Rostocker Mathematisches Kolloquium

Heft 68

Universität Rostock

Institut für Mathematik

2013

Zitat–Kurztitel: Rostock. Math. Kolloq. **68** (2013)

ISSN 0138-3248

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DRUCK: Westarp & Partner Digitaldruck,
Hohenwarsleben

Rostock. Math. Kolloq. **68**, 3 – 12 (2013)

Subject Classification (MSC)
35Q40, 35J50, 46E35

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The limiting Dirac-Sobolev inequality

ABSTRACT. *We prove the critical Dirac-Sobolev inequality for $p \in (1,3)$. It follows that the Dirac Sobolev spaces are equivalent to classical Sobolev spaces if and only if $p \in (1,3)$. We prove the cocompactness of $L^{p^*}(\mathbb{R}^3)$ in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$.* As an application, we prove the existence of minimizers to a class of isoperimetric problems.

KEY WORDS AND PHRASES. cocompact imbeddings, concentration compactness, Dirac operator, minimizers, Sobolev imbeddings, critical exponent

1 Introduction

In [1], Balinsky, Evans and Saito introduced an L^p-seminorm $\|(\alpha \cdot \mathbf{p})u\|_{p,\Omega}$ of a $\mathbb{C}^4$-valued function on an open subset of Ω of $\mathbb{R}^3$ relevant to a massless Dirac operator

$$\alpha \cdot \mathbf{p} = \sum_{j=1}^{3} \alpha_j(-i\partial_j). \tag{1.1}$$

Here $\mathbf{p} = -\mathbf{i}\nabla$, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix} \quad j = 1, 2, 3$$

that use the 2×2 zero matrix 0_2 and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad \sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They proved a family of inequalities for this seminorm, called Dirac-Sobolev inequalities, in order to obtain L^p-estimates of the *zero modes*, i.e. generalized eigenfunctions associated with the eigenvalue 0 of the Dirac operator $(\alpha \cdot \mathbf{p}) + \mathbf{Q}$, where $Q(x)$ is a 4×4 Hermitian matrix-valued potential decaying at infinity.

Let Ω be an open subset of $\mathbb{R}^3$. The first order Dirac-Sobolev space $\mathbf{H}_0^{1,p}(\Omega;\mathbb{C}^4)=\mathbf{H}_0^{1,p}(\Omega)$, $1\le p<\infty$, is the completion of $C_0^\infty(\Omega;\mathbb{C}^4)$ with respect to the norm

$$\|u\|_{D,1,p,\Omega} := \int_\Omega (|u(x)|_p^p + |(\alpha\cdot\mathbf{p})u(x)|_p^p)\mathbf{d}x \tag{1.2}$$

where the p-norm of a vector $a=(a_1,a_2,a_{3,}a_4)^T\in\mathbb{C}^4$ is defined as

$$|a|_p = (\sum_{i=1}^4 |a_i|^p)^{1/p},$$

$u(x)=(u_1(x),u_2(x),u_3(x),u_4(x))^T$, and

$$(\alpha\cdot\mathbf{p})u(x) := \sum_{j=1}^3 \alpha_j p_j u(x) = \sum_{j=1}^3 (-i\alpha_j\partial_j u(x)).$$

A completion of $C^\infty(\Omega;\mathbb{C})^4$ with respect to the same norm will be denoted $\mathbf{H}^{1,p}(\Omega)$.

Let β be the fourth Dirac matrix given by

$$\beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix},$$

where 1_2 is the 2×2 identity matrix. It is known that the free massless Dirac operator $\alpha\cdot\mathbf{p}$ as well as the free Dirac operator $\alpha\cdot\mathbf{p}+m\beta$ with positive mass m and the relativistic Schrodinger operator $\sqrt{m^2-\Delta}$ have similar embedding properties in L^2 but not necessarily in L^p for $p\neq 2$. It is also known that for $1<p<\infty$, the usual $W^{1,p}(\Omega)$ Sobolev norm $(\|\psi\|_p^p+\|\nabla\psi\|_p^p)^{1/p}$ is equivalent to the norm $\|\sqrt{1-\Delta}\psi\|_p$, where $\psi:\mathbb{R}^3\mapsto\mathbb{C}$ [14].

In [5] the authors explore the relationship of $\mathbf{H}_0^{1,p}(\Omega)$ with the classical Sobolev spaces $W_0^{1,p}(\Omega;\mathbb{C}^4)$ when Ω is a bounded domain. In particular, it is shown that $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ are dense subspaces of $H_0^{1,p}(\Omega)$ and $H^{1,p}(\Omega)$ respectively. The maps

$$\begin{cases} J_{\Omega,0}: & W_0^{1,p}(\Omega)\ni u\mapsto u\in\mathbf{H}_0^{1,p}(\Omega) \\ J_\Omega: & W^{1,p}(\Omega)\ni u\mapsto u\in\mathbf{H^{1,p}(\Omega)} \end{cases}$$

are one to one and continuous for $1\le p<\infty$. They showed that the map $J_{\Omega,0}$ is onto with continuous inverse if $1<p<\infty$ so that the spaces $W_0^{1,p}(\Omega)$ and the space $H_0^{1,p}(\Omega)$ are the same. If $p=1$, the map $J_{\Omega,0}$ is not onto.

In this paper we prove the limiting Dirac-Sobolev inequality on the whole space,

$$\int_{\mathbb{R}^3} |u|^{p^*}\mathrm{d}x \le C_p\left(\int_{\mathbb{R}^3}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x\right)^{\frac{p^*}{p}}, \tag{1.3}$$

where C_p is a positive constant, $p \in (1,3)$, and $p^* = \frac{3p}{3-p}$. It follows that the map J_Ω is onto for $p \in (1,3)$ if Ω is an extension domain. An extension domain is a domain for which every $u \in \mathbf{H^{1,p}(\Omega)}$ there is a $\tilde{u} \in \mathbf{H_0^{1,p}(\Omega')}$ such that $u = \tilde{u}|_\Omega$ where $\Omega \subset \Omega'$. We then prove cocompactness of the embedding $L^{p^*}(\mathbb{R}^3) \subset \dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$, where $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |(\alpha\cdot\mathbf{p})u|^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{1.4}$$

See Remark 2.2. We apply this result to show existance of minimizers to isoperimetric problems involving oscillatory nonlinearities with critical growth.

2 A Dirac-Sobolev inequality and the space $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$

In this section we prove inequality (1.3).

Theorem 2.1 *Let $p \in (1,3)$. Then there exists a constant $C_p > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$, (1.3) holds.*

Proof. Let us use the inequality (3.10) of [1], with the choice of parameters $k = p$, $r = 1$ and $\theta = \frac{3p-3}{4p-3}$:

$$\|u\|_{p,B_1}^p \le C\|(\alpha\cdot\mathbf{p})u\|_{p,B_1}^{p\theta}\|u\|_{1,B_1}^{p(1-\theta)},\ u \in C_0^\infty(B_1;\mathbb{C}^4). \tag{2.1}$$

Using an elementary inequality $s^\theta t^{(1-\theta)} \le C(\lambda s + \lambda^{-\gamma} t)$, $\frac{1}{\gamma} = \frac{1}{\theta} - 1$, that holds for all positive t, s, and λ, and setting $\lambda = \rho^p$, $s = \|u\|_{p,B_1}^p$, and $t = \|u\|_{1,B_1}^p$, one deduces from (2.1), for all positive ρ,

$$\|u\|_{p,B_1}^p \le C\left(\rho^p\|(\alpha\cdot\mathbf{p})u\|_{p,B_1}^p + \rho^{3-3p}\|u\|_{1,B_1}^p\right)\ u \in C_0^\infty(B_1;\mathbb{C}^4). \tag{2.2}$$

By choosing $\rho' = R\rho$ and rescaling the integration domain we will have, for any positive ρ', renamed ρ,

$$\|u\|_{p,B_R}^p \le C\left(\rho^p\|(\alpha\cdot\mathbf{p})u\|_{p,B_R}^p + \rho^{3-3p}\|u\|_{1,B_R}^p\right),\ u \in C_0^\infty(B_R;\mathbb{C}^4), \tag{2.3}$$

for any $R > 0$. We conclude that for any positive ρ,

$$\|u\|_{p,\mathbb{R}^3}^p \le C\left(\rho^p\|(\alpha\cdot\mathbf{p})u\|_{p,\mathbb{R}^3}^p + \rho^{3-3p}\|u\|_{1,\mathbb{R}^3}^p\right),\ u \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4). \tag{2.4}$$

Let us apply (2.4) to functions $\chi_j(|u|)$, where $\chi_j(t) = 2^{-j}\chi(2^j t)$, $j \in \mathbb{Z}$ and $\chi \in C_0^\infty((\frac{1}{2},4),[0,3])$, such that $\chi(t) = t$ whenever $t \in [1,2]$ and $|\chi'| \le 2$. Then we obtain, with the values

$\rho = \rho_j$ to be determined,

$$\begin{aligned}\int_{|u|\in[2^j,2^{j+1}]}|u|^p\mathrm{d}x &\leq \int \chi_j(u)^p\mathrm{d}x \\ &\leq C\left(\rho_j^p\int_{|u|\in[2^{j-1},2^{j+2}]}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x+\right. \\ &\left.\rho_j^{3-3p}\left(\int_{|u|\in[2^{j-1},2^{j+2}]}|u|\mathrm{d}x\right)^p\right).\end{aligned}$$

Taking into account the upper and lower bounds of $|u|$ on the respective sets of integration, we have

$$\begin{aligned}2^{(p-p^*)j}\int_{|u|\in[2^j,2^{j+1}]}|u|^{p^*}\mathrm{d}x &\leq C\rho_j^p\int_{|u|\in[2^{j-1},2^{j+2}]}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x + \\ &C2^{p(1-p^*)(j-1)}\rho_j^{3-3p}\left(\int_{|u|\in[2^{j-1},2^{j+2}]}|u|^{p^*}\mathrm{d}x\right)^p.\end{aligned}$$

If we substitute $\rho_j = 2^{-\frac{p^3(1-p)j+pp^*-p}{3-3p}}\rho$, take the sum over $j\in\mathbb{Z}$, and note that each of the intervals $[2^{j-1},2^{j+2}]$, $j\in\mathbb{Z}$, overlaps with the others not more than four times, we get

$$\int|u|^{p^*}\mathrm{d}x \leq C\left(\rho^p\int_{\mathbb{R}^3}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x+\rho^{3-3p}\left(\int_{\mathbb{R}^3}|u|^{p^*}\mathrm{d}x\right)^p\right)$$

Setting $\rho = \left(\frac{1}{2C}\right)^{\frac{1}{3-3p}}\left(\int u^{p^*}\right)^{\frac{1}{3}}$ and collecting similar terms we arrive at (1.3). □

Inequality (1.3) defines a continuous imbedding of $L^{p^*}(\mathbb{R}^3;\mathbb{C}^4)$ into $\dot{H}^{1,p}_D(\mathbb{R}^3)$.

Remark 2.2 Note that (1.4) does indeed define a norm on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$, since $(\alpha\cdot\mathbf{p})u = 0$ implies $|\nabla u|^2 = 0$ which yields $u = $ const. Since $u = 0$ outside of a compact set, the value of this constant is zero. We have therefore a Banach space $\dot{H}^{1,p}_D(\mathbb{R}^3)$ into which $L^{p^*}(\mathbb{R}^3)$ is continuously imbedded. It should be noted, however, that the space $\dot{H}^{1,p}_D(\mathbb{R}^3)$ is equivalent to the usual gradient-norm space $\mathcal{D}^{1,p}(\mathbb{R}^3;\mathbb{C}^4)$ *if and only if* $p\in(1,3)$. If $p>1$, consider the gradient norm and the Dirac-gradient norm (1.4) on $C_0^\infty(B_R;\mathbb{C}^4)$, which are equivalent Sobolev norms in $W_0^{1,p}(B_R;\mathbb{C}^4)$ and $\mathbf{H}_0^{1,p}(B_R)$ respectively. Since these norms are scale-invariant, they are equivalent (by Theorem 1.3 (ii) of [5]) on the balls B_R with bounds independent of R and thus, these norms are equivalent on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$, and, consequently $\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4) = \dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$. As a further consequence, the map J_Ω defined in the introduction is onto if $p\in(1,3)$ and Ω is an extension domain. From this observation, we obtain

Corollary 2.3 *Let $p\in(1,3)$, $\Omega\subset\mathbb{R}^3$ be and extension domain, and $q\in[p,p^*]$ then*

$$\|u\|_{q,\Omega} \leq C_{p,q}\left(\int_\Omega(|(\alpha\cdot\mathbf{p})u|^p+|u|^p)\mathrm{d}x\right)^{1/p}. \tag{2.5}$$

Remark 2.4 If $p = 1$, by Proposition 4.4 of [5] $\dot{H}^{1,1}(\mathbb{R}^3;\mathbb{C}^4)$ is strictly smaller than $\dot{\mathbf{H}}^{1,1}(\mathbb{R}^3)$.

3 Cocompactness of Dirac-Sobolev imbeddings

We recall the following definitions:

Definition 3.1 *Let u_k be a sequence in a Banach space E and D be a set of linear isometries acting on E. We say that u_k converges $D-$weakly to u, which we denote*

$$u_k \overset{D}{\rightharpoonup} u,$$

if for all ϕ in E',

$$\lim_{k\to\infty} \sup_{g\in D}(g\phi, u_k - u) = 0.$$

Remark 3.2 It follows immediately from 3.1 that if a bounded sequence u_k is not $D-$weakly convergent to 0, then there exists a sequence $g_k \in D$ and a $w \neq 0 \in E$ such that $g_k^* u_k \rightharpoonup w$.

Definition 3.3 *Let B be a Banach space continuously embedded in E. We say that B is* cocompact *in E with respect to D if $u_k \overset{D}{\rightharpoonup} u$ in E implies $u_k \to u$ in B.*

Let $\delta_{\mathbb{R}}$ be the group of dilations,

$$h_s u(x) = p^{\frac{3-p}{p}s} u(p^s x),$$

let D_G be the group of translations,

$$g_y u = u(\cdot - y),\ y \in \mathbb{R}^3,$$

and let

$$D := \delta_{\mathbb{R}} \times D_G.$$

We will denote by $D_{\mathbb{Z}}$ the subgroup, $s \in \mathbb{Z}$, $y \in \mathbb{Z}^3$. Note that both $\|u\|_{p^*}$and $\|u\|_{\dot{\mathbf{H}}}$are invariant under D and $D_{\mathbb{Z}}$. Furthermore, cocompactness with respect to D is equivalent to cocompactness with respect to $D_{\mathbb{Z}}$ (Lemma 5.3, [15]).

Theorem 3.4 *Let $p \in (1,3)$. Then $L^{p^*}(\mathbb{R}^3)$is cocompactly embedded in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$ with respect to D.*

Proof. Assume u_k is D-weakly convergent to zero in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$. Since $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ is dense in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$ and the latter is continuously imbedded into $L^{p^*}(\mathbb{R}^3)$, we may assume without loss of generality that $u_k \in C_0^\infty(\mathbb{R}^3)$. Let $\chi \in C_0^\infty((\frac{1}{p}, p^2); [0, p^2 - 1])$, be such that $\chi(t) = t$ for $t \in [1, p]$ and $|\chi'| \leq \frac{p}{p-1}$. By the Dirac-Sobolev inequality (2.5), for every $y \in \mathbb{Z}^3$,

$\left(\int_{(0,1)^3+y}\chi(|u_k|)^{p^*}\mathrm{d}x\right)^{p/p^*} \leq C\int_{(0,1)^3+y}(|(\alpha\cdot\mathbf{p})u_k|^p+\chi(u_k)^p)\mathrm{d}x.$

Since $\chi(t)^{p^*}\leq Ct^p$, this gives

$$\begin{aligned}&\int_{(0,1)^3+y}\chi(|u_k|)^{p^*}\mathrm{d}x\\ \leq\ & C\left(\int_{(0,1)^3+y}(|(\alpha\cdot\mathbf{p})u_k|^p+\chi(u_k)^p)\mathrm{d}x\right)\left(\int_{(0,1)^3+y}\chi(|u_k|)^{p^*}\mathrm{d}x\right)^{1-p/p^*}\\ \leq\ & C\left(\int_{(0,1)^3+y}(|(\alpha\cdot\mathbf{p})u_k|^p+\chi(u_k)^p)\mathrm{d}x\right)\left(\int_{(0,1)^3+y}u_k^p\mathrm{d}x\right)^{1-p/p^*}.\end{aligned}$$

Summing the above inequalities over all $y\in\mathbb{Z}^3$, and noting that by (1.3) $\|u_k\|_{p^*}\leq C$, therefore $|\left\{u_k\geq\frac{1}{p}\right\}|\leq C$ from which we can conclude $\int_{\mathbb{R}^3}\chi(u_k)^p\leq C$, we obtain

$$\int_{\mathbb{R}^3}\chi(|u_k|)^{p^*}\leq C\sup_{y\in\mathbb{Z}^3}\left(\int_{(0,1)^3+y}|u_k|^p\right)^{1-p/p^*}. \tag{3.1}$$

Let $y_k\in\mathbb{Z}^3$ be such that

$$\sup_{y\in\mathbb{Z}^3}\left(\int_{(0,1)^3+y}|u_k|^p\right)^{1-p/p^*}\leq 2\left(\int_{(0,1)^3+y_k}|u_k|^p\right)^{1-p/p^*}.$$

Since u_k converges to zero D-weakly, $u_k(\cdot-y_k)\rightharpoonup 0$ in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$, and thus it follows from Theorem 1.3 (ii) in [5] and the fact that $(0,1)^3$ is an extension domain that $u_k(\cdot-y)\to 0$ in $L^p((0,1)^3;\mathbb{C}^4)$. Therefore,

$$\int_{(0,1)^3+y_k}|u_k|^p=\int_{(0,1)^3}|u_k(\cdot-y_k)|^p\to 0.$$

Substituting into (3.1), we obtain

$$\int_{\mathbb{R}^3}\chi(|u_k|)^{p^*}\mathrm{d}x\to 0.$$

Let

$$\chi_j(t)=p^j\chi(p^{-j}t)),\ \ j\in\mathbb{Z}.$$

Since for any sequence $j\in\mathbb{Z}$, $h_{j_k}u_k$ converges to zero D-weakly, we have also, with arbitrary $j_k\in\mathbb{Z}$,

$$\int_{\mathbb{R}^3}\chi_{j_k}(|u_k|)^{p^*}\mathrm{d}x\to 0. \tag{3.2}$$

For $j\in\mathbb{Z}$, we have

$$\left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*} \mathrm{d}x\right)^{p/p^*} \le C \int_{\{p^{j-1}\le|u_k|\le p^{j+2}\}} |(\alpha\cdot\mathbf{p})u_k|^p \mathrm{d}x,$$

which can be rewritten as

$$\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*} \mathrm{d}x \le C \int_{\{p^{j-1}\le|u_k|\le p^{j+2}\}} |(\alpha\cdot\mathbf{p})u_k|^p \mathrm{d}x \left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*} \mathrm{d}x\right)^{1-\frac{p}{p^*}}. \tag{3.3}$$

Adding the inequalities (3.3) over $j \in \mathbb{Z}$ and taking into account that the sets $\{x \in \mathbb{R}^3 : 2^{j-1} \le |u_k| \le 2^{j+2}\}$ cover $\mathbb{R}^3$ with uniformly finite multiplicity, we obtain

$$\int_{\mathbb{R}^3} |u_k|^{p^*} \mathrm{d}x \le C \int_{\mathbb{R}^3} |(\alpha\cdot\mathbf{p})u_k|^p \mathrm{d}x \sup_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*}\right)^{1-p/p^*}. \tag{3.4}$$

Let j_k be such that

$$\sup_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*}\right)^{1-p/p^*} \le 2\left(\int_{\mathbb{R}^3} \chi_{j_k}(|u_k|)^{p^*}\right)^{1-p/p^*}.$$

Using the previous estimate and (3.2) we see that the right hand side of (3.4) converges to zero. Thus $u_k \to 0$ in L^{p^*}. □

4 Existence of minimizers

We consider the class of functions $F \in C_{\mathrm{loc}}(\mathbb{R})$satisfying

$$F(p^{\frac{3-p}{p}j}s) = p^{3j}F(s),\ s\in\mathbb{R}, j\in\mathbb{Z}. \tag{4.1}$$

This class is characterized by continuous functions on the intervals $[1, p^{\frac{3-p}{p}}]$ and $[-p^{\frac{3-p}{p}}, -1]$ satisfying $F(p^{\frac{3-p}{p}}) = p^3F(1)$ and $F(-p^{\frac{3-p}{p}}) = p^3F(-1)$, extended to $(0,\infty)$ and $(-\infty,0)$ by (4.1). It is immediate that there exists positive constants C_1 and C_2 such that

$$C_1|s|^{p^*} \le |F(s)| \le C_2|s|^{p^*}. \tag{4.2}$$

It also follows from (4.1) that for $h_j \in \delta_{\mathbb{Z}}$,

$$\int_{\mathbb{R}^3} F(h_j u)\mathrm{d}x = \int_{\mathbb{R}^3} F(u)\mathrm{d}x,\ \text{for j} \in \mathbb{Z},\ \mathrm{u} \in \mathrm{L}^{\mathrm{p}^*}(\mathbb{R}^3).$$

The functional

$$G(u) = \int_{\mathbb{R}^3} F(u)\mathrm{d}x$$

is continuous on $L^{p^*}(\mathbb{R}^3)$ and thus on $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$.

Theorem 4.1 *There exists a minimizer to the following isoperimetric problem.*

$$\inf_{G(u)=1} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \mathrm{d}x \tag{4.3}$$

in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$.

