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# The existence and $C^1$-smoothness of local center-unstable manifolds for differential equations with state-dependent delay

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ABSTRACT. The purpose of this work is to construct $C^1$-smooth local center-unstable manifolds at a stationary point for a class of functional differential equations of the form $\dot{x}(t) = f(x_t)$. Here the function $f$ under consideration is defined on an open subset of the space $C^1([-h, 0], \mathbb{R}^n)$, $h > 0$, and satisfies some mild smoothness conditions which are often fulfilled when $f$ represents the right-hand side of a differential equation with state-dependent delay.

KEY WORDS. Center-unstable manifold, functional differential equation, state-dependent delay

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## 1 Introduction

The interest in delay differential equations (abbreviated by DDE, respectively DDEs) dates back at least to the work [10] of Poisson from the year 1806. Even so, the general theory started to be systematically developed only at the beginning of the second half of the last century. During the 60th and 70th the theory of DDEs became an established field of mathematical research. In that progress, the development of another, more abstract class of differential equations, namely the so-called retarded functional differential equations (abbreviated by RFDE, respectively RFDEs), was essential. The development of the theory of RFDEs has also been started in the second half of the last century. We point out the fundamental work [3] and the newer edition [4] of Hale. Great parts of the theory of RFDEs is now as well understood as that for ordinary differential equations as presented in the monographs [2, 5].

Different DDEs with constant as well as with time- or state-dependent delay can be represented in the more abstract form of an RFDE. Accordingly, after carrying out such a transformation, one may ask whether basic or even far-reaching results for RFDEs may be used to study the original differential equation with delay. It turns out that the solution of this question is essentially dependent on the involved delays of the considered DDE. The reason is that the representation of a DDE in the more abstract form of an RFDE may lead to a loss of smoothness of the right-hand side if the involved delays are not constant. Therefore, the theory of RFDEs is in general not applicable to study DDEs with state-dependent delays and a lot of problems such as linearization and invariant manifolds for differential equations with state-dependent delay at a stationary point stayed open for many years.

In recent times, Walther introduced a modified class of functional differential equations and developed the fundamental theory in the series [13–15] of works under mild smoothness hypothesis. The main idea of Walther's approach is to study an abstract functional differential equation only on a smooth submanifold, the so-called solution manifold, of a function space. He proved that under mild smoothness assumptions the Cauchy problem is well-posed on the solution manifold, and the solutions generate a continuous semiflow with continuously differentiable solution operators. In particular, this framework seems to be often applicable in cases where the corresponding functional differential equation represents a DDE with state-dependent delay. Additionally, in cases of applicability it solves the difficulties concerning the linearization of a semiflow generated by differential equations with state-dependent delays. As long as the problem of linearization had not been solved, heuristical methods based on formal linearization were used for considerations as local stability and instability of stationary points. The work [1] of Cooke and Huang is indicative for such an approach.

In connection with the semiflow from the framework in [13–15] the existence of different types of local invariant manifolds at a stationary point is also well know by now. For instance, in [7] Krisztin considers an abstract class of functional differential equations and proves the existence of local unstable manifolds under a hyperbolicity condition but without knowledge of a semiflow. However, the result in [7] is also applicable in the situation of the semiflow discussed in [13–15]. Additionally, [7] discusses the construction of so-called fast or strong unstable manifolds without the hyperbolicity condition. A proof of the existence of continuously differentiable local stable and local center manifolds at stationary points is contained in the survey paper [6] of Hartung et al. and in the work [8] of Krisztin. The occurrence of continuously differentiable local center-stable manifolds is confirmed by Qesmi and Walther in the recent work [11].

The aim of this work is to prove the existence and $C^1$-smoothness of local center-unstable manifolds at stationary points for the semiflow from [13–15]. For this purpose, we first follow the approach used in Hartung et al. [6] for the construction of local center manifolds, and apply a modification of the Lyapunov-Perron method contained in Diekmann et al. [2] to establish the existence of Lipschitz continuous local center-unstable manifolds. Hereafter, we employ the techniques from Krisztin [8] to prove $C^1$-smoothness.

## 2 The Main Result

Let $h > 0$, $n \in \mathbb{N}$ and $\|\cdot\|_{\mathbb{R}^n}$ a norm in $\mathbb{R}^n$. For abbreviation, let us denote by $C$ the set of all continuous functions from the interval $[-h, 0]$ into $\mathbb{R}^n$, equipped with the norm

$$\|\varphi\|_C := \max_{s\in[-h,0]} \|\varphi(s)\|_{\mathbb{R}^n}$$

of uniform convergence. Analogously, we write $C^1$ for the Banach space of all continuously differentiable functions $\varphi : [-h, 0] \longrightarrow \mathbb{R}^n$, provided with the norm $\|\varphi\|_{C^1} := \|\varphi\|_C + \|\varphi'\|_C$.

For a given function $x : I \longrightarrow \mathbb{R}^n$ defined on some interval $I \subseteq \mathbb{R}$, and $t \in \mathbb{R}$ with $[t-h, t] \subset I$, the **segment** $x_t$ of $x$ at $t$ is defined by the relation $x_t(\vartheta) := x(t + \vartheta)$, $\vartheta \in [-h, 0]$; that is, by $x_t$ we restrict the function $x$ to $[t - h, t]$ and shift it back to $[-h, 0]$. In particular, if the function $x$ is continuous, then clearly $x_t \in C$.

Let $U \subseteq C^1$ be an open neighborhood of the origin $0 \in C^1$ and a function $f : U \longrightarrow \mathbb{R}^n$ with $f(0) = 0$ be given. Throughout this paper, we consider the functional differential equation

$$\dot{x}(t) = f(x_t) \tag{1}$$

under the following conditions on the right-hand side:

(S 1) $f$ is continuously differentiable, and

(S 2) each derivative $Df(\varphi)$, $\varphi \in U$, extends to a linear map

$$D_e f(\varphi) : C \longrightarrow \mathbb{R}^n,$$

and the induced map

$$U \times C \ni (\varphi, \chi) \longmapsto D_e f(\varphi)\,\chi$$

is continuous.

By a **solution** of the differential equation (1) we understand either a continuously differentiable function $x : [t_0 - h, t_e) \longrightarrow \mathbb{R}^n$ with $t_0 < t_e \leq \infty$ such that $x_t \in U$ for $t_0 \leq t < t_e$ and Eq. (1) holds for $t_0 < t < t_e$, or a continuously differentiable function $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $x_t \in U$ and Eq. (1) holds everywhere in $\mathbb{R}$. Additionally, we will consider solutions on unbounded, right-closed intervals $(-\infty, t_e]$, $-\infty < t_e$, which are defined in an analogous way.

By assumption $x(t) = 0$, $t \in \mathbb{R}$, is a solution of Eq. (1) as $f(0) = 0$. Therefore, the closed subset

$$X_f := \{\varphi \in U \mid \varphi'(0) = f(\varphi)\}$$

of $C^1$ is not empty. Under the above conditions on $f$ the framework developed in [13–15] implies the following fundamental results. The **solution manifold** $X_f$ is a $C^1$-submanifold of $U \subseteq C^1$ with codimension $n$. Each $\varphi \in X_f$ uniquely defines a constant $t_+(\varphi) > 0$ and a (in the forward time direction) non-continuable solution $x^\varphi : [-h, t_+(\varphi)) \longrightarrow \mathbb{R}^n$ of Eq. (1) with initial value $x_0^\varphi = \varphi$. All segments $x_t^\varphi$, $0 \leq t < t_+(\varphi)$ and $\varphi \in X_f$, belong to $X_f$ and the equations

$$F(t, \varphi) = x_t^\varphi$$

define a continuous semiflow $F : \Omega \longrightarrow X_f$ on the solution manifold $X_f$ where

$$\Omega = \{(t, \varphi) \in [0, \infty) \times X_f \mid 0 \leq t < t_+(\varphi)\}\,.$$

For every $t \geq 0$ the solution map at time $t$, that is, the map

$$F_t : \{\psi \in X_f |\ 0 \leq t < t_+(\psi)\} \ni \varphi \longmapsto F(t, \varphi) \in X_f,$$

is continuously differentiable, and for each $\varphi \in X_f$ the tangent space of $X_f$ at $\varphi$ is

$$T_\varphi X_f = \left\{\chi \in C^1 \mid \chi'(0) = Df(\varphi)\,\chi\right\}.$$

For all $(t, \varphi) \in \Omega$ and all $\chi \in T_\varphi X_f$ the derivative

$$DF_t\varphi : T_\varphi X_f \longrightarrow T_{F_t(\varphi)} X_f$$

satisfies the equations

$$DF_t(\varphi)\,\chi = v_t^{\varphi,\chi},$$

where $v^{\varphi,\chi} : [-h, t_+(\varphi)) \longrightarrow \mathbb{R}^n$ is the solution of the (linear) initial value problem

$$\begin{cases} \dot{v}(t) = Df(F(t, \varphi))\, v_t \\ v_0 = \chi \end{cases} \tag{2}$$

for $\chi \in T_\varphi X_f$. Here a solution of the Cauchy problem (2) is a continuously differentiable function $v : [-h, t_e(\varphi)) \longrightarrow \mathbb{R}$ such that $v_0 = \chi$, $v_t \in T_{F(t,\varphi)} X_f$ for all $0 \leq t < t_e(\varphi)$ and $v$ satisfies the differential equation for all $0 < t < t_e(\varphi)$.

Obviously, we have $F(t,0)=0$ for all $t\in\mathbb{R}$; that is, $\varphi_0 := 0\in X_f$ is a stationary point of the semiflow $F$. As discussed in Hartung et al. [6] the linearization of $F$ at $\varphi_0=0$ is the strongly continuous semigroup $T=\{T(t)\}_{t\geq 0}$ of bounded linear operators $T(t)=D_2F(t,0)$, $t\geq 0$, on the Banach space

$$T_0X_f=\left\{\chi\in C^1\mid \chi'(0)=Df(0)\,\chi\right\},$$

equipped with the norm $\|\cdot\|_{C^1}$ of $C^1$. For any $t\geq 0$ the action of $T(t)$ on an element $\chi\in T_0X_f$ is determined by the relation $T(t)\,\chi=v_t^{\chi}$, where $v^{\chi}:[-h,\infty)\longrightarrow\mathbb{R}^n$ is the unique solution of the variational equation

$$\dot v(t)=Df(0)\,v_t \tag{3}$$

with initial value $v_0=\chi$. The infinitesimal generator $G$ of $T$ is given by the linear operator

$$G:\mathcal{D}(G)\ni\chi\longmapsto\chi'\in T_0X_f$$

with domain

$$\mathcal{D}(G)=\left\{\chi\in C^2\,\middle|\,\chi'(0)=Df(0)\,\chi,\quad \chi''(0)=Df(0)\,\chi'\right\},$$

where $C^2$ denotes the set of all twice continuously differentiable functions from $[-h,0]$ into $\mathbb{R}^n$.

**Remark 2.1** For the convenience of the reader we repeat that an RFDE on some open subset $V\subset\mathbb{R}\times C$ is an equation of the form

$$\dot x(t)=f_e(t,x_t) \tag{4}$$

with a function $f_e:V\longrightarrow\mathbb{R}^n$. A function $x$ is a solution of Eq. (4) on the interval $[t_0-h,t_+)$, if there are $t_0\in\mathbb{R}$ and $t_+>t_0$ such that $x:[t_0-h,t_+)\longrightarrow\mathbb{R}^n$ is continuous, $(t,x_t)\in V$ for all $t_0\leq t<t_+$, and $x$ satisfies Eq. (4) for all $t_0<t<t_+$. Solutions on unbounded intervals $(-\infty,t_+)$ or $(-\infty,t_+]$ for some $t_+>-\infty$ are defined in an analogous way.

By assumption (S 2) on $f$ the linear operator $Df(0)$ may be extended to a bounded linear operator $D_ef(0)$ on the larger space $C$. The operator $L_e:=Df_e(0)$ induces the linear autonomous RFDE

$$\dot v(t)=L_e\,v_t$$

and the solutions of the associated initial value problem

$$\begin{cases}\dot v(t)=L_e\,v_t\\ v_0=\chi\end{cases} \tag{5}$$

for initial values $\chi \in C$ define a strongly continuous semigroup $T_e = \{T_e(t)\}_{t \geq 0}$ on $C$ as shown, for instance, in Diekmann et al. [2]. The infinitesimal generator of $T_e$ is

$$G_e : \mathcal{D}(G_e) \ni \chi \longmapsto \chi' \in C$$

with the domain

$$\mathcal{D}(G_e) = \left\{ \chi \in C^1 \,\middle|\, \chi'(0) = L_e \, \chi \right\}$$

which particularly coincides with $T_0 X_f$. We have $T(t)\,\varphi = T_e(t)\,\varphi$ for all $\varphi \in \mathcal{D}(G_e)$ and $t \geq 0$.

For the spectra $\sigma(G_e), \sigma(G) \subset \mathbb{C}$ of the generators $G_e, G$ of both semigroups we have

$$\sigma(G_e) = \sigma(G)$$

by [6]. The spectrum $\sigma(G_e)$ is given by the zeros of a familiar characteristic equation, is discrete and contains only eigenvalues of finite rank, that is, the generalized eigenspaces are finite-dimensional. Setting

$$\sigma_u(G_e) := \{\lambda \in \sigma(G_e) \mid \mathrm{Re}(\lambda) > 0\},$$
$$\sigma_c(G_e) := \{\lambda \in \sigma(G_e) \mid \mathrm{Re}(\lambda) = 0\}$$

and

$$\sigma_s(G_e) := \{\lambda \in \sigma(G_e) \mid \mathrm{Re}(\lambda) < 0\},$$

we obtain the decomposition

$$\sigma(G_e) = \sigma_u(G_e) \cup \sigma_c(G_e) \cup \sigma_s(G_e).$$

As proven in Hale and Verduyn Lunel [5] or in Diekmann et al. [2], for each $\beta \in \mathbb{R}$ the half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > \beta\}$ of $\mathbb{C}$ contains at most a finite number of elements of $\sigma(G_e)$, so that spectral parts $\sigma_u(G_e)$, $\sigma_c(G_e)$ are empty or finite. Hence, the associated realified generalized eigenspaces $C_u$ and $C_c$, which are called the **unstable** and the **center space** of $G_e$, respectively, are finite dimensional subspaces of $C$. In contrast, the **stable space** $C_s \subset C$ of $G_e$, that is, the realified generalized eigenspace associated to the spectral part $\sigma_s(G_e)$, is infinite-dimensional. The subspaces $C_u$, $C_c$ and $C_s$ are closed, invariant under $T_e(t)$, $t \geq 0$, and provide a decomposition

$$C = C_u \oplus C_c \oplus C_s \tag{6}$$

of $C$. The restriction of $T_e$ to the finite dimensional spaces $C_u$, $C_c$ has a bounded generator so that $T_e$ may be extended to a one-parameter group in each case.

As a consequence of the above decomposition of $C$ we obtain also a decomposition of the smaller Banach space $C^1$, namely

$$C^1 = C_u \oplus C_c \oplus C_s^1 \tag{7}$$

with the closed subspace $C_s^1 := C_s \cap C^1$ of $C^1$.

The sets $C_u, C_c$ lie in $\mathcal{D}(G_e) = T_0 X_f$ and coincide with the unstable and the center space of $G$, respectively. The stable space of $G$ is $C_s \cap T_0 X_f$. Consequently, we have the decomposition

$$T_0 X_f = C_u \oplus C_c \oplus (C_s \cap T_0 X_f).$$

All spaces are closed subspaces of $T_0 X_f$ and positively invariant under the operators $T(t)$, $t \geq 0$, and $T$ forms a one-parameter group on each of the finite-dimensional subspaces $C_u$ and $C_c$.

Using the notation $C_{cu} := C_u \oplus C_c$ for the **center-unstable space** of $G$, we are now able to state our result on the existence of local center-unstable manifolds for the semiflow $F$ at the stationary point $\varphi_0 = 0$.

**Theorem 1 (Existence of Local Center-Unstable Manifold)** *Suppose in addition to the previous assumptions on $f$ that $\{\lambda \in \sigma(G_e) \mid Re(\lambda) \geq 0\} \neq \emptyset$ or, equivalently, $C_{cu} \neq \{0\}$. Then there are open neighborhoods $C_{cu,0}$ of $0$ in $C_{cu}$ and $C_{s,0}^1$ of $0$ in $C_s^1$ with $N_{cu} := C_{cu,0} + C_{s,0}^1 \subseteq U$, and a Lipschitz continuous map $w_{cu} : C_{cu,0} \longrightarrow C_{s,0}^1$ with $w_{cu}(0) = 0$, such that the graph*

$$W_{cu} := \left\{ \varphi + w_{cu}(\varphi) \mid \varphi \in C_{cu,0} \right\}$$

*has the following properties.*

(i) *The set $W_{cu}$ belongs to the solution manifold $X_f$ of Eq. (1). Moreover, $W_{cu}$ is a $k$-dimensional Lipschitz submanifold of $X_f$ where $k := \dim C_{cu}$.*

(ii) *For each solution $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ of Eq. (1) on $(-\infty, 0]$, we have*

$$\left\{ x_t \mid t \leq 0 \right\} \subseteq N_{cu} \quad \Longrightarrow \quad \left\{ x_t \mid t \leq 0 \right\} \subseteq W_{cu}.$$

(iii) *The graph $W_{cu}$ is positively invariant with respect to the semiflow $F$ relative to $N_{cu}$; that is, if $\varphi \in W_{cu}$ and $t > 0$ then*

$$\left\{ F(s, \varphi) \mid 0 \leq s \leq t \right\} \subset N_{cu} \quad \Longrightarrow \quad \left\{ F(s, \varphi) \mid 0 \leq s \leq t \right\} \subset W_{cu}.$$

The submanifold $W_{cu}$ of $X_f$ is called a **local center-unstable manifold** of $F$ at the stationary point $\varphi_0 = 0$. It is $C^1$-smooth and passes $\varphi_0$ tangentially to the center-unstable space $C_{cu}$ as we shall have established by our next theorem.

**Theorem 2** **($C^1$-Smoothness of Local Center-Unstable Manifold)**
*The map*

$$w_{cu} : C_{cu,0} \longrightarrow C^1_{s,0}$$

*obtained in Theorem 1 is continuously differentiable and* $Dw_{cu}(0) = 0$.

In the next three sections we prove the above theorems. Even though the proofs are quite long and at certain points technical, they are nevertheless not difficult to understand. As mentioned in the introduction, we follow the construction of local center manifolds in Hartung et al. [6] and apply the Lyapunov-Perron method to obtain the existence of local center-unstable manifolds as claimed in Theorem 1. The basic idea of this method is to transform the differential equation (1), or more precisely, a smoothed modification of it, into an integral equation such that the corresponding integral operator forms a parameter-dependent contraction in an appropriate Banach space of continuous functions. The fixed points of this contraction define a mapping whose graph forms the desired invariant manifold.

After the described construction, we follow the procedure in Krisztin [8] and show the $C^1$-dependence of the obtained fixed points on the parameter which leads to the continuous differentiability of the manifolds asserted in Theorem 2.

## 3 Preliminaries for the Proof of Existence

For the transformation of the considered differential equation into an integral form we will employ a variation-of-constants formula, which is established in Diekmann et al. [2] and involves duality and adjoint semigroups. For the convenience of the reader and to make our exposition self-contained, we repeat some of the relevant material from Diekmann et al. [2] without proofs. Afterwards we discuss some preparatory results.

### Duality and Sun-Reflexivity

Recall that for a Banach space $X$ over $\mathbb{R}$ the **dual space** $X^*$ is the set of all continuous linear functionals on $X$, that is, $X^*$ consists of all continuous linear maps from $X$ into $\mathbb{R}$. We write $x^*$ for elements of $X^*$, and for $x^* \in X^*$ and $x \in X$ we use the notation $\langle x^*, x\rangle \in \mathbb{R}$ instead of $x^*(x)$. Provided with the norm

$$\|x^*\|_{X^*} := \sup_{\|x\|_X \leq 1} |\langle x^*, x\rangle| ,$$

where $\| \cdot \|_X$ denotes the norm on $X$, the dual space $X^*$ becomes also a Banach space over $\mathbb{R}$.

If $A : \mathcal{D}(A) \longrightarrow X$ is a linear operator defined on some dense linear subspace $\mathcal{D}(A)$ in $X$, then its **adjoint** $A^*$ is defined by

$$\mathcal{D}(A^*) = \Big\{ x^* \in X^* \Big| \ \exists\, y^* \in X^* \text{ with } \langle y^*, x\rangle = \langle x^*, A\,x\rangle \text{ for all } x \in \mathcal{D}(A) \Big\}$$

and then for $x^* \in \mathcal{D}(A^*)$

$$A^* x^* = y^*.$$

If $A : X \longrightarrow X$ is a bounded linear operator, then for each $x^* \in X^*$ the induced map $X \ni x \longmapsto \langle x^*, A\,x\rangle \in \mathbb{K}$ is linear and bounded. Thus, in this case, the relations

$$\langle A^* x^*, x\rangle = \langle x^*, A\,x\rangle$$

for all $x \in X$ and $x^* \in X^*$ uniquely define a bounded linear operator $A^* : X^* \longrightarrow X^*$. In particular, we have $\|A\| = \|A^*\|$.

Consider now the Banach space $C$ and the strongly continuous semigroup $T_e = \{T_e(t)\}_{t\geq 0}$ of bounded linear operators defined by the solutions of the initial value problem (5). For every $t \geq 0$ the adjoint $T_e^*(t)$ of $T_e(t)$ is a linear operator with norm $\|T_e^*(t)\| = \|T_e(t)\|$ on the dual space $C^*$ of $C$ and the family $T_e^* = \{T_e^*(t)\}_{t\geq 0}$ obviously constitutes a semigroup of operators on $C^*$. We also have $T_e^*(0)\,\varphi^* = \varphi^*$ for all $\varphi^* \in C^*$, but $T_e^*$ is in general not a strongly continuous semigroup. Indeed, if $C^*$ is equipped with the topology given by the norm $\|\cdot\|_{C^*}$, it is not difficult to see that for $\varphi^* \in C^*$ the induced curve

$$[0,\infty) \ni t \longmapsto T^*(t)\,\varphi^* \in C^* \tag{8}$$

is not necessarily continuous. However, the set of all functions $\varphi^\odot \in C^*$ for which the curve (8) is continuous, in other words, $\varphi^\odot \in C^*$ with the property $\|T_e^*(t)\,\varphi^\odot - \varphi^\odot\|_{C^*} \to 0$ as $t \searrow 0$, forms a closed subspace $C^\odot$ of $C^*$. Furthermore, $T_e^*(t)(C^\odot) \subset C^\odot$ for all $t \geq 0$ so that the family of operators

$$T_e^\odot(t) : C^\odot \ni \varphi^\odot \longmapsto T_e^*(t)\,\varphi^\odot \in C^\odot$$

constitutes a strongly continuous semigroup $T_e^\odot$ on $C^\odot$.

**Remark 3.1** It is worth to mention that the family $T_e^*$ of linear operators on $C^*$ is a weak* continuous semigroup, and $G_e^*$ the associated weak* generator. More precisely, if the dual space $C^*$ of $C$ is equipped with the so-called *weak* topology*, that is, the coarsest topology on $C^*$ such that for all $\varphi \in C$ the functions $C^* \ni \varphi^* \longmapsto \langle \varphi^*, \varphi\rangle \in \mathbb{R}$ are continuous, then for each $\varphi^* \in C^*$ the induced curve (8) is continuous. In this way, $T_e^*$ becomes a continuous semigroup and $G_e^*$ its generator.

Similarly, we can repeat the above process with the Banach space $C^\odot$ and the strongly continuous semigroup $T_e^\odot$. At first, we introduce again the adjoint operators $T_e^{\odot *}(t)$ of $T_e^\odot(t)$, $t \geq 0$, on the dual space $C^{\odot *}$ of $C^\odot$, and afterwards we restrict the semigroup $T_e^{\odot *} := \{T_e^{\odot *}(t)\}_{t\geq 0}$ to the closed subspace $C^{\odot\odot}$, for which the semigroup is strongly continuous.

The original Banach space $C$ together with the strongly continuous semigroup $T_e$ is $\odot$-reflexive in the sense that there is an isometric linear map $j : C \longrightarrow C^{\odot *}$ with $jC = C^{\odot\odot}$ and $T_e^{\odot *}(t)(j\varphi) = j(T_e(t)\,\varphi)$ for all $\varphi \in C$ and $t \geq 0$. We omit the embedding operator $j$ of $C$ in $C^{\odot *}$ and simply identify the Banach space $C$ with $C^{\odot\odot}$ as usual

The spectrum $\sigma(G_e^{\odot *})$ of the generator $G_e^{\odot *}$ for the semigroup $T_e^{\odot *}$ coincides with $\sigma(G_e)$, and the decomposition (6) of $C$ results in the decomposition

$$C^{\odot *} = C_u \oplus C_c \oplus C_s^{\odot *} \tag{9}$$

of $C^{\odot *}$, where $C_u$, $C_c$, and $C_s^{\odot *}$ are closed and invariant under $T_e^{\odot *}$. Furthermore, there are constants $K \geq 1$, $c_s < 0 < c_u$ and $c_c > 0$ with $c_c < \min\{-c_s, c_u\}$ so that the asymptotic behavior of $T^{\odot *}$ on these subspaces is given by

$$\begin{aligned} \|T_e(t)\,\varphi\|_C &\leq K e^{c_u t}\|\varphi\|_C, && t \leq 0,\ \varphi \in C_u, \\ \|T_e(t)\,\varphi\|_C &\leq K e^{c_c |t|}\|\varphi\|_C, && t \in \mathbb{R},\ \varphi \in C_c, \\ \|T_e^{\odot *}(t)\,\varphi^{\odot *}\|_{C^{\odot *}} &\leq K e^{c_s t}\|\varphi^{\odot *}\|_{C^{\odot *}}, && t \geq 0,\ \varphi^{\odot *} \in C_s^{\odot *}. \end{aligned} \tag{10}$$

The decompositions (7), (9) of $C^1$ and $C^{\odot *}$ induce continuous projections $P_u$, $P_c$, $P_s$ and analogously $P_u^{\odot *}$, $P_c^{\odot *}$, $P_s^{\odot *}$ onto subspaces $C_u$, $C_c$, $C_s^1$, and $C_u$, $C_c$, $C_s^{\odot *}$, respectively. Also, using the identification of $C$ with $C^{\odot\odot}$ we see at once $C_s^1 = C^1 \cap C_s^{\odot *}$.

### The Variation-of-Constants Formula

Next, we proceed with recalling the variation-of-constant formula for solutions of the inhomogeneous linear RFDE

$$\dot{x}(t) = L_e\, x_t + q(t) \tag{11}$$

with given function $q : I \longrightarrow \mathbb{R}^n$ on some interval $I \subset \mathbb{R}$. For this purpose, let $L^\infty([-h,0],\mathbb{R}^n)$ denote the Banach space of all measurable and essentially bounded functions from $[-h,0]$ into $\mathbb{R}^n$, provided with the norm $\|\cdot\|_{L^\infty}$ of essential least upper bound. With the norm

$$\|(\alpha,\varphi)\|_{\mathbb{R}^n\times L^\infty} := \max\{\|\alpha\|_{\mathbb{R}^n}, \|\varphi\|_{L^\infty}\},$$

the product space $\mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$ becomes also a Banach space, which is in particular isometrically isomorphic to the space $C^{\odot *}$. Using the temporary notation $k : C^{\odot *} \longrightarrow \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$ for a norm-preserving isomorphism from $C^{\odot *}$ onto $\mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$, we define elements $r_i^{\odot *} := k^{-1}(e_i, 0) \in C^{\odot *}$, $i = 1, \ldots, n$, where $e_i$ is the $i$-th canonical basis vector of $\mathbb{R}^n$. Clearly, the family $\{r_1^{\odot *}, \ldots, r_n^{\odot *}\}$ constitutes a basis of the linear subspace $Y^{\odot *} := k^{-1}(\mathbb{R}^n \times \{0\})$ of $C^{\odot *}$, and the requirement $l(e_i) = r_i^{\odot *}$ for $i = 1, \ldots, n$ uniquely determines a linear bijective mapping $l : \mathbb{R}^n \longrightarrow Y^{\odot *}$ with $\|l\| = \|l^{-1}\| = 1$.

For reals $a \leq b \leq c$ and a (norm) continuous function $w : [a, b] \longrightarrow C^{\odot *}$ the **weak* integral**

$$\int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau \in C^{\odot *} \tag{12}$$

is defined by

$$\langle \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau, \varphi^{\odot} \rangle := \int_a^b \langle T_e^{\odot *}(c - \tau)\, w(\tau), \varphi^{\odot} \rangle\, d\tau$$

for $\varphi^{\odot} \in C^{\odot}$. Furthermore, set

$$\int_b^a T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau := -\int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau$$

as usual. It turns out that, under the above condition on $w$, this weak* integral belongs to $C$ (more precisely, to $C^{\odot\odot} = j(C)$). Additionally, one obtains the formulas

$$T_e^{\odot *}(t) \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau = \int_a^b T_e^{\odot *}(c + t - \tau)\, w(\tau)\, d\tau \tag{13}$$

for all $t \geq 0$,

$$P_\lambda^{\odot *} \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau = \int_a^b T_e^{\odot *}(c - \tau)\, P_\lambda^{\odot *}\, w(\tau)\, d\tau \tag{14}$$

with $\lambda \in \{s, c, u\}$, and finally the inequality

$$\left\| \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau \right\|_{C^{\odot *}} \leq \int_a^b \left\| T_e^{\odot *}(c - \tau)\, w(\tau) \right\|_{C^{\odot *}} d\tau. \tag{15}$$

If $q : I \longrightarrow \mathbb{R}^n$ is a continuous function defined on some interval $I \subseteq \mathbb{R}$ and if the function $x : I + [-h, 0] \longrightarrow \mathbb{R}^n$ is a solution of the inhomogeneous RFDE (11), then the curve $u : I \ni t \longmapsto x_t \in C$ satisfies the abstract integral equation

$$u(t) = T_e(t - s)\, u(s) + \int_s^t T_e^{\odot *}(t - \tau)\, Q(\tau)\, d\tau \tag{16}$$

for all $s, t \in I$ with $s \leq t$, where $Q : [s, t] \ni \tau \longmapsto l(q(\tau)) \in Y^{\odot *}$. On the other hand, if $Q : I \longrightarrow Y^{\odot *}$ is continuous, and if $u : I \longrightarrow C$ is a solution of Eq. (16) then there is a continuous function $x : I + [-h, 0] \longrightarrow \mathbb{R}^n$ with $x_t = u(t)$, $t \in I$, solving the differential equation (11) for the inhomogeneity $q : I \ni \tau \longmapsto l^{-1}(Q(\tau)) \in \mathbb{R}^n$. In this sense we have a one-to-one correspondence between solutions for Eq.s (11) and (16).

**Preliminary Results on Inhomogeneous Linear Equations**

As the last step to prepare the construction of local center-unstable manifolds for Eq. (1), we establish the existence and some properties of special solutions of the integral equation (16). In doing so, we will need certain Banach spaces which are introduced below.

Let $X$ be a Banach space with norm $\|\cdot\|_X$. For every $\eta \geq 0$ we define the linear space

$$C_\eta((-\infty,0],X) = \left\{ g \in C((-\infty,0],X) \,\middle|\, \sup_{s\in(-\infty,0]} e^{\eta s}\,\|g(s)\|_X < \infty \right\}$$

where $C((-\infty,0],X)$ denotes the Banach space of all continuous functions from the interval $(-\infty,0]$ into $X$. Providing $C_\eta((-\infty,0],X)$ with the weighted supremum norm given by

$$\|g\|_{C_\eta} = \sup_{s\in(-\infty,0]} e^{\eta s}\|g(t)\|_X,$$

we obtain a one-parameter family of Banach spaces with the scaling property

$$C_{\eta_1}((-\infty,0],X) \subseteq C_{\eta_2}((-\infty,0],X)$$

for all $\eta_1 \leq \eta_2$ and

$$\|g\|_{C_{\eta_1}} \geq \|g\|_{C_{\eta_2}}$$

for all $g \in C_{\eta_1}((-\infty,0],X)$. To simplify notation, we use the abbreviations $Y_\eta$, $C^0_\eta$, and $C^1_\eta$, for the spaces $C_\eta((-\infty,0],Y^{\odot *})$, $C_\eta((-\infty,0],C)$, and $C_\eta((-\infty,0],C^1)$, respectively, which are mainly regarded in the sequel.

