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Weighted exponential Dichotomy of the Solutions of linear impulsive differential Equations in a Banach Space

ABSTRACT. In the paper a dependence is established between the ψ-exponential dichotomy of a homogeneous impulsive differential equation in a Banach space and the existence of ψ-bounded solution of the appropriate nonhomogeneous impulsive equation.

KEY WORDS AND PHRASES. Exponential dichotomy for impulsive differential equations, ψ-dichotomy, ψ-boundedness

1 Introduction

The impulsive differential equations are an adequate mathematical apparatus for simulation of numerous processes and phenomena in biology, physics, chemistry and control theory, e.t.c. which during their evolutionary development are subject to short time perturbations in the form of impulses. The qualitative investigation of these processes began with the work of Mil'man and Myshkis [17]. For the first time such equations were considered in an arbitrary Banach space in [2, 3, 18, 19].

The problem of ψ-boundedness and ψ-stability of the solutions of differential equations in finite dimensional Euclidean spaces, introduced for the first time by Akinyele [1] has been studied later by many authors. A beautiful explanation about the benefits of such a use of weighted stability and boundedness can be found for example in [15].

Inspired by the famous monographs of Coppel [6], Daleckii and Krein [7] as well as Massera and Schaeffer [16], where the important notion of exponential and ordinary dichotomy for ordinary differential equations is considered in details, Diamandescu [8]-[10] and Boi [4]-[5] introduced and studied the ψ-dichotomy for linear differential equations in finite dimensional Euclidean space, where ψ is a nonnegative continuous diagonal matrix function. The concept of ψ-dichotomy for arbitrary Banach spaces is introduced and studied in [11] and [12]. In this case $\psi(t)$ is an arbitrary bounded invertible linear operator for all t.

The goal of the present paper is to study such a weighted dichotomy for linear differential equations with impulse effect in arbitrary Banach spaces. We will establish a dependence between the ψ-exponential dichotomy of a homogeneous impulsive equation in a Banach space and the existence of a solution of the corresponding nonhomogeneous impulsive equation which is ψ-bounded on the semi-axis R_+.

The first investigation in this direction was made in [20] for the particular case of ψ-ordinary dichotomy.

It must be mentioned that in [13, 14] the attempt to introduce ψ-exponential dichotomy for impulsive differential equations in finite dimensional spaces is a real disaster - due to the meaningless use of the fundamental matrix there even the definitions are wrong.

2 Preliminaries

Let X be an arbitrary Banach space with norm $|.|$ and let $LB(X)$ be the space of all linear bounded operators acting in X with the norm $||.||$ and identity I. Denote $R_+ = [0, \infty)$.

We consider the nonhomogeneous impulsive equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t) \quad (t \neq t_n) \tag{1}$$

$$x(t_n + 0) = Q_n x(t_n) + h_n \quad (n = 1, 2, 3, ...) \tag{2}$$

where the operator valued function $A(.) : R_+ \to LB(X)$ and the function $f(.) : R_+ \to X$ are strongly measurable and Bochner integrable on the finite subintervals of R_+, $\{Q_n\}_{n=1}^{\infty}$ is a sequence of impulsive operators $Q_n \in LB(X)$ $(n = 1, 2, 3, ...)$, $T = \{t_n\}_{n=1}^{\infty}$ is a sequence of points on the semi-axis R_+ satisfying the condition

$$0 < t_1 < t_2 < ..., \lim_{n\to\infty} t_n = \infty$$

and $\{h_n\}_{n=1}^{\infty}$ is a sequence of elements $h_n \in X$. The corresponding homogeneous linear impulsive equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \quad (t \neq t_n) \tag{3}$$

$$x(t_n + 0) = Q_n x(t_n) \quad (t = 1, 2, 3, ...). \tag{4}$$

Definition 1 *By a solution of the impulsive equation (1), (2) (or (3), (4)) we shall call a function $x(t)$ which for $t \neq t_n$ satisfies equation (1) (or (3)), for $t = t_n$ satisfies condition (2) (or (4)) and is continuous from the left.*

It is known (see [18], [3]) that for the impulsive equation (3), (4) there exists an evolutionary Cauchy operator associating with any element $\xi \in X$ a solution $x(t)$ of the impulsive equation which satisfies the initial condition $x(s) = \xi$ $(0 \le s \le t < \infty)$.

Lemma 1 ([3]) *Let the conditions $A(t), Q_n \in LB(X)$ hold, where $t \in R_+$ $(n = 1, 2, \dots)$. Then the evolutionary operator $V(t,s)$ $(0 \le s \le t < \infty)$ of the impulsive equation (3), (4) has the form*

$$V(t,s) = \begin{cases} V_0(t,s), \ t_n < s \le t \le t_{n+1} \\ V_0(t,t_n)\left(\prod_{j=n}^{k+1} Q_j V_0(t_j,t_{j-1})\right) Q_k V_0(t_k,s), \\ t_{k-1} < s \le t_k < t_n < t \le t_{n+1} \end{cases}$$

where $V_0(t,s)$ $(0 \le s \le t < \infty)$ is the evolutionary operator of equation (3).

The operator-valued function $V(t,s)$ satisfies the equalities

$$V(t,t) = I \ (0 \le t < \infty), \tag{5}$$

$$V(t,s) = V(t,\tau)V(\tau,s) \ (0 \le s \le \tau \le t < \infty). \tag{6}$$

Moreover, it is differentiable at the points $t \in (t_{j-1}, t_j]$ $(j = 1, 2, 3, \dots)$ and $s \in [t_{j-1}, t_j)$ $(j = 1, 2, 3, \dots)$, and it is

$$\frac{\mathrm{d}V(t,s)}{\mathrm{d}t} = A(t)V(t,s), \quad \frac{\mathrm{d}V(t,s)}{\mathrm{d}s} = V(t,s)A(s). \tag{7}$$

At the points t_n $(n = 1, 2, 3, \dots)$ the following equalities are staisfied:

$$V(t_n + 0, s) = Q_n V(t_n, s) \ (0 \le s \le t_n < \infty). \tag{8}$$

Lemma 2 ([3]) *Let the following conditions hold:*

1. $A(t), Q_n \in LB(X)$, where $t \in R_+ (n = 1, 2, \dots)$.

2. The operators Q_n have continuous inverses Q_n^{-1} $(n = 1, 2, 3, \dots)$.

Then the evolutionary operator $V(t,s)$ $(0 \le t, s < \infty)$ of the impulsive equation (3), (4) has the form

$$V(t,s) = \begin{cases} V_0(t,s), \ t_n < s, t \le t_{n+1} \\ V_0(t,t_n)\left(\prod_{j=n}^{k+1} Q_j V_0(t_j,t_{j-1})\right) Q_k V_0(t_k,s), \\ t_{k-1} < s \le t_k < t_n < t \le t_{n+1} \\ V_0(t,t_n)\left(\prod_{j=n}^{k-1} Q_j^{-1} V_0(t_j,t_{j+1})\right) Q_k^{-1} V_0(t_k,s), \\ t_{n-1} < t \le t_n < t_k < s \le t_{k+1} \end{cases}$$

where $V_0(t,s)$ $(0 \le s,t < \infty)$ *is the evolutionary operator of the equation* (3).

If the conditions of Lemma 2 are satisfied, then the following equalities hold:

$$V(t,s) = V^{-1}(s,t), \quad V(t,s) = V(t,\tau)V(\tau,s) \ (0 \le s,\tau,t < \infty), \tag{9}$$

$$V(t_n+0,s) = Q_n V(t_n,s) \ (0 \le s,t_n < \infty). \tag{10}$$

Let $RL(X)$ be the subspace of all invertible operators in $LB(X)$ whose inverse operators are bounded, too. Let $\psi(t) : R_+ \to RL(X)$ be a continuous operator-function with respect to $t \in R_+$.

Definition 2 *A function* $u(.) : R_+ \to X$ *is said to be* ψ*-bounded on* R_+ *if* $\psi(t)u(t)$ *is bounded on* R_+.

Definition 3 *A function* $f(.) : R_+ \to X$ *is said to be* ψ*-integrally bounded on* R_+ *if it is measurable and there exists a positive constant* m *such that* $\int_t^{t+1} |\psi(\tau)f(\tau)|d\tau \le m$ *for all* $t \in R_+$.

Definition 4 *A sequence of points* $h = \{h_n\}_{n=1}^{\infty}$ *is said to be* ψ*-bounded on* R_+ *if* $\sup\limits_{n=1,2,3,\dots} |\psi(t_n)h_n| < \infty, h_n \in X, t_n \in T (n = 1,2,3,\dots)$.

Let $C_\psi(X,T)$ denote the space of all functions with values in X and ψ-bounded on R_+ which are continuous for $t \ne t_n$, have discontinuities of the first kind for $t = t_n$ and are continuous from the left which is a Banach space with the norm

$$|||f|||_{C_\psi} = \sup_{t \in R_+} |\psi(t)f(t)|.$$

Let $M_\psi(X,T)$ denote the Banach space of all functions with values in X and ψ-integrally bounded which are continuous for $t \ne t_n$, have discontinuities of the first kind for $t = t_n$ and are continuous from the left for $t = t_n$ with the norm

$$|||f|||_{M_\psi} = \sup_{t \in R_+} \int_t^{t+1} |\psi(s)f(s)| \mathrm{d}s.$$

Let $H_\psi(X,T)$ denote the space of all ψ-bounded sequences $h = \{h_n\}_{n=1}^{\infty}$ in X, i.e.

$$H_\psi(X,T) = \{h : \sup_{n=1,2,3,\dots} |\psi(t_n)h_n| < \infty, h_n \in X, t_n \in T, n = 1,2,3,\dots\}$$

with the norm

$$|||h|||_{H_\psi} = \sup_{n=1,2,3,\dots} |\psi(t_n)h_n|.$$

Definition 5 *The homogeneous impulsive equation (3), (4) is said to be ψ-exponential dichotomous on R_+ if there exist a pair P_1 and $P_2 = I - P_1$ of mutually complementary projections in X and numbers $M, \delta > 0$ for which the inequalities*

$$||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)|| \leq Me^{-\delta(t-s)} \qquad (0 \leq s \leq t < \infty), \tag{11}$$

$$||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)|| \leq Me^{-\delta(s-t)} \qquad (0 \leq t \leq s < \infty) \tag{12}$$

hold, where $V(t) = V(t,0)$ and $V(t,s)$ $(0 \leq s,t < \infty)$ is the Cauchy evolutionary operator of the impulsive equation (3), (4).

The equation (3), (4) is said to have a ψ-ordinary dichotomy on R_+ if (11) and (12) hold with $\delta = 0$.

Lemma 3 *Equation (3), (4) has a ψ-exponential dichotomy on R_+ with positive constants ν_1 and ν_2 if and only if there exist a pair of mutually complementary projections P_1 and $P_2 = I - P_1$ and positive constants $M, \tilde{N}_1, \tilde{N}_2$ such that following inequalities are fulfilled:*

$$|\psi(t)V(t)P_1\xi| \leq \tilde{N}_1 e^{-\nu_1(t-s)}|\psi(s)V(s)P_1\xi| \quad (\xi \in X, 0 \leq s \leq t), \tag{13}$$

$$|\psi(t)V(t)P_2\xi| \leq \tilde{N}_2 e^{-\nu_2(s-t)}|\psi(s)V(s)P_2\xi| \quad (\xi \in X, 0 \leq t \leq s), \tag{14}$$

$$\|\psi(t)V(t)P_1V^{-1}(t)\psi^{-1}(t)\| \leq M \quad (t \geq 0). \tag{15}$$

The proof of the lemma is similar as the proof of Lemma 3.1 in [11] for equations without impulses and will be omitted.

Definition 6 *The homogeneous impulsive equation (3), (4) is said to have a ψ-bounded growth on R_+ if for some fixed $l > 0$ there exists a constant $c \geq 1$ such that every solution $x(t)$ of (3), (4) satisfies*

$$|\psi(t)x(t)| \leq c|\psi(s)x(s)| \quad (0 \leq s \leq t \leq s + l). \tag{16}$$

Lemma 4 *Equation (3), (4) has ψ-bounded growth on R_+ if and only if there exist positive constants $K \geq 1$ and $\alpha > 0$ such that*

$$\|\psi(t)V(t)V^{-1}(s)\psi^{-1}(s)\| \leq Ke^{\alpha(t-s)} \quad (0 \leq s \leq t). \tag{17}$$

The proof of the lemma is similar as the proof of Lemma 3.2 in [11] for equations without impulses and will be omitted.

Remark 1 It is easy to see that the condition for ψ-bounded growth (and for bounded growth) of (3), (4) is independent of the choice of l. Hence we will use the Definition 6 with fixed $l = 1$.

Lemma 5 *If (3), (4) has ψ-bounded growth on R_+, then (15) is a consequence of (13) and (14).*

The proof of the lemma is similar as the proof of Lemma 3.5 in [11] for equations without impulses and will be omitted.

3. Main results

We shall say that condition (H) is satisfied if the following conditions hold:

H1. $A(t), Q_n \in LB(X)$, where $t \in R_+ (n = 1, 2, 3, \dots)$.

H2. $Q_n \in RL(X) \quad (n = 1, 2, 3, \dots)$.

H3. $\psi(t) : R_+ \to RL(X)$ is a continuous operator-function with respect to $t \in R_+$.

Theorem 2 *Let us assume the following:*

1. Condition (H) is satisfied.

2. Equation (3), (4) is ψ-exponential dichotomous.

3. There exist a number $l > 0$ and a positive integer λ such that each interval on R_+ with length l contains not more than λ points of the sequence T.

Then for any function $f \in C_\psi(X, T)$ and any sequence $h \in H_\psi(X, T)$ there exists a solution of the nonhomogeneous equation (1), (2) which is ψ-bounded on R_+.

Proof. Consider the function

$$\begin{aligned} \tilde{x}(t) = \int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s \\ + \sum_{t_j < t} \psi(t)V(t)P_1V^{-1}(t_j + 0)h_j - \sum_{t_j \geq t} \psi(t)V(t)P_2V^{-1}(t_j + 0)h_j \end{aligned} \tag{18}$$

In order to prove the boundedness of $\tilde{x}(t)$ we shall estimate the norms of the summands in (18). By (11) and (12) we have

$$\begin{aligned} &|\int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s| = \\ &= |\int_0^t \psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\ &\leq \int_0^t ||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)|| \; |\psi(s)f(s)|\mathrm{d}s \\ &\leq Me^{-\delta t} \int_0^t e^{\delta s}\mathrm{d}s \; |||f|||_{C_\psi} \leq \frac{M}{\delta}|||f|||_{C_\psi} \end{aligned} \tag{19}$$

and

$$
\begin{aligned}
&|\int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s| \\
&\qquad = |\int_t^\infty \psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\
&\qquad \leq \int_t^\infty ||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s \\
&\qquad \leq Me^{\delta t}\int_t^\infty e^{-\delta s}\mathrm{d}s\ |||f|||_{C_\psi} \leq \frac{M}{\delta}|||f|||_{C_\psi}.
\end{aligned} \tag{20}
$$

Analogously having in mind also the conditions 3 and H3 we obtain for the next summands

$$
\begin{aligned}
&|\sum_{t_j<t}\psi(t)V(t)P_1V^{-1}(t_j+0)h_j| \\
&\qquad = |\sum_{t_j<t}\psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j+0)h_j| \\
&\qquad = |\sum_{t_j<t}\psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j)h_j| \\
&\qquad \leq \sum_{t_j<t}||\psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)||\ |\psi(t_j)h_j| \\
&\qquad \leq M\left(\sum_{t_j<t}e^{\delta(t_j-t)}\right)|||h|||_{H_\psi} \leq \frac{M\lambda}{1-e^{-\delta l}}\ |||h|||_{H_\psi}
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
&|\sum_{t\leq t_j}\psi(t)V(t)P_2V^{-1}(t_j+0)h_j| \\
&\qquad = |\sum_{t\leq t_j}\psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j+0)h_j| \\
&\qquad = |\sum_{t\leq t_j}\psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j)h_j| \\
&\qquad \leq \sum_{t\leq t_j}||\psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)||\ |\psi(t_j)h_j| \\
&\qquad \leq M\left(\sum_{t\leq t_j}e^{\delta(t-t_j)}\right)|||h|||_{H_\psi} \leq \frac{M\lambda}{1-e^{-\delta l}}\ |||h|||_{H_\psi}.
\end{aligned} \tag{22}
$$

From (18) - (22) it follows that $\tilde{x}(t)$ is bounded on R_+ and satisfies for $t\in R_+$ the inequality

$$|\tilde{x}(t)| \leq \frac{2M}{\delta}|||f|||_{C_\psi} + \frac{2M\lambda}{1-e^{-\delta l}}\ |||h|||_{H_\psi}$$

Let be $x(t) = \psi^{-1}(t)\tilde{x}(t)$. Obviously $x(t)$ is ψ-bounded on R_+. It is immediately verified that the function $x(t)$ is continuous for $t \neq t_n$ and that the limit values $x(t_n+0)$ $(n = 1, 2, ...)$

exist. We shall show that the function $x(t)$ satisfies the impulsive equation (1), (2) using the equalities (7) and (10).

We differentiate $x(t)$ by $t \neq t_n$ and get

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= A(t)\int_0^t V(t)P_1V^{-1}(s)f(s)\mathrm{d}s + V(t)P_1V^{-1}(t)f(t)\\ &+ V(t)P_2V^{-1}(t)f(t) - A(t)\int_t^\infty V(t)P_2V^{-1}(s)f(s)\mathrm{d}s\\ &+ \sum_{t_j<t} A(t)V(t)P_1V^{-1}(t_j+0)h_j - \sum_{t_j\geq t} A(t)V(t)P_2V^{-1}(t_j+0)h_j\\ &= A(t)x(t) + V(t)P_1V^{-1}(t)f(t) + V(t)P_2V^{-1}(t)f(t)\\ &= A(t)x(t) + f(t).\end{aligned}$$

Analogously we obtain for $t = t_n \quad (n = 1, 2, ...)$ taking into account (10)

$$\begin{aligned}&x(t_n+0)\\ &\quad= \int_0^{t_n} V(t_n+0)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_{t_n}^\infty V(t_n+0)P_2V^{-1}(s)f(s)\mathrm{d}s\\ &\quad+ \sum_{t_j\leq t_n} V(t_n+0)P_1V^{-1}(t_j+0)h_j - \sum_{t_j>t_n} V(t_n+0)P_2V^{-1}(t_j+0)h_j\\ &\quad= Q_n\int_0^{t_n} V(t_n)P_1V^{-1}(s)f(s)\mathrm{d}s - Q_n\int_{t_n}^\infty V(t_n)P_2V^{-1}(s)f(s)\mathrm{d}s\\ &\quad+ Q_n\sum_{t_j<t_n} V(t_n)P_1V^{-1}(t_j+0)h_j - Q_n\sum_{t_j\geq t_n} V(t_n)P_1V^{-1}(t_j+0)h_j\\ &\quad+ V(t_n+0)P_1V^{-1}(t_n+0)h_n + V(t_n+0)P_2V^{-1}(t_n+0)h_n\\ &\quad= Q_nx(t_n) + h_n.\end{aligned}$$

Hence the function $x(t)$ is a ψ-bounded solution of the nonhomogeneous impulsive equation (1), (2) on R_+. Theorem 2 is proved. □

Remark 3 Theorem 2 still holds, if the condition $f \in C_\psi(X,T)$ is replaced by the weaker condition $f \in M_\psi(X,T)$.

Proof. In the case $f \in M_\psi(X,T)$ the estimates (19) and (20) can be replaced by the following estimates

$$
\begin{aligned}
&|\int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s| \\
&\quad = |\int_0^t \psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\
&\quad \leq \int_0^t ||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s \\
&\quad \leq M\int_0^t e^{-\delta(t-s)}\ |\psi(s)f(s)|\mathrm{d}s \leq M|||f|||_{M_\psi}\sum_{k=0}^{\infty} e^{-\delta k} \\
&\quad \leq \frac{M}{1-e^{-\delta}}|||f|||_{M_\psi},
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&|\int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s| \\
&\quad = |\int_t^\infty \psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\
&\quad \leq \int_t^\infty ||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s \\
&\quad \leq M\int_t^\infty e^{-\delta(s-t)}\ |\psi(s)f(s)|\mathrm{d}s \leq M|||f|||_{M_\psi}\sum_{k=0}^{\infty} e^{-\delta k} \\
&\quad \leq \frac{M}{1-e^{-\delta}}|||f|||_{M_\psi}.
\end{aligned}
\tag{24}
$$

□

Remark 4 Theorem 2 obviously holds without condition 3 if we consider inhomogeneous equations with $h = 0$. In this case the ψ-bounded solutions lie in the subspace $C^0_\psi(X,T)$ of the space $C_\psi(X,T)$ which consists of the functions satisfying the condition

$$x(t_n + 0) = Q_n x(t_n) \quad (n = 1, 2, 3, \dots). \tag{25}$$

Let X_1 be the linear manifold of all $\xi \in X$ for which the functions $V(t)\xi$ $(t \in R_+)$ are ψ-bounded.

For our next main result we will need the following lemma.

Lemma 6 ([20]) *Assume the following:*

1. *Condition (H) is satisfied.*

2. *$B_\psi(X)$ is an arbitrary Banach space of functions $f(.) : R_+ \to X$ and for any function $f \in B_\psi(X)$ the nonhomogeneous equation (1), (2) has at least one ψ-bounded on R_+ solution $x \in C_\psi(X,T)$.*

3. The set X_1 *is a complementary subspace of* X *and* X_2 *is a complement of it* ($X_1+X_2 = X$).

Then to each function $f(t) \in B_\psi(X)$ *there corresponds a unique solution* $x(t)$ *which is* ψ*-bounded on* R_+ *and starts from* X_2*, i.e.* $x(0) \in X_2$.