Proof. Let u_k be a minimizing sequence. By (4.2) and (2.5), u_k is bounded. By Theorem 3.4 and (4.2), u_k cannot converge $D-$weakly to 0. By Theorem 2 in [13] (see also [12]), (4.2), and using the facts: $\|gw\|^p_{\dot{\mathbf{H}}} = \|w\|^p_{\dot{\mathbf{H}}}$ and $G(gw) = G(w)$, we may write (in our notation) $\|u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)}\|_{L^{p^*}} \to 0$ with $g_k^{(n)} \in D_{\mathbb{Z}}$, $w^{(n)} \in \dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$,

$$\|u_k\|^p_{\dot{\mathbf{H}}} \geq \sum_{n\in\mathbb{N}} \|w^{(n)}\|^p_H, \text{ and} \tag{4.4}$$

$$1 = G(u_k) = \sum_{n\in\mathbb{N}} G(w^{(n)}) + o(1). \tag{4.5}$$

Since $G(u_k) = 1$, (4.5) implies that at least one $w^{(n)} \neq 0$. We will denote this $w^{(n)}$ by w. From the proof of Theorem 2 in [13] it is immediate that

$$\|u_k\|^p_H = \|w\|^p_H + \|u_k - w\|^p_H + o(1). \tag{4.6}$$

From (4.5) we deduce that

$$G(u_k) = G(w) + G(u_k - w) + o(1). \tag{4.7}$$

Assume $G(w) = \lambda$. We imbed problem (4.3) in the continuous family of problems

$$\alpha(t) := \inf_{G(u)=t} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \mathrm{d}x.$$

From the change of variables $u(t^{1/3}\cdot)$, we see that $\alpha(t) = \inf_{G(u)=1} t^{(1-p/3)} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \mathrm{d}x = t^{(1-p/3)}\alpha(1)$, so $\alpha(t)$ is a strictly concave function. From (4.6), we deduce that $\alpha(1) = \alpha(\lambda) + \alpha(1-\lambda)$. Since $\alpha(t)$ is strictly concave, this is only possible if $\lambda = 1$. Therefore $G(w) = 1$ and w solves problem (4.3). $\square$

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received: January 28, 2014

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Rostock. Math. Kolloq. **68**, 13 – 43 (2013)

Subject Classification (AMS)
47H04, 47H09, 65JXX

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Projection kernels of linear operators and convergence considerations

ABSTRACT. In the study of iterative methods used to solve linear operator equations sequences of linear iteration operators (T_k) occur which have a nontrivial projection kernel, that is a linear projector $P \neq O$ satisfying $P = T_k P = P T_k$ for all natural k. The convergence proof for (T_k) or some related operator sequences is simplified if such P is known. It is investigated when projection kernels exist and how they can be determined. Besides, special types of projection kernels are considered.

KEY WORDS. Linear operators, Fejér monotone operators, nonexpansive operators, projectors, orthoprojectors, relaxation of orthoprojectors

1 Introduction

It is remarkable that sequences (T_k) of linear (bounded) operators occuring in iterative methods for linear operator equations or in ergodic theory often have the following property:

(*) There is a projector $P \neq O$ with $T_k P = P T_k = P$ for all $k \in \mathrm{N}$.

Such a projector P is called a (nontrivial) *projection kernel* of (T_k). E.g., if a linear bounded operator T acting on a (real) Banach space X is *asymptotically convergent*, that is, if the power sequence (T^k) is convergent (to a linear bounded operator $T^\infty \neq O$), then (*) is fulfilled for $P = T^\infty$ and $T_k = T^k$. If $T^\infty = O$, then only the trivial projection kernel $P = O$ exists. In both cases the decomposition

$$X = \mathbb{N}(I - T) \oplus \overline{\mathbb{R}(I - T)}, \quad \mathbb{R}(P) = \mathbb{N}(I - T), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - T)}$$

holds, where I is the identity operator. Reversely, if such a projection kernel P is known, the convergence investigation of (T^k) can be reduced to the invariant subspace $\mathbb{N}(P)$, while $\mathbb{R}(P)$ is the fixed point set of T. More generally the knowledge of a projection kernel P simplifies the convergence proof for (T_k) or for other related sequences. In this section we investigate, when sequences or sets of operators possess projection kernels and how they can be determined. Later we specify also orthogonal, maximal, optimal and attainable projection kernels. The starting point of these investigations is my paper [11]. In the mean time some new aspects, examples and results can be presented.

For motivation we state some results concerning the iterative solution of linear equations with operators acting in Banach spaces X and Y. Let $\mathcal{L}(X,Y)$ be the algebra of linear bounded operators from X into Y, let

$$Ax = b, \quad A \in \mathcal{L}(X,Y), \quad b \in Y \tag{1.1}$$

be an equation with unknowns $x \in X$ and let (D_k) be a given operator sequence with $D_k \in \mathcal{L}(Y,X)$. Then linear iterative methods

$$x_{k+1} := T_k x_k + D_k b, \quad T_k := I - D_k A, \quad x_0 \in X \text{ arbitrary} \tag{1.2}$$

for the solution of (1.1) can be constructed. The defects $r_k := b - Ax_k$ are obtained by

$$r_{k+1} := S_k r_k, \quad S_k := I - AD_k, \quad r_0 := b - Ax_0 \in Y. \tag{1.3}$$

Explicitly we have the representations

$$x_{k+1} = T_{k,0}x_0 + B_k b, \quad B_k := \sum_{i=0}^{k} T_{k,i+1} D_i, \quad r_{k+1} = S_{k,0} r_0, \tag{1.4}$$

where the product notation $U_{i,j} := U_i \dots U_{j+1} U_j$ for $i \geq j$ is used (see e.g. [1], [12]). If $D_k = D$ is constant for all k, then we get from (1.2) and (1.3) the stationary method

$$x_{k+1} := Tx_k + Db = T^{k+1}x_0 + \sum_{i=0}^{k} T^i Db, \quad r_{k+1} := Sr_k = S^{k+1} r_0 \tag{1.5}$$

with $T := I - DA$ and $S := I - AD$. If (D_k) is cyclic and iteration is considered in cycles, then again stationary methods of type (1.5) arise.

We state now some examples for $X = \mathrm{R}^n$ and $Y = \mathrm{R}^m$ (finite-dimensional case). Then equation (1.1) is a system of m linear equations with n scalar unknowns. Further, the operator A can be identified with a matrix $A \in \mathrm{R}^{m,n}$. The adjoint operator A^* is realized by the transpose A^t of the corresponding matrix A.

Example 1.1 (Stationary iteration) If the method (1.5) is investigated, the convergence of the power sequences (T^k) and (S^k) as well as of the Neumann series $(\sum_{i=0}^{k} T^i)$ is of interest.

Example 1.2 (PSH method) see [8] and [6]: p. 53f. We start with matrices E_k which select one or more linearly independent rows of the matrix A in steps k in such a way that each (non-vanishing) row is selected at least once in certain step sections uniformly bounded for all k (as cycles if (E_k) is cyclic). Defining matrices

$$D_k := A^* E_k^* (E_k A A^* E_k^*)^{-1} E_k$$

the corresponding iterative method (1.2) projects in each step k orthogonally onto subspaces of R^n formed by intersection of the hyperplanes corresponding to the rows in $E_k A$. Further, the following can be shown:

a) The sequence (T_k) of orthoprojectors $T_k := I - D_k A$ has the orthogonal projection kernel P with $\mathbb{R}(P) = \mathbb{N}(A)$ and $\mathbb{N}(P) = \mathbb{R}(A^*)$. The product sequence $(T_{k,0})$ converges to this P.

b) The sequence (S_k) of operators $S_k := I - AD_k$ has a projection kernel Q with $\mathbb{N}(Q) = \mathbb{R}(A)$. The product sequence $(S_{k,0})$ converges to this Q.

Example 1.3 (SPA method) see [7] and [6]: p. 38f. We start with matrices F_k which select one or more linearly independent columns of the matrix A in steps k in such a way that each (non-vanishing) column is selected at least once in certain step sections uniformly bounded for all k. Defining matrices

$$D_k := F_k (F_k^* A^* A F_k)^{-1} F_k^* A^*$$

the corresponding iterative method (1.3) projects in each step k orthogonally onto subspaces of R^m spanned by the rows in AF_k. Further, the following can be shown:

a) The sequence (S_k) of orthoprojectors $S_k := I - AD_k$ has the orthogonal projection kernel Q with $\mathbb{R}(Q) = \mathbb{N}(A^*)$ and $\mathbb{N}(Q) = \mathbb{R}(A)$. The product sequence $(S_{k,0})$ converges to this Q.

b) The sequence (T_k) of operators $T_k := I - D_k A$ has a projection kernel P with $\mathbb{R}(P) = \mathbb{N}(A)$. The product sequence $(T_{k,0})$ converges to this P.

The methods described in Example 1.2 and Example 1.3 can be generalized in various ways by conservation of the main results (see [16]).

Example 1.4 (A gradient method for regular systems) see [2]. Let A be a regular quadratic matrix ($m = n$). We consider row vectors H_k containing the signs of the k-th columns $\vec{a}_k$ of A, i.e. $H_k := (\text{sign } \vec{a}_k)^*$. Now we define matrices

$$D_k := A^* H_k^* (H_k A A^* H_k^*)^{-1} H_k.$$

Then the operators of the iterative methods (1.2) and (1.3) have the following properties:

a) The sequences (T_k) and (S_k) have only the trivial projection kernel O.

b) The product sequences $(T_{k,0})$ and $(S_{k,0})$ converge to O.

Example 1.5 (A general case with operator relations) see [6]: p. 32 and [1]. We consider the general iterative method described in (1.2) and (1.3). If the operator sequence (B_k) occurring in (1.4) converges, say $\lim_{k\to\infty} B_k = B_\infty$, and if moreover

$$D_k A B_\infty = B_\infty A D_k = D_k \quad \text{for all } k,$$

then the following holds:

a) (T_k) has the projection kernel $P = I - B_\infty A$ and $(T_{k,0})$ converges to P.

b) (S_k) has the projection kernel $Q = I - AB_\infty$ and $(S_{k,0})$ converges to Q.

2 Projection kernels of operator sets

Let X be a (real) Banach space. In the following we consider projectors $P \in \mathcal{L}(X)$, that means $P^2 = P$, and sets $\mathcal{T}$ of operators $T \in \mathcal{L}(X)$. We start with a well-known fact.

Proposition 2.1 *A linear projector P is bounded (continuous) and induces the space decomposition*

$$X = \mathbb{R}(P) \oplus \mathbb{N}(P) = \mathbb{N}(I-P) \oplus \mathbb{R}(I-P),$$

where ranges and nullspaces are (closed) linear subspaces of X. Moreover, P is uniquely determined by this decomposition. The operator $I - P$ is a projector, too, with analogue properties.
A projector P is an orthoprojector ($\mathbb{R}(P) \perp \mathbb{N}(P)$) iff P is self-adjoint ($P = P^*$). An orthoprojector P is uniquely determined by its range $\mathbb{R}(P)$ (see e.g. [10], section 5.6).

Now the main concept of the paper is introduced.

Definition 2.1 *The projector P is said to be a*

- left projection kernel *of $\mathcal{T}$ if $P = PT$ for all $T \in \mathcal{T}$ ($P \in \mathbb{K}_l(\mathcal{T})$).*
- right projection kernel *of $\mathcal{T}$ if $P = TP$ for all $T \in \mathcal{T}$ ($P \in \mathbb{K}_r(\mathcal{T})$).*
- projection kernel *of $\mathcal{T}$ if $P = PT = TP$ for all $T \in \mathcal{T}$ ($P \in \mathbb{K}(\mathcal{T})$).*

In brackets the short notations are given. Another expression for $P \in \mathbb{K}(\mathcal{T})$ is that $\mathcal{T}$ has the projection kernel P.

Remark 2.1 If sequences (T_k) are involved, we write simply (T_k) instead of the set notation $\mathcal{T} = \{T_k : k \in \mathrm{N}\}$. If $\mathcal{T} = \{T\}$ contains only one operator T we often write simply T instead.

Trivially, O is a projection kernel of all operator sets $\mathcal{T}$ ($O \in \mathbb{K}(\mathcal{T})$). For completion we define that P is a projection kernel of $\emptyset$, the empty set in $\mathcal{L}(X)$. Although the identity $P = I$ is a projector, it is no projection kernel of $\mathcal{T}$ if $\mathcal{T}$ contains operators $T \neq I$.

By definition P is a projection kernel of $\mathcal{T}$ iff it is both a left and a right projection kernel of $\mathcal{T}$. The following example shows that indeed left or right projection kernels need not to be projection kernels.

Example 2.1 Consider the matrices

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathrm{R}^{3,3}$. Then the following relations are satisfied:

$$P^2 = P = PT = TP = T^*P = PT^*,$$
$$Q^2 = Q = TQ = QT^*, \quad Q \neq QT, \quad Q \neq QT^*.$$

Hence, P is a projection kernel of T and T^*, while Q is neither a projection kernel of T nor of T^*. But Q is a right projection kernel of T and a left projection kernel of T^*.

Now we list some simple statements about projection kernels. If a proof is missing it is either obvious or it is a simple consequence of more general statements given later.

Proposition 2.2 *Let $\mathcal{T} \setminus \{I\} \neq \emptyset$. If P is a projection kernel of $\mathcal{T}$, then $I - P$ is not.*

Proof: We assume $P \in \mathbb{K}(\mathcal{T})$. Then the projector $I - P$ satisfies

$$(I - P)T = T(I - P) = T - P \neq I - P$$

for $T \neq I$. But such T are supposed to be in $\mathcal{T}$ by assumption. ■

Proposition 2.3

a) *If P is a projection kernel of $\mathcal{T}_1$, then also of $\mathcal{T}_2 \subseteq \mathcal{T}_1$.*

b) *If P is a projection kernel of both $\mathcal{T}_1$ and $\mathcal{T}_2$, then also of $\mathcal{T}_1 \cup \mathcal{T}_2$.*

c) *Each projector P is a projection kernel of itself* ($P \in \mathbb{K}(P)$).

d) *Each projector P is a projection kernel of I* ($P \in \mathbb{K}(I)$).

e) *If P is a projection kernel of $\mathcal{T}$, then also of $\mathcal{T} \cup \{I\}$.*

f) *If $P \neq O$ is a projection kernel of $\mathcal{T}$, then $O \notin \mathcal{T}$.*

The next statements refer to operations conserving projection kernels.

Lemma 2.1 *If P is a (left, right) projection kernel of both T_1 and T_2, then P is also a (left, right) projection kernel of the products $T_1 \cdot T_2$ and $T_2 \cdot T_1$ as well as of the linear combinations $\lambda_1 T_1 + \lambda_2 T_2$ with $\lambda_1 + \lambda_2 = 1$.*

Proof: We assume that $P \in \mathbb{K}(\{T_1, T_2\})$. By the way, the proofs for $P \in \mathbb{K}_l(\{T_1, T_2\})$ and $P \in \mathbb{K}_r(\{T_1, T_2\})$ are included as parts. Since

$$P = PT_i = T_i P \quad (i = 1, 2)$$

holds, we have for $T := T_1 T_2$:

$$PT = PT_1 T_2 = PT_2 = P, \quad TP = T_1 T_2 P = T_1 P = P.$$

Hence T is a projection kernel of P. Analogously this can be shown for $T := T_2 T_1$. If $T := \lambda_1 T_1 + \lambda_2 T_2$ and $\lambda_1 + \lambda_2 = 1$, then

$$PT = P(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 PT_1 + \lambda_2 PT_2 = \lambda_1 P + \lambda_2 P = P.$$

Analogously $TP = P$ is proven for this T. ■

Corollary 2.1 *If P is a projection kernel of $\mathcal{T}$, then P is the projection kernel of the generated multiplicative semi-group $[\mathcal{T}](I)$ with identity I and of the affine hull* $\mathrm{aff}(\mathcal{T})$.

The next statement considers the aspect of regular (invertible) transformations in $\mathcal{L}(X)$.

Proposition 2.4 *Let S be regular. If P is a (left, right) projection kernel of $\mathcal{T}$, then P_S is a (left, right) projection kernel of $\mathcal{T}_S$, where $P_S := S^{-1}PS$ and $\mathcal{T}_S := S^{-1}\mathcal{T}S$.*

Proof: Under the given assumptions it is

$$P_S T_S = S^{-1}PS \cdot S^{-1}TS = S^{-1}PTS = S^{-1}PS = P_S,$$
$$T_S P_S = S^{-1}TS \cdot S^{-1}PS = S^{-1}TPS = S^{-1}PS = P_S$$

for all $T_S \in \mathcal{T}_S$. ■

Proposition 2.5 *If P is a (left, right) projection kernel of $\mathcal{T}$, then the dual (adjoint) P^* is a (right, left) projection kernel of $\mathcal{T}^*$.*

Proof: The assertion follows from the equations

$$(P^2)^* = (P^*)^2, \quad (PT)^* = T^* P^*, \quad (TP)^* = P^* T^*. \ ■$$

Now examples for projection kernels of operator sets are given.

Example 2.2 We consider $X = \mathrm{R}^n$ and the sets

$$\mathcal{T}_m = \left\{ \begin{pmatrix} I_{m,m} & O_{m,n-m} \\ O_{n-m,m} & T_{n-m,n-m} \end{pmatrix} \right\}, \quad \mathcal{P}_l = \left\{ \begin{pmatrix} I_{l,l} & O_{l,n-l} \\ O_{n-l,l} & O_{n-l,n-l} \end{pmatrix} \right\}$$

of matrices in $\mathrm{R}^{n,n}$, where l,m, n are natural numbers with $1 \leq l \leq m \leq n$ and m, n fixed. The indices indicate the size of the submatrices. Further, indexed I stands for identity submatrices and indexed O for zero submatrices. The matrices act as linear operators on R^n. The set $\mathcal{T}_m$ is a subring and a subalgebra of $\mathrm{R}^{n,n}$ containing the identity (matrix).
It is easy to check that each operator $P_l \in \mathcal{P}_l$ is a projection kernel of the set $\mathcal{T}_m$. Hence, there are different projection kernels for the same operator set.
Let us fix an operator $T_m = T \in \mathcal{T}_m$. Then each $P_l \in \mathcal{P}_l$ is also a projection kernel of the power sequence (T^k), where obviously $T^k \in \mathcal{T}_m$ for all $k \in \mathrm{N}$.

The example presents matrices in a canonical form. We can produce many other examples applying a regular matrix $S \in \mathrm{R}^{n,n}$ and its inverse, namely

$$\mathcal{P}_l^S = S^{-1}\mathcal{P}_l S, \quad \mathcal{T}_m^S = S^{-1}\mathcal{T}_m S$$

(see Proposition 2.4). Reversely, for an operator set $\mathcal{T}$ and a projection kernel P we can look for regular matrices S transforming the operators into a canonical form.

Example 2.3 Let us consider the matrices

$$P = \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \dots & \vec{e}_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathrm{R}^{n,n},$$

$$T = \begin{pmatrix} \vec{e}_1 & \vec{t}_2 & \dots & \vec{t}_n \end{pmatrix} = \begin{pmatrix} 1 & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & t_{n2} & \dots & t_{nn} \end{pmatrix} \in \mathrm{R}^{n,n} :$$

$$\sum_{i=1}^{n} t_{ij} = 1 \; (1 < j \leq n),$$

where $\vec{e}_1$ is the first column of the identity matrix $I \in \mathrm{R}^{n,n}$ and the sums of the columns $\vec{t}_j = (t_{ij})$ are equal to 1. Then P is a projection kernel and also an element of the set $\mathcal{T}$ of all such operators T. By the way, $\mathcal{T}$ is a noncommutative semi-group with respect to matrix multiplication. Further, $P^* = P^t$, the matrix with first row elements 1 and other elements 0, is a projection kernel of $\mathcal{T}^*$ whose operators T^* have the same first row as I and the (other) row sums are always equal to 1.

Example 2.4 Let X be a (real) Hilbert space and $T \in \mathcal{L}(X)$ nonexpansive. Then the orthoprojector P defined by $\mathbb{R}(P) = \mathbb{N}(I - T)$ is a projection kernel of T, its powers T^k ($k \in \mathrm{N}$) and their affine combinations (see Section 6 and Corollary 2.1).

3 Properties of projection kernels

Now we look for simple conditions to determine projection kernels. Obviously the relation $P \in \mathbb{K}(\mathcal{T})$ can be characterized by the behavior of operators $T \in \mathcal{T}$ on $\mathbb{R}(P)$ and $\mathbb{N}(P)$. We introduce the abbreviations

$$\mathbb{N}(I - \mathcal{T}) := \bigcap_{T \in \mathcal{T}} \mathbb{N}(I - T), \quad \overline{\mathbb{R}(I - \mathcal{T})} := \operatorname{span} \bigcup_{T \in \mathcal{T}} \mathbb{R}(I - T). \tag{3.1}$$

Both defined sets are (closed) linear subspaces. The set $\mathbb{N}(I - \mathcal{T})$ is the common fixed point set $\mathbb{F}(\mathcal{T})$ of $\mathcal{T}$. The span operation contains the closure of the corresponding set. In finite-dimensional spaces the closure operation can be omitted.

Lemma 3.1 *The following conditions are equivalent for an operator P and operators in a set $\mathcal{T}$:*

a1) $P = TP$ *for all* $T \in \mathcal{T}$,

b1) $T = P + T(I - P)$ *for all* $T \in \mathcal{T}$,

c1) $T \,|\, \mathbb{R}(P) = I \,|\, \mathbb{R}(P)$ *for all* $T \in \mathcal{T}$,

d1) $\mathbb{R}(P) \subseteq \mathbb{N}(I - \mathcal{T})$.

Proof: The equivalence of a1), b1) and c1) is obvious. Besides, a1) is fulfilled iff the equation $(I - T)P = O$, that means $\mathbb{R}(P) \subseteq \mathbb{N}(I - T)$, holds for all $T \in \mathcal{T}$. Hence also a1) and d1) are equivalent. ∎

Lemma 3.2 *The following conditions are equivalent for an operator P and operators in a set $\mathcal{T}$:*

a2) $P = PT$ *for all* $T \in \mathcal{T}$,

b2) $T = P + (I - P)T$ *for all* $T \in \mathcal{T}$,

d2) $\mathbb{N}(P) \supseteq \overline{\mathbb{R}(I - \mathcal{T})}$.

From each of these conditions follows

c2) $T\,\mathbb{N}(P) \subseteq \mathbb{N}(P)$ *for all* $T \in \mathcal{T}$.

Proof: The equivalence of a2) and b2) is obvious. Further a2) is fulfilled iff $P(I-T)=O$, that is $\mathbb{N}(P)\supseteq\mathbb{R}(I-T)$, holds for all $T\in\mathcal{T}$. Since $\mathbb{N}(P)$ is a closed linear subspace, also a2) and d2) are equivalent. Finally, supposing a2), $Px=0$ supplies $PTx=0$ for all $x\in X$. But this is c2). ■

Theorem 3.1 *The following conditions are equivalent for a projector P and operators in a set $\mathcal{T}$:*

a) $P=TP=TP$ *for all* $T\in\mathcal{T}$, *that is* $P\in\mathbb{K}(\mathcal{T})$,

a') $(I-T)P=P(I-T)=O$ *for all* $T\in\mathcal{T}$,

b) $T=I\,|\,\mathbb{R}(P)\oplus T\,|\,\mathbb{N}(P)$ *for all* $T\in\mathcal{T}$,

c) $T\,|\,\mathbb{R}(P)=I\,|\,\mathbb{R}(P)$, $T\,\mathbb{N}(P)\subseteq\mathbb{N}(P)$ *for all* $T\in\mathcal{T}$,

d) $\mathbb{R}(P)\subseteq\mathbb{N}(I-\mathcal{T})$ *and* $\mathbb{N}(P)\supseteq\overline{\mathbb{R}(I-\mathcal{T})}$.

Proof: A great part of the assertions is obtained by combination of Lemma 3.1 and Lemma 3.2. Thus a) and d) are equivalent. Further a) and a') are equivalent because a') can be written as $P-TP=P-PT=O$. Since P is a projector $T(I-P)=(I-P)T$ means that $\mathbb{R}(P)$ and $\mathbb{N}(P)$ are invariant linear subspaces of T. Now the equivalence of a), b) and c) is obvious. ■

The conditions b) and c) play an important part for considering convergence of operators. The condition d) is especially useful for determining suitable projection kernels.