From now on, let us denote by $P^{\odot *}_{cu}$ the projection of $C^{\odot *}$ along $C^{\odot *}_s$ onto the center-unstable space $C_{cu}$, that is, $P^{\odot *}_{cu} := P^{\odot *}_u + P^{\odot *}_c$. For a given function $Q : (-\infty,0] \longrightarrow Y^{\odot *}$ we formally introduce a mapping $\mathcal{K}^{cu}\,Q$ from $(-\infty,0]$ into $C^{\odot *}$ by

$$(\mathcal{K}^{cu}\,Q)(t) = \int_0^t T^{\odot *}_e(t-\tau)\,P^{\odot *}_{cu}\,Q(\tau)\,d\tau + \int_{-\infty}^t T^{\odot *}_e(t-\tau)\,P^{\odot *}_s\,Q(\tau)\,d\tau \tag{17}$$

for $t \leq 0$. Note that the right-hand side of Eq. (17) may not be well-defined for arbitrary $Q$. However, in our next result we show that for maps $Q \in Y_\eta$ with $\eta \in \mathbb{R}$ such that $c_c < \eta < \min\{-c_s, c_u\}$ the integrals in (17) do not only exist, but the functions $\mathcal{K}^{cu}\,Q$ form also solutions for the abstract integral equation (16).

**Proposition 3.2** *Let $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$ be given. Then Eq. (17) induces a bounded linear map*

$$\tilde{\mathcal{K}} : Y_\eta \ni Q \longmapsto \mathcal{K}^{cu}\,Q \in C^0_\eta.$$

*In addition, for every $Q \in Y_\eta$ the function $u = \tilde{\mathcal{K}}\,Q$ is a solution of the integral equation*

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T^{\odot *}_e(t-\tau)\,Q(\tau)\,d\tau \tag{18}$$

*for $-\infty < s \leq t \leq 0$, and the only one in $C^0_\eta$ satisfying $P^{\odot *}_{cu}\,u(0) = 0$.*

**Proof:** The proof falls naturally into three parts. In the first one, we show that, under the stated assumption on $\eta \in \mathbb{R}$, the formal expression (17) forms indeed a well-defined mapping $\mathcal{K}^{cu} Q$ from $(-\infty, 0]$ into $C$ for all $Q \in Y_\eta$. Afterwards we prove that $\tilde{\mathcal{K}}$ is a bounded linear operator and finally we conclude the part of the proposition concerning the abstract integral equation. From now on to the end of the proof, we fix $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$.

1. In order to see $(\mathcal{K}^{cu} Q)(t) \in C$ for all $Q \in Y_\eta$ and $t \leq 0$, recall that for given $Q \in Y_\eta$ and $t \leq 0$ both

$$\int_0^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau = -\int_t^0 T_e^{\odot *}(-\tau)\, T_e^{\odot *}(t)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau$$

and

$$I(s) := \int_s^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau$$

with $s \leq t$ belong to $C$. Hence, it remains to prove the convergence of $I(s)$ in $C$ as $s \to -\infty$. To show this, we assume $\{s_k\}_{k\in\mathbb{N}} \subset (-\infty, t]$ with $s_k \to -\infty$ as $k \to \infty$. Then, by inequality (15) and the estimate (10) for the action of $T_e^{\odot *}$ on the center space,

$$\begin{aligned}
\|I(s_{k_2}) - I(s_{k_1})\|_{C^{\odot *}} &= \left\| \int_{s_{k_2}}^{s_{k_1}} T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \right\|_{C^{\odot *}} \\
&\leq \int_{s_{k_2}}^{s_{k_1}} \left\| T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau) \right\|_{C^{\odot *}} d\tau \\
&\leq K \left\| P_s^{\odot *} \right\| \int_{s_{k_2}}^{s_{k_1}} e^{c_s(t-\tau)} \left\| Q(\tau) \right\|_{C^{\odot *}} d\tau \\
&\leq e^{c_s t} K \left\| P_s^{\odot *} \right\| \int_{s_{k_2}}^{s_{k_1}} e^{-(c_s+\eta)\tau} e^{\eta\tau} \|Q(\tau)\|_{C^{\odot *}} d\tau \\
&\leq e^{c_s t} K \|P_s^{\odot *}\| \|Q\|_{Y_\eta} \int_{s_{k_2}}^{s_{k_1}} e^{-(c_s+\eta)\tau}\, d\tau \\
&\leq \frac{-e^{c_s t}}{c_s+\eta} K \|P_s^{\odot *}\| \|Q\|_{Y_\eta} \left[ e^{-(c_s+\eta)s_{k_1}} - e^{-(c_s+\eta)s_{k_2}} \right] \\
&\leq \frac{-e^{c_s t}}{c_s+\eta} K \|P_s^{\odot *}\| \|Q\|_{Y_\eta} e^{-(c_s+\eta)s_{k_1}}
\end{aligned}$$

for all $k_1, k_2 \in \mathbb{N}$ with $s_{k_1} \geq s_{k_2}$. Thus, $\{I(s_k)\}_{k\in\mathbb{N}}$ constitutes a Cauchy sequence in $C$. In particular, $I := \lim_{k\to\infty} I(s_k)$ exists. Furthermore, in the same manner we see that for any another given sequence $\{\tilde{s}_k\}_{k\in\mathbb{N}} \subset (-\infty, t]$ of reals with $\tilde{s}_k \to -\infty$, we also have $\|I(\tilde{s}_k) - I\|_{C^{\odot *}} \to 0$ as $k \to \infty$. This implies the desired conclusion $I = \lim_{s\to-\infty} I(s)$. Hence, $(\mathcal{K}^{cu} Q)(t) \in C$ for all $Q \in Y_\eta$ and $t \leq 0$.

2. The technical results in Diekmann et al. [2, Chapter III.2] on the continuous dependence of the weak* star integral on parameters and estimates (10) enable to show that the induced

curve $(-\infty, 0] \ni t \longmapsto (\mathcal{K}^{cu} Q)(t) \in C$ is continuous for every $Q \in Y_\eta$. Consequently, Eq. (17) defines by $Q \longmapsto \mathcal{K}^{cu} Q$ a mapping from $Y_\eta$ into $C((-\infty, 0], C)$. This map is also linear. In addition, we claim $\mathcal{K}^{cu} Q \in C^0_\eta$ for all $Q \in Y_\eta$. To this end, consider the apparent inequality

$$\begin{aligned} e^{\eta t} \|(\mathcal{K}^{cu} Q)(t)\|_{C^{\odot *}} \leq e^{\eta t} & \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_c^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} \\ & + e^{\eta t} \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_u^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} \\ & + e^{\eta t} \left\| \int_{-\infty}^t T_e^{\odot *}(t-\tau) \, P_s^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} \end{aligned}$$

for fixed $Q \in Y_\eta$ and $t \leq 0$. Using the inequalities (15) and (10) as in the part above, we estimate the first term on the right-hand side by

$$\begin{aligned} e^{\eta t} \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_c^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} & \leq -e^{\eta t} \int_0^t \left\| T_e^{\odot *}(t-\tau) \, P_c^{\odot *} \, Q(\tau) \right\|_{C^{\odot *}} \, d\tau \\ & \leq -K e^{\eta t} \int_0^t e^{c_c |t-\tau|} \|P_c^{\odot *} \, Q(\tau)\|_{C^{\odot *}} d\tau \\ & = -K \int_0^t e^{(c_c - \eta)(\tau - t)} e^{\eta \tau} \left\| P_c^{\odot *} \, Q(\tau) \right\|_{C^{\odot *}} \, d\tau \\ & \leq -K \|P_c^{\odot *}\| \int_0^t e^{(c_c - \eta)(\tau - t)} e^{\eta \tau} \|Q(\tau)\|_{C^{\odot *}} d\tau \\ & \leq K \|P_c^{\odot *}\| \|Q\|_{Y_\eta} \int_t^0 e^{(c_c - \eta)(\tau - t)} \, d\tau \\ & \leq K \|P_c^{\odot *}\| \|Q\|_{Y_\eta} \frac{1}{\eta - c_c} \, . \end{aligned}$$

In the same manner we can see that

$$e^{\eta t} \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_u^{\odot *} \, Q(\tau) \, d\tau \right\|_{Y^{\odot *}} \leq K \|P_u^{\odot *}\| \|Q\|_{Y_\eta} \frac{1}{c_u + \eta}$$

and

$$e^{\eta t} \left\| \int_{-\infty}^t T_e^{\odot *}(t-\tau) \, P_s^{\odot *} \, Q(\tau) \, d\tau \right\|_{Y^{\odot *}} \leq K \|P_s^{\odot *}\| \|Q\|_{Y_\eta} \frac{1}{-c_s - \eta}.$$

Summarizing, we get

$$e^{\eta t} \|(\mathcal{K}^{cu} Q)(t)\|_{Y^{\odot *}} \leq K \|Q\|_{Y_\eta} \left( \frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta} \right), \tag{19}$$

and thus $\mathcal{K}^{cu} Q \in C^0_\eta$. It follows that $Q \longmapsto \mathcal{K}^{cu} Q$ forms a linear mapping $\tilde{\mathcal{K}}$ from $Y_\eta$ into $C^0_\eta$, which in particular is bounded as claimed.

3. Given any $Q \in Y_\eta$ define $\delta(t,s) := (\mathcal{K}^{cu} Q)(t) - T_e(t-s)\big((\mathcal{K}^{cu} Q)(s)\big)$ for all reals $-\infty < s \le t \le 0$. Then, by the linearity and formula (13), we get

$$\begin{aligned}
\delta(t,s) &= \int_0^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&\quad - T_e(t-s)\left( \int_0^s T_e^{\odot *}(s-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_{-\infty}^s T_e^{\odot *}(s-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \right) \\
&= \int_0^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&\quad - \int_0^s T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau - \int_{-\infty}^s T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&= \int_s^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_s^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&= \int_s^t T_e^{\odot *}(t-\tau)\, Q(\tau)\, d\tau,
\end{aligned}$$

which yields that $u := \mathcal{K}^{cu} Q$ satisfies Eq. (18) for all $-\infty < s \le t \le 0$. Moreover, in view of Eq. (14) for the relation of the weak* integrals and projections on the decomposition of $C^{\odot *}$, for $t = 0$ we have

$$\begin{aligned}
u(0) &= (\mathcal{K}^{cu} Q)(0) \\
&= \int_{-\infty}^0 T_e^{\odot *}(-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&= P_s^{\odot *}\left( \int_{-\infty}^0 T_e^{\odot *}(-\tau)\; Q(\tau)\, d\tau \right)
\end{aligned}$$

implying $P_{cu}^{\odot *}\, u(0) = 0$.

So the assertion of the proposition follows if we are able to prove that $u$ is the only solution of Eq. (18) in $C_\eta^0$ with vanishing $C_{cu}$ component at $t = 0$. For this purpose, suppose $v \in C_\eta^0$ is also a solution of (18) for $-\infty < s \le t \le 0$ with $P_{cu}^{\odot *} v(0) = 0$. Then the difference $w = u - v$ belongs to $C_\eta^0$, has a vanishing $C_{cu}$ component at $t = 0$, and satisfies the equation

$$w(t) = T_e(t-s)\, w(s) \tag{20}$$

for all $-\infty < s \le t \le 0$. Furthermore, $w$ can be extended by

$$t \longmapsto \begin{cases} w(t), & \text{for } t \le 0, \\ T_e(t)\, w(0), & \text{for } t \ge 0 \end{cases}$$

to a solution $\tilde{w} : \mathbb{R} \longrightarrow C$ of Eq. (20) for all $-\infty < s \leq t < \infty$. Since

$$\begin{aligned}\sup_{t\geq 0} e^{-\eta t}\|w(t)\|_C &= \sup_{t\geq 0} e^{-\eta t}\|T_e(t)\, w(0)\|_C \\ &\leq K \sup_{t\geq 0} e^{-\eta t} e^{c_s t}\|w(0)\|_C \\ &= K\|w(0)\|_C\end{aligned}$$

due to $(c_s - \eta) < 0$ we get

$$\begin{aligned}\sup_{t\in\mathbb{R}} e^{-\eta|t|}\|\tilde{w}(t)\|_C &\leq \sup_{t\leq 0} e^{\eta t}\|\tilde{w}(t)\|_C + \sup_{t\geq 0} e^{-\eta t}\|\tilde{w}(t)\|_C \\ &= \|w\|_{C^0_\eta} + K\|w(0)\|_C < \infty.\end{aligned}$$

Now from Diekmann et al. [2, Lemma 2.4 in Section IX.2] it follows $w(0) \in C_u$ and $\tilde{w}(0) \in C_c$. As $w(0) = \tilde{w}(0)$ and $C_u \cap C_c = \{0\}$, we conclude $\tilde{w}(0) = w(0) = 0$, and so by Eq. (20),

$$0 = T_e(s)w(0) = T_e(s)\, T_e(-s)\, w(s) = T_e(0)\, w(s) = u(s) - v(s)$$

for all $-\infty < s \leq 0$. This completes the proof. □

Next, we prove a smoothing property of the integral equation (21). This property will be useful in combination with our preceding result.

**Proposition 3.3** *Suppose that $Q \in Y_\eta$ for some $\eta \geq 0$. If $u \in C^0_\eta$ satisfies the abstract integral equation*

$$u(t) = T_e(t-s)\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, Q(\tau)\, d\tau \tag{21}$$

*for all $-\infty < s \leq t \leq 0$, then $u \in C^1_\eta$ and*

$$\|u\|_{C^1_\eta} \leq \big(1 + e^{\eta h}\|L_e\|\big)\|u\|_{C^0_\eta} + e^{\eta h}\|Q\|_{Y_\eta}.$$

**Proof:** Consider the mapping $q : (-\infty, 0] \longrightarrow \mathbb{R}^n$ defined by $q(t) = l^{-1}(Q(t))$, $-\infty < t \leq 0$. Of course, $q \in C((-\infty, 0], \mathbb{R}^n)$. Moreover, since

$$\begin{aligned}\sup_{t\in(-\infty,0]} e^{\eta t}\|q(t)\|_{\mathbb{R}^n} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|l^{-1}(Q(t))\|_{\mathbb{R}^n} \\ &= \sup_{t\in(-\infty,0]} e^{\eta t}\|Q(t)\|_{Y^{\odot *}} \\ &= \|Q\|_{Y_\eta}\end{aligned}$$

we see at once $q \in C_\eta((-\infty, 0], \mathbb{R}^n)$ with $\|q\|_{C_\eta} = \|Q\|_{Y_\eta}$.

By assumption, $u$ satisfies Eq. (21) such that, taking into account our discussion about the one-to-one correspondence between solutions for (11) and (16), the function $x : (\infty, 0] \longrightarrow \mathbb{R}^n$ given by $x(t) = u(t)(0)$ is a solution of the differential equation

$$\dot{x}(t) = L_e\, x_t + q(t)$$

for all $-\infty < t \leq 0$. Accordingly, $x$ is everywhere continuously differentiable, $x_t$ belongs to $C^1$ for all $-\infty < t \leq 0$, and the map $(-\infty, 0] \ni t \longmapsto u(t) = x_t \in C^1$ is continuous. Furthermore, by the differential equation for $x$ and the estimate for $q$, we have

$$\begin{aligned}
\|\dot{x}(t)\|_{\mathbb{R}^n} &\leq \|L_e\| \|x_t\|_C + \|q(t)\|_{\mathbb{R}^n} \\
&\leq \|L_e\| \|u(t)\|_C + e^{-\eta t} \|q\|_{C_\eta} \\
&\leq e^{-\eta t} (\|L_e\| \|u\|_{C^0_\eta} + \|Q\|_{Y_\eta})
\end{aligned}$$

and therefore

$$\begin{aligned}
\sup_{t \in (-\infty, 0]} e^{\eta t} \|\dot{x}_t\|_C &= \sup_{t \in (-\infty, 0]} \left( e^{\eta t} \sup_{\vartheta \in [-h, 0]} \|\dot{x}(t + \vartheta)\|_{\mathbb{R}^n} \right) \\
&\leq (\|L_e\| \|u\|_{C^0_\eta} + \|Q\|_{Y_\eta}) \sup_{t \in (-\infty, 0]} \left( e^{\eta t} \sup_{\vartheta \in [-h, 0]} e^{-\eta (t + \vartheta)} \right) \\
&\leq e^{\eta h} (\|L_e\| \|u\|_{C^0_\eta} + \|Q\|_{Y_\eta}),
\end{aligned}$$

for all $-\infty < t \leq 0$. From this, it follows that $u \in C^1_\eta$ and

$$\begin{aligned}
\|u\|_{C^1_\eta} &= \sup_{t \in (-\infty, 0]} e^{\eta t} \|u(t)\|_{C^1} \\
&= \sup_{t \in (-\infty, 0]} e^{\eta t} \|x_t\|_{C^1} \\
&= \sup_{t \in (-\infty, 0]} e^{\eta t} (\|x_t\|_C + \|\dot{x}_t\|_C) \\
&\leq \|u\|_{C^0_\eta} + e^{\eta h} (\|L_e\| \|u\|_{C^0_\eta} + \|Q\|_{Y_\eta})
\end{aligned}$$

as claimed. □

As an easy consequence of the last two results we conclude that the formal definition (17) generates a bounded linear mapping from the Banach space $Y_\eta$ into $C^1_\eta$ for $c_c < \eta < \min\{-c_s, c_u\}$.

**Corollary 3.4** *For each $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$, relation (17) defines a bounded linear mapping*

$$\mathcal{K}_\eta : Y_\eta \ni Q \longmapsto \mathcal{K}^{cu} Q \in C^1_\eta$$

*with*

$$\|\mathcal{K}_\eta\| \leq K (1 + e^{\eta h} \|L_e\|) \left( \frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta} \right) + e^{\eta h}.$$

*Moreover, for all $Q \in Y_\eta$ the function $u = \mathcal{K}_\eta Q$ is a solution of*

$$u(t) = T_e(t - s)\, u(s) + \int_s^t T_e^{\odot *}(t - \tau)\, Q(\tau)\, d\tau$$

*for $-\infty < s \leq t \leq 0$, and the only one in $C^1_\eta$ with $P_{cu}^{\odot *} u(0) = 0$.*

**Proof:** Apply Propositions 3.2 and 3.3, taking into account the estimate (19) for the bound of the linear map $\tilde{\mathcal{K}}$. □

**Remark 3.5** Observe that the bounds of the linear maps $\mathcal{K}_\eta$ in the above corollary are given by a continuous function in $\eta$. This will be a crucial point in the proof of Theorem 2.

## 4 The Construction of Local Center-Unstable Manifolds

This section is devoted to the actual proof of Theorem 1 about the existence of local center-unstable manifolds for Eq. (1). Throughout the proof, we consider the differential equation (1) in the equivalent form

$$\dot{x}(t) = L\,x_t + r(x_t) \tag{22}$$

with the linear part

$$L := Df(0)$$

and the nonlinearity

$$r : U \ni \varphi \longmapsto f(\varphi) - L\,\varphi \in \mathbb{R}^n. \tag{23}$$

Obviously, $r$ also satisfies the same smoothness conditions (S 1) and (S 2) as $f$ and we have $r(0) = 0$ and $Dr(0) = 0$.

The proof is organized as follows. In the first part, we modify the nonlinearity $r$ outside a small neighborhood of the origin and assign the resulting differential equation to an abstract integral equation by the variation-of-constants formula. Then, using the changes on the nonlinearity in combination with the auxiliary conclusions of the last section, we show that the associated integral operator forms a parameter-dependent contraction in $C^1_\eta$ for an appropriate $\eta > 0$. In the final step, we prove that the graph of this contraction is an invariant manifold for the modified differential equation and that a part of this graph also satisfies the assertions of Theorem 1.

### Smoothing Modification of the Nonlinearity

As the Banach space $C_{cu}$ is finite-dimensional, there exists a norm $\|\cdot\|_{cu}$ on $C_{cu}$ being infinitely often continuously differentiable on $C_{cu}\setminus\{0\}$. Introducing the projection operator $P_{cu} := P_c + P_u$ of $C^1$ along $C^1_s$ onto the center-unstable space $C_{cu}$ and defining

$$\|\varphi\|_1 = \max\left\{\|P_{cu}\,\varphi\|_{cu}, \|P_s\,\varphi\|_{C^1}\right\} \tag{24}$$

for $\varphi \in C^1$, we get a second norm on $C^1$, which is equivalent to $\|\cdot\|_{C^1}$.

Let $\varrho : [0,\infty) \longrightarrow \mathbb{R}$ be a $C^\infty$-smooth function with $\varrho(t) = 1$ for $0 \leq t \leq 1$, $0 < \varrho(t) < 1$ for $1 < t < 2$, and $\varrho(t) = 0$ for all $t \geq 2$. Further, let the map $\hat{r} : C^1 \longrightarrow \mathbb{R}^n$ be given by

$$\hat{r}(\varphi) = \begin{cases} r(\varphi), & \text{for } \varphi \in U, \\ 0, & \text{for } \varphi \notin U. \end{cases}$$

Using these two functions, we introduce for all $\delta > 0$ the smoothing modification

$$r_\delta : C^1 \ni \varphi \longmapsto \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\varphi) \in \mathbb{R}^n$$

of the nonlinearity $r$, where we write $\varphi_{cu}$, $\varphi_s$ for the components $P_{cu}\,\varphi$, $P_s\,\varphi$ of $\varphi$, respectively. For every $\gamma > 0$ let $B_\gamma(0) = \{\varphi \in C^1 \,|\, \|\varphi\|_1 < \gamma\}$ denote the open ball in $C^1$ of radius $\gamma$ with respect to the $\|\cdot\|_1$-norm and centered at the origin. Since $U \subset C^1$ is open and $r$ continuously differentiable due to property (S 1), we find a sufficiently small $\delta_0 > 0$ with $B_{2\delta_0}(0) \subset U$, so that the restriction $r|_{B_{2\delta_0}(0)}$ of $r$ to $B_{2\delta_0}(0)$ together with the associated derivative $Dr|_{B_{2\delta_0}(0)}$ are both bounded. Subsequently, for small reals $\delta > 0$, the modifications of $r$ in a neighborhood of the origin are also bounded and continuously differentiable with bounded derivatives. More precisely, the following result holds.

**Corollary 4.1** *For all reals $0 < \delta < \delta_0$ the restriction of the map $r_\delta$ to the strip*

$$S := \left\{\psi \in C^1 \middle|\, \|\psi_s\|_1 < \delta\right\}$$

*in $C^1$ is a bounded, $C^1$-smooth function with bounded derivative. Moreover,*

$$r_\delta(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot r(\varphi)$$

*for all $\varphi \in S$.*

**Proof:** Given any positive constant $0 < \delta < \delta_0$ suppose that $\varphi \in S$. Then, by definition of $r_\delta$ in combination with the inequality $\|\varphi_s\|_{C^1} \leq \|\varphi_s\|_1$ we get

$$r_\delta(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot r(\varphi).$$

Consequently, we have $r_\delta(\varphi) = r(\varphi)$ for all $\varphi \in S$ with $\|\varphi\|_1 \leq \delta$, and $r_\delta(\varphi) = 0$ for all $\varphi \in S$ with $\|\varphi\|_1 \geq 2\delta$. Since $r$, $\varrho$ are $C^1$-smooth and the norm $\|\cdot\|_1$ continuously differentiable on $C_{cu} \setminus \{0\}$ by assumption, the restriction of $r_\delta$ to the strip $S$ is clearly also continuously differentiable. Moreover, using the above expressions for $r_\delta$ on $S$ together with the boundedness of $r$ and $Dr$ on $B_{2\delta_0}(0) \subset U$, we conclude that both $r_\delta$ and $Dr_\delta$ are bounded on $S$ as claimed. $\square$

For sufficiently small $\delta > 0$, the functions $r_\delta$ are even globally bounded and Lipschitz continuous with constants continuously depending on $\delta$, as proved in [9].

**Proposition 4.2** [*Proposition* II.2 *in Krisztin et al.* [9]] *Under the above assumptions there exists* $\delta_1 \in (0, \delta_0)$ *and a monotone increasing* $\lambda : [0, \delta_1] \longrightarrow [0, 1]$ *with* $\lambda(0) = 0$ *and* $\lambda(\delta) \searrow 0$ *as* $\delta \searrow 0$ *such that*

$$\|r_\delta(\varphi)\|_{\mathbb{R}^n} \leq \delta \cdot \lambda(\delta)$$

*and*

$$\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} \leq \lambda(\delta) \cdot \|\varphi - \psi\|_{C^1}$$

*for all* $0 < \delta \leq \delta_1$ *and* $\varphi, \psi \in C^1$.

Using the modification $r_\delta$ of the nonlinearity $r$, we introduce for each $0 < \delta \leq \delta_1$ the retarded functional differential equation

$$\dot{x}(t) = L\, x_t + r_\delta(x_t), \qquad -\infty < t \leq 0, \tag{25}$$

and the associated abstract integral equations

$$u(t) = T_e(t-s)\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(u(\tau)))\, d\tau, \qquad -\infty < s \leq t \leq 0. \tag{26}$$

We have now a one-to-one correspondence in the following sense: If $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ is a continuously differentiable solution of RFDE (25), then $u : (-\infty, 0] \longmapsto x_t \in C^1$ is a solution of Eq. (26). On the other hand, for a continuous mapping $u : (-\infty, 0] \longrightarrow C^1$ satisfying integral equation (26), the function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ defined by $x(t) = u(t)(0)$, $-\infty < t \leq 0$, forms a continuously differentiable solution of(25).

### Center-Unstable Manifolds of the Smoothed Equation

Until the end of this section fix $\eta \in \mathbb{R}$ satisfying the estimate

$$c_c < \eta < \min\{-c_s, c_u\}. \tag{27}$$

Then we find a constant $0 < \delta < \delta_1$ with

$$\|\mathcal{K}_\eta\|\, \lambda(\delta) < \frac{1}{2} \tag{28}$$

where the mappings $\mathcal{K}_\eta$ and $\lambda$ are defined in Corollary 3.4 and Proposition 4.2, respectively. Below, we construct a parameter-dependent contraction on the Banach space $C^1_\eta$, such that the fixed points will form solutions for the abstract integral equation (26). For this purpose, we assign to Eq. (26) an integral operator. We begin with the nonlinear part.

**Corollary 4.3** *Let $R$ denote the map, which assigns to $u \in C((-\infty, 0], C^1)$ the mapping $(-\infty, 0] \ni s \longmapsto l(r_\delta(u(s))) \in Y^{\odot*}$ in $C((-\infty, 0], Y^{\odot*})$. Then $R$ maps $C^1_\eta$ into $Y_\eta$, and the induced mapping $R_{\delta\eta} : C^1_\eta \ni u \longmapsto R(u) \in Y_\eta$ satisfies*

$$\|R_{\delta\eta}(u)\|_{Y_\eta} \leq \delta\, \lambda(\delta) \tag{29}$$

*and*

$$\|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta} \leq \lambda(\delta)\|u - v\|_{C^1_\eta} \tag{30}$$

*for all $u, v \in C^1$.*

**Proof:** First, note that $R$ indeed assigns a continuous function from $(-\infty, 0]$ into $Y^{\odot*}$ to a function $u \in C((-\infty, 0], C^1)$, as the mappings $l$ and $r_\delta$ are continuous. Given $u \in C^1_\eta$, Proposition 4.2 implies

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\|R(u)(t)\|_{Y^{\odot*}} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|l(r_\delta(u(t)))\|_{Y^{\odot*}} \\
&= \sup_{t\in(-\infty,0]} e^{\eta t}\|r_\delta(u(t))\|_{\mathbb{R}^n} \\
&\leq \sup_{t\in(-\infty,0]} e^{\eta t}\delta\lambda(\delta) \\
&= \delta\lambda(\delta).
\end{aligned}$$

This shows $R(C^1_\eta) \subset Y_\eta$ and in particular the boundedness of $R_{\delta\eta}$ by $\delta\lambda(\delta)$ as claimed. Using the Lipschitz continuity of $r_\delta$ from Proposition 4.2, we also see that $R_{\delta\eta}$ is Lipschitz continuous with Lipschitz constant $\lambda(\delta)$, and the corollary follows. □

**Remark 4.4** The mapping $R : C((-\infty, 0], C^1) \longrightarrow C((-\infty, 0], Y^{\odot*})$ in the last result is called the **substitution** or the **Nemitsky operator** of the map $C^1 \ni \varphi \longmapsto l(r_\delta(\varphi)) \in Y^{\odot*}$ on $(-\infty, 0]$.

Next, we consider the linear part of the integral equation (26) and prove that it constitutes a bounded linear operator from the center-unstable space into $C^1_\eta$.

**Corollary 4.5** *For each $\varphi \in C_{cu}$, the curve $(-\infty, 0] \ni t \longmapsto T_e(t)\,\varphi \in C^1$ belongs to $C^1_\eta$, and $S_\eta : C^1 \supset C_{cu} \longrightarrow C^1_\eta$ defined by $(S_\eta\,\varphi)(t) = T_e(t)\varphi$ for $\varphi \in C_{cu}$ and $t \leq 0$ is a bounded linear operator with*

$$\|S_\eta\| \leq K\big(\|P_c^{\odot*}\| + \|P_u^{\odot*}\|\big). \tag{31}$$

**Proof:** To start with, recall that $T_e$ defines a group on $C_{cu} \subset C^1$ and coincides with $T$. Thus, for all $\varphi \in C_{cu}$, the curve $(-\infty, 0] \ni t \longmapsto T_e(t)\,\varphi \in C_{cu}$ takes values in $C^1$ and is in fact a continuous map from $(-\infty, 0]$ into $C^1$. Furthermore, we have

$$\|T_e(t)\,\varphi\|_{C^1} = \|T_e(t)\,\varphi\|_C + \big\|\tfrac{d}{dt}T_e(t)\,\varphi\big\|_C$$

and

$$\tfrac{d}{dt}(T_e(t)\,\varphi) = T_e(t)\,G_e\,\varphi = T_e(t)\,\varphi'$$

for $\varphi \in C_{cu}$. Hence, by the exponential trichotomy under our assumption (27), it follows

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\,\varphi\|_{C^1} &= \sup_{t\in(-\infty,0]} e^{\eta t}\Big(\|T_e(t)\,\varphi\|_C + \|T_e(t)\,\varphi'\|_C\Big) \\
&\le \sup_{t\in(-\infty,0]} e^{\eta t}\,\Big(\|T_e(t)\,P_c^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi\|_C \\
&\qquad + \|T_e(t)\,P_c^{\odot *}\,\varphi'\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi'\|_C\Big) \\
&\le \sup_{t\in(-\infty,0]} e^{\eta t}\Big(\|T_e(t)\,P_c^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_c^{\odot *}\,\varphi'\|_C\Big) \\
&\quad + \sup_{t\in(-\infty,0]} e^{\eta t}\Big(\|T_e(t)\,P_u^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi'\|_C\Big) \\
&\le K \sup_{t\in(-\infty,0]} e^{-(c_c-\eta)t}\Big(\|P_c^{\odot *}\,\varphi\|_C + \|P_c^{\odot *}\,\varphi'\|_C\Big) \\
&\qquad + K \sup_{t\in(-\infty,0]} e^{(\eta+c_u)t}\Big(\|P_u^{\odot *}\,\varphi\|_C + \|P_u^{\odot *}\,\varphi'\|_C\Big) \\
&\le K\|P_c^{\odot *}\|\big(|\varphi\|_C + \|\varphi'|_C\big) + \\
&\qquad\quad K\|P_u^{\odot *}\|\big(\|\varphi\|_C + \|\varphi'\|_C\big) \\
&= K\big(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|\big)\|\varphi\|_{C^1}.
\end{aligned}$$

Accordingly, $S_\eta\varphi \in C^1_\eta$ for $\varphi \in C_{cu}$, and thus $S_\eta$ is well-defined. In addition, the mapping $S_\eta$ is obviously linear by definition, and

$$\|S_\eta\varphi\|_{C^1_\eta} \le K(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|)$$

for $\|\varphi\|_{C^1} \le 1$. Therefore, inequality (31) holds and this completes the proof. □

Using Corollaries 3.4, 4.3, and 4.5 to guarantee the well-definedness, we introduce the mapping $\mathcal{G}_\eta$ from the product space $C^1_\eta \times C_{cu}$ into $C^1_\eta$ given by

$$\mathcal{G}_\eta(u,\varphi) := S_\eta\,\varphi + \mathcal{K}_\eta \circ R_{\delta\eta}(u). \tag{32}$$

In the next proposition we prove that each function $\varphi \in C_{cu}$ uniquely determines a solution of $u = \mathcal{G}_\eta(u,\varphi)$ in $C^1_\eta$.