This solution satisfies the estimate

$$|||x|||_{C_\psi} \leq k|||f|||_{B_\psi}, \tag{26}$$

where $k > 0$ *is a constant not depending on* f.

Now we are ready for our second main result - a theorem, which is like an inverse of Theorem 2.

Theorem 5 *Let us assume the following:*

1. Condition (H) is satisfied.

2. The homogeneous impulsive equation (3), (4) *has a* ψ*-bounded growth on* R_+.

3. The linear manifold

$$X_1 = \{\xi \in X : \sup_{0\leq t<\infty} |\psi(t)V(t)\xi| < \infty\} \tag{27}$$

is a complementary subspace (i.e. there exists a subspace X_2 *of* X *for which* $X = X_1 + X_2$*).*

4. For each function $f \in C_\psi(X, T)$ *the nonhomogeneous impulsive equation* (1), (2) *for* $h = \{h_n\}_{n=1}^\infty = 0$ *has at least one solution belonging to the subspace* $C^0_\psi(X, T)$.

Then the impulsive equation (3), (4) *is* ψ*-exponential dichotomous.*

Proof. Let $x(t)$ be a nontrivial ψ-bounded solution of the impulsive equation (3), (4) with initial value $x(0) \in X_1$. Set

$$y(t) = x(t)\int_0^t \chi(\tau)|\psi(\tau)x(\tau)|^{-1}\mathrm{d}\tau,$$

where

$$\chi(t) = \begin{cases} 1 : & 0 \leq t \leq t_0 + \tau \\ 1 - (t - t_0 - \tau) : & t_0 + \tau < t \leq t_0 + \tau + 1 \\ 0 : & t_0 + \tau + 1 \leq t \end{cases}$$

It is not hard to check that the function $y(t)$ is a solution of the nonhomogeneous impulsive equation (1), (2) for $h = 0$ and for

$$f(t) = \chi(t)\frac{x(t)}{|\psi(t)x(t)|}.$$

Obviously $f \in C_\psi(X,T)$ and $|||f|||_{C_\psi} = 1$. But $y(0) = 0 \in X_2$, and applying Lemma 6 it follows

$$|||y|||_{C_\psi} = \sup_{t\in R_+} |\psi(t)y(t)| \le k|||f|||_{C_\psi} = k$$

from (26). Hence

$$|\psi(t)y(t)| = |\psi(t)x(t)| \int_0^t \chi(s)|\psi(s)x(s)|^{-1}\mathrm{d}s \le k \quad (t \in R_+).$$

By $t = t_0 + \tau$ we obtain the inequality

$$|\psi(t_0+\tau)y(t_0+\tau)| = |\psi(t_0+\tau)x(t_0+\tau)| \int_0^{t_0+\tau} |\psi(s)x(s)|^{-1}\mathrm{d}s \le k. \tag{28}$$

Let consider the function

$$\varphi(t) = \int_0^t |\psi(s)x(s)|^{-1}\mathrm{d}s.$$

From (28) it follows

$$\frac{\varphi'(t_0+\tau)}{\varphi(t_0+\tau)} \ge \frac{1}{k}.$$

After integrating the inequality with respect to τ on $[1,\tau]$ this implies the estimate

$$\varphi(t_0+\tau) \ge \varphi(t_0+1)e^{\frac{(\tau-1)}{k}} \quad (\tau \ge 1). \tag{29}$$

From condition 2 of the theorem it follows for $s \in [t_0, t_0+1]$ that there exists a constant $c > 1$ such that

$$|\psi(s)x(s)| \le c|\psi(t_0)x(t_0)|$$

and that is why

$$\varphi(t_0+1) = \int_{t_0}^{t_0+1} |\psi(s)x(s)|^{-1}\mathrm{d}s \ge c^{-1}|\psi(t_0)x(t_0)|^{-1}.$$

From here, taking into account the estimates (28) and (29) we obtain for $\tau \ge 1$ the relation

$$|\psi(t_0+\tau)x(t_0+\tau)| \le \frac{k}{\varphi(t_0+\tau)} \le \frac{ke^{-\frac{\tau-1}{k}}}{\varphi(t_0+1)} \le kce^{\frac{1}{k}}e^{-\frac{\tau}{k}}|\psi(t_0)x(t_0)|.$$

For $\tau \le 1$ we have

$$|\psi(t_0+\tau)x(t_0+\tau)| \le c|\psi(t_0)x(t_0)| \le ce^{\frac{1-\tau}{k}}|\psi(t_0)x(t_0)|.$$

Hence we obtain the estimate

$$|\psi(t)x(t)| \le Ne^{-\nu(t-t_0)}|\psi(t_0)x(t_0)|, \tag{30}$$

where $\nu = \frac{1}{k}$ and $N = \max\{ce^{\frac{1}{k}}, kce^{\frac{1}{k}}\}$, i.e. the inequality (13).

Analogously we consider the case if the solution $x(t)$ of the impulsive equation (3), (4) has an initial value $x(0) \in X_2$. Then we will consider the function

$$\tilde{y}(t) = x(t) \int_t^\infty \chi(s)|\psi(s)x(s)|^{-1}\mathrm{d}s$$

instead of $y(t)$. It is easy to check that the function $\tilde{y}(t)$ is a solution of the nonhomogeneous impulsive equation (1), (2) for $h = 0$ and for

$$\tilde{f}(t) = -\chi(t)\frac{x(t)}{|\psi(t)x(t)|}.$$

The solution $\tilde{y}(t)$ is ψ-bounded because $\tilde{y}(t) = 0$ for $t \geq t_0 + \tau + 1$. But $\tilde{y}(0) \in X_2$ and obviously $\tilde{f} \in C_\psi(X,T)$. Now we can apply Lemma 6, and from (26), taking into account that $|||\tilde{f}|||_{C_\psi} = 1$, it follows

$$|\psi(t)\tilde{y}(t)| = |\psi(t)x(t)| \int_t^\infty \chi(s)|\psi(s)x(s)|^{-1}\mathrm{d}s \leq k|||\tilde{f}|||_{C_\psi} = k.$$

By $\tau \to \infty$ we find the inequality

$$\int_t^\infty |\psi(s)x(s)|^{-1}\mathrm{d}s \leq k|\psi(t)x(t)|^{-1}. \tag{31}$$

Setting

$$\tilde{\varphi}(t) = \int_t^\infty |\psi(s)x(s)|^{-1}\mathrm{d}s$$

we obtain

$$\tilde{\varphi}'(t) \leq \frac{1}{k}\tilde{\varphi}(t).$$

By integration the estimate

$$\tilde{\varphi}(t) \leq \tilde{\varphi}(t_0)e^{\frac{t-t_0}{k}} \tag{32}$$

follows. Now let $\tau \geq t$. From $x(\tau) = V(\tau)V^{-1}x(t)$ it arises

$$\psi(\tau)x(\tau) = \psi(\tau)V(\tau)V^{-1}\psi^{-1}(t)\psi(t)x(t)$$

and

$$|\psi(\tau)x(\tau)| = ||\psi(\tau)V(\tau)V^{-1}\psi^{-1}(t)||\ |\psi(t)x(t)|.$$

Condition 2 of the theorem and Lemma 4 imply that there exist constants $K \geq 1, \alpha > 0$ such that

$$|\psi(\tau)x(\tau)| = Ke^{\alpha(\tau-t)}|\psi(t)x(t)|.$$

Then

$$\begin{aligned}|\psi(t)x(t)|\tilde{\varphi}(t) &= |\psi(t)x(t)| \int_t^\infty |\psi(s)x(s)|^{-1}\mathrm{d}s \\ &\geq \int_t^\infty |\psi(s)x(s)|\frac{e^{-\alpha(s-t)}}{K}|\psi(s)x(s)|^{-1}\mathrm{d}s = \frac{1}{K}\int_t^\infty e^{-\alpha(s-t)}\mathrm{d}s = \frac{1}{K\alpha}.\end{aligned}$$

Having in mind (31) and (32) it follows

$$|\psi(t)x(t)| \geq \frac{(K\alpha)^{-1}}{\tilde{\varphi}(t)} \geq \frac{(K\alpha)^{-1}}{\tilde{\varphi}(t_0)} e^{\frac{1}{k}(t-t_0)} \geq \frac{(K\alpha)^{-1}}{k} e^{\frac{1}{k}(t-t_0)} |\psi(t_0)x(t_0)|.$$

This inequality is from the same type as the desired estimate (14). From condition 2 of the theorem and Lemma 5 and Lemma 3 it follows that the impulsive equation (3), (4) is ψ-exponential dichotomous. Hence Theorem 5 is proved. □

Remark 6 Theorem 5 holds without condition 3 if the space X is finite dimensional.

References

[1] **Akinyele, O. :** *On partial stability and boundedness of degree k.* Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 8, **65**, (1978), 259 – 264

[2] **Bajnov, D., Kostadinov, S.**, and **Myshkis, A. :** *Bounded and periodic solutions of differential equations with impulse effect in a Banach space.* Differential and Integral Equations, **1** (1988), 223 – 230

[3] **Bajnov, D., Kostadinov, S.**, and **Zabrejko, P. :** *Exponential dichotomy of linear impulsive differential equations in a Banach space.* Int. J. Theor. Phys., **28** (1989), No. 7, 797 – 814

[4] **Boi, P. :** *On the ψ-dichotomy for homogeneous linear differential equations.* Electron. J. Differ. Equ., **2006**, No. 40, 1 – 12

[5] **Boi, P. :** *Existence of ψ-bounded solutions for nonhomogeneous linear differential equations.* Electron. J. Differ. Equ., **2007**, No. 52, 1 – 10

[6] **Coppel, W. :** *Dichotomies in stability theory.* Lectures Notes in Mathematics, Vol. 629, Springer Verlag, Berlin (1978)

[7] **Daleckii, J.**, and **Krein, M. :** *Stability of Solutions of Differential Equations in Banach space.* American Mathematical Society, Providence, Rhode Island (1974)

[8] **Diamandescu, A. :** *Note on the Ψ-boundedness of the solutions of a system of differential equations.* Acta Math. Univ. Comen., New Ser. **73**, (2004), No. 2, 223 – 233

[9] **Diamandescu, A. :** *Existence of ψ-bounded solutions for a system of differential equations.* Electron. J. Differ. Equ., **2004**, No. 63, 1 – 6

[10] **Diamandescu, A. :** *Existence of Ψ-bounded solutions for nonhomogeneous linear difference equations.* Appl. Math. E-Notes, **10**, (2010), 94 – 102

[11] **Georgieva, A., Kiskinov, H. , Kostadinov, S.**, and **Zahariev, A. :** *Psi-exponential dichotomy for linear differential equations in a Banach space.* Electron. J. Differ. Equ., **2013**, No. 153, 1 – 13

[12] **Georgieva, A., Kiskinov, H. , Kostadinov, S.**, and **Zahariev, A. :** *Existence of solutions of nonlinear differential equations with Psi-exponential or Psi-ordinary dichotomous linear part in a Banach space.* Electron. J. Qual. Theory Differ. Equ., **2014**, No. 2, 1 – 10

[13] **Gupta, B.**, and **Srivastava, S. :** *On the ψ-dichotomy for impulsive homogeneous linear differential equations.* Int. J. of Math. Anal. (Ruse), **2**, (2008), No. 25, 1241 – 1248

[14] **Gupta, B.**, and **Srivastava, S. :** *Existence of ψ-bounded solution for a system of impulsive differential equations.* Int. J. of Math. Anal. (Ruse), **2**, (2008), No. 25, 1249 – 1256

[15] **Hristova, S.**, and **Proytcheva, V. :** *Weighted exponential stability for generalized delay functional differential equations with bounded delay.* Advances in Difference Equations **2014**, 2014:185

[16] **Massera, J.**, and **Schaeffer, J. :** *Linear Differential Equations and Function Spaces.* Academic Press, (1966)

[17] **Mil'man, V.**, and **Myshkis, A. :** *On the stability of motion in the presence of impulses.* Sib. Math. J, **1**, (1960), 233 – 237

[18] **Zabrejko, P., Bajnov, D.**, and **Kostadinov, S. :** *Characteristic exponents of impulsive differential equations in a Banach space.* Int. J. Theor. Phys., **27** (1988), No. 6, 731 – 743

[19] **Zabrejko, P., Bajnov, D.**, and **Kostadinov, S. :** *Stability of linear equations with impulse effect.* Tamkang J. Math., **18** (1987), No. 4, 57 – 63

[20] **S. Zlatev :** *Psi-ordinary dichotomy of the solutions of impulse differential equations in a Banach space.* Int. J. Pure and Appl. Math., **92**, (2014), No. 4, 609 – 618

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Estimate of the validity interval for the Antimaximum Principle and application to a non-cooperative system

ABSTRACT. We are concerned with the sign of the solutions of non-cooperative systems when the parameter varies near a Principal eigenvalue of the system. With this aim we give precise estimates of the validity interval for the Antimaximum Principle for an equation and an example. We apply these results to a non-cooperative system. Finally a counterexample shows that our hypotheses are necessary. The Maximum Principle remains true only for a restricted positive cone.

KEY WORDS. Maximum Principle, Antimaximum Principle, Elliptic Equations and Systems, Non cooperative systems, Principal Eigenvalue.

1 Introduction

In this paper we use ideas concerning the Anti-Maximum Principle due to Clement and Peletier [5] and later to Arcoya Gámez [3] to obtain in Section 2 precise estimates concerning the validity interval for the Antimaximum Principle for one equation. An example shows that this estimate is sharp.

The Maximum Principle and then the Antimaximum Principle for the case of a single equation have been extensively studied later for cooperative elliptic systems (see the references ([1],[6],[7],[8],[10],[12]). The results in [10], are still valid for systems(with constant coefficients) involving the p-Laplacian. Some results for non-cooperative systems can be found *e.g.* in [4],[11]. Very general results concerning the Maximum Principle for equations and cooperative systems for different classes (classical, weak, very weak) of solutions were given by Amann in a long paper [2], in particular the Maximum Principle was shown to be equivalent to the positivity of the principal eigenvalue.

Here in Section 3, we consider a non-cooperative 2×2 system with constant coefficients depending on a real parameter μ having two real principal eigenvalues $\mu_1^- < \mu_1^+$. We obtain some theorems of Antimaximum Principle type concerning the behavior of different cones of couples of functions having positivity (or negativity) properties. We give several results of this type for values of $\mu_1^- < \mu$ but close to μ_1^- by combining the usual Maximum Principle and the results for the Antimaximum Principle in Section 2.

Finally a counterexample is given showing that the Maximum Principle does not hold in general for non cooperative systems, but a (partial, under an additional assumption) Maximum Principle for $\mu < \mu_1^-$ is also obtained.

2 Estimate of the validity interval for the Anti-maximum Principle

Let Ω be a smooth bounded domain in $I\!R^N$. We consider the following Dirichlet boundary value problem

$$-\Delta z \,=\, \mu z + \,h \text{ in } \Omega \,, \quad z = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where μ is a real parameter. We associate to (2.1) the eigenvalue problem

$$-\Delta\varphi \,=\, \lambda\varphi \text{ in } \Omega \,, \quad \varphi = 0 \text{ on } \partial\Omega. \tag{2.2}$$

We denote by λ_k, $k \in I\!N^*$ the eigenvalues $(0 < \lambda_1 < \lambda_2 \leq ...)$ and by φ_k a set of orthonormal associated eigenfunctions. We choose $\varphi_1 > 0$.

Hypothesis (H_0)**:** We write

$$h = \alpha\varphi_1 + h^\perp \tag{2.3}$$

where $\int_\Omega h^\perp\varphi_1 = 0$ and we assume $\alpha > 0$ and $h \in L^q$, $q > N$ if $N \geq 2$ and $q = 2$ if $N = 1$.

Theorem 1 *We assume* (H_0) *and* $\lambda_1 < \mu \leq \Lambda < \lambda_2$. *There exists a constant* K *depending only on* Ω, Λ *and* q *such that, for* $\lambda_1 < \mu < \lambda_1 + \delta(h)$ *with*

$$\delta(h) = \frac{K\alpha}{\|h^\perp\|_{L^q}}, \tag{2.4}$$

the solution z *to* (2.1) *satisfies the Antimaximum Principle, that is*

$$z < 0 \text{ in } \Omega; \quad \partial z/\partial\nu > 0 \text{ on } \partial\Omega, \tag{2.5}$$

where $\partial/\partial\nu$ *denotes the outward normal derivative.*

Remark 2.1 The Antimaximum Principle of Theorem 1, assuming $\alpha > 0$, is in the line of the version given by Arcoya- Gámez [3].

Lemma 2.1 *We assume* $\lambda_1 < \mu \leq \Lambda < \lambda_2$ *and* $h \in L^q$, $q > N \geq 2$. *We suppose that there exists a constant* C_1 *depending only on* $\Omega, q,$ *and* Λ *such that* z *satisfying* (2.1) *is such that*

$$\|z\|_{L^2} \leq C_1\|h\|_{L^2}. \tag{2.6}$$

Then there exist constants C_2 and C_3, depending only on Ω, q and Λ such that

$$\|z\|_{\mathcal{C}^1} \leq C_2\|h\|_{L^q} \text{ and } \|z\|_{L^q} \leq C_3\|h\|_{L^q}. \tag{2.7}$$

Remark 2.2 Hypothesis (2.6) cannot hold, unless h is orthogonal to φ_1. Indeed, letting μ go to λ_1, (2.6) implies the existence of a solution to (2.1) with $\mu = \lambda_1$. Note that in the proof of Theorem 1, Lemma 2.1 is used for h (and hence z) orthogonal to φ_1 .

2.1 Proof of Lemma 2.1

All constants in this proof depend only on Ω, Λ and q.

Claim: $\|z\|_{L^q} \leq C_3\|h\|_{L^q}$.

If the claim is verified then, by regularity results for the Laplace operator combined with Sobolev imbeddings

$$\|z\|_{\mathcal{C}^1} \leq C_4\|z\|_{W^{2,q}} \leq C_5(\Lambda\|z\|_{L^q} + \|h\|_{L^q}). \tag{2.8}$$

From the claim and regularity results we deduce (2.7).

Proof of the claim:

- Step 1 We consider the sequence $p_j = 2 + \frac{8j}{N}$ for $j \in I\!N$. Observe that for any j, $W^{2,p_j} \hookrightarrow L^{p_{j+1}}$ and that there exists a constant $H(j)$ such that

$$\forall v \in W^{2,p_j},\ \|v\|_{L^{p_{j+1}}} \leq H(j)\|v\|_{W^{2,p_j}}. \tag{2.9}$$

The relation (2.9) is obvious if $2p_j \geq N$ and for $2p_j < N$ we have

$$\frac{Np_j}{N-2p_j} - p_{j+1} = \frac{2p_jp_{j+1} - 8}{N - 2p_j} > 0$$

and the result follows by classical Sobolev imbedding.

- Step 2 We consider z satisfying (2.1). For $j = 0$, we derive from (2.6) and Hölder inequality that

$$\|z\|_{L^2} \leq C_5\|h\|_{L^q}. \tag{2.10}$$

By induction we assume that $z \in L^{p_j}$ with $p_j < q$ and that

$$\|z\|_{L^{p_j}} \leq K(j)\|h\|_{L^q}. \tag{2.11}$$

By Hölder inequality,

$$\|\mu z + h\|_{L^{p_j}} \leq \Lambda\|z\|_{L^{p_j}} + |\Omega|^{\frac{q-p_j}{qp_j}}\|h\|_{L^q}.$$

By regularity results for the Laplace operator:

$$\|z\|_{W^{2,p_j}} \;\le\; C(j)(\Lambda\|z\|_{L^{p_j}} + |\Omega|^{\frac{q-p_j}{qp_j}}\|h\|_{L^q}) \;\le\; C(j)(\Lambda K(j) + |\Omega|^{\frac{q-p_j}{qp_j}})\|h\|_{L^q}.$$

Using (2.9) the relation (2.11) holds for $j+1$ and the induction is proved.

- Step 3 Let J be such that $p_{J+1} \ge q > p_J$. After J iterations we get by (2.11)

$$\|z\|_{L^q} \;\le\; C_6\|z\|_{L^{p_{J+1}}} \;\le\; C_6K(J+1)\|z\|_{W^{2,p}} \le$$

$$C_7K(J+1)\|\mu z + h\|_{L^{p_J}} \le C_8(\Lambda\|h\|_{L^q} + \|h\|_{L^{p_J}}) \le C_9\|h\|_{L^q},$$

which is the claim. □

2.2 Proof of Theorem 1

- Step 1: We prove the following inequality:

$$\|z^\perp\|_{\mathcal{C}^1} \;\le\; C_2\|h^\perp\|_{L^q}. \tag{2.12}$$

We derive from (2.3)

$$z \;=\; \frac{\alpha}{\lambda_1-\mu}\varphi_1 \;+\; z^\perp, \tag{2.13}$$

with $z^\perp$ solution of

$$-\Delta z^\perp \;=\; \mu z^\perp + h^\perp \;\text{ in } \Omega\,; \;\; z^\perp = 0 \text{ on } \partial\Omega. \tag{2.14}$$

By the variational characterization of λ_2:

$$\lambda_2\int_\Omega |z^\perp|^2 \le \int_\Omega |\nabla z^\perp|^2 \;=\; \mu\int_\Omega |z^\perp|^2 \;+\; \int_\Omega z^\perp h^\perp.$$

Hence

$$\|z^\perp\|_{L^2} \;\le\; \frac{1}{\lambda_2-\Lambda}\|h^\perp\|_{L^2}.$$

By Lemma 2.1, we derive (2.12).

- Step 2: *Close to the boundary:*

We show now that on the boundary $\frac{\partial z}{\partial\nu}(x) > 0$. and near the boundary $z < 0$.

