) &= \frac{1}{\Gamma(n-\alpha)} \left( \int_0^t (t-s)^{n-\alpha-1} x(s) ds \right)^{(n)} \\ &= \frac{1}{\Gamma(n-\alpha)} \left( \int_0^t (t-s)^{n-\alpha-1} \left( \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} s^{\alpha-v} + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} p_2(u) F_2(u, x(u)) du \right) ds \right)^{(n)} \\ &= \frac{\left( \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} \int_0^t (t-s)^{n-\alpha-1} s^{\alpha-v} ds + \int_0^t (t-s)^{n-\alpha-1} \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} p_2(u) F_2(u, x(u)) du ds \right)^{(n)}}{\Gamma(n-\alpha)} \\ &= \frac{1}{\Gamma(n-\alpha)} \left( \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{n-v} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw \right. \\ &\quad \left. + \int_0^t \int_u^t (t-s)^{n-\alpha-1} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} ds p_2(u) F_2(u, x(u)) du \right)^{(n)} \\ &= \frac{1}{\Gamma(n-\alpha)} \left( \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{n-v} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw \right. \\ &\quad \left. + \int_0^t (t-u)^{n-1} \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw p_2(u) F_2(u, x(u)) du \right)^{(n)} = p_2(t) F_2(t, x(t)). \end{aligned}$$

So  $x \in C_{n-\alpha}[0, 1]$  is a solution of IVP (3.0.2). The proof is completed.  $\square$

**Theorem 3.2.2** *Suppose that (A2), (H2) hold. Then*

(i) *IVP (3.0.2) has a unique solution.*

(ii) *If there exist constants  $k > -1, l \leq 0$  with  $l > \max\{-\alpha, -n-k\}$  and  $M \geq 0$  such that  $|F(t)| \leq Mt^k(1-t)^l$  for all  $t \in (0, 1)$ , then following special problem*

$$\begin{cases} {}^{RL}D_{0+}^{\alpha} x(t) = F(t), t \in (0, 1], \\ \lim_{t \rightarrow 0^+} t^{n-\alpha} x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}, \\ \lim_{t \rightarrow 0^+} {}^{RL}D_{0+}^{\alpha-j} x(t) = \eta_j, j \in \mathbb{N}[1, n-1] \end{cases} \quad (3.2.4)$$

has a unique solution

$$x(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds, \quad t \in (0, 1]. \quad (3.2.5)$$

**Proof:** (i) From Lemmas 3.2.1, 3.2.2 and 3.2.3, and Theorem 3.1.1, we see that IVP (3.0.2) has a unique solution.

(ii) From the assumptions mentioned and (i), we know (3.2.4) has a unique solution. We get

$$x(t) \equiv \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds$$

by replacing  $p_2(t)F_2(t, x(t))$  with  $F(t)$ . Then we get  $x$  satisfies (3.2.5). The proof is completed.  $\square$

### 3.3 Existence and uniqueness of solutions of (3.0.3)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v}, \quad t \in (1, e],$$

$$\phi_i(t) = \phi_0(t) + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_{i-1}(s)) \frac{ds}{s}, \quad t \in (1, e], i = 1, 2, \dots$$

**Lemma 3.3.1** Suppose that (A3), (H3) hold. Then  $\phi_i \in LC_{n-\alpha}(1, e]$ .

**Proof:** In fact, we have  $\phi_0 \in LC_{n-\alpha}[1, e]$ . Since  $F_3$  is a **III-Carathéodory function**, we have

$$\begin{aligned} M_0 &= \sup_{t \in (1, e]} |F_3(t, x(t))| = \sup_{t \in (1, e]} |F_3(t, (\log t)^{\alpha-n} (\log t)^{n-\alpha} x(t))| \\ &\leq \sup_{t \in (1, e], |x| \leq \|\phi_0\|} |F_3(t, (\log t)^{\alpha-n} x)| \leq +\infty. \end{aligned}$$

and

$$\begin{aligned} &(\log t)^{n-\alpha} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_0(s)) \frac{ds}{s} \right| \\ &\leq M_0 (\log t)^{n-\alpha} \int_1^t (\log \frac{t}{s})^{\alpha-1} (\log s)^k (1 - \log s)^l \frac{ds}{s} \\ &\leq M_0 (\log t)^{n-\alpha} \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k \frac{ds}{s} \\ &= M_0 (\log t)^{n+k+l} \mathbf{B}(\alpha+l, k+1) \rightarrow 0 \text{ as } t \rightarrow 0^+, \end{aligned}$$

we know that

$$t \rightarrow \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_0(s)) \frac{ds}{s}$$

is continuous on  $(1, e]$  and  $\lim_{t \rightarrow 0^+} (\log t)^{n-\alpha} \phi_1(t)$  exists. Then  $\phi_1 \in LC_{n-\alpha}[1, e]$ . By mathematical induction method, we can show  $\phi_i \in LC_{n-\alpha}[1, e]$ .  $\square$

**Lemma 3.3.2** *Suppose that (A3), (H3) hold. Then  $\{t \rightarrow (\log t)^{n-\alpha} \phi_i(t)\}$  is convergent uniformly on  $[1, e]$ .*

**Proof:** In fact we have for  $t \in [1, e]$  similarly to Lemma 3.3.1

$$\begin{aligned} (\log t)^{n-\alpha} |\phi_1(t) - \phi_0(t)| &= \frac{1}{\Gamma(\alpha)} (\log t)^{n-\alpha} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_0(s)) \frac{ds}{s} \right| \\ &\leq M_0 (\log t)^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

So

$$\begin{aligned} (\log t)^{n-\alpha} |\phi_2(t) - \phi_1(t)| &= \frac{1}{\Gamma(\alpha)} (\log t)^{n-\alpha} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) [F_3(s, \phi_1(s)) - F_3(s, \phi_0(s))] \frac{ds}{s} \right| \\ &\leq L_3 \frac{1}{\Gamma(\alpha)} (\log t)^{n-\alpha} \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k |\phi_1(s) - \phi_0(s)| \frac{ds}{s} \\ &\leq L_3 \frac{1}{\Gamma(\alpha)} (\log t)^{n-\alpha} \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k (\log s)^{\alpha-n} M_0 (\log s)^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \frac{ds}{s} \\ &\leq M_0 L_3 \frac{1}{\Gamma(\alpha)} (\log t)^{n-\alpha} \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^{\alpha+2k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \frac{ds}{s} \\ &= M_0 L_3 (\log t)^{\alpha+n+2k+2l} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

Similarly by the mathematical induction method, we get for every  $i = 1, 2, \dots$  that

$$\begin{aligned} &(\log t)^{n-\alpha} |\phi_i(t) - \phi_{i-1}(t)| \\ &\leq M_0 L_3^{i-1} (\log t)^{(i-1)\alpha+n+ik+il} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)} \\ &\leq M_0 L_3^{i-1} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)}, t \in (1, e]. \end{aligned}$$

Similarly we can prove that

$$\sum_{i=1}^{+\infty} u_i = \sum_{i=1}^{+\infty} M_0 L_3^{i-1} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)}$$

is convergent. Hence

$$(\log t)^{n-\alpha} \phi_0(t) + (\log t)^{n-\alpha} [\phi_1(t) - \phi_0(t)] + \dots + (\log t)^{n-\alpha} [\phi_i(t) - \phi_{i-1}(t)] + \dots, t \in (1, e]$$

is uniformly convergent. Then  $\{t \rightarrow (\log t)^{n-\alpha} \phi_i(t)\}$  is convergent uniformly on  $(1, e]$ .  $\square$

**Lemma 3.3.3** Suppose that **(A3)**, **(H3)** hold. Then  $\phi(t) = (\log t)^{\alpha-n} \lim_{i \rightarrow +\infty} (\log t)^{n-\alpha} \phi_i(t)$  defined on  $(1, e)$  is a unique continuous solution of the integral equation

$$x(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, x(s)) \frac{ds}{s}, t \in (1, e). \quad (3.3.1)$$

**Proof:** By  $\lim_{i \rightarrow +\infty} (\log t)^{n-\alpha} \phi_i(t) = (\log t)^{n-\alpha} \phi(t)$  and the uniform convergence, we see  $\phi(t)$  is continuous on  $(1, e]$ . From

$$\begin{aligned} & (\log t)^{n-\alpha} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_p(s)) \frac{ds}{s} - \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_q(s)) \frac{ds}{s} \right| \\ & \leq L_3 \|\phi_p - \phi_q\| (\log t)^{n-\alpha} \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k (\log s)^{\alpha-n} \frac{ds}{s} \\ & \leq L_3 \|\phi_p - \phi_q\| (\log t)^{n-\alpha} \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^{\alpha+k-n} \frac{ds}{s} \\ & \leq L_3 \|\phi_p - \phi_q\| (\log t)^{\alpha+k+l} \mathbf{B}(\alpha+l, \alpha+k-n+1) \\ & \leq L_3 \|\phi_p - \phi_q\| \mathbf{B}(\alpha+l, \alpha+k-n+1) \rightarrow 0 \text{ uniformly as } p, q \rightarrow +\infty, \end{aligned}$$

we know that

$$\begin{aligned} \phi(t) &= (\log t)^{\alpha-n} \lim_{i \rightarrow +\infty} (\log t)^{n-\alpha} \phi_i(t) \\ &= \lim_{i \rightarrow +\infty} \left[ \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_{i-1}(s)) \frac{ds}{s} \right] \\ &= \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \lim_{i \rightarrow +\infty} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi_{i-1}(s)) \frac{ds}{s} \\ &= \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \phi(s)) \frac{ds}{s}. \end{aligned}$$

Then  $\phi$  is a continuous solution of (3.3.1) defined on  $(1, e]$ .

Suppose that  $\psi$  defined on  $(1, e]$  is also a solution of (3.3.1). Then

$$\psi(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} p_3(s) F_3(s, \psi(s)) \frac{ds}{s}, t \in (1, e].$$

We need to prove that  $\phi(t) \equiv \psi(t)$  on  $(1, e]$ . Since  $F_3$  is a **III-Carathéodory function**, we have

$$\begin{aligned} M_0 &= \sup_{t \in (1, e]} |F_2(t, \psi(t))| = \sup_{t \in (1, e]} |F_3(t, (\log t)^{\alpha-n} (\log t)^{n-\alpha} \psi(t))| \\ &\leq \sup_{t \in (1, e], |x| \leq \|\psi\|} |F_3(t, (\log t)^{\alpha-n} x)| \leq +\infty. \end{aligned}$$

Then

$$\begin{aligned} (\log t)^{n-\alpha} |\psi(t) - \phi_0(t)| &= (\log t)^{n-\alpha} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) F_3(s, \psi(s)) \frac{ds}{s} \right| \\ &\leq M_0 (\log t)^{n+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} (\log t)^{n-\alpha} |\psi(t) - \phi_1(t)| &= (\log t)^{n-\alpha} \frac{1}{\Gamma(\alpha)} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) [F_3(s, \psi(s)) - F_3(s, \phi_0(s))] \frac{ds}{s} \right| \\ &\leq M_0 L_3 (\log t)^{\alpha+n+2k+2l} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

By mathematical induction method, we can get that

$$\begin{aligned} &(\log t)^{1-\alpha} |\psi(t) - \phi_{i-1}(t)| \\ &= (\log t)^{n-\alpha} \frac{1}{\Gamma(\alpha)} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_3(s) [F_3(s, \psi(s)) - F_3(s, \phi_{i-2}(s))] ds \right| \\ &\leq M_0 L_3^{i-1} (\log t)^{(i-1)\alpha+n+ik+il} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)} \\ &\leq M_0 L_3^{i-1} \prod_{j=1}^{i-1} \frac{\mathbf{B}(\alpha+l, (j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)}, t \in (1, e], \text{ for all } i = 1, 2, \dots. \end{aligned}$$

Similarly we have  $\lim_{i \rightarrow +\infty} (\log t)^{n-\alpha} \phi_i(t) = (\log t)^{n-\alpha} \psi(t)$  uniformly on  $(1, e]$ . Then  $\phi(t) \equiv \psi(t)$  on  $(1, e]$ . Then (3.2.1) has a unique solution  $\phi$ . The proof is completed.  $\square$

**Theorem 3.3.1** *Suppose that (A3), (H3) hold. Then  $x$  is a solution of IVP (3.0.3) if and only if  $x \in LC_{n-\alpha}(1, e]$  is a solution of the integral equation (3.3.1).*

**Proof:** Suppose that  $x$  is a solution of IVP (3.0.3). Then  $t \rightarrow (\log t)^{n-\alpha} x(t)$  is continuous on  $[1, e]$  by defining  $(\log t)^{n-\alpha} x(t)|_{t=1} = \lim_{t \rightarrow 1^+} (\log t)^{n-\alpha} x(t)$  and  $\|x\| = r < +\infty$ . Since  $F_3$  is a **III-Carathéodory function**, we know

$$\begin{aligned} M_r &= \sup_{t \in (1, e]} |F_2(t, x(t))| = \sup_{t \in (1, e]} |F_3(t, (\log t)^{\alpha-n} (\log t)^{n-\alpha} x(t))| \\ &\leq \sup_{t \in (1, e], |x| \leq r} |F_3(t, (\log t)^{\alpha-n} x)| \leq +\infty. \end{aligned}$$

So

$$\begin{aligned}
\lim_{s \rightarrow 1^+} \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} &= \lim_{s \rightarrow 1^+} \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} (\log w)^{\alpha-n} (\log w)^{n-\alpha} x(w) \frac{dw}{w} \\
&= \lim_{s \rightarrow 1^+} (\log \xi)^{n-\alpha} x(\xi) \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} (\log w)^{\alpha-n} \frac{dw}{w} \\
&\quad \text{by mean value theorem of integral, } \xi \in (1, s) \\
&= \lim_{s \rightarrow 1^+} (\log \xi)^{n-\alpha} x(\xi) \int_0^1 (1-u)^{n-\alpha-1} u^{\alpha-n} du \text{ by } \frac{\log w}{\log s} = u \\
&= \frac{\eta_n}{\Gamma(\alpha-n+1)} \mathbf{B}(n-\alpha, \alpha-n+1).
\end{aligned}$$

and for  $v \in \mathbb{N}[1, n-1]$  we have

$$\begin{aligned}
\lim_{t \rightarrow 1^+} \left(s \frac{d}{ds}\right)^{n-v} \left(\int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w}\right) &= \Gamma(n-v-(\alpha-v)) \lim_{t \rightarrow 1^+} {}^{RLH}D_{0^+}^{\alpha-v} x(t) \\
&= \Gamma(n-\alpha) \eta_v.
\end{aligned}$$

From (A3), we have

$$\begin{aligned}
& \left| (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, x(s)) \frac{ds}{s} \right| \\
& \leq (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} M_r (\log s)^k (1-\log s)^l \frac{ds}{s} \\
& \leq M_r (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l-1} (\log s)^k \frac{ds}{s} = M_r (\log t)^{n+k+l} \mathbf{B}(\alpha+l, k+1).
\end{aligned}$$

So  $t \rightarrow (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, x(s)) \frac{ds}{s}$  is defined on  $(1, e]$  and

$$\lim_{t \rightarrow 1^+} (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, x(s)) \frac{ds}{s} = 0. \quad (3.3.2)$$

Then  $t \rightarrow (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, x(s)) \frac{ds}{s}$  is continuous on  $[1, e]$  by defining

$$\left. (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, x(s)) \frac{ds}{s} \right|_{t=1} = 0. \quad (3.3.3)$$

We have  ${}^H I_{1+}^{\alpha} {}^{RLH} D_{1+}^{\alpha} x(t) = {}^H I_{1+}^{\alpha} p_3(t) F_3(t, x(t))$ . So

$$\begin{aligned}
& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, x(s)) \frac{ds}{s} = {}^H I_{1+}^{\alpha} [B(t)x(t) + G(t)] = {}^H I_{1+}^{\alpha} {}^{RLH} D_{1+}^{\alpha} x(t) \\
& = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(s \frac{d}{ds}\right)^n \left( \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right) \frac{ds}{s} \\
& = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} d \left[ \left(s \frac{d}{ds}\right)^{n-1} \left( \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right) \right] \\
& = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left[ \left(\log \frac{t}{s}\right)^{\alpha-1} \left(s \frac{d}{ds}\right)^{n-1} \left( \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right) \Big|_1^t \right. \\
& \quad \left. + (\alpha - 1) \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-2} \left(s \frac{d}{ds}\right)^{n-1} \left( \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right) \frac{ds}{s} \right] \\
& = -\frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} (\log t)^{\alpha-1} \lim_{t \rightarrow 1^+} \left(s \frac{d}{ds}\right)^{n-1} \left( \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right) \\
& \quad + \frac{1}{\Gamma(\alpha-1)} \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-2} \left(s \frac{d}{ds}\right)^{n-1} \left( \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right) \frac{ds}{s} \\
& = -\frac{\eta_1}{\Gamma(\alpha)} (\log t)^{\alpha-1} \quad + \frac{1}{\Gamma(\alpha-1)} \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-2} \left(s \frac{d}{ds}\right)^{n-1} \left( \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right) \frac{ds}{s} \\
& = \dots\dots\dots \\
& = -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha-n+1)} \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-n} \left( \int_1^s \left(\log \frac{s}{w}\right)^{-\alpha} x(w) \frac{dw}{w} \right)' ds \\
& = -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha-n+2)} \frac{1}{\Gamma(n-\alpha)} t \left[ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-n+1} \left( \int_1^s \left(\log \frac{s}{w}\right)^{-\alpha} x(w) \frac{dw}{w} \right)' ds \right]' \\
& = -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} t \left[ \left(\log \frac{t}{s}\right)^{\alpha-n+1} \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \Big|_1^t \right. \\
& \quad \left. + (\alpha - n + 1) \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-n} \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \frac{ds}{s} \right]' \\
& = -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} t \left[ \lim_{t \rightarrow 1^+} (\log t)^{\alpha-n+1} \int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1} x(w) \frac{dw}{w} \right. \\
& \quad \left. + (\alpha - n + 1) \int_1^t \int_u^t \left(\log \frac{t}{s}\right)^{\alpha-n} \left(\log \frac{s}{w}\right)^{n-\alpha-1} \frac{ds}{s} x(w) \frac{dw}{w} \right]'
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} t \left[ \frac{\eta_n}{\Gamma(\alpha-n+1)} \mathbf{B}(n-\alpha, \alpha-n+1) (\log t)^{\alpha-n+1} \right. \\
&\quad \left. + (\alpha-n+1) \int_1^t \int_0^1 (1-u)^{\alpha-n} u^{n-\alpha-1} du x(w) \frac{dw}{w} \right]' = x(t) - \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v}.
\end{aligned}$$

Then  $x \in LC_{n-\alpha}[1, e]$  is a solution of (3.3.1).

On the other hand, suppose that  $x$  is a solution of (3.3.1). Together with Lemmas 3.3.1, 3.3.2, 3.3.3 and the result at the beginning of the proof, it follows that  $\lim_{t \rightarrow 1^+} (\log t)^{n-\alpha} x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}$  and  ${}^{RLH}D_{1^+}^{\alpha-\nu} x(1) = \eta_\nu, \nu \in \mathbb{N}[1, n-1]$ . Furthermore, we have

$$\begin{aligned}
{}^{RLH}D_{1^+}^{\alpha} x(t) &= \frac{1}{\Gamma(n-\alpha)} \left( t \frac{d}{dt} \right)^n \left( \int_1^t (\log \frac{t}{s})^{n-\alpha-1} x(s) \frac{ds}{s} \right) \\
&= \frac{1}{\Gamma(n-\alpha)} \left( t \frac{d}{dt} \right)^n \left[ \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log s)^{\alpha-v} \right. \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\alpha)} \int_1^s (\log \frac{s}{w})^{\alpha-1} p_3(w) F_3(w, x(w)) \frac{dw}{w} \right) \frac{ds}{s} \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} \left( t \frac{d}{dt} \right)^n \int_1^t (\log \frac{t}{s})^{n-\alpha-1} (\log s)^{\alpha-v} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left( t \frac{d}{dt} \right)^n \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \int_1^s (\log \frac{s}{w})^{\alpha-1} p_3(w) F_3(w, x(w)) \frac{dw}{w} \frac{ds}{s} \\
&= \frac{1}{\Gamma(n-\alpha)} \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} \left( t \frac{d}{dt} \right)^n (\log t)^{n-v} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw \\
&\quad + \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left( t \frac{d}{dt} \right)^n \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \int_1^s (\log \frac{s}{w})^{\alpha-1} p_3(w) F_3(w, x(w)) \frac{dw}{w} \frac{ds}{s} \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left( t \frac{d}{dt} \right)^n \int_1^t \int_u^t (\log \frac{t}{s})^{n-\alpha-1} (\log \frac{s}{w})^{\alpha-1} \frac{ds}{s} p_3(w) F_3(w, x(w)) \frac{dw}{w} \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left( t \frac{d}{dt} \right)^n \int_1^t (\log \frac{t}{w})^{n-1} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-1} dw p_3(w) F_3(w, x(w)) \frac{dw}{w} \\
&= \frac{1}{\Gamma(n)} \left( t \frac{d}{dt} \right)^n \int_1^t (\log \frac{t}{w})^{n-1} p_3(w) F_3(w, x(w)) \frac{dw}{w} = p_3(t) F_3(t, x(t)).
\end{aligned}$$

So  $x \in LC_{n-\alpha}(1, e]$  is a solution of IVP (3.0.3). The proof is completed.  $\square$

**Theorem 3.3.2** *Suppose that (A3), (H3) hold. Then*

(i) *IVP (3.0.3) has a unique solution.*

(ii) If there exist constants  $k > -1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -n - k\}$  and  $M \geq 0$  such that  $|G(t)| \leq M(\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ , then following special problem

$$\begin{cases} {}^{RLH}D_{1^+}^\alpha x(t) = G(t), t \in (1, e], \\ \lim_{t \rightarrow 1^+} (\log t)^{n-\alpha} x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}, \\ \lim_{t \rightarrow 1^+} {}^{RLH}D_{1^+}^{\alpha-j} x(t) = \eta_j, j \in \mathbb{N}[1, n-1] \end{cases} \quad (3.3.4)$$

has a unique solution

$$x(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s}, t \in (1, e]. \quad (3.3.5)$$

**Proof:** (i) From Lemma 3.3.1, 3.3.2 and 3.3.3, IVP (3.0.3) has a unique solution.

(ii) From the assumption mentioned and (i), IVP (3.3.4) has a unique solution. We get

$$x(t) \equiv \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} G(s) \frac{ds}{s}.$$

By replacing  $p_3(t)F_3(t, x(t))$  with  $G(t)$ . Then  $x$  satisfies (3.3.5). The proof is completed.  $\square$

### 3.4 Existence and uniqueness of solutions of (3.0.4)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j, t \in [1, e],$$

$$\phi_i(t) = \phi_0(t) + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \phi_{i-1}(s)) \frac{ds}{s}, t \in [1, e], i = 1, 2, \dots.$$

**Lemma 3.4.1** Suppose that (A4), (H4) hold. Then  $\phi_i \in C[1, e]$ .

**Proof:** On sees  $\phi_0 \in C[1, e]$ . Since  $F_4$  is a **IV-Carathéodory function**, we know

$$M_0 = \sup_{t \in (1, e]} |F_4(t, \phi_0(t))| = \sup_{t \in (1, e]} \leq \sup_{t \in (1, e], |x| \leq \|\phi_0\|} |F_4(t, x)| \leq +\infty.$$

By

$$\begin{aligned} & \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \phi_0(s)) \frac{ds}{s} \right| \\ & \leq \int_1^t (\log \frac{t}{s})^{\alpha-1} M_0 (\log s)^k (1 - \log s)^l \frac{ds}{s} \\ & \leq M_0 \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k \frac{ds}{s} \\ & = M_0 (\log t)^{\alpha+k+l} \mathbf{B}(\alpha+l, k+1) \rightarrow 0 \text{ as } t \rightarrow 1^+, \end{aligned}$$

we get that  $\lim_{t \rightarrow 1^+} \phi_1(t)$  exists and  $\phi_1$  is continuous on  $[1, e]$ . Then  $\phi_1 \in C[1, e]$ . By mathematical induction method, we see that  $\phi_n \in C(1, e]$ .  $\square$