**Proposition 4.6** *For each $\varphi \in C_{cu}$, the mapping $\mathcal{G}_\eta(\,\cdot\,,\varphi) : C^1_\eta \longrightarrow C^1_\eta$ has exactly one fixed point $u = u(\varphi)$. Moreover, the associated solution operator*

$$\tilde{u}_\eta : C_{cu} \ni \varphi \longmapsto u(\varphi) \in C^1_\eta \tag{33}$$

*of $u = \mathcal{G}_\eta(u,\varphi)$ is (globally) Lipschitz continuous.*

**Proof:** We begin with the claim that, for given $\varphi \in C_{cu}$, $\mathcal{G}_\eta(\,\cdot\,,\varphi)$ maps sufficiently large closed balls centered at the origin into themselves. Indeed, for fixed $\varphi \in C_{cu}$ we find a positive real $\gamma > 0$ with $2\|S_\eta\|\,\|\varphi\|_{C^1} \leq \gamma$ so that both estimates (28) and (30) together imply

$$\begin{aligned}\|\mathcal{G}_\eta(u,\varphi)\|_{C^1_\eta} &= \|S_\eta\,\varphi + \mathcal{K}_\eta \circ R_{\delta\eta}(u)\|_{C^1_\eta}\\ &\leq \|S_\eta\,\varphi\|_{C^1_\eta} + \|\mathcal{K}_\eta \circ R_{\delta\eta}(u)\|_{C^1_\eta}\\ &\leq \|S_\eta\|\|\varphi\|_{C^1} + \lambda(\delta)\|\mathcal{K}_\eta\|\,\|u\|_{C^1_\eta}\\ &\leq \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma\end{aligned}$$

for all $u \in C^1_\eta$ with $\|u\|_{C^1_\eta} \leq \gamma$. Hence, $\mathcal{G}_\eta(\,\cdot\,,\varphi)$ maps $\left\{u \in C^1_\eta \middle|\, \|u\|_{C^1_\eta} \leq \gamma\right\}$ into itself. The mapping $\mathcal{G}_\eta(\,\cdot\,,\varphi)$, $\varphi \in C_{cu}$, is also a contraction since, by application of (28) and (30),

$$\begin{aligned}\|\mathcal{G}_\eta(u,\varphi) - \mathcal{G}_\eta(v,\varphi)\|_{C^1_\eta} &= \|\mathcal{K}_\eta \circ R_{\delta\eta}(u) - \mathcal{K}_\eta \circ R_{\delta\eta}(v)\|_{C^1_\eta}\\ &\leq \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta}\\ &\leq \lambda(\delta)\|\mathcal{K}_\eta\|\,\|u-v\|_{C^1_\eta}\\ &\leq \frac{1}{2}\|u-v\|_{C^1_\eta}\end{aligned}$$

for all $u,v \in C^1_\eta$. Consequently, using the Banach contraction principle, we find a unique $u(\varphi) \in C^1_\eta$ satisfying $u = \mathcal{G}_\eta(u,\varphi)$.

To see the global Lipschitz continuity of $\tilde{u}_\eta : C_{cu} \ni \varphi \longmapsto u(\varphi) \in C^1_\eta$, assume $\varphi,\psi \in C_{cu}$. Using the two inequalities (28) and (30) once more, we see

$$\begin{aligned}\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta} &= \|\mathcal{G}_\eta(\tilde{u}_\eta(\varphi),\varphi) - \mathcal{G}_\eta(\tilde{u}_\eta(\psi),\psi)\|_{C^1_\eta}\\ &= \|S_\eta(\varphi-\psi) + \mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\varphi)) - \mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\psi))\|_{C^1_\eta}\\ &\leq \|S_\eta\|\,\|\varphi-\psi\|_{C^1} + \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(\tilde{u}_\eta(\varphi)) - R_{\delta\eta}(\tilde{u}_\eta(\psi))\|_{Y_\eta}\\ &\leq \|S_\eta\|\,\|\varphi-\psi\|_{C^1} + \lambda(\delta)\,\|\mathcal{K}_\eta\|\,\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}\\ &\leq \|S_\eta\|\,\|\varphi-\psi\|_{C^1} + \frac{1}{2}\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}.\end{aligned}$$

Therefore

$$\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta} \leq 2\|S_\eta\|\,\|\varphi-\psi\|_{C^1},$$

which completes the proof. □

For all $\varphi \in C_{cu}$, the associated fixed point $\tilde{u}(\varphi)$ of the last proposition forms a solution of Eq. (26) in $C^1_\eta$ with the property that its component in the center-unstable space at $t=0$ is just given by $\varphi$, as shown in the following.

**Corollary 4.7** *For all $\varphi \in C_{cu}$ the mapping $\tilde{u}_\eta(\varphi)$ is a solution of the abstract integral equation (26) with $P_{cu}(\tilde{u}_\eta(\varphi)(0)) = \varphi$.*

**Proof:** The proof is straightforward. Given $\varphi \in C_{cu}$ define $z = \tilde{u}_\eta(\varphi) - S_\eta\,\varphi$. By Corollary 3.4, we have

$$z(t) = T_e(t-s)\,z(s) + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau, \qquad -\infty < s \le t \le 0,$$

and $P_{cu}\,z(0) = P_{cu}^{\odot *}\,z(0) = 0$. From this we conclude

$$\begin{aligned}
\tilde{u}_\eta(\varphi)(t) - T_e(t)\,\varphi &= \tilde{u}_\eta(\varphi)(t) - (S_\eta\,\varphi)(t)\\
&= z(t)\\
&= T_e(t-s)\,z(s) + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau\\
&= T_e(t-s)\,\tilde{u}_\eta(\varphi)(s) - T_e(t-s)(S_\eta\varphi)(s)\\
&\qquad + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau\\
&= T_e(t-s)\,\tilde{u}_\eta(\varphi)(s) - T_e(t)\,\varphi + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau
\end{aligned}$$

for all $-\infty < s \le t \le 0$ and

$$\begin{aligned}
P_{cu}(\tilde{u}_\eta(\varphi)(0)) - \varphi &= P_{cu}(\tilde{u}_\eta(\varphi)(0)) - P_{cu}\,\varphi\\
&= P_{cu}(\tilde{u}_\eta(\varphi)(0)) - P_{cu}((S_\eta\varphi)(0))\\
&= P_{cu}\,z(0) = 0
\end{aligned}$$

Adding $T_e(t)\,\varphi$ and $\varphi$, respectively, yields the assertion. □

By the discussed one-to-one correspondence of solutions for the differential equation (25) and the associated abstract integral equation (26), the above corollary shows that for all $\varphi \in C_{cu}$ there exists a continuously differentiable function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ satisfying $x_t = \tilde{u}(\varphi)(t)$ for $-\infty < t \le 0$ and solving Eq. (26) on $(-\infty, 0]$. The set $W^\eta$ consisting of all segments of these solutions at time $t = 0$, that is, the set

$$W^\eta := \Big\{\tilde{u}_\eta(\varphi)(0) \mid \varphi \in C_{cu}\Big\},$$

is called the **global center-unstable manifold** of RFDE (25) at the stationary point $0 \in C^1$. Note that $W^\eta$ can also be represented as the graph of the operator

$$w^\eta : C_{cu} \ni \varphi \longmapsto P_s(\tilde{u}_\eta(\varphi)(0)) \in C_s^1.$$

Indeed, applying Corollary 4.7, we see at once

$$W^\eta = \big\{\varphi + w^\eta(\varphi) \mid \varphi \in C_{cu}\big\}.$$

We close this subsection with the conclusion that the values of every solution $v \in C_\eta^1$ of the abstract integral equation (26) belong to the global center-unstable manifold $W^\eta$.

**Proposition 4.8** *Suppose that $v \in C^1_\eta$ is a solution of Eq. (26). Then*

$$v(t) \in W^\eta$$

*for all $t \le 0$.*

**Proof:** Assuming $v \in C^1_\eta$ satisfies the abstract integral equation

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T_e^{\odot *}(t-\tau)\,l(r_\delta(u(\tau)))\,d\tau$$

for $-\infty < s \le t \le 0$, we begin with the claim that $v(0) \in W^\eta$. In order to see this, let $z : (-\infty, 0] \longrightarrow C^1$ be defined by $z(t) = v(t) - T_e(t)\,P_{cu}\,v(0)$. As

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\|z(t)\|_{C^1} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|v(t) - T_e(t)\,P_{cu}\,v(0)\|_{C^1} \\
&\le \sup_{t\in(-\infty,0]} e^{\eta t}\|v(t)\|_{C^1} \\
&\qquad + \sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\,P_{cu}\,v(0)\|_{C^1} \\
&\le \|v\|_{C^1_\eta} + \sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\,P_c\,v(0)\|_{C^1} \\
&\qquad + \sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\,P_u\,v(0)\|_{C^1} \\
&\le \|v\|_{C^1_\eta} + K \sup_{t\in(-\infty,0]} e^{-(c_c-\eta)t}\|P_c\,v(0)\|_{C^1} \\
&\qquad + K \sup_{t\in(-\infty,0]} e^{(c_u+\eta)t}\|P_u\,v(0)\|_{C^1} \\
&\le \|v\|_{C^1_\eta} + K\|P_c\|\,\|v(0)\|_{C^1} + K\|P_u\|\|v(0)\|_{C^1} \\
&\le \big(1 + K\|P_c\| + K\|P_u\|\big)\|v\|_{C^1_\eta} < \infty,
\end{aligned}$$

we have $z \in C^1_\eta$. Moreover, for all $s \le t \le 0$, we have

$$\begin{aligned}
z(t) &= v(t) - T_e(t)\,P_{cu}\,v(0) \\
&= T_e(t-s)\,v(s) + \int_s^t T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau - T_e(t)\,P_{cu}\,v(0) \\
&= T_e(t-s)\,v(s) - T_e(t-s)\,T_e(s)\,P_{cu}\,v(0) + \int_s^t T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&= T_e(t-s)\,z(s) + \int_s^t T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau.
\end{aligned}$$

Since furthermore $R_{\delta\eta}(v) \in Y_\eta$ by Corollary 4.3 and $P_{cu}^{\odot *}\,z(0) = P_{cu}\,z(0) = 0$, we obtain $z = \mathcal{K} \circ R_{\delta\eta}(v)$ due to Corollary 3.4. Hence, by definition

$$v(t) = z(t) + T_e(t)\,P_{cu}\,v(0) = (\mathcal{K}_\eta \circ R_{\delta\eta}(v))(t) + T_e(t)\,P_{cu}\,v(0)$$

for all $t \leq 0$, or equivalently,

$$v = \mathcal{K}_\eta \circ R_{\delta\eta}(v) + S_\eta(P_{cu}\, v(0)) = \mathcal{G}(v, P_{cu}\, v(0)).$$

This implies $v(0) = \mathcal{G}(v, P_{cu}\, v(0))(0) = \tilde{u}_\eta(P_{cu}\, v(0))(0) \in W^\eta$ as claimed.

The proof of $v(t) \in W^\eta$ as $t < 0$ may now be reduced to the above claim as follows. For given $t_0 < 0$ consider the translation

$$\hat{v} : (-\infty, 0] \ni s \longmapsto v(t_0 + s) \in C^1.$$

Obviously, we have $\hat{v} \in C^1_\eta$ and $\hat{v}$ is a solution of Eq. (26). Therefore $v(-t_0) = \hat{v}(0) \in W^\eta$ by the above claim. This completes the proof. $\square$

**Remark 4.9** Note that by application of the above result we easily deduce the identity

$$\tilde{u}_\eta(\varphi)(t) = \tilde{u}_\eta(P_{cu}\, \tilde{u}_\eta(\varphi)(t))(0)$$

for all $\varphi \in C_{cu}$ and $t \leq 0$.

### Proof of Theorem 1

In this final part of the present section we complete the proof of Theorem 1 on the existence of Lipschitz continuous local center-unstable manifolds. We conclude that in a neighborhood of the origin, the global center-unstable manifold $W^\eta$ of Eq. (25) has the properties asserted in Theorem 1.

Our proof starts with the following series of definitions depending on the constant $\delta > 0$ from condition (28):

$$\begin{aligned} C_{cu,0} &:= \Big\{ \varphi \in C_{cu} \mid \|\varphi\|_1 < \delta \Big\}, \\ C^1_{s,0} &:= \Big\{ \varphi \in C^1_s \mid \|\varphi\|_1 < \delta \Big\}, \\ N_{cu} &:= C_{cu,0} + C^1_{s,0}, \\ w_{cu} &:= w^\eta\big|_{C_{cu,0}}, \end{aligned}$$

and

$$W_{cu} := \Big\{ \varphi + w_{cu}(\varphi) \mid \varphi \in C_{cu,0} \Big\}.$$

Given an open neighborhood $V$ of 0 in $X_f$, note that one may choose $\delta > 0$ with $W_{cu} \subset V$. Applying Corollary 3.4 and estimate (29) of Corollary 4.3, we obtain for all $\varphi \in C_{cu,0}$

$$\begin{aligned}
\|w_{cu}(\varphi)\|_1 &= \|w^\eta(\varphi)\|_1 \\
&= \|P_s(\tilde{u}_\eta(\varphi)(0))\|_{C^1} \\
&= \|\tilde{u}_\eta(\varphi)(0) - P_{cu}(\tilde{u}_\eta(\varphi)(0))\|_{C^1} \\
&= \|\mathcal{G}_\eta(\tilde{u}_\eta(\varphi), \varphi)(0) - P_{cu}(\mathcal{G}_\eta(\tilde{u}_\eta(\varphi), \varphi)(0))\|_{C^1} \\
&= \|(S_\eta\,\varphi)(0) + (\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0) - P_{cu}((S_\eta\,\varphi)(0)) \\
&\qquad - P_{cu}((\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0))\|_{C^1} \\
&= \|(\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0)\|_{C^1} \\
&\leq \|\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi))\|_{C^1_\eta} \\
&\leq \|\mathcal{K}_\eta\|\|R_{\delta\eta}(\tilde{u}(\varphi))\|_{Y_\eta} \\
&\leq \|\mathcal{K}_\eta\|\delta\lambda(\delta),
\end{aligned} \tag{34}$$

and thus, $w_{cu}(C_{cu,0}) \subset C^1_{s,0}$ by assumption (28). The mapping $w_{cu}$ is also Lipschitz continuous, because for all $\varphi, \psi \in C_{cu,0}$ we have

$$\begin{aligned}
\|w_{cu}(\varphi) - w_{cu}(\psi)\|_{C^1} &= \|w^\eta(\varphi) - w^\eta(\psi)\|_{C^1} \\
&= \|P_s(\tilde{u}_\eta(\varphi)(0)) - P_s(\tilde{u}_\eta(\psi)(0))\|_{C^1} \\
&\leq \|P_s\|\|\tilde{u}_\eta(\varphi)(0) - \tilde{u}_\eta(\psi)(0)\|_{C^1} \\
&\leq \|P_s\|\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}
\end{aligned}$$

and the operator $\tilde{u}_\eta$ is (globally) Lipschitz continuous due to Proposition 4.6. Moreover, since $\mathcal{G}_\eta(0,0) = 0$ by definition, we have $\tilde{u}_\eta(0) = 0$ and hence $w_{cu}(0) = 0$. Consequently, Theorem 1 follows if we verify properties (i) - (iii) for $W_{cu}$, which is done below.

**Proof of Assertion (ii):** Assuming that $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ is a solution of the differential equation (1) with $x_t \in N_{cu}$, $t \leq 0$, we have to show $x_t \in W_{cu}$ for all $t \leq 0$. To this end, notice that by definition $\|P_{cu}\,x_t\|_1 < \delta$ and $\|P_s\,x_t\|_1 < \delta$ so that Corollary 4.1 yields $r(x_t) = r_\delta(x_t)$ for all $t \leq 0$. Therefore $x$ satisfies the smoothed differential equation (25) as well. Setting $u(t) = x_t$, $t \leq 0$, we consequently obtain a solution of the smoothed abstract integral equation (26). In particular, as $u$ is bounded on $(-\infty, 0]$, we conclude that $u \in C^1_\eta$, and hence $u(t) \in W^\eta$, $t \leq 0$, by Proposition 4.8. This implies $x_t \in W_{cu}$ for all $t \leq 0$, which is the desired conclusion. □

**Proof of Assertion (iii):** Assume that for a function $\varphi \in W_{cu}$ and $t_N > 0$ we have $\{F(t,\varphi) \mid 0 \leq s \leq t_N\} \subset N_{cu}$. To deduce $\{F(t,\varphi) \mid 0 \leq s \leq t_N\} \subset W_{cu}$ from this, consider the function

$$v(t) = \begin{cases} \tilde{u}_\eta(P_{cu}\,\varphi)(t_N + t), & \text{for } t \leq -t_N, \\ F(t_N + t, \varphi), & \text{for } -t_N \leq t \leq 0, \end{cases}$$

where $\tilde{u}_\eta(P_{cu}\,\varphi) \in C^1_\eta$ is the solution of Eq. (26) with $\tilde{u}_\eta(P_{cu}\,\varphi)(0) = \varphi$ from Corollary 4.7. As $v$ takes values in $C^1$, it is continuous at the questionable point $t = -t_N$ in view of the limits

$$\lim_{t \nearrow -t_N} v(t) = \lim_{t \nearrow -t_N} \tilde{u}_\eta(P_{cu}\,\varphi)(t_N + t) = \tilde{u}_\eta(P_{cu}\,\varphi)(0) = \varphi$$

and

$$\lim_{t \searrow -t_N} v(t) = \lim_{t \searrow -t_N} F(t_N + t, \varphi) = F(0, \varphi) = \varphi.$$

In addition, $v$ is bounded in the $\|\cdot\|_{C^1_\eta}$-norm due to

$$\sup_{t \in (-\infty, 0]} e^{\eta t} \|v(t)\|_{C^1} \leq \max\left\{ \|\tilde{u}_\eta(P_{cu}\,\varphi)\|_{C^1_\eta},\ \max_{t \in [0, t_N]} \|F(t, \varphi)\|_{C^1} \right\} < \infty,$$

we have $v \in C^1_\eta$. Moreover, we claim that $v$ is also a solution of Eq. (26). Indeed, suppose $s, t \in (-\infty, 0]$ with $s \leq t$. Then the cases $s \leq t \leq -t_N < 0$ and $-t_N \leq s \leq t \leq 0$ are obvious, whereas in the situation $s \leq -t_N \leq t \leq 0$, we get

$$\begin{aligned}
v(t) - T_e(t - s)\, v(s) &= v(t) - T_e(t + t_N)\, T_e(-t_N - s)\, v(s) \\
&= T_e(t + t_N)\, v(-t_N) + \int_{-t_N}^{t} T_e^{\odot *}(t - \tau)\, l(r_\delta(v(\tau)))\, d\tau \\
&\qquad - T_e(t + t_N)\, T_e(-t_N - s)\, v(s) \\
&= T_e(t + t_N)\Big( v(-t_N) - T_e(-t_N - s)\, v(s) \Big) \\
&\qquad + \int_{-t_N}^{t} T_e^{\odot *}(t - \tau)\, l(r_\delta(v(\tau)))\, d\tau \\
&= T_e(t + t_N) \int_{s}^{-t_N} T_e^{\odot *}(-t_N - \tau)\, l(r_\delta(v(\tau)))\, d\tau \\
&\qquad + \int_{-t_N}^{t} T_e^{\odot *}(-t_N - \tau)\, l(r_\delta(v(\tau)))\, d\tau \\
&= \int_{s}^{-t_N} T_e^{\odot *}(t - \tau)\, l(r_\delta(v(\tau)))\, d\tau + \int_{-t_N}^{t} T_e^{\odot *}(t - \tau)\, l(r_\delta(v(\tau)))\, d\tau \\
&= \int_{s}^{t} T_e^{\odot *}(t - \tau)\, l(r_\delta(v(\tau)))\, d\tau\,.
\end{aligned}$$

Thus, $v$ is a solution of Eq. (26) in $C^1_\eta$ as claimed.

Now Proposition 4.8 shows $v(t) \in W^\eta$ for all $t \leq 0$. Consequently, for constants $0 \leq t \leq t_N$ we have

$$F(t, \varphi) = v(t - t_N) \in N_{cu} \cap W^\eta,$$

and hence $F(t, \varphi) \in W_{cu}$, which proves our assertion. □

**Proof of Assertion (i):** It remains to prove that $W_{cu}$ is contained in the solution manifold $X_f$ of Eq. (1), and that $W_{cu}$ forms a Lipschitz submanifold of dimension $\dim C_{cu}$. For the first part, let $\varphi \in W_{cu}$ be given. Then from Corollary 4.7 it follows that the equations $x_t = \tilde{u}_\eta(P_{cu}\,\varphi)(t)$, $t \leq 0$, define a continuously differentiable function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ satisfying the smoothed differential equation (26) on $(-\infty, 0]$ and $x_0 = \varphi$. In particular, $\dot{\varphi}(0) = L\,\varphi + r_\delta(\varphi)$. As $\varphi \in W_{cu} \subset N_{cu}$ and in addition $r_\delta = r$ on $N_{cu}$ due to Corollary 4.1 we conclude

$$\dot{\varphi}(0) = L\,\varphi + r(\varphi) = f(\varphi) \in X_f.$$

This proves $W_{cu} \subset X_f$.

To see the second part of the assertion, we consider an $n$-dimensional complementary space $E$ of $Y = T_0 X_f$ in the Banach space $C^1$. We claim that there is no loss of generality in assuming $E \subset C^1_s$. In fact, let $\{e_1, \dots, e_n\}$ denote a basis of $E$. Then by the decomposition $C^1 = C_{cu} \oplus C^1_s$ according to Eq. (7) we get for each $i = 1, \dots, n$

$$e_i = u_i + s_i$$

with uniquely determined $u_i \in C_{cu}$ and $s_i \in C^1_s$. As the center-unstable space $C_{cu}$ is contained in $Y$, we conclude that $s_i \notin Y$ for all $i = 1, \dots, n$.

Define vectors $\hat{e}_i = e_i - u_i$ for $i = 1, \dots, n$ and suppose we have

$$\sum_{i=1}^{n} \lambda_i\,\hat{e}_i = 0$$

with reals $\lambda_i$, $i = 1, \dots, n$. Using the definition of $\hat{e}_i$, we obtain

$$E \ni \sum_{i=1}^{n} \lambda_i\,e_i = \sum_{i=1}^{n} \lambda_i\,u_i \in C_{cu}.$$

Since $C_{cu} \cap E = \{0\}$ it follows $\lambda_i = 0$ for all $i \in \{1, \dots, n\}$. Thus, the elements $\hat{e}_i$, $i = 1, \dots, n$, generate an $n$-dimensional subspace $\hat{E}$ of $C^1$, which is complementary to $Y$ in $C^1$. In particular, $\hat{E} \subset C^1_s$.

In view of the above, we suppose now that indeed $E \subset C^1_s$, which leads to

$$C^1_s = E \oplus (C^1_s \cap Y),$$
$$Y = C_{cu} \oplus (C^1_s \cap Y),$$

and

$$C^1 = E \oplus (C^1_s \cap Y) \oplus C_{cu} = E \oplus Y.$$

Let $P_Y : C^1 \longrightarrow C^1$ denote the projection operator of the Banach space $C^1$ onto $Y$ along $E$. Then we find an open neighborhood $V$ of $0$ in $X_f$ such that the restriction of $P_Y$ to $V$ forms a manifold chart of $X_f$ with a $C^1$-smooth inverse mapping from $Y_0 = P_Y(V)$ onto

$V$. Additionally, we may assume that $\delta > 0$ is sufficient small such that $W_{cu} \subset V$ and $P_Y W_{cu} \subset Y_0$. Consequently, we shall have established the assertion if we prove that $P_Y W_{cu}$ is an $\dim C_{cu}$-dimensional Lipschitz submanifold of the Banach space $Y$. But this is clear, since

$$P_Y W_{cu} = \left\{ P_Y(\varphi + w_{cu}(\varphi)) \,|\, \varphi \in C_{cu,0} \right\} = \left\{ \varphi + P_Y\, w_{cu}(\varphi) \mid \varphi \in C_{cu,0} \right\}$$

and $w_{cu}(\varphi) \in C_s^1$ for all $\varphi \in C_{cu,0}$. Therefore, for every $\varphi \in C_{cu,0}$ we obviously have $P_Y w_{cu}(\varphi) \in C_s^1 \cap Y$, so that $P_Y W_{cu}$ is the graph of the map

$$\left\{ \varphi \in C_{cu} \,|\, \|\varphi\|_1 < \delta \right\} \ni \chi \longmapsto P_Y\, w_{cu}(\chi) \in C_s^1 \cap Y.$$

In particular, the above map is Lipschitz continuous. This finishes the proof of the assertion (i) and so of Theorem 1 as a whole. □

## 5 The $C^1$-Smoothness of Local Center-Unstable Manifolds

Having proved the existence of local center-unstable manifolds in the last section, below we establish Theorem 2, asserting the $C^1$-smoothness of these manifolds. For this purpose, we follow very closely the procedure in the proof of smoothness of local center manifolds in Krisztin [8] and show that the technique also works in our situation.

### Auxiliary Results

The main idea of the proof for Theorem 2 is to employ the following abstract lemma stating under which conditions the fixed points of a parameter-dependent contraction form a $C^1$-smooth mapping of the involved parameter.

**Lemma 5.1 (Lemma II.8 in Krisztin et al. [9])** *Let $X$, $\Lambda$ denote two Banach spaces over $\mathbb{R}$, let $\mathcal{P} \subset \Lambda$ be open, and let a map $\xi : X \times \mathcal{P} \longrightarrow X$ and a real $\kappa \in [0, 1)$ be given satisfying*

$$\|\xi(x, p) - \xi(\tilde{x}, p)\|_X \leq \kappa \|x - \tilde{x}\|_X$$

*for all $x, \tilde{x} \in X$ and all $p \in \mathcal{P}$. Consider a convex subset $\mathcal{M}$ of $X$ and a map $\Phi : \mathcal{P} \longrightarrow \mathcal{M}$ with the property that for every $p \in \mathcal{P}$, the element $\Phi(p)$ is the unique fixed point of the induced map $\xi(\cdot, p) : X \longrightarrow X$. Furthermore, suppose that the following hypotheses hold.*

(i) *The restriction $\xi_0 = \xi\big|_{\mathcal{M}\times\mathcal{P}}$ of the mapping $\xi$ has a partial derivative*

$$D_2\xi_0 : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(\Lambda, X),$$

*and $D_2\xi_0$ is continuous.*

(ii) *There exist a Banach space $X_1$ over $\mathbb{R}$ and a continuous injective map $j : X \longrightarrow X_1$ such that the composed map $k = j \circ \xi_0$ is continuously differentiable with respect to $\mathcal{M}$ in the sense that there is a continuous map*

$$B : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X, X_1)$$

*such that for every $(x,p) \in \mathcal{M}\times\mathcal{P}$ and every $\varepsilon^* > 0$ one finds a real $\delta^* > 0$ guaranteeing*

$$\big\| k(\tilde{x}, p) - k(x,p) - B(x,p)\big(\tilde{x} - x\big) \big\|_{X_1} \leq \varepsilon^* \, \|\tilde{x} - x\|_X$$

*for all $\tilde{x} \in \mathcal{M}$ with $\|\tilde{x} - x\|_X \leq \tilde{\delta}$.*

(iii) *There exist maps*

$$\xi^{(1)} : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X, X)$$

*and*

$$\xi_1^{(1)} : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X_1, X_1)$$

*such that*

$$B(x,p)\,\tilde{x} = \big(j \circ \xi^{(1)}(x,p)\big)(\tilde{x}) = \big(\xi_1^{(1)}(x,p) \circ j\big)(\tilde{x})$$

*for all $(x, p, \tilde{x}) \in \mathcal{M} \times \mathcal{P} \times X$ and*

$$\big\| \xi^{(1)}(x,p) \big\| \leq \kappa$$

*as well as*

$$\Big\| \xi_1^{(1)}(x,p) \Big\| \leq \kappa$$

*on $\mathcal{M} \times \mathcal{P}$.*

(iv) *The map*

$$\mathcal{M} \times \mathcal{P} \ni (x,p) \longmapsto j \circ \xi^{(1)}(x,p) \in \mathcal{L}(X, X_1)$$

*is continuous.*

*Then the map $j \circ \Phi : \mathcal{P} \longrightarrow X_1$ is continuously differentiable and its derivative satisfies*

$$D(j \circ \Phi)(p) = \xi_1^{(1)}(\Phi(p), p) \circ D(j \circ \Phi)(p) + j \circ D_2\xi_0(\Phi(p), p)$$

*for all $p \in \mathcal{P}$.*

To verify the hypotheses of the last lemma in our situation, we will need another auxiliary result on some smoothness properties of Nemitsky operators between scaled Banach spaces. This result is a negligible modification of Lemma II.6 in Krisztin et al. [9] and Lemma 3.1 in Krisztin [8].

**Lemma 5.2** *Given any two Banach spaces $E$, $F$ over $\mathbb{R}$, consider for a real $\eta \geq 0$ the scaled Banach spaces $E_\eta := C_\eta((-\infty,0],E)$ and $F_\eta := C_\eta((-\infty,0],F)$. Further, let $q : U \longrightarrow F$ be a continuous and bounded map defined on some subset $U \subset E$ and let $\mathfrak{M}((-\infty,0],U)$, $\mathfrak{M}((-\infty,0],F)$ denote the sets of all mappings from the interval $(-\infty,0]$ into $U$, $F$, respectively. Then for the induced substitution operator*

$$\tilde{q} : \mathfrak{M}((-\infty,0],U) \longrightarrow \mathfrak{M}((-\infty,0],F)$$

*defined by*

$$\tilde{q}(u)(t) = q(u(t))$$

*for all $u \in \mathfrak{M}((-\infty,0],U)$ and $t \leq 0$ the following holds.*

(i) *If $\eta, \tilde{\eta} \geq 0$, then $\tilde{q}(\mathfrak{M}((-\infty,0],U) \cap E_\eta) \subset F_{\tilde{\eta}}$.*

(ii) *If $U$ is open, if $q$ is continuously differentiable with a bounded derivative $Dq$ and $0 \leq \eta \leq \tilde{\eta}$, then, for all $u \in C((-\infty,0],U)$, the linear map*

$$A(u) : \mathfrak{M}((-\infty,0],E) \longrightarrow \mathfrak{M}((-\infty,0],F),$$

*given by*

$$A(u)(v)(t) := Dq(u(t))v(t)$$

*for $v \in \mathfrak{M}((-\infty,0],E)$ and $t \leq 0$, satisfies*

$$A(u)(E_\eta) \subset F_{\tilde{\eta}}$$

*and*

$$\sup_{\|v\|_{E_\eta} \leq 1} \|A(u)(v)\|_{F_{\tilde{\eta}}} \leq \sup_{x \in U} \|Dq(x)\|,$$

*the induced linear maps*

$$A_{\eta\tilde{\eta}}(u) : E_\eta \longrightarrow F_{\tilde{\eta}}$$

*are continuous and in case $\eta < \tilde{\eta}$, the map*

$$A_{\eta\tilde{\eta}} : \big(C((-\infty,0],U) \cap E_\eta\big) \ni u \longmapsto A_{\eta\tilde{\eta}}(u) \in \mathcal{L}(E_\eta, F_{\tilde{\eta}})$$

*is continuous as well.*

(iii) *If additionally to the hypothesis stated above there holds $\eta < \tilde{\eta}$ and the set $U$ is convex, then for every $\tilde{\varepsilon} > 0$ and $u \in C((-\infty,0],U) \cap E_\eta$ there exists $\tilde{\delta} > 0$ such that for every $v \in C((-\infty,0],U) \cap E_\eta$ with $\|v-u\|_{E_\eta} < \tilde{\delta}$ we have*

$$\|\tilde{q}(v) - \tilde{q}(u) - A_{\eta\tilde{\eta}}(u)(v-u)\|_{F_{\tilde{\eta}}} \leq \tilde{\varepsilon}\,\|v-u\|_{E_\eta}.$$

**Proof:** We adopt the proof of Lemma 3.1 in Krisztin [8] which falls naturally into three steps.

1. *The proof of (i).* Assuming $u \in (\mathfrak{M}((-\infty,0],U) \cap E_\eta)$, we see at once that the continuity of $u$ and $q$ implies the one of

$$(-\infty,0] \ni t \longmapsto \tilde{q}(u)(t) = q(u(t)) \in F.$$

Moreover, the boundedness of $q$ leads to

$$\sup_{t\in(-\infty,0]} e^{\tilde{\eta}t}\|q(u(t))\|_F \leq \sup_{t\in(-\infty,0]} e^{\tilde{\eta}t} \sup_{t\in(-\infty,0]} \|q(u(t))\|_F \leq \sup_{x\in U}\|q(x)\|_F < \infty,$$

and thus $\|\tilde{q}(u)\|_{F_{\tilde{\eta}}} < \infty$. Consequently, we have $\tilde{q}(u) \in F_{\tilde{\eta}}$, which is the desired conclusion.