Since $\partial\varphi_1/\partial\nu < 0$ on $\partial\Omega$, we set

$$A := min_{\partial\Omega}|\partial\varphi_1/\partial\nu| > 0. \tag{2.15}$$

By a continuity argument there exists $\varepsilon > 0$ such that

$$dist(x,\partial\Omega) < \varepsilon \;\Rightarrow\; \partial\varphi_1/\partial\nu(x) \le -A/2. \tag{2.16}$$

Hence by (2.12) to (2.16) , for any $x \in \Omega$ such that $dist(x, \partial\Omega) < \varepsilon$, and if

$$0 < \mu - \lambda_1 < \frac{\alpha A}{4C_2\|h^\perp\|_{L^q}},$$

we have

$$\frac{\partial z}{\partial \nu}(x) = \frac{\alpha}{\lambda_1 - \mu}\frac{\partial \varphi_1}{\partial \nu}(x) + \frac{\partial z^\perp}{\partial \nu}(x) \geq \frac{\alpha}{\lambda_1 - \mu}\frac{\partial \varphi_1}{\partial \nu}(x) - C_2\|h^\perp\|_{L^q},$$

hence

$$\frac{\partial z}{\partial \nu}(x) \geq \frac{\alpha}{2(\lambda_1 - \mu)}\frac{\partial \varphi_1}{\partial \nu}(x) > 0. \tag{2.17}$$

Therefore $\frac{\partial z}{\partial \nu}(x) > 0$ on $\partial\Omega$. Moreover since $z = \varphi_1 = 0$ on $\partial\Omega$, we deduce from (2.17) that, for $x \in \Omega$ with $dist(x, \partial\Omega) < \varepsilon' \leq \varepsilon/2$ (ε' small enough),

$$z(x) \leq \frac{\alpha}{2(\lambda_1 - \mu)}\varphi_1(x) < 0,$$

where ε' does not depend on μ.

- Step 3: *Inside* Ω*:*

We consider now $\Omega_{\varepsilon'} := \{x \in \Omega,\ dist(x, \partial\Omega) > \varepsilon'\}$. Set

$$B := \min_{\Omega_{\varepsilon'}} \varphi_1(x) > 0.$$

We have in $\Omega_{\varepsilon'}$ by (2.12) and (2.13)

$$z(x) = \frac{\alpha}{\lambda_1 - \mu}\varphi_1(x) + z^\perp(x) \leq \frac{\alpha}{\lambda_1 - \mu}B + C_2\|h^\perp\|_{L^q} < 0$$

if we choose

$$\mu - \lambda_1 < \frac{\alpha \min(B, A/2)}{C_2\|h^\perp\|_{L^q}}.$$

We derive now Theorem 1. □

2.3 An example

Let $N = 1$, $\Omega =]0, 1[$ and $h = h_1\varphi_1 + h_2\varphi_2$ with $h_1 > 0$, $h_2 > 0$. We note that

$$\varphi_1(x) - s\varphi_2(x) = sin\pi x(1 - 2scos\pi x) > 0 \tag{2.18}$$

in Ω implies $s \leq 1/2$. For this example, taking $\mu = \lambda_1 + \varepsilon, \varepsilon > 0$, we have:

$$z = \frac{h_1}{\lambda_1 - \mu}\varphi_1 + \frac{h_2}{\lambda_2 - \mu}\varphi_2 = -\frac{h_1}{\varepsilon}\left(\varphi_1 - \frac{\varepsilon h_2}{h_1(\lambda_2 - \lambda_1 - \varepsilon)}\varphi_2\right).$$

If the Antimaximum Principle holds, $z < 0$ in Ω, and by (2.18), we have

$$\frac{\varepsilon h_2}{h_1(\lambda_2 - \lambda_1 - \varepsilon)} \leq \frac{1}{2},$$

hence

$$\varepsilon \leq \frac{h_1(\lambda_2 - \lambda_1)}{2h_2(1 + \frac{h_1}{2h_2})} \leq \frac{h_1(\lambda_2 - \lambda_1)}{2h_2}.$$

We obtain an estimate of $\delta(h)$ similar to that in Theorem 1.

3 A non-cooperative system

Now we will consider the 2×2 non-cooperative system depending on a real parameter μ:

$$-\Delta u \;=\; au \;+\; bv \;+\; \mu u \;+\; f \;\text{ in } \Omega, \tag{S_1}$$

$$-\Delta v \;=\; cu \;+\; dv \;+\; \mu v \;+\; g \;\text{ in } \Omega, \tag{S_2}$$

$$u = v = 0 \text{ on } \partial\Omega. \tag{S_3}$$

or shortly

$$-\Delta U = AU + \mu U + F \,\text{in}\, \Omega\,,\; U = 0\,\text{on}\,\partial\Omega. \tag{S}$$

Hypothesis (H_1) We assume $b > 0\,, c < 0,$ and

$$D := (a-d)^2 + 4bc > 0. \tag{3.1}$$

3.1 Eigenvalues of the system

As usual we say that μ is an eigenvalue of System (S) if $(S_1)-(S_3)$ has a non trivial solution $U = (u, v) \neq 0$ for $F \equiv 0$ and we say that μ is a principal eigenvalue of System (S) if there exists $U = (u, v)$ with $u > 0, v > 0$ solution to (S) with $F \equiv 0$.

Notice that, since (S) is not cooperative, it is not necessarily true that there is a lowest principal eigenvalue μ_1 and that the Maximum Principle holds if and only if $\mu_1 > 0$ (Amann [2]).

We seek solutions $u = p\varphi_1$, $v = q\varphi_1$ to the eigenvalue problem where, as above, (λ_1, φ_1) is the principal eigenpair for $-\Delta$ with Dirichlet boundary conditions.

Principal eigenvalues correspond to solutions with $p, q > 0$. The associated linear system is

$$(a + \mu - \lambda_1)p \;+\; bq \;=\; 0,$$

$$cp \;+\; (d + \mu - \lambda_1)q \;=\; 0,$$

and it follows from (H_1) that $(a + \mu - \lambda_1)$ and $(d + \mu - \lambda_1)$ should have opposite signs. We should have

$$Det(A + (\mu - \lambda_1)I) = (a + \mu - \lambda_1)(d + \mu - \lambda_1) - bc = 0,$$

which implies by (H_1) that the condition on signs is satisfied and this whatever the sign of μ could be. (Notice that $D > 0$ implies that both roots are real and that $D = 0$ gives a real double root).

We have then shown directly that our system has (at least) two principal eigenvalues. Their signs will depend on the coefficients. If, for example, $a < \lambda_1$, $d < \lambda_1$, the largest one is positive. We will denote the two principal eigenvalues by μ_1^- and μ_1^+ where

$$\mu_1^- := \lambda_1 - \xi_1 < \mu_1^+ := \lambda_1 - \xi_2, \tag{3.2}$$

where the eigenvalues of Matrix A are:

$$\xi_1 = \frac{a+d+\sqrt{D}}{2} > \xi_2 = \frac{a+d-\sqrt{D}}{2}.$$

Remark 3.1 Usually the Maximum Principle holds if and only if the first eigenvalue is positive. Here by replacing $-\Delta$ by $-\Delta + K$ with $K > 0$ large enough we may get $\mu_1^- > 0$. Nevertheless the Maximum Principle needs an additional condition (see Theorem 4 and its remark).

3.2 Main Theorems

3.2.1 The case $\mu_1^- < \mu < \mu_1^+$

We assume in this subsection that the parameter μ satisfies:

(H_2) $\mu_1^- < \mu < \mu_1^+$.

Theorem 2 *Assume (H_1), (H_2), and*

(H_3)
$$d < a,$$

(H_4)
$$f \geq 0,\ g \geq 0,\ f, g \not\equiv 0,\ f, g \in L^q,\ q > N \, if \, N \geq 2\,;\ q = 2 \, if \, N = 1.$$

Then there exists $\delta > 0$, independent of μ, such that if

(H_5)
$$\mu < \mu_1^- + \delta,$$

we get

$$u < 0,\ v > 0 \ in\ \Omega;\ \frac{\partial u}{\partial \nu} > 0,\ \frac{\partial v}{\partial \nu} < 0 \ on\ \partial\Omega.$$

Remark 3.2 If in the theorem above we reverse signs of f, g, u, v that is $f \leq 0$, $g \leq 0$, $f, g \not\equiv 0$, then for μ satisfying (H_5), we get

$$u > 0,\ v < 0 \ in\ \Omega;\ \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} > 0 \ on\ \partial\Omega.$$

Note that the counterexample in subsection (3.3) shows that for f, g of opposite sign($fg < 0$), u or v may change sign.

Theorem 3 *Assume* (H_1), (H_2), *and*

(H_3') $$a < d,$$

(H_4') $$f \leq 0,\ g \geq 0,\ f, g \not\equiv 0\,,\ f, g \in L^q,\ q > N\, if\ N \geq 2\,;\ q = 2\, if\ N = 1.$$

Then there exists $\delta > 0$, *independent of* μ, *such that if*

(H_5) $$\mu < \mu_1^- + \delta,$$

we obtain

$$u < 0,\ v < 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} > 0,\ \frac{\partial v}{\partial \nu} > 0\ on\ \partial\Omega.$$

Remark 3.3 If in the theorem above we reverse signs of f, g, u, v that is $f \geq 0,\ g \leq 0,\ f, g \not\equiv 0$, then for μ satisfying (H_5), we get

$$u > 0,\ v > 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} < 0\ on\ \partial\Omega.$$

Note that, by the changes used in the proof of the theorem above, the counterexample in subsection (3.3) shows that for f, g with same sign ($fg > 0$), u or v may change sign.

3.2.2 The case $\mu < \mu_1^-$

We assume in this Section that the parameter μ satisfies:

(H_2') $\mu < \mu_1^-$.

Theorem 4 *Assume* (H_1), (H_2'), *and*

(H_3') $$a < d,$$

(H_4'') $$f \geq 0,\ g \geq 0,\ f, g \not\equiv 0,\ f, g \in L^2.$$

Assume also $t^* g - f \geq 0$, $t^* g - f \not\equiv 0$ *with*

$$t^* = \frac{d - a + \sqrt{D}}{-2c}.$$

Then

$$u > 0,\ v > 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} < 0\ on\ \partial\Omega.$$

Remark 3.4 As above we can reverse signs of f, g, u, v .

3.3 Counterexample: $a > d$

We consider the system in 1 dimension

$$-u'' = 4u + v + \mu u + f \text{ in } I :=]0;\pi[,$$

$$-v'' = -u + v + \mu v + g \text{ in } I,$$

$$u(0) = u(\pi) = v(0) = v(\pi) = 0.$$

$\lambda_1 = 1$ and $\lambda_2 = 4$; $\varphi_1 = \sin x$, $\varphi_2 = \sin 2x$. We compute $\mu_1^- = 1 - \frac{5+\sqrt{5}}{2}$. Choose $f = \varphi_1 - \frac{1}{2}\varphi_2 \geq 0$ and $g = kf$ with $k \neq 0$ to be determined later. We obtain

$$u = u_1\varphi_1 + u_2\varphi_2 \text{ and } v = v_1\varphi_1 + v_2\varphi_2,$$

where

$$u_1 = \frac{k-\mu}{\mu^2+3\mu+1}\,,\ u_2 = \frac{\mu-k-3}{2(\mu^2-3\mu+1)},$$

1/ Choosing $\mu = -3 < \mu_1^-$, we get $v_1 = -1$ and $v_2 = \frac{1-3k}{38}$. Therefore

$$-v = \varphi_1 + \frac{3k-1}{38}\varphi_2,$$

and for $\frac{3k-1}{38} > \frac{1}{2}$, v changes sign. Hence Maximum Principle does not hold.

2/ Choosing $\mu_1^- < \mu = \mu_1^- + \epsilon$, $k = \mu_1^- + \epsilon^2$, we have

$$\frac{u_2}{u_1} = \left(\frac{\mu-k-3}{k-\mu}\right)\left(\frac{\mu^2+3\mu+1}{2(\mu^2-3\mu+1)}\right) = \left(\frac{3+\epsilon}{\epsilon}\right)\left(\frac{\sqrt{5}-\epsilon}{(9+3\sqrt{5})-(6+\sqrt{5})\epsilon+\epsilon^2}\right).$$

So that $\frac{u_2}{u_1} \to \infty$ as $\epsilon \to 0$. Hence for these $f > 0$, $g < 0$, u changes sign. □

3.4 Proofs of the main results

3.4.1 Some computations and associate equation

In the following we introduce

$$\gamma_1 = \frac{1}{2}(a+d+2\mu-\sqrt{D}) = \lambda_1 + \mu - \mu_1^+; \tag{3.3}$$

$$\gamma_2 = \frac{1}{2}(a+d+2\mu+\sqrt{D}) = \lambda_1 + \mu - \mu_1^-, \tag{3.4}$$

and some auxiliary results used in the proofs of our results.

Lemma 3.1 *We have*

$(L1)$
$$\mu < \mu_1^+ \Leftrightarrow \gamma_1 < \lambda_1.$$

$(L2)$
$$\mu_1^- < \mu \Leftrightarrow \lambda_1 < \gamma_2.$$

$(L3)$
$$\sqrt{D} < a - d \Leftrightarrow d + \mu < \gamma_1 < \gamma_2 < a + \mu.$$

$(L4)$
$$\sqrt{D} < d - a \Leftrightarrow a + \mu < \gamma_1 < \gamma_2 < d + \mu.$$

$(L5)$
$$\mu < \mu_1^+ + \delta \Leftrightarrow \gamma_1 < \lambda_1 + \delta.$$

$(L6)$
$$\mu < \mu_1^- + \delta \Leftrightarrow \gamma_2 < \lambda_1 + \delta.$$

3.4.2 Proofs of Theorems 2 and 3

Proof of Theorem 2, $a > d$:

We introduce now

$$w = u + tv, \tag{3.5}$$

with

$$t = \frac{a - d + \sqrt{D}}{-2c} = \frac{2b}{a - d - \sqrt{D}} \tag{3.6}$$

so that

$$-\Delta w = \gamma_1 w + f + tg \, in \, \Omega; \tag{3.7}$$

$$w|_{\partial\Omega} = 0.$$

We remark that

$$t = \frac{b}{\gamma_1 - d - \mu} = \frac{b}{a + \mu - \gamma_2} = \frac{\gamma_1 - a - \mu}{c} = \frac{d + \mu - \gamma_2}{c}. \tag{3.8}$$

Note first that Hypothesis (H_3) implies $t > 0$ and $a - d > \sqrt{D}$. By (H_2), (H_4), and $(L1)$ in Lemma 3.1, $\gamma_1 < \lambda_1$, and we apply the Maximum Principle which gives $w > 0$ on Ω and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$. We compute

$$a + \mu - \frac{b}{t} = a + d + 2\mu - \gamma_1 = \gamma_2, \tag{3.9}$$

and since $v = (w - u)/t$, we derive

$$-\Delta u = (a + \mu - \frac{b}{t})u + \frac{b}{t}w + f = \gamma_2 u + \frac{b}{t}w + f,$$

where $\frac{b}{t}w + f > 0$. From (H_5) and $(L6)$, $\gamma_2 \leq \lambda_1 + \delta_1$, where

$$\delta_1 := \delta(\frac{b}{t}w + f), \tag{3.10}$$

we deduce from the Antimaximum Principle that $u < 0$ on Ω and $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$. Hence $cu + g > 0$.

Now (H_2), (L_1) and (L_3) imply $d + \mu < \gamma_1 < \lambda_1$ and the Maximum Principle applied to (S_2) gives $v > 0$ on Ω and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$.

We apply now Section 1 to estimate δ_1.

$$h := \frac{b}{t}w + f = (\gamma_1 - d - \mu)w + f = \sigma\varphi_1 + h^{\perp}. \tag{3.11}$$

First we compute σ:

Set $f = \alpha\varphi_1 + f^{\perp}$, $g = \beta\varphi_1 + g^{\perp}$, $w = \kappa\varphi_1 + w^{\perp}$. Since

$$-\Delta w = \gamma_1 w + f + \frac{b}{\gamma_1 - d - \mu} g,$$

we calculate:

$$\sigma = \alpha + (\gamma_1 - d - \mu)\kappa = \alpha\frac{\lambda_1 - d - \mu}{\lambda_1 - \gamma_1} + \beta\frac{b}{\lambda_1 - \gamma_1}.$$

Now we estimate $\|h^{\perp}\|_{L^2}$.

$$-\Delta w^{\perp} = \gamma_1 w^{\perp} + f^{\perp} + \frac{b}{\gamma_1 - d - \mu} g^{\perp}.$$

The variational characterization of λ_2 gives

$$(\lambda_2 - \gamma_1)\|w^{\perp}\|_{L^2} \leq \|f^{\perp}\|_{L^2} + \frac{b}{\gamma_1 - d - \mu}\|g^{\perp}\|_{L^2}.$$

We derive from (3.11)

$$\|h^{\perp}\|_{L^2} \leq \|f^{\perp}\|_{L^2} + (\gamma_1 - d - \mu)\|w^{\perp}\|_{L^2} \leq \frac{\lambda_2 - d - \mu}{\lambda_2 - \gamma_1}\|f^{\perp}\|_{L^2} + \frac{b}{\lambda_2 - \gamma_1}\|g^{\perp}\|_{L^2}.$$

Reasoning as in Lemma 2.1, we show that there exists a constant C_3 such that

$$\|h^{\perp}\|_{L^q} \leq C_3\left(\frac{\lambda_2 - d - \mu}{\lambda_2 - \gamma_1}\|f^{\perp}\|_{L^q} + \frac{b}{\lambda_2 - \gamma_1}\|g^{\perp}\|_{L^q}\right). \tag{3.12}$$

In fact for proving (3.12) we use the same sequence than that in Lemma 2.1 and we show by induction that

$$\|z^{\perp}\|_{L^{p_j}} \leq K(j)\left(\|f^{\perp}\|_{L^q} + \|g^{\perp}\|_{L^q}\right).$$

Now we apply the Antimaximum Principle to the equation

$$-\Delta u = \gamma_2 u + h.$$

This is possible since by $(L6)$ in Lemma 3.1, $\lambda_1 < \gamma_2 < \lambda_1 + \delta_2 = \lambda_1 + \delta(h)$ where, as in Theorem 1, $\delta(h) = \frac{K\sigma}{\|h^\perp\|_{L^q}}$.

Moreover we notice that $\lambda_1 - \gamma_1 = \mu_1^+ - \mu \le \mu_1^+ - \mu_1^-$ and therefore, since $\alpha > 0$ and $\beta > 0$ by (H_4),

$$\sigma = \alpha \frac{\lambda_1 - d - \mu}{\lambda_1 - \gamma_1} + \beta \frac{b}{\lambda_1 - \gamma_1} \;\ge\; \mathcal{A} := \alpha \frac{\lambda_1 - d - \mu_1^+}{\mu_1^+ - \mu_1^-} + \beta \frac{b}{\mu_1^+ - \mu_1^-},$$

and from (3.12), we obtain

$$\|h^\perp\|_{L^q} \le \mathcal{B} := C_3 \left(\frac{\lambda_2 - d - \mu_1^-}{\lambda_2 - \lambda_1} \|f^\perp\|_{L^q} + \frac{b}{\lambda_2 - \lambda_1} \|g^\perp\|_{L^q} \right).$$

From the computation above we can choose $\delta_2 = \frac{K\mathcal{A}}{\mathcal{B}}$ which does not depend on μ, and the result follows. □

Proof of Theorem 3, $a < d$:
We deduce this theorem from Theorem 2 by change of variables. Set $\hat{a} = d$, $\hat{d} = a$, $\hat{u} = v$, $\hat{v} = -u$ and $\hat{f} = g$, $\hat{g} = -f$. $\hat{f} \ge 0$, $\hat{g} \ge 0$, imply $\hat{u} < 0$, $\hat{v} > 0$. We get Theorem 3. □

3.4.3 Proof of Theorem 4

Since $a < d$, we have $t^* = \frac{d-a+\sqrt{D}}{-2c} > 0$. With now the change of variable $w = -u + t^* v$, as in [4] (see also [11]) , we can write the system as

$$-\Delta u = \gamma_1 u + (b/t^*) w + f \; in\, \Omega, \tag{3.13}$$

$$-\Delta v = \gamma_1 v - cw + g \; in\, \Omega \tag{3.14}$$

$$-\Delta w = \gamma_2 w + (t^* g - f) \; in\, \Omega, \tag{3.15}$$
$$u = v = w = 0 \; on\; \partial\Omega.$$

Now $\mu < \mu_1^-$, and it follows from $(L2)$ in Lemma 3.1 that $\gamma_1 < \gamma_2 < \lambda_1$. From (3.15) it follows from the Maximum Principle that $w > 0$. Then in (3.14) $-cw + g > 0$, and again by the Maximum Principle $v > 0$. Finally, since $(b/t^*)w + f > 0$ in (3.13), again by the Maximum Principle $u > 0$. □

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References

[1] **Amann, H. :** *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces.* SIAM Re. 18, 4, 1976, p. 620 – 709

[2] **Amann, H. :** *Maximum Principles and Principal Eigenvalues.* Ten Mathematical Essays on Approximation in Analysis and Topology, J. Ferrera, J. López-Gómez, F. R. Ruz del Portal ed., Elsevier, 2005, 1 – 60

[3] **Arcoya, D.**, and **Gámez, J. :** *Bifurcation theory and related problems: anti-maximum principle and resonance.* Comm. Part. Diff. Equat., 26, 2001, p. 1879 – 1911

[4] **Caristi, G.**, and **Mitidieri, E. :** *Maximum principles for a class of non-cooperative elliptic systems.* Delft Progress Rep. 14, 1990, p. 33 – 56

[5] **Clément, P.**, and **Peletier, L. :** *An anti-maximum principle for second order elliptic operators.* J. Diff. Equ. 34, 1979, p. 218 – 229

[6] **de Figueiredo, D. G.**, and **Mitidieri, E. :** *A Maximum Principle for an Elliptic System and Applications to semilinear Problems.* SIAM J. Math and Anal. N17 (1986), 836 – 849

[7] **de Figueiredo, D. G.**, and **Mitidieri, E. :** *Maximum principles for cooperative elliptic systems.* C. R. Acad. Sci. Paris 310, 1990, p. 49 – 52

[8] **de Figueiredo, D. G.**, and **Mitidieri, E. :** *Maximum principles for linear elliptic systems.* Quaterno Mat. 177, Trieste, 1988

[9] **Fleckinger, J., Gossez, J. P., Takác, P.**, and **de Thélin, F. :** *Existence, nonexistence et principe de l'antimaximum pour le p-laplacien.* C. R. Acad. Sci. Paris 321, 1995, p. 731 – 734

[10] **Fleckinger, J., Hernández, J.**, and **de Thélin, F. :** *On maximum principles and existence of positive solutions for some cooperative elliptic systems.* Diff. Int. Eq. 8, 1, 1995, p. 69 – 85

[11] **Lécureux, M. H. :** *Au-delà du principe du maximum pour des systèmes d'opérateurs elliptiques.* Thèse, Université de Toulouse, Toulouse 1, 13 juin 2008

[12] **Protter, M. H.**, and **Weinberger, H. :** *Maximum Principles in Differential Equations.* Springer-Verlag, 1984

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Subdensity as a convenient concept for Bounded Topology

ABSTRACT. A *subdensity space* is a special case of a *density space*, which also occur under the name of *hypernear space* in [17]. Hence, most of *classical* spaces, like topological spaces, uniform spaces, proximity spaces, contiguity spaces or nearness spaces, respectively can be immediately described and studied in this *general* framework. Moreover, the more specific defined subdensity spaces allow us to consider and integrate the *fundamental* species of *b-topological* and *b-near spaces*, too, as presented and studied in [19]. In this paper it is shown that b-proximal spaces also can be involved, and b-topological spaces then have an *alternate* description by *different* corresponding subdensity spaces.