**Lemma 3.4.2** *Suppose that (A4), (H4) hold. Then  $\phi_i$  is convergent uniformly on  $[1, e]$ .*

**Proof:** Let  $M_0$  be defined in the proof of Lemma 3.4.1. In fact we have for  $t \in [1, e]$  that

$$|\phi_1(t) - \phi_0(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \phi_0(s)) \frac{ds}{s} \right| \leq M_0 (\log t)^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.$$

So

$$\begin{aligned} |\phi_2(t) - \phi_1(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) [F_4(s, \phi_1(s)) - F_4(s, \phi_0(s))] \frac{ds}{s} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} (\log s)^k (1 - \log s)^l L_4 |\phi_1(s) - \phi_0(s)| \frac{ds}{s} \\ &\leq M_0 L_4 \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k (\log s)^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \frac{ds}{s} \\ &= M_0 L_4 (\log t)^{2\alpha+2k+2l} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

Similarly by the mathematical induction method, we get for every  $i = 1, 2, \dots$  that

$$\begin{aligned} |\phi_i(t) - \phi_{i-1}(t)| &\leq M_0 L_4^{i-1} (\log t)^{i\alpha+ik+il} \prod_{j=1}^i \frac{\mathbf{B}(\alpha+l, (i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)} \\ &\leq M_0 L_4^{i-1} \prod_{j=1}^i \frac{\mathbf{B}(\alpha+l, (i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)}, t \in (1, e]. \end{aligned}$$

Similarly we can prove that

$$\sum_{i=1}^{+\infty} u_i = \sum_{i=1}^{+\infty} M_0 L_4^{i-1} \prod_{j=1}^i \frac{\mathbf{B}(\alpha+l, (i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)}$$

is convergent. Hence

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \dots + [\phi_i(t) - \phi_{i-1}(t)] + \dots, t \in [1, e]$$

is uniformly convergent. Then  $\{\phi_i(t)\}$  is convergent uniformly on  $[1, e]$ .  $\square$

**Lemma 3.4.3** *Suppose that (A4), (H4) hold. Then  $\phi(t) = \lim_{i \rightarrow +\infty} \phi_i(t)$  defined on  $[1, e]$  is a unique continuous solution of the integral equation*

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s}. \quad (3.4.1)$$

**Proof:** By  $\lim_{i \rightarrow +\infty} \phi_i(t) = \phi(t)$  and the uniform convergence, we see  $\phi(t)$  is continuous on  $[1, e]$ . From

$$\begin{aligned} & \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \phi_p(s)) \frac{ds}{s} - \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \phi_q(s)) \frac{ds}{s} \right| \\ & \leq L_4 \|\phi_p - \phi_q\| \int_1^t (\log \frac{t}{s})^{\alpha-1} (\log s)^k (1 - \log s)^l (\log s)^{\alpha-n} \frac{ds}{s} \\ & \leq L_4 \|\phi_p - \phi_q\| \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^{k+\alpha-n} \frac{ds}{s} \\ & \leq L_4 \|\phi_p - \phi_q\| (\log t)^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, \alpha+k-n+1)}{\Gamma(\alpha)} \rightarrow 0 \text{ uniformly as } p, q \rightarrow +\infty, \end{aligned}$$

we know that

$$\begin{aligned} \phi(t) &= \lim_{i \rightarrow \infty} \phi_i(t) = \lim_{i \rightarrow +\infty} \left[ \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \phi_{i-1}(s)) \frac{ds}{s} \right] \\ &= \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \phi(s)) \frac{ds}{s}. \end{aligned}$$

Then  $\phi$  is a continuous solution of (3.4.1) defined on  $[1, e]$ .

Suppose that  $\psi$  defined on  $[1, e]$  is also a solution of (3.4.1). Then

$$\psi(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} p_4(s) F_4(s, \psi(s)) \frac{ds}{s}, t \in (1, e].$$

We need to prove that  $\phi(t) \equiv \psi(t)$  on  $(0, 1]$ . Since  $F_4$  is a **IV-Carathéodory function**, we know

$$M_0 = \sup_{t \in (1, e]} |F_4(t, \psi(t))| = \sup_{t \in (1, e]} \leq \sup_{t \in (1, e], |x| \leq \|\psi\|} |F_4(t, x)| \leq +\infty.$$

Now we have

$$\begin{aligned} |\psi(t) - \phi_0(t)| &= \frac{1}{\Gamma(\alpha)} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, \psi(s)) \frac{ds}{s} \right| \\ &\leq M_0 (\log t)^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

Furthermore, we have similarly to the proof of Lemma 3.4.2

$$\begin{aligned} |\psi(t) - \phi_1(t)| &= \frac{1}{\Gamma(\alpha)} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) [F_4(s, \psi(s)) - F_4(s, \phi_0(s))] \frac{ds}{s} \right| \\ &\leq M_0 L_4 (\log t)^{2\alpha+2k+2l} \frac{\mathbf{B}(\alpha+l, \alpha+2k+l+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

By mathematical induction method, we can get that

$$\begin{aligned} |\psi(t) - \phi_{i-1}(t)| &= \frac{1}{\Gamma(\alpha)} \left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) [F_4(s, \psi(s)) - F_4(s, \phi_{i-2}(s))] \frac{ds}{s} \right| \\ &M_0 L_4^{i-1} (\log t)^{i\alpha+ik+il} \prod_{j=1}^i \frac{\mathbf{B}(\alpha+l, (i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)} \\ &\leq M_0 L_4^{i-1} \prod_{j=1}^i \frac{\mathbf{B}(\alpha+l, (i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)}, t \in (1, e]. \end{aligned}$$

Similarly we have  $\lim_{i \rightarrow +\infty} \phi_i(t) = \psi(t)$  uniformly on  $[1, e]$ . Then  $\phi(t) \equiv \psi(t)$  on  $[1, e]$ . Then (3.4.1) has a unique solution  $\phi$ . The proof is completed.  $\square$

**Theorem 3.4.1** *Suppose that (A4), (H4) hold. Then  $x \in C[1, e]$  is a solution of IVP (3.0.4) if and only if  $x \in C[1, e]$  is a solution of the integral equation (3.4.1).*

**Proof:** Suppose that  $x \in C(1, e]$  is a solution of IVP (3.0.4). Then  $x$  is continuous on  $[1, e]$  by defining  $x(t)|_{t=1} = \lim_{t \rightarrow 1^+} x(t)$  and  $\|x\| = r < +\infty$ . Since  $F_4$  is a **IV-Carathéodory function**, we know

$$M_r = \sup_{t \in (1, e]} |F_4(t, x(t))| = \sup_{t \in (1, e]} \leq \sup_{t \in (1, e], |x| \leq r} |F_4(t, x)| \leq +\infty.$$

One can see that

$$\begin{aligned} &\int_1^t (\log \frac{t}{s})^{\alpha-1} (\log s)^k (1 - \log s)^l \frac{ds}{s} \\ &\leq \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k \frac{ds}{s} \text{ by } \frac{\log s}{\log t} = u \\ &= (\log t)^{\alpha+k+l} \int_0^1 (1-u)^{\alpha+l-1} u^k du \\ &\leq (\log t)^{\alpha+k+l} \int_0^1 (1-u)^{\alpha+l-1} u^k du \\ &= (\log t)^{\alpha+k+l} \mathbf{B}(\alpha+l, k+1). \end{aligned}$$

From (A4), we have for  $t \in [1, e]$  that

$$\begin{aligned} &\left| \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s} \right| \\ &\leq \int_1^t (\log \frac{t}{s})^{\alpha-1} M_r (\log s)^k (1 - \log s)^l \frac{ds}{s} \\ &\leq M_r \int_1^t (\log \frac{t}{s})^{\alpha+l-1} (\log s)^k \frac{ds}{s} = M_r (\log t)^{\alpha+k+l} \mathbf{B}(\alpha+l, k+1). \end{aligned}$$

So  $t \rightarrow \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s}$  is defined on  $(1, e]$  and

$$\lim_{t \rightarrow 1^+} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s} = 0. \quad (3.4.2)$$

Furthermore, we have  $t \rightarrow \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s}$  is continuous on  $(1, e]$ . So

$$t \rightarrow \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s}$$

is continuous on  $[1, e]$  by defining

$$\int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s} \Big|_{t=1} = 0. \quad (3.4.3)$$

We have  ${}^H I_{1+}^\alpha {}^{CH} D_{1+}^\alpha x(t) = {}^H I_{1+}^\alpha p_4(t) F_4(t, x(t))$ . So

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s} = {}^H I_{1+}^\alpha p_4(t) F_4(t, x(t)) \\ &= {}^H I_{1+}^\alpha {}^{CH} D_{1+}^\alpha x(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \left[ \frac{1}{\Gamma(n-\alpha)} \int_1^s (\log \frac{s}{w})^{n-\alpha-1} \left( w \frac{d}{dw} \right)^n x(w) \frac{dw}{w} \right] \frac{ds}{s} \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t \int_u^t (\log \frac{t}{s})^{\alpha-1} (\log \frac{s}{w})^{n-\alpha-1} \frac{ds}{s} \left( w \frac{d}{dw} \right)^n x(w) \frac{dw}{w} \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{w})^{n-1} \int_0^1 (1-u)^{\alpha-1} u^{n-\alpha-1} du \left( w \frac{d}{dw} \right)^n x(w) \frac{dw}{w} \\ &= \frac{1}{(n-1)!} \int_1^t (\log \frac{t}{w})^{n-1} \left( w \frac{d}{dw} \right)^n x(w) \frac{dw}{w} = \int_1^t (\log \frac{t}{w})^{n-1} d \left[ \left( w \frac{d}{dw} \right)^{n-1} x(w) \right] \\ &= \frac{1}{(n-1)!} (\log \frac{t}{w})^{n-1} \left[ \left( w \frac{d}{dw} \right)^{n-1} x(w) \right] \Big|_1^t + \frac{1}{(n-2)!} \int_1^t (\log \frac{t}{w})^{n-2} \left[ \left( w \frac{d}{dw} \right)^{n-1} x(w) \right] \frac{dw}{w} \\ &= -\frac{\eta_{n-1}}{(n-1)!} (\log t)^{n-1} + \frac{1}{(n-2)!} \int_1^t (\log \frac{t}{w})^{n-2} \left[ \left( w \frac{d}{dw} \right)^{n-1} x(w) \right] \frac{dw}{w} \\ &= \dots \dots \dots \\ &= -\sum_{j=1}^{n-2} \frac{\eta_{n-j}}{(n-j)!} (\log t)^{n-j} + \int_1^t x'(w) dw = x(t) - \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j. \end{aligned}$$

So

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p_4(s) F_4(s, x(s)) \frac{ds}{s}.$$

Then  $x \in C[1, e]$  is a solution of (3.4.1).

On the other hand, if  $x \in C[1, e]$  is a solution of (3.4.1), we get

$${}^{CH}D_{1^+}^\nu x(t) = \sum_{j=\nu}^{n-1} \frac{\eta_j}{(j-\nu)!} (\log t)^{j-\nu} + \frac{1}{\Gamma(\alpha-\nu)} \int_1^t (\log \frac{t}{s})^{\alpha-\nu-1} p_4(s) F_4(s, x(s)) \frac{ds}{s}.$$

Similar to (3.4.2), we have  $\lim_{t \rightarrow 1^+} x(t) = \eta_0$  and  ${}^{CH}D_{1^+}^\nu x(1) = \eta_\nu, \nu \in \mathbb{N}[1, n-1]$ . Furthermore, we have for  $t \in (1, e)$

$$\begin{aligned} & \left| \int_1^t (\log \frac{t}{s})^{\alpha-n} p_4(s) F_4(s, x(s)) \frac{ds}{s} \right| \leq \int_1^t (\log \frac{t}{s})^{\alpha-n} M_r (\log s)^k (1 - \log s)^l \frac{ds}{s} \\ & \leq M_r (1 - \log t)^l \int_1^t (\log \frac{t}{s})^{\alpha-n} (\log s)^k \frac{ds}{s} \\ & = M_r (1 - \log t)^l (\log t)^{\alpha-n+k+1} \mathbf{B}(\alpha - n + 1, k + 1) \rightarrow 0 \text{ as } t \rightarrow 1^+. \end{aligned}$$

Then

$$\begin{aligned} {}^{CH}D_{1^+}^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( s \frac{d}{ds} \right)^n x(s) \frac{ds}{s} \\ &= \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( s \frac{d}{ds} \right)^n \left( \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log s)^j + \frac{1}{\Gamma(\alpha)} \int_1^s (\log \frac{s}{u})^{\alpha-1} [B(u)x(u) + G(u)] \frac{du}{u} \right) \frac{ds}{s} \\ &= \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( s \frac{d}{ds} \right)^n \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log s)^j \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( s \frac{d}{ds} \right)^n \left( \int_1^s (\log \frac{s}{u})^{\alpha-1} p_4(u) F_4(u, x(u)) \frac{du}{u} \right) \frac{ds}{s} \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( s \frac{d}{ds} \right)^n \left( \int_1^s (\log \frac{s}{u})^{\alpha-1} p_4(u) F_4(u, x(u)) \frac{du}{u} \right) \frac{ds}{s} \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( s \frac{d}{ds} \right)^{n-1} \left( \int_1^s (\log \frac{s}{u})^{\alpha-2} p_4(u) F_4(u, x(u)) \frac{du}{u} \right) \frac{ds}{s} \\ &= \dots \dots \dots \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \left( \int_1^s (\log \frac{s}{u})^{\alpha-n} p_4(u) F_4(u, x(u)) \frac{du}{u} \right)' ds \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha+1)} t \left[ \int_1^t (\log \frac{t}{s})^{n-\alpha} \left( \int_1^s (\log \frac{s}{u})^{\alpha-1} [B(u)x(u) + G(u)] \frac{du}{u} \right)' ds \right]' \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha+1)} t \left[ \left( \log \frac{t}{s} \right)^{n-\alpha} \int_1^s (\log \frac{s}{w})^{\alpha-n} p_4(w) F_4(w, x(w)) \frac{dw}{w} \right]'_1 \\ &\quad + (n - \alpha) \frac{1}{s} \int_1^t (\log \frac{t}{s})^{n-\alpha-1} \int_1^s (\log \frac{s}{w})^{\alpha-n} p_4(w) F_4(w, x(w)) \frac{dw}{w} ds \Big]' \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} t \left[ \int_1^t \int_u^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(\log \frac{s}{w}\right)^{\alpha-n} \frac{ds}{s} p_4(w) F_4(w, x(w)) \frac{dw}{w} \right]' \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} t \left[ \int_1^t \int_u^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(\log \frac{s}{w}\right)^{\alpha-n} \frac{ds}{s} p_4(w) F_4(w, x(w)) \frac{dw}{w} \right]' \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} t \left[ \int_1^t \int_0^1 (1-u)^{n-\alpha-1} u^{\alpha-n} du p_4(w) F_4(w, x(w)) \frac{dw}{w} \right]' = p_4(t) F_4(t, x(t)).
\end{aligned}$$

So  $x \in C[1, e]$  is a solution of IVP (3.0.4). The proof is completed.  $\square$

**Theorem 3.4.2** *Suppose that (A4), (H4) hold. Then*

(i) IVP (3.0.4) has a unique solution.

(ii) If there exist constants  $k > -\alpha + n - 1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -\alpha - k\}$  and  $M \geq 0$  such that  $|G(t)| \leq M(\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ , then following special problem

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = G(t), t \in (1, e], \\ \lim_{t \rightarrow 1^+} \left(t \frac{d}{dt}\right)^j x(t) = \eta_j, j \in \mathbb{N}[0, n-1] \end{cases} \quad (3.4.4)$$

has a unique solution

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} G(s) \frac{ds}{s}, t \in (1, e]. \quad (3.4.5)$$

**Proof:** (i) From Lemmas 3.4.1, 3.4.2 and 3.4.3, Theorem 3.4.1, IVP (3.0.4) has a unique solution.

(ii) From the assumption mentioned and (i), IVP (3.4.4) has a unique solution. We get

$$x(t) \equiv \sum_{j=0}^{n-1} \frac{\eta_j}{j!} (\log t)^j + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} G(s) \frac{ds}{s}$$

by replacing  $p_4(t)F_4(t, x(t))$  with  $G(t)$ . Then  $x$  satisfies (3.4.5). The proof is completed.  $\square$

## 4 Exact piecewise continuous solutions of LFDEs

In this section, we present exact piecewise continuous solutions of the following linear fractional differential equations (LFDEs), respectively:

$${}^C D_{0+}^\alpha x(t) = F_1(t), a.e., t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \quad (4.0.1)$$

$${}^{RL} D_{0+}^\alpha x(t) = F_2(t), a.e., t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \quad (4.0.2)$$

$${}^{RLH} D_{0+}^\alpha x(t) = F_3(t), a.e., t \in (s_i, s_{i+1}], i \in \mathbb{N}[0, m], \quad (4.0.3)$$

and

$${}^{CH}D_{0+}^{\alpha}x(t) = F_4(t), a.e., t \in (s_i, s_{i+1}], i \in \mathbb{N}[0, m], \quad (4.0.4)$$

where  $n - 1 < \alpha < n$ ,  $\lambda \in \mathbb{R}$ ,  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$  in (3.2.1) and (3.2.2) and  $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = e$  in (3.2.3) and (3.2.4). We say that  $x : (0, 1] \rightarrow \mathbb{R}$  is a piecewise solution of (4.0.1) (or (4.0.2)) if  $x \in P_m C(0, 1]$  (or  $P_m C_{n-\alpha}(0, 1]$ ) and satisfies (4.0.1) or (4.0.2). We say that  $x : (1, e] \rightarrow \mathbb{R}$  is a piecewise continuous solutions of (4.0.3) (or (4.0.4)) if  $x \in LP_m C_{n-\alpha}(1, e]$ , (or  $LP_m C(1, e]$ ) and  $x$  satisfies all equations in (4.0.3) (or (4.0.4)).

**(B1)** there exists constants  $k > -\alpha + n - 1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -\alpha - k\}$ ,  $M \geq 0$  such that  $|F_1(t)| \leq Mt^k(1-t)^l$  for all  $t \in (0, 1)$ .

**(B2)** there exists constants  $k > -1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -n - k\}$ ,  $M \geq 0$  such that  $|F_2(t)| \leq t^k(1-t)^l$  for all  $t \in (0, 1)$ .

**(B3)** there exists constants  $k > -1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -n - k\}$ ,  $M \geq 0$  such that  $|F_3(t)| \leq (\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ .

**(B4)** there exists constants  $k > -\alpha + n - 1$ ,  $l \leq 0$  with  $l > \max\{-\alpha, -\alpha - k\}$ ,  $M \geq 0$  such that  $|F_4(t)| \leq (\log t)^k(1 - \log t)^l$  for all  $t \in (1, e)$ .

**Theorem 4.0.1** *Suppose that (B1) holds. Then  $x$  is a piecewise solution of (4.0.1) if and only if there exist constants  $c_{iv}(i \in \mathbb{N}[0, m], v \in \mathbb{N}[0, n - 1]) \in \mathbb{R}$  such that*

$$x(t) = \sum_{\sigma=0}^i \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!} (t - t_{\sigma})^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (4.0.5)$$

**Proof:** We find for  $s \in (0, 1)$

$$\begin{aligned} & \left| \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du \right| \leq \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} u^k (1-u)^l du \\ & \leq (1-s)^l \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} u^k du = (1-s)^l s^{\alpha+k-n+1} \int_0^1 \frac{(1-w)^{\alpha-n}}{\Gamma(\alpha-n+1)} w^k dw \\ & = (1-s)^l s^{\alpha+k-n+1} \frac{\mathbf{B}(\alpha-n+1, k+1)}{\Gamma(\alpha-n+1)} \rightarrow 0 \text{ as } s \rightarrow 0^+. \end{aligned} \quad (4.0.6)$$

We also have for  $t \in (t_i, t_{i+1}]$  that

$$\begin{aligned} & \left| \int_0^t (t-s)^{\alpha-1} F_1(s) ds \right| \leq \int_0^t (t-s)^{\alpha-1} |F_1(s)| ds \leq \int_0^t (t-s)^{\alpha-1} s^k (1-s)^l ds \\ & \leq \int_0^t (t-s)^{\alpha+l-1} s^k ds = t^{\alpha+k+l} \int_0^1 (1-w)^{\alpha+l-1} w^k dw = t^{\alpha+k+l} \mathbf{B}(\alpha+l, k+1). \end{aligned}$$

Then  $\int_0^t (t-s)^{\alpha-1} F_1(s) ds$  is continuous on  $[0, 1]$ .

If  $x$  is a solution of (4.0.5), then we know that  $\lim_{t \rightarrow t_i^+} x(t)$  ( $i \in \mathbb{N}[0, m]$ ) exist and  $x \in P_m C(0, 1]$ .

Now we prove that  $x$  satisfies differential equation in (4.0.1). In fact, for  $t \in (t_0, t_1]$  we have by Theorem 3.1.2  ${}^C D_{0^+}^\alpha x(t) = F(t)$ . For  $t \in (t_i, t_{i+1}]$  ( $i \in \mathbb{N}[1, m]$ ), we have by Definition 2.3

$$\begin{aligned}
{}^C D_{0^+}^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} x^{(n)}(s) ds + \int_{t_i}^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} \left( \sum_{\sigma=0}^j \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!} (s-t_\sigma)^v + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u) du \right)^{(n)} ds \right. \\
&\quad \left. + \int_{t_i}^t (t-s)^{n-\alpha-1} \left( \sum_{\sigma=0}^i \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!} (s-t_\sigma)^v + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u) du \right)^{(n)} ds \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left( \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u) du \right)^{(n)} ds \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left( \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du \right)' ds \\
&= \frac{1}{\Gamma(n-\alpha+1)} \left[ \int_0^t (t-s)^{n-\alpha} \left( \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du \right)' ds \right]' \\
&= \frac{1}{\Gamma(n-\alpha+1)} \left[ (t-s)^{n-\alpha} \left( \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du \right) \Big|_0^t \text{ using (4.0.6)} \right. \\
&\quad \left. + (n-\alpha) \int_0^t (t-s)^{n-\alpha-1} \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du ds \right]' \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t (t-s)^{n-\alpha-1} \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du ds \right]' \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t \int_u^t (t-s)^{n-\alpha-1} \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} ds F_1(u) du \right]' \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-n}}{\Gamma(\alpha-n+1)} dw F_1(u) du \right]' \text{ by } \frac{s-u}{t-u} = w \\
&= F_1(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].
\end{aligned}$$

We have done that  $x$  satisfies (4.0.1) if  $x$  satisfies (4.0.5).