Corollary 3.1 *If P is a projection kernel of $\mathcal{T}$, then all operators $T\in\mathcal{T}$ map for all $x\in X$ the affine subspaces $x+\mathbb{N}(P)$ into itself and the affine subspaces $x+\mathbb{R}(P)$ onto the affine subspaces $Tx+\mathbb{R}(P)$.*

Proof: Let be $P\in\mathbb{K}(\mathcal{T})$. First $(I-P)x\in N(P)$ because of $P^2=P$. Hence

$$x+\mathbb{N}(P)=Px+(I-P)x+\mathbb{N}(P)=Px+\mathbb{N}(P).$$

Having also Theorem 3.1 in mind, we get

$$\begin{aligned}T(x+\mathbb{N}(P))&=T(Px+\mathbb{N}(P))=TPx+T\,\mathbb{N}(P)\\&\subseteq Px+\mathbb{N}(P)=x+\mathbb{N}(P),\\T(x+\mathbb{R}(P))&=Tx+T\,\mathbb{R}(P)=Tx+TP\,\mathbb{R}(P)\\&=Tx+P\,\mathbb{R}(P)=Tx+\mathbb{R}(P).\ \blacksquare\end{aligned}$$

The corollary shows that the operators T map affine subspaces which are parallel to $\mathbb{R}(P)$ again into such subspaces. Further, all images Tx of x remain in the affine subspace $x+\mathbb{N}(P)$.

Relations between ranges and nullspaces of projectors can be used to define a semi-order between projectors.

Definition 3.1 *We write $P \leq Q$ for two projectors P and Q, if $\mathbb{R}(P) \subseteq \mathbb{R}(Q)$ and $\mathbb{N}(P) \supseteq \mathbb{N}(Q)$. We write $P < Q$ if $P \leq Q$ and $P \neq Q$.*

Proposition 3.1 *If P is a projection kernel of the projector Q, then $P \leq Q$ holds.*

Proof: The assumption $P \in \mathbb{K}(Q)$ implies by Theorem 3.1 the relations $\mathbb{R}(P) \subseteq \mathbb{R}(Q)$ and $\mathbb{N}(P) \supseteq \mathbb{N}(Q)$. By Definition 3.1 this is $P \leq Q$. ■

In Example 2.2 the projectors P_l fulfil the relations $P_l < P_{l+1}$ for $1 \leq l \leq n-1$. The following statement shows how we can construct 'smaller' and 'bigger' projection kernels.

Proposition 3.2 *If P_1 and P_2 are commutable projection kernels of $\mathcal{T}$, then $P = P_1P_2$ and $\tilde{P} = P_1 + P_2 - P_1P_2$ are projection kernels of $\mathcal{T}$ satisfying*

$$\begin{aligned} \mathbb{R}(P) &= \mathbb{R}(P_1) \cap \mathbb{R}(P_2), \quad \mathbb{N}(P) = \operatorname{span}\left(\mathbb{N}(P_1) \cup \mathbb{N}(P_2)\right) \\ \mathbb{R}(\tilde{P}) &= \operatorname{span}\left(\mathbb{R}(P_1) \cup \mathbb{R}(P_2)\right), \quad \mathbb{N}(\tilde{P}) = \mathbb{N}(P_1) \cap \mathbb{N}(P_2). \end{aligned}$$

This means $P \leq P_1, P_2 \leq \tilde{P}$ and $P < \tilde{P}$ for $P_1 \neq P_2$.

Proof: The first part is shown in [11], p. 33. The relations between ranges and nullspaces supply

$$\mathbb{R}(P) \subseteq \mathbb{R}(P_i) \subseteq \mathbb{R}(\tilde{P}), \quad \mathbb{N}(P) \supseteq \mathbb{N}(P_i) \supseteq \mathbb{N}(\tilde{P}) \quad (i = 1, 2).$$

Hence, the relations $P \leq P_1, P_2 \leq \tilde{P}$ follow by Definition 3.1. Finally we suppose $P_1 \neq P_2$. In contrary to the assertion we assume $P = \tilde{P}$. By the above relations we get $\mathbb{R}(P_1) = \mathbb{R}(P_2)$ and $\mathbb{N}(P_1) = \mathbb{N}(P_2)$. Proposition 2.1 shows that $P_1 = P_2$. This is a contradiction. Hence, $P < \tilde{P}$ is true. ■

4 Special kinds of projection kernels

If we investigate the convergence behavior of a operator sequence (T_k), we are interested in projection kernels P with maximal range $\mathbb{R}(P)$, where the operators T_k are the identity (see Theorem 3.1). Further, if the limit of (T_k) is P, then P is in the closure of $\{T_k : k \in \mathrm{N}\}$.

Definition 4.1 *Let P be a (left, right) projection kernel of $\mathcal{T}$. Then P is called*

- nontrivial *iff $P \neq O$,*
- maximal *iff there is no other projection kernel Q of $\mathcal{T}$ with $\mathbb{R}(Q) \supset \mathbb{R}(P)$,*

- orthogonal *iff P is an orthoprojector (in a* HILBERT *space X),*
- attainable *(r.t. operator topology τ), iff P is in the (τ-) closure of $\mathcal{T}$.*

Proposition 4.1 *Generally projection kernels P of an operator set $\mathcal{T}$ are neither uniquely determined nor maximal, orthogonal or attainable.*

This can be seen by the examples. The next example also exposes that the method of projection kernels has limitations.

Example 4.1 For $X = \mathrm{R}^3$ we investigate operators

$$T(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} : \quad c \in \mathrm{R}.$$

The matrices $T(c)$ have the determinant 1 and inverses $T(-c)$. Further, it holds

$$\mathbb{N}(I - T(c)) = \mathbb{N}(I - T(1)) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \ (c \neq 0),$$

$$\mathbb{N}(I - T(0)) = \mathbb{N}(O) = \mathrm{R}^3,$$

$$\mathbb{R}(I - T(c)) = \mathbb{R}(I - T(1)) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \ (c \neq 0),$$

$$\mathbb{R}(I - T(0)) = \mathbb{R}(O) = \{0\}.$$

The set $\mathcal{T}_0$ of all such operators $T(c)$ is a multiplicative commutative group. Now we consider subsets $\mathcal{T}$ only assuming $\mathcal{T} \setminus \{I\} \neq \emptyset$. Hence, $\mathcal{T}$ contains at least one operator $T(c)$ with $c \neq 0$. Then we get

$$\mathbb{N}(I - \mathcal{T}) = \bigcap_{T(c) \in \mathcal{T}} \mathbb{N}(I - T(c)) = \mathbb{N}(I - T(1)),$$

$$\overline{\mathbb{R}(I - \mathcal{T})} = \operatorname{span} \bigcup_{T(c) \in \mathcal{T}} \mathbb{R}(I - T(c)) = \mathbb{R}(I - T(1)).$$

It is easy to check that the set $\mathcal{P} = \mathbb{K}(\mathcal{T})$ of all projection kernels consists of the matrices

$$P(a, b) = \begin{pmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ ab & b & 0 \end{pmatrix} : \quad a, b \in \mathrm{R}.$$

The ranges and nullspaces are

$$\mathbb{R}(P(a,b)) = \text{span}\left\{\begin{pmatrix}0\\1\\b\end{pmatrix}\right\},\ \mathbb{N}(P(a,b)) = \text{span}\left\{\begin{pmatrix}1\\-a\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

These results show the relations

$$\mathbb{R}(P(a,b)) \subset \mathbb{N}(I-\mathcal{T}), \quad \mathbb{N}(P(a,b)) \supset \mathbb{R}(I-\mathcal{T})$$

such that condition d) in Theorem 3.1 is fulfilled properly, not reaching set equality. For all operators $P(a,b)$ the range is one-dimensional. Hence all these projection kernels are nontrivial and even maximal. Since $\mathcal{T}$ is a set whose closure does not contain operators of $\mathcal{P}$, all projection kernels are not attainable. The power sequence $(T(1)^k) = (T(k))$ is divergent, but $(T_k) = (T(0.5^k))$ tends to I, which is no projection kernel. The constant sequence $(T_k) = (T(1))$ tends to $T(1)$ which is even no projector $(T(1)^2 = T(2) \neq T(1))$. Further, only the projection kernel $P(0,0)$ is orthogonal (self-adjoint).

We turn to the question if always maximal projection kernels exist.

Theorem 4.1 *Each set $\mathcal{T}$ of operators with finite-dimensional subspace $\mathbb{N}(I-\mathcal{T})$ has at least one maximal projection kernel.*

Proof: Because of $O \in \mathbb{K}(\mathcal{T})$ it is $\mathbb{K}(\mathcal{T}) \neq \emptyset$. For $P \in \mathbb{K}(\mathcal{T})$ it holds $\mathbb{R}(P) \subseteq \mathbb{N}(I-\mathcal{T})$ and therefore $\dim \mathbb{R}(P) \leq \dim \mathbb{N}(I-\mathcal{T}) =: n < \infty$. Hence, there is a $\tilde{P} \in \mathbb{K}(\mathcal{T})$ with $n \geq k := \dim \mathbb{R}(\tilde{P}) \geq \dim \mathbb{R}(P)$ for all $P \in \mathbb{K}(\mathcal{T})$. This $\tilde{P}$ is maximal, since the assumption $\mathbb{R}(P) \supset \mathbb{R}(\tilde{P})$ leads to the contradiction $\dim \mathbb{R}(P) > \dim \mathbb{R}(\tilde{P})$. ■

Condition d) of Theorem 3.1 is of special importance for convergence, if equality of the sets is reached in the subset relation. Remember that this was not the case in Example 4.1.

Definition 4.2

- *A (right) projection kernel P of $\mathcal{T}$ is said to be* right optimal *if* $\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T})$.
- *A (left) projection kernel P of $\mathcal{T}$ is said to be* left optimal *if* $\mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})}$.
- *A projection kernel P of $\mathcal{T}$ is said to be* optimal *if* $\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T})$ *as well as* $\mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})}$.

Theorem 4.2 *If P_s is a (left, right) optimal projection kernel of $\mathcal{T}$, then P_s is maximal.*

Proof: a) Let us assume that $P_s \in \mathbb{K}(\mathcal{T})$ is right optimal. Then $\mathbb{R}(P_s) = \mathbb{N}(I-\mathcal{T})$. Since $\mathbb{R}(P) \subseteq \mathbb{N}(I-\mathcal{T})$ for all $P \in \mathbb{K}(\mathcal{T})$ by Theorem 3.1, this P_s is maximal.

b) Let us assume that $P_s \in \mathbb{K}(\mathcal{T})$ is left optimal. Then $\mathbb{N}(P_s) = \overline{\mathbb{R}(I - \mathcal{T})}$. Supposing that P_s is not maximal there is a $\tilde{P} \in \mathbb{K}(\mathcal{T})$ with $\mathbb{R}(\tilde{P}) \supset \mathbb{R}(P_s)$. Considering again Theorem 3.1 it is also $\mathbb{N}(\tilde{P}) \supseteq \mathbb{N}(P_s)$. Since both $\tilde{P}$ and P_s are projectors we get

$$\mathbb{R}(\tilde{P}) \oplus \mathbb{N}(\tilde{P}) \supset X = \mathbb{R}(P_s) \oplus \mathbb{N}(P_s).$$

This is a contradiction. ■

Theorem 4.3 *The operator set $\mathcal{T}$ has an optimal projection kernel iff*

$$X = \mathbb{N}(I - \mathcal{T}) \oplus \overline{\mathbb{R}(I - \mathcal{T})}.$$

In this case the projector P with $\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T})$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})}$ is the optimal and also the unique maximal projection kernel of $\mathcal{T}$.

Proof: a) Let P be an optimal projection kernel of $\mathcal{T}$. Then $\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T})$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})}$ by definition. Hence

$$X = \mathbb{R}(P) \oplus \mathbb{N}(P) = \mathbb{N}(I - \mathcal{T}) \oplus \overline{\mathbb{R}(I - \mathcal{T})}.$$

b) Let be $X = \mathbb{N}(I - \mathcal{T}) \oplus \mathbb{R}(I - \mathcal{T})$. Then we consider the projector P with $\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T})$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})}$. Consequently, P is an optimal projection kernel of $\mathcal{T}$ by definition. By Theorem 4.2 this P is also maximal. Assuming another optimal or maximal projection kernel $\tilde{P} \neq P$ we would get $\mathbb{R}(\tilde{P}) = \mathbb{R}(P)$ and $\mathbb{N}(\tilde{P}) = \mathbb{N}(P)$. For projectors this means $\tilde{P} = P$ by Proposition 2.1 in contradiction with the assumption. ■

Remark 4.1 The optimal projection kernel P of $\mathcal{T}$ is shortly denoted by $P = \mathbb{K}_o(\mathcal{T})$. If $\mathcal{T}$ has more than one maximal projection kernel, then $\mathcal{T}$ has no optimal projection kernel ($\mathbb{K}_o(\mathcal{T}) = \emptyset$). This shows that the set $\mathcal{T}$ in Example 4.1 has no optimal projection kernel. If P is the optimal projection kernel of $\mathcal{T}$, where $\mathcal{T} \setminus \{I\} \neq \emptyset$, then $I - P$ is the projector with $\mathbb{R}(I - P) = \overline{\mathbb{R}(I - \mathcal{T})}$ and $\mathbb{N}(I - P) = \mathbb{N}(I - \mathcal{T})$. This projector is no projection kernel of $\mathcal{T}$ (see Proposition 2.2).

5 Optimal projection kernels

Now we consider optimal projection kernels of an operator set in more detail.

Lemma 5.1 *If P is the optimal projection kernel of $\mathcal{T}_1$ and a projection kernel of $\mathcal{T}_2$, then P is the optimal projection kernel of $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2$.*

Proof: By Theorem 3.1 we have

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}_1) \subseteq \mathbb{N}(I - \mathcal{T}_2), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T}_1)} \supseteq \overline{\mathbb{R}(I - \mathcal{T}_2)}.$$

It follows

$$\mathbb{N}(I-\mathcal{T}) = \mathbb{N}(I-\mathcal{T}_1)\cap\mathbb{N}(I-\mathcal{T}_2) = \mathbb{N}(I-\mathcal{T}_1) = \mathbb{R}(P),$$
$$\overline{\mathbb{R}(I-\mathcal{T})} = \overline{\operatorname{span}(\mathbb{R}(I-\mathcal{T}_1)}\cup\overline{\mathbb{R}(I-\mathcal{T}_2))} = \overline{\mathbb{R}(I-\mathcal{T}_1)} = \mathbb{N}(P).$$

This is the assertion. ■

Corollary 5.1 *If P is a projection kernel of $\mathcal{T}$ and $P\in\mathcal{T}$, then P is the optimal projection kernel of $\mathcal{T}$.*

Proof: By Proposition 2.1 it holds

$$X = \mathbb{R}(P)\oplus\mathbb{N}(P) = \mathbb{N}(I-P)\oplus\mathbb{R}(I-P).$$

Hence P is the optimal projection kernel of itself. Since P is a projection kernel of $\mathcal{T}$, then P is the optimal projection kernel of $\mathcal{T}\cup\{P\} = \mathcal{T}$ by Lemma 5.1. ■

Lemma 5.2 *If P is the optimal projection kernel of $\mathcal{T}$, then P is the optimal projection kernel of the generated semi-group $[\mathcal{T}](I)$ with identity I and of the affine hull $\operatorname{aff}(\mathcal{T})$.*

Proof: Let $P = \mathbb{K}_o(\mathcal{T})$. Consequently $P\in\mathbb{K}(\mathcal{T})$. By Corollary 2.1 we have $P\in\mathbb{K}([\mathcal{T}](I))$ and $P\in\mathbb{K}(\operatorname{aff}(\mathcal{T}))$. Since $\mathcal{T}_1 := \mathcal{T}$ is a subset of both $\mathcal{T}_2 := [\mathcal{T}](I)$ and $\mathcal{T}_3 := \operatorname{aff}(\mathcal{T})$ the assertion follows now by Lemma 5.1. ■

Corollary 5.2 *An operator T as well as the corresponding sets $\mathcal{T} := \{T^k : k\in\mathrm{N}\}$ and $\mathcal{S} := \operatorname{aff}(\mathcal{T})$ have an optimal projection kernel iff*

$$X = \mathbb{N}(I-T)\oplus\overline{\mathbb{R}(I-T)}.$$

In this case the projector P with $\mathbb{R}(P) = \mathbb{N}(I-T)$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I-T)}$ is the optimal projection kernel of T, $\mathcal{T}$ and $\mathcal{S}$.

Proof: The operators $S\in\mathcal{S}$ have the representations

$$S = S_k(T) = \sum_{i=0}^{k}\alpha_i T^i, \quad \sum_{i=0}^{k}\alpha_i = 1.$$

Considering the coefficient relation of the α_i, each polynomial $P_k(\lambda) := 1 - S_k(\lambda)$ has the zero 1. Hence, in each operator $I-S$ a factor $I-T$ can be separated. This implies

$$\mathbb{N}(I-T)\subseteq\mathbb{N}(I-S), \quad \mathbb{R}(I-T)\supseteq\mathbb{R}(I-S)$$

for all $S\in\mathcal{S}$. Observing $\{T\}\subseteq\mathcal{T}\subseteq\mathcal{S}$ we obtain

$$\mathbb{N}(I-T) = \mathbb{N}(I-\mathcal{T}) = \mathbb{N}(I-\mathcal{S}), \quad \overline{\mathbb{R}(I-T)} = \mathbb{R}(I-\mathcal{T}) = \mathbb{R}(I-\mathcal{S}).$$

Now Theorem 4.3 shows the assertions. ■

Remark 5.1 In Corollary 5.2 the space decomposition

$$X = \mathbb{N}(I - T) \oplus \overline{\mathbb{R}(I - T)}$$

occurs. Operators $I - T$ with this property are called *decomposition regular* (short: d-regular) in [14]. This paper contains more material about such operators. The given decomposition of X is also necessary for the convergence of (T^k) (see Section 9).

Example 5.1 Now we come back to Example 2.2 discussing matrices

$$T_m = \begin{pmatrix} I_{m,m} & O_{m,n-m} \\ O_{n-m,m} & T_{n-m,n-m} \end{pmatrix}, \quad P_l = \begin{pmatrix} I_{l,l} & O_{l,n-l} \\ O_{n-l,l} & O_{n-l,n-l} \end{pmatrix},$$
$$1 \le l \le m \le n$$

in $\mathrm{R}^{n,n}$, where the corresponding operators P_l are stated to be projection kernels of the corresponding operators T_m. Let us choose $m < n$. Further let $\mathcal{T}$ be a set of matrices T_m, where at least one $T_m = \hat{T}_m$ has rank n. Then, using the coordinate unit vectors $\vec{e}_i$ $(i = 1, 2, \ldots, n)$, in other words the columns of the identity $I_{n,n}$, and the linear subspaces

$$\mathrm{V}_{i,j} := \operatorname{span}\{\vec{e}_i, \ldots, \vec{e}_j\}, \quad 1 \le i \le j \le n,$$

we get for T_m the relations

$$\mathbb{R}(I - T_m) \subseteq \mathbb{R}(I - \hat{T}_m) = \mathrm{V}_{m+1,n}, \quad \mathbb{N}(I - T_m) \supseteq \mathbb{N}(I - \hat{T}_m) = \mathrm{V}_{1,m}$$

and finally for the set $\mathcal{T}$ the result

$$\overline{\mathbb{R}(I - \mathcal{T})} = \mathrm{V}_{m+1,n}, \quad \mathbb{N}(I - \mathcal{T}) = \mathrm{V}_{1,m}.$$

Further, it is

$$\mathbb{R}(P_l) = \mathrm{V}_{1,l}, \quad \mathbb{N}(P_l) = \mathrm{V}_{l+1,n}, \quad l \le m.$$

Hence, we have a chain of orthogonal projection kernels P_l, where the maximal one, namely P_m, is the optimal one. Moreover, P_m is attainable iff $O_{n-m,n-m}$ is in the closure of the set of submatrices $T_{n-m,n-m}$ belonging to the matrices $T_m \in \mathcal{T}$. For instance, this is the case if $\mathcal{T}$ consists of all possible T_m, because $O_{n-m,n-m}$ is a submatrix of $P_m \in \mathcal{T}$.

Example 5.2 It is interesting to discuss Example 2.3 in more detail. There is stated that the multiplicative semi-group $\mathcal{T}$ of all matrices

$$T = \begin{pmatrix} \vec{e}_1 & \vec{t}_2 & \ldots & \vec{t}_n \end{pmatrix} \in \mathrm{R}^{n,n} : \quad \sum_{i=1}^{n} t_{ij} = 1 \quad (1 < j \le n)$$

has the projection kernel

$$P = \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \ldots & \vec{e}_1 \end{pmatrix} \in \mathcal{T}.$$

It can be shown that

$$\mathbb{R}(P) = \text{span}\{\vec{e}_1\}, \quad \mathbb{N}(P) = \text{span}\{\vec{e}_2 - \vec{e}_1,\ \vec{e}_3 - \vec{e}_1,\ \dots,\ \vec{e}_n - \vec{e}_1\}.$$

The nullspace of P contains all vectors with coordinate sums 0. Further, each vector $\vec{x} \in \mathbb{N}(P)$ has the basis representation

$$\vec{x} = x_2\,(\vec{e}_2 - \vec{e}_1) + x_3\,(\vec{e}_3 - \vec{e}_1) + \ldots + x_n\,(\vec{e}_n - \vec{e}_1).$$

The matrices $S := I - T$ have the form

$$\begin{pmatrix} \vec{0} & \vec{s}_2 & \dots & \vec{s}_n \end{pmatrix} \in \mathrm{R}^{n,n} : \quad \sum_{i=1}^{n} s_{ij} = 0 \quad (1 < j \le n).$$

Since P is a projection kernel of all operators T, we have by Theorem 3.1

$$\mathbb{R}(I - T) = \mathbb{R}(S) \subseteq \mathbb{N}(P), \quad \mathbb{N}(I - T) = \mathbb{N}(S) \supseteq \mathbb{R}(P).$$

Indeed, these relations are also a consequence of the above results. Now we consider a subset $\mathcal{T}_s$ of $\mathcal{T}$. If

$$\dim \mathbb{R}(I - \mathcal{T}_s) = n - 1,$$

then it holds

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}_s), \quad \mathbb{N}(P) = \mathbb{R}(I - \mathcal{T}_s).$$

Hence, P is the optimal projection kernel of $\mathcal{T}_s$. Especially this is the case if there is a matrix $\hat{T} \in \mathcal{T}_s$ with rank $(I - \hat{T}) = n - 1$. Such a matrix is

$$\hat{T} = \begin{pmatrix} \vec{e}_1 & 2\vec{e}_2 - \vec{e}_1 & 2\vec{e}_3 - \vec{e}_1 & \dots & 2\vec{e}_n - \vec{e}_1 \end{pmatrix} \in \mathcal{T}$$

with full rank n since

$$I - \hat{T} = \begin{pmatrix} \vec{0} & \vec{e}_1 - \vec{e}_2 & \vec{e}_1 - \vec{e}_3 & \dots & \vec{e}_1 - \vec{e}_n \end{pmatrix}$$

has indeed rank $n - 1$. This means also that P is the optimal projection kernel of $\mathcal{T}$. Additionally P is then attainable, because $P \in \mathcal{T}$. The operator $\hat{T}$ has interesting properties, for instance

$$(I - \hat{T})^2 = -(I - \hat{T}) = I - P.$$

Hence $\hat{T} - I$ is a projector. But observe that $I - \hat{T}$ and $I - P$ are not in $\mathcal{T}$ and are no projection kernels of $\mathcal{T}$. A simple consideration shows

$$\hat{T}^n = P - 2^n(I - \hat{T}) \in \mathcal{T}$$

for all integers n. Hence, the sequence $(\hat{T}^n)$ is divergent (for natural n) while the sequence $(\hat{T}^{-n})$ of the inverses converges to P. Since

$$I - \hat{T}^{-1} = -\frac{1}{2}\,(I - \hat{T})$$

has also rank $n - 1$, the sequence $(\hat{T}^{-n})$ has the optimal and attainable projection kernel P.