At last, we establish a one-to-one correspondence between suitable subdensity spaces and their related *strict* topological extensions [1]. This relationship generalizes the *one* of LODATO, studied by him in the realm of generalized proximity spaces [20].

KEY WORDS AND PHRASES. Bounded Topology; b-topological space; b-proximal space; strict topological extension

1 Basic Concepts

As usual $\underline{P}X$ denotes the power set of a set X, and we call $\mathcal{B}^X \subset \underline{P}X$ a *bornology* (on X) [8], if it possesses the following properties, i.e.

(b_0) $\emptyset \in \mathcal{B}^X$;

(b_1) $B_2 \subset B_1 \in \mathcal{B}^X$ imply $B_2 \in \mathcal{B}^X$;

(b_2) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;

(b_3) $B_1, B_2 \in \mathcal{B}^X$ imply $B_1 \cup B_2 \in \mathcal{B}^X$.

The elements of $\mathcal{B}^X$ are called *bounded sets*. Then, for bornologies $\mathcal{B}^X, \mathcal{B}^Y$ a function $f : X \longrightarrow Y$ is called *bi-bounded* iff f satisfies (bib$_1$) $f\mathcal{B}^X := \{f[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^Y$;

(bib$_2$) $f^{-1}\mathcal{B}^Y := \{f^{-1}[D] : D \in \mathcal{B}^Y\} \subset \mathcal{B}^X$.

Evidently, for corresponding power sets each map $f : X \longrightarrow Y$ is bi-bounded. As an *instructive* example we consider for sets X, Y as bornologies in each case the set of all *finite* subsets of those. Then, for each map $f : X \longrightarrow Y$ and some $B \in \mathcal{B}^X_{fi} := \{D \subset X : D$ is finite$\}$ we look at the power set on B and consider the restriction $f|_B$ of f on B. Then $f|_B$ is bi-bounded.

Then we make use of the following notations: For collections $\rho, \rho_1, \rho_2 \subset \underline{P}X$ we put:

$$\rho_2 << \rho_1 \text{ iff } \forall F_2 \in \rho_2 \exists F_1 \in \rho_1 \; F_1 \subset F_2;$$
$$\rho_1 \vee \rho_2 := \{F_1 \cup F_2 : F_1 \in \rho_1, F_2 \in \rho_2\};$$
$$\sec \rho := \{D \subset X : \forall F \in \rho D \cap F \neq \emptyset\}.$$

Definition 1.1 *We call a triple $(X, \mathcal{B}^X, N)$ consisting of a set X, bornology $\mathcal{B}^X$ and a function $N : \mathcal{B}^X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ an* episd-space *(shortly* esd-space*) iff the following axioms are satisfied:*

(esd$_1$) $\rho_2 << \rho_1 \in N(B), B \in \mathcal{B}^X, \rho_2 \subset \underline{P}X$ *imply* $\rho_2 \in N(B)$;

(esd$_2$) $B \in \mathcal{B}^X$ *implies* $\mathcal{B}^X \notin N(B) \neq \emptyset$;

(esd$_3$) $\rho \in N(\emptyset)$ *implies* $\rho = \emptyset$;

(esd$_4$) $x \in X$ *implies* $\{\{x\}\} \in N(\{x\})$;

(esd$_5$) $\emptyset \neq B_2 \subset B_1 \in \mathcal{B}^X$ *imply* $N(B_2) \subset N(B_1)$;

(esd$_6$) $\{cl_N(F) : F \in \rho\} \in N(B), \rho \subset \underline{P}X, B \in \mathcal{B}^X$ *imply* $\rho \in N(B)$, *where* $cl_N(F) := \{x \in X : \{F\} \in N(\{x\})\}$;

(esd$_7$) $\rho_1 \vee \rho_2 \in N(B), \rho_1, \rho_2 \subset \underline{P}X, B \in \mathcal{B}^X$ *imply* $\rho_1 \in N(B)$ *or* $\rho_2 \in N(B)$;

(esd$_8$) $B \in \mathcal{B}^X$ *implies* $cl_N(B) \in \mathcal{B}^X$;

(esd$_9$) $\rho \cap \mathcal{B}^X \in N(B), B \in \mathcal{B}^X \backslash \{\emptyset\}, \rho \subset \underline{P}X$ *imply* $\rho \in N(B)$.

If $\rho \in N(B)$ for some $B \in \mathcal{B}^X$, then we call ρ a B-collection *(in N). For esd-spaces $(X, \mathcal{B}^X, N)$, $(Y, \mathcal{B}^Y, M)$ a function $f : X \longrightarrow Y$ is called* bi-bounded sd-map *(shortly* bibsd-map *iff it satisfies* (bib$_1$), (bib$_2$) *and*

(sd) $B \in \mathcal{B}^X$ *and* $\rho \in N(B)$ *imply* $f\rho := \{f[F] : F \in \rho\} \in M(f[B])$.

We denote by ESD the corresponding category.

Remark 1.2 In a former paper [19] it was shown, that the category b-TOP of b-topological spaces and b-continuous maps as well as the category b-NEAR of b-nearness spaces and b-near maps can be *fully embedded* into ESD. In our following research we will establish a *further equivalent* description of b-topological spaces by means of *different* esd-spaces resulting into an alternate description of the category TOP, if the given bornology $\mathcal{B}^X$ of the considered esd-space is *saturated*, which means X is an element of $\mathcal{B}^X$. Moreover, we focus our attention on so called *b-proximal spaces* which also can be integrated into the above defined concept. Then, in a *natural* way, we will characterize those esd-spaces which can be *extended* to a certain topological one. In case of *saturation* this *new* established *connection* deliver us the well-known famous theorem of LODATO [20] up to isomorphism.

Definition 1.3 *For a set X let $\mathcal{B}^X$ be a bornology. A function $t : \mathcal{B}^X \longrightarrow \underline{P}X$ is called a* b-topological operator (b-topology) *(on $\mathcal{B}^X$) iff the following axioms are satisfied, i.e.*

(b-t$_1$) $B \in \mathcal{B}^X$ *implies* $t(B) \in \mathcal{B}^X$*;*

(b-t$_2$) $t(\emptyset) = \emptyset$*;*

(b-t$_3$) $B \in \mathcal{B}^X$ *implies* $B \subset t(B)$*;*

(b-t$_4$) $B_1 \subset B_2 \in \mathcal{B}^X$ *imply* $t(B_1) \subset t(B_2)$*;*

(b-t$_5$) $B \in \mathcal{B}^X$ *implies* $t(t(B)) \subset t(B)$*;*

(b-t$_6$) $B_1, B_2 \in \mathcal{B}^X$ *imply* $t(B_1 \cup B_2) \subset t(B_1) \subset t(B_2)$.

Then the triple $(X, \mathcal{B}^X, t)$ is called a b-topological space. *For b-topological spaces $(X, \mathcal{B}^X, t^X)$, $(Y, \mathcal{B}^Y, t^Y)$ a function $f : X \longrightarrow Y$ is called* b-continuous map *iff it is bi-bounded and satisfies the following condition, i.e.*

(cont) $B \in \mathcal{B}^X$ *implies* $f[t^X(B)] \subset t^Y(f[B])$.

We denote by b-TOP the corresponding category [19].

Example 1.4 For a set X let $\mathcal{B}_f^X$ be denote the set of *all* finite subsets of X. Thus, $\mathcal{B}_f^X$ defines a bornology on X. Then, for a fixed set $D \in \mathcal{B}_f^X$ we establish a b-topology $t^D : \mathcal{B}^X \longrightarrow \underline{P}X$ by setting $t^D(\emptyset) := \emptyset$ and $t^D(B) := B \cup D$, otherwise.

Remark 1.5 If $\mathcal{B}^X$ is saturated, then a b-topological space can be considered as topological space and vice versa. Moreover, if for bornologies $\mathcal{B}^X, \mathcal{B}^Y$ with saturated $\mathcal{B}^X f : X \longrightarrow Y$ is constant map, then f is *automatically* b-continuous.

Lemma 1.6 *For a b-topological space $(X, \mathcal{B}^X, t)$ we set: $N_t(\emptyset) := \{\emptyset\}$ and $N_t(B) := \{\rho \subset \underline{P}X : B \in \sec\{t(F) : F \in \rho \cap \mathcal{B}^X\}\}$, otherwise.*

Then $(X, \mathcal{B}^X, N_t)$ is an esd-space such that $t = cl_{N_t}$ (see also Chapter 2).

Proof: Firstly, we have to verify that N_t is satisfying the axioms (esd$_1$) to (esd$_9$).

to (esd$_1$): $\rho_2 << \rho_1 \in N_t(B), \rho \subset \underline{P}X, B \in \mathcal{B}^X\backslash\{\emptyset\}$ and $F \in \rho_2 \cap \mathcal{B}^X$ imply the existence of $F_1 \in \rho_1$ with $F_1 \subset F_2$. Hence $F_1 \in \rho_1 \cap \mathcal{B}^X$ follows by applying (b$_1$), and $B \cap t(F_1) \neq \emptyset$ results by hypothesis. Consequently, $B \cap t(F_2) \neq \emptyset$ is valid according to (b-t$_4$), resulting into $\rho_2 \in N_t(B)$.

to (esd$_2$): Let $B \in \mathcal{B}^X$; in first case if $B = \emptyset$ we have $\emptyset \in N_t(B)$ by definition. In second case if $B \neq \emptyset$ we get $\{B\} \in N_t(B)$, since $B \cap t(B) \neq \emptyset$ is valid.

Further suppose $\mathcal{B}^X \in N_t(B)$, and without restriction $B \neq \emptyset$, otherwise $B = \emptyset$ contradicts. Then $B \in \sec\{t(F) : F \in B^X\}$ implies $B \cap t(\emptyset) \neq \emptyset$, which contradicts too. Hence $\mathcal{B}^X \notin N_t(B)$ follows.

to (esd$_3$): evident by definition of N_t.

to (esd$_4$): see especially proof of (esd$_2$).

to (esd$_5$): evident.

to (esd$_6$): For $\{cl_{N_t}(F) : F \in \rho\} \in N_t(B), \rho \subset \underline{P}X, B \in \mathcal{B}^X$ let $A \in \rho \cap \mathcal{B}^X$, we have to verify $B \cap t(A) \neq \emptyset$. Since $cl_{N_t}(A) \in \{cl_{N_t}(F) : F \in \rho\}$ we get $B \cap t(cl_{N_t}(A)) \neq \emptyset$ by hypothesis. Note, that $cl_{N_t}(A) \subset t(A) \in \mathcal{B}^X$ is valid. Consequently $B \cap t(t(A)) \neq \emptyset$ follows, and $B \cap t(A) \neq \emptyset$ results according to (b-t$_5$), showing our made assertion.

to (esd$_7$): $\rho_1 \vee \rho_2 \in N_t(B)$ and without restriction $B \neq \emptyset$ with $\rho_1 \neq \emptyset \neq \rho_2$ imply $B \in \sec\{t(F) : F \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X\}$. Now, let us suppose $\rho_1, \rho_2 \notin N_t(B)$. Hence there exists $F_1 \in \rho_1 \cap \mathcal{B}^X B \cap t(F_1) = \emptyset$ and $F_2 \in \rho_2 \cap \mathcal{B}^X$ $B \cap t(F_2) = \emptyset$. But $F_1 \cup F_2 \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X$, since $\mathcal{B}^X$ is bornology and

$$\emptyset = (B \cap t(F_1)) \cup (B \cap t(F_2)) = B \cap (t(F_1) \cup t(F_2)) = B \cap t(F_1 \cup F_2)$$

according to (b-t$_4$) and (b-t$_6$), respectively which contradicts.

to (esd$_8$): evident.

to (esd$_9$): $B \in \mathcal{B}^X\backslash\{\emptyset\}$ and $\rho \cap \mathcal{B}^X \in N_t(B), \rho \subset \underline{P}X$ imply $B \in \sec\{t(F) : F \in (\rho \cap \mathcal{B}^X) \cap \mathcal{B}^X\}$, and $\rho \in N_t(B)$ results. To show the equality $t = cl_{N_t}$ is valid let without restriction $B \in \mathcal{B}^X\backslash\{\emptyset\}$, then $x \in cl_{N_t}(B)$ is equivalent to the statement $\{B\} \in N_t(\{x\})$, which is further equivalent to $\{x\} \in \sec\{t(F) : F \in B \cap \mathcal{B}^X\}$, at last resulting into the statement $x \in t(B)$ as equivalent to above.

□

Remark 1.7 As an interpretation of this Lemma we keep hold that every b-topological space is induced by a certain esd-space.

As a next step in our research we will introduce the concept of b-proximal spaces and related facts.

Definition 1.8 *For a bornology $\mathcal{B}^X$ a relation $\delta \subset \mathcal{B}^X \times \mathcal{B}^X$ is called* b-proximal*, and the triple $(X, \mathcal{B}^X, \delta)$ a* b-proximal space *iff δ satisfies the following conditions, i.e.*

(b-p$_1$) *$B \in \mathcal{B}^X$ implies $cl_\delta(B) \in \mathcal{B}^X$, where $cl_\delta(B) := \{x \in X : \{x\}\delta B\}$;*

(b-p$_2$) *$\emptyset\bar{\delta}D$ and $B\bar{\delta}\emptyset$ for each $B, D \in \mathcal{B}^X$;*

(b-p$_3$) *$B\delta(D_1 \cup D_2)$ iff $B\delta D_1$ or $B\delta D_2$ for each $B, D_1, D_2 \in \mathcal{B}^X$;*

(b-p$_4$) *$x \in X$ implies $\{x\}\delta\{x\}$;*

(b-p$_5$) *$B_1 \subset B \in \mathcal{B}^X$ and $B_1\delta D$ imply $B\delta D$ for each $D \in \mathcal{B}^X$;*

(b-p$_6$) *$B_1\delta D$ and $D \subset cl_\delta(B), B \in \mathcal{B}^X$ imply $B_1\delta B$.*

(Hereby, $\bar{\delta}$ denotes the negation *of δ). For b-proximal spaces $(X, \mathcal{B}^X, \delta)$, $(Y, \mathcal{B}^Y, \gamma)$ a function $f : X \longrightarrow Y$ is called* b-proximal map *iff f is bi-bounded and satisfies the following condition, i.e.*

(prox) *$B\delta D$ implies $f[B]\gamma f[D]$. We denote by b-PX the corresponding category.*

Remark 1.9 If $\mathcal{B}^X$ is saturated, then a b-proximal space $(X, \mathcal{B}^X, \delta)$ may be considered as a *generalized proximity space* and vice versa [14]. In *special* cases LEADER proximities as well as LODATO proximities then can be easily recovered.

Proposition 1.10 *For a b-topological space $(X, \mathcal{B}^X, t)$ we set: $B\delta_t D$ iff $B \cap t(D) \neq \emptyset$ for each $B, D \in \mathcal{B}^X$. Then $(X, \mathcal{B}^X, \delta_t)$ defines a b-proximal space which additionally is* additive *by satisfying*

(add) *$(B_1 \cup B_2)\delta D, B_1, B_2, D \in \mathcal{B}^X$ imply $B_1\delta D$ or $B_2\delta D$.*

Proof: straight forward. □

Definition 1.11 *A b-proximal space $(X, \mathcal{B}^X, \delta)$ is called* symmetric *iff in addition holds*

(s) *$B_1\delta B_2$ implies $B_2\delta B_1$ for each $B_1, B_2 \in \mathcal{B}^X$.*

Remark 1.12 Here, we only note that if $\mathcal{B}^X$ is saturated, then $(X, \mathcal{B}^X, \delta)$ can be *essentially* considered as a LODATO proximity space [20] and vice versa. We denote by b-SPX the corresponding full subcategory of b-PX.

2 b-TOP, b-PX and b-SPX as fully embedded subcategories of ESD

Now, firstly let us start with the objects of b-PX.

Lemma 2.1 *For a b-proximal space* $(X, \mathcal{B}^X, \delta)$ *we set:* $N_\delta(\emptyset) := \{\emptyset\}$ *and* $N_\delta(B) := \{\rho \subset \underline{P}X : \rho \cap \mathcal{B}^X \subset \delta(B)\}$*, where* $\delta(B) := \{D \in \mathcal{B}^X : B\delta D\}$*, otherwise. Then* $(X, \mathcal{B}^X, N_\delta)$ *is an esd-space.*

Proof: Straight forward. Here, we only will verify the validity of the axioms (esd_6), (esd_7) and (esd_8) in definition 1.1.

to (esd_6): For $\rho \subset \underline{P}X$ let $\{cl_{N_\delta}(F) : F \in \delta\} \in N_\delta(B)$, we have to verify $\rho \cap \mathcal{B}^X \subset \delta(B)$. $A \in \rho \cap \mathcal{B}^X$ implies $cl_{N_\delta}(A) \in \{cl_{N_\delta}(F) : F \in \rho\}$. Since $A \in \mathcal{B}^X$ we claim $cl_{N_\delta}(A) \subset cl_\delta(A)$, hence $cl_{N_\delta}(A) \in \mathcal{B}^X$. By hypothesis $cl_{N_\delta}(A) \in \delta(B)$ follows, showing that $B\delta cl_{N_\delta}(A) \subset cl_\delta(A)$ is valid. But δ is satisfying (b-p6), and $B\delta A$ results, hence $A \in \delta(B)$ follows.

to (esd_7): Without restriction let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\rho_1 \vee \rho_2 \in N_\delta(B)$, $\rho_1 \neq \emptyset \neq \rho_2$. If supposing $\rho_1, \rho_2 \notin N_\delta(B)$ we get $F_1, F_2 \notin \delta(B)$ for some $F_1 \in \rho_1 \cap \mathcal{B}^X$ and $F_2 \in \rho_2 \cap \mathcal{B}^X$. Hence $B\bar{\delta}F_1$ and $B\bar{\delta}F_2$ implying $B\bar{\delta}(F_1 \cup F_2)$ according to (b-p_3), note that $\mathcal{B}^X$ is bornology. But $F_1 \cup F_2 \in (\rho_1 \cup \rho_2) \cap \mathcal{B}^X$ leads us to a contradiction.

to (esd_8): $B \in \mathcal{B}^X$ implies $cl_\delta(B) \in \mathcal{B}^X$. We will show that $cl_{N_\delta}(B) \subset cl_\delta(B)$, then by ($\text{b}_1$) we get the desired result. $x \in cl_{N_\delta}(B)$ implies $\{B\} \in N_\delta(\{x\})$, hence $\{B\} \subset \delta(\{x\})$, and $\{x\}\delta B$ results, showing that $x \in cl_\delta(B)$ is valid.

□

Definition 2.2 *An esd-space* $(X, \mathcal{B}^X, N)$ *is called* conic *iff* N *satisfies the condition*

(con) $B \in \mathcal{B}^X$ *implies* $\bigcup\{\rho \subset \underline{P}X : \rho \in N(B)\} \in N(B)$.

Example 2.3 According to Lemma 1.6 we state that the esd-space $(X, \mathcal{B}^X, N_t)$ is conic.

Remark 2.4 Here, we note that the esd-space $(X, \mathcal{B}^X, N_\delta)$ is conic, too. But in *general* this property must not be *necessary* fulfilled, if, par example we look at the near subdensity spaces considered in [19].