Now, we suppose that  $x$  is a solution of (4.0.1). We will prove that  $x$  satisfies (4.0.5) by

mathematical induction method. Since  $x$  is continuous on  $(t_i, t_{i+1}]$  and the limit  $\lim_{t \rightarrow t_i^+} x(t)$  ( $i \in \mathbb{N}[0, m]$ ) exists, then  $x \in P_m C(0, 1]$ . For  $t \in (t_0, t_1]$ , we know from Theorem 3.1.2 that there exists  $c_{0v} \in \mathbb{R}$  such that

$$x(t) = \sum_{v=0}^{n-1} \frac{c_{0v}}{v!} t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(s) ds, t \in (t_0, t_1].$$

Then (4.0.5) holds for  $i = 0$ . We suppose that (4.0.5) holds for all  $i \in \mathbb{N}[0, j], j \leq m - 1$ . We derive the expression of  $x$  on  $(t_{j+1}, t_{j+2}]$ . In order to get the expression of  $x$  on  $(t_{j+1}, t_{j+2}]$ , suppose that

$$x(t) = \Phi(t) + \sum_{\sigma=0}^j \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!} (t - t_\sigma)^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(s) ds, t \in (t_{j+1}, t_{j+2}]. \quad (4.0.7)$$

By  ${}^C D_{0+}^\alpha x(t) = F_1(t), t \in (t_{j+1}, t_{j+2}]$ , we get

$$\begin{aligned} F_1(t) &= {}^C D_{0+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds \\ &= \sum_{\rho=0}^j \int_{t_\rho}^{t_{\rho+1}} \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} \left( \sum_{\sigma=0}^{\rho} \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!} (s - t_\sigma)^v + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u) du \right)^{(n)} ds \\ &\quad + \int_{t_{j+1}}^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} \left( \Phi(s) + \sum_{\sigma=0}^j \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!} (s - t_\sigma)^v + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u) du \right)^{(n)} ds \\ &= {}^C D_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left( \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u) du \right)^{(n)} ds \\ &= {}^C D_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left( \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du \right)' ds \\ &= {}^C D_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\Gamma(n-\alpha+1)} \left[ \int_0^t (t-s)^{n-\alpha} \left( \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u) du \right)' ds \right]'. \end{aligned}$$

By a similar computation, we get

$$F_1(t) = F_1(t) + {}^C D_{t_{j+1}^+}^\alpha \Phi(t).$$

It follows that  ${}^C D_{t_{j+1}^+}^\alpha \Phi(t) = 0$  for all  $t \in (t_{j+1}, t_{j+2}]$ . By Theorem 3.1.2, we know that there exists  $c_{j+1v} \in \mathbb{R}$  ( $v \in \mathbb{N}[0, n - 1]$ ) such that  $\Phi(t) = \sum_{v=0}^{n-1} \frac{c_{j+1v}}{v!} (t - t_{j+1})^v$  for  $t \in (t_{j+1}, t_{j+2}]$ . Substituting  $\Phi$  into (4.0.7), we get that (4.0.5) holds for  $i = j + 1$ . By the mathematical induction method, we know that  $x$  satisfies (4.0.5) and  $x|_{(t_i, t_{i+1}]}$  is continuous and  $\lim_{t \rightarrow t_i^+} x(t)$  exists. The proof is completed.  $\square$

**Remark 4.0.1** It is easy to see from (4.0.5) that  $x$  is a solution of (4.0.1) if and only if  $x$  satisfies that there exist numbers  $C_{iv}(i \in \mathbb{N}[0, m], v \in \mathbb{N}[0, n - 1])$  such that

$$x(t) = \sum_{v=0}^{n-1} C_{iv} t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

In [73], authors proved the similar result under a strong assumption that  $F_1$  is continuous on  $[0, 1]$ . So our result generalizes the result in [73]. One sees that (4.0.5) is more convenient for use. For example, when consider the following impulsive problem

$${}^C D_{0+}^{\alpha} x(t) = F_1(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m],$$

$$\Delta x^{(j)}(t_i) = a_{ij}, i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, n - 1],$$

$$x^{(j)}(0) = a_{0j}, j \in \mathbb{N}[0, n - 1]$$

where  $\alpha \in (n - 1, n)$ ,  $F_1$  satisfies **(B1)**. Then we can easily get by (4.0.5)

$$c_{0v} = a_{0v}, v \in \mathbb{N}[0, n - 1], c_{iv} = a_{iv}, v \in \mathbb{N}[0, n - 1], i \in \mathbb{N}[1, m].$$

So

$$x(t) = \sum_{\sigma=1}^i \sum_{v=0}^{n-1} \frac{a_{\sigma v}}{v!} (t - t_{\sigma})^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

**Theorem 4.0.2** Suppose that **(B2)** holds. Then  $x$  is a solution of (4.0.2) if and only if there exist constants  $c_{\sigma v}(\sigma \in \mathbb{N}[0, m], v \in \mathbb{N}[1, n]) \in \mathbb{R}$  such that

$$x(t) = \sum_{\sigma=0}^i \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} (t - t_{\sigma})^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_2(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (4.0.8)$$

**Proof:** For  $t \in (t_j, t_{j+1}]$  ( $j \in \mathbb{N}[0, m]$ ), similarly to the beginning of the proof of Theorem 4.0.1 we know that

$$\begin{aligned} t^{n-\alpha} \left| \int_0^t (t-s)^{\alpha-1} F_2(s) ds \right| &\leq \int_0^t (t-s)^{\alpha-1} |F_2(s)| ds \\ &\leq t^{n-\alpha} \int_0^t (t-s)^{\alpha-1} s^k (1-s)^l ds \leq t^{n-\alpha} \int_0^t (t-s)^{\alpha+l-1} s^k ds \\ &= t^{n+k+l} \int_0^1 (1-w)^{\alpha+l-1} w^k dw = t^{n+k+l} \mathbf{B}(\alpha+l, k+1). \end{aligned}$$

So  $t^{n-\alpha} \int_0^t (t-s)^{\alpha-1} F_2(s) ds$  is continuous on  $[0, 1]$ .

If  $x$  is a solution of (4.0.8), we have  $x \in P_m C_{1-\alpha}[0, 1]$ . It follows for  $t \in (t_i, t_{i+1}]$  and from

Definition 2.2 that

$$\begin{aligned}
{}^{RL}D_{0+}^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t (t-s)^{n-\alpha-1} x(s) ds \right]^{(n)} \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} \sum_{\sigma=0}^j \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} (s-t_{\sigma})^{\alpha-v} ds \right]^{(n)} \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \left[ \int_{t_i}^t (t-s)^{n-\alpha-1} \sum_{\sigma=0}^i \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} (s-t_{\sigma})^{\alpha-v} ds \right]^{(n)} \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t (t-s)^{n-\alpha-1} \int_0^s \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} F_2(u) du ds \right]^{(n)} \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{j=0}^{i-1} \sum_{\sigma=0}^j \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} (s-t_{\sigma})^{\alpha-v} ds \right]^{(n)} \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{\sigma=0}^i \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} \int_{t_i}^t (t-s)^{n-\alpha-1} (s-t_{\sigma})^{\alpha-v} ds \right]^{(n)} \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t \int_u^t (t-s)^{n-\alpha-1} \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} ds F_2(u) du \right]^{(n)} \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{j=0}^{i-1} \sum_{\sigma=0}^j \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} (t-t_{\sigma})^{n-v} \int_{\frac{t_j-t_{\sigma}}{t-t_{\sigma}}}^{\frac{t_{j+1}-t_{\sigma}}{t-t_{\sigma}}} (1-w)^{n-\alpha-1} w^{\alpha-v} dw \right]^{(n)} \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{\sigma=0}^i \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} \int_{\frac{t_i-t_{\sigma}}{t-t_{\sigma}}}^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw \right]^{(n)} \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \left[ \int_0^t (t-u)^{n-1} \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw F_2(u) du \right]^{(n)} \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{\sigma=0}^{i-1} \sum_{j=\sigma}^{i-1} \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} (t-t_{\sigma})^{n-v} \int_{\frac{t_j-t_{\sigma}}{t-t_{\sigma}}}^{\frac{t_{j+1}-t_{\sigma}}{t-t_{\sigma}}} (1-w)^{n-\alpha-1} w^{\alpha-v} dw \right]^{(n)} \\
&\quad + \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{\sigma=0}^i \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} \int_{\frac{t_i-t_{\sigma}}{t-t_{\sigma}}}^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw \right]^{(n)} \\
&\quad + \left[ \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} F_2(u) du \right]^{(n)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{\sigma=0}^i \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} (t-t_\sigma)^{n-v} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw \right]^{(n)} + F_2(t) \\
&= F_2(t), t \in (t_i, t_{i+1}].
\end{aligned}$$

It follows that  $x$  is a solution of (4.0.2).

Now we prove that if  $x$  is a solution of (4.0.2), then  $x$  satisfies (4.0.8) and  $x \in P_m C_{n-\alpha}[0, 1]$  by mathematical induction method. By Theorem 3.2.2, we know that there exists a constant  $c_{0v} \in \mathbb{R} (v \in \mathbb{N}[1, n])$  such that

$$x(t) = \sum_{v=1}^n \frac{c_{0v}}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_2(s) ds, t \in (t_0, t_1].$$

Hence (4.0.8) holds for  $i = 0$ . Assume that (4.0.8) holds for  $i = 0, 1, 2, \dots, j \leq m-1$ , we will prove that (4.0.8) holds for  $i = j+1$ . Suppose that

$$x(t) = \Phi(t) + \sum_{\sigma=0}^j \sum_{v=1}^n \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)} (t-t_\sigma)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_2(s) ds, t \in (t_{j+1}, t_{j+2}].$$

Then for  $t \in (t_{j+1}, t_{j+2}]$  we have

$$F(t) = {}^{RL}D_{0+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left[ \sum_{\rho=0}^j \int_{t_\rho}^{t_{\rho+1}} (t-s)^{n-\alpha-1} x(s) ds + \int_{t_{j+1}}^t (t-s)^{n-\alpha-1} x(s) ds \right]^{(n)}.$$

Similarly to the above discussion we can get

$$F(t) = {}^{RL}D_{0+}^\alpha x(t) = F(t) + {}^{RL}D_{t_{j+1}^+}^\alpha \Phi(t).$$

So  ${}^{RL}D_{t_{j+1}^+}^\alpha \Phi(t) = 0$  on  $(t_{j+1}, t_{j+2}]$ . Then Theorem 3.2.2 implies that there exists a constant  $c_{j+1v} \in \mathbb{R}$  such that  $\Phi(t) = \sum_{v=1}^n \frac{c_{j+1v}}{\Gamma(\alpha-v+1)} (t-t_{i+1})^{\alpha-v} (t-t_{j+1})^\alpha$  on  $(t_{j+1}, t_{j+2}]$ . Hence

$$x(t) = \sum_{\rho=0}^{j+1} \sum_{v=1}^n \frac{c_{\rho v}}{\Gamma(\alpha-v+1)} (t-t_\rho)^{\alpha-v} + \int_1^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_2(s) ds, t \in (t_{j+1}, t_{j+2}].$$

By mathematical induction method, we know that (4.0.8) holds for  $j \in \mathbb{N}[0-, m]$ . The proof is completed.  $\square$

**Theorem 4.0.3** *Suppose that (B3) holds. Then  $x$  is a solution of (4.0.3) if and only if there exist constants  $c_{jv} \in \mathbb{R} (j \in \mathbb{N}[0, m], v \in \mathbb{N}[1, n])$  such that*

$$x(t) = \sum_{j=0}^i \sum_{v=1}^n \frac{c_{jv}}{\Gamma(\alpha-v+1)} \left( \log \frac{t}{t_j} \right)^{\alpha-v} + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} F_3(s) \frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (4.0.9)$$

**Proof:** The proof is similar to the proof of Theorem 4.0.2 and is omitted.  $\square$

**Theorem 4.0.4** *Suppose that (B4) holds. Then  $x$  is a piecewise solution of (4.0.4) if and only if there exist constants  $c_{jv} \in \mathbb{R}$  ( $j \in \mathbb{N}[0, m], v \in \mathbb{N}[1, n]$ ) such that*

$$x(t) = \sum_{\rho=0}^j \sum_{v=0}^{n-1} \frac{c_{\rho v}}{v!} \left(\log \frac{t}{t_\rho}\right)^v + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} F_4(s) \frac{ds}{s}, t \in (t_j, t_{j+1}], j \in \mathbb{N}[0, m]. \quad (4.0.10)$$

**Proof:** The proof is similar to the proof of Theorem 4.0.1 and is omitted.  $\square$

## 5 Main results

In this section, we establish existence results for solutions of BVP (1.0.i) ( $i = 8, 9, 10, 11$ ) respectively. The following two well known fixed point theorems will be of use in the sections to follow. In particular, the Nonlinear Alternative ([28]: Theorem 5.1, p. 61) will be employed.

**Lemma 5.0.1** [Nonlinear Alternative] *Let  $X$  be a normed space with  $C$  a convex subset of  $X$ . Let  $U$  be an open subset of  $C$  with  $0 \in C$  and consider a compact map  $H : \bar{U} \rightarrow C$ . If*

$$u \neq \lambda Hu \quad \text{for all } u \in \partial U \text{ and for all } \lambda \in [0, 1]$$

*then  $H$  has at least one fixed-point.*

### 5.1 Solvability of BVP (1.0.8)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.8).

**Lemma 5.1.1** *Suppose that  $\sigma : (0, 1) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > 1 - \beta$  and  $l \leq 0$  with  $l > \max\{-\beta, -\beta - k\}$  such that  $|\sigma(t)| \leq t^k(1 - t)^l$  for all  $t \in (0, 1)$ . The  $x$  is a solutions of*

$$\begin{cases} {}^C D_{0+}^\beta x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ x(0) = a, \quad x'(1) = b, \\ \Delta x(t_i) = I_i, \quad \Delta x'(t_i) = J_i, \quad i \in \mathbb{N}[1, m] \end{cases} \quad (5.1.1)$$

if and only if  $x$  and

$$\begin{aligned} x(t) = & a + \left[ b - \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \sigma(s) ds - \sum_{\sigma=1}^m J_\sigma \right] t \\ & + \sum_{\sigma=1}^i [I_\sigma + (t - t_\sigma) J_\sigma] + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \end{aligned} \quad (5.1.2)$$

**Proof:** Let  $x$  be a solution of (5.1.1). We know by Theorem 4.0.1 that there exist numbers  $c_{\sigma 0}, c_{\sigma 1} \in \mathbb{R}$  ( $\sigma \in \mathbb{N}[0, m]$ ) such that

$$x(t) = \sum_{\sigma=0}^i [c_{\sigma 0} + c_{\sigma 1}(t - t_\sigma)] + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (5.1.3)$$

So

$$x'(t) = \sum_{\sigma=0}^i c_{\sigma 1} + \int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} \sigma(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (5.1.4)$$

By

$$\begin{aligned} \left| \int_0^t (t-s)^{\beta-1} \sigma(s) ds \right| & \leq \int_0^t (t-s)^{\beta-1} s^k (1-s)^l ds \\ & \leq \int_0^t (t-s)^{\beta+l-1} s^k ds = t^{\beta+k+l} \mathbf{B}(\beta+l, k+1) \rightarrow 0 \text{ as } t \rightarrow 0^+, \end{aligned}$$

together with (5.1.3), (5.1.4) and the boundary conditions and the impulse assumption in (5.1.1) that

$$c_{00} = a, \quad c_{\sigma 0} = I_\sigma, \quad c_{\sigma 1} = J_\sigma, \quad \sigma \in \mathbb{N}[1, m],$$

$$\sum_{\sigma=0}^m c_{\sigma 1} + \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \sigma(s) ds = b.$$

Then

$$c_{01} = b - \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \sigma(s) ds - \sum_{\sigma=1}^m J_\sigma. \quad (5.1.5)$$

Substituting  $c_{\sigma 0}, c_{\sigma 1}$  ( $\sigma \in \mathbb{N}[0, m]$ ) into (5.1.3), we get (5.1.2) obviously.

On the other hand, if  $x$  satisfies (5.1.2), then  $x|_{(t_i, t_{i+1}]}$  ( $i \in \mathbb{N}[0, m]$ ) are continuous and the limits  $\lim_{t \rightarrow t_i^+} x(t)$  ( $i \in \mathbb{N}[0, m]$ ) exist. So  $x \in P_m C(0, 1]$ . Using (5.1.5) and  $c_{00} = a$ ,  $c_{\sigma 0} = I_\sigma + t_\sigma J_\sigma$ ,  $c_{\sigma 1} = J_\sigma$ ,  $\sigma \in \mathbb{N}[1, m]$ , we rewrite  $x$  by

$$x(t) = \sum_{\sigma=0}^i [c_{\sigma 0} + c_{\sigma 1}(t - t_\sigma)] + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, 1].$$

Since  $\sigma$  is continuous on  $(0, 1)$  and  $|\sigma(t)| \leq t^k (1-t)^l$ , one can show easily that  $x$  is continuous on  $(t_i, t_{i+1}]$  ( $i = 0, 1$ ) and the limits  $\lim_{t \rightarrow t_i^+} x(t)$  ( $i \in \mathbb{N}[0, m]$ ) exist. So  $x \in P_m C(0, 1]$ .

Furthermore, by direct computation, we have  $x(0) = a, x'(1) = b, \lim_{t \rightarrow t_i^+} x(t) - x(t_i) = I_i$  and  $\lim_{t \rightarrow t_i^+} x(t) - x(t_i) = J_i$ . Furthermore, we have by Theorem 4.0.1

$${}^C D_{0+}^\beta x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

So  $x$  is a solution of (5.1.1). The proof is completed.  $\square$

Define the nonlinear operator  $Q$  on  $P_m C(0, 1]$  by  $Qx$  for  $x \in P_m C(0, 1]$  with

$$\begin{aligned} (Qx)(t) &= \int_0^1 \phi(s)G(s, x(s))ds \\ &+ \left[ \int_0^1 \psi(s)H(s, x(s))ds - \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} p(s)f(s, x(s))ds - \sum_{\sigma=1}^m J(t_\sigma, x(t_\sigma)) \right] t \\ &+ \sum_{\sigma=1}^i [I(t_\sigma, x(t_\sigma)) + (t - t_\sigma)J(t_\sigma, x(t_\sigma))] \\ &+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} p(s)f(s, x(s))ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \end{aligned}$$

**Lemma 5.1.2** *Suppose that (a)-(e) hold, and  $f, G, H$  are impulsive I-Carathéodory functions,  $I, J$  discrete I-Carathéodory functions. Then  $Q : P_m C(0, 1] \rightarrow P_m C(0, 1]$  is well defined and is completely continuous,  $x \in P_m C(0, 1]$  is a solution of BVP (1.0.8) if and only if  $x \in P_m C(0, 1]$  is a fixed point of  $Q$ .*

**Proof.** The proof is similar to that of Theorem 3.1 in [85] and is omitted.

**Theorem 5.1.1** *Suppose that (a)-(e) hold,  $f, G, H$  are impulsive I-Carathéodory functions,  $I, J$  discrete I-Carathéodory functions and*

(C1) *there exist nondecreasing functions  $M_f, M_g, M_h, M_I, M_J : [0, +\infty) \rightarrow [0, +\infty)$  such that*

$$|f(t, x)| \leq M_f(|x|), t \in (0, 1), x \in \mathbb{R},$$

$$|G(t, x)| \leq M_G(|x|), t \in (0, 1), x \in \mathbb{R},$$

$$|H(t, x)| \leq M_H(|x|), t \in (0, 1), x \in \mathbb{R},$$

$$|I(t_i, x)| \leq M_I(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R},$$

$$|J(t_i, x)| \leq M_J(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}.$$

Then BVP (1.0.8) has at least one solution if there exists  $r_0 > 0$  such that

$$\begin{aligned} & \|\phi\|_1 M_G(r_0) + \|\psi\|_1 M_H(r_0) + m M_I(r_0) + 2m M_J(r_0) \\ & + \left[ \frac{\mathbf{B}(\beta+l-1, k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0. \end{aligned} \quad (5.1.6)$$

**Proof:** From Lemma 5.1.1, Lemma 5.1.2 and the definition of  $Q$ ,  $x \in P_m C(0, 1]$  is a solution of BVP (1.0.7) if and only if  $x \in P_m C(0, 1]$  is a fixed point of  $Q$ . Lemma 5.1.2 implies that  $Q$  is a completely continuous operator. From (C1), we have for  $x \in P_m C(0, 1]$  that

$$|f(t, x(t))| \leq M_f(|x(t)|) \leq M_f(\|x\|), t \in (0, 1),$$

$$|G(t, x(t))| \leq M_G(\|x\|), t \in (0, 1),$$

$$|H(t, x(t))| \leq M_H(\|x\|), t \in (0, 1),$$

$$|I(t_i, x(t_i))| \leq M_I(\|x\|), i \in \mathbb{N}[1, m],$$

$$|J(t_i, x(t_i))| \leq M_J(\|x\|), i \in \mathbb{N}[1, m].$$

We consider the set  $\Omega = \{x \in P_m C(0, 1] : x = \lambda(Tx), \text{ for some } \lambda \in [0, 1]\}$ . For  $x \in \Omega$ , we have for  $t \in (t_i, t_{i+1}]$  that

$$\begin{aligned} |(Qx)(t)| & \leq \|\phi\|_1 M_G(\|x\|) + \|\psi\|_1 M_H(\|x\|) + \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} s^k (1-s)^l ds M_f(\|x\|) + m M_J(\|x\|) \\ & \quad + m M_I(\|x\|) + m M_J(\|x\|) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s^k (1-s)^l ds M_f(\|x\|) \\ & \leq \|\phi\|_1 M_G(\|x\|) + \|\psi\|_1 M_H(\|x\|) + m M_I(\|x\|) + 2m M_J(\|x\|) \\ & \quad + \left[ \frac{\mathbf{B}(\beta+l-1, k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|). \end{aligned}$$

It follows that

$$\begin{aligned} \|x\| & = \lambda \|Tx\| < \|Tx\| \leq \|\phi\|_1 M_G(\|x\|) + \|\psi\|_1 M_H(\|x\|) + m M_I(\|x\|) + 2m M_J(\|x\|) \\ & \quad + \left[ \frac{\mathbf{B}(\beta+l-1, k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|). \end{aligned}$$

From (5.1.6), we choose  $\Omega = \{x \in P_m C(0, 1] : \|x\| < r_0\}$  and  $C = X$ . Then  $\Omega$  is a open subset of  $C$  with  $0 \in C$ .

If there exists  $x \in \partial\Omega$  such that  $x = Qx$ , then  $x$  is a solution of BVP (1.0.7). If there exist some  $x \in \partial\Omega$  and  $\lambda \in [0, 1)$  such that  $x = \lambda Qx$ , then

$$\begin{aligned} r_0 = \|x\| &= \lambda \|Qx\| < \|Qx\| \leq \|\phi\|_1 M_G(r_0) + \|\psi\|_1 M_H(r_0) + m M_I(r_0) + 2m M_J(r_0) \\ &+ \left[ \frac{\mathbf{B}(\beta+l-1, k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0, \end{aligned}$$

a contradiction. As a consequence of Lemma 5.0.1, we deduce that  $Q$  has a fixed point which is a solution of the problem BVP (1.0.8). The proof is completed. The proof of Theorem 5.1.1 is completed.  $\square$

**Corollary 5.1.1** *Suppose that (a)-(e), and (C1) hold. Then BVP (1.0.8) has at least one solution if*

$$\inf_{r \in (0, +\infty)} \frac{\|\phi\|_1 M_G(r) + \|\psi\|_1 M_H(r) + m M_I(r) + 2m M_J(r) + \left[ \frac{\mathbf{B}(\beta+l-1, k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r)}{r} < 1.$$

**Proof:** From the assumption, we know that there exists  $r_0 > 0$  such that (5.1.6) holds. By Theorem 5.1.1, BVP (1.0.8) has at least one solution. The proof is omitted.  $\square$

**Corollary 5.1.2** *Suppose that (a)-(e), and (C1) hold. Then BVP (1.0.8) has at least one solution if*

$$\begin{aligned} \lim_{r \rightarrow +\infty} \frac{\|\phi\|_1 M_G(r) + \|\psi\|_1 M_H(r) + m M_I(r) + 2m M_J(r) + \left[ \frac{\mathbf{B}(\beta+l-1, k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r)}{r} &= 0 \text{ or} \\ \lim_{r \rightarrow 0} \frac{\|\phi\|_1 M_G(r) + \|\psi\|_1 M_H(r) + m M_I(r) + 2m M_J(r) + \left[ \frac{\mathbf{B}(\beta+l-1, k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r)}{r} &= 0. \end{aligned}$$

**Proof:** From the assumption, we know that there exists  $r_0 > 0$  such that (5.1.6) holds. By Theorem 5.1.1, BVP (1.0.7) has at least one solution. The proof is omitted.  $\square$

## 5.2 Solvability of BVP (1.0.9)

In this subsection, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.9).