Example 5.3 Let us investigate a more general approach to the set $\mathcal{T}$ of all matrices

$$T = \begin{pmatrix} \vec{e}_1 & \vec{t}_2 & \dots & \vec{t}_n \end{pmatrix} \in \mathrm{R}^{n,n} : \quad \sum_{i=1}^{n} t_{ij} = 1 \quad (1 < j \leq n)$$

just investigated in Example 5.2. If subsets $\mathcal{T}_s$ of $\mathcal{T}$ are considered, then possibly P is not the optimal projection kernel. On the other hand, if $Q \in \mathcal{T}$ is any projector, then we can find subsets with Q as an optimal projection kernel. Let us look at the special projectors

$$P_k = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_k & \vec{e}_1 & \dots & \vec{e}_1 \end{pmatrix} \in \mathcal{T} \quad (1 \leq k < n).$$

For $k = 1$ we have $P_k = P$ (see Example 5.2). The case $k = n$ supplies $P_n = I$ which is not of interest. For arbitrary k we get

$$\mathbb{R}(P_k) = \mathrm{span}\,\{\vec{e}_1,\ \vec{e}_2,\ \dots,\ \vec{e}_k\},$$
$$\mathbb{N}(P_k) = \mathrm{span}\,\{\vec{e}_1 - \vec{e}_{k+1},\ \vec{e}_1 - \vec{e}_{k+2},\ \dots,\ \vec{e}_1 - \vec{e}_n\}.$$

The sets $\mathcal{T}_k$ of matrices

$$T_k = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_k & \vec{t}_{k+1} & \dots & \vec{t}_n \end{pmatrix},$$
$$\vec{e}_i - \vec{t}_i \in \mathbb{N}(P_k) \quad (1 \leq k < n,\ k+1 \leq i \leq n)$$

are again semi-groups of operators containing P_k. Now $\mathcal{T}_k$ has the projection kernel P_k because of

$$I - T_k = \begin{pmatrix} \vec{0} & \vec{0} & \dots & \vec{0} & \vec{e}_{k+1} - \vec{t}_{k+1} & \dots & \vec{e}_n - \vec{t}_n \end{pmatrix}$$

and

$$\mathbb{R}(P_k) \subseteq \mathbb{N}(I - T_k), \quad \mathbb{N}(P_k) \supseteq \mathbb{R}(I - T_k).$$

There are special matrices

$$\hat{T}_k = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_k & 2\vec{e}_{k+1} - \vec{e}_1 & \dots & 2\vec{e}_n - \vec{e}_1 \end{pmatrix}$$

with

$$I - \hat{T}_k = \begin{pmatrix} \vec{0} & \vec{0} & \dots & \vec{0} & \vec{e}_1 - \vec{e}_{k+1} & \dots & \vec{e}_1 - \vec{e}_n \end{pmatrix}$$

and

$$\mathbb{R}(P_k) = \mathbb{N}(I - \hat{T}_k), \quad \mathbb{N}(P_k) = \mathbb{R}(I - \hat{T}_k).$$

Hence, P_k is the optimal projection kernel of $\mathcal{T}_k$. Besides, the relations

$$\mathcal{T} \supseteq \mathcal{T}_k \supset \mathcal{T}_{k+1}, \quad \mathbb{R}(P) \subseteq \mathbb{R}(P_k) \subset \mathbb{R}(P_{k+1}), \quad \mathbb{N}(P) \supseteq \mathbb{N}(P_k) \supset \mathbb{N}(P_{k+1})$$

are fulfilled.

Let X be a (real) HILBERT space. We turn to optimal projection kernels which are orthogonal.

Proposition 5.1 *Let $\mathcal{T}$ possess the optimal projection kernel P. Then the following conditions are equivalent:*

a) *P is orthogonal ($P = P^*$),* b) $\mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T}^*)}$, c) $\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}^*)$.

Proof: Using Theorem 4.3 the assumption $P \in \mathbb{K}_o(\mathcal{T})$ implies

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})}.$$

Then it follows

$$\mathbb{N}(P^*) = \mathbb{R}(P)^\perp = \mathbb{N}(I - \mathcal{T})^\perp = \overline{\mathbb{R}(I - \mathcal{T}^*)},$$
$$\mathbb{R}(P^*) = \mathbb{N}(P)^\perp = \overline{\mathbb{R}(I - \mathcal{T})}^\perp = \mathbb{N}(I - \mathcal{T}^*).$$

Because of the equivalences

$$P = P^* \quad \Leftrightarrow \quad \mathbb{N}(P) = \mathbb{N}(P^*) \quad \Leftrightarrow \quad \mathbb{R}(P) = \mathbb{R}(P^*)$$

the assertion is true. ■

Self-adjoint operators $T = T^*$ trivially satisfy $\mathbb{N}(I - T) = \mathbb{N}(I - T^*)$. But all operators with this property have an outstanding property.

Theorem 5.1 *If the operators T in $\mathcal{T}$ have the property $\mathbb{N}(I - T) = \mathbb{N}(I - T^*)$, then $\mathcal{T}$ has an orthogonal optimal projection kernel, namely the orthoprojector P with*

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})} = \overline{\mathbb{R}(I - \mathcal{T}^*)}.$$

Proof: It is known that the linear subspaces

$$\mathbb{N} := \mathbb{N}(I - \mathcal{T}), \quad \mathbb{R} := \overline{\mathbb{R}(I - \mathcal{T}^*)}$$

are orthogonal complements (see e.g. [13]). Hence an orthogonal projector P is defined by $R(P) = \mathbb{N}$ and $\mathbb{N}(P) = \mathbb{R}$. The property $\mathbb{N}(I - T) = \mathbb{N}(I - T^*)$ implies

$$\overline{\mathbb{R}(I - T)} = \mathbb{N}(I - T^*)^\perp = \mathbb{N}(I - T)^\perp = \overline{\mathbb{R}(I - T^*)}.$$

This means also

$$\overline{\mathbb{R}(I - \mathcal{T}^*)} = \overline{\mathbb{R}(I - \mathcal{T})}.$$

Now Theorem 4.3 shows that P is the optimal projection kernel of $\mathcal{T}$. ■

Remark 5.2 The property $\mathbb{N}(I-T)=\mathbb{N}(I-T^*)$ is equivalent to the orthogonal decomposition

$$X=\mathbb{N}(I-T)\oplus\overline{\mathbb{R}(I-T)},\quad \overline{\mathbb{R}(I-T)}=\mathbb{N}(I-T)^{\perp},$$

considering the orthogonality relation $\mathbb{N}(I-T^*)^{\perp}=\overline{\mathbb{R}(I-T)}$ (see e.g. [13]). Not only self-adjoint, but also *normal operators* T, defined by the commutation relation $TT^*=T^*T$, have the property $\mathbb{N}(I-T)=\mathbb{N}(I-T^*)$ (see e.g. [21], p. 331 – 332). Moreover all products $T=P_kP_{k-1}\dots P_1$ of orthoprojectors P_i $(i=1,2,\dots,k)$ fulfil this condition. Here it is

$$\begin{aligned}\mathbb{N}(I-T)=\mathbb{N}(I-P_kP_{k-1}\dots P_1)&=\bigcap_{i=1}^{k}\mathbb{R}(P_i)=\bigcap_{i=1}^{k}\mathbb{R}(P_i^*)\\&=\mathbb{N}(I-P_1^*\dots P_{k-1}^*P_k^*)=\mathbb{N}(I-T^*).\end{aligned}$$

Moreover, T is nonexpansive. In [17], p. 183f. a more general result is proven, namely if so-called relaxations T_i of orthoprojectors P_i replace P_i. In section 6 we will see that arbitrary nonexpansive operators T satisfy $\mathbb{N}(I-T)=\mathbb{N}(I-T^*)$ (see Remark 6.1 and the text after it).

Finally, the operators $T\in\mathcal{T}$ itself can be orthoprojectors.

Corollary 5.3 *If the operators $T\in\mathcal{T}$ are orthoprojectors ($T^2=T=T^*$), then $\mathcal{T}$ has an orthogonal optimal projection kernel, namely the orthoprojector P with*

$$\mathbb{R}(P)=\bigcap_{T\in\mathcal{T}}\mathbb{R}(T),\quad \mathbb{N}(P)=\operatorname{span}\bigcup_{T\in\mathcal{T}}\mathbb{N}(T).$$

Proof: If the operators T are orthoprojectors, then we get

$$\mathbb{R}(T)=\mathbb{N}(I-T)=\mathbb{N}(I-T^*),\quad \mathbb{N}(T)=\mathbb{R}(I-T)=\mathbb{R}(I-T^*),$$

where both $\mathbb{R}(T)$ and $\mathbb{N}(T)$ are closed. This means also

$$\mathbb{N}(I-\mathcal{T})=\bigcap_{T\in\mathcal{T}}\mathbb{R}(T),\quad \overline{\mathbb{R}(I-\mathcal{T})}=\operatorname{span}\bigcup_{T\in\mathcal{T}}\mathbb{N}(T).$$

Now the assertion follows if we take Theorem 5.1 into account. ■

The assumptions and hence the assertions of Corollary 5.3 are fulfilled in Example 1.2 for the set $\mathcal{T}$ of sequence members T_k and in Example 1.3 for the set $\mathcal{T}$ of sequence members S_k. It turns out that these assertions also hold for a bigger class than that of orthoprojectors. We follow this topic in Section 6.

Proposition 5.2 *Let $\mathcal{T}$ be a set of orthoprojectors and P be a further orthoprojector. Then the following conditions are equivalent:*

a) *P is a projection kernel of $\mathcal{T}$.*

b) *P is a left projection kernel of $\mathcal{T}$.*

c) *P is a right projection kernel of $\mathcal{T}$.*

Proof: Obviously the conditions b) and c) follow from condition a). Now we want to show the reversions. Starting with the relations

$$P^2 = P, \quad P = PT, \quad P = TP \quad \text{for all } T \in \mathcal{T}$$

the transition to the adjoint operators supplies

$$(P^*)^2 = P^*, \quad P^* = T^*P^*, \quad P^* = P^*T^* \quad \text{for all } T \in \mathcal{T}.$$

Observing the assumptions $P^* = P$ and $T^* = T$ for all $T \in \mathcal{T}$ we get correspondingly

$$P^2 = P, \quad P = TP, \quad P = PT \quad \text{for all } T \in \mathcal{T}.$$

Hence, a left (right) projection kernel of $\mathcal{T}$ is also a right (left) projection kernel of $\mathcal{T}$ and consequently also a projection kernel of $\mathcal{T}$. ■

6 Nonexpansive operators and projection kernels

Let X be a (real) Hilbert space. Nonexpansive operators play an important part in the fixed point theory. Here we study the linear case.

Definition 6.1 *A linear operator T is called*

a) nonexpansive, *if $\|Tx\| \leq \|x\|$ for all x.*

b) isometric, *if $\|Tx\| = \|x\|$ for all x.*

c) contractive, *if $\|Tx\| \leq k\,\|x\|$ for all x and a number $k < 1$.*

d) Fejér monotone, *if $\|Tx\| < \|x\|$ for all $x \notin \mathbb{N}(I-T)$.*

e) strongly Fejér monotone, *if $\|Tx\| \leq k\,\|x\|$ for all $x \in \mathbb{N}(I-T)^{\perp}$ and a number $k < 1$.*

These concepts are also defined for nonlinear operators (see e.g. [4], [24], [22]). In my papers [17] and [18] linear (strongly) Fejér monotone operators are called (strong) *relaxations*. The concepts are renamed to get a better coordination between linear and nonlinear theory.

Remark 6.1 Nonexpansive operators T are characterized by $\|T\| \le 1$. They induce via $I - T$ the orthoprojector $P = P(T)$, where

$$X = \mathbb{R}(P) \oplus \mathbb{N}(P), \quad \mathbb{R}(P) \perp \mathbb{N}(P),$$
$$\mathbb{R}(P) = \mathbb{N}(I - T) = \overline{R(I - T)}^{\perp} = \mathbb{N}(I - T^*),$$
$$\mathbb{N}(P) = \overline{R(I - T)} = \mathbb{N}(I - T)^{\perp} = \overline{R(I - T^*)}$$

is the corresponding decomposition of X (see [17]: p. 182). By Corollary 5.2 and the property $\mathbb{R}(P) \perp \mathbb{N}(P)$ the projector P is the orthogonal optimal projection kernel of T. Moreover, $\mathbb{N}(P) = \mathbb{N}(I-T)^{\perp}$ is an invariant linear subspace under T (for another proof see [17]: p. 180). Further

$$\|T - P\| = \|T(I - P)\| = \|T \,|\, \mathbb{R}(I - P)\| = \|T \,|\, \mathbb{N}(P)\| =: \nu \le 1,$$

where the number ν measures the deviation of T from P. We call $P = P(T)$ also the *eigenprojection* of T.

Theorem 6.1 *If $\mathcal{T}$ consists of nonexpansive operators T, then $\mathcal{T}$ has an orthogonal optimal projection kernel, namely the orthoprojector P with*

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})} = \overline{\mathbb{R}(I - \mathcal{T}^*)}.$$

Proof: By Remark 6.1 nonexpansive operators T satisfy $\mathbb{N}(I - T) = \mathbb{N}(I - T^*)$. Hence, the assertion follows immediately from Theorem 5.1. ■

The set of all nonexpansive operators is a multiplicative semi-group with identity. This set can be divided again into semi-groups of nonexpansive operators with the same eigenprojection P.

Isometric operators T satisfy $\|T\| = 1$. Contractive operators T are norm reducing for $x \ne 0$ and fulfil $\|T\| \le k < 1$.

Fejér monotone operators T are nonexpansive, but not isometric, since they are norm reducing outside their fixed point sets $\mathbb{N}(I - T)$. Strongly Fejér monotone operators T are contractive on the invariant subspace $\mathbb{N}(I - T)^{\perp}$ (see Definition 6.1). Here it is

$$\nu = \nu(T) := \|T \,|\, \mathbb{N}(I - T)^{\perp}\| \le k < 1.$$

For $\mathbb{N}(I - T) = \{0\}$ strongly Fejér monotone operators and contractive operators coincide. In [17] and [18] a (strongly) Fejér monotone T with eigenprojection P is said to be a (strong) relaxation of its carrier P. In [18] you can find an example of T which is Fejér monotone, but not strong.

Since Fejér monotone operators play an important part in a certain class of iterative solution methods (see [16]), we mention some further facts about them.

Example 6.1 Let $P \neq I$ be an orthoprojector. Then the operators

$$T = (1-\lambda)I + \lambda P, \quad |1-\lambda| < 1$$

are self-adjoint strongly Fejér monotone operators with the same eigenprojection P (*scalar relaxations*). Here it is $\nu(T) = |1-\lambda|$.

If T is in one of the operator classes of Definition 6.1, then the same is true for T^*. We state this for one class.

Proposition 6.1 ([17]: p. 182, [18]: p. 33, p. 37) *T is (strongly) Fejér monotone iff T^* is (strongly) Fejér monotone. Thereby both have the same eigenprojection.*

Proposition 6.2 ([17]: p. 184, [18]: p. 40) *If T is (strongly) Fejér monotone, then T^k, TT^* and T^*T are (strongly) Fejér monotone with the same eigenprojection as T.*

Theorem 6.2 ([17]: p. 183) *Let $\mathcal{T}$ be a set of Fejér monotone operators T with eigenprojections $P = P(T)$. Then each projection kernel of $\mathcal{T}$ is a projection kernel of $\mathcal{P} := \{P = P(T) : T \in \mathcal{T}\}$ and vice versa.*

This statement also holds if $\mathcal{T}$ is a set of nonexpansive operators. The proof is the same as in the paper [17].

7 Attainable projection kernels

For simplicity we use in $\mathcal{L}(X)$ the strong operator topology τ_s. Then $\mathcal{L}(X)$ becomes a local convex topological vector space (see e.g. [23]: p. 110). But, corresponding results can also be obtained for the uniform and the weak operator topology.
First we investigate the relation between attainable and optimal projection kernels. In this section we use generalized sequences $(T_\alpha)_{\alpha\in J}$ of operators, where α is an element of an index set J. Shortly we write (T_α). Such sequences are also called Moore-Smith sequences. Further, we use the notation $\mathcal{T}_J := \{T_\alpha : \alpha \in J\}$ for the set of sequence members.

Lemma 7.1 *If $\mathcal{T} \subseteq \mathcal{L}(X)$ and $S \in \overline{\mathcal{T}}$, then*

$$\mathbb{N}(S) \subseteq \mathbb{R}(I-S) \subseteq \overline{\mathbb{R}(I-\mathcal{T})}, \quad \mathbb{R}(S) \supseteq \mathbb{N}(I-S) \supseteq \mathbb{N}(I-\mathcal{T}).$$

Proof: Since S belongs to $\overline{\mathcal{T}}$, there is a generalized sequence (T_α) in $\mathcal{T}$ with limit S (see e.g. [24]: p. 205). Further, we have for arbitrary $x \in X$ and all $\alpha \in J$

$$(I-T_\alpha)\,x \in \mathbb{R} := \overline{\mathbb{R}(I-\mathcal{T})}.$$

Hence, it holds also

$$(I - S)\,x = \lim_{\alpha}(I - T_\alpha)\,x \in \mathbb{R}$$

because $\mathbb{R}$ is closed. Moreover, we get

$$(I - S)\,x = \lim_{\alpha}(I - T_\alpha)\,x = 0 \quad \text{for all } x \in \mathbb{N} := \mathbb{N}(I - \mathcal{T}).$$

This implies $\mathbb{R}(I - S) \subseteq \mathbb{R}$ and $\mathbb{N}(I - S) \supseteq \mathbb{N}$. The relations $\mathbb{N}(S) \subseteq \mathbb{R}(I - S)$ and $\mathbb{R}(S) \supseteq \mathbb{N}(I - S)$ are obvious. ■

Theorem 7.1 *Each attainable (left, right) projection kernel P of $\mathcal{T}$ is a (left, right) optimal (left, right) projection kernel of $\mathcal{T}$.*

Proof: According to Lemma 3.2 and Lemma 3.1, respectively, we have

$$\mathbb{N}(P) \supseteq \overline{\mathbb{R}(I - \mathcal{T})}, \quad \mathbb{R}(P) \subseteq \mathbb{N}(I - \mathcal{T})$$

for a left (right) projection kernel P of $\mathcal{T}$, respectively. Lemma 7.1 supplies for $S = P$ the relations

$$\mathbb{N}(P) \subseteq \overline{\mathbb{R}(I - \mathcal{T})}, \quad \mathbb{R}(P) \supseteq \mathbb{N}(I - \mathcal{T}),$$

respectively. This shows

$$\mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})}, \quad \mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}),$$

respectively. Hence, the assertions hold by Definition 4.2. ■

Remark 7.1 Theorem 7.1 shows that an operator set $\mathcal{T}$ has at most one attainable projection kernel P, because there is at least one optimal projection kernel.

Now we turn to the question under which conditions the limits T_∞ of generalized sequences (T_α) are projection kernels of these sequences.

Proposition 7.1 *If the limit T_∞ of (T_α) exists, then the following equivalences hold:*

a) $T_\alpha T_\infty = T_\infty$ *for all* $\alpha \in J \Leftrightarrow T_\infty \in \mathbb{K}_+(\mathcal{T}_J) \;\Leftrightarrow\; \mathbb{R}(T_\infty) = \mathbb{N}(I - \mathcal{T}_J)$,

b) $T_\infty T_\alpha = T_\infty$ *for all* $\alpha \in J \Leftrightarrow T_\infty \in \mathbb{K}_-(\mathcal{T}_J) \;\Leftrightarrow\; \mathbb{N}(T_\infty) = \overline{\mathbb{R}(I - \mathcal{T}_J)}$.

Proof: a) We start with the first part and conclude cyclically. The relation $T_\alpha T_\infty = T_\infty$ implies $T_\infty^2 = T_\infty$ by limit transition. Hence T_∞ is a right projection kernel of $\mathcal{T}_J$, that is $T_\infty \in \mathbb{K}_+(\mathcal{T}_J)$. Since T_∞ is attainable by assumption, Theorem 7.1 shows that T_∞ is right optimal, that is $\mathbb{R}(T_\infty) = \mathbb{N}(I - \mathcal{T}_J)$ by Definition 4.2. This relation implies again $T_\alpha T_\infty = T_\infty$ for all α by Lemma 3.1. Hence, the cycle is closed.
b) This part can be obtained analogously. ■

Theorem 7.2 *If the limit T_∞ of (T_α) exists, then the following statements are equivalent:*

1) $T_\alpha T_\infty = T_\infty T_\alpha = T_\infty$ *for all* $\alpha \in J$,
2) $T_\infty \in \mathbb{K}(\mathcal{T}_J)$,
3) $\mathbb{R}(T_\infty) = \mathbb{N}(I - \mathcal{T}_J)$, $\quad \mathbb{N}(T_\infty) = \overline{\mathbb{R}(I - \mathcal{T}_J)}$.

Proof: Combining a) and b) in Proposition 7.1 we arrive at the assertion. ■

Theorem 7.2 shows: if the limit T_∞ of (T_α) is a projection kernel, then it is an optimal one.

Example 7.1 (Semi-group of operators) Let $\mathcal{T}$ be a semi-group of operators $T \in \mathcal{L}(X)$ with identity I. Further, let exist a generalized sequence (T_α) of operators $T_\alpha \in \mathcal{L}(X)$ with the following properties:

a) (T_α) is (uniformly) bounded,
b) $T_\alpha x \in \overline{\mathrm{co}}(\{Tx : T \in \mathcal{T}\})$ for all α and for all $x \in X$,
c) $\lim_\alpha T_\alpha x$ exists for all $x \in X$,
d) $\lim_\alpha (I - T) T_\alpha x = \lim_\alpha T_\alpha (I - T) x = 0$ for all $x \in X$ and for all $T \in \mathcal{T}$.