Lemma 2.5 *For a conic esd-space* $(Y, \mathcal{B}^Y, M)$ *we put* $B\gamma_M D$ *iff* $\{D\} \in M(B)$ *for sets* $B, D \in \mathcal{B}^Y$*. Then* $(Y, \mathcal{B}^Y, \gamma_M)$ *is a b-proximal space such that* $N_{\gamma_M} = M$*.*

Proof: Straight forward. Here, we only will verify the validity of axiom (b-p6) in definition 1.8.

to (b-p6): $B_1 \delta D$ and $D \subset cl_{\gamma_M}(B)$, $B \in \mathcal{B}^Y$ imply $\{D\} \in M(B_1)$, hence $\{cl_M(B)\} << \{cl_{\gamma_M}(B)\} << \{D\}$ follows, and $\{cl_M(B)\} \in M(B_1)$ is valid. We get $\{B\} \in M(B_1)$, according to (esd_6) which results in $B_1 \gamma_M B$. It remains to prove the equality $N_{\gamma_M} = M$. Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\rho \in N_{\gamma_M}(B)$, hence $\rho \cap \mathcal{B}^X \subset \gamma_M(B)$. Now, we will show that $\gamma_M(B) \subset \bigcup\{\sigma : \sigma \in M(B)\}$ holds. $D \in \gamma_M(B)$ implies $B \gamma_M D$, hence $\{D\} \in M(B)$ is valid with $D \in \{D\}$, and $D \in \bigcup\{\sigma : \sigma \in M(B)\}$ follows. Consequently, $\rho \cap \mathcal{B}^X \in M(B)$ can be deduced by applying (esd_1), resulting into $\rho \in M(B)$ according to (esd_9). The reverse case is easily to verify.

□

Theorem 2.6 *The full subcategory CON-ESD of ESD, whose objects are the conic esd-spaces is isomorphic to the category b-PX.*

Proof: Taking into account former results we further note that for a given b-proximal space $(X, \mathcal{B}^X, \delta)$ the equality $\gamma_{N_\delta} = \delta$ is valid. Moreover, we claim that for each b-proximal map f between b-proximal spaces f is bibsd-map between the corresponding esd-spaces and vice versa. □

Definition 2.7 *A conic esd- space $(X, \mathcal{B}^X, N)$ is called* proximal *iff N satisfies the condition*

(px) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ *and* $\rho \in N(B)$ *imply* $\{B\} \in \bigcap\{N(F) : F \in \rho \cap \mathcal{B}^X\}$.

Remark 2.8 Here, we note that for a given symmetric b-proximal space $(X, \mathcal{B}^X, \delta)$ the corresponding esd-space $(X, \mathcal{B}^X, N_\delta)$ is proximal. Because for $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\rho \in N_\delta(B)$ we have $\rho \cap \mathcal{B}^X \subset \delta(B)$. Then, $F \in \rho \cap \mathcal{B}^X$ implies $\{B\} \in N_\delta(F)$. Since by hypothesis $B \delta F$ is valid $F \delta B$ results, because δ is symmetric.

Corollary 2.9 *The full subcategory **PX-ESD** of **CON-ESD**, whose objects are the proximal esd-spaces is isomorphic to the category **b-SPX**.*

Proof: Here, we only note that for a given proximal esd-space the corresponding b-proximal space is symmetric. □

Proposition 2.10 *Every proximal esd-space $(X, \mathcal{B}^X, N)$ is* closed *by satisfying*

(clo) $B \in \mathcal{B}^X$ *implies* $N(cl_N(B)) = N(B)$.

Proof: Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\rho \in N(cl_N(B))$, we will show that $\rho \cap \mathcal{B}^X \subset \cup\{\sigma : \sigma \in N(B)\}$ is valid. $F \in \rho \cap \mathcal{B}^X$ implies $\{cl_N(B)\} \in N(F)$, since $(X, \mathcal{B}^X, N)$ is proximal. Then $\{B\} \in N(F)$ follows by applying (esd$_6$), and $\{F\} \in N(B)$ results with respect to (px). Consequently, $F \in \cup\{\sigma : \sigma \in N(B)\}$ is valid, showing that $\rho \cap \mathcal{B}^X \in N(B)$, according to (esd$_1$). But this induce $\rho \in N(B)$ by applying (esd$_9$). The reverse inclusion then can be easily verified with respect to (esd$_5$). □

Proposition 2.11 *Every proximal esd-space* $(X, \mathcal{B}^X, N)$ *is* linked *by satisfying*

(lik) $\rho \in N(B_1 \cup B_2), B_1, B_2 \in \mathcal{B}^X$ *imply* $\{F\} \in N(B_1) \cup N(B_2) \forall F \in \rho \cap \mathcal{B}^X$.

Proof: evident. □

Definition 2.12 *A conic esd-space* $(X, \mathcal{B}^X, N)$ *is called* covered *iff* N *satisfies the condition*

(cov) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ *and* $\rho \in N(B)$ *imply* $B \in \sec\{cl_N(F) : F \in \rho \cap \mathcal{B}^X\}$.

Example 2.13 With respect to example 2.3 we note that $(X, \mathcal{B}^X, N_t)$ is a covered esd-space.

Lemma 2.14 *For a covered esd-space* $(X, \mathcal{B}^X, N)$ *the restriction of* cl_M *on* $\mathcal{B}^X$*, denoted by* cl^b_M *is a b-topology on* $\mathcal{B}^X$ *such that* $N_{cl^b_M} = M$.

Proof: Firstly, we only will verify the validity of the axioms (b-t$_5$) and (b-t$_6$), respectively in definition 1.3. Then, the remaining is clear.

to (b-t$_5$): $x \in cl^b_M(cl^b_M(B)), B \in \mathcal{B}^X$ imply $\{cl^b_M(B)\} \in M(\{x\})$, hence $\{cl_M(B)\} \in M(\{x\})$ is valid, and $\{B\} \in M(\{x\})$ results, according to (esd$_6$). But then $x \in cl_M(B) = cl^b_M(B)$ follows.

to (b-t$_6$): $B_1, B_2 \in \mathcal{B}^X$ and without restriction let $B_1 \neq \emptyset \neq B_2 \cdot x \in cl^b_M(B_1 \cup B_2)$ implies $\{B_1 \cup B_2\} \in M(\{x\})$, by paying attention to the fact that $\mathcal{B}^X$ is bornology. Since $\{B_1\} \vee \{B_2\} = \{B_1 \cup B_2\}$, we get $\{B_1\} \in M(\{x\})$ or $\{B_2\} \in M(\{x\})$ by applying (esd$_7$), resulting into $x \in cl^b_M(B_1) \cup cl^b_M(B_2)$. In showing the equality $N_{cl^b_M} = M$ let without restriction $B \in \mathcal{B}^X \backslash \{\emptyset\}$. $\rho \in N_{cl^b_M}(B)$ implies $B \in \sec\{cl^b_M(F) : F \in \rho \cap \mathcal{B}^X\}$, which is the same as $B \in \sec\{cl_M(F) : F \in \rho \cap \mathcal{B}^X\}$. Since $(X, \mathcal{B}^X, M)$ is conic, we know that $\bigcup\{\sigma : \sigma \in M(B)\} \in M(B)$. Thus, it remains to verify $\rho \cap \mathcal{B}^X \subset \cup\{\sigma : \sigma \in M(B)\}$, because then $\rho \cap \mathcal{B}^X \in M(B)$ follows, according to (esd$_1$), and $\rho \in M(B)$ is valid by applying (esd$_9$). $F \in \rho \cap \mathcal{B}^X$ implies $B \cap cl_M(F) \neq \emptyset$, hence $x \in cl_M(F)$ for some $x \in B$. Consequently, $\{F\} \in M(\{x\}) \subset M(B)$ follows, showing that $F \in \bigcup\{\sigma : \sigma \in M(B)\}$, which put an end of this. Then, the reverse inclusion is easily to verify.

□

Theorem 2.15 *The full subcategory* ***COV-ESD*** *of* ***CON-ESD****, whose objects are the covered esd-spaces is isomorphic to the category* ***b-TOP****.*

Proof: Taking into account former results we further note that for each b-continuous map f between b-topological spaces f is bibsd-map between the corresponding esd-spaces and vice versa. $\square$

Theorem 2.16 *The category* ***CON-ESD*** *is bireflective in* ***ESD****.*

Proof: For an esd-space $(X, \mathcal{B}^X, N)$ we set: $N^C(\emptyset) := \{\emptyset\}$ and $N^C(B) := \{\mathcal{A} \subset \underline{P}X : \{cl_N(A) : A \in \mathcal{A} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}\}$, otherwise. Then $(X, \mathcal{B}^X, N^C)$ is conic esd-space, and $\underline{1}_X : (X, \mathcal{B}^X, N) \longrightarrow (X, \mathcal{B}^X, N^C)$ is bibsd-map. In the following we only will verify the validity of the axioms (esd_6), (esd_7) in definition 1.1 and that of axiom (con) in definition 2.2. Then the remaining statements are obvious.

to (esd_6): $\{cl_{N^C}(A) : A \in \mathcal{A}\} \in N^C(B)$, $B \in \mathcal{B}^X \backslash \{\emptyset\}, \mathcal{A} \subset \underline{P}X$ imply $\{cl_N(F) : F \in \{cl_{N^C}(A) : A \in \mathcal{A}\} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$. We will show that $\{cl_N(A) : A \in \mathcal{A} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$. $A \in \mathcal{A} \cap \mathcal{B}^X$ implies $cl_N(cl_{N^C}(A)) \in \bigcup\{\rho : \rho \in N(B)\}$, since $cl_{N^C}(A) \in \mathcal{B}^X$. Further we have the inclusion $cl_{N^C}(A) \subset cl_N(A)$ is valid: $x \in cl_{N^C}(A)$ implies $\{A\} \in N^C(\{x\})$, hence $cl_N(A) \in \rho$ for some $\rho \in N(\{x\})$. $\{cl_N(A)\} \in N(\{x\})$ holds by applying (esd_1), and $\{A\} \in N(\{x\})$ results according to (esd_6), hence $x \in cl_N(A)$ follows. By hypothesis $cl_N(cl_{N^C}(A)) \in \sigma$ for some $\sigma \in N(B)$, and $\{cl_N(A)\} \in N(B)$ follows by applying (esd_6), again. Consequently our assertion holds.

to (esd_7): Let $\mathcal{A}_1 \vee \mathcal{A}_2 \in N^C(B)$ and without restriction $B \in \mathcal{B}^X \backslash \{\emptyset\}$ with $\mathcal{A}_1 \neq \emptyset \neq \mathcal{A}_2$. Then $\{cl_N(A) : A \in (\mathcal{A}_1 \vee \mathcal{A}_2) \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$ follows. If supposing $\mathcal{A}_1, \mathcal{A}_2 \notin N^C(B)$ we can choose $A_1 \in \mathcal{A}_1 \cap \mathcal{B}^X$ with $cl_N(A_1) \notin \bigcup\{\rho : \rho \in N(B)\}$ and $A_2 \in \mathcal{A}_2 \cap \mathcal{B}^X$ with $cl_N(A_2) \notin \bigcup\{\rho : \rho \in N(B)\}$. Consequently, $A_1 \cup A_2 \in (\mathcal{A}_1 \vee \mathcal{A}_2) \cap \mathcal{B}^X$ follows, since $\mathcal{B}^X$ is bornology. By hypothesis $cl_N(A_1 \cup A_2) \in \mathcal{A}$ for some $\mathcal{A} \in N(B)$, hence $\{cl_N(A_1 \cup A_2)\} \in N(B)$ is valid. But $\{cl_N(A_1)\} \vee \{cl_N(A_2)\} = \{cl_N(A_1 \cup A_2)\}$ is holding, and consequently $\{cl_N(A_1)\} \in N(B)$ or $\{cl_N(A_2)\} \in N(B)$ follows by applying (esd_7) which contradicts.

to (con): Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$. We have to verify $\bigcup\{\mathcal{A} : \mathcal{A} \in N^C(B)\} \in N^C(B)$, which means that $\{cl_N(F) : F \in \bigcup\{\mathcal{A} : \mathcal{A} \in N^C(B)\} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$. Now, let $cl_N(F)$ be given for $F \in \bigcup\{\mathcal{A} : \mathcal{A} \in N^C(B)\} \cap \mathcal{B}^X$ hence $F \in \mathcal{A}$ for some $\mathcal{A} \in N^C(B)$. By hypothesis there exists $\rho \in N(B)$ with $cl_N(F) \in \rho'$, and $cl_N(F) \in \bigcup\{\rho : \rho \in N(B)\}$ results. Now, let $(Y, \mathcal{B}^Y, M)$ be a conic esd-space and $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$ be a bibsd-map, we have to show $f:(X,\mathcal{B}^X,N^C)\longrightarrow(Y,\mathcal{B}^Y,M)$ is bibsd-map, too. Since by hypothesis f is bi-bounded, we will now verify the validity of axiom (sd) in definition 1.1.

to (sd): Without restriction let $B\in\mathcal{B}^X\backslash\{\emptyset\}$ and $\mathcal{A}\in N^C(B)$, hence by definition $\{cl_N(A):A\in\mathcal{A}\cap\mathcal{B}^X\}\subset\bigcup\{\rho:\rho\in N(B)\}$ is valid. It suffices to show $f\mathcal{A}\cap\mathcal{B}^Y\in M(f[B])$. Therefore its being enough to verify the validity of the inclusion $f\mathcal{A}\cap\mathcal{B}^Y\subset\bigcup\{\mathcal{M}:\mathcal{M}\in M(f[B])\}$. $D\in f\mathcal{A}\cap\mathcal{B}^Y$ implies $D=f[A]$ for some $A\in\mathcal{A}$. Then $A\subset f^{-1}[f[A]]=f^{-1}[D]\in\mathcal{B}^X$, and $A\in\mathcal{B}^X$ follows. Hence $cl_N(A)\in\rho$ for some $\rho\in N(B)$ by hypothesis. Consequently, $f\rho\in M(f[B])$ follows with $f[cl_N(A)]\in f\rho$. Since $cl_M(f[A])\supset f[cl_N(A)]$ we get $\{cl_M(f[A])\}\in M(f[B])$, and $\{D\}=\{f[A]\}\in M(f[B])$ results, according to (esd_6). But then $f\mathcal{A}\cap\mathcal{B}^Y\in M(f[A])$ is valid, since by hypothesis $(Y,\mathcal{B}^Y,M)$ is conic, and at last $f\mathcal{A}\in M(f[B])$ can be deduced by applying (esd_9).

□

Theorem 2.17 *The category **COV-ESD** is bicoreflective in **CON-ESD**.*

Proof: For a conic esd-space $(X,\mathcal{B}^X,N)$ we set: $N^{CV}(\emptyset):=\{\emptyset\}$ and $N^{CV}(B):=\{\rho\subset\underline{P}X: B\in\sec\{cl_N(F):F\in\rho\cap\mathcal{B}^X\}\}$, otherwise. Then $(X,\mathcal{B}^X,N^{CV})$ is a covered esd-space, and $\underline{1}_X:(X,\mathcal{B}^X,N^{CV})\longrightarrow(X,\mathcal{B}^X,N)$ is bibsd-map. It is straight forward to verify that $(X,\mathcal{B}^X,N^{CV})$ is a covered esd-space. In showing that $\underline{1}_X$ is bibsd-map let $\rho\in N^{CV}(B)$ and without restriction $B\in\mathcal{B}^X\backslash\{\emptyset\}$. Consequently, $B\in\sec\{cl_N(F):F\in\rho\cap\mathcal{B}^X\}$ holds by definition of N^{CV}. Now, we will verify that $\rho\cap\mathcal{B}^X$ is a subset of $\bigcup\{\mathcal{A}:\mathcal{A}\in N(B)\}$. $F\in\rho\cap\mathcal{B}^X$ implies the existence of an element $x\in B$ with $x\in cl_N(F)$. Hence $\{F\}\in N(\{x\})\subset N(B)$ follows, showing that $F\in\bigcup\{\mathcal{A}:\mathcal{A}\in N(B)\}$ is valid. Now, let $(Y,\mathcal{B}^Y,M)$ be a covered esd-space and $f:(Y,\mathcal{B}^Y,M)\longrightarrow(X,\mathcal{B}^X,N)$ be a bibsd-map, we have to show $f:(Y,\mathcal{B}^Y,M)\longrightarrow(X,\mathcal{B}^X,N^{CV})$ is bibsd-map, too. Since by hypothesis f is bi-bounded we will verify the validity of axiom (sd) in definition 1.1. Without restriction let $B\in\mathcal{B}^Y\backslash\{\emptyset\}$ and $\rho\in M(B)$, hence $B\in\sec\{cl_M(F):F\in\rho\cap\mathcal{B}^Y\}$. For $A\in f\rho\cap\mathcal{B}^X$ we have $A=f[F]$ for some $F\in\rho$ with $f^{-1}[A]\in\mathcal{B}^Y$, since f is bi-bounded. Consequently, $F\in\mathcal{B}^Y$ is valid, and we can choose $y\in cl_M(F)$ for some $y\in B$ by hypothesis. But f also satisfies (sd) in definition 1.1, hence $f(y)\in cl_N(A)\cap f[B]$ results, concluding the proof. □

3 Topological extensions and related esd-spaces

We will now consider the problem for finding a *one-to-one correspondence* between certain topological extensions and their related esd-spaces. This question arises from a problem formulated by LODATO in 1966 as follows:

He asked for an axiomatization of the following binary nearness relation on the power set of a set X: there exists an embedding of X into a topological space Y such that subsets A and B are *near* in X iff their closures meet in Y.

Now, we will generalize and solve this problem for esd-spaces, *involving* also LODATO's result as a special case. At first, we define the category ***BTEXT*** of so-called *bornotopological extensions* – shortly btop-extensions – and related morphisms (see also [19]).

Definition 3.1 *Objects of* ***BTEXT*** *are triples* $E:=(e, \mathcal{B}^X, Y)$*, where* $X:=(X, t_X)$*,* $Y:=(Y, t_Y)$ *are topological spaces (given by closure operators* t_X *respectively* t_Y*) with bornology* $\mathcal{B}^X$*, so that iff* $B \in \mathcal{B}^X$ *then* $t_X(B) \in \mathcal{B}^X$ *also holds.*

$e: X \longrightarrow Y$ *is a function satisfying the following conditions:*

(btx$_1$) $B \in \mathcal{B}^X$ *implies* $t_X(B) = e^{-1}[t_y(e[B])]$*, where* e^{-1} *denotes the* inverse *image under* e*;*

(btx$_1$) $t_Y(e[X]) = Y$*, which means that the image of* X *under* e *is* dense *in* Y*.*

Morphisms *in* ***BTEXT*** *have the form* $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$*, where* $f : X \longrightarrow X'$ $g : Y \longrightarrow Y'$ *are continuous maps such that* f *is bi-bounded, and the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{e'} & Y' \end{array} .$$

If $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ *and* $(f', g') : (e', \mathcal{B}^{X'}, Y') \longrightarrow (e'', \mathcal{B}^{X''}, Y'')$ *are* ***BTEXT****-morphisms, then they can be* composed *according to the rule* $(f', g') \circ (f, g) := (f' \circ f, g' \circ g) : (e, \mathcal{B}^X, Y) \longrightarrow (e'', \mathcal{B}^{X''}, Y'')$*, where "*$\circ$*" denotes the* composition *of maps.*

Remark 3.2 Observe, that axiom (btx$_1$) in this definition is *automatically* satisfied if $e : X \longrightarrow Y$ is a *topological embedding.* Moreover, we admit an *ordinary* bornology $\mathcal{B}^X$, which need *not* be necessary *coincide* with the power set $\underline{P}X$.

Definition 3.3 *We call such an extension* $E:=(e, \mathcal{B}^X, Y)$

(i) strict *iff* E *satisfies the condition*

(st) $\{t_Y(e[A]) : A \subset X\}$ *forms a base for the* closed *subsets of* Y *[1];*

(ii) symmetric *iff* E *satisfies the condition*

(sy) $x \in X$ *and* $y \in t_Y(\{e(x)\})$ *imply* $e(x) \in t_Y(\{y\})$ *[3].*

Example 3.4 For a symmetric bornotopological extension $E:=(e, \mathcal{B}^X, Y)$ we consider the triple $(X, \mathcal{B}^X, N^e)$, where N^e is defined by setting:

$N^e(\emptyset) := \{\emptyset\}$ and
$N^e(B) := \{\rho \subset \underline{P}X : t_Y(e[B]) \in \sec\{t_Y(e[F]) : F \in \rho \cap \mathcal{B}^X\}\}$, otherwise.

Then $(X, \mathcal{B}^X, N^e)$ is a proximal esd-space such that for each $B \in \mathcal{B}^X$ $cl_{N^e}(B) = t_X(B)$.

Proof: Firstly, we will verify the above cited equality. Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$.

to "$\subset$": $x \in cl_{N^e}(B)$ implies $\{B\} \in N^e(\{x\})$, hence $t_Y(\{e(x)\}) \cap t_Y(e[B]) \neq \emptyset$. Then we can choose $y \in t_Y(e[B])$ with $y \in t_Y(\{e(x)\})$. Since by hypothesis E is symmetric, we get $e(x) \in t_Y(\{y\})$. But then $e(x) \in t_Y(e[B])$ is valid, because t is topological. Consequently, $x \in t_X(B)$ follows by applying (btx_1) in definition 3.1.

to "$\supset$": $x \in t_X(B)$ implies $e(x) \in t_Y(e[B])$ according to (btx_1), hence $\{B\} \in N^e(\{x\})$ follows, resulting into $x \in cl_{N^e}(B)$. Further, we only will verify the validity of the axioms (esd_6) and (esd_7), respectively. Then the remaining statements are clear.

to (esd_6): $\{cl_{N^e}(F) : F \in \rho\} \in N^e(B), B \in \mathcal{B}^X \backslash \{\emptyset\}, \rho \subset \underline{P}X$ imply $t_Y(e[B]) \in \sec\{t_Y(e[A]) : A \in \{cl_{N^e}(F) : F \in \rho\} \cap \mathcal{B}^X\}$. Then $F' \in \rho \cap \mathcal{B}^X$ implies $cl_{N^e}(F') \in \{cl_{N^e}(F) : F \in \rho\} \cap \mathcal{B}^X$, because $cl_{N^e}(F') = t_X(F') \in \mathcal{B}^X$ by definition 3.1. By hypothesis $t_Y(e[B]) \cap t_Y(e[t_X(F')]) \neq \emptyset$ follows. But $e[t_X(F')] \subset t_Y(e[F'])$ holds by applying (btx_1), and $t_Y(e[t_X(F')]) \subset t_Y(e[F'])$ can be deduced, since t_Y is topological, resulting into $\rho \in N^e(B)$.

to (esd_7): Let $\rho_1 \vee \rho_2 \in N^e(B)$ and without restriction $\rho_1 \neq \emptyset \neq \rho_2, B \neq \emptyset$. By definition we get $t_Y(e[B]) \in \sec\{t_Y(e[F]) : F \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X\}$. If supposing $\rho_1, \rho_2 \notin N^e(B)$. Then we can choose $F_1 \in \rho_1 \cap \mathcal{B}^X$ with $t_Y(e[B]) \cap t_Y(e[F_1]) = \emptyset$ and $F_2 \in \rho_2 \cap \mathcal{B}^X$ with $t_Y(e[B]) \cap t_Y(e[F_2]) = \emptyset$. Hence $F_1 \cup F_2 \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X$, since $\mathcal{B}^X$ is bornology. Consequently, $t_Y(e[B]) \cap t_Y(e[F_1 \cup F_2]) \neq \emptyset$ results. On the other hand we have $\emptyset = t_Y(e[B]) \cap (t_Y(e[F_1]) \cup t_Y(e[F_2])) = t_Y(e[B]) \cap t_Y(e[F_1] \cup e[F_2]) = t_Y(e[B]) \cap t_Y(e[F_1 \cup F_2])$, which contradicts.