2. *The proof of (ii).* We begin with the observation that for all elements $u \in C((-\infty,0],U)$ the map $A(u)$ is well-defined, linear and that under the stated assumption the image $A(u)v \in \mathfrak{M}((-\infty,0],F)$ of an element $v \in E_\eta$, that is, the map

$$[0,\infty) \ni t \longmapsto Dq(u(t))\,v(t) \in F,$$

is continuous. As in this situation we also have

$$\begin{aligned} e^{\tilde{\eta}t}\,\|Dq(u(t))\,v(t)\|_F &\leq e^{(\tilde{\eta}-\eta)t}e^{\eta t}\,\|v(t)\|_E \sup_{x\in U}\|Dq(x)\| \\ &\leq \sup_{t\in(-\infty,0]} e^{\eta t}\|v(t)\|_E \sup_{x\in U}\|Dq(x)\| \\ &\leq \|v\|_{E_\eta}\sup_{x\in U}\|Dq(x)\| < \infty \end{aligned}$$

due to the boundedness of $Dq$ on $U$, we conclude $A(u)(E_\eta) \subset F_{\tilde{\eta}}$ and additionally

$$\sup_{\|v\|_{E_\eta}\leq 1}\|A(u)v\|_{F_{\tilde{\eta}}} \leq \sup_{x\in U}\|Dq(x)\|.$$

In particular, this shows the continuity of the maps $A_{\eta\tilde{\eta}} : E_\eta \longmapsto F_{\tilde{\eta}}$.

The only point remaining of assertion (ii) concerns the continuity of the map

$$A_{\eta\tilde{\eta}} : C((-\infty,0],U) \cap E_\eta \ni u \longmapsto A_{\eta\tilde{\eta}}(u) \in \mathcal{L}(E_\eta, F_{\tilde{\eta}})$$

in case $\eta < \tilde{\eta}$. To see this, choose $u \in C((-\infty,0],U) \cap E_\eta$ and let $\tilde{\varepsilon} > 0$ be given. As $\eta < \tilde{\eta}$ and $Dq$ is bounded on $U$, there clearly is a real $t_0 < 0$ satisfying

$$2e^{(\tilde{\eta}-\eta)t}\sup_{x\in U}\|Dq(x)\| < \tilde{\varepsilon}$$

for all $t \leq t_0$. Furthermore, in view of the continuity of $u$ and $Dq$ we find a constant $\tilde{\delta} > 0$ such that

$$B_t(u) = \left\{ y \in E \,\middle|\, \|y - u(t)\|_E < \tilde{\delta}e^{-\eta t_0} \right\} \subset U$$

as $t_0 \le t \le 0$ and such that additionally

$$\|Dq(y) - Dq(u(t))\| < \tilde{\varepsilon}$$

holds for all $y \in B_t$. Consequently, if $\tilde{u} \in C((-\infty, 0], U) \cap E_\eta$ with $\|\tilde{u} - u\|_{E_\eta} < \tilde{\delta}$, and if $v \in E_\eta$ with $\|v\|_{E_\eta} \le 1$, then the above estimates yield

$$e^{\tilde{\eta} t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big) v(t) \right\|_F \le \tilde{\varepsilon}$$

for all $t \le 0$. Indeed, in case $t \le t_0$ we see

$$\begin{aligned} e^{\tilde{\eta} t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big) v(t) \right\|_F &\le 2 e^{(\tilde{\eta} - \eta)t} e^{\eta t} \|v(t)\|_E \sup_{x \in U} \|Dq(x)\| \\ &\le 2 e^{(\tilde{\eta} - \eta)t} \|v\|_{E_\eta} \sup_{x \in U} \|Dq(x)\| \\ &< \tilde{\varepsilon}, \end{aligned}$$

whereas, for $t_0 < t \le 0$, we first conclude

$$\|\tilde{u}(t) - u(t)\|_E < \tilde{\delta} e^{-\eta t} < \tilde{\delta} e^{-\eta t_0}$$

and hence

$$\begin{aligned} e^{\tilde{\eta} t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big) v(t) \right\|_F &\le e^{(\tilde{\eta} - \eta)t} e^{\eta t} \|v(t)\|_E \|Dq(\tilde{u}(t)) - Dq(u(t))\| \\ &\le \|v\|_{E_\eta} \|Dq(\tilde{u}(t)) - Dq(u(t))\| \\ &< \tilde{\varepsilon}. \end{aligned}$$

This shows

$$\|A_{\eta\tilde{\eta}}(\tilde{u}) - A_{\eta\tilde{\eta}}(u)\| \le \tilde{\varepsilon},$$

and the continuity of $A_{\eta\tilde{\eta}}$ is proved.

3. *The proof of (iii).* Note that from the additional assumption on the convexity of the open set $U$ in $E$ it is easy to check that the set $C((-\infty, 0], U) \cap E_\eta$ is convex as well. Hence, for all $u, v \in C((-\infty, 0], U) \cap E_\eta$ and all $t \le 0$ we have

$$\begin{aligned} & e^{\tilde{\eta} t} \left\| q(v(t)) - q(u(t)) - Dq(u(t))\big(v(t) - u(t)\big) \right\|_F \\ &\quad = e^{\tilde{\eta} t} \left\| \int_0^1 \Big( Dq\big(sv(t) + (1-s)u(t)\big) - Dq\big(u(t)\big) \Big) \big(v(t) - u(t)\big)\, ds \right\|_F \\ &\quad \le e^{(\tilde{\eta} - \eta)t} e^{\eta t} \|v(t) - u(t)\|_E \\ &\qquad\qquad \cdot \max_{s \in [0,1]} \left\| Dq\big(sv(t) + (1-s)u(t)\big) - Dq\big(u(t)\big) \right\| \\ &\quad \le e^{(\tilde{\eta} - \eta)t} \|v - u\|_{E_\eta} \\ &\qquad\qquad \cdot \max_{s \in [0,1]} \left\| Dq\big(s\, v(t) + (1-s)\, u(t)\big) - Dq\big(u(t)\big) \right\|. \end{aligned} \tag{35}$$

Fix $u \in C((-\infty,0],E) \cap E_\eta$ and $\tilde{\varepsilon} > 0$. Then, using $\eta < \tilde{\eta}$, we find constants $t_0 < 0$ and $\tilde{\delta} \geq 0$ as in the last part. Let now an arbitrary $v \in C((-\infty,0],U) \cap E_\eta$ with $\|v-u\|_{E_\eta} < \tilde{\delta}$ be given. Then, in the situation $t \leq t_0$, the estimate (35) and the choice of the real $t_0$ yield

$$\begin{aligned} e^{\tilde{\eta}t} &\big\| q(v(t)) - q(u(t)) - Dq(u(t))\big(v(t)-u(t)\big)\big\|_F \\ &\leq e^{(\tilde{\eta}-\eta)t}\|v-u\|_{E_\eta} \\ &\qquad \cdot \max_{s\in[0,1]} \big\|Dq\big(sv(t)+(1-s)u(t)\big) - Dq\big(u(t)\big)\big\| \\ &\leq 2e^{(\tilde{\eta}-\eta)t} \max_{x\in U} \|Dq(x)\|\|v-u\|_{E_\eta} \\ &< \tilde{\varepsilon}\|v-u\|_{E_\eta} \end{aligned}$$

On the other hand, if $t_0 < t \leq 0$, then we have

$$\|v(t)-u(t)\|_E \leq \tilde{\delta}e^{-\eta t} < \tilde{\delta}e^{-\eta t_0}.$$

This implies $sv(t)+(1-s)u(t) \in B_t(u)$ for all $0 \leq s \leq 1$ and hence, by inequality (35), we get again

$$\begin{aligned} e^{\tilde{\eta}t} &\big\| q(v(t)) - q(u(t)) - Dq(u(t))\big(v(t)-u(t)\big)\big\|_F \\ &\leq e^{(\tilde{\eta}-\eta)t}\|v-u\|_{E_\eta} \\ &\qquad \cdot \max_{s\in[0,1]} \big\|Dq\big(sv(t)+(1-s)u(t)\big) - Dq\big(u(t)\big)\big\| \\ &< \tilde{\varepsilon}e^{(\tilde{\eta}-\eta)t}\|v-u\|_{E_\eta} \\ &< \tilde{\varepsilon}\|v-u\|_{E_\eta}. \end{aligned}$$

Combining these yields

$$\big\| \tilde{q}(v) - \tilde{q}(u) - A_{\eta\tilde{\eta}}(u)\big(v-u\big)\big\|_{F_{\tilde{\eta}}} \leq \tilde{\varepsilon}\, \|v-u\|_{E_\eta}\,,$$

and the proof is complete. □

### Proof of Theorem 2

After the preparatory results above, we return to the local center-unstable manifolds from the last section and prove Theorem 2.

We start our proof with the observation that an important, but probably inconspicuous point of our construction of the invariant manifolds in the foregoing section was the choice of a constant $\eta > 0$ satisfying condition (27), that is,

$$c_c < \eta < \min\{-c_s, c_u\},$$

and hereafter the choice of a second constant $0 < \delta < \delta_1$ satisfying condition (28), that is,

$$\|\mathcal{K}_\eta\|\lambda(\delta) < \frac{1}{2}.$$

Now, recall from Corollary 3.4 that $\mathcal{K}_\eta$ is a bounded linear map from the Banach space $Y_\eta$ into $C^1_\eta$. Moreover, the bound of $\mathcal{K}_\eta$ satisfies the inequality

$$\|\mathcal{K}_\eta\| < c(\eta) \tag{36}$$

with the continuous map $c : (c_c, \min\{-c_s, c_u\}) \longrightarrow [0,\infty)$ given by

$$c(\eta) = K\Big(1 + e^{\eta h}\|L_e\|\Big)\left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta}\right) + e^{\eta h}.$$

Hence, fixing a constant $\eta_1 > 0$ with $c_c < \eta_1 < \min\{-c_u, c_s\}$ and additionally a constant $0 < \delta < \delta_1$ with

$$c(\eta_1)\lambda(\delta) < \frac{1}{2},$$

we clearly find a real $c_c < \eta_0 < \eta_1$ such that the estimate

$$c(\eta)\lambda(\delta) < \frac{1}{2} \tag{37}$$

is fulfilled for all $\eta_0 \leq \eta \leq \eta_1$. As an immediate consequence, we see that for any $\eta_0 \leq \eta \leq \eta_1$ the pair $(\eta, \delta)$ satisfies both conditions (27), (28), and thus the construction in the last section works for any such choice of constants.

Below, we show the assertion of Theorem 2 for the map $w^{\eta_1}$. Hereby, remember that $w^{\eta_1}$ may be also written as the composition

$$w^{\eta_1} = P_s \circ \mathrm{ev}_0 \circ \tilde{u}_{\eta_1}$$

with the projection operator $P_s$ of $C^1$ along the center-unstable space $C_{cu}$ onto $C^1_s$, the evaluation map

$$\mathrm{ev}_0 : C^1_{\eta_1} \ni u \longmapsto u(0) \in C^1$$

and the fixed point operator $\tilde{u}_{\eta_1} : C_{cu} \longrightarrow C^1_{\eta_1}$ defined by (33). Since $P_s$ and $\mathrm{ev}_0$ are both bounded linear maps, for a conclusion on the $C^1$-smoothness of $w^{\eta_1}$ we are obviously reduced to proving the continuous differentiability of $\tilde{u}_{\eta_1}$ on $C_{cu}$. By application of Lemmata 5.1, 5.2, we show that $\tilde{u}_{\eta_1}$ is indeed continuously differentiable on $C_{cu}$ in the following.

Consider the open neighborhood

$$O_\delta := \big\{\psi \in C^1 \mid \|P_s\psi\|_1 < \delta\big\}$$

of the origin in $C^1$. The set $O_\delta$ is clearly convex, and from Corollary 4.1 and Proposition 4.2 we see that the restriction of the function $r_\delta$ to $O_\delta$ is bounded, $C^1$-smooth and has a bounded derivative with

$$\sup_{\varphi \in O_\delta} \|Dr_\delta(\varphi)\| \leq \lambda(\delta).$$

Additionally, we claim

$$\big\{\tilde{u}_\eta(\varphi)(t)\,\big|\,\varphi\in C_{cu}, t\le 0\big\}\subset O_\delta$$

for all $\eta_0\le\eta\le\eta_1$. Indeed, combining the inequalities (29), (36) and (37) yields

$$\begin{aligned}\|w^\eta(\varphi)\|_1 &= \|P_s\,\tilde{u}_\eta(\varphi)(0)\|_{C^1}\\ &= \big\|\big(\mathcal{K}_\eta\circ R_{\delta\eta}(\tilde{u}_\eta(\varphi))\big)(0)\big\|_{C^1}\\ &\le \big\|\big(\mathcal{K}_\eta\circ R_{\delta\eta}(\tilde{u}_\eta(\varphi))\big)\big\|_{C^1_\eta}\\ &\le \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(\tilde{u}_\eta(\varphi))\|_{Y_\eta}\\ &\le c(\eta)\,\delta\,\lambda(\delta)\\ &<\delta\end{aligned}$$

as $\varphi\in C_{cu}$ and $\eta_0\le\eta\le\eta_1$. Thus, in view of Remark 4.9 we obtain

$$\|P_s\,\tilde{u}_\eta(\varphi)(t)\|_1=\|P_s\,\tilde{u}_\eta(P_{cu}\,\tilde{u}_\eta(\varphi)(t))(0)\|_1=\|w^\eta(P_{cu}\,\tilde{u}_\eta(\varphi)(t))\|_1<\delta$$

for all $(\varphi,\eta,t)\in C_{cu}\times[\eta_0,\eta_1]\times(-\infty,0]$, as claimed. Now, setting $E:=C^1$, $F:=Y^{\odot*}$, $O:=O_\delta$, $q:=l\circ r_\delta$, $\eta:=\eta_0$, $\tilde{\eta}:=\eta_1$ and applying Lemma 5.2, we conclude that the linear maps

$$A(u):\mathfrak{M}((-\infty,0],C^1)\longrightarrow\mathfrak{M}((-\infty,0],Y^{\odot*})$$

define a continuous map $A_{\eta_0\eta_1}$ from the convex set

$$\mathcal{M}:=\Big\{u\in C^1_{\eta_0}\mid u(t)\in O_\delta\text{ for all }t\in(-\infty,0]\Big\}$$

into the Banach space $\mathcal{L}(C^1_{\eta_0},Y_{\eta_1})$. In addition, we see that $A_{\eta_0\eta_1}$ has the property that for every point $u\in\mathcal{M}$ and every real $\tilde{\varepsilon}>0$ there is a constant $\tilde{\delta}(\tilde{\varepsilon})>0$ such that for all $v\in\mathcal{M}$ with $\|v-u\|_{C^1_{\eta_0}}\le\tilde{\delta}$ we have $R_{\delta\eta_1}(u),R_{\delta\eta_1}(v)\in Y_{\eta_1}$ and

$$\big\|R_{\delta\eta_1}(u)-R_{\delta\eta_1}(v)-A_{\eta_0\eta_1}(u)\big(v-u\big)\big\|_{Y_{\eta_1}}\le\tilde{\varepsilon}\|v-u\|_{C^1_{\eta_0}}. \tag{38}$$

Next, we are going to employ Lemma 5.1. To this end, we regard the inclusion map

$$j_{\eta_0\eta_1}:C^1_{\eta_0}\ni u\longmapsto u\in C^1_{\eta_1}.$$

As $\eta_0<\eta_1$, this map obviously is well-defined and is trivially linear and bounded. Moreover, for all $\varphi\in C_{cu}$, $j_{\eta_0\eta_1}$ maps the fixed point $\tilde{u}_{\eta_0}(\varphi)$ of $\mathcal{G}_{\eta_0}(\,\cdot\,,\varphi)$ defined in Proposition 4.6 onto the fixed point $\tilde{u}_{\eta_1}(\varphi)$ of $\mathcal{G}_{\eta_1}(\,\cdot\,,\varphi)$. Indeed, since for a given $\varphi\in C_{cu}$ we have

$$\begin{aligned}\mathcal{G}_{\eta_1}(j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)),\varphi) &= S_{\eta_1}\,\varphi+\mathcal{K}_{\eta_1}\circ R_{\delta\eta_1}(j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)))\\ &= T_e(\,\cdot\,)\,\varphi+\mathcal{K}^{cu}\,R(\tilde{u}_{\eta_0}(\varphi))\\ &= j_{\eta_0\eta_1}\big(S_{\eta_0}\,\varphi+\mathcal{K}_{\eta_0}\circ R_{\delta\eta_0}(\tilde{u}_{\eta_0}(\varphi))\big)\\ &= j_{\eta_0\eta_1}\big(\mathcal{G}_{\eta_0}(\tilde{u}_{\eta_0}(\varphi),\varphi)\big)\\ &= j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)),\end{aligned}$$

$j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi))$ is a fixed point of $\mathcal{G}_{\eta_1}(\,\cdot\,,\varphi) : C^1_{\eta_1} \longrightarrow C^1_{\eta_1}$ and from the uniqueness of the fixed point there actually follows

$$j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)) = \tilde{u}_{\eta_1}(\varphi).$$

Set $X := C^1_{\eta_0}$, $X_1 := C^1_{\eta_1}$, $\Lambda := \mathcal{P} = C_{cu}$, $\xi := \mathcal{G}_{\eta_0}$, $j := j_{\eta_0\eta_1}$ and $\kappa := 1/2$. Then we see at once that $\tilde{u}_{\eta_0}(P) \subset \mathcal{M}$, and this implies that the unique fixed point of $\xi(\,\cdot\,,\varphi) : X \longrightarrow X$ is given by the value $\Phi(\varphi)$ of the map

$$\Phi : \mathcal{P} \ni \varphi \longmapsto \tilde{u}_{\eta_0}(\varphi) \in \mathcal{M}.$$

Additionally, for each $\varphi \in C_{cu}$ the map $\xi(\,\cdot\,,\varphi) = \mathcal{G}_{\eta_0}(\,\cdot\,,\varphi)$ is Lipschitz continuous with Lipschitz constant $\kappa$ due to the proof of Proposition 4.6. Thus, for an application of Lemma 5.1 with the above choice of spaces, maps and reals it remains to confirm conditions (i) - (iv). This point is done below in detail.

**Verification of hypothesis (i):** Observe that for the restriction $\xi_0$ of the map $\xi$ to $\mathcal{M} \times \mathcal{P}$ we have

$$\xi_0(u,\varphi) = \mathcal{G}_{\eta_0}(u,\varphi) = S_{\eta_0}\,\varphi + \mathcal{K}_{\eta_0} \circ R_{\delta\eta_0}(u).$$

Consequently, $\xi_0$ is partially differentiable with respect to the second variable, and for every $(u,\varphi) \in \mathcal{M} \times \mathcal{P}$ its derivative $D_2\xi_0(u,\varphi) \in \mathcal{L}(\Lambda, X)$ is given by

$$D_2\xi_0(u,\varphi)\psi = S_{\eta_0}\psi$$

for all $\psi \in C_{cu}$. Obviously, $D_2\xi_0 : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(\Lambda, X)$ is a constant map and thus in particular continuous. This shows hypothesis (i) of Lemma 5.1.

**Verification of hypothesis (ii):** The mapping $k = j \circ \xi_0$ reads

$$k(u,\varphi) = S_{\eta_1}\,\varphi + \mathcal{K}_{\eta_1} \circ R_{\delta\eta_1}(j(u)),$$

and the map

$$B : \mathcal{M} \times \mathcal{P} \ni (u,\varphi) \longmapsto \mathcal{K}_{\eta_1} \circ (A_{\eta_0\eta_1}(u)) \in \mathcal{L}(X, X_1)$$

is of course continuous as $\mathcal{K}_{\eta_1}$, $A_{\eta_0\eta_1}$ are so. Consider next an arbitrary point $(u,\varphi) \in \mathcal{M} \times \mathcal{P}$ and $\varepsilon^* > 0$. Choosing

$$\delta^* = \tilde{\delta}\left(\frac{\varepsilon^*}{1 + \|\mathcal{K}_{\eta_1}\|}\right)$$

with the constant $\tilde{\delta}$ from estimate (38), we find that for all points $v \in \mathcal{M}$ with $\|v - u\|_{C^1_{\eta_0}} < \delta^*$

we have

$$\begin{aligned}
&\big\|k(v,\varphi)-k(u,\varphi)-B(u,\varphi)\big(v-u\big)\big\|_{X^1}\\
&\qquad= \big\|\mathcal{K}_{\eta_1}(R(v))-\mathcal{K}_{\eta_1}(R(u))-\mathcal{K}_{\eta_1}\big(A_{\eta_0\eta_1}(u)(v-u)\big)\big\|_{C^1_{\eta_1}}\\
&\qquad\le \|\mathcal{K}_{\eta_1}\|\,\big\|R(v)-R(u)-A_{\eta_0\eta_1}(u)\big(v-u\big)\big\|_{Y_{\eta_1}}\\
&\qquad\le \|\mathcal{K}_{\eta_1}\|\,\frac{\varepsilon^*}{1+\|\mathcal{K}_{\eta_1}\|}\|v-u\|_{C^1_{\eta_0}}\\
&\qquad\le \varepsilon^*\|v-u\|_{C^1_{\eta_0}}.
\end{aligned}$$

Thus, condition (ii) is satisfied.

**Verification of hypothesis (iii):** Next we note that for every $u\in\mathcal{M}$ and all $v\in X$ we have

$$\begin{aligned}
A(u)(v)(t) &= Dq(u(t))v(t)\\
&= D(l\circ r_\delta)(u(t))v(t)\\
&= Dl(r_\delta(u(t)))\circ Dr_\delta(u(t))v(t)\\
&= l\circ Dr_\delta(u(t))v(t)
\end{aligned}$$

for $t\le 0$. Since $\sup_{\varphi\in O_\delta}\|Dr_\delta(\varphi)\|\le\lambda(\delta)$ and $\|\mathcal{K}_{\eta_0}\|\le c(\eta_0)$, and $\|l\|=1$, it is obvious that for every $u\in\mathcal{M}$, the induced map

$$\mathcal{K}_{\eta_0}\circ(A_{\eta_0\eta_0}(u))\in\mathcal{L}(X,X)$$

satisfies

$$\|\mathcal{K}_{\eta_0}\circ(A_{\eta_0\eta_0}(u))\|\le c(\eta_0)\lambda(\delta).$$

In the same manner we see that for all $u\in\mathcal{M}$

$$\mathcal{K}_{\eta_1}\circ(A_{\eta_1\eta_1}(u))\in\mathcal{L}(X_1,X_1)$$

with

$$\|\mathcal{K}_{\eta_0}\circ(A_{\eta_1\eta_1}(u))\|\le c(\eta_1)\lambda(\delta).$$

Define

$$\xi^{(1)}:\mathcal{M}\times\mathcal{P}\ni(u,\varphi)\longmapsto\mathcal{K}_{\eta_0}\circ(A_{\eta_0\eta_0}(u))\in\mathcal{L}(X,X)$$

and

$$\xi^{(1)}_1:\mathcal{M}\times\mathcal{P}\ni(u,\varphi)\longmapsto\mathcal{K}_{\eta_1}\circ(A_{\eta_1\eta_1}(u))\in\mathcal{L}(X_1,X_1).$$

Then, for all $(u,\varphi,v)\in\mathcal{M}\times\mathcal{P}\times X$, we get

$$\begin{aligned}
B(u,\varphi)v &= \big(\mathcal{K}_{\eta_1}\circ(A_{\eta_0\eta_1}(u))\big)(v)\\
&= \mathcal{K}^{cu}(A(u)v)\\
&= j\big(\xi^{(1)}(u,\varphi)v\big)\\
&= \xi^{(1)}_1(u,\varphi)\big(j(v)\big).
\end{aligned}$$

Moreover, in view of the choice of $\eta_0, \eta_1$ and $\delta$ due to Eq. (37) we have

$$\left\|\xi^{(1)}(u,\varphi)\right\|_X \leq \kappa$$

and

$$\left\|\xi_1^{(1)}(u,\varphi)\right\|_{X_1} \leq \kappa$$

for all $(u,\varphi) \in \mathcal{M} \times \mathcal{P}$. This shows that hypothesis (iii) is valid too.

**Verification of hypothesis (iv):** Finally, we find that the map

$$\mathcal{M} \times \mathcal{P} \ni (x,p) \longmapsto j \circ \xi^{(1)}(x,p) \in \mathcal{L}(X, X_1)$$

satisfies

$$j\big(\xi^{(1)}(u,\varphi)v\big) = \big(j \circ \mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u))\big)(v) = \mathcal{K}^{cu}(A(u)v) = B(u,\varphi)v$$

for all $(u,\varphi,v) \in \mathcal{M} \times \mathcal{P} \times X$. As $B$ is continuous, the continuity of the map

$$\mathcal{M} \times \mathcal{P} \ni (x,p) \longrightarrow j \circ \xi^{(1)}(x,p) \in \mathcal{L}(X, X_1)$$

follows, and this is precisely condition (iv) of Lemma 5.1.

As by the above all assumptions of Lemma 5.1 are fulfilled, we conclude that the map

$$\tilde{u}_{\eta_1} = j \circ \Phi : C_{cu} \longrightarrow C^1_{\eta_1}$$

is in fact continuously differentiable. So, if we prove that additionally we have $Dw_{cu}(0) = 0$, the assertion of Theorem 2 follows. But this is easily seen in consideration of the formula

$$D\tilde{u}_{\eta_1}(\varphi) = \xi_1^{(1)}(\tilde{u}_{\eta_0}(\varphi),\varphi) \circ D\tilde{u}_{\eta_1}(\varphi) + j \circ D_2\xi_0(\tilde{u}_{\eta_0}(\varphi),\varphi)$$

for the derivative of $\tilde{u}_{\eta_1}$ at $\varphi \in C_{cu}$. Indeed, by $Dr_\delta(0) = 0$, we first obtain $A(0) = 0$ and $\xi_1^{(1)}(0,0) = 0$. Thus, in consideration of $\tilde{u}_{\eta_0}(0) = 0$ we get

$$D\tilde{u}_{\eta_1}(0)\psi = j \circ D_2\xi_0(0,0)\psi = S_{\eta_1}\psi$$

for all $\psi \in C_{cu}$. This implies

$$Dw^{\eta_1}(0)\psi = \big(P_s \circ \mathrm{ev}_0 \circ D\tilde{u}_{\eta_1}(0)\big)(\psi) = P_s\psi = 0$$

on $C_{cu}$. Consequently, we get

$$Dw^{\eta_1}(0) = 0$$

and this completes the proof of Theorem 2.

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# The partial derivatives of de Rham's singular function and power sums of binary digital sums

ABSTRACT. This note is a supplement to the paper [9] on the partial derivatives $T_n$ of de Rham's function $R_a(x)$ with respect to the parameter $a$ at $a = 1/2$. In particular, $T_0(x) = x$ and $T_1(x) = 2T(x)$ where $T$ is Takagi's continuous nowhere differentiable function. We present a new representation of $T_n$. From this we derive a limit relation at dyadic rational points. Moreover, we show that real linear combinations of $T_n$ with $n \geq 1$ are nowhere differentiable. Thus we are able to prove that the functions which appear e.g. in the well known formula of Coquet for power sums of binary digital sums are nowhere differentiable. Finally, we derive a corresponding formula for power sums of the number of zeros.

KEY WORDS. De Rham's singular function, Takagi's continuous nowhere differentiable function, functional equations, binary digital sums, number of zeros, Stirling numbers.

## 1 Introduction

For a fixed parameter $a \in (0, 1)$ the system of functional equations

$$\left.\begin{array}{rcl} f\left(\frac{x}{2}\right) & = & af(x), \\ f\left(\frac{x+1}{2}\right) & = & a + (1-a)f(x) \end{array}\right\} \qquad (x \in [0,1]) \tag{1.1}$$

has a unique bounded solution $f = R_a(x)$ with $R_a(0) = 0$ and $R_a(1) = 1$, cf. [6]. It is $R_{1/2}(x) = x$, but for $a \neq \frac{1}{2}$ de Rham's function $R_a(x)$ is a strictly singular function which is also called Lebesgue singular function, cf. e.g. [1]. In [2] it was shown that for $\ell \in \mathbb{N}$ and $n = 0, 1, \ldots, 2^\ell$ it holds

$$R_a\left(\frac{n}{2^\ell}\right) = a^\ell \sum_{j=0}^{n-1} q^{s(j)} \tag{1.2}$$

where $q = (1-a)/a$ and where $s(j)$ denotes the number of ones in the binary representation of $j$. As consequence of (1.2) it was shown in [9] that for $q > 0$ it holds

$$\sum_{j=0}^{N-1} q^{s(j)} = N^\alpha G_q(\log_2 N) \tag{1.3}$$

where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is a continuous, 1-periodic function which is connected with de Rham's function by

$$G_q(u) = a^u R_a(2^u) \qquad (u \le 0) \tag{1.4}$$

where $a = \frac{1}{1+q}$. Formula (1.3) was the start point for the proof of explicit formulas for digital sums. For the binomial sum

$$B_k(N) = \sum_{j=0}^{N-1} \binom{s(j)}{k} \tag{1.5}$$

with integer $k \ge 1$ it holds the formula ([9])

$$\frac{1}{N} B_k(N) = \frac{1}{k!}\left(\frac{\log_2 N}{2}\right)^k + \frac{1}{k!}\sum_{\ell=0}^{k-1} (\log_2 N)^\ell F_{k,\ell}(\log_2 N) \tag{1.6}$$

and for the power sum

$$S_k(N) = \sum_{j=0}^{N-1} s(j)^k \tag{1.7}$$

with $k \ge 1$ it holds the formula of Coquet [3], (cf. also [5], [11] and [9])

$$\frac{1}{N} S_k(N) = \left(\frac{\log_2 N}{2}\right)^k + \sum_{\ell=0}^{k-1} (\log_2 N)^\ell G_{k,\ell}(\log_2 N) \tag{1.8}$$

where $F_{k,\ell}(u)$ and $G_{k,\ell}(u)$ are continuous, 1-periodic functions. In this note we show that the functions $F_{k,\ell}(u)$ and $G_{k,\ell}(u)$ are nowhere differentiable. (For $G_{k,\ell}(u)$ this is already known from [5]). In case $k = 1$ both formulas yield the well-known formula of Trollope-Delange ([13], [4]) for the sum of digits

$$\frac{1}{N}\sum_{j=0}^{N-1} s(j) = \frac{1}{2}\log_2 N + F_1(\log_2 N) \tag{1.9}$$

where the 1-periodic function $F_1(u)$ is connected with Takagi's function $T(x)$ by

$$F_1(u) = -\frac{u}{2} - \frac{1}{2^{u+1}} T(2^u) \qquad (u \le 0), \tag{1.10}$$

cf. [8, Theorem 2.1]. In [9] the functions $F_{k,\ell}(u)$ and $G_{k,\ell}(u)$ were expressed by means of the partial derivatives of de Rham's function $R_a(x)$ with respect to the parameter $a$ at $a = \frac{1}{2}$, i.e.

$$T_n(x) = \frac{\partial^n}{\partial a^n} R_a(x)\Big|_{a=1/2} \qquad (x \in [0,1]). \tag{1.11}$$

In particular, $T_0(x) = x$ and $T_1(x) = 2T(x)$ where $T$ is the Takagi function, cf. [9]. We show that for $0 < x \le 1$ we have

$$\frac{1}{x} T_n(x) = (-2)^n (\log_2 x)^n + \sum_{\nu=0}^{n-1} (\log_2 x)^\nu g_{n,\nu}(\log_2 x)$$

where the functions $g_{n,\nu}(u)$ are 1-periodic, continuous and nowhere differentiable. At dyadic points $x = \frac{k}{2^\ell}$ it hold the one-sided limits

$$\lim_{h\to+0} \frac{T_n(x+h)-T_n(x)}{h(\log_2 \frac{1}{h})^n} = 2^n$$

and

$$\lim_{h\to-0} \frac{T_n(x+h)-T_n(x)}{|h|(\log_2 \frac{1}{|h|})^n} = (-1)^{n+1}2^n.$$

Finally, if $s_0(j)$ denotes the number of zeros in the binary expansion of $j$ then

$$\frac{1}{N}\sum_{j=1}^{N-1} s_0(j)^k = \left(\frac{\log_2 N}{2}\right)^k + \frac{(-1)^{k-1}}{N} + \sum_{\ell=0}^{k-1}(\log_2 N)^\ell H_{k,\ell}(\log_2 N) \tag{1.12}$$

where $H_{k,\ell}(u)$ are 1-periodic continuous, nowhere differentiable functions.