Then the limit operator P defined by $Px := \lim_\alpha T_\alpha x$ is a projection kernel of $\mathrm{co}(\mathcal{T})$. This result can be derived from [5], p. 220 − 222. In our context, P is moreover the attainable and optimal projection kernel of $\mathrm{co}(\mathcal{T})$.

Theorem 7.3 *Let P be a projection kernel of $\mathcal{T}_J = \{T_\alpha : \alpha \in J\}$. Then the following conditions are equivalent:*

a) (T_α) *converges to* P.
b) (T_α) *converges on* $\mathbb{N}(P)$ *to the null operator* O.

In both cases P is the optimal projection kernel of $\mathcal{T}_J$, that means

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}_J), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T}_J)}.$$

Proof: It is supposed that $P \in \mathbb{K}(\mathcal{T}_J)$. According to Theorem 3.1 the operators T_α have the direct sum representation

$$T_\alpha = I \,|\, \mathbb{R}(P) \oplus T_\alpha \,|\, \mathbb{N}(P) = P \,|\, \mathbb{R}(P) \oplus T_\alpha \,|\, \mathbb{N}(P).$$

Hence, a) and b) are equivalent. Under the condition a) P is attainable. Consequently P is by Theorem 7.1 the optimal projection kernel of $\mathcal{T}_J$, where the given range and nullspace follow by Definition 4.2. ■

8 Projection kernels of operator products

Considering iterative methods, beside operators T_k also product operators

$$T_{k,0} := T_k \dots T_1 T_0$$

occur. Hence, especially the limit behavior of $(T_{k,0})$ is of interest. Finally, we introduce the set notations

$$\mathcal{T}_N := \{T_k : \ k \in \mathrm{N}\}, \quad \mathcal{T}_{N,0} := \{T_{k,0} : \ k \in \mathrm{N}\}$$

for the corresponding sequences (T_k) and $(T_{k,0})$.

Theorem 8.1 *If the product sequence $(T_{k,0})$ converges to a (left, right) projection kernel P of (T_k), then P is a (left, right) optimal (left, right) projection kernel of (T_k) and $(T_{k,0})$.*

Proof: Let P be a (left, right) projection kernel of (T_k) with $\lim_{k\to\infty} T_{k,0} = P$. Let us consider the identities

$$I - T_{k,0} = \sum_{i=0}^{k} T_k \dots T_{i+1}(I - T_i) = \sum_{i=0}^{k} (I - T_i)\, T_{i-1} \dots T_0.$$

Since $x \in \mathbb{N}(I - \mathcal{T}_N)$ implies $(I - T_k)x = 0$ for all k, it implies also $(I - T_{k,0})x = 0$ for all k. Hence, by limit transition it is $(I - P)x = 0$. This shows

$$\mathbb{R}(P) = \mathbb{N}(I - P) \supseteq \mathbb{N}(I - \mathcal{T}_{N,0}) \supseteq \mathbb{N}(I - \mathcal{T}_N).$$

On the other hand the identities verify $\mathbb{R}(I - T_{k,0}) \subseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}$ for all k and by limit transition also $\mathbb{R}(I - P) \subseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}$. In more detail, we have even

$$\mathbb{N}(P) = \mathbb{R}(I - P) \subseteq \overline{\mathbb{R}(I - \mathcal{T}_{N,0})} \subseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}.$$

But, by Lemma 3.1 and Lemma 3.2 it holds

$$\mathbb{R}(P) \subseteq \mathbb{N}(I - \mathcal{T}_N), \quad \mathbb{N}(P) \supseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}$$

for a left and right projection kernel of $\mathcal{T}_N$, respectively.
Consequently, the assertion is true. ■

Proposition 8.1 *If the limit $T_{\infty,0}$ of $(T_{k,0})$ exists, then it follows*

a) $T_k T_{\infty,0} = T_{\infty,0}$ *for all* $k \Leftrightarrow T_{\infty,0} \in \mathbb{K}_+(\mathcal{T}_N) \ \Leftrightarrow \ \mathbb{R}(T_{\infty,0}) = \mathbb{N}(I - \mathcal{T}_N)$,

b) $T_{\infty,0} T_k = T_{\infty,0}$ *for all* $k \Leftrightarrow T_{\infty,0} \in \mathbb{K}_-(\mathcal{T}_N) \ \Leftrightarrow \ \mathbb{N}(T_{\infty,0}) = \overline{\mathbb{R}(I - \mathcal{T}_N)}$.

Proof: a) The relation $T_k T_{\infty,0} = T_{\infty,0}$ for all k implies $T_{k,0} T_{\infty,0} = T_{\infty,0}$ for all k. By limit transition we get $T_{\infty,0}^2 = T_{\infty,0}$. This means $T_{\infty,0} \in \mathbb{K}_+(\mathcal{T}_N)$ considering the first relation. Since $T_{\infty,0}$ is attainable, Theorem 7.1 shows that $T_{\infty,0}$ is right optimal. By Definition 4.2 the relation $\mathbb{R}(T_{\infty,0}) = \mathbb{N}(I - \mathcal{T}_N)$ holds in this case.
Assertion b) is shown analogously. ■

Theorem 8.2 *If the limit $T_{\infty,0}$ of $(T_{k,0})$ exists, then the following conditions are equivalent:*

1) $T_k T_{\infty,0} = T_{\infty,0} T_k = T_{\infty,0}$ *for all* k,
2) $T_{\infty,0} \in \mathbb{K}(\mathcal{T}_N)$,
3) $\mathbb{R}(T_{\infty,0}) = \mathbb{N}(I - \mathcal{T}_N), \quad \mathbb{N}(T_{\infty,0}) = \overline{\mathbb{R}(I - \mathcal{T}_N)}$.

Proof: The assertion follows by combination of a) and b) in Proposition 8.1. ■

Theorem 8.3 *Let P be a projection kernel of $\mathcal{T}_N$. Then the following conditions are equivalent:*

a) $(T_{k,0})$ *converges to* P.
b) $(T_{k,0})$ *converges on* $\mathbb{N}(P)$ *to the null operator* O.

In both cases P is the optimal projection kernel of $\mathcal{T}_N$, that means

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}_N), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T}_N)}.$$

Proof: By assumption it is $P \in \mathbb{K}(\mathcal{T}_N)$. Then $P \in \mathbb{K}(\mathcal{T}_{N,0})$ follows using Lemma 2.1 or Corollary 2.1. Applying Theorem 7.3 with $(T_{k,0})$ instead of (T_α), the statements a) and b) are shown to be equivalent. If a) or b) are supposed, Theorem 8.1 supplies that the projection kernel P is optimal. By definition P has the stated range and nullspace. ■

Remark 8.1 If $(T_{k,0})$ converges to O, then there is only the trivial projection kernel $P = O$. Further, it is

$$\mathbb{N}(I - \mathcal{T}_N) = \mathbb{R}(O) = \{0\}, \quad \overline{\mathbb{R}(I - \mathcal{T}_N)} = \mathbb{N}(O) = X.$$

Reversely, if these space conditions hold for (T_k), then $(T_{k,0})$ converges to O.

Theorem 8.4 *If the operators T_k are* FEJÉR *monotone with eigenprojections P_k ($k = 1, \ldots, m$), then the product $T_{m,1} = T_m \ldots T_2 T_1$ is* FEJÉR *monotone with eigenprojection P defined by*

$$\mathbb{R}(P) = \bigcap_{k=1}^{m} \mathbb{R}(P_k), \quad \mathbb{N}(P) = \operatorname{span} \bigcup_{k=1}^{m} \mathbb{N}(P_k).$$

Thereby P is the orthogonal optimal projection kernel of both (P_k) and (T_k).

Proof: The first part is shown in [17], p. 183. The last part follows by Theorem 5.1, if the relations

$$\mathbb{N}(I - T_k) = \mathbb{N}(I - T_k^*) = \mathbb{R}(P_k) = \mathbb{N}(I - P_k) = \mathbb{N}(I - P_k^*), \quad \mathbb{R}(I - P_k) = \mathbb{N}(P_k)$$

are observed (see also Corollary 5.3). ■

9 Power sequences and related series

Now we turn to the special case $T_k = T$ for all k. Then $T_{k,0} = T^{k+1}$. Thus we arrive at power sequences (T^k) and their convergence properties.

Lemma 9.1 *The following statements are equivalent:*

a) (T^k) *converges strongly (to an operator* $T^\infty \in \mathcal{L}(X)$*).*

b) (T^k) *converges strongly to the projector* P *given by* $\mathbb{R}(P) = \mathbb{N}(I - T)$ *and* $\mathbb{N}(P) = \overline{\mathbb{R}(I - T)}$.

c) *There is a projector* P *with* $\mathbb{R}(P) = \mathbb{N}(I - T)$ *such that* $\tilde{T} := T|\mathbb{N}(P) \in \mathcal{L}(\mathbb{N}(P))$ *and* $(\tilde{T}^k)$ *converges strongly to* $O \in \mathbb{N}(P)$.

Proof: The equivalence of b) and c) is a consequence of Theorem 8.3. It remains to show that the limit operator of (T^k) is a projector in $\mathcal{L}(X)$ with given range and nullspace. This is done e.g. in [15], pp. 6 – 8. ■

Remark 9.1 Several authors have proven in different ways and in different spaces that the limit of a convergent power sequence (T^k) is a projector P (see e.g. [1]: p. 367, [3]: p. 567, [9], [19]: p. 179, [20]: p. 351). This projector P in the above lemma is the optimal projection kernel of T. Hence, the existence of an optimal projection kernel for T is necessary for the convergence of (T^k). In other words, $I - T$ has to be decomposition regular (see [14]):

$$X = \mathbb{N}(I - T) \oplus \overline{\mathbb{R}(I - T)}.$$

Proposition 9.1 ([18], p. 35 – 36) *Let* T *be a strongly* FEJÉR *monotone operator. Then the sequence* (T^k) *converges (uniformly, r.t. the operator norm) to the eigenprojection* $P(T)$ *of* T*.*

Lemma 9.1 shows that powers sequences (T^k) of strongly FEJÉR monotone operators converge uniformly to O on $\mathbb{N}(P) = \mathbb{R}(I - T)$, where the range of $I - T$ is closed in this case. If $\mathbb{R}(I - T) = X$, then (T^k) converges uniformly to O. The latter statement fits to the following well-known facts.

Proposition 9.2 *These conditions are equivalent:*

a) *The sequence* (T^k) *converges uniformly to* O.

b) *There is a natural* n *such that* $\|T^n\| < 1$.

c) *The* NEUMANN *series* $\left(\sum_{i=0}^{k} T^i\right)$ *converges uniformly.*

Theorem 9.1 *The following two statements are equivalent:*

a) *The sequence* (T^k) *converges.*

b) *The sequence* (T^{nk}) *converges for a fixed* n *and all* n*-th roots of unity which are different from* 1 *are no eigenvalues of* T.

Under one of these conditions a) or b) both sequences converge to their optimal projection kernel P *with* $\mathbb{R}(P) = \mathbb{N}(I - T)$ *and* $\mathbb{N}(P) = \overline{\mathbb{R}(I - T)}$.

Proof: The equivalence of a) and b) is given in [15]: p. 11 and in [3]: p. 568 for the special case $n = 2$. The consequence is shown by Lemma 9.1. ■

Corollary 9.1 *If the natural power of an operator* T *is a projector* P*, say* $T^n = P$*, and all* n*-th roots of unity which are different from* 1 *are no eigenvalues of* T*, then it holds also* $T^k = P$ *for all members of* (T^k) *with* $k \geq n$*. Thereby* P *is the optimal projection kernel of* (T^k).

Proof: By assumption we have $T^n = P$. Considering $T^{nk} = P$ for all k and Theorem 9.1 both sequences (T^k) and (T^{nk}) converge to the common optimal projection kernel P such that also $PT = TP = P$ holds. Hence,

$$T^{n+1} = T \cdot T^n = T \cdot P = P.$$

By induction we get $T^k = P$ for all $k \geq n$. ■

Example 9.1 Let us choose

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where P is a projector. Then it is $T^3 = P$. The eigenvalues 0, 1, i and $-$i of T are no third roots of unity except for 1. Thus Corollary 9.1 can be applied for $n = 3$. This implies $T^k = P$ for $k \geq 3$. Direct computation also confirms the result. Further P is the optimal projection kernel of (T^k), i.e.

$$\mathbb{R}(P) = \mathbb{N}(I - T), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - T)} = \mathbb{R}(I - T).$$

This can also be shown directly. Observe that the given matrices fits Example 5.2. The matrix

$$I - T = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

has rank 3. This again proves the optimality of P in the referred context.

Example 9.2 We consider an operator

$$T := I + B, \quad B \in \mathrm{R}^{n,n}, \quad B \neq O, \quad B^2 = O.$$

Then we get

$$T^k = I + kB.$$

Hence, the limit T^∞ of (T^k) does not exist. Therefore $I - T$ and B are not decomposition regular (see Remark 9.1). A projector $P \in \mathrm{R}^{n,n}$ is a projection kernel of (T^k) iff $PB = BP = O$. But such a projector cannot be optimal (see again Remark 9.1). Example 4.1 shows that the conditions for B given above can be fulfilled. We choose

$$B := c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $T(c) := I + B$, where $B^2 = O$ and $BP(a, b) = P(a, b)B = O$ holds for the projection kernels $P(a, b)$ of operators $T(c)$.

The following example shows that the optimal projection kernel of T are sometimes obtained by limits of more general sequences which can converge if (T^k) diverges.

Example 9.3 (Means of operator powers) Let X be a reflexive BANACH space and $T \in \mathcal{L}(X)$ an operator with a (uniformly) bounded sequence (T^k) of powers. Then the sequence of CESÀRO means

$$T_k := \frac{1}{k+1} \sum_{i=0}^{k} T^i$$

converges strongly to the optimal projection kernel P of (T^k) (see also [23]: p. 214).

Similar results can be obtained also by other means of operator powers.

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received: March 3, 2014

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Rostock. Math. Kolloq. **68**, 45 – 64 (2013)

Subject Classification (AMS)
39A13, 11A63, 11B68, 26A30

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Two-Scale Difference Equations and Power Sums related to Digital Sequences

ABSTRACT. This paper uncovers a connection between two-scale difference equations and the representation of sums of sequences which satisfy a certain multiplicative recurrence formula. For certain digital power sums related with such a sequence we derive a formula which in case of usual power sums yields the known representation of power sums by means of Bernoulli polynomials.

KEY WORDS. Two-scale difference equations, digital sums, Bernoulli polynomials, Appell polynomials, generating functions

1 Introduction

Let $p > 1$ be an integer and C_n the sequence which is given by the p initial values $C_0 = 1$, $C_1, \dots, C_{p-1}$ such that

$$C := C_0 + \cdots + C_{p-1} > 0 \tag{1.1}$$

and which satisfies the recurrence formula

$$C_{kp+r} = C_k C_r \qquad (k \in \mathbb{N},\ r = 0, \dots, p-1). \tag{1.2}$$

In this paper we derive a formula for the sum

$$S_m(N) = \sum_{n=0}^{N-1} C_n n^m \tag{1.3}$$

where $m \in \mathbb{N}_0$. In the simple case $C_n = 1$ for all n we have the usual power sum which can be expressed by means of the Bernoulli polynomials $B_m(t)$ in the form

$$\sum_{n=0}^{N-1} n^m = \tilde{B}_m(N) \tag{1.4}$$

where

$$\tilde{B}_m(t) = \frac{1}{m+1}\{B_{m+1}(t) - B_{m+1}\}. \tag{1.5}$$

Digital sums were investigated by many authors, cf. e.g. [4], [13], [3], [12], [5], [6], [10].

Under the condition

$$|C_r| < C \quad \text{for} \quad r = 0, 1, \ldots, p-1 \tag{1.6}$$

we show that for the digital sum (1.3) it holds

$$\sum_{n=0}^{N-1} C_n n^m = N^\alpha \sum_{\mu=0}^{m} N^\mu F_{m,\mu}\left(\log_p N\right) \tag{1.7}$$

with $\alpha = \log_p C$ and 1-periodic continuous functions $F_{m,\mu}$ which can be expressed by means of the solutions of certain two-scale difference equations (Theorem 4.1).

In order to derive formula (1.7) we quote some facts on the two-scale difference equation

$$\lambda\varphi\left(\frac{t}{p}\right) = \sum_{r=0}^{p-1} c_r \varphi(t-r) \qquad (t \in \mathbb{R}) \tag{1.8}$$

with $\lambda \neq 0$ and complex coefficients c_r where $c_0 \neq 0$ and

$$\sum_{r=0}^{p-1} c_r = 1, \tag{1.9}$$

cf. [11] where equation (1.8) with $\lambda = 1$ was studied in detail. In [7] and [8] it was investigated a system of simple functional equations which is equivalent to equation (1.8) with $\lambda = 1$, cf. [11, p. 60]. It is known that under the condition $|c_r| < 1$ equation (1.8) with $\lambda = 1$ has a continuous solution φ_0 satisfying

$$\varphi_0(t) = 0 \quad \text{for} \quad t < 0, \qquad \varphi_0(t) = 1 \quad \text{for} \quad t > 1 \tag{1.10}$$

and that φ_0 is even Hölder continuous cf. [11, Theorem 3.6]. The solution $\varphi = \varphi_0$ has the Laplace transform

$$\mathcal{L}\{\varphi_0\} = \frac{1}{z}\Phi(z) \tag{1.11}$$

where

$$\Phi(z) = \prod_{j=1}^{\infty} P\left(e^{-z/p^j}\right) \tag{1.12}$$

with the polynomial

$$P(w) = \sum_{r=0}^{p-1} c_r w^r, \tag{1.13}$$

cf. [1], [2].

The iterated integrals φ_n $(n \in \mathbb{N})$ of φ_0, defined recursively by

$$\varphi_n(t) = \int_0^t \varphi_{n-1}(\tau)d\tau$$

are solutions of (1.8) with $\lambda = p^n$. For $t > 1$ the solution φ_n is a polynomial

$$\varphi_n(t) = p_n(t) \qquad (t > 1) \tag{1.14}$$

of degree n with the main term $\frac{1}{n!}t^n$. We remark that the polynomials p_n have the property $p_n'(t) = p_{n-1}(t)$, i.e. they are Appell polynomials, cf. [1], [2]. The generating function reads

$$e^{tz}\Phi(z) = \sum_{n=0}^{\infty} p_n(t)z^n \qquad (t \in \mathbb{R}) \tag{1.15}$$

with Φ from (1.12). The coefficients of the power series

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.16}$$

can be calculated recursively by $a_0 = \Phi(0) = 1$ and

$$a_n = \frac{1}{p^n - 1}\sum_{k=1}^{n}(-1)^k \frac{a_{n-k}}{k!}\sum_{r=1}^{p-1} r^k c_r \quad (n \in \mathbb{N}) \tag{1.17}$$

cf. [2, Proposition 2.6] where $p = 2$, and the polynomials p_n in (1.15) have the representation

$$p_n(t) = \sum_{k=0}^{n} \frac{a_{n-k}}{k!}t^k. \tag{1.18}$$

We also need the power series

$$\frac{1}{\Phi(z)} = \sum_{n=0}^{\infty} b_n z^n \tag{1.19}$$

where the coefficients b_n are determined by $b_0 = 1$ and the equations

$$a_n b_0 + a_{n-1}b_1 + \cdots + a_0 b_n = 0 \qquad (n > 1). \tag{1.20}$$

The corresponding Appell polynomials

$$q_n(t) = \sum_{k=0}^{n} \frac{b_{n-k}}{k!}t^k \tag{1.21}$$

have the generating function

$$\frac{e^{tz}}{\Phi(z)} = \sum_{n=0}^{\infty} q_n(t)z^n. \tag{1.22}$$

This paper is organized as follows: At first we show that the solution $\varphi = \varphi_n$ of the two-scale difference equation (1.8) with $\lambda = p^n$ has for $k \le p^\ell$ the representation

$$\varphi_n\left(\frac{k}{p^\ell}\right) = \frac{c_0^\ell}{p^{n\ell}}\sum_{j=1}^{k} C_{k-j}p_n(j) \tag{1.23}$$

where p_n are the polynomials (1.18), (Theorem 2.1). This formula is the start point for the representation (1.7) of digital power sums. In Section 3 we prove (1.7) in the case $m = 0$, i.e.

$$S_0(N) = \sum_{n=0}^{N-1} C_n = N^\alpha F_0(\log_p N) \tag{1.24}$$

(Theorem 3.2), and give some properties of the 1-periodic continuous function F_0 under the condition (1.6), for instance that F_0 is Hölder continuous and that F_0 is differentiable almost everywhere if $p|C_0C_1\cdots C_{p-1}|^{1/p} < C$, (Proposition 3.5). By means of a Toeplitz theorem we prove the convergence of the arithmetical mean

$$\frac{1}{p^n}\sum_{N=1}^{p^n} \frac{1}{N^\alpha} S_0(N) \tag{1.25}$$

as $n \to \infty$ (Proposition 3.7). In Section 4 we prove the main result of this paper, namely the representation (1.7), (Theorem 4.1). In the simple case $C_n = 1$ for all n formula (1.7) turns over into the known representation (1.4) for the usual power sums, cf. Remark 4.2. For the specific power sums (1.3) where N is a power of p we have the representation

$$S_m(p^k) = (-1)^m m! p^{\alpha k} \sum_{\mu=0}^{m} p^{\mu k} a_\mu b_{m-\mu} \tag{1.26}$$

with $\alpha = \log_p C$ and the coefficients a_n from (1.16) and b_n from (1.19), (Proposition 5.2), and we prove for positive integers k, ℓ

$$S_m(p^{k+\ell}) = \sum_{\mu=0}^{m} \binom{m}{\mu} p^{k\mu} S_\mu(p^\ell) S_{m-\mu}(p^k) \tag{1.27}$$

(Proposition 5.6).

2 Functional relations

For given coefficients $c_0, c_1, \ldots, c_{p-1}$ of the two-scale difference equation (1.8) we define a sequence C_n by $C_n = \frac{c_n}{c_0}$ for $n = 0, 1, \ldots, p-1$ and for $n \geq p$ by the recursion

$$C_{kp+r} = C_k C_r \qquad (k \geq 1, r \in \{0, 1, \ldots, p-1\}). \tag{2.1}$$

If n has the p-adic representation

$$n = \sum n_i p^i, \qquad (n_i \in \{0, 1, \ldots, p-1\}) \tag{2.2}$$

then we have

$$C_n = \prod_{r=1}^{p-1} C_r^{s_r(n)} \tag{2.3}$$

where $s_r(n)$ denotes the total number of occurrences of the digit r in the representation (2.2) of n, cf. [11, p. 63].

The numbers C_n have the generating function

$$G(z) = \prod_{j=0}^{\infty} \frac{1}{c_0} P\left(z^{p^j}\right) = \sum_{n=0}^{\infty} C_n z^n \tag{2.4}$$

which converges for $|z| < 1$, cf. [11, Remark 2.2.1.].

In the following we want to generalize Proposition 2.3 from [11] for φ_n.