□

Definition 3.5 *For a proximal esd-space $(X, \mathcal{B}^X, N)$ and for $B \in \mathcal{B}^X \sigma \subset \underline{P}X$ is called* B-bunch *in N iff σ satisfies the following conditions:*

(bun_1) $\emptyset \notin \sigma$;

(bun_2) $F_1 \cup F_2 \in \sigma$ *iff* $F_1 \in \sigma$ *or* $F_2 \in \sigma$;

(bun_3) $B \in \sigma \in N(B)$;

(bun_4) $A \in \sigma$ *and* $A \subset cl_N(F) : F \in \mathcal{B}^X$ *imply* $F \in \sigma$;

(bun_5) $A \in \sigma \cap \mathcal{B}^X$ *implies* $\{A\} \in \bigcap\{N(F) : F \in \sigma \cap \mathcal{B}^X\}$.

Proposition 3.6 *For a proximal esd-space $(X, \mathcal{B}^X, N)$ and for $B \in \mathcal{B}^X \setminus \{\emptyset\}$ with $x \in B$ $x_N := \{A \subset X : x \in cl_N(A)\}$ is a B-bunch in N. Moreover, x_N is maximal element in $N(\{x\}) \setminus \{\emptyset\}$, ordered by inclusion.*

Proof: Evidently, x_N is satisfying (bun$_1$) and (bun$_2$). $B \in x_N$, since $\{B\} << \{\{x\}\} \in N(\{x\}) \subset N(B)$ and (esd$_6$) are holding.

to (bun$_4$): $A \in x_N$ and $A \subset cl_N(F), F \in \mathcal{B}^X$ imply $x \in cl_N(A)$, hence $x \in cl_N(F)$ follows, showing that $F \in x_N$ is valid.

to (bun$_5$): $A \in x_N \cap \mathcal{B}^X$ and $F \in x_N \cap \mathcal{B}^X$ imply $\{A\} \in N(\{x\}) \subset N(cl_N(F)) = N(F)$, according to proposition 2.10.

Now, let $\sigma \in N(\{x\}) \setminus \{\emptyset\}$ with $x_N \subset \sigma$. For $F \in \sigma$ we have $\{F\} \in N(\{x\})$, and $x \in cl_N(F)$ follows, showing that $\sigma = x_N$ holds. □

Definition 3.7 *A proximal esd-space $(X, \mathcal{B}^X, N)$ is called a* bunch space *iff N satisfies the condition*

(bun) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ *and* $\rho \in N(B)$ *imply* $\forall F \in \rho \cap \mathcal{B}^X \exists$ *B-bunch* σ *in* N *with* $F \in \sigma$.

Proposition 3.8 *The esd-space $(X, \mathcal{B}^X, N^e)$ is a bunch space.*

Proof: For $B \in \mathcal{B}^X \setminus \{\emptyset\}, \rho \in N^e(B)$ let $F \in \rho \cap \mathcal{B}^X$, hence by definition $t_Y(e[B]) \cap t_Y(e[F]) \neq \emptyset$ holds, so that we can choose $y_F \in t_Y(e[B]) \cap t_Y(e[F])$. Now, we put $t(y_F) := \{A \subset X : y_F \in t_Y(e[A])\}$, hence $F \in t(y_F) \cdot t(y_F)$ is a B-bunch in N^e, since $\emptyset \notin t(y_F)$, and for $A_1 \cup A_2 \in t(y_F)$ we have $y_F \in t_Y(A_1 \cup A_2) = t_Y(A_1) \cup t_Y(A_2)$, showing that $A_1 \in t(y_F)$ or $A_2 \in t(y_F)$ is valid. If $A_1 \in t(y_F)$ and $A_1 \subset A_2 \subset X$, then $y_F \in t_y(e[A_1])$ is valid with $t_Y(e[A_1]) \subset t_Y(e[A_2])$, and consequently $y_F \in t_Y(e[A_2])$ follows, resulting into $A_2 \in t(y_F)$. By definition $B \in t(y_F)$ holds, and $t(y_F) \in N^e(B)$, because for $A \in t(y_F) \cap \mathcal{B}^X$ we have $y_F \in t_Y(e[A]) \cap t_Y(e[B])$. Now, let $A \in t(y_F)$ and $A \subset cl_{N^e}(F), F \in \mathcal{B}^X$, hence $y_F \in t_y(e[A]) \subset t_Y(e[cl_{N^e}(F)]) = t_Y(e[t_X(F)]) \subset t_Y(e[F])$ follows by applying (btx$_1$). Consequently, $F \in t(y_F)$ results. At last let $A \in t(y_F) \cap \mathcal{B}^X$ and $F \in t(y_F) \cap \mathcal{B}^X$, then $\{A\} \in N^e(F)$ follows, because $y_F \in t_Y(e[A]) \cap t_Y(e[F])$ is valid. The above arguments are showing that $(X, \mathcal{B}^X, N^e)$ is bunch space. □

Convention 3.9 By ***SYBTEXT*** we denote the full subcategory of ***BTEXT***, whose objects are the symmetric btop-extensions and by ***BUN*** the full subcategory of ***PX-ESD*** whose objects are the bunch spaces.

Theorem 3.10 *Let H : **SYBTEXT** $\longrightarrow$ **BUN** be defined by*

(a) *for a **SYBTEXT**-object $E := (e, \mathcal{B}^X, Y)$ we put $H(E) := (X, \mathcal{B}^X, N^e)$;*

(b) *for a* $\boldsymbol{BTEXT}$*-morphism* $(f,g) : E \longrightarrow E'$ *we put* $H(f,g) := f$.

Then $H : \boldsymbol{SYBTEXT} \longrightarrow \boldsymbol{BUN}$ *is a functor.*

Proof: We already know that the image of H lies in $\boldsymbol{BUN}$. Now, let $(f,g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ be a $\boldsymbol{BTEXT}$-morphism: it has to be shown that f is bibsd-map.

By hypothesis f is bi-bounded. Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\rho \in N^e(B)$, we have to verify that $f\rho \in N^{e'}(f[B])$. For showing this statement let $A \in f\rho \cap \mathcal{B}^{X'}$, then we claim $t_{Y'}(e'[f[B]]) \cap t_{Y'}(e'[A]) \neq \emptyset$, which would prove our assertion. We have $A \in \mathcal{B}^{X'}$ with $A = f[F]$ for some $F \in \rho$. By hypothesis we get $t_Y(e[B]) \cap t_Y(e[F]) \neq \emptyset$. Note, that F is also an element of $\mathcal{B}^X$, since $F \subset f^{-1}[f[F]] = f^{-1}[A] \in \mathcal{B}^X$ is valid, and f is bi-bounded. Now, we can choose an element $y \in t_Y(e[B]) \cap t_Y(e[F])$. Consequently, $g(y) \in g[t_Y(e[B])] \cap g[t_Y(e[F])]$ follows.

But the proposed diagram (see 3.1) commutes so that $t_{Y'}(g[e[B]]) = t_{Y'}(e'[f[B]])$ and $t_{Y'}(e'[A]) = t_{Y'}(g[e[F]] = t_{Y'}(e'[f[F]])$ are valid, which put an end of this. Evidently, H fulfills the remaining properties for being a functor. □

4 Strict bornotopological extensions

In the previous section we have found a functor H from $\boldsymbol{SYBTEXT}$ to $\boldsymbol{BUN}$. Now, we are going to introduce a related one in the *opposite* direction.

Lemma 4.1 *Let* $(X, \mathcal{B}^X, N)$ *be a proximal esd-space. We set:* $X^b := \{\sigma \subset \underline{P}X : \sigma$ *is* B*-bunch in* N *for some* $B \in \mathcal{B}^X \backslash \{\emptyset\}\}$*, and for each* $A^b \subset X^b$ *we put:* $t_{X^b}(A^b) := \{\sigma \in X^b : \triangle A^b \subset \sigma\}$*, where* $\triangle A^b := \{F \in \mathcal{B}^X : \forall \sigma \in A^b F \in \sigma\}$*. (By* convention $\triangle A^b = \mathcal{B}^X$ *if* $A^b = \emptyset$*). Then* $t_{X^b} : \underline{P}X^b \longrightarrow \underline{P}X^b$ *is a topological closure operator.*

Proof: Firstly, we note that $t_{X^b}(\emptyset) = \emptyset$, since $\emptyset \notin \sigma$ for each $\sigma \in X^b$. Now, let A^b be a subset of X^b and consider $\sigma \in A^b$. Then $F \in \triangle A^b$ implies $F \in \sigma$, hence $A^b \subset t_{X^b}(A^b)$ is valid. If$A_1^b \subset A_2^b$, then $\triangle A_2^b \subset \triangle_1^b$ implying $t_{X^b}(A_1^b) \subset t_{X^b}(A_2^b)$. For arbitrary subsets $A_1^b, A_2^b \subset X^b$ we consider an element $\sigma \in X^b$ such that $\sigma \notin t_{X^b}(A_1^b) \cup t_{X^b}(A_2^b)$. Then we get $\triangle A_1^b \not\subset \sigma$ and $\triangle A_2^b \not\subset \sigma$. We can choose $F_1 \in \triangle A_1^b$ with $F_1 \notin \sigma$ and $F_2 \in \triangle A_2^b$ with $F_2 \notin \sigma$. By (bun_2) we get $F_1 \cup F_2 \notin \sigma$. On the other hand $F_1 \cup F_2 \in \mathcal{B}^X$, since $\mathcal{B}^X$ is bornology, and $F_1 \cup F_2 \in \triangle A_1^b \cap \triangle A_2^b \subset \triangle(A_1^b \cup A_2^b)$ imply $\sigma \notin t_{X^b}(A_1^b \cup A_2^b)$. At last, let σ be an element of $t_{X^b}(t_{X^b}(A^b))$, $A^b \subset X^b$, and suppose $\sigma \notin t_{X^b}(A^b)$. We can choose $F \in \triangle A^b$, with $F \notin \sigma$. By assumption we have $\triangle t_{X^b}(A^b) \subset \sigma$, hence $F \notin \triangle t_{X^b}(A^b)$. Consequently, there exists $\sigma_1 \in t_{X^b}(A^b)$ with $F \notin \sigma_1$. But this implies $\triangle A^b \subset \sigma_1$, and $F \in \sigma_1$ results, which contradicts. □

Theorem 4.2 *For proximal esd-spaces* $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ *let* $f : X \longrightarrow Y$ *be a bibsd-map. Define a function* $f^b : X^b \longrightarrow Y^b$ *by setting for each* $\sigma \in X^b : f^b(\sigma) := \{D \subset Y : f^{-1}[cl_M(D)] \in \sigma\}$. *Then the following statements are valid:*

(1) f^b *is a continuous map from* (X^b, t_{X^b}) *to* (Y^b, t_{Y^b});

(2) *the composites* $f^b \circ e_X$ *and* $e_Y \circ f$ *coincide, where* $e_X : X \longrightarrow X^b$ *denotes that function which assigns the* $\{x\}$*-bunch* x_N *to each* $x \in X$.

Proof: First, let σ be a B-bunch in N. We will show that $f^b(\sigma)$ is a $f[B]$-bunch in M. It is easy to verify that $f^b(\sigma)$ satisfies the conditions (bun$_1$) and (bun$_2$), respectively (see 3.4). In order to establish (bun$_3$) we observe that $B \in \sigma \in N(B)$ is valid by hypothesis. Since $cl_M(f[B]) \supset f[B]$ we have $f^{-1}[cl_M(f[B])] \supset f^{-1}[f[B]] \supset B$. Then $f[B] \in f^b(\sigma)$ results by applying (bun$_1$). In showing $f^b(\sigma) \in M(f[B])$, we will verify that $\{cl_M(D) : D \in f^b(\sigma)\} << f\sigma$ (note, that f is satisfying (sd) in definition 1.1). For any $D \in f^b(\sigma)$ we have $f^{-1}[cl_M(D)] \in \sigma$, hence $cl_M(D) \supset f[f^{-1}[cl_M(D)]] \in f\sigma$. By applying (esd$_6$) we obtain the desired result. Now, let $D \in f^b(\sigma)$ and $D \subset cl_M(F), F \in \mathcal{B}^Y$. We have to show that $f^{-1}[cl_M(F)] \in \sigma$. By hypothesis $f^{-1}[cl_M(D)] \in \sigma$ is valid. $f^{-1}[cl_M(F)] \in \mathcal{B}^X$ holds by applying (esd$_8$) and since f is bi-bounded. Consequently, $f^{-1}[cl_M(D)] \subset cl_N(f^{-1}[cl_M(D)]) \subset cl_N(f^{-1}[cl_M(F)])$ follows, leading us to the desired result by applying (bun$_4$) for σ. At last let $D \in f^b(\sigma) \cap \mathcal{B}^Y$. For $F \in f^b(\sigma) \cap \mathcal{B}^Y$ we have to show that $\{D\} \in M(F)$ is valid. Since M is proximal, therefore it suffices to prove $\{F\} \in M(D)$. By hypothesis $f^{-1}[cl_M(D)] \in \sigma \cap \mathcal{B}^X$, note that f is bi-bounded. On the other hand if $F \in f^b(\sigma) \cap \mathcal{B}^Y$ we also have $f^{-1}[cl_M(F)] \in \sigma \cap \mathcal{B}^X$. But σ satisfies (bun$_5$), hence $\{f^{-1}[cl_M(F)]\} \in N(f^{-1}[cl_M(D)])$ is valid. Consequently, $\{cl_M(F)\} \in M(cl_M(D))$ follows, since f satisfies (sd) and by applying (esd$_5$). But then $\{F\} \in M(D)$ results according to (esd$_6$) and proposition 2.10. Taking all these facts into account we conclude that $f^b(\sigma)$ defines a $f[B]$-bunch in M, and thus $f^b(\sigma) \in Y^b$ is valid.

to (1): Let $A^b \subset X^b, \sigma \in t_{X^b}(A^b)$ and suppose $f(\sigma) \notin t_{Y^b}(f^b[A^b])$. Then $\triangle f^b[A^b] \not\subset f^b(\sigma)$, hence $D \notin f^b(\sigma)$ for some $D \in \triangle f^b[A^b]$, which means $f^{-1}[cl_M(D)] \notin \sigma$. But $\triangle A^b \subset \sigma$ implies $f^{-1}[cl_M(D)] \notin \sigma_1$ for some $\sigma_1 \in A^b$. Consequently, $D \notin f^b(\sigma_1)$ results, which contradicts, because $D \in \triangle f^b[A^b]$ is valid.

to (2): Now, let x be an element of X. We will prove the validity of the equation $f^b(e_X(x)) = e_Y(f(x))$. To this end let $D \in e_Y(f(x))$. Then $f(x) \in cl_M(D)$ follows, and $x \in f^{-1}[cl_M(D)]$ is valid. Consequently, $f^{-1}[cl_M(D)] \in x_N = e_X(x)$ holds, and $D \in f^b(e_X(x))$ results, proving the inclusion $e_Y(f(x)) \subset f^b(e_X(x))$. Conversely, we note that $\{cl_M(D) : D \in f^b(e_X(x))\} << fe_X(x) \in M(\{f(x)\})$, since by supposition f satisfies (sd). But $e_Y(f(x))$ is maximal in $M(\{f(x)\}) \setminus \{\emptyset\}$, and thus we obtain the desired result.

□

Theorem 4.3 *We obtain a functor G : **BUN** to **SYBTEXT** by setting:*

(a) $G(X, \mathcal{B}^X, N) := (e_X, \mathcal{B}^X, X^b)$ *for any bunch space* $(X, \mathcal{B}^X, N)$ *with* $X := (X, cl_N)$ *and* $X^b := (X^b, t_{X^b})$*;*

(b) $G(f) := (f, f^b)$ *for any bibsd-map* $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$*.*

Proof: With respect to (esd$_6$), cl_N is topological closure operator, and by Lemma 4.1 this also holds for t_{X^b}. Therefore we get topological spaces with bornology $\mathcal{B}^X$, and $e_X : X \longrightarrow X^b$ is a map according to theorem 4.2. Moreover, e_X is a function satisfying (btx$_1$) and (btx$_2$), respectively.

To establish (btx$_1$) let $B \in \mathcal{B}^X$ and suppose $x \in cl_N(B)$. Then we get $\triangle e_X[B] \subset x_N$, hence $e_X(x) \in t_{X^b}(e_X[B])$, which means $x \in e_X^{-1}[t_{X^b}(e_X[B])$. Conversely, let x be an element of $e_X^{-1}[t_{X^b}(e_X[B])]$. Then by definition we have $\triangle e_X[B] \subset x_N$. Since $B \in \triangle e_X[B]$ we get $x \in cl_N(B)$. To establish (btx$_2$) let $\sigma \in X^b$ and suppose $\sigma \notin t_{X^b}(e_X[X])$. By definition we get $\triangle e_X[X] \not\subset \sigma$, so that there exists a set $F \in \triangle e_X[X]$ with $F \notin \sigma$. But then $X \subset cl_N(F)$ follows. Since $B \in \sigma$ for some $B \in \mathcal{B}^X \setminus \{\emptyset\}$ we get $B \subset cl_N(F)$, hence $F \in \sigma$, because σ is satisfying (bun$_4$). But this contradicts, and $\sigma \in t_{X^b}(e_X[X])$ is valid. Moreover, we have that f and f^b are continuous maps (see also theorem 4.2), and the diagram

$$\begin{array}{ccc} X & \xrightarrow{e_X} & X^b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^b} \\ Y & \xrightarrow{e_Y} & Y^b \end{array} \quad \text{commutes.}$$

Finally, this establishes that the composition of bibsd-maps is preserved by G. In showing $(e_X, \mathcal{B}^X, X^b)$ is symmetric, let x be an element of X such that $\sigma \in t_{X^b}(\{e_X(x)\})$. We have to prove $x_N \in t_{X^b}(\{\sigma\})$. By hypothesis we have $x_N \cap \mathcal{B} \subset \sigma$ and must show that $\triangle\{\sigma\} \subset x_N$. To this end let $F \in \triangle\{\sigma\}$, hence $F \in \sigma \cap \mathcal{B}^X$ follows. We already know that $\{x\} \in \sigma$ is valid, and consequently $\{F\} \in N(\{x\})$ follows by applying (bun$_5$). But this implies $x \in cl_N(F)$, and $F \in x_N$ results. At last we will show that the image of G also is contained in ***ST-SYBTEXT*** the full subcategory of ***SYBTEXT***, whose objects are the *strict* symmetric bornotopological extensions. □

Corollary 4.4 *The image of G is contained in **ST-SYBTEXT**.*

Proof: Consider $\sigma \notin X^b$ and let A^b be closed in X^b with $\sigma \notin A^b$. Then $\sigma \notin t_{X^b}(A^b)$, hence $\triangle A^b \not\subset \sigma$. We can find some $F \in \triangle A^b$ such that $F \notin \sigma$. Now, for each $\sigma_1 \in A^b$ we have $F \in \sigma_1$, which implies $\triangle e_X[F] \subset \sigma_1$, because $D \in \triangle e_X[F]$ implies $F \subset cl_N(D)$ with $D \in \mathcal{B}^X$, and σ_1 satisfies (bun$_4$). Therefore we conclude $\sigma_1 \in t_{X^b}(e_X[F])$, and $A^b \subset t_{X^b}(e_X[F])$

results. On the other hand, since $F \notin \sigma$ we have $\triangle e_X[F] \not\subset \sigma$, hence $\sigma \notin t_{X^b}(e_X[F])$, and $t_{X^b}(e_X[F]) \subset A^b$ results, which put an end of this. □

Theorem 4.5 *Let $H : \boldsymbol{SYBTEXT} \longrightarrow \boldsymbol{BUN}$ and $G : \boldsymbol{BUN} \longrightarrow \boldsymbol{SYBTEXT}$ be the above defined functors. For each object $(X, \mathcal{B}^X, N)$ of $\boldsymbol{BUN}$ let $t_{(\mathcal{B}^X, N)}$ denote the identity map $id_X : H(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$. Then $t : H \circ G \longrightarrow 1_{\boldsymbol{BUN}}$ is* natural equivalence *from $H \circ G$ to the identity functor $1_{\boldsymbol{BUN}}$, i.e. $id_X : H(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$ is bibsd-map in* both *directions for each object $(X, \mathcal{B}^X, N)$, and the following diagram commutes for each bibsd-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$:*

$$\begin{array}{ccc} H(G(X, \mathcal{B}^X, N)) & \xrightarrow{id_X} & (X, \mathcal{B}^X, N) \\ {\scriptstyle H(G(f))}\downarrow & & \downarrow{\scriptstyle f} \\ H(G(Y, \mathcal{B}^Y, M)) & \xrightarrow{id_Y} & (Y, \mathcal{B}^Y, M) \end{array}$$

Proof: The commutativity of the diagram is obvious, because of $H(G(f)) = f$. It remains to prove that $id_X : H(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$ is bibsd-map in *both* directions. Since $H(G(X, \mathcal{B}^X, N)) = (X, \mathcal{B}^X, N^{ex})$ by definition of G respectively H, it suffices to show that for each $B \in \mathcal{B}^X \backslash \{\emptyset\}$ we have $N^{ex}(B) \subset N(B) \subset N^{ex}(B)$. To this end assume $\rho \in N^{ex}(B), B \neq \emptyset$. Then $t_{X^b}(e_X[B]) \in \sec\{t_{X^b}(F) : F \in \rho \cap \mathcal{B}^X\}$. Now, we will show that $\rho \cap \mathcal{B}^X$ is subset of $\bigcup\{\mathcal{A} : \mathcal{A} \in N(B)\}$. Note, that $(X, \mathcal{B}^X, N)$ is conic by assumption. $F \in \rho \cap \mathcal{B}^X$ implies the existence of $\sigma \in t_{X^b}(B) \cap t_{X^b}(F)$, hence $\triangle e_X[B], \triangle e_X[F] \subset \sigma$ are valid. Consequently, $B, F \in \sigma \cap \mathcal{B}^X$ result, and $\{F\} \in N(B)$ follows, since σ satisfies (bun_5). Consequently, $F \in \bigcup\{\mathcal{A} : \mathcal{A} \in N(B)\}$ is valid, showing that $\rho \cap \mathcal{B}^X \in N(B)$. But then $\rho \in N(B)$ follows by applying (esd_9). Conversely, let $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\rho \in N(B)$. We have to verify $t_{X^b}(e_X[B]) \in \sec\{t_{X^b}(F) : F \in \rho \cap \mathcal{B}^X\} \cdot F \in \rho \cap \mathcal{B}^X$ implies the existence of a B-bunch σ in N with $F \in \sigma$, according to (bun). Now, we claim that the following statements are valid, i.e.