In this note we use the Stirling numbers of first and second kind $s^{(1)}_{k,\ell}$, $s^{(2)}_{k,\ell}$ given by

$$k!\binom{x}{k} = \sum_{\ell=0}^{k} s^{(1)}_{k,\ell} x^\ell \tag{1.13}$$

and

$$x^k = \sum_{\ell=0}^{k} s^{(2)}_{k,\ell}\, \ell!\binom{x}{\ell}. \tag{1.14}$$

These numbers are integers. In particular, $s^{(1)}_{k,0} = s^{(2)}_{k,0} = 0$ for $k \geq 1$ and $s^{(1)}_{k,k} = s^{(2)}_{k,k} = 1$ for $k \geq 0$.

## 2 Partial derivatives

In [9] were introduced the partial derivatives of de Rham's function $R_a(x)$ at $a = \frac{1}{2}$, i.e.

$$T_n(x) = \frac{\partial^n}{\partial a^n} R_a(x)\bigg|_{a=1/2} \qquad (x \in [0,1]). \tag{2.1}$$

Thus $T_0(x) = x$ and $T_1(x) = 2T(x)$ where $T$ is Takagi's function. For $n \geq 1$ the function $T_n$ is continuous and has the symmetry property

$$T_n(1-x) = (-1)^{n+1}T_n(x) \tag{2.2}$$

and for $n \geq 2$ it satisfies the functional equations

$$\left.\begin{aligned} T_n\left(\tfrac{x}{2}\right) &= nT_{n-1}(x) + \tfrac{1}{2}T_n(x) \\ T_n\left(\tfrac{x+1}{2}\right) &= -nT_{n-1}(x) + \tfrac{1}{2}T_n(x) \end{aligned}\right\} \qquad (x \in [0,1]). \tag{2.3}$$

In [1] were investigated the functions

$$\tilde{T}_n(x) = \frac{1}{n!} T_n(x), \tag{2.4}$$

there with the notation $T_n(x)$. For every $\varepsilon > 0$ there exist constants $C_{n,\varepsilon}$ such that if $0 \le x < x + y \le 1$, then

$$|\tilde{T}_n(x+y) - \tilde{T}_n(x)| \le C_{n,\varepsilon} y^{1-\varepsilon}, \tag{2.5}$$

cf. [1]. By [9, Proposition 4.2] we know that for $n \ge 1$ the derivatives (2.1) of de Rham's function $R_a$ satisfy the functional relations

$$T_n\left(\frac{k+x}{2^\ell}\right) = T_n\left(\frac{k}{2^\ell}\right) + \sum_{\nu=0}^{n} a_\nu T_\nu(x) \tag{2.6}$$

where $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$, $x \in [0,1]$, $T_0(x) = x$ and where $a_\nu$ are the constants

$$a_\nu = \binom{n}{\nu} \frac{\partial^{n-\nu}}{\partial a^{n-\nu}} a^{\ell - s(k)} (1-a)^{s(k)} \Big|_{a=1/2} \tag{2.7}$$

which depend on $n$, $k$ and $\ell$. In particular, $a_n = 1/2^\ell$. Moreover, for $k = 0, 1, \ldots, 2^\ell$ it holds

$$T_n\left(\frac{k}{2^\ell}\right) = \frac{n!}{2^{\ell-n}} \sum_{j=0}^{k-1} \sum_{r=0}^{n} (-1)^r \binom{s(j)}{r} \binom{\ell - s(j)}{n-r}. \tag{2.8}$$

**Proposition 2.1** *For $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$, $x \in [0,1]$ we have*

$$T_n\left(\frac{k-x}{2^\ell}\right) = T_n\left(\frac{k}{2^\ell}\right) + \sum_{\nu=0}^{n} b_\nu T_\nu(x) \tag{2.9}$$

*where $b_\nu$ are the constants*

$$b_\nu = (-1)^{\nu+1} \binom{n}{\nu} \frac{\partial^{n-\nu}}{\partial a^{n-\nu}} a^{\ell - s(k-1)} (1-a)^{s(k-1)} \Big|_{a=1/2} \tag{2.10}$$

*which depend on $n$, $k$ and $\ell$. In particular, $b_n = (-1)^{n+1}/2^\ell$.*

**Proof:** If we denote the coefficients (2.7) more precisely by $a_{\nu,k}$ (for fixed $n$ and $\ell$) then from (2.6) with $k-1$ instead of $k$ and $1-x$ instead of $x$ we get

$$\begin{aligned} T_n\left(\frac{k-x}{2^\ell}\right) &= T_n\left(\frac{k-1}{2^\ell}\right) + \sum_{\nu=0}^{n} a_{\nu,k-1} T_\nu(1-x) \\ &= T_n\left(\frac{k-1}{2^\ell}\right) + a_{0,k-1} + \sum_{\nu=0}^{n} (-1)^{\nu+1} a_{\nu,k-1} T_\nu(x) \end{aligned}$$

where we have used (2.2) and $T_0(x) = x$. For $x = 0$ it follows

$$T_n\left(\frac{k}{2^\ell}\right) = T_n\left(\frac{k-1}{2^\ell}\right) + a_{0,k-1}$$

and hence (2.9) with the coefficients $b_\nu$ given by (2.10). $\square$

## 3 Non-differentiability of linear combinations of $T_n$

The following proposition is a generalization of [1, Theorem 1.5] to linear combinations

$$f_n(x) = \sum_{\nu=1}^{n} c_\nu \tilde{T}_\nu(x) = \sum_{\nu=1}^{n} \frac{c_\nu}{\nu!} T_\nu(x) \qquad (x \in [0,1]) \tag{3.1}$$

with certain constants $c_1, \dots, c_n$. We will modify a bit the nice proof in [1] where we use largely the same notations.

**Proposition 3.1** *If $c_n \neq 0$ then the function $f_n(x)$ from (3.1) is nowhere differentiable.*

**Proof:** For $x_0 \in [0,1)$ and positive integers $k$ we put $j_k = [2^k x_0]$ such that $0 \leq j_k \leq 2^k - 1$ and

$$\frac{j_k}{2^k} \leq x_0 < \frac{j_k + 1}{2^k}, \qquad k \in \mathbb{N}. \tag{3.2}$$

Observe that $j_{k+1} = 2j_k$ or $j_{k+1} = 2j_k + 1$ where $A = \{k : j_{k+1} = 2j_k\}$ is always infinite and $\mathbb{N} \setminus A = \{k : j_{k+1} = 2j_k + 1\}$ is finite if and only if $x_0$ is dyadic rational.

For an arbitrary function $f : [0,1] \mapsto \mathbb{R}$ we define

$$\Delta_f(k,j) := \frac{f((j+1) \cdot 2^{-k}) - f(j \cdot 2^{-k})}{2^{-k}} \qquad k \in \mathbb{N}, \quad j = 0, 1, \dots, 2^k - 1. \tag{3.3}$$

Let be $K_n$ the set of all functions (3.1) with $c_n \neq 0$. We show by induction on $n$ that for no $f \in K_n$ the limit

$$\lim_{k \to \infty} \Delta_f(k, j_k) \tag{3.4}$$

exists. For $n = 1$ this is true since each $f \in K_1$ has the form $f(x) = c_1 \tilde{T}_1(x) = 2c_1 T(x)$ with $c_1 \neq 0$ and the Takagi function $T(x)$ for which the nonexistence of the limit is well known (cf. [12]). Assume for a fixed $n \geq 2$ that for no $f \in K_{n-1}$ the limit (3.4) exists. Now we consider the function $f_n(x)$ from (3.1) with $c_n \neq 0$ which belongs to $K_n$ and assume that there exists a finite number $\lambda$ such that

$$\lim_{k \to \infty} \Delta_{f_n}(k, j_k) = \lambda. \tag{3.5}$$

It follows

$$\lim_{k \to \infty, k \in A} \Delta_{f_n}(k+1, 2j_k) = \lambda \tag{3.6}$$

and

$$\lim_{k \to \infty, k \notin A} \Delta_{f_n}(k+1, 2j_k + 1) = \lambda \tag{3.7}$$

whenever $\mathbb{N} \setminus A$ is infinite, cf. [1].

Put $\Delta_\nu(k,j) = \Delta_{\tilde{T}_\nu}(k,j)$ then $\Delta_0(k,j) = 1$ since $\tilde{T}_0(x) = x$ and by (3.1) we have

$$\Delta_{f_n}(k,j) = \sum_{\nu=1}^{n} c_\nu \Delta_\nu(k,j).$$

In view of

$$\Delta_\nu(k+1,2j) - \Delta_\nu(k+1,2j+1) = 4\Delta_{\nu-1}(k,j), \qquad (\nu \geq 1) \tag{3.8}$$

cf. [1], we find

$$\begin{aligned} \Delta_{f_n}(k+1,2j_k) - \Delta_{f_n}(k+1,2j_k+1) &= \sum_{\nu=1}^{n} 4c_\nu \Delta_{\nu-1}(k,j_k) \\ &= 4c_1\Delta_0(k,j_k) + \sum_{\mu=1}^{n-1} 4c_{\mu+1}\Delta_\mu(k,j_k) \end{aligned}$$

and hence

$$\Delta_{f_n}(k+1,2j_k) - \Delta_{f_n}(k+1,2j_k+1) = 4c_1 + \Delta_f(k,j_k) \tag{3.9}$$

where $f$ is the function

$$f(x) = 4c_2\tilde{T}_1(x) + \cdots + 4c_n\tilde{T}_{n-1}(x). \tag{3.10}$$

Obviously,

$$\Delta_{f_n}(k+1,2j_k) + \Delta_{f_n}(k+1,2j_k+1) = 2\Delta_{f_n}(k,j_k). \tag{3.11}$$

Now we consider two cases:

**1.** If $x_0$ is not dyadic rational, i.e. $\mathbb{N} \setminus A$ is infinite, then (3.5), (3.6) and (3.7) imply

$$\lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k) = \lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k+1) = \lambda.$$

**2.** If $x_0$ is dyadic rational, i.e. $\mathbb{N} \setminus A$ is finite, then there exists $k_0$ such that $j_{k+1} = 2j_k$ for $k > k_0$ and (3.6) can be written as

$$\lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k) = \lambda. \tag{3.12}$$

Now, (3.11), (3.5) and (3.12) imply

$$\lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k+1) = \lambda.$$

So in both cases from (3.9) we get $\lim_{k\to\infty} \Delta_f(k,j_k) = -4c_1$ for $f$ from (3.10) which belongs to $K_{n-1}$ since $c_n \neq 0$. This is a contradiction to the induction hypothesis. Thus $f_n(x)$ with $c_n \neq 0$ is not differentiable at $x_0 \in [0,1)$ which is valid also at $x_0 = 1$ in view of (2.2). □

**Remark 3.2** The proof makes use of the recursion (3.8) which in [1] was derived by a system of infinitely many difference equations for the functions $\tilde{T}_n(x)$, cf. [1, Corollary 2.5].

**Theorem 3.3** *If $g_\nu(x)$ $(\nu = 1, \ldots, n)$ are differentiable functions for $x \in [0,1]$ then the function*

$$f(x) = \sum_{\nu=1}^{n} g_\nu(x) T_\nu(x) \qquad (x \in [0,1])$$

*is differentiable at a point $x_0$ if and only if $g_\nu(x_0) = 0$ for $\nu = 1, \ldots, n$.*

**Proof:** For $x_0 \in [0,1]$ we consider $h \neq 0$ such that also $x_0 + h \in [0,1]$. Obviously,

$$\frac{f(x_0+h) - f(x_0)}{h} = \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = \sum_{\nu=1}^{n} \frac{g_\nu(x_0+h) - g_\nu(x_0)}{h} T_\nu(x_0+h), \qquad \Sigma_2 = \sum_{\nu=1}^{n} g_\nu(x_0) \frac{T_\nu(x_0+h) - T_\nu(x_0)}{h}.$$

Note that $\Sigma_1$ converges as $h \to 0$ since $g_\nu(x)$ is differentiable and $T_\nu(x)$ is continuous and that $\Sigma_2$ is convergent by Proposition 3.1 if and only if $g_\nu(x_0) = 0$ for all $\nu = 1, \ldots, n$. $\square$

## 4 Relations to periodic functions

In [9] were introduced the continuous, 1-periodic functions $F_k(u)$ given for $u \leq 0$ by

$$F_k(u) = \frac{\partial^k}{\partial q^k} a^u R_a(2^u) \bigg|_{q=1} \qquad (u \leq 0). \tag{4.1}$$

In particular, $F_0(u) = 1$ and $F_1(u)$ is the function from (1.10) which appears in the formula (1.9) of Trollope-Delange. For $k \geq 1$ the 1-periodic functions $F_k(u)$ have the representations

$$F_k(u) = \frac{1}{2^{u+k}} \sum_{\ell=0}^{k} \frac{P_{k,\ell}(u)}{2^\ell} T_\ell(2^u) \qquad (u \leq 0) \tag{4.2}$$

with the binomial polynomials

$$P_{k,\ell}(u) = (-1)^k \frac{k!}{\ell!} \binom{u+k-1}{k-\ell} \qquad (0 \leq \ell \leq k) \tag{4.3}$$

of degree $k - \ell$ and the partial derivatives $T_\ell$ from (2.1). In particular,

$$P_{k,0}(u) = (-1)^k u(u+1) \cdots (u+k-1), \qquad P_{k,k}(u) = (-1)^k, \tag{4.4}$$

cf. [9, Proposition 5.1]. From (2.4), (2.5) and (4.2) it follows

**Proposition 4.1** *For $h > 0$ and $\varepsilon > 0$ we have*

$$|F_k(u+h) - F_k(u)| \leq A_{k,\varepsilon} h^{1-\varepsilon}$$

*with a constant $A_{k,\varepsilon}$.*

A consequence of Theorem 3.3 and (4.2) is the following

**Proposition 4.2** *If the functions $h_k(u)$ are differentiable then*

$$F(u) = \sum_{k=1}^{n} h_k(u) F_k(u)$$

*is differentiable at $u_0$ if and only if $h_k(u_0) = 0$ for all $k \in \{1, 2, \ldots, n\}$.*

If we put $P_{k,\ell}(u) = 0$ for $\ell > k$ then for $n \in \mathbb{N}$ equation (4.2) can also be written in the matrix form

$$(1, 2F_1(u), \ldots, 2^n F_n(u))^\top = \mathbf{A}_n \left( \frac{1}{2^u}, \frac{1}{2^{u+1}} T_1(u), \ldots, \frac{1}{2^{u+n}} T_n(u) \right)^\top \tag{4.5}$$

with the lower triangular matrix $\mathbf{A}_n = (P_{k,\ell}(u))$, $0 \le k, \ell \le n$.

**Lemma 4.3** *For arbitrary integer $n \ge 1$ the matrix $\mathbf{A}_n$ is invertible and for the inverse matrix it holds $\mathbf{A}_n^{-1} = \mathbf{A}_n$.*

**Proof:** We have to show that $\mathbf{B}_n = (b_{k,\ell}) = \mathbf{A}_n^2$ is the unit matrix, i.e. $b_{k,\ell} = \delta_{k,\ell}$. We have

$$b_{k,\ell} = \sum_{j=0}^{n} P_{k,j}(u) P_{j,\ell}(u) = \sum_{j=\ell}^{k} P_{k,j}(u) P_{j,\ell}(u)$$

and hence $b_{k,\ell} = 0$ for $0 \le k \le \ell - 1$. In view of $P_{\ell,\ell}(u) = (-1)^\ell$ we get $b_{\ell,\ell} = 1$. Now let be $k \ge \ell + 1$. Note that

$$P_{k,\ell}(u) = (-1)^k \binom{k}{\ell} (u + k - 1)(u + k - 2) \cdots (u + \ell)$$

so that

$$P_{k,j}(u) P_{j,\ell}(u) = (-1)^{k+j} \binom{k}{j} \binom{j}{\ell} (u + k - 1)(u + k - 2) \cdots (u - \ell)$$

and therefore

$$b_{k,\ell} = (-1)^k (u - k - 1)(u - k - 2) \cdots (u - \ell) \sum_{j=\ell}^{k} (-1)^j \binom{k}{j} \binom{j}{\ell}.$$

Now

$$\binom{k}{j} \binom{j}{\ell} = \binom{k}{\ell} \binom{k - \ell}{j - \ell}$$

and

$$\sum_{j=\ell}^{k} (-1)^j \binom{k - \ell}{j - \ell} = (-1)^\ell (1 - 1)^{k-\ell} = 0.$$

Hence $b_{k,\ell} = 0$ for $k \ge \ell + 1$. □

As consequence we get from (4.5)

**Proposition 4.4** *The partial derivatives (2.1) of de Rham's function $R_a(x)$ have the representations*

$$\frac{1}{2^{u+k}} T_k(2^u) = \sum_{\ell=0}^{k} P_{k,\ell}(u) 2^\ell F_\ell(u) \qquad (u \le 0) \tag{4.6}$$

*with the polynomials (4.3) and the 1-periodic functions (4.1).*

**Remark 4.5** According to $P_{1,0}(u) = -u$, $P_{1,1}(u) = -1$, $F_0(u) = 1$ and $F_1(u)$ in (1.9) we get

$$\frac{1}{2^{u+1}} T_1(2^u) = -u - 2F_1(u) \qquad (u \le 0).$$

Putting $x = 2^u$ and using the fact that $T_1(x) = 2T(x)$ where $T(x)$ is the Takagi function, we find

$$\frac{1}{x} T(x) = -\log_2 x - 2F_1(\log_2 x) \qquad (0 < x \le 1), \tag{4.7}$$

cf. [8, Formula (2.5)].

By means of (4.6) we can give a new representation of $T_n$ using the explicit representation of the polynomials $P_{k,\ell}(u)$ of degree $k - \ell$

$$P_{k,\ell}(u) = \sum_{j=0}^{k-\ell} c_{k,\ell,j} u^j. \tag{4.8}$$

In view of (4.3) and the Stirling numbers of first kind $s^{(1)}_{k,\ell}$ given by (1.13) it is easy to compute the coefficients

$$c_{k,\ell,j} = (-1)^k \binom{k}{\ell} \sum_{r=0}^{k-\ell-j} s^{(1)}_{k-\ell,j+r} \binom{j+r}{r} (k-1)^r. \tag{4.9}$$

In particular, the coefficient of $u^{k-\ell}$ reads

$$c_{k,\ell,k-\ell} = (-1)^k \binom{k}{\ell} \tag{4.10}$$

which can be seen directly from (4.3).

**Theorem 4.6** *For $n \ge 1$ the derivatives (2.1) of de Rham's function $R_a$ have the representations*

$$\frac{1}{x} T_n(x) = (-2)^n (\log_2 x)^n + \sum_{\nu=0}^{n-1} (\log_2 x)^\nu g_{n,\nu}(\log_2 x) \qquad (0 < x \le 1) \tag{4.11}$$

*where $g_{n,\nu}(u)$ are 1-periodic functions given by*

$$g_{n,\nu}(u) = 2^n \sum_{\ell=0}^{n-\nu} c_{n,\ell,\nu} 2^\ell F_\ell(u) \tag{4.12}$$

*with the coefficients from (4.9). They are continuous and nowhere differentiable.*

**Proof:** For $u \le 0$ we have by (4.6) and (4.8)

$$\begin{aligned}\frac{1}{2^{u+k}}T_k(2^u) &= \sum_{\ell=0}^{k}\sum_{j=0}^{k-\ell} c_{k,\ell,j}u^j 2^\ell F_\ell(u)\\ &= \sum_{j=0}^{k}\sum_{\ell=0}^{k-j} c_{k,\ell,j}u^j 2^\ell F_\ell(u).\end{aligned}$$

For $k = n$ we get

$$\begin{aligned}\frac{1}{2^{u+n}}T_n(2^u) &= \sum_{\nu=0}^{n} u^\nu \sum_{\ell=0}^{n-\nu} c_{n,\ell,\nu}2^\ell F_\ell(u)\\ &= (-1)^n u^n + \sum_{\nu=0}^{n-1} u^\nu \sum_{\ell=0}^{n-\nu} c_{n,\ell,\nu}2^\ell F_\ell(u)\end{aligned}$$

where we have used that $c_{n,0,n} = (-1)^n$ and $F_0(u) = 1$. With $u = \log_2 x$ it follows (4.11) with (4.12). Obviously, the function $g_{n,\nu}(u)$ is 1-periodic and continuous. By (4.12) we have

$$g_{n,\nu}(u) = 2^{2n-\nu}c_{n,n-\nu,\nu}F_{n-\nu}(u) + 2^n \sum_{\ell=0}^{n-\nu-1} c_{n,\ell,\nu}2^\ell F_\ell(u)$$

where according to (4.10) it is $c_{n,n-\nu,\nu} = (-1)^n\binom{n}{\nu} \neq 0$. Therefore, by Proposition 4.2 the function $g_{n,\nu}(u)$ is nowhere differentiable. □

## 5 Limit relations

For the Takagi function $T$ it is known that at each dyadic point $x = \frac{k}{2^\ell}$ it holds

$$\lim_{h\to 0} \frac{T(x+h) - T(x)}{h\log_2 \frac{1}{h}} = 1, \tag{5.1}$$

cf. [7, Proposition 3.2]. We remember $T_1(x) = 2T(x)$ so that the following result is a generalization of (5.1).

**Proposition 5.1** *For $n \ge 1$ the derivatives (2.1) of de Rham's function $R_a$ satisfy at each dyadic rational point $x = \frac{k}{2^\ell}$ the limit relations*

$$\lim_{h\to +0} \frac{T_n(x+h) - T_n(x)}{h(\log_2 \frac{1}{h})^n} = 2^n \tag{5.2}$$

*and*

$$\lim_{h\to -0} \frac{T_n(x+h) - T_n(x)}{|h|(\log_2 \frac{1}{|h|})^n} = (-1)^{n+1}2^n. \tag{5.3}$$

**Proof:** For $x = 0$ equation (5.2) is a consequence of Theorem 4.6. Let $x = \frac{k}{2^\ell}$ and $0 < h < 1/2^\ell$. According to (2.6) we have

$$T_n(x+h) - T_n(x) = \sum_{\nu=0}^{n} a_\nu T_\nu(2^\ell h)$$

where $a_n = 1/2^\ell$ so that

$$\frac{T_n(x+h) - T_n(x)}{h(\log_2 \frac{1}{h})^n} = \frac{T_n(2^\ell h)}{2^\ell h(\log_2 \frac{1}{h})^n} + \sum_{\nu=0}^{n-1} a_\nu \frac{T_\nu(2^\ell h)}{h(\log_2 \frac{1}{h})^n}.$$

In view of $(\log_2 \frac{1}{h})^\nu \sim (\log_2 \frac{1}{2^\ell h})^\nu$ as $h \to 0$ it follows (5.2) by Proposition 4.6.

According to (2.9) we have

$$T_n(x-h) - T_n(x) = \sum_{\nu=0}^{n} b_\nu T_\nu(2^\ell h)$$

where $b_n = (-1)^{n+1}/2^\ell$ and hence

$$\frac{T_n(x-h) - T_n(x)}{h(\log_2 \frac{1}{h})^n} = (-1)^{n+1} \frac{T_n(2^\ell h)}{2^\ell h(\log_2 \frac{1}{h})^n} + \sum_{\nu=0}^{n-1} b_\nu \frac{T_\nu(2^\ell h)}{h(\log_2 \frac{1}{h})^n}$$

which implies (5.3). $\square$

**Remark 5.2** Relations (5.2) and (5.3) imply that at dyadic rational points $x = \frac{k}{2^\ell}$ there exists the improper derivative

$$\lim_{h\to 0} \frac{T_n(x+h) - T_n(x)}{h} = +\infty,$$

whenever $n \geq 2$ is even, whereas for odd $n$ it holds

$$\lim_{h\to 0} \frac{T_n(x+h) - T_n(x)}{|h|} = +\infty,$$

i.e. $T_n$ with odd $n$ has at $x$ a local minimum. Note that in case $n = 3$ there are further points $x$ where $T_3$ has a local minimum, cf. Theorem 6.24 in [1].

Start point for the proof of (5.1) in [7] was the fact that for $0 < x \leq \frac{1}{2}$ the Takagi function $T$ satisfies the estimate

$$x \log_2 \frac{1}{x} \leq T(x) \leq x \log_2 \frac{1}{x} + cx \tag{5.4}$$

with a constant $c < \frac{2}{3}$, cf. [7, Lemma 3.1]. By [10, Lemma 2.1] the estimate (5.4) is valid for $0 < x \leq 1$.

**Proposition 5.3** *The Takagi function $T$ satisfies for $0 < x \le 1$ the estimate (5.4) with the optimal constant $c = 2 - \log_2 3 = 0,415\ldots$ where on the right-hand side we have equality if and only if $x = \frac{1}{3} \cdot 2^{1-\ell}$ $(\ell = 0, 1, 2, \ldots)$.*

**Proof:** For the Takagi function $T$ we know that

$$\frac{1}{x} T(x) = -\log_2 x - 2F_1(\log_2 x) \qquad (0 < x \le 1)$$

where $F_1(u)$ is the the fractal function in (1.9), cf. (4.7). The assertion follows by Proposition 2.2 and Proposition 2.5 in [8] in view of $c = -2 \min F_1(.) = -2(\frac{\log 3}{\log 4} - 1) = 2 - \log_2 3$. $\square$

**Proposition 5.4** *For $n \ge 1$ the 1-periodic functions $F_n(u)$ given by (4.2) for $u \le 0$ satisfy at each point $u$ with $2^u = \frac{k}{2^\ell}$ the limit relations*

$$\lim_{h \to +0} \frac{F_n(u+h) - F_n(u)}{h(\log_2 \frac{1}{h})^n} = \frac{(-1)^n}{2^n} \ln 2 \tag{5.5}$$

*and*

$$\lim_{h \to -0} \frac{F_n(u+h) - F_n(u)}{|h|(\log_2 \frac{1}{|h|})^n} = \frac{-1}{2^n} \ln 2. \tag{5.6}$$

**Proof:** For $2^u = \frac{k}{2^\ell} < 1$ and $h > 0$ such that $2^{u+h} \le 1$ we have

$$\frac{1}{2^{u+h}} T_n(2^{u+h}) - \frac{1}{2^u} T_n(2^u) = \frac{1}{2^u} \left\{T_n(2^{u+h}) - T_n(2^u)\right\} + \frac{1}{2^u}\left(\frac{1}{2^h} - 1\right) T_n(2^{u+h})$$

and by (5.2) the asymptotic relation

$$\frac{1}{2^{u+h}} T_n(2^{u+h}) - \frac{1}{2^u} T_n(2^u) \sim 2^n(2^h - 1)\left(\log_2 \frac{1}{2^{u+h} - 2^u}\right)^n \qquad (h \to +0).$$

In view of $(2^h - 1)/h \to \ln 2$ as $h \to 0$ as well as

$$\log_2 \frac{1}{2^{u+h} - 2^u} = -u + \log_2 \frac{1}{2^h - 1}$$

and

$$\log_2 \frac{1}{2^h - 1} = \log_2 \frac{h}{2^h - 1} + \log_2 \frac{1}{h} \sim \log_2 \frac{1}{h} \qquad (h \to +0)$$

we get

$$\frac{1}{2^{u+h}} T_n(2^{u+h}) - \frac{1}{2^u} T_n(2^u) \sim 2^n h \ln 2 \left(\log_2 \frac{1}{h}\right)^n \qquad (h \to +0).$$

By (4.2) we have

$$F_n(u) = \frac{1}{2^{u+n}} \frac{(-1)^n}{2^n} T_n(2^u) + \frac{1}{2^{u+n}} \sum_{\ell=0}^{n-1} \frac{P_{n,\ell}(u)}{2^\ell} T_\ell(2^u) \qquad (u \le 0)$$

and it follows

$$\frac{F_n(u+h) - F_n(u)}{h(\log_2 \frac{1}{h})^n} \sim \frac{(-1)^n}{2^n} \ln 2 \frac{T_n(2^{u+h}) - T_n(2^u)}{h(\log_2 \frac{1}{h})^n} \qquad (h \to +0).$$

Hence (5.2) implies (5.5) at $u$ with $2^u = \frac{k}{2^\ell} < 1$ which is true for arbitrary $u$ with $2^u = \frac{k}{2^\ell}$ since $F_k(u)$ is an 1-periodic function. $\square$

## 6 Binomial and Power sums

In [9] it was shown that for integer $k \geq 1$ it holds

$$\frac{\partial^k}{\partial q^k} N^\alpha = \frac{N^\alpha}{(1+q)^k} \sum_{\ell=1}^{k} (\log_2 N)^\ell a_{k,\ell} \tag{6.1}$$

with certain coefficients $a_{k,\ell}$ which satisfy a recurrence relation. However, we have overlooked that $a_{k,\ell}$ is the Stirling number $s^{(1)}_{k,\ell}$ of first kind, given by (1.13). By a hint of L. Berg this can be seen as follows: We have $N^\alpha = (1+q)^\beta$ with $\beta = \log_2 N$ and hence

$$\frac{\partial^k}{\partial q^k} N^\alpha = \beta(\beta-1)\cdots(\beta-k+1)(1+q)^{\beta-k}.$$

In view of (1.13) it follows (6.1) with

$$a_{k,\ell} = s^{(1)}_{k,\ell}. \tag{6.2}$$

**Theorem 6.1** *For the binary binomial sum (1.5) with integer $k \geq 1$ we have the explicit formula*

$$\frac{1}{N} B_k(N) = \frac{1}{k!} \left( \frac{\log_2 N}{2} \right)^k + \frac{1}{k!} \sum_{\ell=0}^{k-1} (\log_2 N)^\ell F_{k,\ell}(\log_2 N) \tag{6.3}$$

*where*

$$F_{k,\ell}(u) = \frac{1}{2^\ell} \binom{k}{\ell} F_{k-\ell}(u) + \sum_{j=0}^{k-\ell-1} \binom{k}{j} \frac{s^{(1)}_{k-j,\ell}}{2^{k-j}} F_j(u) \tag{6.4}$$

*with the Stirling numbers of first kind $s^{(1)}_{k,\ell}$ and the 1-periodic functions $F_k(u)$ from (4.1). In particular, $F_{k,0}(u) = F_k(u)$ and $F_{k,k}(u) = 1/2^k$. For $\ell < k$ the functions $F_{k,\ell}(u)$ are continuous, nowhere differentiable and of period 1.*

**Proof:** In view of (6.2) and $s^{(1)}_{\ell,\ell} = 1$ the representation (6.3) with (6.4) is already proved in [9, Theorem 5.3] where $F_{k,\ell}(u)$ $(\ell < k)$ is continuous and of period 1. By Proposition 4.2 the function $F_{k,\ell}(u)$ is nowhere differentiable since the coefficient of $F_{k-\ell}(u)$ is different from zero. □

**Remarks 6.2** **1.** By Proposition 5.4 it holds that if $2^u$ is dyadic rational then for $\ell < k$ the functions $F_{k,\ell}$ from (6.4) satisfy the limit relations

$$\lim_{h \to +0} \frac{F_{k,\ell}(u+h) - F_{k,\ell}(u)}{h(\log_2 \frac{1}{h})^{k-\ell}} = \frac{(-1)^{k-\ell}}{2^k} \binom{k}{\ell} \ln 2 \tag{6.5}$$

and

$$\lim_{h \to -0} \frac{F_{k,\ell}(u+h) - F_{k,\ell}(u)}{|h|(\log_2 \frac{1}{|h|})^{k-\ell}} = \frac{-1}{2^k} \binom{k}{\ell} \ln 2. \tag{6.6}$$

**2.** In case $k = 1$ formula (6.3) yields the formula (1.9) of Trollope-Delange and in case $k = 2$ we get

$$\frac{1}{N}B_2(N) = \frac{1}{2}\left(\frac{\log_2 N}{2}\right)^2 + \frac{\log_2 N}{2}\left\{-\frac{1}{4} + F_1(\log_2 N)\right\} + \frac{1}{2}F_2(\log_2 N).$$

(In the corresponding formula in [9, p. $70_2$] the term $\frac{1}{2}F_1(L)$ is to cancel and in the previous formula the term $\binom{m}{2}F_1(u)$ is to replace by $\binom{m-1}{2}F_1(u)$).