Theorem 2.1 *For $\ell \in \mathbb{N}$ and non-negative integers $k < p^\ell$ the solution $\varphi = \varphi_n$ of (1.8) with $\lambda = p^n$ satisfies the equations*

$$\varphi_n\left(\frac{k+t}{p^\ell}\right) = \frac{c_0^\ell}{p^{n\ell}} \sum_{j=0}^{k} C_j \varphi_n(t+k-j) \qquad (0 \le t \le 1). \tag{2.5}$$

Moreover, for $k \le p^\ell$ we have

$$\varphi_n\left(\frac{k}{p^\ell}\right) = \frac{c_0^\ell}{p^{n\ell}} \sum_{j=1}^{k} C_{k-j} p_n(j) \tag{2.6}$$

where p_n are the polynomials (1.18).

Proof: In (1.8) with $\lambda = p^n$ we replace t by $k+t$ with $0 \le k \le p-1$ and get in view of $C_r = \frac{c_r}{c_0}$ for $0 \le r \le p-1$

$$\begin{aligned}
\varphi_n\left(\frac{k+t}{p}\right) &= \frac{1}{p^n} \sum_{r=0}^{p-1} c_r \varphi_n(k+t-r) \\
&= \frac{c_0}{p^n} \sum_{r=0}^{p-1} C_r \varphi_n(k+t-r) \\
&= \frac{c_0}{p^n} \sum_{j=0}^{k} C_j \varphi_n(k+t-j)
\end{aligned}$$

since $\varphi_n(t) = 0$ for $t \le 0$. So (2.5) is true for $\ell = 1$. Assume that (2.5) is valid for a fixed ℓ. Replace t by $\frac{s+t}{p}$ with $0 \le s \le p-1$ we get

$$\begin{aligned}
\varphi_n\left(\frac{kp+s+t}{p^{\ell+1}}\right) &= \frac{c_0^\ell}{p^{n\ell}} \sum_{j=0}^{k} C_j \varphi_n\left(\frac{p(k-j)+s+t}{p}\right) \\
&= \frac{c_0^\ell}{p^{n\ell+n}} \sum_{j=0}^{k} \sum_{r=0}^{p-1} C_j c_r \varphi_n(pk+s-pj-r+t) \\
&= \frac{c_0^{\ell+1}}{p^{n(\ell+1)}} \sum_{j=0}^{k} \sum_{r=0}^{p-1} C_{jp+r} \varphi_n(t+kp+s-pj-r).
\end{aligned}$$

So (2.5) is proved by induction. Formula (2.6) follows by summation in view of $\varphi_n(0) = 0$ and (1.14) for the polynomials $p_n(t)$ from (1.18). □

Remark 2.2 Formula (2.6) yields in case $n = 0$ the known representations

$$\varphi_0\left(\frac{k+t}{p^\ell}\right) = \varphi_0\left(\frac{k}{p^\ell}\right) + c_0^\ell C_k \varphi_0(t) \quad (0 \le t \le 1) \tag{2.7}$$

and

$$\varphi_0\left(\frac{k}{p^\ell}\right) = c_0^\ell \sum_{j=0}^{k-1} C_j \tag{2.8}$$

for the solution $\varphi = \varphi_0$ of equation (1.8) with $\lambda = 1$, cf. [11].

From (2.5) and (2.6) we get in view of (1.14) the following result.

Corollary 2.3 *For $\ell \in \mathbb{N}_0$ and non-negative integers $k < p^\ell$ the solution $\varphi = \varphi_n$ of (1.8) with $\lambda = p^n$ satisfies*

$$\varphi_n\left(\frac{k+t}{p^\ell}\right) = \frac{c_0^\ell C_k}{p^{n\ell}} \varphi_n(t) + \frac{c_0^\ell}{p^{n\ell}} p_{nk}(t) \qquad (0 \le t \le 1) \tag{2.9}$$

with the polynomials

$$p_{nk}(t) = \sum_{j=1}^{k} C_{k-j} p_n(j+t) \tag{2.10}$$

and $p_n(t)$ from (1.18).

We remark that (2.9) with (2.10) is already known for the iterated integrals of de Rham's function, cf. [2, (3.16) and Theorem 3.1].

3 Digital sums

Let C_n be an arbitrary sequence with the properties $C_0 = 1$, (1.1) and (1.2). In order to obtain a formula for the sum (1.3) with $m = 0$, i.e.

$$S_0(N) = \sum_{n=0}^{N-1} C_n \tag{3.1}$$

we consider the two-scale difference equation

$$\varphi\left(\frac{t}{p}\right) = \frac{1}{C} \sum_{r=0}^{p-1} C_r \varphi(t-r) \tag{3.2}$$

with C from (1.1). In the following we assume that (1.6) is satisfies so that equation (3.2) has a continuous solution $\varphi = \varphi_0$ satisfying (1.10) since the quotients $c_r = \frac{C_r}{C}$ satisfy (1.9) and $|c_r| < 1$. For $0 \le t \le 1$ we have in view of $C_0 = 1$ and (1.10)

$$\varphi_0\left(\frac{t}{p}\right) = \frac{1}{C}\varphi_0(t) \qquad (0 \le t \le 1).$$

We put

$$\alpha := \log_p C \tag{3.3}$$

so that $p^\alpha = C$ and

$$\frac{\varphi_0(\frac{t}{p})}{(\frac{t}{p})^\alpha} = \frac{\varphi_0(t)}{t^\alpha} \qquad (0 < t \le 1). \tag{3.4}$$

Hence, the function

$$f_0(t) := \frac{\varphi_0(t)}{t^\alpha} \qquad (0 < t \le 1) \tag{3.5}$$

has the property: $f_0(\frac{t}{p}) = f_0(t)$ so that it can be extended for all $t > 0$ by

$$f_0(pt) = f_0(t) \tag{3.6}$$

where $f_0(t)$ is continuous for $t > 0$.

Proposition 3.1 *If (1.6) is satisfies then for $N \in \mathbb{N}$ the sum $S_0(N)$ from (3.1) can be represented as*

$$S_0(N) = N^\alpha f_0(N) \tag{3.7}$$

with α from (3.3) and the continuous function f_0 from (3.5) and (3.6).

Proof: Because of (1.6) equation (3.2) has a continuous solution φ_0 satisfying (1.10). For $N \le p^\ell$ we have by (2.8) the formula

$$S_0(N) = C^\ell \varphi_0\left(\frac{N}{p^\ell}\right). \tag{3.8}$$

For arbitrary N we choose ℓ so large that $p^\ell > N$. In view of (3.8), (3.3) and (3.5) we have

$$S_0(N) = C^\ell \varphi_0\left(\frac{N}{p^\ell}\right) = N^\alpha \left(\frac{p^\ell}{N}\right)^\alpha \varphi_0\left(\frac{N}{p^\ell}\right) = N^\alpha f_0\left(\frac{N}{p^\ell}\right).$$

Owing to (3.6) it follows (3.7). □

According to (3.6) the function

$$F_0(u) := f_0(p^u) \qquad (u \in \mathbb{R}) \tag{3.9}$$

has the period 1 and in virtue of (3.5) we have by Proposition 3.1 :

Theorem 3.2 *If (1.6) is satisfies then for $N \in \mathbb{N}$ the sum $S_0(N)$ from (3.1) can be represented as*

$$S_0(N) = N^\alpha F_0(\log_p N) \tag{3.10}$$

with α from (3.3) and an -1 periodic continuous function F_0 which is given by

$$F_0(u) = \frac{\varphi_0(p^u)}{p^{\alpha u}} = C^{-u}\varphi_0(p^u) \qquad (u \leq 0) \tag{3.11}$$

where φ_0 is the solution of (3.2) satisfying (1.10).

Remark 3.3 Note that from (3.10) and (3.11) for $N = p^k$ we get in view of $F_0(k) = F_0(0) = 1$ that

$$S_0(p^k) = \sum_{n=0}^{p^k-1} C_n = p^{k\alpha} = C^k \tag{3.12}$$

with C from (1.1).

Remark 3.4 In the case $C_r = 1$ for all $r = 0, 1, \ldots, p-1$ we have $C = p$ and $\alpha = 1$. Equation (3.2) has the trivial solution $\varphi_0(t) = t$ for $0 \leq t \leq 1$, $f_0(t) = 1$ for $t > 0$, $F_0(u) = 1$ for all $u \in \mathbb{R}$ and we get $S_0(N) = N$ for the sum (3.1).

In the following we exclude the trivial case $C_n = 1$ for all n.

Proposition 3.5 *If (1.6) is satisfies then the 1-periodic continuous function $F_0(u)$ from (3.11) has the following properties:*

1. F_0 is Hölder continuous.
2. *If $pM_0 < C$ where $M_0 = |C_0 C_1 \cdots C_{p-1}|^{1/p}$ then F_0 is differentiable almost everywhere and if $pM_0 \geq C$ then it is almost nowhere differentiable.*
3. *F_0 has finite total variation on $[0, 1]$ if and only if $C_r \geq 0$ for $r = 0, 1, \ldots, p-1$. In this case we have*

$$\bigvee_0^1 (F_0) \leq 2C - 2. \tag{3.13}$$

Proof: It is known that in case $|c_r| < 1$ the solution $\varphi = \varphi_0$ of (1.8) with $\lambda = 1$ is Hölder continuous, cf. [11, Theorem 3.6]. This implies in view of $c_r = \frac{C_r}{C}$ with C from (1.1), (3.5) and (3.9) the first property of F_0. Analogously, the second property is a consequence of [11, Theorem 4.12].

In order to prove the third property first we consider the case $C_r \geq 0$ where the solution $\varphi = \varphi_0$ of (3.2) is increasing, cf. [11, Proposition 5.1]. We show that for f_0 from (3.5) it holds

$$\bigvee_{1/p}^1 (f_0) \leq 2C - 2. \tag{3.14}$$

Let $\frac{1}{p} = t_0 < t_1 < \ldots < t_n = 1$ be some decomposition of $[\frac{1}{p}, 1]$. Because of the identity

$$2(aA - bB) = (a+b)(A-B) + (A+B)(a-b) \tag{3.15}$$

it holds

$$2|aA - bB| \leq |a+b||A-B| + |A+B||a-b|.$$

Using this inequality with $a = \frac{1}{t_i^\alpha}$, $b = \frac{1}{t_{i+1}^\alpha}$, $A = \varphi_0(t_i)$ and $B = \varphi_0(t_{i+1})$ we have in view of $\max|\varphi_0(t)| = \varphi_0(1) = 1$ and (3.5)

$$\begin{aligned} 2|f_0(t_i) - f_0(t_{i+1})| &\leq \left|\frac{1}{t_i^\alpha} + \frac{1}{t_{i+1}^\alpha}\right| |\varphi_0(t_i) - \varphi_0(t_{i+1})| + 2\left|\frac{1}{t_i^\alpha} - \frac{1}{t_{i+1}^\alpha}\right| \\ &\leq 2\max\{p^\alpha, 1\}|\varphi_0(t_i) - \varphi_0(t_{i+1})| + 2\left|\frac{1}{t_i^\alpha} - \frac{1}{t_{i+1}^\alpha}\right|. \end{aligned}$$

Since $p^\alpha = C > 1$ and $\varphi_0(.)$ is increasing we get by summation

$$\bigvee_{1/p}^{1}(f_0) \leq C\left(\varphi_0(1) - \varphi_0\left(\frac{1}{p}\right)\right) + |p^\alpha - 1| = C\left(1 - \frac{1}{C}\right) + (C-1)$$

where we have used $\varphi_0(1) = 1$, $\varphi_0(\frac{1}{p}) = \frac{1}{p^\alpha} = \frac{1}{C}$, cf. (3.4) with $t = 1$, and (3.3). So we have proved (3.14) which implies (3.13) in virtue of (3.9).

Now we consider the case that $C_r \geq 0$ is not true for all $r = 0, 1, \ldots, p-1$. Then by [11, Proposition 2.6] the solution $\varphi = \varphi_0$ of (3.2) does not have finite total variation on $[0, 1]$. According to (2.7) this is true also for the subinterval $[\frac{k}{p}, \frac{k+1}{p}]$ if $C_k \neq 0$. This implies

$$\bigvee_{1/p}^{1}(\varphi_0) = \infty \tag{3.16}$$

since in view of (1.6) it is impossible that $C_r = 0$ for all $r = 1, 2, \ldots, p-1$.

From (3.15) we get

$$2|aA - bB| \geq |a+b||A-B| - |A+B||a-b|$$

and with the same notations as before

$$2|f_0(t_i) - f_0(t_{i+1})| \geq 2\min\{p^\alpha, 1\}|\varphi_0(t_i) - \varphi_0(t_{i+1})| - 2M\left|\frac{1}{t_i^\alpha} - \frac{1}{t_{i+1}^\alpha}\right|$$

where $M = \max\{|\varphi_0(t)|\}$ for $\frac{1}{p} \leq t \leq 1$. In view of $p^\alpha > 1$ it follows

$$\sum_{i=0}^{n-1}|f_0(t_i) - f_0(t_{i+1})| \geq \sum_{i=0}^{n-1}|\varphi_0(t_i) - \varphi_0(t_{i+1})| - M(p^\alpha - 1)$$

which implies

$$\bigvee_{1/p}^{1}(f_0)=\infty$$

according to (3.16). Finally, (3.9) yields that F_0 does not have finite total variation on [0,1]. □

Remark 3.6 Note that according to (2.7) the solution φ_0 is constant on $[\frac{k}{p},\frac{k+1}{p}]$ if $C_k=0$ for some $k\le p-1$. We remark that the suppositions of Proposition 2.6 in [11] are to add by $c_j\neq 0$ for all $j=0,1,\ldots,p-1$.

Proposition 3.7 *If (1.6) is satisfies then for the sums $S_0(N)$ from (3.1) we have*

$$\frac{1}{p^n}\sum_{N=1}^{p^n}\frac{1}{N^\alpha}S_0(N)\to c \qquad (n\to\infty) \tag{3.17}$$

where

$$c=\int_{1/p}^{1}f_0(t)dt \tag{3.18}$$

with f_0 from (3.5) and (3.6).

Proof: The sum in (3.17) can be written as

$$\frac{1}{p^n}\sum_{N=1}^{p^n}\frac{1}{N^\alpha}S_0(N)=\sum_{m=0}^{n}t_{n,m}A_m \tag{3.19}$$

with

$$t_{n,0}:=\frac{1}{p^n},\qquad t_{n,m}:=\frac{p^m-p^{m-1}}{p^n}\qquad (1\le m\le n)$$

and

$$A_0:=1,\qquad A_m:=\frac{1}{p^m-p^{m-1}}\sum_{N=p^{m-1}+1}^{p^m}\frac{1}{N^\alpha}S_0(N)\qquad (1\le m\le n).$$

For the numbers $t_{n,m}$ we have $t_{n,m}>0$, $t_{n,0}+t_{n,1}+\cdots+t_{n,n}=1$ and $t_{n,m}\to 0$ as $n\to\infty$ for fixed m, so that by a known Toeplitz theorem the sum (3.19) converges to c from (3.18) if

$$A_m\to\int_{1/p}^{1}f_0(t)dt\qquad (m\to\infty). \tag{3.20}$$

According to (3.7) with the continuous function f_0 from (3.5) and (3.6) we have for $m\ge 1$

$$\begin{aligned} A_m &= \frac{1}{p^m-p^{m-1}}\sum_{N=p^{m-1}+1}^{p^m}f_0(N) \\ &= \frac{1}{p^m-p^{m-1}}\sum_{N=p^{m-1}+1}^{p^m}f_0\left(\frac{N}{p^m}\right) \end{aligned}$$

where we have used (3.6). With the substitution $k = N - p^{m-1}$ we get

$$A_m = \frac{1}{p^m - p^{m-1}} \sum_{k=1}^{p^m - p^{m-1}} f_0\left(\frac{1}{p} + \frac{k}{p^m}\right)$$

and in view of the continuity of f_0 it follows (3.20). □

Example 3.8 (*Digital exponential sums*) We consider the sequence $C_n = q^{s(n)}$ with $q > 0$, where $s(n)$ denotes the number of ones in the binary representation of n. This sequence satisfies relation (1.2) with $p = 2$, $C_0 = 1$ and $C_1 = q$. The corresponding two-scale difference equation (3.2) reads

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-a)\varphi(t-1) \qquad (t \in \mathbb{R}) \tag{3.21}$$

with $a = \frac{1}{1+q}$ and the solution $\varphi = \varphi_0$ satisfying (1.10) which clearly depend on the parameter a. (cf. de Rham's function [10]). By Theorem 3.2 we have for the sum

$$S_0(N) = \sum_{n=0}^{N-1} q^{s(n)} \tag{3.22}$$

the exact formula

$$S_0(N) = N^\alpha F_0(\log_2 N)$$

where $\alpha = \log_2(1+q)$ and where $F_0(u)$ is a continuous, 1-periodic function which is connected with de Rham's function φ_0, i.e. the solution of (3.21), by

$$F_0(u) = a^u \varphi_0(2^u) \qquad (u \leq 0),$$

cf. also [10, Theorem 2.1]. Let us mention that in case $q = 2$ the sum (3.22) is equal to the number of odd binomial coefficients in the first N rows of Pascal's triangle and that the sum (3.22) was already investigated by many authors, cf. e.g. [12], [6], [10].

Example 3.9 (*Cantor's function*) We consider the sequence C_n where $C_n = 0$ if the triadic representation of n contains the digit 1, elsewhere $C_n = 0$. This sequence satisfies relation (1.2) with $p = 3$, $C_0 = 1$, $C_1 = 0$ and $C_2 = 1$. Note that for the generating function (2.4) we have

$$G(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{k=0}^{\infty} z^{\gamma_k} = 1 + z^2 + z^6 + z^8 + z^{18} + z^{20} + z^{24} + z^{26} + \cdots$$

with strictly increasing exponents $\gamma_0 = 0, \gamma_1 = 2, \gamma_2 = 6, \gamma_3 = 8$ and so on, where it holds with $\varepsilon_\mu \in \{0, 1\}$:

$$n = \sum_{\mu=0}^{m} \varepsilon_\mu 2^\mu \qquad \Longrightarrow \qquad \gamma_n = 2\sum_{\mu=0}^{m} \varepsilon_\mu 3^\mu, \tag{3.23}$$

cf. [11, Formula (5.9)]. For the sum (3.1) it follows

$$S_0(N) = \sum_{n=0}^{N-1} C_n = k+1 \qquad \text{for} \quad \gamma_k + 1 \le N < \gamma_{k+1}. \tag{3.24}$$

By means of Theorem 3.2 this sum can also represented by means of Cantor's function. Cantor's function is the solution φ_0 of (3.2) restricted to [0,1] with $p = 3$, $C_0 = 1$, $C_1 = 0$, $C_2 = 1$ and $C = 2$, i.e. $\varphi = \varphi_0$ is solution of

$$\varphi\left(\frac{t}{3}\right) = \frac{1}{2}\varphi(t) + \frac{1}{2}\varphi(t-2) \qquad (t \in \mathbb{R})$$

satisfying (1.10), cf. [9, Section 5], [11, Example 5.6]. By Theorem 3.2 the sum (3.24) can be expressed as follows:

$$S_0(N) = N^{\alpha} F_0(\log_3 N) \tag{3.25}$$

where $\alpha = \log_3 2$ and where F_0 is a continuous periodic function with period 1 which is given by

$$F_0(u) = \frac{1}{2^u}\varphi_0(2^u) \qquad (u \le 0) \tag{3.26}$$

with Cantor's function φ_0.

It is remarkable that the intervals $J_{m,n}$, where Cantor's function φ_0 is constant, have the form

$$J_{m,n} = \left(\frac{\gamma_{m-1}+1}{3^n}, \frac{\gamma_m}{3^n}\right) \qquad (n = 1,2,3,\ldots, \quad m = 1,2,\ldots,2^n)$$

with $\varphi_0(t) = \frac{m}{2^n}$ for $t \in J_{m,n}$, cf. [11, Formula (5.11)]. Let us mention that in [6, Section 5] it was considered a sequence $h(n)$, defined by

$$h\left(\sum_i 2^{e_i}\right) = \sum_i 3^{e_i}$$

with strictly increasing exponents e_i, and in virtue of (3.23) we see that $h(n) = \frac{1}{2}\gamma_n$. In [6] it was mentioned that $h(1) < h(2) < \cdots < h(n)$ is the "minimal" sequence of n positive integers not containing an arithmetic progression. By means of the Mellin transformation it was shown [6, Theorem 5.1]:

$$H(N) := \sum_{n<N} h(n) = N^{\rho+1} F(\log_2 N) - \frac{1}{4}N$$

where $\rho = \log_2 3$ and where $F(u)$ is an 1-periodic function which has the Fourier series

$$F(u) = \frac{1}{3\log 2}\sum_{k\in\mathbb{Z}} \zeta(\rho+\chi_k)\frac{e^{2\pi iku}}{(\rho+\chi_k)(\rho+\chi_k+1)}$$

with $\chi_k = 2\pi ik/\log 2$ and Riemann's zeta function $\zeta(.)$.

4 Power sums related to digital sequences

Now we investigate the sum

$$S_m(N) = \sum_{n=0}^{N-1} C_n n^m \tag{4.1}$$

with $m \in \mathbb{N}_0$, where C_n is an arbitrary sequence with $C_0 = 1$ and (1.1) satisfying (1.2). For this we consider the two-scale difference equations (1.8) with $\lambda = p^n$ $(n \in \mathbb{N}_0)$ and $c_r = \frac{C_r}{C}$ with C from (1.1). By Theorem 2.1 we have for the solutions $\varphi(t) = \varphi_n(t)$ that

$$\varphi_n\left(\frac{t}{p}\right) = \frac{1}{Cp^n}\varphi_n(t) \qquad (0 \le t \le 1)$$

since $\varphi_n(t) = 0$ for $t < 0$. Choosing α_n so that $p^{\alpha_n} = Cp^n$ i.e.

$$\alpha_n = n + \log_p C \tag{4.2}$$

then

$$\frac{\varphi_n(\frac{t}{p})}{(\frac{t}{p})^{\alpha_n}} = \frac{\varphi_n(t)}{t^{\alpha_n}} \qquad (0 < t \le 1).$$

Hence, the functions

$$f_n(t) := \frac{\varphi_n(t)}{t^{\alpha_n}} \qquad (0 < t \le 1) \tag{4.3}$$

have the property $f_n(\frac{t}{p}) = f_n(t)$ so that they can be extended for all $t > 0$ by

$$f_n(pt) = f_n(t) \qquad (t > 0). \tag{4.4}$$

Theorem 4.1 *If (1.6) is satisfies then for $N \in \mathbb{N}$ the sum $S_m(N)$ from (4.1) can be represented as*

$$S_m(N) = N^\alpha \sum_{\mu=0}^{m} N^\mu F_{m,\mu}(\log_p N) \tag{4.5}$$

where $\alpha = \log_p C$ and where $F_{m,\mu}(u)$ are 1-periodic continuous functions which have the representations

$$F_{m,\mu}(u) = (-1)^m m!\, b_{m-\mu} \sum_{\nu=0}^{\mu} \frac{(-1)^\nu}{\nu!} f_{\mu-\nu}(p^u) \tag{4.6}$$

with the coefficients b_n from (1.19) and $f_n(.)$ from (4.3) and (4.4).