**Lemma 5.2.1** *Suppose that  $\sigma : (0, 1) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > -1$  and  $\max\{-\beta, -2 - k\} < l \leq 0$  such that  $|\sigma(t)| \leq t^k(1-t)^l$  for all  $t \in (0, 1)$ .*

The  $x$  is a solutions of

$$\begin{cases} {}^{RL}D_{0+}^{\beta}x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \lim_{t \rightarrow 0^+} t^{2-\beta}x(t) = a, \quad x(1) = b, \\ \lim_{t \rightarrow t_i^+} (t - t_i)^{2-\beta}x(t) = I_i, i \in \mathbb{N}[1, m], \\ \Delta {}^{RL}D_{0+}^{\beta-1}x(t_i) = J_i, i \in \mathbb{N}[1, m], \end{cases} \quad (5.2.1)$$

if and only if  $x$  satisfies

$$\begin{aligned} x(t) &= t^{\beta-1} \left[ b - a - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds - \sum_{\sigma=1}^m \left( \frac{J_{\sigma}(1-t_{\sigma})^{\beta-1}}{\Gamma(\beta)} + I_{\sigma}(1-t_{\sigma})^{\beta-2} \right) \right] + at^{\beta-2} \\ &+ \sum_{\sigma=1}^i \left[ \frac{(t-t_{\sigma})^{\beta-1}}{\Gamma(\beta)} J_{\sigma} + (t-t_{\sigma})^{\beta-2} I_{\sigma} \right] + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \end{aligned} \quad (5.2.2)$$

**Proof:** Let  $x$  be a solution of (5.2.1). By Theorem 4.0.2, we know that there exist numbers  $c_{\sigma 0}, c_{\sigma 1} \in \mathbb{R}(\sigma \in \mathbb{N}[1, n])$  such that

$$x(t) = \sum_{\sigma=0}^i \left[ \frac{c_{\sigma 1}(t-t_{\sigma})^{\beta-1}}{\Gamma(\beta)} + \frac{c_{\sigma 2}(t-t_{\sigma})^{\beta-2}}{\Gamma(\beta-1)} \right] + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (5.2.3)$$

It follows that

$${}^{RL}D_{0+}^{\beta-1}x(t) = \sum_{\sigma=0}^i c_{\sigma 1} + \int_0^t \sigma(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (5.2.4)$$

It follows from the boundary conditions and the impulse assumption in (3.3.1) that  $c_{02} = \Gamma(\beta-1)a$ ,  $c_{\sigma 2} = \Gamma(\beta-1)I_{\sigma}(\sigma \in \mathbb{N}[1, m])$ ,  $c_{\sigma 1} = J_{\sigma}(\sigma \in \mathbb{N}[1, m])$  and

$$\sum_{\sigma=0}^m \left[ \frac{c_{\sigma 1}(1-t_{\sigma})^{\beta-1}}{\Gamma(\beta)} + \frac{c_{\sigma 2}(1-t_{\sigma})^{\beta-2}}{\Gamma(\beta-1)} \right] + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds = b.$$

Then

$$c_{0,1} = \Gamma(\beta) \left[ b - a - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \sigma(s) ds - \sum_{\sigma=1}^m \left( \frac{J_{\sigma}(1-t_{\sigma})^{\beta-1}}{\Gamma(\beta)} + I_{\sigma}(1-t_{\sigma})^{\beta-2} \right) \right]. \quad (5.2.5)$$

Substituting  $c_{\sigma 1}, c_{\sigma 2}(\sigma \in \mathbb{N}[0, m])$  into (5.2.3), we get (5.2.2) obviously.

On the other hand, if  $x$  satisfies (3.3.2), then  $x|_{(t_i, t_{i+1}]}(i \in \mathbb{N}[0, m])$  are continuous and the limits  $\lim_{t \rightarrow t_i^+} (t - t_i)^{2-\beta}x(t)(i \in \mathbb{N}[0, m])$  exist. So  $x \in P_m C_{2-\beta}(0, 1]$ . Using (3.3.5) and  $c_{02} = \Gamma(\beta-1)a$ ,  $c_{\sigma 2} = \Gamma(\beta-1)I_{\sigma}(\sigma \in \mathbb{N}[1, m])$ ,  $c_{\sigma 1} = J_{\sigma}(\sigma \in \mathbb{N}[1, m])$ , we rewrite  $x$  by (5.2.3). Since  $\sigma$  is continuous on  $(0, 1)$  and  $|\sigma(t)| \leq t^k(1-t)^l$ , one can show easily that  $x$  is continuous on  $(t_i, t_{i+1}](i = 0, 1)$  and using the method at the beginning of the proof of this lemma,

we know that both the limits  $\lim_{t \rightarrow t_i^+} (t - t_i)^{2-\beta} x(t) (i \in \mathbb{N}[0, m])$  exist. So  $x \in P_m C_{2-\beta}(0, 1]$ . Furthermore, by direct computation, we have  $x(1) = b$ , and  $\lim_{t \rightarrow 0^+} t^{2-\beta} x(t) = a$ . One have from Theorem 4.0.2 similarly that  $D_{0^+}^\beta x(t) = \sigma(t)$  and for  $t \in (t_i, t_{i+1}] (i \in \mathbb{N}_0)$ . So  $x$  is a solution of (5.2.1). The proof is completed.  $\square$

Define the nonlinear operator  $T$  on  $P_m C_{2-\beta}(0, 1]$  for  $x \in P_m C_{2-\beta}(0, 1]$  by

$$\begin{aligned} (Tx)(t) &= t^{\beta-1} \left[ \int_0^1 \psi(s) H(s, x(s)) ds - \int_0^1 \phi(s) G(s, x(s)) ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} q(s) f(s, x(s)) ds \right. \\ &\quad \left. - \sum_{\sigma=1}^m \left( \frac{J(t_\sigma, x(t_\sigma))(1-t_\sigma)^{\beta-1}}{\Gamma(\beta)} + I(t_\sigma, x(t_\sigma))(1-t_\sigma)^{\beta-2} \right) \right] + t^{\beta-2} \int_0^1 \phi(s) G(s, x(s)) ds \\ &\quad + \sum_{\sigma=1}^i \left[ \frac{(t-t_\sigma)^{\beta-1}}{\Gamma(\beta)} J(t_\sigma, x(t_\sigma)) + (t-t_\sigma)^{\beta-2} I(t_\sigma, x(t_\sigma)) \right] \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} q(s) f(s, x(s)) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \end{aligned}$$

**Lemma 5.2.2** *Suppose that (a)-(e) hold, and  $f, G, H$  are impulsive II-Carathéodory functions,  $I, J$  discrete II-Carathéodory functions. Then  $T : P_m C_{2-\beta}(0, 1] \rightarrow P_m C_{2-\beta}(0, 1]$  is well defined and is completely continuous,  $x \in P_m C_{2-\beta}(0, 1]$  is a solution of BVP (1.0.9) if and only if  $x \in P_m C_{2-\beta}(0, 1]$  is a fixed point of  $T$ .*

**Proof:** The proof is similar to that of the proof of Lemma 3.1 in [81] and is omitted.  $\square$

**Theorem 5.2.1** *Suppose that (f)-(h),  $f, G, H$  are impulsive II-Carathéodory functions,  $I, J$  discrete II-Carathéodory functions and*

**(C2)** *there exist nondecreasing functions  $M_f, M_g, M_h, M_I, M_J : [0, +\infty) \rightarrow [0, +\infty)$  such that*

$$|f(t, (t - t_i)^{\beta-2} x)| \leq M_f(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|G(t, (t - t_i)^{\beta-2} x)| \leq M_G(|x|), t \in (t_i, t_{i1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|H(t, (t - t_i)^{\beta-2} x)| \leq M_H(|x|), t \in (t_i, t_{i1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|I(t_i, (t_i - t_{i-1})^{\beta-2} x)| \leq M_I(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R},$$

$$|J(t_i, (t_i - t_{i-1})^{\beta-2} x)| \leq M_J(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}.$$

Then BVP (1.0.9) has at least one solution if there exists a  $r_0 > 0$  such that

$$\begin{aligned} & \|\psi\|_1 M_H(r_0) + 2\|\phi\|_1 M_G(r_0) + \frac{2m}{\Gamma(\beta)} M_J(r_0) \\ & + \left[ m + \sum_{\sigma=1}^m (1-t_\sigma)^{\beta-2} \right] M_I(r_0) + \left[ \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0. \end{aligned} \quad (5.2.6)$$

**Proof:** From Lemma 5.2.1, the definition of  $T$ ,  $x \in P_m C_{2-\beta}(0, 1]$  is a solution of BVP (1.0.8) if and only if  $x \in P_m C_{2-\beta}(0, 1]$  is a fixed point of  $T$  in  $P_m C_{2-\beta}(0, 1]$ . Lemma 5.2.2 implies that  $T$  is a completely continuous operator. From (C2), we have for  $x \in P_m C_{2-\beta}(0, 1]$  that

$$\begin{aligned} |f(t, x(t))| &= |f(t, (t-t_i)^{\beta-2}(t-t_i)^{2-\beta}x(t))| \\ &\leq M_f(|(t-t_i)^{2-\beta}x(t)|) \leq M_f(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ |G(t, x(t))| &\leq M_G(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ |H(t, x(t))| &\leq M_H(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ |I(t_i, x(t_i))| &= |I(t_i, (t_i-t_{i-1})^{\beta-2}(t_i-t_{i-1})^{2-\beta}x(t_i))| \\ &\leq M_I((t_i-t_{i-1})^{2-\beta}|x(t_i)|) \leq M_I(\|x\|), i \in \mathbb{N}[1, m], \\ |J(t_i, x(t_i))| &\leq M_J(\|x\|), i \in \mathbb{N}[1, m]. \end{aligned}$$

We consider the set  $\Omega = \{x \in P_m C_{2-\beta}(0, 1] : x = \lambda(Tx), \text{ for some } \lambda \in [0, 1]\}$ . For  $x \in \Omega$ , we have for  $t \in (t_i, t_{i+1}]$

$$\begin{aligned} (t-t_i)^{2-\beta}|(Tx)(t)| &\leq (t-t_i)^{2-\beta}t^{\beta-1} [\|\psi\|_1 M_H(\|x\|) + \|\phi\|_1 M_G(\|x\|) \\ &+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} s^k (1-s)^l ds M_f(\|x\|) + \sum_{\sigma=1}^m \left( \frac{M_J(\|x\|)(1-t_\sigma)^{\beta-1}}{\Gamma(\beta)} + M_I(\|x\|)(1-t_\sigma)^{\beta-2} \right)] \\ &+ (t-t_i)^{2-\beta}t^{\beta-2} \|\phi\|_1 M_G(\|x\|) + (t-t_i)^{2-\beta} \sum_{\sigma=1}^i \left[ \frac{(t-t_\sigma)^{\beta-1}}{\Gamma(\beta)} M_J(\|x\|) + (t-t_\sigma)^{\beta-2} M_I(\|x\|) \right] \\ &+ (t-t_i)^{2-\beta} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s^k (1-s)^l ds M_f(\|x\|) \\ &\leq \|\psi\|_1 M_H(\|x\|) + 2\|\phi\|_1 M_G(\|x\|) + \frac{2m}{\Gamma(\beta)} M_J(\|x\|) \\ &+ \left[ m + \sum_{\sigma=1}^m (1-t_\sigma)^{\beta-2} \right] M_I(\|x\|) + \left[ \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|). \end{aligned}$$

It follows that

$$\begin{aligned} \|x\| &= \lambda \|Tx\| < \|Tx\| \leq \|\psi\|_1 M_H(\|x\|) + 2\|\phi\|_1 M_G(\|x\|) + \frac{2m}{\Gamma(\beta)} M_J(\|x\|) \\ &+ \left[ m + \sum_{\sigma=1}^m (1-t_\sigma)^{\beta-2} \right] M_I(\|x\|) + \left[ \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|). \end{aligned}$$

From (5.2.6), we choose  $\Omega = \{x \in P_1 C_{2-\beta}(0, 1) : \|x\| \leq r_0\}$ . For  $x \in \Omega$ , we get  $x \neq \lambda(Tx)$  for any  $\lambda \in [0, 1)$  and  $x \in \partial\Omega$ . In fact, if  $x = \lambda(Tx)$  for some  $\lambda \in [0, 1)$  and  $x \in \partial\Omega$ , then

$$\begin{aligned} r_0 &= \|x\| < \|\psi\|_1 M_H(r_0) + 2\|\phi\|_1 M_G(r_0) + \frac{2m}{\Gamma(\beta)} M_J(r_0) \\ &+ \left[ m + \sum_{\sigma=1}^m (1-t_\sigma)^{\beta-2} \right] M_I(r_0) + \left[ \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0, \end{aligned}$$

which is a contradiction. As a consequence of Lemma 5.0.1, we deduce that  $T$  has a fixed point which is a solution of the problem BVP (1.0.9). The proof is completed. The proof of Theorem 5.2.1 is completed.  $\square$

**Corollary 5.2.1** *Suppose that (f)-(h), and (C2) hold. Then BVP (1.0.9) has at least one solution if*

$$\begin{aligned} &\inf_{r \in (0, +\infty)} \frac{1}{r} \left[ \|\psi\|_1 M_H(r) + 2\|\phi\|_1 M_G(r) + \frac{2m}{\Gamma(\beta)} M_J(r) \right. \\ &\left. + \left[ m + \sum_{\sigma=1}^m (1-t_\sigma)^{\beta-2} \right] M_I(r) + \left[ \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r) \right] < 1. \end{aligned}$$

**Proof:** From the assumption, we know that there exists  $r_0 > 0$  such that (5.2.6) holds. By Theorem 4.0.1, BVP (1.0.9) has at least one solution. The proof is omitted.  $\square$

### 5.3 Solvability of BVP (1.0.10)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.10).

**Lemma 5.3.1** *Suppose that  $\sigma : (0, 1) \rightarrow \mathbb{R}$  is continuous and there exist numbers  $k > -1$  and  $\max\{-\beta, -2 - k\} < l \leq 0$  such that  $|\sigma(t)| \leq (\log t)^k (1 - \log t)^l$  for all  $t \in (1, e)$ . Then*

$x$  is a solutions of

$$\left\{ \begin{array}{l} {}^{RLH}D_{1+}^{\beta}x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \lim_{t \rightarrow 1^+} (\log t)^{2-\beta}x(t) + x(e) = a, \\ \lim_{t \rightarrow 1^+} {}^{RLH}D_{1+}^{\beta-1}x(t) + {}^{RLH}D_{1+}^{\beta-1}x(e) = b, \\ \lim_{t \rightarrow t_i^+} (\log t - \log t_i)^{2-\beta}x(t) = I_i, \Delta^{RL}D_{1+}^{\beta-1}x(t_i) = J_i, i \in \mathbb{N}[1, m], \\ \Delta^{RL}D_{1+}^{\beta-1}x(t_i) = J_i, i \in \mathbb{N}[1, m], \end{array} \right. \quad (5.3.1)$$

if and only if  $x$  satisfies

$$\begin{aligned} x(t) &= \frac{1}{2\Gamma(\beta)} \left[ b - \sum_{\sigma=1}^m J_{\sigma} - \int_1^e \sigma(s) \frac{ds}{s} \right] (\log t)^{\beta-1} \\ &+ \frac{1}{2} \left[ a - \frac{1}{\Gamma(\beta)} \int_1^e (\log \frac{t}{s})^{\beta-1} \sigma(s) \frac{ds}{s} - \sum_{\sigma=1}^m \left( \frac{J_{\sigma}}{\Gamma(\beta)} \left( \log \frac{e}{t_{\sigma}} \right)^{\beta-1} + I_{\sigma} \left( \log \frac{e}{t_{\sigma}} \right)^{\beta-2} \right) \right. \\ &\left. - \frac{1}{2\Gamma(\beta)} \left( b - \sum_{\sigma=1}^m J_{\sigma} - \int_1^e \sigma(s) \frac{ds}{s} \right) \right] (\log t)^{\beta-2} + \frac{1}{\Gamma(\beta)} \sum_{\sigma=1}^i \left( \log \frac{t}{t_{\sigma}} \right)^{\beta-1} J_{\sigma} \\ &+ \sum_{\sigma=1}^i \left( \log \frac{t}{t_{\sigma}} \right)^{\beta-2} I_{\sigma} + \frac{1}{\Gamma(\beta)} \int_1^t (\log \frac{t}{s})^{\beta-1} \sigma(s) \frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \end{aligned} \quad (5.3.2)$$

**Proof:** Let  $x$  be a solution of (5.3.1). We know from Theorem 4.0.3 that there exist numbers  $c_{\sigma 1}, c_{\sigma 2} \in \mathbb{R}$  such that

$$\begin{aligned} x(t) &= \sum_{\sigma=0}^i \left[ \frac{c_{\sigma 1}}{\Gamma(\beta)} \left( \log \frac{t}{t_{\sigma}} \right)^{\beta-1} + \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \left( \log \frac{t}{t_{\sigma}} \right)^{\beta-2} \right] \\ &+ \frac{1}{\Gamma(\beta)} \int_1^t (\log \frac{t}{s})^{\beta-1} \sigma(s) \frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \end{aligned} \quad (5.3.3)$$

By direct computation, we have for  $t \in (t_i, t_{i+1}]$

$$\begin{aligned}
{}^{RLH}D_{1+}^{\beta-1}x(t) &= \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \int_1^t \left(\log \frac{t}{s}\right)^{1-\beta} x(s) \frac{ds}{s} \\
&= \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{v=0}^{i-1} \int_{t_v}^{t_{v+1}} \left(\log \frac{t}{s}\right)^{1-\beta} \left[ \sum_{\sigma=0}^v \left( \frac{c_{\sigma 1}}{\Gamma(\beta)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-1} + \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-2} \right) \right. \\
&\quad \left. + \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \right] \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \int_{t_i}^t \left(\log \frac{t}{s}\right)^{1-\beta} \left[ \sum_{\sigma=0}^i \left( \frac{c_{\sigma 1}}{\Gamma(\beta)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-1} + \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-2} \right) \right. \\
&\quad \left. + \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \right] \frac{ds}{s} \\
&= \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{v=0}^{i-1} \sum_{\sigma=0}^v \frac{c_{\sigma 1}}{\Gamma(\beta)} \int_{t_v}^{t_{v+1}} \left(\log \frac{t}{s}\right)^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-1} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{v=0}^{i-1} \sum_{\sigma=0}^v \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \int_{t_v}^{t_{v+1}} \left(\log \frac{t}{s}\right)^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-2} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 1}}{\Gamma(\beta)} \int_{t_i}^t \left(\log \frac{t}{s}\right)^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-1} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \int_{t_i}^t \left(\log \frac{t}{s}\right)^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-2} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\beta)} \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \int_1^t \left(\log \frac{t}{s}\right)^{1-\beta} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \frac{ds}{s}
\end{aligned}$$

changing the order of sums and integral,  $\frac{\log s - \log t_\sigma}{\log t - \log t_\sigma} = w$  or  $\frac{\log s - \log u}{\log t - \log u} = w$

$$\begin{aligned}
&= \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^{i-1} \sum_{v=\sigma}^{i-1} \frac{c_{\sigma 1}}{\Gamma(\beta)} \int_{t_v}^{t_{v+1}} \left(\log \frac{t}{s}\right)^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-1} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^{i-1} \sum_{v=\sigma}^{i-1} \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \int_{t_v}^{t_{v+1}} \left(\log \frac{t}{s}\right)^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-2} \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 1}}{\Gamma(\beta)} \int_{t_i}^t \left(\log \frac{t}{s}\right)^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-1} \frac{ds}{s}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \int_{t_i}^t (\log \frac{t}{s})^{1-\beta} \left(\log \frac{s}{t_\sigma}\right)^{\beta-2} \frac{ds}{s} \\
& + \frac{1}{\Gamma(\beta)} \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \int_1^t \int_u^t (\log \frac{t}{s})^{1-\beta} (\log \frac{s}{u})^{\beta-1} \frac{ds}{s} \sigma(u) \frac{du}{u} \\
& = \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^{i-1} \frac{c_{\sigma 1}}{\Gamma(\beta)} \sum_{v=\sigma}^{i-1} \left(\log \frac{t}{t_\sigma}\right) \int_{\frac{\log t_v - \log t_\sigma}{\log t - \log t_\sigma}}^{\frac{\log t_{v+1} - \log t_\sigma}{\log t - \log t_\sigma}} (1-w)^{1-\beta} w^{\beta-1} dw \\
& + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^{i-1} \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \sum_{v=\sigma}^{i-1} \int_{\frac{\log t_v - \log t_\sigma}{\log t - \log t_\sigma}}^{\frac{\log t_{v+1} - \log t_\sigma}{\log t - \log t_\sigma}} (1-w)^{1-\beta} w^{\beta-2} dw \\
& + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 1}}{\Gamma(\beta)} \left(\log \frac{t}{t_\sigma}\right) \int_{\frac{\log t_i - \log t_\sigma}{\log t - \log t_\sigma}}^1 (1-w)^{1-\beta} w^{\beta-1} dw \\
& + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \int_{\frac{\log t_i - \log t_\sigma}{\log t - \log t_\sigma}}^1 (1-w)^{1-\beta} w^{\beta-2} dw \\
& + \frac{1}{\Gamma(\beta)} \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \int_1^t (\log \frac{t}{u}) \int_0^1 (1-w)^{1-\beta} w^{\beta-1} dw \sigma(u) \frac{du}{u} \\
& = \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 1}}{\Gamma(\beta)} \left(\log \frac{t}{t_\sigma}\right) \int_0^1 (1-w)^{1-\beta} w^{\beta-1} dw \\
& + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \sum_{\sigma=0}^i \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \int_0^1 (1-w)^{1-\beta} w^{\beta-2} dw \\
& + \frac{1}{\Gamma(\beta)} \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right) \int_1^t (\log \frac{t}{u}) \int_0^1 (1-w)^{1-\beta} w^{\beta-1} dw \sigma(u) \frac{du}{u} \\
& = \sum_{\sigma=0}^i c_{\sigma 1} + \int_1^t \sigma(u) \frac{du}{u}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].
\end{aligned}$$

We have

$${}^{RLH}D_{1^+}^{\beta-1} x(t) = \sum_{\sigma=0}^i c_{\sigma 1} + \int_1^t \sigma(u) \frac{du}{u}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \quad (5.3.4)$$

One sees that

$$\begin{aligned}
& (\log t)^{2-\beta} \left| \int_1^t (\log \frac{t}{s})^{\beta-1} \sigma(s) \frac{ds}{s} \right| \leq (\log t)^{2-\beta} \int_1^t (\log \frac{t}{s})^{\beta-1} (\log s)^k (\log \frac{e}{s})^l \frac{ds}{s} \\
& \leq (\log t)^{2-\beta} \int_1^t (\log \frac{t}{s})^{\beta+l-1} (\log s)^k \frac{ds}{s} \\
& = (\log t)^{2-\beta} (\log t)^{\beta+k+l} \int_0^1 (1-w)^{\beta+l-1} w^k dw \rightarrow 0 \text{ as } t \rightarrow 1^+.
\end{aligned}$$

Similarly we have

$$\begin{aligned} \left| \int_1^t \sigma(s) \frac{ds}{s} \right| &\leq \int_1^t (\log s)^k \left( \log \frac{e}{s} \right)^l \frac{ds}{s} \leq \int_1^t \left( \log \frac{t}{s} \right)^l (\log s)^k \frac{ds}{s} \\ &= (\log t)^{k+l+1} \int_0^1 (1-w)^l w^k dw = (\log t)^{k+l+1} \mathbf{B}(l+1, k+1) \rightarrow 0 \text{ as } t \rightarrow 1^+. \end{aligned}$$

It follows from (5.3.3), (5.3.4), the boundary conditions and the impulse assumption in (5.3.1) that

$$\begin{aligned} \frac{1}{\Gamma(\beta-1)} c_{02} + \sum_{\sigma=0}^m \left[ \frac{c_{\sigma 1}}{\Gamma(\beta)} \left( \log \frac{e}{t_\sigma} \right)^{\beta-1} + \frac{c_{\sigma 2}}{\Gamma(\beta-1)} \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} \right] + \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{t}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s} &= a, \\ c_{01} + \sum_{\sigma=0}^m c_{\sigma 1} + \int_1^e \sigma(u) \frac{du}{u} &= b. \end{aligned}$$

and  $c_{i2} = \Gamma(\beta-1)I_i (i \in \mathbb{N}[1, m])$ ,  $c_{i1} = J_i (i \in \mathbb{N}[1, m])$ . Then

$$\begin{aligned} c_{01} &= \frac{1}{2} \left[ b - \sum_{\sigma=1}^m J_\sigma - \int_1^e \sigma(s) \frac{ds}{s} \right], \\ c_{02} &= \frac{\Gamma(\beta-1)}{2} \left[ a - \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{t}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s} - \sum_{\sigma=1}^m \left( \frac{J_\sigma}{\Gamma(\beta)} \left( \log \frac{e}{t_\sigma} \right)^{\beta-1} + I_\sigma \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} \right) \right. \\ &\quad \left. - \frac{1}{2\Gamma(\beta)} \left( b - \sum_{\sigma=1}^m J_\sigma - \int_1^e \sigma(s) \frac{ds}{s} \right) \right]. \end{aligned} \tag{5.3.5}$$

Substituting  $c_{\sigma v} (\sigma \in \mathbb{N}[0, m], v \in \mathbb{N}[1, 2])$  into (5.3.3), we get (5.3.2) obviously.