Next, we consider the formula (1.8) of Coquet for the sum of digital power sums.

**Theorem 6.3** *For the power sum (1.7) it holds the formula of Coquet*

$$\frac{1}{N}S_k(N) = \left(\frac{\log_2 N}{2}\right)^k + \sum_{\ell=0}^{k-1}(\log_2 N)^\ell G_{k,\ell}(\log_2 N) \tag{6.7}$$

*where*

$$G_{k,\ell}(u) = \sum_{j=0}^{k-\ell}\sum_{n=\ell+j}^{k}\binom{n}{j}\frac{s^{(1)}_{n-j,\ell}}{2^{n-j}}s^{(2)}_{k,n}F_j(u) \tag{6.8}$$

*with the Stirling numbers of the first and second kind given by (1.13), (1.14) and the 1-periodic functions $F_j(u)$ from (4.1). So $G_{k,k}(u) = 1/2^k$ and for $\ell < k$ they are continuous, nowhere differentiable 1-periodic functions which can be written as*

$$G_{k,\ell}(u) = \frac{1}{2^\ell}\binom{k}{\ell}F_{k-\ell}(u) + \sum_{j=0}^{k-\ell-1}a_jF_j(u) \tag{6.9}$$

*with certain constants $a_j$ which depend on $k$ and $\ell$.*

**Proof:** In view of (6.2) the representation (6.7) with (6.8) is already proved in [9, Theorem 6.1] where $G_{k,\ell}(u)$ is continuous and of period 1. Obviously, the function $G_{k,\ell}(u)$ has the form

$$G_{k,\ell}(u) = \sum_{j=0}^{k-\ell}a_jF_j(u)$$

where the constants $a_j$ depend on $k$ and $\ell$. From (6.8) we get for the main coefficient $a_{k-\ell}$ the term

$$a_{k-\ell} = \binom{k}{\ell}\frac{s^{(1)}_{\ell,\ell}}{2^\ell}s^{(2)}_{k,k} = \frac{1}{2^\ell}\binom{k}{\ell}$$

which yields representation (6.9). By Proposition 4.2 the function $G_{k,\ell}(u)$ ($\ell < k$) is nowhere differentiable since $a_{k-\ell} \neq 0$. $\square$

**Remarks 6.4** **1.** In view of (6.9) the statements for $F_{k,\ell}$ in Remarks 6.2/1. are valid also for the functions $G_{k,\ell}$.

**2.** In case $k=1$ formula (6.7) yields the formula of Trollope-Delange (1.9) and in case $k=2$ we get the formula of Coquet [3]

$$\frac{1}{N}S_2(N) = \left(\frac{\log_2 N}{2}\right)^2 + \log_2 N\left\{\frac{1}{4} + F_1(\log_2 N)\right\} + G(\log_2 N)$$

where $G(u) = F_1(u) + F_2(u)$.

**Proposition 6.5** *For every integer $k \geq 1$ we have*

$$\frac{\partial^k}{\partial t^k} N^\alpha G_q(\log_2 N)\Big|_{t=0} = N\left(\frac{\log_2 N}{2}\right)^k + N\sum_{\ell=0}^{k-1}(\log_2 N)^\ell G_{k,\ell}(\log_2 N)$$

*where $q = e^t$ and $\alpha = \log_2(1+e^t)$.*

**Proof:** With $q = e^t$ we get from (1.3)

$$\sum_{j=0}^{N-1} e^{ts(j)} = N^\alpha G_q(\log_2 N) \tag{6.10}$$

where $\alpha = \log_2(1+e^t)$ and where the 1-periodic function $G_q$ is connected with de Rham's function by (1.4) with $a = \frac{1}{1+q}$. It follows

$$\sum_{j=0}^{N-1} s(j)^k = \frac{\partial^k}{\partial t^k} N^\alpha G_q(\log_2 N)\Big|_{t=0}$$

and by (6.7) the assertion. □

## 7 The number of zeros

If $2^n \leq j < 2^{n+1}$ then the number of zeros is $s_0(j) = n+1-s(j)$ where $s(j)$ denotes the number of ones.

**Lemma 7.1** *For $q > 0$ and $2^n \leq N < 2^{n+1}$ we have*

$$\sum_{j=1}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{q^{n+1}}N^\alpha G_q(\log_2 N) - q + \left(q - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)^n \tag{7.1}$$

*where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is a continuous, 1-periodic function given by (1.4).*

**Proof:** By formula (1.2) we get for $2^n \le N < 2^{n+1}$

$$\begin{aligned} R_a\left(\frac{N}{2^{n+1}}\right) - R_a\left(\frac{1}{2}\right) &= a^{n+1}\sum_{j=2^n}^{N-1} q^{s(j)} \\ &= a^{n+1}q^{n+1}\sum_{j=2^n}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)}. \end{aligned}$$

Moreover, (1.2) yields $R_a(1/2^r) = a^r$. If $2^{r-1} \le j < 2^r$ the number of zeros is $s_0(j) = r - s(j)$ and by (1.2) we get

$$\begin{aligned} R_a\left(\frac{2^r}{2^n}\right) - R_a\left(\frac{2^{r-1}}{2^n}\right) &= a^n \sum_{j=2^{r-1}}^{2^r-1} q^{s(j)} \\ &= a^n q^r \sum_{j=2^{r-1}}^{2^r-1}\left(\frac{1}{q}\right)^{s_0(j)} \end{aligned}$$

and hence

$$\sum_{j=2^{r-1}}^{2^r-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{a^\ell q^r}\left(a^{n-r} - a^{n-r+1}\right) = \frac{1-a}{(aq)^r} = \frac{1}{(aq)^{r-1}}.$$

In view of $aq = 1 - a$ and

$$\sum_{r=1}^{n}\frac{1}{(aq)^{r-1}} = \frac{1-\frac{1}{(aq)^n}}{1-\frac{1}{aq}} = -q\left(1-\frac{1}{(aq)^n}\right) = -q + \frac{1}{a^n q^{n-1}}$$

we get

$$\sum_{j=1}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{(1-a)^{n+1}}R_a\left(\frac{N}{2^{n+1}}\right) - \frac{1}{a^n q^{n+1}} - q + \frac{1}{a^n q^{n-1}}$$

i.e.

$$\sum_{j=1}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{q^{n+1}a^{n+1}}R_a\left(\frac{N}{2^{n+1}}\right) - q + \frac{q^2+1}{a^n q^{n-1}}.$$

Hence

$$\sum_{j=1}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{q^{n+1}}N^\alpha G_q(\log_2 N) - q + \frac{q^2+1}{a^n q^{n-1}}$$

with $a = \frac{1}{1+q}$ which yields the representation (7.1). $\square$

With $q = e^t$ we get from (7.1)

$$\sum_{j=1}^{N-1} e^{-ts_0(j)} = e^{-t(n+1)} N^\alpha G_q(\log_2 N) - e^t + (e^t - e^{-t})(1 + e^{-t})^n \tag{7.2}$$

where $\alpha = \log_2(1 + e^t)$ and $n = [\log_2 N]$ since $2^n \le N \le 2^{n+1} - 1$ and it follows for every integer $k \ge 1$

$$(-1)^k \sum_{j=1}^{N-1} s_0(j)^k = A_k(N) + B_k(N) - 1 \tag{7.3}$$

where

$$A_k(N) = \frac{\partial^k}{\partial t^k} \left[ e^{-t(n+1)} N^\alpha G_q(\log_2 N) \right] \Big|_{t=0} \tag{7.4}$$

and

$$B_k(N) = \frac{\partial^k}{\partial t^k} \left[ (e^t - e^{-t})(1 + e^{-t})^n \right] \Big|_{t=0} . \tag{7.5}$$

**Lemma 7.2** *For (7.4) we have the representations*

$$A_k(N) = (-1)^k N \left( \frac{\log_2 N}{2} \right)^k + N \sum_{\ell=0}^{k-1} (\log_2 N)^\ell A_{k,\ell}(\log_2 N) \tag{7.6}$$

*where $A_{k,\ell}(u)$ are 1-periodic function given for $0 \le u < 1$ by*

$$A_{k,\ell}(u) = \sum_{i=0}^{\ell} (-1)^i \sum_{m=i}^{k} \binom{k}{m} \binom{m}{i} (u-1)^{m-i} G_{k-m,\ell-i}(u) \tag{7.7}$$

*with the functions $G_{k,\ell}(u)$ from (6.8).*

**Proof:** We put $L = \log_2 N$. Observe that

$$\frac{\partial^k}{\partial t^k} \left[ e^{-t(n+1)} N^\alpha G_q(L) \right] = \sum_{m=0}^{k} \binom{k}{m} (-n-1)^m e^{-t(n+1)} \frac{\partial^{k-m}}{\partial t^{k-m}} \left[ N^\alpha G_q(L) \right] .$$

It follows by (7.4) and Proposition 6.5 with $n = [\log_2 N]$

$$A_k(N) = \sum_{m=0}^{k} \binom{k}{m} (-n-1)^m N \sum_{j=0}^{k-m} L^j G_{k-m,j}(L)$$

with the 1-periodic functions $G_{k-m,j}(u)$ from (6.8). For $2^n \le N \le 2^{n+1} - 1$ we write $N = 2^{n+u_N}$ with $0 \le u_N < 1$. In view of $L = \log_2 N = n + u_N$ we have $G_{k-m,j}(L) = G_{k-m,j}(u_N)$ and

$$A_k(N) = N \sum_{m=0}^{k} \binom{k}{m} (u_N - 1 - L)^m \sum_{j=0}^{k-m} L^j G_{k-m,j}(u_N).$$

We want to sort the right-hand side by powers of $L = \log_2 N$. From

$$A_k(N) = N \sum_{m=0}^{k} \binom{k}{m} \sum_{i=0}^{m} \binom{m}{i} (u_N - 1)^{m-i} (-L)^i \sum_{j=0}^{k-m} L^j G_{k-m,j}(u_N)$$

we get

$$A_k(N) = N \sum_{\ell=0}^{k} L^\ell A_{k,\ell}(u_N)$$

with

$$A_{k,\ell}(u) = \sum_{i+j=\ell} (-1)^i \sum_{m=i}^{k} \binom{k}{m} \binom{m}{i} (u-1)^{m-i} G_{k-m,j}(u)$$

which can be written as (7.7). In particular,

$$A_{k,k}(u) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} G_{k-i,k-i}(u) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{2^{k-i}} = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} (-2)^i = \frac{(-1)^k}{2^k}$$

where we have used (6.9) and $F_0(u) = 1$. If we continue the functions $A_{k,\ell}(u)$ to 1-periodic functions on $\mathbb{R}$ then we also get $A_{k,\ell}(u_N) = A_{k,\ell}(L)$ since $u_N = L - n$, and it follows (7.6). $\square$

**Remark 7.3** In particular, for $0 \leq u < 1$ we get by (7.7) in case $k = 1$

$$A_{1,0}(u) = u - 1 + F_1(u)$$

and in case $k = 2$

$$A_{2,0}(u) = u^2 - 2u + 2 + (1 - 2u) F_1(u) + F_2(u),$$
$$A_{2,1}(u) = \frac{1}{4} - (u - 1) - F_1(u)$$

where we have used (6.8) with the 1-periodic functions $F_j(u)$ from (4.1).

Now, for integer $k \geq 1$ we compute (7.5). Applying Leibniz formula it is easy to see that

$$B_k(N) = 2^n \sum_{i=0}^{k-1} b_{k,i} n^i \tag{7.8}$$

with certain coefficients $b_{k,i}$. The first sums read

$$B_1(N) = 2 \cdot 2^n, \quad B_2(N) = -2n \cdot 2^n, \qquad B_3(N) = (n^2 + 2n + 2) 2^n. \tag{7.9}$$

**Lemma 7.4** *For (7.5) we have the representations*

$$B_k(N) = N \sum_{\ell=0}^{k-1} (\log_2 N)^\ell B_{k,\ell}(\log_2 N) \tag{7.10}$$

*where $B_{k,\ell}(u)$ are 1-periodic functions given for $0 \leq u < 1$ by*

$$B_{k,\ell}(u) = \frac{1}{2^u} \sum_{i=\ell}^{k-1} b_{k,i} \binom{i}{\ell} (-u)^{i-\ell} \tag{7.11}$$

*with the numbers $b_{k,i}$ from (7.8).*

**Proof:** Starting with (7.8) we prove (7.10) with (7.11). As before we write $N = 2^{n+u_N}$ with $0 \leq u_N < 1$ so that $L = \log_2 N = n + u_N$, $2^n = 2^{L-u_N} = N/2^{u_N}$ and

$$n^i = (L - u_N)^i = \sum_{\ell=0}^{i} \binom{i}{\ell} L^\ell (-u_N)^{i-\ell}.$$

From (7.8) we get

$$B_k(N) = N \sum_{\ell=0}^{k-1} (\log_2 N)^\ell B_{k,\ell}(u_N)$$

with $B_{k,\ell}(u)$ from (7.11) for $0 \leq u < 1$. If we $B_{k,\ell}$ continue to 1-periodic functions on $\mathbb{R}$ then we have $B_{k,\ell}(\log_2 N) = B_{k,\ell}(u_N)$ since $N = 2^{n+u_N}$. So we get (7.10) with (7.11). □

**Remark 7.5** In particular, for $0 \geq u < 1$ we get by (7.11), (7.8) and (7.9) in case $k = 1$

$$B_{1,0}(u) = 2 \cdot \frac{1}{2^u}$$

and in case $k = 2$

$$B_{2,0}(u) = \frac{u}{2^{u-1}}, \quad B_{2,1}(u) = -\frac{1}{2^{u-1}}.$$

**Lemma 7.6** *For $\ell < k$ the 1-periodic function $A_{k,\ell}(u)$ given for $0 \leq u < 1$ by (7.7) is nowhere differentiable.*

**Proof:** We apply Proposition 4.2. According to (7.7) and (6.9) the function $A_{k,\ell}(u)$ has the form

$$A_{k,\ell}(u) = \sum_{j=0}^{k-\ell} h_j(u) F_j(u) \qquad (0 \leq u < 1)$$

where

$$h_{k-\ell}(u) = \sum_{i=0}^{\ell} (-1)^i \binom{k}{i} \frac{1}{2^{k-i}} \binom{k-i}{\ell-i}.$$

In view of

$$\binom{k}{i}\binom{k-i}{\ell-i} = \binom{k}{\ell}\binom{\ell}{i}$$

we get

$$h_{k-\ell}(u) = \frac{1}{2^k}\binom{k}{\ell}\sum_{i=0}^{\ell}(-2)^i\binom{\ell}{i} = (-1)^\ell \frac{1}{2^k}\binom{k}{\ell}$$

such that $h_{k-\ell}(u) \neq 0$ for $0 \leq u < 1$. By Proposition 4.2 the function $A_{k,\ell}(u)$ is nowhere differentiable. $\square$

**Theorem 7.7** *If $s_0(j)$ denotes the number of zeros in the binary expansion of the integer $j$ then for integer $k \geq 1$ we have*

$$\frac{1}{N}\sum_{j=1}^{N-1} s_0(j)^k = \left(\frac{\log_2 N}{2}\right)^k + \frac{(-1)^{k-1}}{N} + \sum_{\ell=0}^{k-1}(\log_2 N)^\ell H_{k,\ell}(\log_2 N) \tag{7.12}$$

*where*

$$H_{k,\ell}(u) = (-1)^k A_{k,\ell}(u) + (-1)^k B_{k,\ell}(u) \tag{7.13}$$

*with the functions $A_{k,\ell}$ from (7.6) and $B_{k,\ell}$ from (7.10). They are 1-periodic functions which are continuous and nowhere differentiable.*

**Proof:** The representation (7.12) follows from (7.3) in view of (7.6), (7.10) and (7.13) where $H_{k,k}(u) = 1/2^k$ since $B_{k,k}(u) = 0$. For $\ell < k$ the functions $A_{k,\ell}(u)$ are nowhere differentiable (Lemma 7.6) and $B_{k,\ell}(u)$ from (7.11) are differentiable in [0,1) so that $H_{k,\ell}(u)$ are nowhere differentiable. By Lemma 7.2 we know that the 1-periodic functions $H_{k,\ell}(u)$ are continuous in $[0,1)$ and that $H_{k,\ell}(1-0)$ there exist. It remains to show that $H_{k,\ell}(1-0) = H_{k,\ell}(1)$. For that we show that for integer $n$ it holds

$$S(n) = \sum_{\ell=0}^{k} n^\ell\{H_{k,\ell}(1) - H_{k,\ell}(1-0)\} = o(1) \qquad (n\to\infty)$$

which is possible only if $H_{k,\ell}(1) - H_{k,\ell}(1-0) = 0$ for $\ell = k, k-1, \ldots, 0$. We write $S(n) = \Sigma_1(n) + \Sigma_2(n)$ where

$$\Sigma_1(n) = \sum_{\ell=0}^{k} n^\ell\{H_{k,\ell}(1) - H_{k,\ell}(1+\log_2(1-2^{-n})\},$$

$$\Sigma_2(n) = \sum_{\ell=0}^{k} n^\ell\{H_{k,\ell}(1+\log_2(1-2^{-n}) - H_{k,\ell}(1-0)\}$$

and investigate both sums separately.

**1.** Using (7.12) we get for $s_0(N-1)^k$ the representation

$$\sum_{\ell=0}^{k}\left\{N(\log_2 N)^\ell H_{k,\ell}(\log_2 N)-(N-1)(\log_2(N-1))^\ell H_{k,\ell}(\log_2(N-1))\right\}.$$

As $N\to\infty$ we get the asymptotic equation

$$\frac{1}{N}s_0(N-1)^k=\sum_{\ell=0}^{k}(\log_2 N)^\ell\left\{H_{k,\ell}(\log_2 N)-H_{k,\ell}(\log_2(N-1))\right\}+o(1)$$

since in view of

$$(\log_2(N-1))^\ell=(\log_2 N+\log_2(1-1/N))^\ell=(\log_2 N)^\ell+\frac{(\log_2 N)^{\ell-1}}{N}O(1)$$

and $(\log_2 N)^{\ell-1}/N\to 0$ we have

$$(\log_2(N-1))^\ell H_{k,\ell}(\log_2(N-1))=(\log_2 N)^\ell H_{k,\ell}(\log_2(N-1))+o(1).$$

We choose $N=2^n$ with integer $n$. Note that $s_0(2^n-1)=0$ so that

$$0=\sum_{\ell=0}^{k}n^\ell\{H_{k,\ell}(n)-H_{k,\ell}(\log_2(2^n-1))\}+o(1)\qquad(n\to\infty),$$

and in view of $\log_2(2^n-1)=n+\log_2(1-2^{-n})$ and $H_{k,\ell}(u+1)=H_{k,\ell}(u)$ we get $\Sigma_1(n)=o(1)$ as $n\to\infty$.

**2.** Now, we consider the sum $\Sigma_2(n)$. In view of (7.13), (7.6), (7.7), (6.8) and the fact that $B_{k,\ell}(u)$ are continuous differentiable in $[0,1)$ (Lemma 7.2) we conclude that each function $H_{k,\ell}$ can be written as

$$H_{k,\ell}(u)=\sum_{j=0}^{k-\ell}f_j(u)F_j(u)\qquad(0\le u<1)$$

with certain continuous differentiable functions $f_j(u)$ which depend on $k$ and $\ell$. By Proposition 4.1 the functions $F_j(u)$ are Hölder continuous with Hölder exponents $1-\varepsilon$ where $\varepsilon>0$. It follows that for $0\le u<1$ the function $H_{k,\ell}(u)$ is Hölder continuous which is true for $0\le u\le 1$ if we choose $H_{k,\ell}(1-0)$ for $u=1$. So we get

$$|H_{k,\ell}(1-0)-H_{k,\ell}(1+\log_2(1-2^{-n}))|\le C_\varepsilon|\log_2(1-2^{-n})|^{1-\varepsilon}$$

with $\varepsilon>0$ and in view of $|\log_2(1-2^{-n})|\sim 2^{-n}$ and $n^\ell/2^{n(1-\varepsilon)}=o(1)$ as $n\to\infty$ we get $\Sigma_2(n)=o(n)$.

Consequently, $S(n)=o(n)$ as $n\to\infty$ and the functions $H_{k,\ell}(u)$ are continuous. □

**Remark 7.8** In view of Remarks 7.3 and 7.5 we get in case $k = 1$ the known representation

$$\frac{1}{N}\sum_{j=1}^{N-1} s_0(j) = \frac{1}{2}\log_2 N + \frac{1}{N} + H_{1,0}(\log_2 N)$$

with the 1-periodic function $H_{1,0}(u)$, given for $0 \le u < 1$ by

$$H_{1,0}(u) = \frac{1-u}{2} - 2^{1-u} + \frac{1}{2^u}T(2^{u-1})$$

cf. [8, Theorem 3.2], and in case $k = 2$

$$\frac{1}{N}\sum_{j=1}^{N-1}(s_0(j))^2 = \left(\frac{1}{2}\log_2 N\right)^2 - \frac{1}{N} + H_{2,0}(\log_2 N) + \log_2 N H_{2,1}(\log_2 N)$$

with the 1-periodic functions $H_{2,0}(u)$, $H_{2,1}(u)$, given for $0 \le u < 1$ by

$$H_{2,0}(u) = u^2 - 2u + 2 + (1-2u)F_1(u) + F_2(u) + \frac{u}{2^{u-1}}$$

and

$$H_{2,1}(u) = \frac{1}{4} - (u-1) - \frac{1}{2^{u-1}} - F_1(u).$$

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# Compactness in function spaces with splitting topologies

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## 1 Introduction

Let $(X, \tau)$, $(Y, \sigma)$ be topological spaces, and let be $\emptyset \neq \mathfrak{A} \subseteq \mathfrak{P}(X)$. We consider the set-open topology $\tau_{\mathfrak{A}}$ for $Y^X$ or for $C(X, Y)$, generated by the family $\mathfrak{A}$, and we assume that $\tau_p \subseteq \tau_{\mathfrak{A}}$ holds, where $\tau_p$ denotes the pointwise topology. For $H \subseteq C(X, Y)$ we want to characterize the $\tau_{\mathfrak{A}}$-compactness of $H$. We will need the condition that $H$ is evenly continuous on each $A \in \mathfrak{A}$. Hence we consider both sets $C(X, Y)$ and $C(A, Y)$ and of course we can link these spaces by the map $q_A : q_A(f) := f_{|A}$, the restriction of $f$ to the subspace $(A, \tau_{|A})$ of $(X, \tau)$. So we have $q_A : C(X, Y) \to C(A, Y)$.

Using these maps, we give a new and interesting proof of a "final" kind of the Ascoli theorem, as former derived by use of hyperspaces in [1].

Most notions used here are standard and explanations can be found in standard books on general topology such as [3], [4], [5]. Concerning some more special notions we refer to [2], more explanations can be found in [6] and [1], too.

## 2 The continuity of the map $q_A$

Now let be $B \subseteq X$ with $\emptyset \neq B \neq X$; let $\mathfrak{A} \subseteq \mathfrak{P}(X)$, $\mathfrak{B} \subseteq \mathfrak{P}(B)$, $\mathfrak{A} \neq \emptyset$ and $\mathfrak{B} \neq \emptyset$. Then we can consider the set-open topologies $\tau_{\mathfrak{A}}$ on $Y^X$ and $\tau_{\mathfrak{B}}$ on $Y^B$ respectively, and for fixed $B$ we have our map $q_B : Y^X \to Y^B : q_B(f) := f_{|B}$. Here at first the question arises, when is $q_B : (Y^X, \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ continuous? (Remark: If $q_B : (Y^X, \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ is continuous, then $q_B : (C(X, Y), \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ is continuous, and we know that $q_B(C(X, Y)) \subseteq C(B, Y)$ so we find $q_B : (C(X, Y), \tau_{\mathfrak{A}}) \to (C(B, Y), \tau_{\mathfrak{B}})$ being continuous.)

**Proposition 2.1** *If $\mathfrak{B} \subseteq \mathfrak{A}$ holds, then $q_B : (Y^X, \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ is continuous.*

**Proof:** For the generating subbase-elements of our topologies we use the symbols $(Z,V)_B := \{g \in Y^B |\ g(Z) \subseteq V\}$ and $(Z,V)_X := \{f \in Y^X |\ f(Z) \subseteq V\}$ with elements $Z$ of $\mathfrak{B}$ or $\mathfrak{A}$, respectively, and open subsets $V$ of $Y$.

To prove continuity of $q_B$, it is enough to show that the preimage of every subbase element of $\tau_{\mathfrak{B}}$ belongs to $\tau_{\mathfrak{A}}$, so let $Z \in \mathfrak{B} \subseteq \mathfrak{P}(B)$ and $V \in \sigma$ be given. Then we have $q_B^{-1}((Z,V)_B) = \{f \in Y^X |\ f_{|B} \in (Z,V)_B\} = \{f \in Y^X |\ f_{|B}(Z) \subseteq V\} = \{f \in Y^X |\ f(Z) \subseteq V\} = (Z,V)_X \in \tau_{\mathfrak{A}}$. ■

Some options to define suitable families $\mathfrak{A}, \mathfrak{B}$:

1. Let $\mathcal{E}$ be a property, which is defined for subsets of topological spaces (such as compactness, relative compactness or closedness, for example; but even such "non-topological" defined things as finiteness may be considered). The family of all subsets of a topological space $(X,\tau)$ having property $\mathcal{E}$ w.r.t. $\tau$ is denoted by $\mathcal{E}(X,\tau)$.[1] Then we can define $\mathfrak{A} := \mathcal{E}(X,\tau)$ and $\mathfrak{B} := \mathcal{E}(B,\tau_{|B})$.

2. We start with a family $\mathfrak{A} \subseteq \mathfrak{P}(X)$ and define $\forall B \in \mathfrak{A} : \mathfrak{A}_B := \{A \in \mathfrak{A} \mid A \subseteq B\}$.

## 3 Basic lemmas

We provide a few lemmas, which are very useful for our considerations.

**Lemma 3.1** *Let $(X,\tau)$ a topological space, $(Y,\sigma)$ a Hausdorff topological space. Let $\zeta$ be a topology (*$\lim$ *a convergence structure) on $C(X,Y)$ with $\tau_p \leq \zeta$ ($\tau_p \leq \lim$) and let $\mathcal{H} \subseteq C(X,Y)$ be compact w.r.t. $\zeta$ (resp.* $\lim$*). The $\mathcal{H}$ is $\tau_p$-closed in $Y^X$.*

**Proof:** Because of $\tau_p \leq \zeta$ ($\tau_p \leq \lim$) the compactness of $\mathcal{H}$ w.r.t. $\tau_p$ follows from assumtion. So, $\mathcal{H}$ is $\tau_p$-closed in $Y^X$ as a compact subset of the Hausdorff-space $(Y^X, \tau_p)$. ■

**Lemma 3.2** *Let $(X,\tau), (Y,\sigma)$ topological spaces; let $\emptyset \neq B \subseteq X$ and $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{P}(B)$ be given with the properties:*

(1) *$\forall Z \subseteq B : Z$ is $\tau_{|B}$-closed $\implies Z \in \mathfrak{B}$ and*

(2) *$\forall f \in C(B,Y) : f(B)$ is a $T_3$-subspace of $Y$.*

*Then the set-open topology $\tau_{\mathfrak{B}}$ is conjoining for $C(B,Y)$.*

[1] Although any dependence of our property $\mathcal{E}$ on $\tau$ is not required, it remains still allowed, so, we respect $\tau$ as a parameter. Somewhat more precise: such an "property" $\mathcal{E}$ is just a map from the class of all topological spaces to the class of all sets fulfilling the condition, that the image $\mathcal{E}(X,\tau)$ of every topological space $(X,\tau)$ is a subset of $\mathfrak{P}(X)$.

**Proof:** We will show, that the evaluation map

$$\omega : B \times C(B,Y) \to Y : \omega(x,f) := f(x)$$

is continuous w.r.t. $\tau \times \tau_{\mathfrak{B}}, \sigma$. For arbitrary $x \in B$ and $f \in C(B,Y)$ let $V \in \sigma$ be given with $\omega(x,f) \in V$. Because $f(B)$ is $T_3$ by assumtion and $V \cap f(B)$ is open in $f(B)$, there exist a closed subset $Z$ of $f(B)$ and an open subset $W$ of $f(B)$ such that

$$f(x) \in W \subseteq Z \subseteq f(B) \cap V \ .$$

since $f : B \to (Y,\sigma)$ ist continuous, it is continuous, too, viewed as a map from $B$ onto $f(B)$ w.r.t. $\sigma_{|f(B)}$. Thus $f^{-1}(Z)$ is closed and $f^{-1}(W)$ is open in $B$, and of course, $x \in f^{-1}(W)$ holds. So, by assumption (1), we have $f^{-1}(Z) \in \mathfrak{B}$ and consequently $(f^{-1}(Z), V) \in \tau_{\mathfrak{B}}$. Now, $f(f^{-1}(Z)) \subseteq Z \subseteq V$ implies $f \in (f^{-1}(Z), V)$, so $(f^{-1}(Z), V)$ is an open $\tau_{\mathfrak{B}}$-neighborhood of $f$ in $C(B,Y)$ and obviously, $f^{-1}(W)$ is an open neighborhood of $x$ in $B$. Now we have $\omega(f^{-1}(W) \times (f^{-1}(Z), V)) \subseteq V$, thus $\omega$ is continuous. ■

**Lemma 3.3** *Let $(X,\tau), (Y,\sigma)$ be topological spaces; let $\emptyset \neq \mathfrak{A} \subseteq \mathfrak{P}(X)$ be given and for every $B \in \mathfrak{A}$ let $\mathfrak{A}_B$ be a subset of $\mathfrak{P}(B)$ such that $B \in \mathfrak{A}_B$. Now we consider a filter $\mathcal{F}$ on $Y^X$ and a function $f \in Y^X$. Assume*

$$\forall B \in \mathfrak{A} : \ q_B(\mathcal{F}) \xrightarrow{\tau_{\mathfrak{A}_B}} f_{|B} \ .$$

*Then we have $\mathcal{F} \xrightarrow{\tau_{\mathfrak{A}}} f$ in $Y^X$.*

**Proof:** The sets $(B,V)_X$ with $B \in \mathfrak{A}$ and $V \in \sigma$ form a subbase of $\tau_{\mathfrak{A}}$, so we have to show, that $\mathcal{F}$ contains all such neighborhoods of $f$.