Proof: For given $N \in \mathbb{N}$ we choose ℓ such that $p^\ell \ge N$. From (2.6) with $n = m$ and $k = N$ we get

$$\varphi_m\left(\frac{N}{p^\ell}\right) = \frac{c_0^\ell}{p^{m\ell}} \sum_{j=1}^{N} C_{N-j} p_m(j)$$

where φ_m is the continuous solution of (1.8) with $\lambda = p^m$ satisfying (1.10). With $j = N - n$ it follows in view of $c_0 = \frac{1}{C}$ and $p^m C = p^{\alpha_m}$

$$\sum_{n=0}^{N-1} C_n p_m(N-n) = \frac{p^{m\ell}}{c_0^\ell} \varphi_m\left(\frac{N}{p^\ell}\right) = N^{\alpha_m} \left(\frac{p^\ell}{N}\right)^{\alpha_m} \varphi_m\left(\frac{N}{p^\ell}\right).$$

In virtue of (4.2) and (4.4) it follows

$$\sum_{n=0}^{N-1} C_n p_m(N-n) = N^{\alpha+m} f_m(N)\ . \tag{4.7}$$

Next we write $p_m(N-n)$ as polynomial with respect to n. By Taylor's formula

$$p_m(N-n) = \sum_{\mu=0}^{m} p_m^{(\mu)}(N)\frac{(-n)^\mu}{\mu!} = \sum_{\mu=0}^{m} \frac{(-1)^\mu}{\mu!} p_{m-\mu}(N) n^\nu$$

where we have used that $p_m(t)$ are Appell polynomials. It follows

$$\sum_{n=0}^{N-1} C_n p_m(N-n) = \sum_{\mu=0}^{m} \frac{(-1)^\mu}{\mu!} p_{m-\mu}(N) \sum_{n=0}^{N-1} C_n n^\mu$$

and comparison with (4.7) yields in view of (4.1) that

$$N^{\alpha+m} f_m(N) = \sum_{\mu=0}^{m} \frac{(-1)^\mu}{\mu!} p_{m-\mu}(N) S_\mu(N).$$

Multiplication by z^m and summation over m we get in view of the Cauchy product and (1.15)

$$\begin{aligned}\sum_{m=0}^{\infty} N^{\alpha+m} f_m(N) z^m &= \sum_{n=0}^{\infty} p_n(N) z^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} S_m(N) z^m \\ &= e^{Nz}\Phi(z) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} S_m(N) z^m.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} S_m(N) z^m &= \frac{e^{-Nz}}{\Phi(z)} \sum_{m=0}^{\infty} N^{\alpha+m} f_m(N) z^m \\ &= \sum_{n=0}^{\infty} q_n(-N) z^n \sum_{m=0}^{\infty} N^{\alpha+m} f_m(N) z^m\end{aligned}$$

where we have used (1.22) with $t = -N$. Comparison of coefficients implies in view of the Cauchy product

$$\frac{(-1)^m}{m!} S_m(N) = \sum_{n=0}^{m} q_{m-n}(-N) N^{\alpha+n} f_n(N).$$

Moreover, for the Appell polynomials $q_n(t)$ we have by (1.21) the representation

$$q_{m-n}(-N) = \sum_{k=0}^{m-n} \frac{b_{m-n-k}}{k!}(-1)^k N^k$$

so that with the substitution $\mu = n + k$ we get

$$\begin{aligned} \frac{(-1)^m}{m!} S_m(N) &= N^\alpha \sum_{n=0}^{m} \sum_{k=0}^{m-n} \frac{b_{m-n-k}}{k!}(-1)^k N^{n+k} f_n(N) \\ &= N^\alpha \sum_{\mu=0}^{m} \sum_{n=0}^{\mu} (-1)^{\mu-n} \frac{b_{m-\mu}}{(\mu-n)!} N^\mu f_n(N) \\ &= N^\alpha \sum_{\mu=0}^{m} b_{m-\mu} N^\mu \sum_{n=0}^{\mu} (-1)^{\mu-n} \frac{1}{(\mu-n)!} f_n(N) \end{aligned}$$

and it follows (4.5) with (4.6). □

Remark 4.2 In the simple case $C_n = 1$ for all $n \in \mathbb{N}_0$ the sum (4.1) is the usual power sum. In this case equation (1.8) with $\lambda = 1$ has the solution $\varphi_0(t) = t$ for $0 \leq t \leq 1$ so that the iterated integrals are $\varphi_n(t) = \frac{1}{(n+1)!} t^{n+1}$ in $[0, 1]$. From (4.2) we get $\alpha_n = n + 1$ so that $f_n(t) = \frac{1}{(n+1)!}$ for all $t > 0$. Hence, the functions $F_{m,\mu}$ from (4.6) are constant and it easy to see that (4.5) yields the known representation (1.4) with the Bernoulli polynomials.

In the following we again exclude the trivial case $C_n = 1$ for all n.

Proposition 4.3 *If (1.6) is satisfies then the 1-periodic continuous functions $F_{m,\mu}(u)$ from (4.6) have the following properties:*

1. *Each of the functions $F_{m,\mu}$ is Hölder continuous.*
2. *If $pM_0 < C$ where $M_0 = |C_0 C_1 \cdots C_{p-1}|^{1/p}$ then each $F_{m,\mu}$ is differentiable almost everywhere and if $pM_0 \geq C$ then each $F_{m,\mu}$ is almost nowhere differentiable.*
3. *Each of the functions $F_{m,\mu}$ has finite total variation on $[0, 1]$ if and only if $C_r \geq 0$ for all $r = 0, 1, \ldots, p - 1$.*

Proof: Owing to (4.6) and (4.3) we see in view of the fact that φ_n are the iterated integrals of φ_0, that the analytic properties as differentiability of $F_{m,\mu}$ are determined by the function f_0. So the assertions are consequences of Proposition 3.5. □

5 Specific power sums

We consider the sum (4.1) for $N = p^k$, i.e.

$$S_m(p^k) = \sum_{n=0}^{p^k-1} C_n n^m. \tag{5.1}$$

In order to get a simple formula for this sum we need the following lemma.

Lemma 5.1 *For the 1-periodic function $F_{m,\mu}(.)$ from (4.6) we have*

$$F_{m,\mu}(0) = (-1)^m m!\, b_{m-\mu} a_\mu \tag{5.2}$$

with the coefficients a_n from (1.16) and b_n from (1.19).

Proof: From (4.6) with $u = 0$ we get

$$F_{m,\mu}(0) = (-1)^m m!\, b_{m-\mu} d_\mu$$

with

$$d_\mu = \sum_{\nu=0}^{\mu} \frac{(-1)^\nu}{\nu!} f_{\mu-\nu}(1). \tag{5.3}$$

Multiplication by t^μ and summation over μ yields in view of the Cauchy product

$$\begin{aligned} \sum_{\mu=0}^{\infty} d_\mu z^\mu &= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} z^\nu \sum_{\mu=0}^{\infty} f_\mu(1) z^\mu \\ &= e^{-z} \sum_{\mu=0}^{\infty} f_\mu(1) z^\mu. \end{aligned}$$

Further, by (4.3) we have $f_n(1) = \varphi_n(1)$ and by (1.14) also $\varphi_n(1) = p_n(1)$. Hence, in view of (1.15) with $t = 1$ we get

$$\sum_{\mu=0}^{\infty} f_\mu(1) z^\mu = \sum_{n=0}^{\infty} p_n(1) z^n = e^z \Phi(z).$$

It follows

$$\sum_{\mu=0}^{\infty} d_\mu z^\mu = \Phi(z)$$

so that $d_\mu = a_\mu$ according to (1.16). □

Theorem 4.1 and Lemma 5.1 imply

Proposition 5.2 *The sum (4.1) for $N = p^k$ with $k \in \mathbb{N}$ reads*

$$S_m(p^k) = (-1)^m m! p^{\alpha k} \sum_{\mu=0}^{m} p^{\mu k} a_\mu b_{m-\mu} \tag{5.4}$$

where $\alpha = \log_p C$ with C from (1.1), a_n from (1.16) and b_n from (1.19).

Remark 5.3 Formula (5.4) for $m = 0$ yields

$$S_0(p^k) = \sum_{n=0}^{p^k-1} C_n = p^{\alpha k} = C^k$$

in accordance with (3.12).

If we introduce the polynomials

$$P_m(t) := \sum_{\mu=0}^{m} t^\mu a_\mu b_{m-\mu} \tag{5.5}$$

then in virtue of (5.4) we have

$$S_m(p^k) = (-1)^m m! p^{\alpha k} P_m(p^k). \tag{5.6}$$

Lemma 5.4 *The polynomials $P_m(t)$ have the generating function*

$$\sum_{m=0}^{\infty} P_m(t) z^m = \frac{\Phi(tz)}{\Phi(z)}. \tag{5.7}$$

with Φ from (1.16), cf. also (1.12).

Proof: By multiplication of the power series (1.16) with tz in place of z and (1.19) we get by means of the Cauchy product

$$\begin{aligned} \Phi(tz)\frac{1}{\Phi(z)} &= \sum_{n=0}^{\infty} a_n (tz)^n \sum_{n=0}^{\infty} b_n z^n \\ &= \sum_{m=0}^{\infty} \left(\sum_{\mu=0}^{m} t^\mu a_\mu b_{m-\mu} \right) z^m \end{aligned}$$

and in view of (5.5) it follows (5.7). □

Proposition 5.5 *The polynomials P_m from (5.5) satisfy the relation*

$$P_m(st) = \sum_{\mu=0}^{m} s^\mu P_\mu(t) P_{m-\mu}(s). \tag{5.8}$$

Proof: By repeated application of (5.7) we get

$$\begin{aligned} \sum_{m=0}^{\infty} P_m(st) z^m &= \frac{\Phi(stz)}{\Phi(z)} = \frac{\Phi(stz)}{\Phi(sz)} \cdot \frac{\Phi(sz)}{\Phi(z)} \\ &= \sum_{m=0}^{\infty} P_m(t)(sz)^m \sum_{m=0}^{\infty} P_m(s) z^m \\ &= \sum_{m=0}^{\infty} \left(\sum_{\mu=0}^{m} s^\mu P_\mu(t) P_{m-\mu}(s) \right) z^m \end{aligned}$$

where we have used the Cauchy product. Comparison of coefficients yields (5.8). □

Proposition 5.6 *For positive integers k, ℓ the sums (5.1) satisfy the relation*

$$S_m(p^{k+\ell}) = \sum_{\mu=0}^{m} \binom{m}{\mu} p^{k\mu} S_\mu(p^\ell) S_{m-\mu}(p^k). \tag{5.9}$$

Proof: Applying (5.6) and (5.8) we get

$$\begin{aligned}
\frac{(-1)^m}{m!} S_m(p^{k+\ell}) &= p^{\alpha(k+\ell)} P_m(p^{k+\ell}) \\
&= p^{\alpha(k+\ell)} \sum_{\mu=0}^{m} p^{k\mu} P_\mu(p^\ell) P_{m-\mu}(p^k) \\
&= \sum_{\mu=0}^{m} p^{\mu} \frac{(-1)^\mu}{\mu!} S_\mu(p^\ell) \frac{(-1)^{m-\mu}}{(m-\mu)!} S_{m-\mu}(p^k)
\end{aligned}$$

which implies (5.9). □

Remark 5.7 Let us mention that in the simple case $C_n = 1$ for all n the polynomials (5.5) can be represented as

$$P_m(t) = \frac{(-1)^m}{m!} \cdot \frac{1}{t} \tilde{B}_m(t) \tag{5.10}$$

with the polynomials $\tilde{B}_m(t)$ from (1.5) which as is known have the generating function

$$\frac{e^{tz} - 1}{e^z - 1} = \sum_{m=0}^{\infty} \frac{\tilde{B}_m(t)}{m!} z^m \qquad (|z| < 2\pi). \tag{5.11}$$

In order to see (5.10) we note that in case $C_n = 1$ for all n the polynomial (1.13) has the form

$$P(w) = \frac{1}{p}(1 + w + \cdots + w^{p-1}) = \frac{1 - w^p}{p(1-w)}$$

so that for Φ from (1.12) we obtain

$$\Phi(z) = \frac{1 - e^{-z}}{z}.$$

Therefore

$$\frac{\Phi(-tz)}{\Phi(-z)} = \frac{e^{tz} - 1}{t(e^z - 1)}$$

and in virtue of (5.11) and Lemma 5.4 it follows (5.10).

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received: March 10, 2014

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Rostock. Math. Kolloq. **68**, 65 – 69 (2013)

Subject Classification (AMS)
46A20, 46B50

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An abstract Alaoglu theorem

1 Introduction

We start with the paper [6] and repeat the definition of the dual space X^d of the space X w. r. t. another space Y. By Corollary 3.3 of this paper was given a (strong) generalization of the Alaoglu theorem for normed spaces. The Alaoglu theorem concerns the compactness of subsets $H \subseteq X^d$, where for H the pointwise topology τ_p is considered. Of course for the classical Alaoglu theorem we have:

$$(X, \|\cdot\|) \text{ is a Banachspace}, \quad Y = \mathbb{K}, \quad X^d = X', \quad H = B(X') \subseteq X',$$

the norm-closed ball, and the topology τ_p here is nothing else than the weak-star topology.

What is the aim of this paper? We prove a general τ_p-compactness theorem for special subsets $H \subseteq X^d$ (Theorem 2.3). This theorem includes both the generalized Alaoglu theorem of [6], Corollary 3.3 and the Alaoglu theorem for locally convex topological vector spaces (see for instance [5]).

2 The τ_p-compactness theorem

Definition 2.1 *Let (X, τ), (Y, σ) be topological spaces. Moreover let X, Y be spaces with finitely many algebraic operations such that X, Y belong to the same class of such spaces. We assume that we can assign to each algebraic operation in X an algebraic operation in Y (in a naturel manner).*

$X^d = \{h : X \to Y |$ h is continuous and h is a homomorphism with respect to each pair of corresponding algebraic operations in X and in Y respectively$\}$; X^d is called the (first) dual space of X with respect to Y or the Y-dual of X.

Remark 2.2 1. A theory of this general concept of duality is developed in [6], [2], [3] (definitions of X^d and of the second Y-dual space X^{dd} together with the corresponding toolbox).

Hereby in [2] the definition of X^d and that of the other notions (definitions, propositions and theorems occuring in [2]) is presented very precisely using the language of universal algebra.

2. Clearly, a closed subset of a compact topological space is compact too. Hence, we not only look for the τ_p-compactness of some sets $H \subseteq X^d$ but also for the τ_p-closedness of H.

3. We still provide a notion of relative compactness we need:

   if X is a topological space, $A \subseteq X$ is called relatively compact in X iff holds: each open cover of X has a finite subcover which covers A, or equivalently:

   for each ultrafilter π on X:

$$A \in \pi \Longrightarrow \exists\, x \in X,\ \pi \longrightarrow x.$$

   For the definition and especially the properties of this notion see [8], [1]. Sometimes one defines: A is relatively compact iff the closure $\overline{A}$ is compact, a notion somewhat stronger than the first one.

   For regular spaces the two notions coincide.

Theorem 2.3 *Let (X,τ), (Y,σ) be topological spaces with algebraic structure according to definition 2.1 and let us consider X^d. We assume that (Y,σ) is Hausdorff such that all algebraic operations in Y are continuous with respect to σ. Moreover there is a bornology $\underline{B}$ in Y.*

We assume that there exists a family $(K_x)_{x\in X}$ such that $\forall\, x \in X$, $K_x \in \underline{B}$ and $K_x \neq \emptyset$; for the product $\Pi\{K_x|x \in X\}$ we consider the Tychonoff-topology; thus $\Pi\{K_x|x \in X\}$ is a subspace of $(Y^X;\tau_p)$.

$$H = \{h \in X^d | \forall\, x \in X :\ h(x) \in K_x\};$$

identifying

$$h \equiv (h(x))_{x\in X} \quad \textit{we get} \quad H \subseteq \Pi\{K_x|x \in X\}.$$

Finally, we assume

1. *For $(Y;\underline{B})$ holds: $\forall\, B \in \underline{B}$: B is relatively compact in Y.*
2. *Either*

   (a) *H is τ_p-closed in Y^X, or*

(b) $\forall\, x \in X$: *K_x is closed and H is closed in* $\Pi\{K_x | x \in X\}$.

Then H is compact and Hausdorff in (Y^X, τ_p) and hence in (X^d, τ_p) too.

Proof: $\forall\, x \in X$, $K_x \subseteq Y$ is relatively compact in $Y \Longrightarrow \Pi\{K_x|x \in X\}$ is relatively compact in $(Y^X; \tau_p)$ by the Tychonoff theorem. Y Hausdorff $\Longrightarrow \forall\, x \in X : K_x$ is Hausdorff $\Longrightarrow \Pi\{K_x|x \in X\}$ is Hausdorff. By 2. (a) H is τ_p-closed in Y^X and since $H \subseteq \Pi\{K_x|x \in X\}$, H is relatively compact in $(Y^X; \tau_p)$ too and hence H is compact and Hausdorff in (Y^X, τ_p).

We have $H \subseteq X^d \subseteq Y^X$ and hence H is Hausdorff and compact in $(X^d; \tau_p)$, too.

By 2. (b) each K_x is relatively compact and closed yielding that K_x is compact and thus $\Pi\{K_x|x \in X\}$ is a Hausdorff and compact topological space again by Tychonoff. Then H being closed in $\Pi\{K_x|x \in X\}$, is compact in $\Pi\{K_x|x \in X\}$ and hence H is Hausdorff and compact in (Y^X, τ_p) and in (X^d, τ_p) respectively.

Corollary 2.4 *Let X, Y be Hausdorff locally convex topological vector spaces (shortly: l. c. s.); $\underline{B} = \{B \subseteq Y | B$ is bounded$\}$; clearly $\underline{B}$ is a bornology. Only the vector space operations are the algebraic operations in X and Y and these operations are continuous w. r. t. the topologies of X and Y. Now let Y be a Montel space, which means a Hausdorff barreled l. c. s. such that $\forall\, B \subseteq Y : B$ bounded $\Longrightarrow B$ is relatively compact.*

With these spaces X, Y, their properties and with the other assumptions from Theorem 2.3 the assertions of Theorem 2.3 hold here.

3 Application

We will deduce from Theorem 2.3 or Corollary 2.4 respectively the classical Alaoglu theorem for l. c. s. (sometimes called Alaoglu-Bourbaki theorem) and the generalized τ_p-theorem in Corollary 3.3 of [6].

A. Let X be a Hausdorff l. c. s. and $Y = \mathbb{K}$; $\mathbb{R}$ and $\mathbb{C}$ are Montel spaces (as normed Euclidian spaces) and thus the vector space operations in $\mathbb{K}$ are continuous. $X^d = \{h : X \to \mathbb{K}\,|\, h$ linear and continuous$\} = X'$. Now let U be a neighborhood of $o \in X$ and $U^o = \{h \in X^d | \forall\, x \in U : |h(x)| \leq 1\}$ the polar of U; let $H = U^o$; $\forall\, x \in U$,

$$K_x = \{y \in \mathbb{K}\,|\, |y| \leq 1\},$$

where $|\cdot|$ is the $\mathbb{K}$-norm; $\forall\, z \in X \backslash U$, $\{z\}$ is bounded in X and hence there exists $\lambda_z > 0 :\ z \in \lambda_z U$.

Hence $\forall\, z \in X\backslash U$, $\{\lambda > 0 \,|\, z \in \lambda U\} \neq \emptyset$; by the axiom of choice there exists a vector $(\lambda_z)_{z\in X\backslash U} : \forall\, z \in X\backslash U : z \in \lambda_z U$.

$$\forall\, x \in U : \;\lambda_x := 1;\; \forall\, x \in X : \;K_x = \{y \in \mathbb{K} \,|\, |y| \leq \lambda_x\};$$

each K_x is bounded and closed in $\mathbb{K}$ and hence compact implying that K_x is relatively compact too, of course each K_x is Hausdorff.

We show:

1. $H = U^o \subseteq \Pi\{K_x \,|\, x \in X\}$.
2. U^o is closed in $(\mathbb{K}^X; \tau_p)$.

Then Corollary 2.4 shows:

U^o is τ_p-compact and Hausdorff in $\mathbb{K}^X$ and in X^d too, thus showing the assertion of the Alaoglu-Bourbaki theorem.

1. $\forall\, h \in U^o : \forall\, x \in U$, $|h(x)| \leq 1 = \lambda_x \Longrightarrow h(x) \in K_x$; $x \in X\backslash U \Longrightarrow x \in \lambda_x U \Longrightarrow x = \lambda_x u$, $u \in U \Longrightarrow h(x) = \lambda_x h(u) \Longrightarrow |h(x)| = \lambda_x |h(u)| \leq \lambda_x \Longrightarrow h(x) \in K_x$. Hence $U^o \subseteq \Pi\{K_x | x \in X\}$.
2. We know that the polar set U^o is equicontinuous and hence U^o is evenly continuous; let (h_i) be a net from U^o, $h_i \xrightarrow{\tau_p} h \in \mathbb{K}^X$; $\forall\, i, h_i$ is linear $\Longrightarrow h$ is linear by proposition 3.1 of [6]. Since U^o is evenly continuous and $h_i \xrightarrow{\tau_p} h$ we get $h_i \xrightarrow{c} h$ (continuous convergence) by Theorem 31 of [4] (see also [7]).

   Now $\mathbb{K}$ is a regular topological space and thus h is continuous by Theorem 30 of [4].

   Hence $h \in X^d$; $\forall\, x \in U$, $h_i(x) \longrightarrow h(x)$, $\forall\, i$, $|h_i(x)| \leq 1 \Longrightarrow |h(x)| \leq 1$, meaning that $h \in U^o$ and thus U^o is closed in $(\mathbb{K}^X, \tau_p)$.

B. Now let our l. c. s. X, Y be Banach spaces.

   But that means that $Y = (Y, \|\cdot\|)$ is a finite-dimensional normed space because Y is a Montel space.

In [6] there was defined:

$$\forall\, c \in \mathbb{R},\; c > 0 : H_c = \{h \in X^d \,|\, \|h\| \leq c\},$$

where $\|\cdot\|$ is the operator norm,

$$\forall\, c > 0,\; \forall\, x \in X : K_{x,c} = \{y \in Y |\, \|y\| \leq c\|x\|\}.$$

Then $H_c \subseteq \Pi\{K_{x,c} |\, x \in X\}$ and in Theorem 3.2 of [6] was shown that H_c is closed w. r. t. the Tychonoff-topology in $\Pi\{K_{x,c} |\, x \in X\}$. For a fixed $c > o$, $K_{x,c}$ is bounded and closed in Y and hence compact too, because Y is finite-dimensional.

Theorem 2.3 then shows that H_c is τ_p-compact in Y^X and in X^d respectively. But this is the assertion of Corollary 3.3 of [6].