On the other hand, if  $x$  satisfies (5.3.2), then  $x|_{(t_i, t_{i+1}]} (i \in \mathbb{N}[0, m])$  are continuous and the limits  $\lim_{t \rightarrow t_i^+} (\log t - \log t_i)^{2-\beta} x(t) (i \in \mathbb{N}[0, m])$  exist. So  $x \in LP_m C_{2-\beta}(1, e]$ . Furthermore, by direct computation, we have  $\lim_{t \rightarrow 1^+} (\log t)^{2-\beta} x(t) + x(e) = a$ ,  $\lim_{t \rightarrow 1^+} {}^{RLH}D_{1^+}^{\beta-1} x(t) + {}^{RLH}D_{1^+}^{\beta-1} x(e) = b$ ,  $\lim_{t \rightarrow t_i^+} (\log t - \log t_i)^{2-\beta} x(t) = I_i, i \in \mathbb{N}[1, m]$  and  $\Delta^{RL}D_{1^+}^{\beta-1} x(t_i) = J_i, i \in \mathbb{N}[1, m]$ .

Using (5.3.5) and  $c_{i2} = \Gamma(\beta-1)I_i (i \in \mathbb{N}[1, m])$ ,  $c_{i1} = J_i (i \in \mathbb{N}[1, m])$ , we rewrite  $x$  by (5.3.3).

One has similarly for  $t \in (t_i, t_{i+1}]$

$$\begin{aligned}
{}^{RLH}D_{1+}^\beta x(t) &= \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right)^2 \int_1^t \left(\log \frac{t}{s}\right)^{1-\beta} x(s) \frac{ds}{s} \\
&= \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right)^2 \left[ \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left(\log \frac{t}{s}\right)^{1-\beta} \left( \sum_{\sigma=0}^i \sum_{v=1}^2 \frac{c_{\sigma v}}{\Gamma(\beta-v+1)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-v} \right. \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \right) \frac{ds}{s} \right. \\
&\quad \left. + \int_{t_j}^t \left(\log \frac{t}{s}\right)^{1-\beta} \left( \sum_{\sigma=0}^j \sum_{v=1}^2 \frac{c_{\sigma v}}{\Gamma(\beta-v+1)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-v} \right. \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \right) \frac{ds}{s} \right] \\
&= \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right)^2 \left[ \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \left(\log \frac{t}{s}\right)^{1-\beta} \left( \sum_{\sigma=0}^i \sum_{v=1}^2 \frac{c_{\sigma v}}{\Gamma(\beta-v+1)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-v} \frac{ds}{s} \right) \right. \\
&\quad \left. + \frac{1}{\Gamma(2-\beta)} \left(t \frac{d}{dt}\right)^2 \left[ \int_{t_j}^t \left(\log \frac{t}{s}\right)^{1-\beta} \left( \sum_{\sigma=0}^j \sum_{v=1}^2 \frac{c_{\sigma v}}{\Gamma(\beta-v+1)} \left(\log \frac{s}{t_\sigma}\right)^{\beta-v} \frac{ds}{s} \right) \right. \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(2-\beta)\Gamma(\beta)} \left(t \frac{d}{dt}\right)^2 \left[ \int_1^t \left(\log \frac{t}{s}\right)^{1-\beta} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \right) \frac{ds}{s} \right] \right]
\end{aligned}$$

By using the method above, we have

$${}^{RLH}D_{1+}^\beta x(t) = \sigma(t).$$

So  $x$  is a solution of (5.3.1). The proof is completed.  $\square$

Define the nonlinear operator  $R$  on  $LP_m C_{2-\beta}(1, e]$  for  $x \in LP_m C_{2-\alpha}(1, e]$  by  $(Rx)(t)$  by

$$\begin{aligned}
(Rx)(t) &= \frac{1}{2\Gamma(\beta)} \left[ \int_1^e \psi(s) H(s, x(s)) ds - \sum_{\sigma=1}^m J(t_\sigma, x(t_\sigma)) - \int_1^e q(s) f(s, x(s)) \frac{ds}{s} \right] (\log t)^{\beta-1} \\
&\quad + \frac{1}{2} \left[ \int_1^e \phi(s) G(s, x(s)) ds - \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{t}{s}\right)^{\beta-1} q(s) f(s, x(s)) \frac{ds}{s} \right. \\
&\quad \left. - \sum_{\sigma=1}^m \left( \frac{J(t_\sigma, x(t_\sigma))}{\Gamma(\beta)} \left(\log \frac{e}{t_\sigma}\right)^{\beta-1} + I(t_\sigma, x(t_\sigma)) \left(\log \frac{e}{t_\sigma}\right)^{\beta-2} \right) \right. \\
&\quad \left. - \frac{1}{2\Gamma(\beta)} \left( \int_1^e \psi(s) H(s, x(s)) ds - \sum_{\sigma=1}^m J(t_\sigma, x(t_\sigma)) - \int_1^e q(s) f(s, x(s)) \frac{ds}{s} \right) \right] (\log t)^{\beta-2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\beta)} \sum_{\sigma=1}^i \left( \log \frac{t}{t_\sigma} \right)^{\beta-1} J(t_\sigma, x(t_\sigma)) + \sum_{\sigma=1}^i \left( \log \frac{t}{t_\sigma} \right)^{\beta-2} I(t_\sigma, x(t_\sigma)) \\
& + \frac{1}{\Gamma(\beta)} \int_1^t \left( \log \frac{t}{s} \right)^{\beta-1} q(s) f(s, x(s)) \frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].
\end{aligned}$$

**Lemma 5.3.2** *Suppose that (i)-(k) hold, and  $f, G, H$  are impulsive III-Carathéodory functions,  $I, J$  a discrete III-Carathéodory functions. Then  $R : LP_m C_{2-\beta}(1, e] \rightarrow LP_m C_{2-\alpha}(1, e]$  is well defined and is completely continuous,  $x$  is a solution of BVP (1.0.10) if and only if  $x$  is a fixed point of  $R$  in  $LP_m C_{2-\alpha}(1, e]$ .*

**Proof:** The proof is similar to that of the proof of Lemma 3.3.2 and is omitted.  $\square$

**Theorem 5.3.1** *Suppose that (i), (j) and (k) hold,  $f, G, H$  are impulsive III-Carathéodory functions,  $I, J$  a discrete III-Carathéodory functions and*

**(C3)** *there exist nondecreasing functions  $M_f, M_g, M_h, M_I, M_J : [0, +\infty) \rightarrow [0, +\infty)$  such that*

$$\begin{aligned}
& \left| f \left( t, \left( \log \frac{t}{t_i} \right)^{\beta-2} x \right) \right| \leq M_f(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R}, \\
& \left| G \left( t, \left( \log \frac{t}{t_i} \right)^{\beta-2} x \right) \right| \leq M_g(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R}, \\
& \left| H \left( t, \left( \log \frac{t}{t_i} \right)^{\beta-2} x \right) \right| \leq M_h(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R}, \\
& \left| I \left( t_i, \left( \log \frac{t_i}{t_{i-1}} \right)^{\beta-2} x \right) \right| \leq M_I(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}, \\
& \left| J \left( t_i, \left( \log \frac{t_i}{t_{i-1}} \right)^{\beta-2} x \right) \right| \leq M_J(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}.
\end{aligned}$$

Then BVP (1.0.10) has at least one solution if there exists a constant  $r_0 > 0$  such that

$$\begin{aligned}
& \left[ \frac{\|\psi\|_1}{2\Gamma(\beta)} + \frac{\|\psi\|_1}{4\Gamma(\beta)} \right] M_H(r_0) + \frac{\|\phi\|_1}{2} M_G(r_0) \\
& + \left[ \frac{m}{2\Gamma(\beta)} + \frac{m}{2\Gamma(\beta)} + \frac{m}{4\Gamma(\beta)} + \frac{m}{\Gamma(\beta)} \right] M_J(r_0) + \left[ \frac{1}{2} \sum_{\sigma=1}^m \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} + m \right] M_I(r_0) \quad (5.3.6) \\
& + \left[ \frac{\mathbf{B}(l+1, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(l+1, k+1)}{4\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0.
\end{aligned}$$

**Proof:** From Lemma 5.3.1 and the definition of  $R$ ,  $x \in LP_m C_{2-\beta}(1, e]$  is a solution of BVP (1.0.10) if and only if  $x \in LP_m C_{2-\beta}(1, e]$  is a fixed point of  $R$ . Lemma 5.3.2 implies that  $R$  is a completely continuous operator.

From (C3), we have for  $x \in LP_m C_{2-\beta}(1, e]$  that

$$\begin{aligned}
|f(t, x(t))| &= \left| f \left( t, \left( \log \frac{t}{t_i} \right)^{\beta-2} \left( \log \frac{t}{t_i} \right)^{2-\beta} x(t) \right) \right| \\
&\leq M_f \left( \left| \left( \log \frac{t}{t_i} \right)^{2-\beta} x(t) \right| \right) \leq M_f(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\
|G(t, x(t))| &\leq M_G(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\
|H(t, x(t))| &\leq M_H(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\
|I(t_i, x(t_i))| &= \left| I \left( t_i, \left( \log \frac{t_i}{t_{i-1}} \right)^{\beta-2} \left( \log \frac{t_i}{t_{i-1}} \right)^{2-\beta} x(t) \right) \right| \\
&\leq M_f \left( \left| \left( \log \frac{t_i}{t_{i-1}} \right)^{2-\beta} x(t) \right| \right) \leq M_G(\|x\|), i \in \mathbb{N}[1, m], \\
|I(t_i, x(t_i))| &\leq M_H(\|x\|), i \in \mathbb{N}[1, m].
\end{aligned}$$

We consider the set  $\Omega = \{x \in LP_m C_{2-\beta}(1, e] : x = \lambda(Rx), \text{ for some } \lambda \in [0, 1]\}$ . For  $x \in \Omega$ , we have for  $t \in (t_i, t_{i+1}]$  that

$$\begin{aligned}
&\left( \log \frac{t}{t_i} \right)^{2-\beta} |(Rx)(t)| \\
&\leq \frac{1}{2\Gamma(\beta)} \left[ \|\psi\|_1 M_H(\|x\|) + m M_J(\|x\|) + \int_1^e (\log s)^l (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \right] (\log t)^{\beta-1} \left( \log \frac{t}{t_i} \right)^{2-\beta} \\
&\quad + \frac{1}{2} \left[ \|\phi\|_1 M_G(\|x\|) + \frac{1}{\Gamma(\beta)} \int_1^e (\log \frac{t}{s})^{\beta-1} (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \right. \\
&\quad \left. + \sum_{\sigma=1}^m \left( \frac{1}{\Gamma(\beta)} M_J(\|x\|) + \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} M_I(\|x\|) \right) \right. \\
&\quad \left. + \frac{1}{2\Gamma(\beta)} (\|\psi\|_1 M_H(\|x\|) + m M_J(\|x\|)) \right. \\
&\quad \left. + \int_1^e (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \right] (\log t)^{\beta-2} \left( \log \frac{t}{t_i} \right)^{2-\beta}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\beta)} \sum_{\sigma=1}^i \left( \log \frac{t}{t_i} \right)^{2-\beta} \left( \log \frac{t}{t_\sigma} \right)^{\beta-1} M_J(\|x\|) + \sum_{\sigma=1}^i \left( \log \frac{t}{t_i} \right)^{2-\beta} \left( \log \frac{t}{t_\sigma} \right)^{\beta-2} M_I(\|x\|) \\
& + \frac{1}{\Gamma(\beta)} \int_1^t \left( \log \frac{t}{s} \right)^{\beta-1} (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \\
\leq & \frac{\|\psi\|_1}{2\Gamma(\beta)} M_H(\|x\|) + \frac{m}{2\Gamma(\beta)} M_J(\|x\|) + \frac{\mathbf{B}(l+1, k+1)}{2\Gamma(\beta)} M_f(\|x\|) \\
& + \frac{\|\phi\|_1}{2} M_G(\|x\|) + \frac{1}{2\Gamma(\beta)} \int_1^e \left( \log \frac{t}{s} \right)^{\beta+l-1} (\log s)^k \frac{ds}{s} M_f(\|x\|) \\
& + \frac{m}{2\Gamma(\beta)} M_J(\|x\|) + \frac{1}{2} \sum_{\sigma=1}^m \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} M_I(\|x\|) \\
& + \frac{\|\psi\|_1}{4\Gamma(\beta)} M_H(\|x\|) + \frac{m}{4\Gamma(\beta)} M_J(\|x\|) + \frac{1}{4\Gamma(\beta)} \int_1^e (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \\
& + \frac{m}{\Gamma(\beta)} M_J(\|x\|) + m M_I(\|x\|) + \frac{1}{\Gamma(\beta)} \int_1^t \left( \log \frac{t}{s} \right)^{\beta+l-1} (\log s)^k \frac{ds}{s} M_f(\|x\|) \\
\leq & \left[ \frac{\|\psi\|_1}{2\Gamma(\beta)} + \frac{\|\psi\|_1}{4\Gamma(\beta)} \right] M_H(\|x\|) + \frac{\|\phi\|_1}{2} M_G(\|x\|) \\
& + \left[ \frac{m}{2\Gamma(\beta)} + \frac{m}{2\Gamma(\beta)} + \frac{m}{4\Gamma(\beta)} + \frac{m}{\Gamma(\beta)} \right] M_J(\|x\|) + \left[ \frac{1}{2} \sum_{\sigma=1}^m \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} + m \right] M_I(\|x\|) \\
& + \left[ \frac{\mathbf{B}(l+1, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(l+1, k+1)}{4\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x\| = \lambda \|Rx\| & < \|Rx\| \leq \left[ \frac{\|\psi\|_1}{2\Gamma(\beta)} + \frac{\|\psi\|_1}{4\Gamma(\beta)} \right] M_H(\|x\|) + \frac{\|\phi\|_1}{2} M_G(\|x\|) \\
& + \left[ \frac{m}{2\Gamma(\beta)} + \frac{m}{2\Gamma(\beta)} + \frac{m}{4\Gamma(\beta)} + \frac{m}{\Gamma(\beta)} \right] M_J(\|x\|) + \left[ \frac{1}{2} \sum_{\sigma=1}^m \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} + m \right] M_I(\|x\|) \\
& + \left[ \frac{\mathbf{B}(l+1, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(l+1, k+1)}{4\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|).
\end{aligned}$$

From (5.3.6), we choose  $\Omega = \{x \in LP_m C_{2-\beta}(1, e) : \|x\| \leq r_0\}$ . For  $x \in \partial\Omega$ , we get  $x \neq \lambda(Rx)$  for any  $\lambda \in [0, 1)$ . In fact, if there exists  $x \in \partial\Omega$  such that  $x = \lambda(Rx)$  for some  $\lambda \in [0, 1)$ .

Then

$$\begin{aligned}
r_0 = \|x\| = \lambda \|Rx\| < \|Rx\| &\leq \left[ \frac{\|\psi\|_1}{2\Gamma(\beta)} + \frac{\|\psi\|_1}{4\Gamma(\beta)} \right] M_H(r_0) + \frac{\|\phi\|_1}{2} M_G(r_0) \\
&+ \left[ \frac{m}{2\Gamma(\beta)} + \frac{m}{2\Gamma(\beta)} + \frac{m}{4\Gamma(\beta)} + \frac{m}{\Gamma(\beta)} \right] M_J(r_0) + \left[ \frac{1}{2} \sum_{\sigma=1}^m \left( \log \frac{e}{t_\sigma} \right)^{\beta-2} + m \right] M_I(r_0) \\
&+ \left[ \frac{\mathbf{B}(l+1, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(l+1, k+1)}{4\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0,
\end{aligned}$$

a contradiction.

As a consequence of Lemma 5.0.1, we deduce that  $R$  has a fixed point which is a solution of the problem BVP (1.0.10). The proof is completed. The proof of Theorem 5.3.1 is completed.  $\square$

#### 5.4 Solvability of BVP (1.0.11)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.11).

**Lemma 5.4.1** *Suppose that  $\sigma : (0, 1) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > 1 - \beta$  and  $l \leq 0$  with  $l > \max\{-\beta, -\beta - k\}$  such that  $|\sigma(t)| \leq (\log t)^k (1 - \log t)^l$  for all  $t \in (1, e)$ . The  $x$  is a solutions of*

$$\left\{ \begin{array}{l}
{}^{CH}D_{1^+}^\beta x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\
x(1) - \left(t \frac{d}{dt}\right) x(t) \Big|_{t=1} = a, \\
x(e) + \left(t \frac{d}{dt}\right) x(t) \Big|_{t=e} = b, \\
\lim_{t \rightarrow t_i^+} x(t) - x(t_i) = I_i, i \in \mathbb{N}[1, m], \\
\lim_{t \rightarrow t_i^+} \left(t \frac{d}{dt}\right) x(t) - \left(t \frac{d}{dt}\right) x(t) \Big|_{t=t_i} = J_i, i \in \mathbb{N}[1, m],
\end{array} \right. \quad (5.4.1)$$

if and only if  $x \in LP_m C(1, e]$  and

$$\begin{aligned}
x(t) = & \frac{1}{3} \left[ b + 2a - \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s} - \frac{1}{\Gamma(\beta-1)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-2} \sigma(s) \frac{ds}{s} \right. \\
& \left. - \sum_{\sigma=1}^m I_\sigma - \sum_{\sigma=1}^m \left( 1 + \log \frac{e}{t_\sigma} \right) J_\sigma \right] \\
& + \frac{1}{3} \left[ b - a - \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s} - \frac{1}{\Gamma(\beta-1)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-2} \sigma(s) \frac{ds}{s} \right. \\
& \left. - \sum_{\sigma=1}^m I_\sigma - \sum_{\sigma=1}^m \left( 1 + \log \frac{e}{t_\sigma} \right) J_\sigma \right] \log t \\
& + \sum_{\sigma=1}^i \left[ I_\sigma + J_\sigma \left( \log \frac{t}{t_\sigma} \right) \right] + \frac{1}{\Gamma(\beta)} \int_1^t \left( \log \frac{t}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].
\end{aligned} \tag{5.4.2}$$

**Proof:** Let  $x$  be a solution of (5.4.1). We know by Theorem 4.0.4 that there exist numbers  $c_{\sigma 0}, c_{\sigma 1} \in \mathbb{R} (\sigma \in \mathbb{N}[0, n-1])$  such that

$$x(t) = \sum_{\sigma=0}^i \left[ c_{\sigma 0} + c_{\sigma 1} \left( \log \frac{t}{t_\sigma} \right) \right] + \frac{1}{\Gamma(\beta)} \int_1^t \left( \log \frac{t}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \tag{5.4.3}$$

It follows that

$$\left( t \frac{d}{dt} \right) x(t) = \sum_{\sigma=0}^i c_{\sigma 1} + \frac{1}{\Gamma(\beta-1)} \int_1^t \left( \log \frac{t}{s} \right)^{\beta-2} \sigma(s) \frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \tag{5.4.4}$$

It follows from (5.4.3), (5.4.4), the boundary conditions and the impulse assumption in (5.4.1) that  $c_{\sigma 0} = I_\sigma$ ,  $c_{\sigma 1} = J_\sigma$ ,  $\sigma \in \mathbb{N}[1, m]$ ,  $c_{00} - c_{01} = a$  and

$$\sum_{\sigma=0}^m \left[ c_{\sigma 0} + c_{\sigma 1} \left( \log \frac{e}{t_\sigma} \right) \right] + \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s} + \sum_{\sigma=0}^m c_{\sigma 1} + \frac{1}{\Gamma(\beta-1)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-2} \sigma(s) \frac{ds}{s} = b.$$

Then

$$\begin{aligned}
c_{00} = & \frac{1}{3} \left[ b + 2a - \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s} - \frac{1}{\Gamma(\beta-1)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-2} \sigma(s) \frac{ds}{s} \right. \\
& \left. - \sum_{\sigma=1}^m I_\sigma - \sum_{\sigma=1}^m \left( 1 + \log \frac{e}{t_\sigma} \right) J_\sigma \right],
\end{aligned} \tag{5.4.5}$$

$$\begin{aligned}
c_{01} = & \frac{1}{3} \left[ b - a - \frac{1}{\Gamma(\beta)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-1} \sigma(s) \frac{ds}{s} - \frac{1}{\Gamma(\beta-1)} \int_1^e \left( \log \frac{e}{s} \right)^{\beta-2} \sigma(s) \frac{ds}{s} \right. \\
& \left. - \sum_{\sigma=1}^m I_\sigma - \sum_{\sigma=1}^m \left( 1 + \log \frac{e}{t_\sigma} \right) J_\sigma \right].
\end{aligned}$$

Substituting  $c_{\sigma 0}, c_{\sigma 1} (\sigma \in \mathbb{N}[0, m])$  into (5.4.3), we get (5.4.2) obviously.