(a) $\sigma \in t_{X^b}(e_X[B])$;

(b) $\sigma \in t_{X^b}(e_X[F])$.

to (a): We have to check that the inclusion $\triangle e_X[B] \subset \sigma$ is valid. $A \in \triangle e_X[B]$ implies $B \subset cl_N(A)$. Since $B \in \sigma$ we get $cl_N(A) \in \sigma$, and $A \in \sigma$ results, according to (bun_4). Note, that $A \in \mathcal{B}^X$ by definition.

to (b): We must show that the inclusion $\triangle e_X[F] \subset \sigma$ is valid. But by hypothesis we know that $F \in \sigma$ holds, hence this proving is as above.

□

Corollary 4.6 *For a btop-T_1 extension $E:=(e,\mathcal{B}^X,Y)$, where e is topological embedding and Y T_1-space, then $(X,\mathcal{B}^X,N^e)$ is separated by satisfying*

(sep) $x,z\in X$ *and* $\{\{z\}\}\in N^e(\{x\})$ *imply* $x=z$.

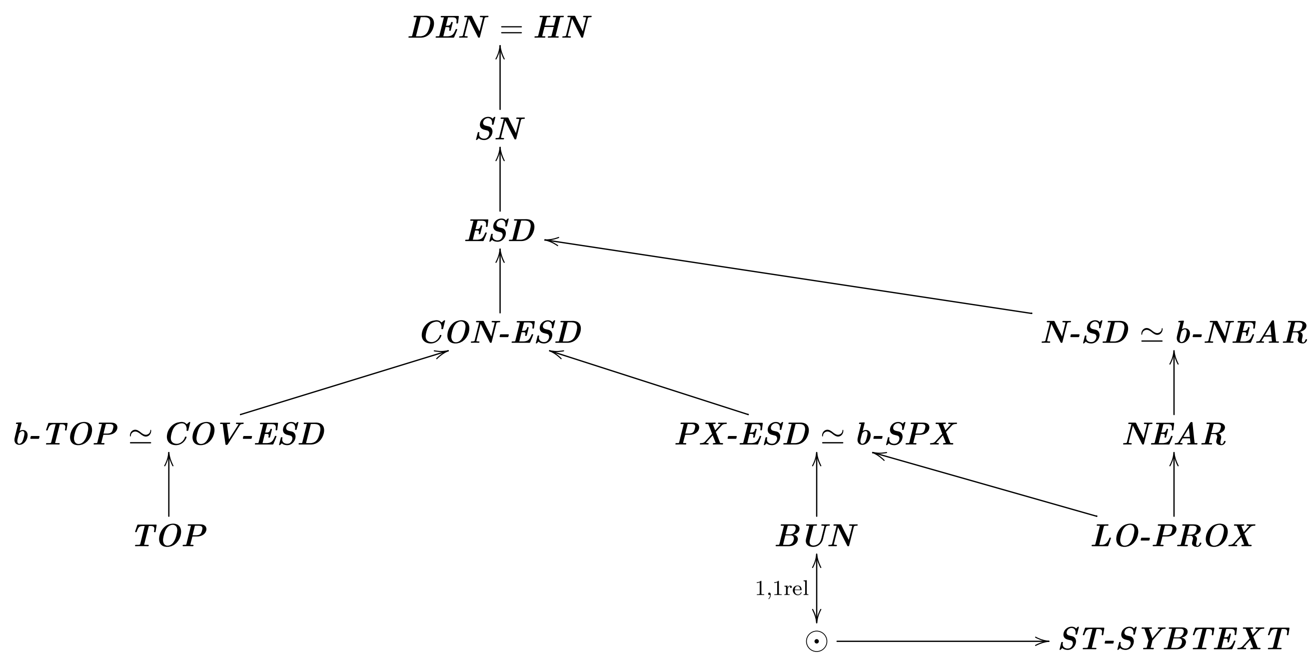
Proof: For $x,z\in X$ with $\{\{z\}\}\in N^e(\{x\})$ there exists $y\in t_Y(\{e(x)\})\cap t_Y(\{e(z)\})$. By hypothesis $e(x)=y=e(z)$ follows, and $x=z$ results, because e is injective. □

Corollary 4.7 *For a separated proximal esd-space $(X,\mathcal{B}^X,N)$ the function $e_X:X\longrightarrow X^b$ is injective.*

Proof: For $x,z\in X$ let $e_X(x)=e_X(z)$, hence $z\in cl_N(\{x\})$, and $\{\{x\}\}\in N(\{z\})$ follows. By hypothesis $x=z$ results. □

Remark 4.8 In making the main theorem of this paper more *transparent* we state that a proximal esd-space $(X,\mathcal{B}^X,N)$ is a bunch space iff it can be considered as subspace of a topological space Y, such that the B-collections in N for non-empty bounded sets B are characterized by the fact that their closures of bounded members in Y meet the closure of B in Y. In case if $\mathcal{B}^X$ is *saturated*, then proximal esd-spaces *essentially* coincide with LODATO proximity spaces up to isomorphism. Hence the main theorem generalizes the one of LODATO, presented by him in the past and where symmetric generalized proximities are playing an important role, especially those arising from a family of bunches on a set X.

Diagram of used categories



legend

$\odot \longrightarrow \odot$ subcategory of	
DEN	$\triangleq$ density spaces
HN	$\triangleq$ hypernear spaces
SN	$\triangleq$ supernear spaces

References

[1] **Banaschewski, B. :** *Extensions of Topological Spaces.* Canadian Math. Bull. 7 (1964), 1 – 23

[2] **Bartsch, R. :** *Compactness properties for some hyperspaces and function spaces.* Shaker Verlag, Aachen (2002)

[3] **Bentley, H. L. :** *Nearness spaces and extension of topological spaces.* In: Studies in Topology, Academic Press, NY (1975), 47 – 66

[4] **Deák, J. :** *Extending a family of screens in a contiguity space.* Topology with applications; Szekszárd, Hungary (1993), 121 – 133

[5] **Dimov, G.**, and **Ivanova, E. :** *On some categories arising in the theory of locally compact extensions.* Math. Pannonica 19/2 (2008), 219 – 239

[6] **Doîtchînov, D. :** *Compactly determined extensions of topological spaces.* SERDICA Bulgarice Math. Pub. 11 (1985), 269 – 286

[7] **Herrlich, H. :** *A concept of nearness.* General Topology and its Appl. 5 (1974), 191 – 212

[8] **Hogbe − Nlend, H. :** *Bornologies and functional analysis.* Amsterdam, North-Holland Pub. Co (1977)

[9] **Hůsek, M. :** *Categorical connections between generalized proximity spaces and compactifications.* Contributions to extension theory of topological structures (Proc. Symp. Berlin 1967) Berlin (1969), 127 – 132

[10] **Ivanova, V. M.**, and **Ivanov, A. :** *Contiguity spaces and bicompact extensions.* Dokl. Akad. Nauk SSSR 127 (1959), 20 – 22

[11] **Kent, D.C.**, and **Min, W. K. :** *Neighbourhood spaces.* IJMMS 32:7 (2002), 387 – 399

[12] **Leseberg, D. :** *Supernearness, a common concept of supertopologies and nearness.* Top. Appl. 123 (2002). 145 – 156

[13] **Leseberg, D. :** *Bounded Topology : a convenient foundation for Topology.* http://www.digibib.tu-bs.de/?docid=00029438, FU Berlin (2009)

[14] **Leseberg, D. :** *Improved nearness research.* Math. Pannonica, preprint

[15] **Leseberg, D. :** *Improved nearness research II.* Rostock. Math. Kolloq. 66 (2011), 87 – 102

[16] **Leseberg, D. :** *Improved nearness research III.* Int. J. of Mathematical Sciences and Applications vol. 1, No2 (May 2011), 1 – 14

[17] **Leseberg, D. :** *Improved nearness research IV. Quaest.* Math. preprint

[18] **Leseberg, D. :** *Bounded proximities and related nearness.* Quaest. Math. 36 (2013), 381 – 387

[19] **Leseberg, D. :** *Extensions in Bounded Topology.* Mathematics for Applications, preprint

[20] **Lodato, M. W. :** *On topologically induced generalized proximity relations II.* Pacific Journal of Math. vol. 17, No 1 (1966), 131 – 135

[21] **Naimpally, S. :** *Proximity approach to problems in Topology and Analysis.* München, Oldenburg (2009)

[22] **Poppe, H. :** *On locally defined topological notions.* Q+A in General Topology 13 (1995), 39 – 53

[23] **Smirnov, Y. M. :** *On the completeness of proximity spaces.* Dokl. Akad. Nauk. SSSR 88 (1953), 761 – 794 (in Russian). MR 15 # 144

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Vietoris hyperspaces as quotients of natural function spaces

ABSTRACT. Hyperspaces form a powerful tool in some branches of mathematics: lots of fractal and other geometric objects can be viewed as fixed points of some functions in suitable hyperspaces - as well as interesting classes of formal languages in theoretical computer sciences, for example (to illustrate the wide scope of this concept). Moreover, there are many connections between hyperspaces and function spaces in topology. Thus results from hyperspaces help to get new results in function spaces and vice versa.

We give here a new description of the Vietoris hyperspace on the family $K(Y)$ of the nonempty compact subsets of a regular topological space Y as quotient of the space $C(\beta D, Y)$, endowed with compact-open topology τ_{co}, where βD is the Stone-Čech-compactification of a discrete space.

1 Preliminary Definitions and Results

For a given set X we denote by $\mathfrak{P}(X)$ the power set of X, by $\mathfrak{P}_0(X)$ the power set without the empty set. By $\mathfrak{F}(X)$ (resp. $\mathfrak{F}_0(X)$) we mean the set of all filters (resp. ultrafilters) on X; if φ is a filter on X, the term $\mathfrak{F}_0(\varphi)$ denotes the set of all ultrafilters on X, which contain φ. For $x \in X$ we denote by $\dot{x} := \{A \subseteq X |\ x \in A\}$ the singleton filter on X, generated by $\{x\}$. With $\mathfrak{S}(X) := \{\dot{x} |\ x \in X\}$ we mean the family of all singleton filters on X.

For families $\mathfrak{A} \subseteq \mathfrak{P}(X)$ and any $M \subseteq X$ we set

$$M^{-\mathfrak{A}} := \{A \in \mathfrak{A} |\ A \cap M \neq \emptyset\}$$

and

$$M^{+\mathfrak{A}} := \{A \in \mathfrak{A} |\ A \cap M = \emptyset\} \ .$$

Then for a topological space (X, τ) on $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ the *lower Vietoris topology* $\tau_{l,\mathfrak{A}}$ is defined by the subbase $\{O^{-\mathfrak{A}} |\ O \in \tau\}$, whereas the *upper Vietoris topology* $\tau_{u,\mathfrak{A}}$ on $\mathfrak{A}$ comes from the

subbase $\{(X \setminus O)^{+\mathfrak{A}} | \, O \in \tau\}$. The *Vietoris topology* on $\mathfrak{A}$ is $\tau_{V,\mathfrak{A}} := \tau_{l,\mathfrak{A}} \vee \tau_{u,\mathfrak{A}}$. In most cases $\mathfrak{A}$ is chosen as the family $Cl(X)$ of the nonempty closed, or $K(X)$ of the nonempty compact subsets of a topological space (X, τ), or as the entire $\mathfrak{P}_0(X)$.
The Vietoris topology on $\mathfrak{A}$ is also generated by the standard *basis* consisting of all sets

$$\langle U_1, ..., U_n \rangle_{\mathfrak{A}} := \left\{ A \in \mathfrak{A} \;\middle|\; A \subseteq \bigcup_{i=1}^{n} U_i \;\wedge\; \forall i = 1, ..., n : A \cap U_i \neq \emptyset \right\}$$

with open $U_1, ..., U_n$.
Whenever there is no doubt about $\mathfrak{A}$, we will omit it as sub- and superscript.

We will need some basic facts about the Stone-Čech-compactification of discrete spaces.

A discrete space (D, δ) clearly is T_4 and Hausdorff, so its Stone-Čech-compactification is homeomorphic to its Wallman extension, consisting in this case just of the set $\mathfrak{F}_0(D)$, where the singleton filters are identified with their generating points via $w : D \to \mathfrak{F}_0(D) : w(x) := \dot{x}$, endowed with the topology generated from the base consisting of all sets $\mathfrak{F}_0(M)$, with $M \subseteq D$ (see [4], p. 176 ff).

Proposition 1.1 *Let (D, δ) be a discrete topological space. Then for its Stone-Čech-compactification $(\beta D, \delta^\beta)$ hold*

(a) *For all $M \subseteq D$ in B the closure $\overline{M}$ is clopen.*

(b) *δ^β has a base consisting of clopen sets.*

(c) *All clopen sets C in $(\beta D, \delta^\beta)$ are of the form $C = \overline{C \cap D}$.*

Proof: We use the homeomorphy of $(\beta D, \delta^\beta)$ to the Wallman extension.

(a) + (b) We have $\overline{w(M)} = \mathfrak{F}_0(M)$: from $\mathfrak{F}_0(M) = \mathfrak{F}_0(D) \setminus \mathfrak{F}_0(D \setminus M)$ we conclude, that $\mathfrak{F}_0(M)$ is closed and of course it contains $w(M)$. So, $\overline{w(M)} \subseteq \mathfrak{F}_0(M)$ follows. If there would be a filter $\varphi \in \mathfrak{F}_0(M)$ which belongs not to $\overline{w(M)}$, then there would exist a base set $\mathfrak{F}_0(S)$, $S \subseteq D$, of δ^β s.t. $\varphi \in \mathfrak{F}_0(S)$ and $\mathfrak{F}_0(S) \cap w(M) = \emptyset$. But this implies $M \cap S = \emptyset$, and thus $\mathfrak{F}_0(S) \cap \mathfrak{F}_0(M) = \emptyset$ - in contradiction to $\varphi \in \mathfrak{F}_0(S) \cap \mathfrak{F}_0(M)$. So, we have indeed $\overline{w(M)} = \mathfrak{F}_0(M)$, which is also open, because it belongs to our defining base of δ^β.

(c) Let $C \subseteq \beta D$ be clopen. Then for all $c \in C$ there exists a basic open set $\mathfrak{F}_0(M_c)$ with $M_c \subseteq D$, s.t. $c \in \mathfrak{F}_0(M_c) \subseteq C$, because C is open. From closedness of C automatically follows compactness, because βD is compact, thus there are finitely many $M_{c_1}, ..., M_{c_n}$ with $C = \bigcup_{i=1}^{n} \mathfrak{F}_0(M_{c_i})$. Now, for such finite union we have generally $\bigcup_{i=1}^{n} \mathfrak{F}_0(M_{c_i}) = \mathfrak{F}_0(\bigcup_{i=1}^{n} M_{c_i})$ and it is clear, that $w(\bigcup_{i=1}^{n} M_{c_i}) = C \cap w(D)$ holds. □

For a topological space (Y, σ) - especially, if it is not T_0 - we define an equivalence relation on Y by

$$x \sim y \;:\Leftrightarrow\; (\forall O \in \sigma : x \in O \leftrightarrow y \in O) \;.$$

Then the quotient space $(Y/{\sim}, \sigma_\sim)$ is obviously T_0; we call it the T-zerofication of (Y, σ). Let $\nu : Y \to Y/{\sim} : \nu(y) := [y]_\sim$ be the canonical surjection. Because ν is continuous, the space $(Y/{\sim}, \sigma_\sim)$ is compact, whenever (Y, σ) is.

Proposition 1.2 *Let (X, τ) be a Tychonoff space, (Y, σ) a compact T_3-space and $f : X \to Y$ a continuous function. Then there exists a continuous extension $F : \beta X \to Y$ with $F_{|X} = f$.*

Proof: The T-zerofication $(Y/{\sim}, \sigma_\sim)$ of (Y, σ) is also T_3 (see [3], p. 191), and of course T_0, so it is T_2. Because it is also compact, from the theorem of Stone-Čech we get a continuous extension $G : \beta X \to (Y/{\sim}, \sigma_\sim)$ of $\nu \circ f : X \to Y/{\sim}$, where ν is the canonical surjection from Y to $Y/{\sim}$. Now, let $\alpha : Y/{\sim} \to Y$ be a choice function, i.e. $\forall [y]_\sim \in Y_\sim : \alpha([y]_\sim) \in [y]_\sim$.

Then

$$F : \beta X \to Y : F(x) := \begin{cases} f(x) & ; \quad x \in X \\ \alpha \circ G(x) & ; \quad x \in \beta X \setminus X \end{cases}$$

is continuous by [3], prop. 4.1.4(4), and is obviously an extension of f. □

Proposition 1.3 *Let (Y, σ) be a topological T_3-space, $K \subseteq Y$ compact and $O \subseteq Y$ open with $K \subseteq O$. Then an open set U exists with $K \subseteq U \subseteq \overline{U} \subseteq O$. Especially, $\overline{K} \subseteq O$ holds.*

Proof: $K \subseteq O$ just means $K \cap (Y \setminus O) = \emptyset$ and $(Y \setminus O)$ is closed. Thus, by T_3, for every element $k \in K$ there are $U_k, V_k \in \sigma$ s.t. $k \in U_k, Y \setminus O \subseteq V_k$ and $U_k \cap V_k = \emptyset$. The U_k's cover K, so by compactness a finite subcover $U_{k_1}, ..., U_{k_n}$ exists. Let $U := \bigcup_{i=1}^{n} U_{k_i}$ and $V := \bigcap_{i=1}^{n} V_{K_i}$, so U, V are open, $U \cap V = \emptyset$, $K \subseteq U$ and $Y \setminus O \subseteq V$ hold, i.e.

$$K \subseteq U \subseteq Y \setminus V \subseteq O \;.$$

Now, $Y \setminus V$ is closed, so we get

$$\overline{K} \subseteq \overline{U} \subseteq \overline{Y \setminus V} = Y \setminus V \subseteq O \;.$$

□

2 Vietoris Hyperstructure as final w.r.t. Function Spaces

Remember a wide class of function space structures, defined for Y^X or $C(X,Y)$: the so called set–open topologies, examined in [1], [5]. According to [5], we use the following convention: Let X and Y be sets and $A \subseteq X$, $B \subseteq Y$; then let be $(A,B) := \{f \in Y^X | \ f(A) \subseteq B\}$. Now let X be a set, (Y,σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Then the topology $\tau_{\mathfrak{A}}$ on Y^X (resp. $C(X,Y)$), which is defined by the open subbase $\{(A,W)| \ A \in \mathfrak{A}, W \in \sigma\}$ is called the *set–open topology, generated by* $\mathfrak{A}$, or shortly the $\mathfrak{A}$*–open topology.*

We know

Lemma 2.1 [cf. [2], lemma 3.4] Let (X,τ), (Y,σ) be topological spaces, let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and $\mathcal{H} \subseteq Y^X$ be endowed with $\tau_{\mathfrak{A}}$. Then the map

$$\mu_X : \mathcal{H} \to \mathfrak{P}_0(Y)^{\mathfrak{A}} : \ f \to \mu_X(f) : \forall A \in \mathfrak{A} : \mu_X(f)(A) := f(A),$$

is open, continuous and bijective onto its image, for $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology σ_V, and $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ with the generated pointwise topology.

Now, the pointwise topology on $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ is just the product topology on $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$ (with all $\mathfrak{P}_0(Y)_A$ being copies of $\mathfrak{P}_0(Y)$). By chosing $\mathcal{H} := C(X,Y)$, $\mathfrak{A} := K(X)$ and consequently replacing $\mathfrak{P}_0(Y)$ by $K(Y)$, we have the following situation:

$$C(X,Y) \xrightarrow{\mu_X} K(Y)^{K(X)} \cong \prod_{A\in K(X)} K(Y)_A \xrightarrow{\pi_A} (K(Y),\sigma_V)$$

Of course, by π_A we mean the canonical projection from the product to the factor $K(Y)_A = K(Y)$.