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received: March 25, 2014

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Rostock. Math. Kolloq. **68**, 71 – 79 (2013)

Subject Classification (MSC)
05A15, 05C05, 05C30

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Ordered and non-ordered non-congruent convex quadrilaterals inscribed in a regular n-gon

ABSTRACT. Using several arguments, some authors showed that the number of non-congruent triangles inscribed in a regular n-gon equals $\{n^2/12\}$, where $\{x\}$ is the nearest integer to x. In this paper, we revisit the same problem, but study the number of ordered and non-ordered non-congruent convex quadrilaterals, for which we give simple closed formulas using Partition Theory. The paper is complemented by a study of two further kinds of quadrilaterals called proper and improper non-congruent convex quadrilaterals, which allows to give a formula that connects the number of triangles and ordered quadrilaterals. This formula can be considered as a new combinatorial interpretation of a certain identity in Partition Theory.

KEY WORDS. Congruent triangles; Congruent quadrilaterals; Ordered quadrilaterals; proper quadrilaterals; Integer partitions.

1 Introduction

In 1938, Anning proposed the following problem [6]: "*From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. How many essentially different possible triangles are there ?*". For any given positive integer $n \geq 3$, let $\Delta(n)$ denote the number of such triangles.

Using a geometric argument, Frame showed that $\Delta(n) = \{n^2/12\}$, where $\{x\}$ is the nearest integer to x. After that, other solutions were proposed by some authors, such as Auluck [2].

In 1978, Reis posed the following natural general problem: *From the vertices of a regular n-gon k are chosen to be the vertices of a k-gon. How many incongruent convex k-gons are there ?*

Let us first specify that two k-gons are called congruent if one k-gon can be moved to the other by rotation or reflection.

For any given positive integers $2 \leq k \leq n$, let $R(n,k)$ denotes the number of such k-gons. In 1979 Gupta [5] gave the solution of Reis's problem, using the Möbius inversion formula.

Theorem 1

$$R(n,k) = \frac{1}{2}\binom{\left\lfloor \frac{n-h_k}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{2k}\sum_{d/\gcd(n,k)} \varphi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1},$$

where $h_k \equiv k \pmod 2$ and $\varphi(n)$ is the Euler function.

One can find the first values of $R(n,k)$ in the Online Encyclopedia of Integer Sequences (OEIS) [7] as A004526 for $k=2$, A001399 for $k=3$, A005232 for $k=4$ and A032279 for $k=5$.

The immediate consequence of both Gupta's and Frame's Theorems is the following identity:

$$\left\{\frac{n^2}{12}\right\} = \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor + \frac{1}{6}\binom{n-1}{2} + \frac{\chi(3/n)}{3},$$

where $\chi(3/n) = 1$ if $n \equiv 0 \,(mod\ 3)$, 0 otherwise.

In 2004, Shevelev gave a short proof of Theorem 1, using a bijection between the set of convex polygons with the tops in the n-gon splitting points and the set of all (0,1)-configurations with the elements in these points [8].

The aim of this paper is to enumerate two kinds of non-congruent convex quadrilaterals, inscribed in a regular n-gon, the ordered ones which have the sequence of their sides's sizes ordered, denoted by $R_O(n,4)$ and those which are non-ordered denoted by $R_{\overline{O}}(n,4)$, using the Partition Theory. As an example, let us consider Figure 1 showing three quadrilaterals inscribed in a regular 12-gon, the first is not convex, the second is ordered while the third is not. Observe that the second quadrilateral generates 1+1+3+3 as a partition of 8 into four parts, that is why it is called ordered.

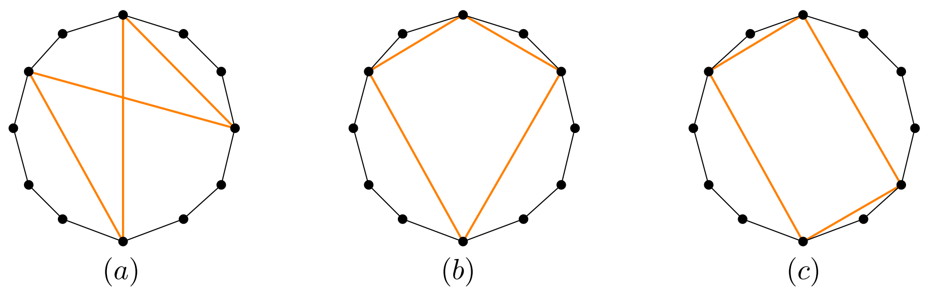

Figure 1

2 Notations and preliminaries

We denote by G_n a regular n-gon and by $\mathbb{N}$ the set of nonnegative integers. The partition of $n \in \mathbb{N}$ into k parts is a tuple $\pi = (\pi_1, \ldots, \pi_k) \in \mathbb{N}^k, k \in \mathbb{N}$, such that

$$n = \pi_1 + \cdots + \pi_k, \; 1 \leq \pi_1 \leq \cdots \leq \pi_k,$$

where the nonnegative integers π_i are called parts. We denote the number of partitions of n into k parts by $p(n,k)$, the number of partitions of n into parts less than or equal to k by $P(n,k)$ and by $q(n,k)$ we denote the number of partitions of n into k distinct parts. We sometimes write a partition of n into k parts $\pi = (\pi_1^{f_1}, \ldots, \pi_s^{f_s})$, where $\sum_{i=1}^{s} f_i = k$, the value of f_i is termed as frequency of the part π_i. For $m \in \mathbb{N}, m \leq k$, we denote the number of partitions of n into k parts $\pi = (\pi_1^{f_1}, \ldots, \pi_s^{f_s})$ for which $1 \leq f_i \leq m$ and $f_j = m$ for at least one $j \in \{1, \ldots, s\}$ by $c_m(n,k)$. For example $c_2(12,4) = 10$, since such partitions are exactly 1128, 1137, 1146, 1155, 1227, 1335, 1344, 2235, 2244, 2334. Let $\delta(n) \equiv n \; (mod\; 2)$, so that $\delta(n) = 1$ or 0, $\lfloor x \rfloor$ the integer part of x and finally $\{x\}$ the nearest integer to x.

3 Main results

In this section, we give the explicit formulas of $R_O(n,4)$ and $R_{\overline{O}}(n,4)$.

Theorem 2 *For $n \geq 4$,*

$$R_O(n,4) = \left\{ \frac{n^3}{144} + \frac{n^2}{48} - \frac{n\delta(n)}{16} \right\}.$$

Proof: First of all, notice that

$$R_O(n,4) = p(n,4). \tag{1}$$

Indeed, each ordered convex quadrilateral $ABCD$ inscribed in G_n can be viewed as a quadruple of integers (x,y,z,t), abbreviated for convenience as a word $xyzt$, such that:

$$\begin{cases} n - 4 = x + y + z + t; \\ 0 \leq x \leq y \leq z \leq t, \end{cases} \tag{2}$$

where x, y, z and t represent the number of vertices between A and B, B and C, C and D and finally between D and A, respectively. It should be noted, that the number of solutions of System (2) equals $p(n,4)$, by setting $x' = x+1$, $y' = y+1$, $z' = z+1$ and $t' = t+1$.

Now, let $g(z)$ be the known generating function of $p(n,4)$ [3]:

$$g(z) = \frac{z^4}{(1-z)(1-z^2)(1-z^3)(1-z^4)}.$$

From expanding $g(z)$ into partial fractions, we obtain

$$g(z) = \frac{1}{32\,(1+z)^2} - \frac{13}{288\,(1-z)^2} - \frac{1}{24\,(1-z)^3} + \frac{1}{24\,(1-z)^4} + \frac{1-z^2}{8\,(1-z^4)} - \frac{1-z}{9\,(1-z^3)}.$$

Via straightforward calculations, it can be proved that

$$g(z) = \sum_{n\geq 0} \left(\frac{(-1)^n\,(n+1)}{32} - \frac{13\,(n+1)}{288} - \frac{(n+1)\,(n+2)}{48} + \frac{\left(1+\frac{11}{6}n+n^2+\frac{1}{6}n^3\right)}{24} + \epsilon\,(n) \right) z^n,$$

where $\epsilon\,(n) \in \left\{-\frac{17}{72}, -\frac{1}{8}, -\frac{1}{9}, -\frac{1}{72}, 0, \frac{1}{72}, \frac{1}{9}, \frac{1}{8}, \frac{17}{72}\right\}.$

Thus, we have

$$g(z) = \sum_{n\geq 0} \left(\frac{n^3}{144} + \frac{n^2}{48} + \frac{((-1)^n - 1)\,n}{32} + \beta\,(n) \right) z^n,$$

where

$\beta\,(n) \in \left\{-\frac{5}{16}, -\frac{1}{4}, -\frac{29}{144}, -\frac{3}{16}, -\frac{5}{36}, -\frac{1}{8}, -\frac{13}{144}, -\frac{11}{144}, -\frac{1}{16}, -\frac{1}{36}, -\frac{1}{72}, 0, \frac{5}{144}, \frac{7}{144}, \frac{1}{9}, \frac{23}{144}, \frac{2}{9}, \frac{7}{72}\right\}.$

Since $p\,(n,4)$ is an integer and $|\beta\,(n)| < 1/2$, we get

$$p\,(n,4) = \left\{ \frac{n^3}{144} + \frac{n^2}{48} + \frac{((-1)^n - 1)\,n}{32} \right\}. \tag{3}$$

Hence, the result follows. □

Remark 3 Andrews and Eriksson said that the method used in the proof above dates back to Cayley and MacMahon [1, p. 58]. Using the same method [1, p. 60], they proved the following formula for $P(n,4)$:

$$P\,(n,4) = \left\{ \frac{(n+1)\,(n^2+23n+85)}{144} - \frac{(n+4)\left\lfloor \frac{n+1}{2} \right\rfloor}{8} \right\}.$$

Because $p(n,k) = P(n-k,k)$ (see for example [4]), it follows:

$$p\,(n,4) = \left\{ \frac{n^3}{144} + \frac{n^2}{12} - \frac{n}{8} - \frac{n\left\lfloor \frac{n-1}{2} \right\rfloor}{8} \right\}. \tag{4}$$

Note that the formula (3) seems a little bit simpler than (4).

To give an explicit formula for $R_{\overline{O}}\,(n,4)$, we need the following lemma.

Lemma 4 *For $n \geq 4$,*

$$c_2\,(n,4) = p\,(n,4) - q\,(n,4) - \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Proof: By definition of $c_m(n,k)$ in Section 2, it easily follows that

$$c_2(n,4) = p(n,4) - (q(n,4) + c_3(n,4) + \chi(4/n)),$$

where $\chi(4/n) = 1$ if $n \equiv 0 \ (mod\ 4)$, 0 otherwise.

Furthermore, $c_3(n,4)$ can be considered as the number of integer solutions of the equation

$$3x + y = n, \text{ with } 1 \le y \ne x \ge 1.$$

Since $x \ne y$, the solution $x = y = n/4$, when 4 divides n, must be removed. Then, by taking $y = 1$, one can get $c_3(n,4) = \left\lfloor \frac{n-1}{3} \right\rfloor - \chi(4|n)$. This completes the proof. □

Now we can derive the following theorem.

Theorem 5 *For $n \ge 4$,*

$$R_{\overline{O}}(n,4) = \left\{ \frac{n^3}{144} + \frac{n^2}{48} - \frac{n\delta(n)}{16} \right\} + \left\{ \frac{(n-6)^3}{144} + \frac{(n-6)^2}{48} - \frac{(n-6)\delta(n)}{16} \right\} - \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Proof: First of all, notice that $q(n,k) = p(n - k(k-1)/2, k)$ [1]. Then from Theorem 2 we get

$$q(n,4) = p(n-6,4) = \left\{ \frac{(n-6)^3}{144} + \frac{(n-6)^2}{48} - \frac{(n-6)\delta(n)}{16} \right\}.$$

Therefore, it is enough to prove that

$$R_{\overline{O}}(n,4) = p(n,4) + q(n,4) - \left\lfloor \frac{n-1}{3} \right\rfloor. \tag{5}$$

In fact, each non-ordered convex quadrilateral may be obtained by permuting exactly two parts of some partition of n into four parts, which is associated from System (2) to a unique ordered convex quadrilateral. For example, in Figure 1 above, the ordered convex quadrilateral (b) assimilated to the solution 1133 of 8 or to the partition 2244 of 12, generates the non-ordered convex quadrilateral (c) via the permutation 1313. Obviously, not every partition of n can generate a non-ordered convex quadrilateral, those having three equal parts or four equal parts cannot. Also, each partition of n into four distinct parts $xyzt$ generates two non-ordered convex quadrilaterals, each one corresponds to one of the two following permutations $xytz$ and $xzyt$. On the other hand, each partition of n into two equal parts, like $xxyz$, with y and z both of them $\ne x$, generates only one non-ordered convex quadrilateral, corresponding to the unique permutation $xyxz$. Thus,

$$R_{\overline{O}}(n,4) = 2q(n,4) + c_2(n,4). \tag{6}$$

Hence, from Lemma 4 the assertion follows. □

Remark 6 By substituting $k = 4$ in Theorem 1, we get

$$R(n,4) = \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor}{2} + \frac{1}{8}\binom{n-1}{3} + \frac{n(1-\delta(n))}{16} + \alpha,$$

where

$$\alpha = \begin{cases} \frac{1}{8} & if\ n \equiv 0\ (mod\ 4), \\ -\frac{1}{8} & if\ n \equiv 2\ (mod\ 4), \\ 0 & otherwise. \end{cases}$$

Knowing furthermore that

$$R(n,4) = R_O(n,4) + R_{\overline{O}}(n,4),$$

the following identity follows according to Theorem 1 and Theorem 5:

$$\begin{aligned} \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor}{2} + \frac{1}{8}\binom{n-1}{3} + \frac{n(1-\delta(n))}{16} + \alpha &= 2\left\{\frac{n^3}{144} + \frac{n^2}{48} - \frac{n\delta(n)}{16}\right\} + \\ &+ \left\{\frac{(n-6)^3}{144} + \frac{(n-6)^2}{48} - \frac{(n-6)\delta(n)}{16}\right\} - \\ &- \left\lfloor \frac{n-1}{3} \right\rfloor. \end{aligned}$$

4 Connecting formula between $\Delta(n)$ and $R_O(n,4)$

There are two further kinds of quadrilaterals inscribed in G_n, the proper ones, those which do not use the sides of G_n and the improper ones, those using them. In Figure 2 below, two quadrilaterals inscribed in G_{12} are shown, the first one is proper while the second is not.

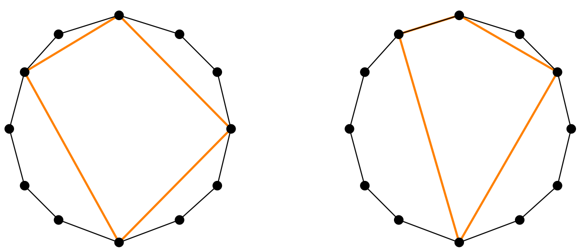

Figure 2

Let denote the number of these two kinds of quadrilaterals by $R_O^P(n,4)$ and $R_O^{\overline{P}}(n,4)$, respectively. The goal of this section is to prove the following theorem.

Theorem 7 *For $n \geq 4$,*

$$\Delta(n) = R_O(n+1,4) - R_O(n-3,4).$$

Proof: Note first that an improper ordered quadrilateral is formed by at least one side of G_n, hence the concatenation of the vertices of one of such sides gives a triangle inscribed in G_{n-1}, as shown in Figure 3.

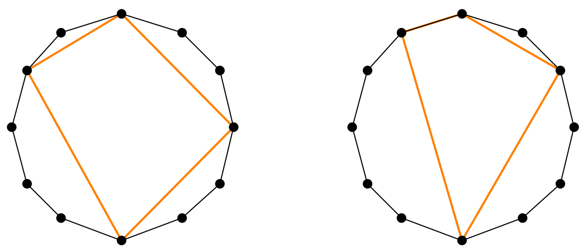

Figure 3

Thus we have

$$R_O^{\overline{P}}(n,4) = \Delta(n-1).$$

On the other hand, it is obvious to see that

$$R_O^{P}(n,4) = p(n-4,4).$$

Then from (1), we get

$$R_O^{P}(n,4) = R_O(n-4,4).$$

Since

$$R_O(n,4) = R_O^{P}(n,4) + R_O^{\overline{P}}(n,4),$$

we obtain

$$R_O(n,4) = R_O(n-4,4) + \Delta(n-1).$$

So, the theorem has been proved by substituting n by $n+1$. □

Remark 8 The well-known recurrence relation [4, p. 373],

$$p(n,k) = p(n+1,k+1) - p(n-k,k+1), \tag{7}$$

implies by setting $k = 3$,

$$p(n,3) = p(n+1,4) - p(n-3,4). \tag{8}$$

Thus, as we can see, the formula of Theorem 7 can be considered as a combinatorial interpretation of identity (8).

For $k \leq n$, we have the following generalization, using the same arguments to prove Theorem 7.

Theorem 9 *For $n \geq k$,*

$$R_O(n, k) = R_O(n+1, k+1) - R_O(n-k, k+1).$$

The formula of Theorem 9 can be considered as a combinatorial interpretation of the recurrence formula (7).

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received: January 19, 2014

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Rostock. Math. Kolloq. **68**, 81 – 82 (2013)

Subject Classification (MSC)
54A05, 54B30, 54D35, 54E17

Dieter Leseberg

Erratum to "Improved nearness research II" [Rostock. Math. Kolloq. 66, 87 – 102 (2011)]

KEY WORDS AND PHRASES. LEADER proximity; supertopological space; LODATO space; supernear space; superclan space; Bounded Topology.

Theorem 2.32 *states that the category CG-SN is bicoreflective in G-SN.*

But this theorem has to be replaced by the following one:

Theorem 2.32 *The category CG-SN is bireflective in G-SN.*

Proof. For a supergrill space $(X, \mathcal{B}^X, N)$ we set for each $B \in \mathcal{B}^X$:

$$N_C(B) := \{\rho \subset \underline{P}X : \{cl_n(F) : F \in \rho\} \subset \bigcup N(B)\}.$$

Then $(X, \mathcal{B}^X, N_C)$ is a conic supergrill space and $1_X : (X, \mathcal{B}^X, N) \to (X, \mathcal{B}^X, N_C)$ to be the bireflection in demand. First, we only show that N_C satisfies (sn$_7$): Let be $\{cl_{N_C}(A).A \in \mathcal{A}\} \in N_C(B)$ for $B \in \mathcal{B}^X, \mathcal{A} \subset \underline{P}X$, we have to verify $\mathcal{A} \in N_C(B)$ which means $cl_N(A) \in \bigcup N(B)$ for each $A \in \mathcal{A}$. $A \in \mathcal{A}$ implies $cl_N(cl_{N_C}(A)) \in \bigcup N(B)$ by hypothesis. We claim now that the statement $cl_{N_C} \subset cl_N(A)$ is valid. $x \in cl_{N_C}(A)$ implies $\{A\} \in N_C(\{x\})$, hence $cl_N(A) \in \bigcup N(\{x\})$. We can find $\rho \in N(\{x\})$ such that $cl_N(A) \in \rho$. Consequently $\{cl_N(A)\} \in N(\{x\})$ follows, which shows $\{A\} \in N(\{x\})$, hence $x \in cl_N(A)$ results. Altogether we get $cl_N(cl_{N_C}(A)) \subset cl_N(A)$ implying $cl_N(A) \in \bigcup N(B)$, because by hypothesis $cl_N(cl_{N_C}(A)) \in \rho_1$ for some $\rho_1 \in N(B)$. Secondly, we prove $\bigcup N_C(B) \in GRL(X)$ for each $B \in \mathcal{B}^X$. Let be given $B \in \mathcal{B}^X$, evidently $\varnothing \notin \bigcup N_C(B)$. Now, if $F_1 \in \bigcup N_C(B)$ and $F_1 \subset F_2 \subset X$, then there exists $\rho \in N_C(B)$ $F_1 \in \rho$. Consequently, $cl_N(F_1) \in \rho_1$ for some $\rho_1 \in N(B)$. By hypothesis we can find $\gamma \in N(B) \cap GRL(X)$ with $cl_N(F_1) \in \gamma$. Consequently $cl_N(F_2) \in \gamma$ follows, and $\{cl_N(F_2)\} \in N(B)$ is valid. Hence $F_2 \in \bigcup N_C(B)$ results. At last let be $F_1 \cup F_2 \in \bigcup N_C(B)$ then there exists $\rho \in N_C(B)$ with $F_1 \cup F_2 \in \rho$. By definition of N_C we get $\{cl_N(F) : F \in \rho\} \subset \bigcup N(B)$. Hence $cl_N(F_1 \cup F_2) \in \mathcal{A}$ for some

$\mathcal{A} \in N(B)$. Moreover we can choose $\gamma \in GRL(X) \cap N(B)$ with $cl_N(F_1) \cup cl_N(F_2) \in \gamma$. Consequently the statement $cl_N(F_1) \in \gamma$ or $cl_N(F_2) \in \gamma$ results. But then $cl_N(F_1) \in \bigcup N(B)$ or $cl_N(F_2) \in \bigcup N(B)$ is valid showing that $\{F_1\} \in N_C(B)$ or $\{F_2\} \in N_C(B)$, which concludes this part of proof. Evidently, $1_X : (X, \mathcal{B}^X, N) \to (X, \mathcal{B}^X, N_C)$ is sn-map. Now, let be given $(Y, \mathcal{B}^Y, M) \in Ob(CG - SN)$ and sn-map $f : (X, \mathcal{B}^X, N) \to (Y, \mathcal{B}^Y, M)$, we have to prove $f : (X, \mathcal{B}^X, N_C) \to (Y, \mathcal{B}^Y, M)$ is sn-map. For $\mathcal{B} \in B^Y$ and $\mathcal{A} \in N_C(B)$ we have to show $f\mathcal{A} \in M(f[B])$. Therefore it suffices to verify that the inclusion $f\mathcal{A} \subset \bigcup M(f[B])$ holds. For $A \in \mathcal{A}$ $cl_N(A) \in \rho$ for some $\rho \in N(B)$. Since f is sn-map we get $f\rho \in M(f[B])$. But $\{cl_M(f[A])\} << \{f[cl_N(A)]\} \in f\rho$. Consequently $\{cl_M(f[A])\} \in M(f[B])$ follows implying $\{f[A]\} \in M(f[B])$. But then $f[A] \in \bigcup M(f[B])$ results.

Definition 2.12 *explains when a given round paranear space* $(X, \mathcal{B}^X, N)$ *is LOproximal.*

The condition (LOp) has to be corrected as follows:

(LOp) $B \in \mathcal{B}^X \setminus \{\varnothing\}$, $\rho \subset p_N(B)$ and $\{B\} \cup \rho \subset \bigcap\{p_N(F) : F \in \rho \cap \mathcal{B}^X\}$ imply $\rho \in N(B)$, where Bp_NA iff $\{A\} \in N(B)$.

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received: September 27, 2013

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