On the other hand, if  $x$  satisfies (5.4.2), then  $x|_{(t_i, t_{i+1}]}$  ( $i \in \mathbb{N}[0, m]$ ) are continuous and the limits  $\lim_{t \rightarrow t_i^+} x(t)$ . So  $x \in LP_m C(1, e]$ . Using (5.4.5),  $c_{\sigma 0} = I_\sigma$ ,  $c_{\sigma 1} = J_\sigma$ ,  $\sigma \in \mathbb{N}[1, m]$ , we rewrite  $x$  by (5.4.3). One have from Theorem 4.0.4 easily for  $t \in (t_0, t_1]$  that  ${}^{CH}D_{1+}^\beta x(t) = \lambda x(t) + \sigma(t)$  and for  $t \in (t_i, t_{i+1}]$  similarly to the proof of Theorem 4.0.1 that

$$\begin{aligned} {}^{CH}D_{1+}^\beta x(t) &= \frac{1}{\Gamma(2-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{1-\alpha} \left(s \frac{d}{ds}\right)^2 x(s) \frac{ds}{s} \\ &= \frac{1}{\Gamma(2-\beta)} \sum_{v=0}^{i-1} \int_{t_v}^{t_{v+1}} \left(\log \frac{t}{s}\right)^{1-\alpha} \left(s \frac{d}{ds}\right)^2 \left[ \sum_{\sigma=0}^v \left( c_{\sigma 0} + c_{\sigma 1} \left(\log \frac{s}{t_\sigma}\right) \right) \right. \\ &\quad \left. + \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \right] \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(2-\beta)} \int_{t_i}^t \left(\log \frac{t}{s}\right)^{1-\alpha} \left(s \frac{d}{ds}\right)^2 \left[ \sum_{\sigma=0}^i \left( c_{\sigma 0} + c_{\sigma 1} \left(\log \frac{s}{t_\sigma}\right) \right) \right. \\ &\quad \left. + \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{u}\right)^{\beta-1} \sigma(u) \frac{du}{u} \right] \frac{ds}{s} = \sigma(t). \end{aligned}$$

So  $x$  is a solution of (5.4.1). The proof is completed.  $\square$

Define the nonlinear operator  $J$  on  $LP_m C(1, e]$  by  $(Jx)$  by

$$\begin{aligned} (Jx)(t) &= \frac{1}{3} \left[ \int_1^e \psi(s) H(s, x(s)) ds + 2 \int_1^e \phi(s) G(s, x(s)) ds - \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-1} p(s) f(s, x(s)) \frac{ds}{s} \right. \\ &\quad \left. - \frac{1}{\Gamma(\beta-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-2} p(s) f(s, x(s)) \frac{ds}{s} - \sum_{\sigma=1}^m I(t_\sigma, x(t_\sigma)) - \sum_{\sigma=1}^m \left(1 + \log \frac{e}{t_\sigma}\right) J(t_\sigma, x(t_\sigma)) \right] \\ &\quad + \frac{1}{3} \left[ \int_1^e \psi(s) H(s, x(s)) ds - \int_1^e \phi(s) G(s, x(s)) ds - \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-1} p(s) f(s, x(s)) \frac{ds}{s} \right. \\ &\quad \left. - \frac{1}{\Gamma(\beta-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-2} p(s) f(s, x(s)) \frac{ds}{s} - \sum_{\sigma=1}^m I(t_\sigma, x(t_\sigma)) - \sum_{\sigma=1}^m \left(1 + \log \frac{e}{t_\sigma}\right) J(t_\sigma, x(t_\sigma)) \right] \log t \\ &\quad + \sum_{\sigma=1}^i \left[ I(t_\sigma, x(t_\sigma)) + J(t_\sigma, x(t_\sigma)) \left(\log \frac{t}{t_\sigma}\right) \right] + \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} p(s) f(s, x(s)) \frac{ds}{s}, \end{aligned}$$

$$t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

**Lemma 5.4.2** *Suppose that (g), (l), (m) and (n) hold, and  $f, G, H$  are impulsive IV-Carathéodory functions,  $I, J$  discrete IV-Carathéodory functions. Then  $R$  :*

$LP_mC(1, e] \rightarrow LP_mC(1, e]$  is well defined and is completely continuous,  $x$  is a solution of BVP (1.0.11) if and only if  $x$  is a fixed point of  $J$  in  $LP_mC(1, e]$ .

**Proof:** The proof is similar to that of the proof of Theorem 3.3.2 and is omitted.  $\square$

**Theorem 5.4.1** Suppose that (1)-(n) hold,  $f, G, H$  are impulsive IV-Carathéodory functions,  $I, J$  discrete IV-Carathéodory functions and

(C4) there exist nondecreasing functions  $M_f, M_g, M_h, M_I, M_J : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$|f(t, x)| \leq M_f(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|G(t, x)| \leq M_g(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|H(t, x)| \leq M_h(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|I(t_i, x)| \leq M_I(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R},$$

$$|J(t_i, x)| \leq M_J(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}.$$

Then BVP (1.0.11) has at least one solution if there exists a constant  $r_0 > 0$  such that

$$\begin{aligned} & \frac{2\|\psi\|_1}{3} M_H(r_0) + \|\phi\|_1 M_G(r_0) + \frac{5m}{3} M_I(r_0) + \frac{7m}{3} M_J(r_0) \\ & + \left[ \frac{2\mathbf{B}(\beta+l, k+1)}{3\Gamma(\beta)} + \frac{2\mathbf{B}(\beta+l-1, k+1)}{3\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0. \end{aligned} \quad (5.4.6)$$

**Proof:** From Lemma 5.4.1, and the definition of  $J$ ,  $x \in LP_mC(1, e]$  is a solution of BVP (1.0.11) if and only if  $x \in LP_mC(1, e]$  is a fixed point of  $R$ . Lemma 5.4.2 implies that  $J$  is a completely continuous operator.

From (C4), we have for  $x \in LP_mC(1, e]$  that

$$|f(t, x(t))| \leq M_f(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m],$$

$$|G(t, x(t))| \leq M_g(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m],$$

$$|H(t, x(t))| \leq M_h(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m],$$

$$|I(t_i, x(t_i))| \leq M_I(\|x\|), i \in \mathbb{N}[1, m]$$

$$|J(t_i, x(t_i))| \leq M_J(\|x\|), i \in \mathbb{N}[1, m].$$

We consider the set  $\Omega = \{x \in LP_m C(1, e) : x = \lambda(Jx), \text{ for some } \lambda \in [0, 1]\}$ . For  $x \in \Omega$ , we have for  $t \in (t_i, t_{i+1}]$  that

$$\begin{aligned}
& |(Jx)(t)| \leq \\
& \frac{1}{3} \left[ \|\psi\|_1 M_H(\|x\|) + 2\|\phi\|_1 M_G(\|x\|) + \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-1} (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \right. \\
& \left. + \frac{1}{\Gamma(\beta-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-2} (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) + m M_I(\|x\|) + 2m M_J(\|x\|) \right] \\
& + \frac{1}{3} \left[ \|\psi\|_1 M_H(\|x\|) + \|\phi\|_1 M_G(\|x\|) + \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-1} (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \right. \\
& \left. + \frac{1}{\Gamma(\beta-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-2} (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) + m M_I(\|x\|) + 2m M_J(\|x\|) \right] \\
& + m M_I(\|x\|) + m M_J(\|x\|) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} (\log s)^k (1 - \log s)^l \frac{ds}{s} M_f(\|x\|) \\
& = \frac{2\|\psi\|_1}{3} M_H(\|x\|) + \|\phi\|_1 M_G(\|x\|) + \frac{5m}{3} M_I(\|x\|) + \frac{7m}{3} M_J(\|x\|) \\
& + \left[ \frac{2\mathbf{B}(\beta+l, k+1)}{3\Gamma(\beta)} + \frac{2\mathbf{B}(\beta+l-1, k+1)}{3\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x\| = \lambda \|Rx\| < \|Rx\| &\leq \frac{2\|\psi\|_1}{3} M_H(\|x\|) + \|\phi\|_1 M_G(\|x\|) + \frac{5m}{3} M_I(\|x\|) + \frac{7m}{3} M_J(\|x\|) \\
& + \left[ \frac{2\mathbf{B}(\beta+l, k+1)}{3\Gamma(\beta)} + \frac{2\mathbf{B}(\beta+l-1, k+1)}{3\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(\|x\|).
\end{aligned}$$

From (5.4.6), we choose  $\Omega = \{x \in LP_m C(1, e) : \|x\| \leq r_0\}$ . For  $x \in \partial\Omega$ , we get  $x \neq \lambda(Jx)$  for any  $\lambda \in [0, 1)$ . In fact, if there exists  $x \in \partial\Omega$  such that  $x = \lambda(Jx)$  for some  $\lambda \in [0, 1)$ . Then

$$\begin{aligned}
r_0 = \|x\| = \lambda \|Jx\| < \|Jx\| &\leq \frac{2\|\psi\|_1}{3} M_H(r_0) + \|\phi\|_1 M_G(r_0) + \frac{5m}{3} M_I(r_0) + \frac{7m}{3} M_J(r_0) \\
& + \left[ \frac{2\mathbf{B}(\beta+l, k+1)}{3\Gamma(\beta)} + \frac{2\mathbf{B}(\beta+l-1, k+1)}{3\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l, k+1)}{\Gamma(\beta)} \right] M_f(r_0) \leq r_0,
\end{aligned}$$

a contradiction.

As a consequence of Lemma 5.0.1, we deduce that  $J$  has a fixed point which is a solution of the problem BVP (1.0.11). The proof is completed. The proof of Theorem 5.4.1 is completed.  $\square$

## 6 Comparison results

In this section, we study the solvability of two class of impulsive boundary value problems of fractional differential equations for showing readers the differences between those equations one's fractional derivative has single start point and the other's has multiple starting points.

**Example 6.0.1.** Let  $0 = t_0 < t_1 < t_2 = 1$ . Consider the following problems:

$$\begin{cases} {}^C D_{t_i^+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, 1], \\ \Delta x(t_1) = I, \quad \Delta x'(t_1) = J, \\ x'(0) = x_0, \quad x'(1) = x_1, \end{cases} \quad (6.0.1)$$

and

$$\begin{cases} {}^C D_{0^+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, 1], \\ \Delta x(t_1) = I, \quad \Delta x'(t_1) = J, \\ x'(0) = x_0, \quad x'(1) = x_1. \end{cases} \quad (6.0.2)$$

It is easy to show that BVP (6.0.1) has solutions

$$x(t) = \begin{cases} c + x_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ c + I + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - t_1 \left[ x_1 - x_0 - \int_{t_1}^1 \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] \\ + t \left[ x_1 - \int_{t_1}^1 \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] + \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in (t_1, 1] \end{cases}$$

if and only if

$$\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds + \int_{t_1}^1 \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds = x_1 - x_0 - J. \quad (6.0.3)$$

However, BVP (6.0.2) has solutions

$$x(t) = \begin{cases} c + x_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ c + I - t_1 J + (x_0 + J)t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in (t_1, 1] \end{cases}$$

if and only if

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds = x_1 - x_0 - J. \quad (6.0.4)$$

One sees that (6.0.3) is different from (6.0.4).  $\square$

**Example 6.0.2.** Let  $0 = t_0 < t_1 < t_2 = 1$ . Consider the following problems:

$$\begin{cases} {}^C D_{t_i^+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, 1], \\ \Delta x(t_1) = ax(t_1), \quad \Delta x'(t_1) = bx'(t_1), \\ x(0) = x_0, \quad x'(1) = x_1, \end{cases} \quad (6.0.5)$$

and

$$\begin{cases} {}^C D_{0^+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, 1], \\ \Delta x(t_1) = ax(t_1), \quad \Delta x'(t_1) = bx'(t_1), \\ x(0) = x_0, \quad x'(1) = x_1. \end{cases} \quad (6.0.6)$$

It is easy to show that

(i) BVP (6.0.5) has no solution if and only if  $1 + b = 0$  and  $x_1 - \int_{t_1}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \neq 0$ ;

(ii) BVP (6.0.5) has a unique solution

$$x(t) = \begin{cases} x_0 + \frac{t}{1+b} \left[ x_1 - \int_{t_1}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - (1+b) \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ \frac{(1+a)t_1}{1+b} \left[ x_1 - \int_{t_1}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - (1+b) \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds + x_0 \right. \\ \left. + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \right] \\ + \left[ x_1 - \int_{t_1}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] t + \int_{t_1}^t \frac{t-s}{\Gamma(\alpha)} p(s) ds, t \in (t_1, 1] \end{cases}$$

if and only if  $1 + b \neq 0$ ;

(iii) BVP (6.0.5) has infinitely many solutions

$$x(t) = \begin{cases} x_0 + dt + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ (1+a) \left[ t_1 d + x_0 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \right] + \left[ x_1 - \int_{t_1}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] t \\ + \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in (t_1, 1] \end{cases}$$

if and only if  $1 + b = 0$  and  $x_1 - \int_{t_1}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds = 0$ ;

(iv) BVP (6.0.6) has no solution if and only if  $1 + b = 0$  and  $x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \neq 0$ ;

(v) BVP (6.0.6) has a unique solution

$$x(t) = \begin{cases} x_0 + \frac{t}{1+b} \left[ x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - b \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ (1+a) \left[ x_0 + \frac{t_1}{1+b} \left( x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - b \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right) \right. \\ \left. + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \right] \\ - t_1 \left[ x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \\ + \left[ x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] t + \int_0^t \frac{t-s}{\Gamma(\alpha)} p(s) ds, t \in (t_1, 1] \end{cases}$$

if and only if  $1 + b \neq 0$ ;

(vi) BVP (6.0.6) has infinitely many solutions

$$x(t) = \begin{cases} x_0 + dt + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ (1+a) \left[ x_0 + dt_1 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \right] - t_1 \left[ x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] \\ - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \\ + \left[ x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right] t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in (t_1, 1] \end{cases}$$

if and only if  $1 + b = 0$  and  $x_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds = 0$ .  $\square$

Now, we consider the existence of solutions of the following nonlinear boundary value problems of fractional impulsive differential equations

$$\begin{cases} {}^C D_{t_i^+}^\alpha x(t) = p(t)f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta x(t_i) = I(t_i, x(t_i)), \Delta x'(t_i) = J(t_i, x(t_i)), i \in \mathbb{N}[1, m], \\ ax(0) - bx'(0) = x_0, cx(1) + dx'(1) = x_1, \end{cases} \quad (6.0.7)$$

and

$$\begin{cases} {}^C D_{0^+}^\alpha x(t) = q(t)g(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta x(t_i) = I(t_i, x(t_i)), \Delta x'(t_i) = J(t_i, x(t_i)), i \in \mathbb{N}[1, m], \\ ax(0) - bx'(0) = x_0, cx(1) + dx'(1) = x_1, \end{cases} \quad (6.0.8)$$

where  $\alpha \in (1, 2)$ ,  ${}^C D_{t_i^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$  with the start points  $t_i, i \in \mathbb{N}[0, m]$ ,  ${}^C D_{0^+}^\alpha$  represents the standard Caputo fractional derivatives of order  $\alpha$  with the single start point 0,  $a, b, c, d \in \mathbb{R}$  with  $\delta = ac + ad + bc \neq 0$ ,  $x_0, x_1 \in \mathbb{R}$ ,  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ ,  $I, J : \{t_i, i \in \mathbb{N}[1, m]\} \times \mathbb{R} \rightarrow \mathbb{R} (i \in \mathbb{N}[1, m])$ ,  $f : \bigcup_{i=0}^m (t_i, t_{i+1}) \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ , and  $p$  is defined on  $\bigcup_{i=0}^m (t_i, t_{i+1})$ ,  $q$  is defined on  $(0, 1)$ .

**Lemma 6.0.1** *Suppose that  $p : \bigcup_{i=0}^m (t_i, t_{i+1}) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > -1$  and  $l > -1$  such that  $|p(t)| \leq (t - t_i)^k (t_{i+1} - t)^l$  for all  $t \in (t_i, t_{i+1}), i \in \mathbb{N}[0, m]$ . Then  $x$  is a solution of the following problem*

$$\begin{cases} {}^C D_{t_i^+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta x(t_i) = I_i, \Delta x'(t_i) = J_i, i \in \mathbb{N}[1, m], \\ ax(0) - bx'(0) = x_0, cx(1) + dx'(1) = x_1 \end{cases} \quad (6.0.9)$$

if and only if

$$x(t) = \begin{cases} c_0 + d_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ c_0 + \sum_{j=1}^i I_j - \sum_{j=1}^i t_j J_j + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - \sum_{j=1}^i t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \\ + \left( d_0 + \sum_{j=1}^i J_j + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right) t \\ + \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[1, m], \end{cases} \quad (6.0.10)$$

where

$$\begin{aligned}
c_0 = \frac{1}{\delta} & \left[ (c+d)x_0 + bx_1 - bc \sum_{j=1}^m I_j - b(c+d) \sum_{j=1}^m J_j + bc \sum_{j=1}^m t_j J_j \right. \\
& - bc \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - bd \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - bc \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \\
& \left. + bc \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - b(c+d) \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right],
\end{aligned} \tag{6.0.11}$$

$$\begin{aligned}
d_0 = \frac{1}{\delta} & \left[ -cx_0 + ax_1 - ac \sum_{j=1}^m I_j - ac \sum_{j=1}^m t_j J_j - a(c+d) \sum_{j=1}^m J_j \right. \\
& - ac \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - ad \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - ac \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds \\
& \left. + ac \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds - a(c+d) \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right].
\end{aligned}$$

**Proof:** Suppose that  $x$  is a solution of (6.0.9). From  ${}^C D_{t_i^+}^\alpha x(t) = p(t)$ ,  $t \in (t_i, t_{i+1}]$ , from Theorem 3.1.2, there exist numbers  $c_i, d_i \in \mathbb{R}$  such that  $x(t) = c_i + d_i(t-t_i) + \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds$ ,  $t \in (t_i, t_{i+1}]$ ,  $i \in \mathbb{N}[0, m]$ . Then we have

$$ac_0 - bd_0 = x_0, \tag{6.0.12}$$

$$cc_m + (c+d)d_m = x_1 - c \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - d \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds.$$

Furthermore, we have

$$c_i - c_{i-1} + (d_i - d_{i-1})t_i = I_i + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds,$$

$$d_i - d_{i-1} = J_i + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds.$$

It follows that

$$c_i = c_0 + \sum_{j=1}^i I_j - \sum_{j=1}^i t_j J_j + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - \sum_{j=1}^i t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds,$$

$$d_i = d_0 + \sum_{j=1}^i J_j + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds.$$

Then

$$c_m = c_0 + \sum_{j=1}^m I_j - \sum_{j=1}^m t_j J_j + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds, \quad (6.0.13)$$

$$d_m = d_0 + \sum_{j=1}^m J_j + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds.$$

Substituting (6.0.13) into (6.0.12), we get  $c_0$  and  $d_0$  defined by (6.0.11). Then  $x$  satisfies (6.0.10). On the other hand, we can prove that  $x$  is a solution of (6.0.9) if  $x$  satisfies (6.0.10). The proof is completed.  $\square$

**Remark 6.0.1** Lemma 6.0.1 generalizes Lemma 2.3 in [108] since we allow  $p$  be singular at  $t = t_i (i \in \mathbb{N}[0, m])$  while  $p \in C[0, 1]$  in [108]. We note that two examples are given at the end of [108]. In Example 4.1 and Example 4.2, the fractional derivative is  $D_{0+}^\alpha$  which has single starting point 0. So these examples are unsuitable.

**Lemma 6.0.2** Suppose that  $q : (0, 1) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > -1$  and  $l > -1$  such that  $|q(t)| \leq t^k(1-t)^l$  for all  $t \in (0, 1)$ . Then  $x$  is a solution of the following problem

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta x(t_i) = I_i, i \in \mathbb{N}[1, m], \\ \Delta x'(t_i) = J_i, i \in \mathbb{N}[1, m], \\ ax(0) - bx'(0) = x_0, cx(1) + dx'(1) = x_1 \end{cases} \quad (6.0.14)$$

if and only if

$$x(t) = \begin{cases} c_0 + d_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in [0, t_1], \\ c_0 + \sum_{j=1}^i I_j - \sum_{j=1}^i t_j J_j + \left( d_0 + \sum_{j=1}^i J_j \right) t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[1, m], \end{cases} \quad (6.0.15)$$

where

$$c_0 = \frac{1}{\delta} \left[ (c+d)x_0 + bx_1 - bc \sum_{j=1}^m I_j + bc \sum_{j=1}^m t_j J_j - b(c+d) \sum_{j=1}^m J_j - bc \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - bd \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right],$$

$$d_0 = \frac{1}{\delta} \left[ -cx_0 + ax_1 - ac \sum_{j=1}^m I_j + ac \sum_{j=1}^m t_j J_j - a(c+d) \sum_{j=1}^m J_j - ac \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds - ad \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) ds \right].$$

**Proof:** From  ${}^C D_{0+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}]$ , from **Remark 4.0.1**, there exist numbers  $c_i, d_i \in \mathbb{R}$  such that  $x(t) = c_i + d_i(t - t_i) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]$ . The remainder of the proof is similar to that of the proof of Lemma 6.0.1 and is omitted.  $\square$

**Remark 6.0.2** In [103], the following problem was studied:

$$\begin{cases} {}^C D_{0+}^q x(t) = w(t)g(t, x(t), x'(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1, m], \\ ax(0) - bx'(0) = 0, cx(1) + dx'(1) = 0, \end{cases} \quad (6.0.16)$$

where  $q \in (1, 2)$ ,  ${}^C D_{0+}^q$  represents the standard Caputo fractional derivatives of order  $q$  with the single start point 0,  $a, b, c, d \in \mathbb{R}$  with  $\delta = ac + ad + bc \neq 0$ ,  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ ,  $I, J \in C(\mathbb{R}, \mathbb{R}) (i \in \mathbb{N}[1, m])$ ,  $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous, and  $w$  is continuous and nonnegative. We find that Theorem 2.1 in [103] is not correct. In fact, in the proof of Theorem 2.1, it claims that there exist  $b_0, b_1, c_0, c_1$  such that

$$x(t) = \begin{cases} -b_0 - b_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds, t \in [0, t_1], \\ -c_0 - c_1 t + \int_{t_1}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds, t \in (t_1, t_2]. \end{cases}$$

However, if  $x$  satisfies above equation, we have for  $t \in (t_1, t_2]$

$$\begin{aligned} D_{0+}^q x(t) &= \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} x''(s) ds = \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} x''(s) ds \\ &\quad + \frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} x''(s) ds \\ &= \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} \left[ -b_0 - b_1 s + \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u) du \right]'' ds \\ &\quad + \frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} \left[ -c_0 - c_1 s + \int_{t_1}^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u) du \right]'' ds \\ &= \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} \left[ \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u) du \right]'' ds + \frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} \left[ \int_{t_1}^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u) du \right]'' ds \\ &= \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} \left[ \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du \right]' ds + \frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} \left[ \int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du \right]' ds \\ &= \frac{1}{\Gamma(3-q)} \left[ \int_0^{t_1} (t-s)^{2-q} \left( \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du \right)' ds \right]' \\ &\quad + \frac{1}{\Gamma(3-q)} \left[ \int_{t_1}^t (t-s)^{2-q} \left( \int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du \right)' ds \right]' \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(3-q)} \left[ (t-s)^{2-q} \left( \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du \right) \Big|_0^{t_1} + (2-q) \int_0^{t_1} (t-s)^{1-q} \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du ds \right]' \\
&\quad + \frac{1}{\Gamma(3-q)} \left[ (t-s)^{2-q} \left( \int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du \right) \Big|_{t_1}^t + (2-q) \int_{t_1}^t (t-s)^{1-q} \int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du ds \right]' \\
&= \frac{1}{\Gamma(3-q)} \left[ (t-t_1)^{2-q} \int_0^{t_1} \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du + (2-q) \int_0^{t_1} \int_u^t (t-s)^{1-q} \frac{(s-u)^{q-2}}{\Gamma(q-1)} ds \sigma(u) du \right]' \\
&\quad + \frac{1}{\Gamma(2-q)} \left[ \int_{t_1}^t \int_u^t (t-s)^{1-q} \frac{(s-u)^{q-2}}{\Gamma(q-1)} ds \sigma(u) du \right]' \\
&= \frac{1}{\Gamma(3-q)} \left[ (t-t_1)^{2-q} \int_0^{t_1} \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du + (2-q) \int_0^{t_1} \int_0^1 (1-w)^{1-q} \frac{w^{q-2}}{\Gamma(q-1)} dw \sigma(u) du \right]' \\
&\quad + \frac{1}{\Gamma(2-q)} \left[ \int_{t_1}^t \int_0^1 (1-w)^{1-q} \frac{w^{q-2}}{\Gamma(q-1)} dw \sigma(u) du \right]' \\
&= \sigma(t) + \frac{1}{\Gamma(2-q)} (t-t_1)^{1-q} \int_0^{t_1} \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u) du \neq \sigma(t).
\end{aligned}$$

Hence Lemma 6.0.2 corrected Theorem 2.3 in [103].  $\square$

We need the following assumption:

**(D1)** there exist nondecreasing functions  $\phi_f \in L^1[0, 1]$ ,  $M_f, M_I, M_J : [0, +\infty) \rightarrow [0, +\infty)$ , numbers  $I_i, J_i \in \mathbb{R}$  such that

$$|f(t, x) - \phi_f(t)| \leq M_f(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|I(t_i, x) - I_i| \leq M_I(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R},$$

$$|J(t_i, x) - J_i| \leq M_J(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}.$$