To do this, let $B \in \mathfrak{A}$, $V \in \sigma$ with $f \in (B,V)_X$ be given; we have $f(B) \subseteq V$ and hence $f_{|B}(B) \subseteq V$; by this way $f_{|B} = q_B(f) \in (B,V)_B = \{h \in Y^B | \ h(B) \subseteq V\}$; since $B \in \mathfrak{A}_B$, $(B,V)_B$ is an open subbase-element of $\tau_{\mathfrak{A}_B}$ in $Y^B$. Since $q_B(\mathcal{F}) \longrightarrow f_{|B}$ w.r.t. $\tau_{\mathfrak{A}_B}$, there exists $A \in \mathcal{F}$ such that $q_B(A) \subseteq (B,V)_B$ and so follows $A \subseteq (B,V)_X$ implying $(B,V)_X \in \mathcal{F}$. ■

## 4 $\tau_{\mathfrak{A}}$-compactness

Now, we want to formulate and prove the compactness criterion.

**Proposition 4.1** *Let $(X,\tau), (Y,\sigma)$ be topological spaces, let $\mathcal{H} \subseteq C(X,Y)$ and let $\emptyset \neq \mathfrak{A} \subseteq \mathfrak{P}(X)$ be given. Moreover, for every $B \in \mathfrak{A}$ let $\mathfrak{B}_B$ be a nonempty subset of $\mathfrak{P}(B)$. Assume $\tau_p \leq \tau_{\mathfrak{A}}$.*

1. *If $\mathcal{H}$ is $\tau_{\mathfrak{A}}$-compact and if*

   (i) $(Y, \sigma)$ *is Hausdorff,*

   (ii) $\forall B \in \mathfrak{A} : \mathfrak{B}_B \subseteq \mathfrak{A}$,

   (iii) $\forall B \in \mathfrak{A}, Z \subseteq B :\ Z\ \tau_{|B}$*-closed* $\implies Z \in \mathfrak{B}_B$,

   (iv) $\forall B \in \mathfrak{A}, f \in C(B, Y) :\ f(B)$ *is a $T_3$-subspace of $Y$*

   *hold, then we have:*

   (a) $\forall x \in X : \mathcal{H}(x)$ *is relatively compact in $Y$.*

   (b) $\mathcal{H}$ *is evenly continuous on each $B \in \mathfrak{A}$.*

   (c) $\mathcal{H}$ *is $\tau_p$-closed in $Y^X$.*

2. *Let (a), (b), (c) be true and let hold*

   (ii) $\forall B \in \mathfrak{A} : \mathfrak{B}_B \subseteq \mathfrak{A}$,

   (v) $\forall B \in \mathfrak{A} : B \in \mathfrak{B}_B$,

   (vi) $\forall B \in \mathfrak{A} :$ *the set-open topology $\tau_{\mathfrak{B}_B}$ is splitting in $C(B, Y)$.*

   *Then $\mathcal{H}$ is $\tau_{\mathfrak{A}}$-compact in $C(X, Y)$.*

**Proof:** (1) By lemma 3.1 we get (c); moreover by the proof of lemma 3.1 we know that $\mathcal{H}$ is $\tau_p$-compact, too, and hence $\mathcal{H}$ is $\tau_p$-relatively compact in $Y^X$, but then we obtain (a) by the Tychonoff-theorem for relatively compact sets (see [2], [1]). Now by condition (ii) and by proposition 2.1 we get: $\forall B \in \mathfrak{A} : q_B(\mathcal{H})$ is $\tau_{\mathfrak{B}_B}$-compact in $C(B, Y)$. (iii) and (iv) yield that $\tau_{\mathfrak{B}_B}$ is conjoining and hence $\mathcal{H}$ is evenly continuous on $B$ since $Y$ is Hausdorff (see theorem 32 in [2]). Thus we got (b).

(2) By (a), $\mathcal{H}$ is $\tau_p$-relatively compact in $Y^X$ and hence $\tau_p$-compact by (c). Let $\mathcal{F}$ be an ultrafilter on $C(X, Y)$ such that $\mathcal{H} \in \mathcal{F}$; by the $\tau_p$-compactness of $\mathcal{H}$ there exists $f \in \mathcal{H}$ with $\mathcal{F} \stackrel{\tau_p}{\to} f$; now, for all $B \in \mathfrak{A}$ the map $q_B : (C(X, Y), \tau_p) \to (C(B, Y), \tau_p)$ is continuous, implying that $q_B(\mathcal{F}) \stackrel{\tau_p}{\to} q_B(f) = f_{|B}$ in $C(B, Y)$ yielding by (b) that $q_B(\mathcal{F}) \stackrel{c}{\to} q_B(f)$ in $C(B, Y)$ holds. By (vi) we get $q_B(\mathcal{F}) \stackrel{\tau_{\mathfrak{B}_B}}{\to} q_B(f)$, thus $\mathcal{F} \stackrel{\tau_{\mathfrak{A}}}{\to} f$, by lemma 3.3 - showing that $\mathcal{H}$ is $\tau_{\mathfrak{A}}$-compact. ■

Assume $\mathfrak{A} := \{A \subseteq X |\ A \text{ compact}\}$ and for all $B \in \mathfrak{A}$ let $\mathfrak{B}_B := \{Z \subseteq B |\ Z \text{ compact}\}$. Then for the families $\mathfrak{A}$, $\mathfrak{B}_B$ the assumptions (ii) ... (vi) are obviously valid. So, we get:

**Corollary 4.2** *Let $(X, \tau), (Y, \sigma)$ be topological spaces, $(Y, \sigma)$ Hausdorff. Let $\mathcal{H} \subseteq C(X, Y)$ be given and consider the compact-open topology $\tau_{co}$ on $C(X, Y)$. Then are equivalent:*

(1) *$\mathcal{H}$ is $\tau_{co}$-compact.*

(2) (a) *$\forall x \in X : \mathcal{H}(x)$ is relatively compact in $Y$,*

(b) *$\mathcal{H}$ is evenly continuous on every compact set $K \subseteq X$,*

(c) *$\mathcal{H}$ is in $Y^X$ $\tau_p$-closed.*

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# Existence of solutions of nonlinear differential equations with generalized dichotomous linear part in a Banach space

ABSTRACT. A generalization of the well known dichotomies for a class of homogeneous differential equations in an arbitrary Banach space is introduced. The aim of this paper is the consideration of the nonlinear differential equation with generalized dichotomous linear part. By the help of the fixpoint principle of Banach and Schauder-Tychonoff are found sufficient conditions for the existence of solutions of the nonlinear equation.

KEY WORDS. Ordinary Differential Equations, Generalized Dichotomy

## 1 Introduction

The notion of exponential and ordinary dichotomy is fundamental in the qualitative theory of ordinary differential equations. It is considered in detail for example in the monographs [2], [3],[6–8].

In the given paper we use a $(M, N, R)$ dichotomy, introduced in [5], which is a generalization of all dichotomies known by the authors.

It is considered a nonlinear differential equation with generalized dichotomous linear part. A nonlinear operator, acting in the phase space is introduced. Sufficient conditions for the existence of fixed point of this operator are found. These fixed points are solutions of the differential equation.

## 2 Problem statement

Let $X$ is an arbitrary Banach space with norm $|.|$ and identity $I$ and let $J = [c, \infty)$ where $c \in \mathbb{R}$. Let $L(X)$ is the space of all linear bounded operators acting in $X$ with the norm $||.||$.

We consider the nonlinear differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + F(t, x), \tag{1}$$

where $A(.) : J \to L(X),\ F(.,.) : J \times X \to X$. Let $F$ is continuous.

By $V(t)$ we will denote the Cauchy operator of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \tag{2}$$

where $A(t) \in L(X), t \in J$.

We consider also the nonhomogeneous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t) \tag{3}$$

where $f(.) : J \to X$ is continuous and bounded.

In this paper we will use the $(M, N, R)$-dichotomy, introduced in [5] with both following theorems.

Let $R(t) : X \to X \quad (t \in J)$ is an arbitrary bounded operator.

**Lemma 1** [5] *The function*

$$x(t) = \int_c^t V(t)R(s)V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty V(t)(I - R(s))V^{-1}(s)f(s)\mathrm{d}s \tag{4}$$

*is a solution of the equation (3) if the integrals in (4) exist.*

Following conditions are introduced

H1. $|\ V(t)R(s)V^{-1}(s)z\ | \leq M(t, s, z), t \geq s, z \in X$

H2. $|\ V(t)(I - R(s))V^{-1}(s)z\ | \leq N(t, s, z), t < s, z \in X$

For all considered cases the right hand part of (H1) and (H2) will have the form

$$\begin{cases} M(t, s, z) = \varphi_1(t)\varphi_2(s) \mid z \mid, \ (t \geq s), \ z \in X \\ N(t, s, z) = \psi_1(t)\psi_2(s) \mid z \mid, \ (t < s), \ z \in X \end{cases} \tag{5}$$

where $\varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t)$ are positive scalar functions. We set

$$\alpha(t) = \max\{\varphi_1(t), \psi_1(t), 1\},$$

$$\mu(t) = \min\{\varphi_1(t), \psi_1(t)\},$$

$$\beta(t) = \max\{\varphi_2(t), \psi_2(t)\} \quad (t \in J).$$

**Definition 1** *We call the equation (2) be a $(M, N, R)$ - dichotomous if the conditions (H1), (H2) are fulfilled.*

Let $a(t)$ is an arbitrary positive scalar function. We consider the following Banach spaces :

$$K_a = \{g : J \to X : \sup_{t \in J} a(t) \int_c^t M(t, s, g(s))\mathrm{d}s < \infty\}$$

with the norm

$$\mid g \mid_{K_a} = \sup_{t \in J} \ a(t) \int_c^t M(t, s, g(s))\mathrm{d}s,$$

$$L_a = \{g : J \to X : \sup_{t \in J} a(t) \int_t^\infty N(t, s, g(s))\mathrm{d}s < \infty\}$$

with the norm

$$\mid g \mid_{L_a} = \sup_{t \in J} \ a(t) \int_t^\infty N(t, s, g(s))\mathrm{d}s,$$

$$C_a = \{g : J \to X : \sup_{t \in J} a(t) \mid g(t) \mid < \infty\}$$

with the norm

$$\mid g \mid_{C_a} = \sup_{t \in J} \ a(t) \mid g(t) \mid$$

and

$$T_a = \{g : J \to X : \int_c^\infty a(s) \mid g(s) \mid \mathrm{d}s < \infty\}$$

with the norm

$$\mid g \mid_{T_a} = \int_c^\infty a(s) \mid g(s) \mid \mathrm{d}s.$$

The case, when $X = \mathbb{R}_+$ will be denoted with $\bar{T}_a$ :

$$\bar{T}_a = \{g : J \to \mathbb{R}_+ : \int_c^\infty a(s)g(s)\mathrm{d}s < \infty\}$$

with the norm

$$\mid g \mid_{\bar{T}_a} = \int_c^\infty a(s)g(s)\mathrm{d}s.$$

**Theorem 1** [5] *Let the equation* (2) *is* $(M, N, R)$ *- dichotomous. Then for every function* $f \in K_a \bigcap L_a$ *the equation* (3) *has a solution in the space* $C_a$.

**Corollary 1** [5] *Let the equation* (1) *is* $(M, N, R)$ *- dichotomous of the form* (5). *Then for every function* $f \in T_\beta$ *the equation* (2) *has a solution in the space* $C_{\alpha^{-1}}$ *and the following estimates hold*

$$\sup_{t \in J} \ \alpha^{-1}(t) \mid x(t) \mid \leq \int_c^t \beta(s) \mid f(s) \mid \mathrm{d}s + \int_t^\infty \beta(s) \mid f(s) \mid \mathrm{d}s < \infty$$

**Theorem 2** [5] *Let the equation* (2) *is* $(M, N, R)$ *- dichotomous.*

*Then following estimates hold*

$$| x_1(t) | \leq M(t, s, x_1(s)), \ t \geq s \geq c \tag{6}$$

*for all solutions* $x_1(t)$ *of (1)* ,$(t \geq c)$, *which started in the set*

$$\bigcap_{s \in J} Fix \ R(s)$$

*and*

$$| x_2(t) | \leq N(t, s, x_2(s)), \ c \leq t < s \tag{7}$$

*for all solutions* $x_2(t)$ *of (1)* ,$(t \geq c)$, *which started in the set*

$$\bigcap_{s \in J} Fix(I - R(s))$$

(By $FixS$ we denote the set of all fixed points of the map $S$, $S : X \to X$.)

**Remark 1** Let $R(t) = P$,where $P : X \to X$ is a projector.

For

$$M(t, s, z) = K_1 e^{-\int_s^t \delta_1(\tau) \mathrm{d}\tau} |z| \ (t \geq s, z \in X)$$
$$N(t, s, z) = K_2 e^{-\int_t^s \delta_1(\tau) \mathrm{d}\tau} |z| \ (s > t, z \in X)$$

where $K_1, K_2$ are positive constants and $\delta_1, \delta_2$ are continuous real-valued functions on $J$, we obtain the exponential dichotomy of [7]:

$$\| V(t)PV^{-1}(s) \| \leq K_1 e^{-\int_s^t \delta_1(\tau) \mathrm{d}\tau} \ (t \geq s)$$
$$\| V(t)(I - P)V^{-1}(s) \| \leq K_2 e^{-\int_t^s \delta_2(\tau) \mathrm{d}\tau} \ (s > t).$$

For $\delta_i(t) = 0$ $(c \leq t < \infty, \ i = 1, 2)$ we obtain the exponential dichotomy of [2], [3], [6], for which case we have $K_a \bigcap L_a = C_a$ by $a(t) \equiv 1$.

For

$$M(t, s, z) = Kh(t)h^{-1}(s)|z| \ (t \geq s \geq c, , z \in X)$$
$$N(t, s, z) = Kk(t)k^{-1}(s)|z| \ (c \leq t \leq s, z \in X)$$

where $K$ is a positive constant and $h, k : [0, \infty) \to (0, \infty)$ are two continuous functions, we obtain the dichotomy of [8–10]:

$$\| V(t)PV^{-1}(s) \| \leq Kh(t)h^{-1}(s) \ , (t \geq s \geq c)$$
$$\| V(t)(I - P)V^{-1}(s) \| \leq Kk(t)k^{-1}(s) \ , (c \leq t \leq s)$$

It may be also noted, that the dichotomies [1], [7–10] are a generalization of the dichotomy in [3].

## 3 Main results

By the help of the fixpoint principle of Banach we will find sufficient conditions for the existence of solutions of the nonlinear equation (1).

Let $r > 0$. We introduce following conditions

H3. There exists a positive function $m \in \bar{T}_\beta$, such that

$$|F(t,x)| \le m(t) \quad (|x| \le r,\ t \in J).$$

H4. There exists a positive function $k \in \bar{T}_\beta$, such that

$$|F(t,x_2) - F(t,x_1)| \le \alpha^{-1}(t)k(t)|x_2 - x_1| \quad (|x_1|, |x_2| \le r,\ t \in J).$$

We set $a_1 = |m|_{\bar{T}_\beta},\ a_2 = |k|_{\bar{T}_\beta}$.

**Definition 2** *We say that the equation (1) belongs to the class $D(a_1, a_2, r)$ if there exists $r > 0$, such that the conditions (H3) and (H4) are fulfilled.*

**Theorem 3** *Let the linear part of (1) is $(M, N, R)$ dichotomous with $R(s)$ $(s \in J)$ be linear and the conditions (H1) and (H2) have the form (5).*

*Then there exist numbers $\bar{a}_1, \bar{a}_2 > 0$ and $\rho < r$ with following property:*

*If the initial value $\xi$ fulfilled $|\xi| \le \rho$ and if the equation (1) belongs to the class $D(a_1, a_2, r)$ for $a_1 \in (0, \bar{a}_1),\ a_2 \in (0, \bar{a}_2)$ then there exists an unique solution $x(t)$ in the ball $|\, x\,|_{C_{\alpha^{-1}}} \le r$, i.e.*

$$\sup_{t \in J}\ \alpha^{-1}(t)|x(t)| \le r$$

**Proof:** First we shall prove, that the operator $Q$, defined by the formula

$$(Qx)(t) = V(t)\xi + \int_c^t V(t)R(s)V^{-1}(s)F(s,x(s))\mathrm{d}s -$$

$$-\int_t^\infty V(t)(I - R(s))V^{-1}(s)F(s,x(s))\mathrm{d}s$$

maps the ball $|\, x\,|_{C_{\alpha^{-1}}} \le r$ into itself. Indeed we have

$$|(Qx)(t)| \le \varphi_1(t)\varphi_2(c)|\xi| + \psi_1(t)\psi_2(c)|\xi| + \int_c^t \varphi_1(t)\varphi_2(s)m(s)\mathrm{d}s + \int_t^\infty \psi_1(t)\psi_2(s)m(s)\mathrm{d}s$$

$$|(Qx)(t)| \le \alpha(t)(\varphi_2(c) + \psi_2(c))|\xi| + \alpha(t)\int_c^\infty \beta(s)m(s)\mathrm{d}s$$

Hence

$$\alpha^{-1}(t)|(Qx)(t)| \leq (\varphi_2(c) + \psi_2(c))\rho + a_1$$

For sufficiently small $\rho$ and $a_1$, $Q$ will map the ball $| \, x \, |_{C_{\alpha^{-1}}} \leq r$ into itself.

Now we shall prove, that the operator $Q$ is a contraction in the ball $| \, x \, |_{C_{\alpha^{-1}}} \leq r$

Indeed, we have

$$\begin{aligned}|(Qx_1)(t)-(Qx_2)(t)| \leq & \int_c^t |V(t)R(s)V^{-1}(s)(F(s,x_1(s)) - F(s,x_2(s)))|\mathrm{d}s+ \\ & + \int_t^\infty |V(t)(I-R(s))V^{-1}(s)(F(s,x_1(s)) - F(s,x_2(s)))|\mathrm{d}s \leq \\ & \leq \int_c^t \varphi_1(t)\varphi_2(s)|F(s,x_1(s)) - F(s,x_2(s))|\mathrm{d}s+ \\ & + \int_t^\infty \psi_1(t)\psi_2(s)|F(s,x_1(s)) - F(s,x_2(s))|\mathrm{d}s \leq \\ & \leq \alpha(t) \int_c^\infty \beta(s)\alpha^{-1}(s)k(s)|x_1(s) - x_2(s)|\mathrm{d}s\end{aligned}$$

We obtain

$$\alpha^{-1}(t)|(Qx_1)(t) - (Qx_2)(t)| \leq \sup_{t\in J} \alpha^{-1}(t)|x_1(t) - x_2(t)| \int_c^\infty \beta(s)k(s)\mathrm{d}s$$

$$|Qx_1 - Qx_2|_{C_{\alpha^{-1}}} \leq |x_1 - x_2|_{C_{\alpha^{-1}}} |k|_{T_{\bar{\beta}}} = |x_1 - x_2|_{C_{\alpha^{-1}}} a_2$$

Hence for sufficiently small $a_2$, the operator $Q$ is a contraction in the ball $| \, x \, |_{C_{\alpha^{-1}}} \leq r$.

The assertion of the theorem follows from the theorem of Banach - Cacciopolli [4]. □

Other sufficient conditions for existence of solution of the equation (1) we will find, using the fixed point principle of Schauder-Tychonoff. In connection with its applying, we will use a generalization of the Arzella-Ascoli's theorem for locally convex spaces.

Let $S(J,X)$ is the linear set of all functions, acting from $J$ in $X$, which are continuous. The set $S(J,X)$ is a locally convex space w.r.t. the metric

$$\rho(u,v) = \sup_{c<T<\infty} (1+T)^{-1} \frac{\max_{c\leq t\leq T} \|u(t) - v(t)\|}{1 + \max_{c\leq t\leq T} \|u(t) - v(t)\|}.$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli's theorem is valid.

**Lemma 2** *The set $H \subset S(J,X)$is relatively compact if the intersections $H(t) = \{h(t) : h \in H\}$ are relatively compact subsets of $X$ for every $t \in J$ and $H$ is equicontinuous on each finite closed interval.*

**Proof:** We apply Arzella-Ascoli's theorem to each finite and closed interval. □

Let $C$ is an unempty subset of $X$ and let

$$\tilde{C} = \{u \in S(J,X) : u(t) \in C, t \in J\}$$

**Lemma 3** *Let $C$ is an unempty, convex and closed subset of $X$ and the operator $F$ maps $\tilde{C}$ into itself and is continuous. Let $F(\tilde{C})$ is relatively compact subset of $\tilde{C}$.*

*Then $F$ has a fixed point in $\tilde{C}$.*

**Proof:** It follows from the fixed point principle of Schauder-Tychonoff [4]. □

Let

$$C(r) = \{x \in S(J,X) : |x|_{C_{\alpha^{-1}}} \leq r\}$$

Obviously $C(r)$ is unempty, convex and closed.

**Theorem 4** *Let the following conditions are fulfilled:*

1. *Let the linear part of (1) is $(M,N,R)$ dichotomous and the conditions (H1) and (H2) have the form (5).*
2. *There exists a number $r > 0$ such that*
$$\sup_{|u|\leq r} |F(t,u)| = m(t), \textit{where } \ m \in \bar{T}_\beta.$$
3. *The function $F(t,u)$ is continuous $(t \in J, |u| \leq r)$.*
4. *The set $K(r) = \{m^{-1}(t)F(t,x) : t \in J, |u| \leq r\}$ is relatively compact.*
5. *$R(t)u$ is continuous for every $u \in X$ by any fixed $t \in J$.*

*Then for sufficient small $|m|_{\bar{T}_\beta}$ and initial value $|\xi| \leq r$ the nonlinear equation (1) has a solution $x \in C(r)$.*

**Proof:** We consider the operator $Q$ defined by the formula

$$(Qx)(t) = V(t)\xi + \int_c^t V(t)R(s)V^{-1}(s)F(s,x(s))\mathrm{d}s -$$

$$- \int_t^\infty V(t)(I - R(s))V^{-1}(s)F(s,x(s))\mathrm{d}s,$$

where $(|\xi| \leq r)$. First we shall prove, that $Q$ maps $C(r)$ into itself. Let $x \in C(r)$. Then

$$|(Qx)(t)| \leq \varphi_1(t)\varphi_2(c)|\xi| + \psi_1(t)\psi_2(c)|\xi| + \int_c^t \varphi_1(t)\varphi_2(s)m(s)\mathrm{d}s + \int_t^\infty \psi_1(t)\psi_2(s)m(s)\mathrm{d}s$$

$$|(Qx)(t)| \leq \alpha(t)(\varphi_2(c) + \psi_2(c))|\xi| + \alpha(t)\int_c^\infty \beta(s)m(s)\mathrm{d}s$$

Hence

$$\alpha^{-1}(t)|(Qx)(t)| \leq (\varphi_2(c) + \psi_2(c))\rho + a_1$$

For sufficiently small $|\xi|$ and $|m|_{\bar{T}_\beta}$ we obtain $\alpha^{-1}(t)|(Qx)(t)| \leq r \quad (t \in J)$, i.e. $Q$ maps $C(r)$ into itself.

Now we shall prove that the set $QC(r)$ is relatively compact in $S(J, X)$. For this aim we shall show, that the functions of $QC(r)$ are equicontinuous on each finite closed interval $[a, b]$.

Let $a$ and $b$ are fixed and $t', t'' \in [a, b],\ t' < t''$. Then for $x \in C(r)$ we have

$$|(Qx)(t') - (Qx)(t'')| \leq I_1 + I_2 + I_3$$

where

$$I_1 = |V(t')\xi - V(t'')\xi|$$

$$I_2 = |\int_c^{t'} V(t')R(s)V^{-1}(s)F(s, x(s))\mathrm{d}s - \int_c^{t'} V(t'')R(s)V^{-1}(s)F(s, x(s))\mathrm{d}s -$$
$$- \int_{t'}^{t''} V(t'')R(s)V^{-1}(s)F(s, x(s))\mathrm{d}s\ |$$

$$I_3 = |\int_{t'}^{\infty} V(t')(I - R(s))V^{-1}(s)F(s, x(s))\mathrm{d}s -$$
$$- \int_{t''}^{\infty} V(t'')(I - R(s))V^{-1}(s)F(s, x(s))\mathrm{d}s\ |$$

For $t'' \to t'$ we have $I_1, I_2 \to 0$, because V(t) is continuous in respect to $t$. For $I_3$ we obtain the estimate

$$\begin{aligned} I_3 \leq & \int\limits_{t'}^{\infty} |V(t')(I - R(s))V^{-1}(s)F(s, x(s)) - \\ & - V(t'')(I - R(s))V^{-1}(s)F(s, x(s))|\mathrm{d}s + \\ & + \int\limits_{t'}^{t''} |V(t'')(I - R(s))V^{-1}(s)F(s, x(s))|\mathrm{d}s \end{aligned} \tag{8}$$

For $t'' \to t'$ the second integral in (8) converges to zero. We will use the Lebesgue's theorem to prove, that the first integral in (8) by $t'' \to t'$ converges to zero too. Because V(t) is continuous in respect to $t$ we have

$$|V(t')(I - R(s))V^{-1}(s)F(s, x(s)) - V(t'')(I - R(s))V^{-1}(s)F(s, x(s))| \underset{t'' \to t'}{\longrightarrow} 0$$

From the estimates

$$\begin{aligned}
&\int_c^\infty |V(t')(I-R(s))V^{-1}(s)F(s,x(s))|\mathrm{d}s+\\
&+\int_c^\infty |V(t'')(I-R(s))V^{-1}(s)F(s,x(s))|\mathrm{d}s\le\\
&\le\int_c^\infty \varphi_1(t')\varphi_2(s)|F(s,x(s))|\mathrm{d}s+\int_c^\infty \psi_1(t'')\psi_2(s)|F(s,x(s))|\mathrm{d}s\le\\
&\le\int_c^\infty \alpha(t')\beta(s)m(s)\mathrm{d}s+\int_c^\infty \alpha(t'')\beta(s)m(s)\mathrm{d}s\le\\
&\le(\alpha(t')+\alpha(t''))|m|_{\bar{T}_\beta}
\end{aligned}$$

and from the Lebesgue's theorem follows, that the first integral in (8) converges to zero.

Let $t\in[a,b]$ be fixed. We shall show, that the set $(Qx)(t)\quad(x\in C(r))$ is relatively compact in $S(J,X)$.

Let $\epsilon>0$ be an arbitrary number. If the numbers $T$ and $N$ are large enough, we obtain the inequality

$$|\int_c^\infty W(t,s)F(s,x(s))\mathrm{d}s-\int_c^T W(t,s)F_N(s,x(s))\mathrm{d}s|<\epsilon$$

where

$$W(t,s)=\begin{cases}V(t)R(s)V^{-1}(s) & t\ge s\\ -V(t)(I-R(s))V^{-1}(s) & t<s\end{cases}$$

and

$$F_N(t,u)=\begin{cases}F(t,u) & m(t)\le N\\ 0 & m(t)>N\end{cases}$$

From condition 4 of the Theorem follows, that for $F(s,x(s))\in NK$ we have the inclusion

$$\int_c^T W(t,s)F(s,x(s))\mathrm{d}s\in TN\bigcup_{c\le s\le T}W(t,s)K \tag{9}$$

The set in the right hand of (9) is compact. Hence the set

$$\{\int_c^T W(t,s)F(s,x(s))\mathrm{d}s\ :\ x\in C(r)\}$$

is compact too. From the theorem of Hausdorff follows the compactness of the set

$$\{\int_c^\infty W(t,s)F(s,x(s))\mathrm{d}s\ :\ x\in C(r)\}$$

Hence the set $QC(r)$ is relatively compact in $S(J,X)$.

Now we shall prove that the operator $Q$ is continuous.

Let $\{z_n(t)\} \subset C(r)$ is an arbitrary sequence which converges to $z(t)$ in $S(J,X)$ and let $t \in J$ is fixed. Then

$$\begin{aligned} |(Qz)(t) - (Qz_n)(t)| \leq \int_c^t & |V(t)R(s)V^{-1}(s)F(s,z(s)) - \\ & - V(t)R(s)V^{-1}(s)F(s,z_n(s))|\mathrm{d}s + \\ & + \int_t^\infty |V(t)(I-R(s))V^{-1}(s)F(s,z(s)) - \\ & - V(t)(I-R(s))V^{-1}(s)F(s,z_n(s))|\mathrm{d}s \end{aligned} \tag{10}$$

Because $F$ and $V(t)R(s)V^{-1}(s)$ are continuous, the first integral in (10) converges to zero, by $n \to \infty$.

Let

$$J_1(s) = |V(t)(I-R(s))V^{-1}(s)F(s,z(s)) - V(t)(I-R(s))V^{-1}(s)F(s,z_n(s))|$$

Because $V(t)(I-R(s))V^{-1}(s)$ is continuous, so we have

$$J_1(s) \underset{n\to\infty}{\longrightarrow} 0 \ \text{ for any } s \geq t.$$

From the estimate

$$\int_c^\infty J_1(s)\mathrm{d}s \leq \int_c^\infty \psi_1(t)\psi_2(s)m(s)\mathrm{d}s \leq \alpha(t)|m|_{\bar{T}_\beta}$$

and the Lebesgue's theorem follows, that the second integral in (10) converges to zero for $n \to \infty$. Because $QC(r)$ is compact it follows, that

$$Qz_n \underset{n\to\infty}{\longrightarrow} Qz \ \text{ in } S(J,X).$$

From the Schauder-Tychonoff theorem [4] it follows the existence of a fixpoint $x$ of the operator $Q$ in the set $C(r)$. □

**Remark 2** By $dimX < \infty$ the condition 4 of Theorem 4 is not necessary.

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# Improved nearness research II

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ABSTRACT. When applying in consequence the new created concept ”Bounded Topology” [8] hence ”classical structures” like nearness structures [5], convergence structures [8] and syntopogenous structures [8] will be analyzed in connexion with neighbourhood structures [11] or supertopologies [4], respectively. In this context ”nearness” is presented as special paranearness, ”convergence” as special $b$-convergence and being ”syntopogenous” as special case of $b$-syntopogenous, leading us accordingly to a general theory of his own! Now, in this paper we will study certain superclan spaces, whichever are in one-to-one correspondence to strict topological extensions. Here, we should mention that the presented concept is not of utmost generality, but then the reader is referred to [9].

KEY WORDS AND PHRASES. LEADER proximity; supertopological space; LODATO space; supernear space; superclan space; Bounded Topology

## 1 Basic concepts

As usual $\underline{P}X$ denotes the power set of a set $X$, and we use $\mathcal{B}^X \subset \underline{P}X$ to denote a collection of bounded subsets of $X$, also known as $\underline{B}$-sets, e.g. $\mathcal{B}^X$ has the following properties:

(b$_1$) $\varnothing \in \mathcal{B}^X$;

(b$_2$) $B_2 \subset B_1 \in \mathcal{B}^X$ imply $B_2 \in \mathcal{B}^X$;

(b$_3$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$.

Then, for $\underline{B}$-sets $\mathcal{B}^X, \mathcal{B}^Y$ a function $f : X \longrightarrow Y$ is called bounded iff $f$ satisfies (b), e.g.

(b) $\{f[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^Y$.