From lemma 2.1 we get the continuity of μ_X, if $C(X,Y)$ is equipped with compact-open topology, thus in this case all compositions $\pi_A \circ \mu_X$ are continuous, too.

Moreover, μ_X is even a homeomorphism onto its image and the product structure is initial w.r.t. the projections. So, the question arises, whether or not the Vietoris topology σ_V on $K(Y)$ is final w.r.t. all $\pi_A \circ \mu_X$.

Lemma 2.2 *Let (X,τ), (Y,σ) be topological spaces and let σ_V be the Vietoris topology on $K(Y)$. Then for every $\mathfrak{O} \in \sigma_V$ and every $A \in K(X)$ the set $(\pi_A \circ \mu_X)^{-1}(\mathfrak{O}) \subseteq C(X,Y)$ is open w.r.t. the compact-open topology.*

Proof: Let $A \in K(X)$ be given and let $F \in Cl(Y)$ be a closed subset of Y. Then $(\pi_A \circ \mu_X)^{-1}(F^+) = \{f \in C(X,Y) |\ f(A) \subseteq Y \setminus F\} = (A, Y \setminus F) \in \tau_{co}$. Let now $O \in \sigma$ be given, then

$(\pi_A \circ \mu_X)^{-1}(O^-) = \{f \in C(X,Y) |\ f(A) \cap O \neq \emptyset\} = \bigcup_{a \in A}(\{a\}, O) \in \tau_{co}$.

So, because the F^+ and O^- form a subbase of σ_V, for $\mathfrak{O} \in \sigma_V$ the preimage $(\pi_A \circ \mu_X)^{-1}(\mathfrak{O})$ is an element of τ_{co}. □

Corollary 2.3 *Let (Y, σ) be a topological space. For every topological space let $C(X,Y)$ be equipped with compact-open topology.*

Then the Vietoris topology σ_V on $K(Y)$ is contained in the final topology w.r.t. all $\pi_A \circ \mu_{(X,\tau)}$, $(X, \tau) \in \mathcal{B}$, $A \in K(X, \tau)$, for every class $\mathcal{B}$ of topological spaces.

Theorem 2.4 *Let (Y, σ) be a T_3-space and let $(K(Y), \sigma_V)$ be its Vietoris Hyperspace of compact subsets. Let furthermore δ be the discrete topology on $Y \times Y$ and denote by (Z, ζ) the Stone-Čech-compactification of $(Y \times Y, \delta)$.*

Then σ_V is the final topology on $K(Y)$ w.r.t. $\pi_Z \circ \mu_Z : C(Z,Y) \to K(Y)$, where $C(Z,Y)$ is endowed with compact-open topology τ_{co}.

Proof: From Lemma 2.2 we know that σ_V is contained in the final topology w.r.t. $\pi_Z \circ \mu_Z$, so we only have to show, that every open set of the final topology also belongs to σ_V. Let $\mathfrak{O}$ be an open set of the final topology, i.e. $(\pi_Z \circ \mu_Z)^{-1}(\mathfrak{O}) \in \tau_{co}$, and let $A \in \mathfrak{O}$.

We want to show, that there exist finitely many open sets $U_1, ..., U_m \in \sigma$ s.t.
$A \in \langle U_1, ..., U_m \rangle_{K(Y)} \subseteq \mathfrak{O}$.

At first, chose any surjection s from Y onto $A \subseteq Y$. Then extend it to a surjection $f_A : Y \times Y \to A$ by $f_A(y_1, y_2) := s(y_1)$, just meaning, that f_A maps such pairs with equal first component to the same image.

Now, endowing $Y \times Y$ with discrete topology, we get f_A being continuous. So, if (Z, ζ) denotes the Stone-Čech-compactification of the discrete $Y \times Y$, there exists a continuous extension $F_A : Z \to A$ of f_A, by proposition 1.2.

Because F_A is an extension of f_A, we have

$$\forall (a,b), (c,d) \in Y \times Y : a = c \implies F_A(a,b) = F_A(c,d)\ . \qquad (1)$$

Because $\mathfrak{O}$ is open in the final topology, there are finitely many compact subsets $K_1, ..., K_n \in K(Z)$ and open subsets $O_1, ..., O_n \in \sigma$ s.t. $F_A \in \bigcap_{i=1}^n (K_i, O_i) \subseteq (\pi_X \circ \mu_X)^{-1}(\mathfrak{O})$.

We will improve the sets K_i and O_i a little in an appropriate manner.

(a) For each K_i and every $k \in K_i$ there is an open neighbourhood U_k of k, s.t. $F_A(U_k) \subseteq O_i$, because F_A is continuous. Now, ζ has a base consisting of clopen sets B of the form $B = \overline{B \cap (Y \times Y)}$. So, there exist always such a clopen $B_k \subseteq U_k$ with $k \in B_k$ and $F_A(B_k) \subseteq O_i$. The family of all $B_k, k \in K_i$ is an open cover of K_i and consequently there is a finite subcover $\{B_{k_1}, ..., B_{k_l}\}$, by compactness of K_i. Now let

$$K_i' := \bigcup_{j=1}^{l} B_{k_j}$$

and observe, that K_i' as a finite union of clopen sets is clopen again, hence it is compact and of the form $K_i' = \overline{K_i' \cap (Y \times Y)}$. Furthermore we have $K_i \subseteq K_i'$ and consequently

$$F_A \in (K_i', O_i) \subseteq (K_i, O_i) \ .$$

(b) We want to have our K's saturated in the sense, that whenever $(a, b) \in K \cap (Y \times Y)$ holds, then $\{a\} \times Y \subseteq K$ also holds. So, let us define

$$D_i := \bigcup_{\substack{a \in Y, \exists b \in Y: \\ (a,b) \in K_i' \cap (Y \times Y)}} \{a\} \times Y$$

and then $K_i'' := \overline{D_i}$. From the continuity of F_A follows

$$F_A(K_i'') = F_A(\overline{D_i}) \subseteq \overline{F_A(D_i)} \tag{2}$$

and from (1) we get

$$F_A(D_i) = F_A\left(K_i' \cap (Y \times Y)\right) \ . \tag{3}$$

Of course, $F_A\left(K_i' \cap (Y \times Y)\right) \subseteq F_A(K_i')$ and $F_A(K_i')$ is compact and fulfills $F_A(K_i') \subseteq O_i$, so by proposition 1.3 we get from (Y, σ) being T_3

$$F_A(K_i'') \subseteq \overline{F_A(D_i)} \subseteq \overline{F_A(K_i')} \subseteq O_i \ . \tag{4}$$

Note, that all K_i'' are compact and clopen again, by construction as a closure of a subset of $Y \times Y$ in the Stone-Čech-compactification (Z, ζ) of the discrete $Y \times Y$. Clearly, $K_i'' \supseteq K_i'$ holds, yielding $(K_i'', O_i) \subseteq (K_i', O_i)$, thus

$$F_A \in \bigcap_{i=1}^{n} (K_i'', O_i) \subseteq \bigcap_{i=1}^{n} (K_i', O_i) \ . \tag{5}$$

(c) To cover Z (resp. A) with our compact sets, we add $K_0'' := Z$ (resp. $O_0 := Y$) and find of course

$$F_A \in \bigcap_{i=1}^{n}(K_i'', O_i) \ = \ \bigcap_{i=0}^{n}(K_i'', O_i) \ .$$

For each $z \in Z$ define

$$I(z) := \{i \in \{0, ..., n\} \mid z \in K_i''\}$$

and then

$$C(z) := \bigcap_{i \in I(z)} K_i'' \setminus \left(\bigcup_{j \in \{0,...,n\} \setminus I(z)} K_j'' \right) \tag{6}$$

as well as

$$V(z) := \bigcap_{i \in I(z)} O_i \ . \tag{7}$$

Obviously for every $z \in Z$ we have

$$F_A(C(z)) \subseteq F_A \left(\bigcap_{i \in I(z)} K_i'' \right) \subseteq \bigcap_{i \in I(z)} O_i \ = V(z)$$

implying $F_A \in (C(z), V(z))$.

The family of all $C(z)$ covers Z, because every $z \in Z$ is contained at least in it's own $C(z)$. Observe, that different $C(z_1)$ and $C(z_2)$ are disjoint: if $y \in C(z_1) \cap C(z_2)$ exists, then $I(z_1) = I(y) = I(z_2)$ follows, implying $C(z_1) = C(z_2)$ by (6).

Obviously, there are only finitely many different sets $C(z), V(z)$, because they are uniquely determined by $I(z)$, which is a subset of $\{0, ..., n\}$ and this set has just finitely many subsets. So, for simplicity, let us denote them by $C_1, ..., C_m$ and $V_1, ..., V_m$, respectively.

It is clear, that the C_j's are clopen (thus compact) and saturated in the sense of paragraph (b), by construction (6) from just clopen saturated K_i'''s.

For $G \in \bigcap_{j=1}^{m}(C_j, V_j) = \bigcap_{z \in Z}(C(z), V(z))$ we find

$$\begin{array}{rclcl} \forall i \in \{0, ..., n\} & : & \forall z \in K_i'' & : & i \in I(z) \\ & & \Longrightarrow & : & G(z) \in V(z) \subseteq O_i \\ \Longrightarrow & : & G(K_i'') \subseteq O_i \ . & & \end{array}$$

Consequently, we have

$$F_A \in \bigcap_{j=1}^{m}(C_j, V_j) \subseteq \bigcap_{i=0}^{n}(K_i'', O_i) \tag{8}$$

(d) At last, let us chose for every $j = 1, ..., m$ an open set $U_j \in \sigma$ s.t. $F_A(C_j) \subseteq U_j \subseteq \overline{U_j} \subseteq V_j$ holds, as provided by proposition 1.3. Of course, we have then automatically $F_A \in (C_j, U_j) \subseteq (C_j, V_j)$.
So, because the C_j's cover Z, the $F_A(C_j)$'s cover A, and so the U_j's do.

With these U_j, $j = 1, ..., m$, we show $A \in \langle U_1, ..., U_j \rangle_{K(Y)} \subseteq \mathfrak{O}$.

$A \in \langle U_1, ..., U_m \rangle_{K(Y)}$ is clear, because the U_j's cover A, as seen in paragraph (d), and $\emptyset \neq F_A(C_j) \subseteq A \cap U_j$ for all $j = 1, ..., m$.

Let

$$B \in \langle U_1, ..., U_j \rangle_{K(Y)} \tag{9}$$

be given.

Because every C_j is nonempty clopen and saturated in the sense of paragraph (b), $C_j \cap (Y \times Y)$ has the cardinality of Y. So, there exists a surjection $t_j : C_j \cap (Y \times Y) \to U_j \cap B$ (the range is not empty by (9)).

Now, define

$$t : (Y \times Y) \to B : \; t(x, y) := t_j(x, y) \text{ for } (x, y) \in C_j$$

This t is well defined, because the C_j's are pairwise disjoint and cover Z by paragraph (c), and it is a surjection onto B, because the U_j's cover B by (9) and the t_j are surjections onto $U_j \cap B$. Our t is continuous w.r.t. the discrete topology on $Y \times Y$, so it extends to a continuous $T : Z \to B$.

By construction we have for each $j \in \{1, ..., m\}$

$$T\left(C_j \cap (Y \times Y)\right) \subseteq U_j \; , \tag{10}$$

implying $T(C_j) = T\left(\overline{C_j \cap (Y \times Y)}\right) \subseteq \overline{T\left(C_j \cap (Y \times Y)\right)} \subseteq \overline{U_j}$ by continuity, thus $T(C_j) \subseteq V_j$ by choice of U_j in paragraph (d).

We find $T \in \bigcap_{j=1}^{m} (C_j, V_j) \subseteq (\pi_Z \circ \mu_Z)^{-1}(\mathfrak{O})$, yielding $B = \pi_Z \circ \mu_Z(T) \in \mathfrak{O}$. This works for every $B \in \langle U_1, ..., U_m \rangle_{K(Y)}$, thus we have indeed $\langle U_1, ..., U_m \rangle_{K(Y)} \subseteq \mathfrak{O}$. Consequently, $\mathfrak{O}$ is a union of Vietoris-open subsets of $K(Y)$, just meaning $\mathfrak{O} \in \sigma_V$. □

Remark 2.5 Of course, $Y \times Y$ with discrete topology is homeomorphic to Y with discrete topology for infinite Y. So, we used $Y \times Y$ here just for convenience concerning the description of the „saturated“ subsets within the proof. Moreover, even for finite Y this proof works fine, but wouldn't do so with Y instead of $Y \times Y$.

Corollary 2.6 *Let (Y, σ) be a T_3-space. For every topological space let $C(X,Y)$ be equipped with compact-open topology. Let $\mathcal{B}$ be a class of topological spaces, that contains the Stone-Čech-compactification of a discrete space with cardinality at least card(Y).*
Then the Vietoris topology σ_V on $K(Y)$ is the final topology w.r.t. all $\pi_A \circ \mu_{(X,\tau)}$, $(X,\tau) \in \mathcal{B}$, $A \in K(X,\tau)$.

This characterization of the Vietoris hyperspace of the nonempty *compact* subsets of a regular space as a quotient (or more generally as a final object of a given class of spaces under certain mappings) includes an easy possibilty to characterize the Vietoris hyperspace of the nonempty *closed* subsets for Hausdorff T_4-spaces.

Lemma 2.7 *Let (Y, σ) be a Hausdorff T_4-space. Then its Vietoris hyperspace on the nonempty closed subsets $(Cl(Y), \sigma_V)$ is homeomorphic to a subspace of the Vietoris hyperspace $(K(\beta Y), \sigma^\beta)$ of compact subsets of the Stone-Čech-compactification of (Y, σ).*

Proof:

(1) The map

$$\alpha : Cl(Y) \to K(\beta Y) : \alpha(A) := \overline{A}^{\beta Y}$$

is injective: Let $A_1 \neq A_2 \in Cl(Y)$ be given, say w.l.o.g. $\exists a \in A_1 \setminus A_2$. Because Y is Tychonoff, we get a continuous $f : Y \to [0,1]$ such that $f(a) = 0$ and $f(A_2) = \{1\}$, and then by the theorem of Stone-Čech a continuous extension $F : \beta Y \to [0,1]$ with $F(a) = f(a) = 0$ and $F(A_2) = f(A_2) = \{1\}$, thus by continuity $F(\overline{A_2}^{\beta Y}) \subseteq \overline{F(A_2)} = \overline{\{1\}} = \{1\}$, implying $a \notin \overline{A_2}^{\beta Y}$ and consequently $\overline{A_1}^{\beta Y} \neq \overline{A_2}^{\beta Y}$.

(2) α is continuous w.r.t. to $\sigma_V, (\sigma^\beta)_V$:

- Let $O \in \sigma^\beta$ and

$$\begin{aligned} A_0 \in \alpha^{-1}\left(O^{-K(\beta Y)}\right) &= \{A \in Cl(Y) \mid \overline{A}^{K(\beta Y)} \cap O \neq \emptyset\} \\ &= \{A \in Cl(Y) \mid A \cap O \neq \emptyset\} \end{aligned}$$

  be given.
  Because Y is a dense subspace of βY, we get $\emptyset \neq O \cap Y \in \sigma$ and $A_0 \in (O \cap Y)^{-Cl(Y)} \subseteq \alpha^{-1}\left(O^{-K(\beta Y)}\right)$. Thus $\alpha^{-1}\left(O^{-K(\beta Y)}\right)$ is open in σ_V.
- Let $O \in \sigma^\beta$ and

$$A_0 \in \alpha^{-1}\left((\beta Y \setminus O)^{+K(\beta Y)}\right) = \{A \in Cl(Y) \mid \overline{A}^{K(\beta Y)} \subseteq O\}$$

  be given.

Now, βY is T_3 and $\overline{A_0}^{K(\beta Y)}$ is compact, so by proposition 1.3 we get an $U_0 \in \sigma^\beta$ with $\overline{A_0}^{K(\beta Y)} \subseteq U_0 \subseteq \overline{U_0}^{K(\beta Y)} \subseteq O$. So, we have $A_0 \subseteq U_0 \cap Y \in \sigma$ and furthermore $\forall A \in (Y \setminus U_0)^{+_{Cl(Y)}} : \overline{A}^{K(\beta Y)} \subseteq \overline{U_0}^{K(\beta Y)} \subseteq O$, yielding $A_0 \in (Y \setminus U_0)^{+_{Cl(Y)}} \subseteq \alpha^{-1}\left((\beta Y \setminus O)^{+_{K(\beta Y)}}\right)$. Consequently, $\alpha^{-1}\left((\beta Y \setminus O)^{+_{K(\beta Y)}}\right)$ is open in σ_V.

Note, that we didn't use T_4 so far.

(3) α is an open map onto its image.
Let $U_1, ..., U_n \in \sigma$ be given.

Let $A \in \langle U_1, ..., U_n \rangle_{Cl(Y)}$.
We have $A \subseteq \bigcup_{i=1}^n U_i \Rightarrow A \cap (\bigcap_{i=1}^n (Y \setminus U_i)) = \emptyset$. So, A and $\bigcap_{i=1}^n (Y \setminus U_i)$ are disjoint closed subsets of Y, which can be separated by a continuous function from Y to $[0,1]$, according to T_4. This function extends to a continuous function from βY to $[0,1]$ by the Stone-Čech theorem, yielding $\emptyset = \overline{A}^{\beta Y} \cap \overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}$, so we have $\overline{A}^{\beta Y} \subseteq \beta Y \setminus \left(\overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}\right)$.

Furthermore, $A \cap U_i \neq \emptyset$ implies $A \not\subseteq Y \setminus U_i$, and this yields by the same argument as in (1), that $\overline{A}^{\beta Y} \not\subseteq \overline{Y \setminus U_i}^{\beta Y}$, thus $\overline{A}^{\beta Y} \cap \left(\beta Y \setminus \left(\overline{Y \setminus U_i}^{\beta Y}\right)\right) \neq \emptyset$.

So, let $V_0 := \beta Y \setminus \left(\overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}\right)$ and for $i = 1, ..., n$ we define $V_i := \beta Y \setminus (\overline{Y \setminus U_i}^{\beta Y}) \in \sigma^\beta$.

Note, that $\bigcup_{i=1}^n \left(\beta Y \setminus \left(\overline{Y \setminus U_i}^{\beta Y}\right)\right) \subseteq \beta Y \setminus \left(\overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}\right)$ holds, i.e.

$$\bigcup_{i=1}^n V_i \subseteq V_0 \ . \tag{11}$$

So we get $\alpha(A) \in \langle V_0, V_1, ..., V_n \rangle_{K(\beta Y)}$ from the above.
This for all $A \in \langle U_1, ..., U_n \rangle_{Cl(Y)}$ yields

$$\alpha(\langle U_1, ..., U_n \rangle_{Cl(Y)}) \subseteq \langle V_0, V_1, ..., V_n \rangle_{K(\beta Y)} \ . \tag{12}$$

If otherwise $A \in Cl(Y)$ is given with $\alpha(A) = \overline{A}^{\beta Y} \in \langle V_0, V_1, ..., V_n \rangle_{K(\beta Y)}$, then for $i = 1, ..., n$ we get from $\emptyset \neq \overline{A}^{\beta Y} \cap V_i = \overline{A}^{\beta Y} \cap \left(\beta Y \setminus (\overline{Y \setminus U_i}^{\beta Y})\right)$, that $\overline{A}^{\beta Y} \not\subseteq \overline{Y \setminus U_i}^{\beta Y}$ and consequently $A \not\subseteq Y \setminus U_i$, thus $A \cap U_i \neq \emptyset$.

Moreover, from $\overline{A}^{\beta Y} \subseteq (\bigcup_{k=0}^n V_k) = V_0$ we get

$$\begin{aligned} A &\subseteq Y \cap \overline{A}^{\beta Y} \subseteq Y \cap \left(\beta Y \setminus \overline{\bigcap_{i=1}^{n} Y \setminus U_i}^{\beta Y} \right) \\ &\subseteq Y \setminus \overline{\bigcap_{i=1}^{n} Y \setminus U_i}^{\beta Y} \subseteq Y \setminus \bigcap_{i=1}^{n} Y \setminus U_i \\ &\subseteq \bigcup_{i=1}^{n} (Y \setminus (Y \setminus U_i)) = \bigcup_{i=1}^{n} U_i \end{aligned}$$

This yields

$$\alpha^{-1}(\langle V_0, V_1, ..., V_n \rangle_{K(\beta Y)}) \subseteq \langle U_1, ..., U_n \rangle_{Cl(Y)} \ . \tag{13}$$

So, from (12) and (13) we get

$$\alpha(\langle U_1, ..., U_n \rangle_{Cl(Y)}) = \alpha(Cl(Y)) \cap \langle V_0, V_1, ..., V_n \rangle_{K(\beta Y)} \ .$$

□

References

[1] **Arens, Richard** , and **Dugundji, James** : *Topologies for function spaces.* In: Pac. J. Math. 1 (1951), S. 5 – 31. http://dx.doi.org/10.2140/pjm.1951.1.5. – DOI 10.2140/pjm.1951.1.5. – ISSN 0030-8730

[2] **Bartsch, René :** *On a nice embedding and the Ascoli theorem.* In: N. Z. J. Math. 33 (2004), Nr. 1, S. 25 - 39. – ISSN 1179-4984/e

[3] **Bartsch, René :** *Allgemeine Topologie.* 2nd ed. Berlin: De Gruyter, 2015. – xvi + 306 S. – ISBN 978-3-11-040617-7

[4] **Engelking, Ryszard :** *General topology.* Berlin: Heldermann Verlag, 1989. – viii + 529 S. – ISBN 3-88538-006-4

[5] **Poppe, Harry :** *Compactness in general function spaces.* Berlin: VEB Deutscher Verlag der Wissenschaften, 1974

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