Denote

$$\Phi(t) = \begin{cases} \bar{c}_0 + \bar{d}_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) \phi_f(s) ds, t \in [0, t_1], \\ \bar{c}_0 + \sum_{j=1}^i I_j - \sum_{j=1}^i t_j J_j + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) \phi_f(s) ds \\ - \sum_{j=1}^i t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds \\ + \left( \bar{d}_0 + \sum_{j=1}^i J_j + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds \right) t \\ + \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) \phi_f(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[1, m], \end{cases}$$

where

$$\begin{aligned}
\bar{c}_0 = \frac{1}{\delta} & \left[ (c+d)x_0 + bx_1 - bc \sum_{j=1}^m I_j - b(c+d) \sum_{j=1}^m J_j + bc \sum_{j=1}^m t_j J_j \right. \\
& - bc \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) \phi_f(s) ds - bd \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds \\
& - bc \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) \phi_f(s) ds \\
& \left. + bc \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds - b(c+d) \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds \right], \\
\bar{d}_0 = \frac{1}{\delta} & \left[ -cx_0 + ax_1 - ac \sum_{j=1}^m I_j - ac \sum_{j=1}^m t_j J_j - a(c+d) \sum_{j=1}^m J_j \right. \\
& - ac \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) \phi_f(s) ds - ad \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds \\
& - ac \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) \phi_f(s) ds \\
& \left. + ac \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds - a(c+d) \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) \phi_f(s) ds \right].
\end{aligned}$$

**Theorem 6.0.1** Suppose that  $p : \bigcup_{i=0}^m (t_i, t_{i+1}) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > -1$  and  $l > -\alpha + 1$  such that  $|p(t)| \leq (t - t_i)^k (t_{i+1} - t)^l$  for all  $t \in (t_i, t_{i+1}), i \in \mathbb{N}[0, m]$ ,  $f$  is a **impulsive I-Carathéodory function**,  $I, J$  **discrete I-Carathéodory functions** and (D1) holds. Then BVP (6.0.7) has at least one solution if there exists  $r_0 > 0$  such that

$$\begin{aligned}
& \left[ m + \frac{m|ac|}{|\delta|} + \frac{m|bc|}{|\delta|} \right] M_I(r_0 + \|\Phi\|) + \left[ 2m + \frac{m[2|ac|+|ad|]}{|\delta|} + \frac{m[2|bc|+|bd|]}{|\delta|} \right] M_J(r_0 + \|\Phi\|) \\
& + \frac{2|\delta|+|ac|+|bc|}{|\delta|} \sum_{j=1}^{m+1} \frac{\mathbf{B}(\alpha+l, k+1)(t_j-t_{j-1})^{\alpha+k+l}}{\Gamma(\alpha)} M_f(r_0 + \|\Phi\|) \\
& + \frac{2|\delta|+2|ac|+2|bc|+|bd|+|ad|}{|\delta|} \sum_{j=1}^{m+1} \frac{\mathbf{B}(\alpha+l-1, k+1)(t_j-t_{j-1})^{\alpha+k+l-1}}{\Gamma(\alpha-1)} M_f(r_0 + \|\Phi\|) \leq r_0.
\end{aligned} \tag{6.0.17}$$

**Proof:** Define the nonlinear operator  $T$  on  $P_m C[0, 1]$  by

$$(Tx)(t) = \begin{cases} c_0 + d_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f(s, x(s)) ds, & t \in [0, t_1], \\ c_0 + \sum_{j=1}^i I(t_j, x(t_j)) - \sum_{j=1}^i t_j J(t_j, x(t_j)) + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f(s, x(s)) ds \\ - \sum_{j=1}^i t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds \\ + \left( d_0 + \sum_{j=1}^i J(t_j, x(t_j)) + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds \right) t \\ + \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f(s, x(s)) ds, & t \in (t_i, t_{i+1}], i \in \mathbb{N}[1, m], \end{cases}$$

where

$$\begin{aligned} c_0 = \frac{1}{\delta} & \left[ (c+d)x_0 + bx_1 - bc \sum_{j=1}^m I(t_j, x(t_j)) - b(c+d) \sum_{j=1}^m J(t_j, x(t_j)) + bc \sum_{j=1}^m t_j J(t_j, x(t_j)) \right. \\ & - bc \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f(s, x(s)) ds - bd \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds \\ & - bc \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f(s, x(s)) ds \\ & \left. + bc \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds - b(c+d) \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds \right], \\ d_0 = \frac{1}{\delta} & \left[ -cx_0 + ax_1 - ac \sum_{j=1}^m I(t_j, x(t_j)) - ac \sum_{j=1}^m t_j J(t_j, x(t_j)) - a(c+d) \sum_{j=1}^m J(t_j, x(t_j)) \right. \\ & - ac \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f(s, x(s)) ds - ad \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds \\ & - ac \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s) f(s, x(s)) ds \\ & \left. + ac \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds - a(c+d) \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s) f(s, x(s)) ds \right]. \end{aligned}$$

By Lemma 6.0.1, we have  $x$  is a solution of BVP (6.0.7) if and only if  $x$  is a fixed point of  $T$  in  $P_m C[0, 1]$ . A standard proof shows that  $T : P_m C[0, 1] \rightarrow P_m C[0, 1]$  is well defined and completely continuous.

Choose  $\Omega = \{x \in P_m C[0, 1] : \|x - \Phi\| \leq r_0\}$ . We will prove that  $T\Omega \subset \Omega$ . For  $x \in \Omega$ , we

have  $\|x\| \leq \|x - \Phi\| + \|\Phi\| \leq r_0 + \|\Phi\|$ . Then (D1) implies

$$|f(t, x(t)) - \phi_f(t)| \leq M_f(|x(t)|) \leq M_f(\|x\|) \leq M_f(r_0 + \|\Phi\|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m],$$

$$|I(t_i, x(t_i)) - I_i| \leq M_I(r_0 + \|\Phi\|), i \in \mathbb{N}[1, m],$$

$$|J(t_i, x(t_i)) - J_i| \leq M_J(r_0 + \|\Phi\|), i \in \mathbb{N}[1, m].$$

Then

$$\begin{aligned} |c_0 - \bar{c}_0| &\leq \frac{1}{|\delta|} \left[ |bc| \sum_{j=1}^m |I(t_j, x(t_j)) - I_j| + |b(c+d)| \sum_{j=1}^m |J(t_j, x(t_j)) - J_j| \right. \\ &\quad + |bc| \sum_{j=1}^m t_j |J(t_j, x(t_j)) - J_j| \\ &\quad + |bc| \int_{t_m}^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds \\ &\quad + |bd| \int_{t_m}^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds \\ &\quad + |bc| \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds \\ &\quad + |bc| \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds \\ &\quad \left. + |b(c+d)| \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds \right] \\ &\leq \frac{1}{|\delta|} [m|bc|M_I(r_0 + \|\Phi\|) + m[|bc| + |bd|]M_J(r_0 + \|\Phi\|) + m|bc|M_J(r_0 + \|\Phi\|) \\ &\quad + |bc| \sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} (s - t_{j-1})^k (t_j - s)^l ds M_f(r_0 + \|\Phi\|) \\ &\quad + [2|bc| + |bd|] \sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} (s - t_{j-1})^k (t_j - s)^l ds M_f(r_0 + \|\Phi\|)] \\ &\leq \frac{1}{|\delta|} [m|bc|M_I(r_0 + \|\Phi\|) + m[2|bc| + |bd|]M_J(r_0 + \|\Phi\|) \\ &\quad + |bc| \sum_{j=1}^{m+1} (t_j - t_{j-1})^{\alpha+k+l} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw M_f(r_0 + \|\Phi\|) \end{aligned}$$

$$\begin{aligned}
 & + [2|bc| + |bd|] \sum_{j=1}^{m+1} (t_j - t_{j-1})^{\alpha+k+l-1} \int_0^1 \frac{(1-w)^{\alpha+l-2}}{\Gamma(\alpha-1)} w^k dw M_f(r_0 + \|\Phi\|) \Big] \\
 & = \frac{m|bc|}{|\delta|} M_I(r_0 + \|\Phi\|) + \frac{m[2|bc|+|bd|]}{|\delta|} M_J(r_0 + \|\Phi\|) \\
 & + \sum_{j=1}^{m+1} \left[ \frac{|bc|}{|\delta|} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} (t_j - t_{j-1})^{\alpha+k+l} + \frac{[2|bc|+|bd|]}{|\delta|} \frac{\mathbf{B}(\alpha+l-1, k+1)}{\Gamma(\alpha-1)} (t_j - t_{j-1})^{\alpha+k+l-1} \right] \\
 & M_f(r_0 + \|\Phi\|).
 \end{aligned}$$

Similarly we have

$$\begin{aligned}
 |d_0 - \bar{d}_0| & \leq \frac{m|ac|}{|\delta|} M_I(r_0 + \|\Phi\|) + \frac{m[2|ac|+|ad|]}{|\delta|} M_J(r_0 + \|\Phi\|) \\
 & + \sum_{j=1}^{m+1} \left[ \frac{|ac|}{|\delta|} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} (t_j - t_{j-1})^{\alpha+k+l} + \frac{2|ac|+|ad|}{|\delta|} \frac{\mathbf{B}(\alpha+l-1, k+1)}{\Gamma(\alpha-1)} (t_j - t_{j-1})^{\alpha+k+l-1} \right] \\
 & M_f(r_0 + \|\Phi\|).
 \end{aligned}$$

Use (6.0.17), we have for  $t \in (t_i, t_{i+1}]$

$$|(Tx)(t) - \Phi(t)| \leq \left\{ \begin{aligned}
 & |c_0 - \bar{c}_0| + |d_0 - \bar{d}_0| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds, \quad i = 0, \\
 & |c_0 - \bar{c}_0| + |d_0 - \bar{d}_0| + \sum_{j=1}^i |I(t_j, x(t_j)) - I_j| + \sum_{j=1}^i t_j |J(t_j, x(t_j)) - J_j| \\
 & + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_j(s)| ds \\
 & + \sum_{j=1}^i t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds \\
 & + \left( \sum_{j=1}^i |J(t_j, x(t_j)) - J_j| \right. \\
 & \left. + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} |p(s)| |f(s, x(s)) \phi_f(s)| ds \right) t \\
 & + \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds, \quad i \in \mathbb{N}[1, m],
 \end{aligned} \right.$$

$$\leq |c_0 - \bar{c}_0| + |d_0 - \bar{d}_0|$$

$$+ \sum_{j=1}^i |I(t_j, x(t_j)) - I_j| + 2 \sum_{j=1}^i |J(t_j, x(t_j)) - J_j|$$

$$+ 2 \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_j(s)| ds$$

$$\begin{aligned}
& +2 \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds + \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds \\
& \leq \sum_{j=1}^i |I(t_j, x(t_j)) - I_j| + 2 \sum_{j=1}^i |J(t_j, x(t_j)) - J_j| \\
& \quad +2 \sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} |p(s)| |f(s, x(s)) - \phi_j(s)| ds \\
& \quad +2 \sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} |p(s)| |f(s, x(s)) - \phi_f(s)| ds + |c_0 - \bar{c}_0| + |d_0 - \bar{d}_0| \\
& \leq mM_I(r_0 + \|\Phi\|) + 2mM_J(r_0 + \|\Phi\|) \\
& \quad +2 \sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} (s - t_{j-1})^k (t_j - s)^l ds M_f(r_0 + \|\Phi\|) \\
& \quad +2 \sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} (s - t_{j-1})^k (t_j - s)^l ds M_f(r_0 + \|\Phi\|) + |c_0 - \bar{c}_0| + |d_0 - \bar{d}_0| \\
& \leq mM_I(r_0 + \|\Phi\|) + 2mM_J(r_0 + \|\Phi\|) \\
& \quad +2 \sum_{j=1}^{m+1} (t_j - t_{j-1})^{\alpha+k+l} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw M_f(r_0 + \|\Phi\|) \\
& \quad +2 \sum_{j=1}^{m+1} (t_j - t_{j-1})^{\alpha+k+l-1} \int_0^1 \frac{(1-w)^{\alpha+l-2}}{\Gamma(\alpha-1)} w^k dw M_f(r_0 + \|\Phi\|) + |c_0 - \bar{c}_0| + |d_0 - \bar{d}_0| \\
& \leq \frac{2|\delta|+|ac|+|bc|}{|\delta|} \sum_{j=1}^{m+1} \frac{\mathbf{B}(\alpha+l, k+1)(t_j-t_{j-1})^{\alpha+k+l}}{\Gamma(\alpha)} M_f(r_0 + \|\Phi\|) \\
& \quad + \frac{2|\delta|+2|ac|+2|bc|+|bd|+|ad|}{|\delta|} \sum_{j=1}^{m+1} \frac{\mathbf{B}(\alpha+l-1, k+1)(t_j-t_{j-1})^{\alpha+k+l-1}}{\Gamma(\alpha-1)} M_f(r_0 + \|\Phi\|) \\
& \quad + \left[ m + \frac{m|ac|}{|\delta|} + \frac{m|bc|}{|\delta|} \right] M_I(r_0 + \|\Phi\|) + \left[ 2m + \frac{m[2|ac|+|ad|]}{|\delta|} + \frac{m[2|bc|+|bd|]}{|\delta|} \right] M_J(r_0 + \|\Phi\|) \leq r_0.
\end{aligned}$$

Hence  $T\Omega \subset \Omega$ . By Schauder's fixed point theorem see Theorem 2.2.1 in [35],  $T$  has a fixed point  $x \in \Omega$  which is a solution of BVP (6.0.7). The proof is completed.  $\square$

Now suppose that

(D2) there exist nondecreasing functions  $\phi_g \in L^1[0, 1]$ ,  $M_g, M_I, M_J : [0, +\infty) \rightarrow [0, +\infty)$ , numbers  $I_i, J_i \in \mathbb{R}$  such that

$$|g(t, x) - \phi_g(t)| \leq M_g(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], x \in \mathbb{R},$$

$$|I(t_i, x) - I_i| \leq M_I(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R},$$

$$|J(t_i, x) - J_i| \leq M_J(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}.$$

Denote

$$\Phi(t) = \begin{cases} c_0 + d_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) \phi_g(s) ds, t \in [0, t_1], \\ c_0 + \sum_{j=1}^i I_j - \sum_{j=1}^i t_j J_j + \left( d_0 + \sum_{j=1}^i J_j \right) t \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) \phi_g(s) ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[1, m], \end{cases}$$

where

$$\bar{c}_0 = \frac{1}{\delta} \left[ (c + d)x_0 + bx_1 - bc \sum_{j=1}^m I_j + bc \sum_{j=1}^m t_j J_j - b(c + d) \sum_{j=1}^m J_j \right.$$

$$\left. -bc \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) \phi_g(s) ds - bd \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} q(s) \phi_g(s) ds \right],$$

$$\bar{d}_0 = \frac{1}{\delta} \left[ -cx_0 + ax_1 - ac \sum_{j=1}^m I_j + ac \sum_{j=1}^m t_j J_j - a(c + d) \sum_{j=1}^m J_j \right.$$

$$\left. -ac \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) \phi_g(s) ds - ad \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} q(s) \phi_g(s) ds \right].$$

**Theorem 6.0.2** Suppose that  $q : (0, 1) \rightarrow \mathbb{R}$  is continuous and satisfies that there exist numbers  $k > -1$  and  $l > -\alpha + 1$  such that  $|q(t)| \leq t^k(1-t)^l$  for all  $t \in (0, 1)$ ,  $f$  is a **impulsive I-Carathéodory function**,  $I, J$  **discrete I-Carathéodory functions** and (D2) holds. Then BVP (6.0.8) has at least one solution if

$$\begin{aligned} & \frac{|\delta|+m|ac|+m|bc|}{|\delta|} M_I(r_0 + \|\Phi\|) + \frac{2m|bc|+2|ac|+|bd|+|ac|}{|\delta|} M_J(r_0 + \|\Phi\|) \\ & + \left[ \frac{|\delta|+|ac|+|bc|}{|\delta|} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} + \frac{|ad|+|bd|}{|\delta|} \frac{\mathbf{B}(\alpha+l-1, k+1)}{\Gamma(\alpha-1)} \right] M_g(r_0 + \|\Phi\|) \leq r_0. \end{aligned} \tag{6.0.18}$$

**Proof:** Define the nonlinear operator  $T$  on  $P_m C[0, 1]$  by

$$(Tx)(t) = \begin{cases} c_0 + d_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)g(s, x(s))ds, t \in [0, t_1], \\ c_0 + \sum_{j=1}^i I(t_j, x(t_j)) - \sum_{j=1}^i t_j J(t_j, x(t_j)) + \left( d_0 + \sum_{j=1}^i J(t_j, x(t_j)) \right) t \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)g(s, x(s))ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[1, m], \end{cases}$$

where

$$\begin{aligned} c_0 &= \frac{1}{\delta} \left[ (c + d)x_0 + bx_1 - bc \sum_{j=1}^m I(t_j, x(t_j)) + bc \sum_{j=1}^m t_j J(t_j, x(t_j)) - b(c + d) \sum_{j=1}^m J(t_j, x(t_j)) \right. \\ &\quad \left. - bc \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)g(s, x(s))ds - bd \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} q(s)g(s, x(s))ds \right], \\ d_0 &= \frac{1}{\delta} \left[ -cx_0 + ax_1 - ac \sum_{j=1}^m I(t_j, x(t_j)) + ac \sum_{j=1}^m t_j J(t_j, x(t_j)) - a(c + d) \sum_{j=1}^m J(t_j, x(t_j)) \right. \\ &\quad \left. - ac \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)g(s, x(s))ds - ad \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} q(s)g(s, x(s))ds \right]. \end{aligned}$$

The remainder of the proof is similar to that of the proof of Theorem 6.0.1 and is omitted.  $\square$

**Remark 6.0.3** In [101], Zhao studied the eigenvalue intervals and positive solutions of integral boundary value problem for the following higher-order nonlinear fractional differential equation with impulses

$$\begin{cases} -D_{0+}^{\alpha} u(t) = \lambda a(t)f(t, u(t)), t \in (0, 1) \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), k \in \mathbb{N}[1, m], \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, u(1) = \int_0^1 u(s)dH(s), \end{cases}$$

where where  $D_{0+}^{\alpha}$  is the standard Riemann-Liouville fractional derivative of order  $n - 1 < \alpha < n$ ,  $n \geq 3$ , the number  $n$  is the smallest integer greater than or equal to  $\alpha$ , the impulsive point sequence  $\{t_i : i \in \mathbb{N}[1, m]\}$  satisfies  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$ ,  $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ ,  $u(t_k^+) = \lim_{h \rightarrow 0^+} u(t_k + h)$  and  $u(t_k^-) = \lim_{h \rightarrow 0^+} u(t_k - h)$  represent the right and left-hand limits of  $u(t)$  at  $t = t_k$ , respectively,  $\lambda \geq 0$  is a parameter,  $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$ ,  $a \in C((0, 1), [0, +\infty))$ ,  $I_k \in C([0, +\infty), [0, \infty))$ , the integral  $\int_0^1 u(s)dH(s)$  is the Riemann-Stieltjes integral with  $H : [0, 1] \rightarrow \mathbb{R}$ . The following result was proved:

**Result Z (Lemma 2.4 [101]).** If  $H : [0, 1] \rightarrow \mathbb{R}$  is a function of bounded variation  $\delta =: \int_0^1 s^{\alpha-1} dH(s) \neq \alpha - 1$  and  $h \in C([0, 1])$ , then the unique solution of

$$\begin{cases} -D_{0+}^\alpha u(t) = h(t), t \in (0, 1) \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), k = 1, 2, \dots, m, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, u(1) = \int_0^1 u(s) dH(s), \end{cases} \quad (6.0.19)$$

is

$$u(t) = \int_0^1 G(t, s) h(s) ds + t^{\alpha-1} \sum_{t \leq t_k < 1} t_k^{1-\alpha} I_k(u(t_k)), t \in (0, 1] \quad (6.0.20)$$

where

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & s \in [0, t], \\ t^{\alpha-1}(1-s)^{\alpha-2}, & s \in [t, 1], \end{cases}$$

$$G_2(t, s) = \frac{t^{\alpha-1}}{\alpha-1-\delta} \int_0^1 G_1(\tau, s) dH(\tau), \quad G(t, s) = G_1(t, s) + G_2(t, s).$$

**Remark 6.0.4** Result Z is wrong. In fact, (6.0.20) can be re-written by

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + t^{\alpha-1} \left[ \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} h(s) ds + \frac{1}{\alpha-1-\delta} \int_0^1 G_1 \left[ \int_0^1 G_1(\tau, s) h(s) ds \right] dH(\tau) \right. \\ &\quad \left. + \sum_{k=1}^m t_k^{1-\alpha} I_k(u(t_k)) \right] - t^{\alpha-1} \sum_{j=1}^{k-1} t_j^{1-\alpha} I_j(u(t_j)) \\ &=: \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + A_0 t^{\alpha-1} + t^{\alpha-1} \sum_{j=1}^{k-1} A_j, t \in (t_{k-1}, t_k], k = 0, 1, \dots, m. \end{aligned}$$

Then by Definition 2.2 we have for  $t \in (t_i, t_{i+1}]$  that

$$\begin{aligned} D_{0+}^\alpha u(t) &= \frac{\left[ \int_0^t (t-s)^{n-\alpha-1} u(s) ds \right]^{(n)}}{\Gamma(n-\alpha)} = \frac{\left[ \sum_{\tau=0}^{i-1} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} u(s) ds + \int_{t_i}^t (t-s)^{n-\alpha-1} u(s) ds \right]^{(n)}}{\Gamma(n-\alpha)} \\ &= \frac{\left[ \sum_{\tau=0}^{i-1} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} \left( \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv + A_0 s^{\alpha-1} + s^{\alpha-1} \sum_{j=1}^{\tau} A_j \right) ds \right]^{(n)}}{\Gamma(n-\alpha)} \\ &\quad + \frac{\left[ \int_{t_i}^t (t-s)^{n-\alpha-1} \left( \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv + A_0 s^{\alpha-1} + s^{\alpha-1} \sum_{j=1}^i A_j \right) ds \right]^{(n)}}{\Gamma(n-\alpha)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\left[ \sum_{\tau=0}^{i-1} \sum_{j=0}^{\tau} A_j \int_{t_{\tau}}^{t_{\tau+1}} (t-s)^{n-\alpha-1} s^{\alpha-1} ds \right]^{(n)}}{\Gamma(n-\alpha)} \\
&+ \frac{\left[ \int_0^t (t-s)^{n-\alpha-1} \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv ds + \sum_{j=0}^i A_j \int_{t_j}^t (t-s)^{n-\alpha-1} s^{\alpha-1} ds \right]^{(n)}}{\Gamma(n-\alpha)} \\
&= \frac{\left[ \sum_{j=0}^{i-1} \sum_{\tau=j}^{i-1} A_j \int_{t_{\tau}}^{t_{\tau+1}} (t-s)^{n-\alpha-1} s^{\alpha-1} ds \right]^{(n)}}{\Gamma(n-\alpha)} \\
&+ \frac{\left[ \int_0^t \int_v^t (t-s)^{n-\alpha-1} \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} ds h(v) dv + \sum_{j=0}^i A_j \int_{t_j}^t (t-s)^{n-\alpha-1} s^{\alpha-1} ds \right]^{(n)}}{\Gamma(n-\alpha)} \\
&= \frac{\left[ \int_0^t (t-v)^{n-1} \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw h(v) dv + \sum_{j=0}^i A_j \int_{t_j}^t (t-s)^{n-\alpha-1} s^{\alpha-1} ds \right]^{(n)}}{\Gamma(n-\alpha)} \\
&= \frac{\left[ \int_0^t (t-v)^{n-1} \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw h(v) dv + \sum_{j=0}^i A_j t^{n-1} \int_{\frac{t_j}{t}}^1 (1-w)^{n-\alpha-1} w^{\alpha-1} dw \right]^{(n)}}{\Gamma(n-\alpha)} \\
&= h(t) + \frac{\sum_{j=0}^i A_j \left[ t^{n-1} \int_{\frac{t_j}{t}}^1 (1-w)^{n-\alpha-1} w^{\alpha-1} dw \right]^{(n)}}{\Gamma(n-\alpha)} \neq h(t) \text{ if } A_j \neq 0.
\end{aligned}$$

This shows us that Result Z is wrong.

We now correct Result Z. Involving Consider the following more general problem:

$$\left\{ \begin{array}{l} -{}^C D_{0+}^{\alpha} u(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta u^{(i)}(t_k) = I_i(t_k, u(t_k)), k \in \mathbb{N}[1, m], i \in \mathbb{N}[0, n-1], \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, u(1) = \int_0^1 u(s) dH(s). \end{array} \right. \quad (6.0.21)$$

**Theorem 6.0.3** *Suppose that  $\int_0^1 \frac{s^{n-1}}{(n-1)!} dH(s) \neq 1$ . Then  $x$  is a solution of (6.0.21) if and only if*

$$\begin{aligned}
 x(t) = & - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=1}^k \sum_{\nu=0}^{n-1} \frac{(t-t_j)^\nu}{\nu!} I_\nu(t_j, x(t_j)) \\
 & + \frac{1}{1 - \int_0^1 \frac{s^{n-1}}{(n-1)!} dH(s)} \left[ \int_0^1 \frac{\alpha(1-s)^{\alpha-1} - (1-s)^\alpha}{\Gamma(\alpha+1)} h(s) ds - \sum_{j=1}^m \sum_{\nu=0}^{n-1} \frac{(1-t_j)^\nu}{\nu!} I_\nu(t_j, x(t_j)) \right. \\
 & \left. + \sum_{j=1}^m \sum_{\nu=0}^{n-1} \int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!} dH(s) I_\nu(t_j, x(t_j)) \right] \frac{t^{n-1}}{(n-1)!}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m].
 \end{aligned}$$

**Proof:** Suppose  $x$  is a solution of (6.0.25). By  $\alpha \in (n-1, n)$ , using Theorem 4.0.2, we know from  $-{}^C D_{0+}^\alpha x(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]$  that there exist constants  $c_{\nu,j} \in \mathbb{R}$  such that

$$x(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=0}^k \sum_{\nu=0}^{n-1} c_{\nu,j} \frac{(t-t_j)^\nu}{\nu!}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m]. \quad (6.0.22)$$

It follows for  $i = 1, 2, \dots, n-1$  that

$$x^{(i)}(t) = - \int_0^t \frac{(t-s)^{\alpha-i-1}}{\Gamma(\alpha-i)} h(s) ds + \sum_{j=0}^k \sum_{\nu=i}^{n-1} c_{\nu,j} \frac{(t-t_j)^{\nu-i}}{(\nu-i)!}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m]. \quad (6.0.23)$$

(i) By  $x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0$  and (6.22)-(6.23), we get  $c_{0,0} = c_{1,0} = \dots = c_{n-2,0} = 0$ .

(ii) By  $\Delta x^{(i)}(t_k) = I_i(t_k, x(t_k)), k \in \mathbb{N}[1, m], i \in \mathbb{N}[0, n-1]$  and (6.22)-(6.23), we get  $c_{i,k} = I_i(t_k, x(t_k)), k = 1, 2, \dots, m, i \in [0, n-1]$ .

(iii) By  $x(1) = \int_0^1 x(s) dH(s)$  and (6.22) and (i)-(ii), we get

$$\begin{aligned}
 & - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=0}^m \sum_{\nu=0}^{n-1} c_{\nu,j} \frac{(1-t_j)^\nu}{\nu!} = \int_0^1 x(s) dH(s) = \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} x(s) dH(s) \\
 & = \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \left( - \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv + \sum_{j=0}^{\tau} \sum_{\nu=0}^{n-1} c_{\nu,j} \frac{(s-t_j)^\nu}{\nu!} \right) dH(s) \\
 & = - \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv ds + \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \sum_{j=0}^{\tau} \sum_{\nu=0}^{n-1} c_{\nu,j} \frac{(s-t_j)^\nu}{\nu!} dH(s) \\
 & = - \int_0^1 \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv ds + \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \sum_{j=0}^{\tau} \sum_{\nu=0}^{n-1} c_{\nu,j} \frac{(s-t_j)^\nu}{\nu!} dH(s) \\
 & = - \int_0^1 \int_v^1 \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} ds h(v) dv + \sum_{j=0}^m \sum_{\nu=0}^{n-1} c_{\nu,j} \int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!} dH(s)
 \end{aligned}$$

$$= - \int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)} h(v) dv + \sum_{j=1}^m \sum_{\nu=0}^{n-1} I_\nu(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!} dH(s) + c_{n-1,0} \int_0^1 \frac{s^{n-1}}{(n-1)!} dH(s).$$

Hence

$$\begin{aligned} & - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=1}^m \sum_{\nu=0}^{n-1} I_\nu(t_j, x(t_j)) \frac{(1-t_j)^\nu}{\nu!} + c_{n-1,0} \\ & = - \int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)} h(v) dv + \sum_{j=1}^m \sum_{\nu=0}^{n-1} I_\nu(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!} dH(s) + c_{n-1,0} \int_0^1 \frac{s^{n-1}}{(n-1)!} dH(s). \end{aligned}$$

It follows that

$$\begin{aligned} c_{n-1,0} = & \frac{1}{1 - \int_0^1 \frac{s^{n-1}}{(n-1)!} dH(s)} \left[ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \sum_{j=1}^m \sum_{\nu=0}^{n-1} \frac{(1-t_j)^\nu}{\nu!} I_\nu(t_j, x(t_j)) \right. \\ & \left. - \int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)} h(v) dv + \sum_{j=1}^m \sum_{\nu=0}^{n-1} \int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!} dH(s) I_\nu(t_j, x(t_j)) \right]. \end{aligned}$$

Substituting  $c_{\nu,j}$  into (6.0.22), we get

$$\begin{aligned} x(t) = & - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=1}^k \sum_{\nu=0}^{n-1} \frac{(t-t_j)^\nu}{\nu!} I_\nu(t_j, x(t_j)) \\ & + \frac{1}{1 - \int_0^1 \frac{s^{n-1}}{(n-1)!} dH(s)} \left[ \int_0^1 \frac{\alpha(1-s)^{\alpha-1} - (1-s)^\alpha}{\Gamma(\alpha+1)} h(s) ds - \sum_{j=1}^m \sum_{\nu=0}^{n-1} \frac{(1-t_j)^\nu}{\nu!} I_\nu(t_j, x(t_j)) \right. \\ & \left. + \sum_{j=1}^m \sum_{\nu=0}^{n-1} \int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!} dH(s) I_\nu(t_j, x(t_j)) \right] \frac{t^{n-1}}{(n-1)!}, \end{aligned} \tag{6.0.24}$$

$$t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m].$$

On the other hand, if  $x$  satisfies (6.0.24), we can prove that  $x$  is a solution of (6.0.21) by direct computation. The proof is completed.  $\square$

Involving the Riemann-Liouville fractional derivative, we consider the following more general

problem:

$$\left\{ \begin{array}{l} {}^{RL}D_{0+}^{\alpha}x(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta {}^{RL}D_{0+}^{\alpha-i}x(t_k) = I_i(t_k, x(t_k)), k \in \mathbb{N}[1, m], i \in \mathbb{N}[1, n-1], \\ \Delta I_{0+}^{n-\alpha}x(t_k) = I_n(t_k, x(t_k)), k \in \mathbb{N}[1, m], \\ I_{0+}^{n-\alpha}x(0) = {}^{RL}D_{0+}^{\alpha-1}x(0) = \dots = {}^{RL}D_{0+}^{\alpha-(n-2)}x(0) = 0, u(1) = \int_0^1 u(s)dH(s). \end{array} \right. \quad (6.0.25)$$

**Theorem 6.0.4** Suppose that  $\int_0^1 s^{\alpha-(n-1)}dH(s) \neq 1$ . Then  $x$  is a solution of (6.0.25) if and only if

$$\begin{aligned} x(t) = & -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=1}^k \sum_{\nu=1}^n \frac{(t-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}I_{\nu}(t_j, x(t_j)) \\ & + \frac{\Gamma(\alpha-n+2)}{1-\int_0^1 s^{\alpha-(n-1)}dH(s)} \left[ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - \sum_{j=1}^m \sum_{\nu=1}^n I_{\nu}(t_j, x(t_j)) \frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} \right. \\ & \left. - \int_0^1 \frac{(1-v)^{\alpha}}{\Gamma(\alpha+1)}h(v)dv + \sum_{j=1}^m \sum_{\nu=1}^{n-1} I_{\nu}(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s) \right] \frac{t^{\alpha-(n-1)}}{\Gamma(\alpha-n+2)}, t \in (t_k, t_{k+1}], \\ & k \in \mathbb{N}[0, m]. \end{aligned}$$

**Proof:** Suppose  $x$  is a solution of (6.0.25). By  $\alpha \in (n-1, n)$ , using Theorem 4.0.1, we know from  $-{}^{RL}D_{0+}^{\alpha}x(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]$  that there exist constants  $c_{\nu,j} \in \mathbb{R}$  such that

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=0}^k \sum_{\nu=1}^n c_{\nu,j} \frac{(t-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m]. \quad (6.0.26)$$

It follows that

$$I_{0+}^{n-\alpha}x(t) = -\int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)}h(s)ds + \sum_{j=0}^k \sum_{\nu=1}^n c_{\nu,j} \frac{(t-t_j)^{n-\nu}}{\Gamma(n-\nu+1)}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m] \quad (6.0.27)$$

and for  $i = 1, 2, \dots, n-1$  that

$${}^{RL}D_{0+}^{\alpha-i}x(t) = -\int_0^t \frac{(t-s)^{i-1}}{\Gamma(i)}h(s)ds + \sum_{j=0}^k \sum_{\nu=1}^i c_{\nu,j} \frac{(t-t_j)^{i-\nu}}{\Gamma(i-\nu+1)}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m]. \quad (6.0.28)$$

(i) By  $I_{0+}^{n-\alpha}x(0) = {}^{RL}D_{0+}^{\alpha-1}x(0) = \dots = {}^{RL}D_{0+}^{\alpha-(n-2)}x(0) = 0$  and (6.27) and (6.0.28), we get  $c_{1,0} = c_{2,0} = \dots = c_{n-2,0} = c_{n,0} = 0$ .

(ii) By  $\Delta^{RL}D_{0+}^{\alpha-i}x(t_k) = I_i(t_k, x(t_k)), k \in \mathbb{N}[1, m], i \in \mathbb{N}[1, n-1]$  and (6.0.28), we get  $c_{i,k} = I_i(t_k, x(t_k)), k \in \mathbb{N}[1, m], i \in \mathbb{N}[1, n-1]$ .

(iii) By  $\Delta I_{0+}^{n-\alpha}x(t_k) = I_n(t_k, x(t_k)), k \in \mathbb{N}[1, m]$  and (6.27), we get  $c_{n,k} = I_n(t_k, x(t_k)), k \in \mathbb{N}[1, m]$ .

(iv) By  $x(1) = \int_0^1 x(s)dH(s)$  and (6.26) and (i)-(iii), we get

$$\begin{aligned}
& - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=0}^m \sum_{\nu=1}^n c_{\nu,j} \frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} = \int_0^1 x(s) dH(s) = \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} x(s) dH(s) \\
& = \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \left( - \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv + \sum_{j=0}^{\tau} \sum_{\nu=1}^n c_{\nu,j} \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} \right) dH(s) \\
& = - \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv ds + \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \sum_{j=0}^{\tau} \sum_{\nu=1}^n c_{\nu,j} \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} dH(s) \\
& = - \int_0^1 \int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} h(v) dv ds + \sum_{\tau=0}^m \int_{t_\tau}^{t_{\tau+1}} \sum_{j=0}^{\tau} \sum_{\nu=1}^n c_{\nu,j} \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} dH(s) \\
& = - \int_0^1 \int_v^1 \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)} ds h(v) dv + \sum_{j=0}^m \sum_{\nu=1}^n c_{\nu,j} \int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} dH(s) \\
& = - \int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)} h(v) dv + \sum_{j=1}^m \sum_{\nu=1}^{n-1} I_\nu(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} dH(s) + c_{n-1,0} \int_0^1 \frac{s^{\alpha-(n-1)}}{\Gamma(\alpha-(n-1)+1)} dH(s).
\end{aligned}$$

Hence

$$\begin{aligned}
& - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=1}^m \sum_{\nu=1}^n I_\nu(t_j, x(t_j)) \frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} + \frac{c_{n-1,0}}{\Gamma(\alpha-n+2)} \\
& = - \int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)} h(v) dv + \sum_{j=1}^m \sum_{\nu=1}^{n-1} I_\nu(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} dH(s) \\
& \quad + c_{n-1,0} \int_0^1 \frac{s^{\alpha-(n-1)}}{\Gamma(\alpha-n+2)} dH(s).
\end{aligned}$$

It follows that

$$\begin{aligned}
c_{n-1,0} & = \frac{\Gamma(\alpha-n+2)}{1 - \int_0^1 s^{\alpha-(n-1)} dH(s)} \left[ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \sum_{j=1}^m \sum_{\nu=1}^n I_\nu(t_j, x(t_j)) \frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} \right. \\
& \quad \left. - \int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)} h(v) dv + \sum_{j=1}^m \sum_{\nu=1}^{n-1} I_\nu(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} dH(s) \right].
\end{aligned}$$

Substituting  $c_{\nu,j}$  into (6.0.26), we get

$$\begin{aligned}
 x(t) = & - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \sum_{j=1}^k \sum_{\nu=1}^n \frac{(t-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} I_\nu(t_j, x(t_j)) \\
 & + \frac{\Gamma(\alpha-n+2)}{1-\int_0^1 s^{\alpha-(n-1)} dH(s)} \left[ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \sum_{j=1}^m \sum_{\nu=1}^n I_\nu(t_j, x(t_j)) \frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} \right. \\
 & \left. - \int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)} h(v) dv + \sum_{j=1}^m \sum_{\nu=1}^{n-1} I_\nu(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} dH(s) \right] \frac{t^{\alpha-(n-1)}}{\Gamma(\alpha-n+2)},
 \end{aligned} \tag{6.0.29}$$

$$t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m].$$

On the other hand, if  $x$  satisfies (6.0.29), we can prove that  $x$  is a solution of (6.0.25) by direct computation. The proof is completed.  $\square$

**Remark 6.0.5** In [80], Wang studied the the existence and uniqueness of solutions of the following impulsive problems:

$$\begin{cases}
 {}^C D_{t_i^+}^q x(t) = f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\
 \Delta x^{(j)}(t_i) = I_{j,i}(x(t_i)), i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, q-1], \\
 x^{(j)}(0) = x_j, j \in \mathbb{N}[0, q-1],
 \end{cases} \tag{6.0.30}$$

where  ${}^C D_{t_i^+}^q x$  is the standard Caputo fractional derivative of order  $q \in (n-1, n)$  with the starting point  $t_i$ . The equivalent integral equation of IVP (6.0.30) was obtained in [80] see Lemma 2.2 [80].

Suppose that  $x$  is a solution of (6.0.30). By Theorem 3.1.2, there exist constants  $c_{i,\nu} \in \mathbb{R}$  ( $\nu \in \mathbb{N}[0, q-1], i \in \mathbb{N}[0, m]$ ) such that

$$x(t) = \int_{t_i}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds + \sum_{\nu=0}^{q-1} c_{i,\nu} \frac{(t-t_i)^\nu}{\Gamma(\nu+1)}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

Then

$$x^{(j)}(t) = \int_{t_i}^t \frac{(t-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + \sum_{\nu=j}^{q-1} c_{i,\nu} \frac{(t-t_i)^{\nu-j}}{\Gamma(\nu-j+1)}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], j \in \mathbb{N}[1, q-1].$$

By  $x^{(j)}(0) = x_j, j \in \mathbb{N}[0, q-1]$ , we get  $c_{0,j} = x_j (j \in \mathbb{N}[0, q-1])$ .

By  $\Delta x^{(j)}(t_i) = I_{j,i}(x(t_i)), i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, q-1]$ , we get

$$\begin{aligned}
 c_{i,j} - & \left[ \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + \sum_{\nu=j}^{q-1} c_{i-1,\nu} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} \right] \\
 = & I_{j,i}(x(t_i)), i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, q-1].
 \end{aligned}$$

It follows that

$$c_{i-1,\nu} = \sum_{\chi=\nu}^{q-1} c_{i-2,\chi} \frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)} + \int_{t_{i-2}}^{t_{i-1}} \frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)} f(s, x(s)) ds + I_{\nu,i-1}(x(t_{i-1}))$$

and

$$\begin{aligned} c_{i,j} &= \sum_{\nu=j}^{q-1} c_{i-1,\nu} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + I_{j,i}(x(t_i)) \\ &= \sum_{\nu=j}^{q-1} \left[ \sum_{\chi=\nu}^{q-1} c_{i-2,\chi} \frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)} + \int_{t_{i-2}}^{t_{i-1}} \frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)} f(s, x(s)) ds + I_{\nu,i-1}(x(t_{i-1})) \right] \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} \\ &\quad + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + I_{j,i}(x(t_i)) \\ &= \sum_{\nu=j}^{q-1} \sum_{\chi=\nu}^{q-1} c_{i-2,\chi} \frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} + \sum_{\nu=j}^{q-1} \int_{t_{i-2}}^{t_{i-1}} \frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)} f(s, x(s)) ds \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} \\ &\quad + \sum_{\nu=j}^{q-1} I_{\nu,i-1}(x(t_{i-1})) \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + I_{j,i}(x(t_i)). \end{aligned}$$

By  $\sum_{\nu=j}^{\chi} \frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} = \frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)}$ , we have

$$\begin{aligned} c_{i,j} &= \sum_{\chi=j}^{q-1} c_{i-2,\chi} \sum_{\nu=j}^{\chi} \frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} + \sum_{\nu=j}^{q-1} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\nu,i-1}(x(t_{i-1})) + I_{j,i}(x(t_i)) \\ &\quad + \sum_{\nu=j}^{q-1} \int_{t_{i-2}}^{t_{i-1}} \frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)} f(s, x(s)) ds \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds \\ &= \sum_{\chi=j}^{q-1} c_{i-2,\chi} \frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)} + \sum_{\nu=j}^{q-1} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\nu,i-1}(x(t_{i-1})) + I_{j,i}(x(t_i)) \\ &\quad + \int_{t_{i-2}}^{t_{i-1}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\chi_1=j}^{q-1} c_{i-3,\chi_1} \sum_{\chi=j}^{\chi_1} \frac{(t_{i-2}-t_{i-3})^{\chi_1-\chi}}{\Gamma(\chi_1-\chi+1)} \frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)} \\
 &+ \sum_{\chi=j}^{q-1} \sum_{\nu=j}^{\chi} \frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\chi,i-2}(x(t_{i-2})) + \sum_{\nu=j}^{q-1} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\nu,i-1}(x(t_{i-1})) + I_{j,i}(x(t_i)) \\
 &+ \int_{t_{i-3}}^{t_{i-2}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + \int_{t_{i-2}}^{t_{i-1}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds \\
 &+ \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds \\
 &= \sum_{\chi_1=j}^{q-1} c_{i-3,\chi_1} \frac{(t_i-t_{i-3})^{\chi_1-j}}{\Gamma(\chi_1-j+1)} + \sum_{\chi=j}^{q-1} \frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)} I_{\chi,i-2}(x(t_{i-2})) + \sum_{\nu=j}^{q-1} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\nu,i-1}(x(t_{i-1})) \\
 &+ I_{j,i}(x(t_i)) + \int_{t_{i-3}}^{t_{i-2}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds \\
 &+ \int_{t_{i-2}}^{t_{i-1}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds \\
 &= \dots\dots\dots \\
 &= \sum_{\chi=j}^{q-1} c_{0,\chi_1} \frac{t_i^{\chi-j}}{\Gamma(\chi-j+1)} + \sum_{\nu=1}^{i-1} \sum_{\chi=j}^{q-1} \frac{(t_i-t_\nu)^{\chi-j}}{\Gamma(\chi-j+1)} I_{\chi,\nu}(x(t_\nu)) + I_{j,i}(x(t_i)) \\
 &+ \int_0^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds.
 \end{aligned}$$

Hence

$$\begin{aligned}
 x(t) &= \int_{t_i}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds + \sum_{j=0}^{q-1} \left[ \sum_{\chi=j}^{q-1} c_{0,\chi_1} \frac{t_i^{\chi-j}}{\Gamma(\chi-j+1)} + \int_0^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s)) ds \right. \\
 &\left. + \sum_{\nu=1}^{i-1} \sum_{\chi=j}^{q-1} \frac{(t_i-t_\nu)^{\chi-j}}{\Gamma(\chi-j+1)} I_{\chi,\nu}(x(t_\nu)) + I_{j,i}(x(t_i)) \right] \frac{(t-t_i)^j}{\Gamma(j+1)}. \quad t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].
 \end{aligned} \tag{6.0.31}$$

On the other hand, we can prove that  $x$  is a solution of (6.0.30) if  $x$  satisfies (6.0.31). We note that Lemma 2.2 in [80] is different from our result. □

We can investigate the following similar problem for impulsive fractional differential equation which involves a single starting point 0:

$$\begin{cases}
 {}^C D_{0+}^q x(t) = f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\
 \Delta x^{(j)}(t_i) = I_{j,i}(x(t_i)), i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, q-1], \\
 x^{(j)}(0) = x_j, j \in \mathbb{N}[0, q-1].
 \end{cases} \tag{6.0.32}$$

In fact, suppose that  $x$  is a solution of IVP(6.0.32). By Theorem 4.0.1, there exist constants  $c_{j,\nu} \in \mathbb{R}(j \in \mathbb{N}[0, m], \nu \in \mathbb{N}[0, n - 1])$  such that

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds + \sum_{j=0}^i \sum_{\nu=0}^{n-1} c_{j,\nu} \frac{(t-t_j)^\nu}{\Gamma(\nu+1)}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

Then

$$x^{(k)}(t) = \int_0^t \frac{(t-s)^{q-k-1}}{\Gamma(q-k)} f(s, x(s)) ds + \sum_{j=0}^i \sum_{\nu=k}^{n-1} c_{j,\nu} \frac{(t-t_j)^{\nu-k}}{\Gamma(\nu-k+1)}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

By  $\Delta x^{(k)}(t_i) = I_{k,i}(x(t_i)), i \in \mathbb{N}[1, m], k \in \mathbb{N}[0, q - 1]$ , we get  $c_{i,k} = I_{k,i}(x(t_i)), i \in \mathbb{N}[1, m], k \in \mathbb{N}[0, q - 1]$ .

By  $x^{(k)}(0) = x_k, k \in \mathbb{N}[0, q - 1]$ , we get  $c_{0,k} = x_k(k \in \mathbb{N}[0, q - 1])$ .

Hence

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds + \sum_{\nu=0}^{n-1} x_\nu \frac{t^\nu}{\Gamma(\nu+1)} \\ &+ \sum_{j=1}^i \sum_{\nu=0}^{q-1} I_{\nu,j}(x(t_j)) \frac{(t-t_j)^\nu}{\Gamma(\nu+1)}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]. \end{aligned} \tag{6.0.33}$$

On the other hand, if  $x$  satisfies (6.0.33), we can prove that  $x$  is a solution of (6.0.32). Then (6.0.32) is equivalent to (6.0.33). The proof is completed.  $\square$

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