**Definition 1.1** *For a set $X$, we call a tripel $(X, \mathcal{B}^X, N)$ consisting of $X$, $\underline{B}$-set $\mathcal{B}^X$ and a near-operator $N : \mathcal{B}^X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ a supernearness space (shortly supernear space) iff the following axioms are satisfied, e.g.*

(sn$_1$) *$B \in \mathcal{B}^X$ and $\rho_2 << \rho_1 \in N(B)$ imply $\rho_2 \in N(B)$, where $\rho_2 << \rho_1$ iff $\forall F_2 \in \rho_2 \exists F_1 \in \rho_1$ $F_2 \supset F_1$;*

(sn$_2$) *$B \in \mathcal{B}^X$ implies $\mathcal{B}^X \notin N(B) \neq \varnothing$;*

(sn$_3$) *$\rho \in N(\varnothing)$ implies $\rho = \varnothing$;*

(sn$_4$) *$x \in X$ implies $\{\{x\}\} \in N(\{x\})$;*

(sn$_5$) *$B_1 \subset B_2 \in \mathcal{B}^X$ imply $N(B_1) \subset N(B_2)$;*

(sn$_6$) *$B \in \mathcal{B}^X$ and $\rho_1 \vee \rho_2 \in N(B)$ imply $\rho_1 \in N(B)$ or $\rho_2 \in N(B)$, where $\rho_1 \vee \rho_2 := \{F_1 \cup F_2 : F_1 \in \rho_1, F_2 \in \rho_2\}$;*

(sn$_7$) *$B \in \mathcal{B}^X, \rho \subset \underline{P}X$ and $\{cl_N(F) : F \in \rho\} \in N(B)$ imply $\rho \in N(B)$, where $cl_N(F) := \{x \in X : \{F\} \in N(\{x\})\}$.*

*If $\rho \in N(B)$ for some $B \in \mathcal{B}^X$, then we call $\rho$ a B-near collection in $N$. For supernear spaces $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ a bounded function $f : X \longrightarrow Y$ is called sn-map iff it satisfies* (sn), *e.g.*

(sn) *$B \in \mathcal{B}^X$ and $\rho \in N(B)$ imply $\{f[F] : F \in \rho\} =: f\rho \in M(f[B])$.*

*We denote by SN the corresponding category.*

**Example 1.2** (i) For a nearness space $(X, \xi)$ let $\mathcal{B}^X$ be $\underline{B}$-set. Then we consider the tripel $(X, \mathcal{B}^X, N_\xi)$, where

$N_\xi(\varnothing) := \{\varnothing\}$ and

$N_\xi(\varnothing) := \{\rho \subset \underline{P}X : \{B\} \cup \rho \in \xi\}$, otherwise.

(ii) For a topological space $(X, t)$ given by closure operator $t$ let $\mathcal{B}^X$ be $\underline{B}$-set. Then we consider the tripel $(X, \mathcal{B}^X, N_t)$, where $N_t(\varnothing) := \{\varnothing\}$ and $N_t(B) := \{\rho \subset \underline{P}X : \exists x \in Bx \in \bigcap\{t(F) : F \in \rho\}\}$, otherwise.

(iii) For a LODATO space $(X, \mathcal{B}^X, \delta)$ with $\delta \subset \mathcal{B}^X \times \underline{P}X$ we consider the tripel $(X, \mathcal{B}^X, N_\delta)$, where $N_\delta(\varnothing) := \{\varnothing\}$ and $N_\delta(B) := \{\rho \subset \underline{P}X : \rho \subset \delta(B)$ and $\{B\} \cup \rho \subset \cap\{\delta(F) : F \in \rho \cap \mathcal{B}^X\}\}$, otherwise, with $\delta(B) := \{A \subset X : B\delta A\}$. Hereby, following conditions must be satisfied:

(bp$_0$) $B \in \mathcal{B}^X$ implies $cl_\delta(B) \in \mathcal{B}^X$, where $cl_\delta(B) := \{x \in X : \{x\}\delta B\}$;

(bp$_1$) $\varnothing\overline{\delta}A$ and $B\overline{\delta}\varnothing$ (e.g. $\varnothing$ is <u>not</u> in relation to $A$, and analogously this is also holding for $B$;

(bp$_2$) $B\delta(A_1 \cup A_2)$ iff $B\delta A_1$ or $B\delta A_2$;

(bp$_3$) $x \in X$ implies $\{x\}\delta\{x\}$;

(bp$_4$) $B_1 \subset B_2 \in \mathcal{B}^X$ and $B_1\delta A$ imply $B_2\delta A$;

(bp$_5$) $B \in \mathcal{B}^X$ and $B\delta A$ with $A \subset cl_\delta(C)$ imply $B\delta C$;

(bp$_6$) $B_1 \cup B_2 \in \mathcal{B}^X$ and $(B_1 \cup B_2)\delta A$ imply $B_1\delta A$ or $B_2\delta A$;

(bp$_7$) $A, B \subset X, cl_\delta(B) \in \mathcal{B}^X$ and $cl_\delta(B)\delta A$ imply $B\delta A$;

(bp$_8$) $B_1, B_2 \in \mathcal{B}^X$ and $B_1\delta B_2$ imply $B_2\delta B_1$.

(iv) For a preLEADER space $(X, \mathcal{B}^X, \delta)$ with $\delta \subset \mathcal{B}^X \times \underline{P}X$ only satisfies (bp$_1$) to (bp$_5$) we consider the tripel $(X, \mathcal{B}^X, N^\delta)$, where $N^\delta(B) := \{\rho \subset \underline{P}X : \rho \subset \delta(B)\}$ for each $B \in \mathcal{B}^X$.

**Definition 1.3** *For preLEADER spaces $(X, \mathcal{B}^X, \delta), (Y, \mathcal{B}^Y, \gamma)$ a bounded function $f : X \longrightarrow Y$ is called <u>p-map</u> iff $f$ satisfied* (p), *e.g.*

(p) *$B \in \mathcal{B}^X, A \subset X$ and $B\delta A$ imply $f[B]\gamma f[A]$. By LOSP respectively pLESP we denote the corresponding categories.*

**Definition 1.4** *TEXT denotes the category, whose objects are triples $E := (e, \mathcal{B}^X, Y)$ - called <u>topological extensions</u> - where $X := (X, cl_X), Y := (Y, cl_Y)$ are topological spaces (given by closure operators) with $\underline{B}$-set $\mathcal{B}^X$, and $e : X \longrightarrow Y$ is a function satisfying the following conditions:*

(tx$_1$) *$A \in \underline{P}X$ implies $cl_X(A) = e^{-1}[cl_Y(e[A])]$, where $e^{-1}$ denotes the <u>inverse</u> image under $e$;*

(tx$_2$) *$cl_Y(e[X]) = Y$, which means the image of $X$ under $e$ is <u>dense</u> in $Y$. Morphisms in TEXT have the form $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$, where $f : X \longrightarrow X', g : Y \longrightarrow Y'$ are <u>continuous maps</u> such that $f$ is <u>bounded</u>, and the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[e']{} & Y' \end{array} .$$

*If* $(f,g):(e,\mathcal{B}^X,Y)\longrightarrow(e',\mathcal{B}^{X'},Y')$ *and* $(f',g'):(e',\mathcal{B}^{X'},Y')\longrightarrow(e'',\mathcal{B}^{X''},Y'')$*, are TEXT-morphisms, then they can be* *composed* *according to the rule:*

$$(f',g')\circ(f,g):=(f'\circ f,g'\circ g):(e,\mathcal{B}^X,Y)\longrightarrow(e'',\mathcal{B}^{X''},Y''),$$

*where "o" denotes the composition of maps.*

**Remark 1.5** Observe, that axiom ($\mathrm{tx}_1$) in this definition is automatically satisfied if $e: X\longrightarrow Y$ is a topological embedding. Moreover, we only admit an ordinary $\underline{B}$-set $\mathcal{B}^X$ on $X$ which need not be necessary coincide with the power $\underline{P}X$. In addition we mention that such an extension is called strict iff it satisfies ($\mathrm{tx}_3$), e.g.

($\mathrm{tx}_3$) $\{cl_Y(e[A]):A\subset X\}$ forms a base for the closed subsets of $Y$ [1].

By STREXT we denote the corresponding full subcategory of TEXT.

(v) For a topological extension $E:=(e,\mathcal{B}^X,Y)$ we consider the tripel $(X,\mathcal{B}^X,N_e)$, where

$N_e(\varnothing):=\{\varnothing\}$ and

$N_e(B):=\{\rho\subset\underline{P}X: y\in\cap\{cl_Y(e[F]):F\in\rho\}$ for some $y\in e[B]\}$, otherwise.

## 2 Some important isomorphisms

With respect to above examples, first let us focus our attention to some special classes of supernear spaces.

**Definition 2.1** *A supernear space* $(X,\mathcal{B}^X,N)$ *is called* *saturated* *iff* $\mathcal{B}^X$ *is, e.g.*

(s) $X\in\mathcal{B}^X$.

**Remark 2.2** Note, that in above case $\mathcal{B}^X$ coincide with the power $\underline{P}X$. (Also compare with examples (i) or (ii), respectively). Moreover, we claim that the full subcategory $\mathrm{SN}^S$ of SN, whose objects are the saturated supernear spaces is bireflective in SN. Concretely, for a supernear space $(X,\mathcal{B}^X,N)$ we put: $N^S(B):=N(B)$ for each $B\in\mathcal{B}^X$ and $N^S(B):=\{\rho\subset\underline{P}X:\exists x\in X\exists B^*\in\mathcal{B}^X(x\in B\supset B^*$ and $\rho\in N(\{x\})\cup N(B^*))\}$ for each $B\in\underline{P}X\setminus\mathcal{B}^X$, hence $(X,\underline{P}X,N^S)$ is saturated supernear space and $1_X:(X,\mathcal{B}^X,N)\longrightarrow(X,\underline{P}X,N^S)$ to be the bireflection in demand!

**Definition 2.3** *A supernear space* $(X,\mathcal{B}^X,N)$ *is called*

(i) *paranearness space* *(paranear space) iff it is symmetric, hence $N$ additionally satisfies* (sy)*, e.g.*

(sy) $B \in \mathcal{B}^X \backslash \{\varnothing\}$ *and* $\rho \in N(B)$ *imply* $\{B\} \cup \rho \in \cap\{N(A) : A \in (\rho \cap \mathcal{B}^X) \cup \{B\}\}$;

(ii) *pointed iff* $N$ *satisfies (pt), e.g.*

(pt) $B \in \mathcal{B}^X \backslash \{\varnothing\}$ *implies* $N(B) = \cup\{N(\{x\}) : x \in B\}$. *By PN respectively PT-SN we denote the corresponding full subcategory of SN.*

**Theorem 2.4** *The category NEAR of nearness spaces and nearness preserving maps is isomorphic to the full subcategory* $PN^S$ *of PN, whose objects are the saturated paranear spaces.*

**Proof:** According to example (i). Conversely, we consider for a saturated paranear space $(Y, \mathcal{B}^Y, M)$:

$$\mu_M := \{\mathcal{A} \subset \underline{P}X : \mathcal{A} \in \cap\{M(A) : A \in \mathcal{A}\}\}.$$

□

**Theorem 2.5** *The category TOP of topological spaces and continuous maps is isomorphic to the full subcategory PT-*$SN^S$ *of PT-SN, whose objects are the saturated pointed supernear spaces.*

**Proof:** According to example (ii) and by respecting ($sn_7$) in definition 1.1. □

**Definition 2.6** *Let be given a supernear space* $(X, \mathcal{B}^X, N)$. *For* $B \in \mathcal{B}^X \mathcal{C} \in GRL(X)$ *is called B-clan in N iff it satisfies*

($cla_1$) $B \in \mathcal{C} \in N(B)$;

($cla_2$) $A \in \mathcal{C}$ *and* $A \subset cl_N(F)$ *imply* $F \in \mathcal{C}$, *where* $GRL(X) := \{\gamma \subset \underline{P}X : \gamma \text{ is grill }\}$, *and* $\gamma \subset \underline{P}X$ *is called grill (Choquet [3]) iff*

($gri_1$) $\varnothing \notin \gamma$;

($gri_2$) $G_1 \cup G_2 \in \gamma$ *iff* $G_1 \in \gamma$ *or* $G_2 \in \gamma$.

*Then* $(X, \mathcal{B}^X, N)$ *is called superclan space iff N satisfies* (cla), *e.g.*

(cla) $B \in \mathcal{B}^X \backslash \{\varnothing\}$ *and* $\rho \in N(B)$ *imply the existence of B-clan* $\mathcal{C} \in GRL(X)\rho \subset \mathcal{C}$.

*Moreover, if* $(X, \mathcal{B}^X, N) \in \underline{P}N$ *satisfies* (cla), *we analogously call it paraclan space!*

**Remark 2.7** Here, we note that each pointed supernear space is always a superclan space by making use of the fact that for each $B \in \mathcal{B}^X$ with $x \in B$ $\{T \subset X : x \in cl_N(T)\} =: x_N$ is B-clan in $N$, and $x_N$ is maximal in $N(\{x\}) \setminus \{\varnothing\}$, ordered by inclusion!

**Theorem 2.8** *The category BUN of bunch-determined nearness spaces and related maps* [2] *is isomorphic to the full subcategory CLA-PN$^S$ of PN$^S$, whose objects are the saturated paraclan spaces.*

**Proof:** Compare with theorem 2.4. □

**Definition 2.9** *A paranear space $(X, \mathcal{B}^X, N)$ is called* <u>*round*</u> *iff it satisfies* (r), *e.g.*

(r) $B \in \mathcal{B}^X$ *implies* $cl_N(B) \in \mathcal{B}^X$.

**Theorem 2.10** *The full subcategory R-PN of PN, whose objects are the round paranear spaces is bireflective in PN.*

**Proof:** For a paranear space $(X, \mathcal{B}^X, N)$ we set:

$\mathcal{B}_N^X := \{\mathcal{D} \subset X : \exists B \in \mathcal{B}^X cl_N(B) \supset \mathcal{D}\}$ and

$N_r(\varnothing) := \{\varnothing\}$ respectively

$N_r(\mathcal{D}) := \{\rho \subset \underline{P}X : \exists B \in \mathcal{B}^X \{D\} \cup \rho \in N(B)\}$, otherwise.

Then the tripel $(X, \mathcal{B}^X, N_r)$ is a round paranear space and $1_X : (X, \mathcal{B}^X, N) \longrightarrow (X, \mathcal{B}^X, N_r)$ to be the bireflection in demand! □

**Corollary 2.11** *If $(X, \mathcal{B}^X, N)$ is paraclan space then $(X, \mathcal{B}_N^X, N_r)$ as well.*

**Definition 2.12** *A round paranear space $(X, \mathcal{B}^X, N)$ is called* <u>*LOproximal*</u> *iff it satisfies* (LOp), *e.g.*

(LOp) $B \in \mathcal{B}^X \backslash \{\varnothing\}, \rho \in p_N(B)$ *and* $\{B\} \cup \rho \subset \cap\{p_N(F) : F \in \rho \cap \mathcal{B}^X\}$ *imply* $\rho \in N(B)$, *where* $B_{P_N} A$ *iff* $\{A\} \in N(B)$.

**Theorem 2.13** *The category LOSP is isomorphic to the full subcategory LO-PN of R-PN, whose objects are the LOproximal paranear spaces.*

**Proof:** According to example (iii). Conversely, we consider the near-relation "$p_N$" as defined in 2.12. Moreover we note that for a paranear space $(X, \mathcal{B}^X, N)$ the near-operator $N$ is <u>dense</u>, e.g. by satisfying $(d) B \subset X$ and $cl_N(B) \in \mathcal{B}^X$ imply $N(cl_N(B)) = N(B)$, and moreover it is <u>connected</u>, e.g. by satisfying

(cnc) $B_1 \cup B_2 \in \mathcal{B}^X$ implies $N(B_1 \cup B_2) = N(B_1) \cup N(B_2)$.

□

**Remark 2.14** Now, we mention that in the "saturated case" LOproximal paranear spaces and LODATO proximity spaces [10] essentially are the same!

**Proposition 2.15** *Let $(Y, t)$ be a symmetric topological space given by closure operator $t$ and $\mathcal{B}^X$ $\underline{B}$-set with $X \subset Y$. We set $\mathcal{B}^X_t := \{D \subset X : \exists B \in \mathcal{B}^X t(B) \supset D\}$ and $D\delta_t A$ iff $t(D) \cap t(A) \neq \emptyset$. Then $(X, \mathcal{B}^X_t, \delta_t)$ is LODATO space.*

**Remark 2.16** Now, surely it seems to be of interest to characterize those LODATO spaces whichever are induced by a topologival space $Y$ as above so that bounded and arbitrary sets are near iff their closures meet in $Y$. But this problem already has been solved under more general conditions in [9].

**Remark 2.17** Returning to nearness spaces we already know that in general subspaces of topological nearness spaces need not to be topological again, hence Bentley [2] has called them subtopological. But now here, we will give an extended description of this definition in term of supernear spaces as follows:

**Definition 2.18** *A supernear space $(X, \mathcal{B}^X, N)$ is called supergrill space if $N$ satisfies* (gri)*, e.g.*

(gri) $B \in \mathcal{B}^X$ *and* $\rho \in N(B)$ *imply the existence of* $\gamma \in GRL(X) \cap N(B)$ *with* $\rho \subset \gamma$.

**Remark 2.19** We point out that this definition generalize that of 2.6. Moreover, if $(X, \mathcal{B}^X, N) \in$ PN satisfies (gri), we analogously call it a paragrill space. By G-SN respectively G-PN we denote the corresponding full subcategory of SN respectively PN.

**Proposition 2.20** *For a nearness space $(X, \xi)$ the following statements are equivalent:*

(i) $(X, \xi)$ *is subtopological;*

(ii) $(X, \underline{P}X, N_\xi)$ *is paragrill space.*

**Remark 2.21** According to example (iv) we also note that $(X, \mathcal{B}^X, N^\delta)$ is a supergrill space.

**Definition 2.22** *A supergrill space $(X, \mathcal{B}^X, N)$ then is called conic iff $N$ satisfies* (c), *e.g.*

(c) $B \in \mathcal{B}^X$ *implies* $\{F \subset X : \exists \rho \in N(B) F \in \rho\} =: \cup N(B) \in N(B)$.

**Theorem 2.23** *The category pLESP is isomorphic to the full subcategory CG-SN of G-SN, whose objects are the conic supergrill spaces.*

**Proof:** According to example (iv) in connexion with the definition of "$p_N$" in 2.12. □

**Definition 2.24** *A preLEADER space* $(X, \mathcal{B}^X, \delta)$ *then is called* <u>*LEADERspace*</u> *iff* $\delta$ *in addition satisfies* (bp$_6$) *in* (iii).

**Remark 2.25** We point out that in the "saturated" case LEADER spaces and LEADER proximity spaces [6] essentially are the same. Moreover, each supertopological space [4] $(X, \mathcal{B}^X, \Theta)$, where $\Theta : \mathcal{B}^X \longrightarrow \mathrm{FIL}(X) := \{\mathcal{F} \subset \underline{P}X : \mathcal{F} \text{ is filter}\}$ satisfies the following conditions, e.g.

(stop$_1$) $\Theta(\varnothing) = \underline{P}X$;

(stop$_2$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply $U \supset B$;

(stop$_3$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply there exists a set $V \in \Theta(B)$ such that always $U \in \Theta(B') \forall B' \in \mathcal{B}^X B' \subset V$ is leading us to the preLEADER space $(X, \mathcal{B}^X, \delta_\Theta)$ by setting $B \delta_\Theta A$ iff $A \in \sec\Theta(B)$. If in addition $(X, \mathcal{B}^X, \Theta) \in$ ASTOP [11], then $(X, \mathcal{B}^X, \delta_\Theta)$ is LEADER space, too. The above assignment now is "bi-functoriell", hence STOP can be considered as a subcategory of CG-SN. In the second case we note that the corresponding supergrill operator $N^{\delta_\Theta}$ is in addition linked, hence it satisfies (l), e.g.

(l) $B_1 \cup B_2 \in \mathcal{B}^X$ and $\rho \in N^{\delta_\Theta}(B_1 \cup B_2)$ imply $\{F\} \in N^{\delta_\Theta}(B_1) \cup N^{\delta_\Theta}(B_2)$ for each $F \in \rho$.

**Definition 2.26** *A conic supergrill space* $(X, \mathcal{B}^X, N)$ *then is called* <u>*LEproximal*</u> *iff* $N$ *is linked. By LE-SN we denote the full subcategory of SN.*

**Theorem 2.27** *The category LE-SN is isomorphic to the full subcategory LESP of pLESP, whose objects are the LEADER spaces.*

**Remark 2.28** According to 2.25 we also note that ASTOP now can be considered as subcategory of LE-SN.

**Proposition 2.29** *Let* $(Y, t)$ *be a topological space given by closure operator* $t$ *and* $\mathcal{B}^X$ $\underline{B}$*-set with* $X \subset Y$. *We set* $B\delta^t A$ *iff* $B \cap t(A) \neq \varnothing$ *for each* $B \in \mathcal{B}^X$ *and* $A \subset X$. *Then* $(X, \mathcal{B}^X, \delta^t)$ *is LEADER space*

**Proof:** straightforward. □

**Remark 2.30** According to 2.16 now it seems to be of interest to characterize those LEADER spaces, whichever are included by a topological space $Y$ as above so that a bounded set $B$ is near to an arbitrary one iff $B$ intersects its closure in $Y$. But we will solve this problem under more <u>general</u> conditions in a <u>forthcoming</u> paper!

**Remark 2.31** Returning to conic supergrill spaces we point out that for such a space $(X, \mathcal{B}^X, N)$ and for each $B \in \mathcal{B}^X \backslash \{\emptyset\} \cup N(B)$ is a B-clan in $N$. hence, we claim that conic supergrill spaces even are superclan spaces!

**Theorem 2.32** *The category CG-SN is bicoreflective in G-SN.*

**Proof:** For a supergrill space $(X, \mathcal{B}^X, N)$ we set for each $B \in \mathcal{B}^X$:

$$N_C(B) := \{\rho \subset \underline{P}X : \{cl_N(F) : F \in \rho\} \subset \cup N(B)\}.$$

Then $(X, \mathcal{B}^X, N_c)$ is a conic supergrill space and $1_X : (X, \mathcal{B}^X, N_c) \longrightarrow (X, \mathcal{B}^X, N)$ to be the bicoreflection in demand. First, we only show that $N_C$ satisfies (sn$_7$): Let be $\{cl_{N_c}(A) : A \in \mathcal{A}\} \in N_c(B)$ for $B \in \mathcal{B}^X$, we have to verify $cl_N(A) \in \cup N(B)$ for each $A \in \mathcal{A}$.

$A \in \mathcal{A}$ implies $cl_N(cl_{N_c}(A)) \in \cup N(B)$ by hypothesis. We claim now that the statement $cl_{N_c}(A) \subset cl_N(A)$ is valid. $x \in cl_{N_c}(A)$ implies $\{A\} \in N_c(\{x\})$, hence $cl_N(A) \in \cup N_c(\{x\})$. We can find $\rho \in N(\{x\})$ such that $cl_N(A) \in \rho$. Consequently $\{cl_N(A)\} \in N(\{x\})$ follows, which shows $\{A\} \in N(\{x\})$, hence $x \in cl_N(A)$ results.

Altogether we get $cl_N(A) \supset cl_N(cl_N(A)) \supset cl_N(cl_{N_c}(A))$ implying $cl_N(A) \in \cup N(B)$, since $\cup N(B) \in \mathrm{GRL}(X)$. Secondly, we prove $\cup N_c(B) \in \mathrm{GRL}(X)$ for each $B \in \mathcal{B}^X$. Let be given $B \in \mathcal{B}^X$, evidently $\emptyset \notin \cup N_c(B)$. Now, if $F_1 \in \cup N_c(B)$ and $F_1 \subset F_2 \subset X$, then there exists $\rho_1 \in N_c(B) F_1 \in \rho_1$. Consequently $\{cl_N(A) : A \in \rho_1\} \subset \cup N(B)$ follows by definition. We put $\rho_2 := \{F_2\}$, hence $\rho_2 \in N_C(B)$, because $\{cl_N(F) : F \in \rho_2\} = \{cl_N(F_2)\}$ and $cl_N(F_2) \supset cl_N(F_1) \in \cup N(B)$ implies $cl_N(F_2) \in \cup N(B)$. But $F_2 \in \{F_2\} = \rho_2$ immediately leading us to $F_2 \in \cup N_c(B)$. At last let be $F_1 \cup F_2 \in \cup N_c(B)$, hence there exists $\rho \in N_c(B) F_1 \cup F_2 \in \rho$ By definition $\{cl_N(F) : F \in \rho\} \subset \cup N(B)$ is valid showing that $cl_N(F_1) \cup cl_N(F_2) \supset cl_N(F_1 \cup F_2) \in \cup N(B)$. Consequently, $cl_N(F_1) \in \cup N(B)$ or $cl_N(F_2) \in \cup N(B)$ results, since $\cup N(B) \in \mathrm{GRL}(X)$. If $cl_N(F_1) \in \cup N(B)$ then we put $\rho_1 := \{F_1\}$,hence $F_1 \in \cup N_c(B)$ results.

Analogously, this also holds in the second case. Evidently, $1_X : (X, \mathcal{B}^X, N_c) \longrightarrow (X, \mathcal{B}^X, N)$ is sn-map. Now, let be given $(Y, \mathcal{B}^Y, M) \in \mathrm{CG\text{-}SN}$ and sn-map $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N)$, we have to prove $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N_c)$ is sn-map. For $B \in \mathcal{B}^Y$ and $\rho \in M(B)$ we must show $f\rho \in N_c(f[B])$, which means $\{cl_N(A) : A \in f\rho\} \subset \cup N(f[B])$. $A \in f\rho$ implies $A = f[F]$ for some $F \in \rho$. By supposition $f\rho \in N(f[B])$ follows, and $cl_N(A) = cl_N(f[F]) \supset f[cl_M(F)] \supset f[F] \in f\rho \in \cup N(f[B])$ is valid. Consequently, $cl_N(A) \in \cup N(f[B])$ results! $\square$

**Remark 2.33** As mentioned in 2.7 we already know, that pointed supernear spaces are superclan spaces as well. Moreover, in the next, we will show that PT-SN can be "nicely embedded" in SN as follows:

**Theorem 2.34** *PT-SN is bicoreflective subcategory of SN.*

**Proof:** For a supernear space $(X, \mathcal{B}^X, N)$ we set:

$N_P(\varnothing) := \{\varnothing\}$ and

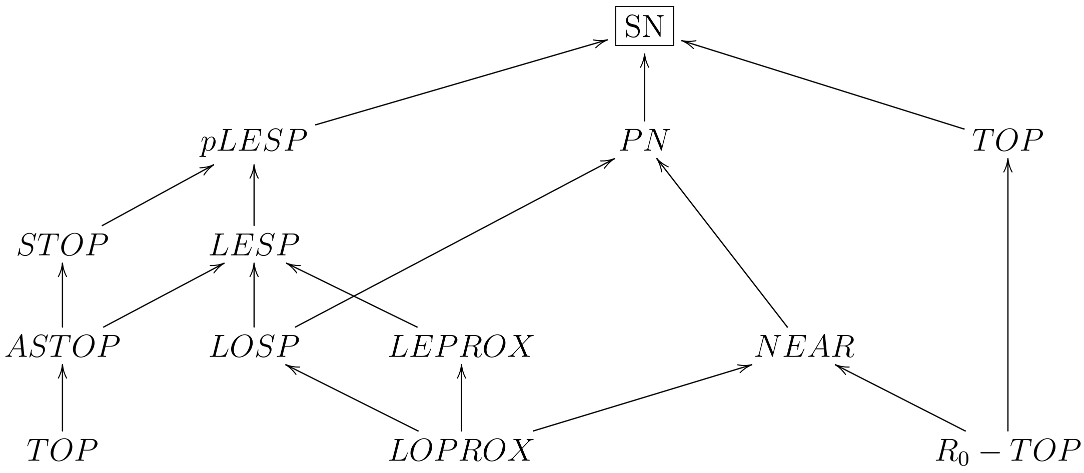
$N_P(B) := [\mathcal{A} \subset \underline{P}X : \exists x \in B \exists \gamma \in N(\{x\}) \cap \mathrm{GRL}(X) \{cl_N(A) : A \in \mathcal{A}\} \subset \gamma\}$, otherwise.

Then $(X, \mathcal{B}^X, N_P)$ is pointed supernear space and $1_X : (X, \mathcal{B}^X, N_P) \longrightarrow (X, \mathcal{B}^X, N)$ to be the bicoreflection in demand. First, we will show that $N_P$ satisfies $(\mathrm{sn}_7)$. Let be $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\{cl_{N_P}(A) : A \in \mathcal{A}\} \in N_P(B)$, then we can choose $x \in B$ and $\gamma \in N(\{x\}) \cap \mathrm{GRL}(X)$ such that $\{cl_N(F) : F \in \{cl_{N_P}(A) \in \mathcal{A}\}\} \subset \gamma$. In showing $\mathcal{A} \in N_P(B)$ we have to verify $cl_N(A) \in \gamma$ for each $A \in \mathcal{A}$ : $A \in \mathcal{A}$ implies $cl_N(cl_{N_P}(A)) \in \gamma$ by hypothesis. Now, we claim that $cl_{N_P}(A) \subset cl_N(A)$, because $x \in cl_{N_P}(A)$ implies $\{A\} \in N_P(\{x\})$, hence there exists $\gamma' \in N(\{x\}) \cap \mathrm{GRL}(X) \{cl_N(A)\} \subset \gamma'$. Then $\{cl_N(A)\} \in N(\{x\})$ is valid, and consequently $\{A\} \in N(\{x\})$ follows which shows $x \in cl_N(A)$. Altogether we have $cl_N(A) \supset cl_N(cl_N(A)) \supset cl_N(cl_{N_P}(A)) \in \gamma$, hence $cl_N(A) \in \gamma$ results! Evidently, $N_P$ fulfills the axioms $(\mathrm{sn}_1)$ to $(\mathrm{sn}_5)$.

to $(\mathrm{sn}_6)$: $\mathcal{A}_1 \vee \mathcal{A}_2 \in N_P(B)$ for $B \in \mathcal{B}^X \backslash \{\varnothing\}$ implies the existence of $x \in B$ and $\gamma \in N(\{x\}) \cap \mathrm{GRL}(X)$ so that $\{cl_N(A) : A \in \mathcal{A}_1 \vee \mathcal{A}_2\} \subset \gamma$. If supposing $\mathcal{A}_1, \mathcal{A}_2 \notin N_P(B)$ we get $\{cl_N(A_1) : A_1 \in \mathcal{A}_1\} \not\subset \gamma$ and $\{cl_N(A_2) : A_2 \in \mathcal{A}_2\} \not\subset \gamma$, hence there exist $A_1 \in \mathcal{A}_1 cl_N(A_1) \notin \gamma$ and $A_2 \in \mathcal{A}_2 cl_N(A_2) \notin \gamma$ implying $A_1 \cup A_2 \in \mathcal{A}$ and $cl_N(A_1) \cup cl_N(A_2) \notin \gamma$. Consequently $cl_N(A_1 \cup A_2) \notin \gamma$ follows, since $\gamma \in \mathrm{GRL}(X)$. On the other hand $cl_N(A_1 \cup A_2) \in \gamma$ by hypothesis is leading us to a contradiction! By definition $N_P$ is pointed and $1_X : (X, \mathcal{B}^X, N_P) \longrightarrow (X, \mathcal{B}^X, N))$ sn-map. Now, let be given a pointed supernear space $(Y, \mathcal{B}^Y, M)$ and sn-map $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N)$, we will show that $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N_P)$ is sn-map as well. Without restriction let be $B \in \mathcal{B}^Y \setminus \{\varnothing\}$ and $\mathcal{A} \in M(B)$, hence by hypothesis there exists $y \in B$ such that $\mathcal{A} \in M(\{y\})$. Since $f$ is sn-map $f\mathcal{A} \in N(\{f(y)\})$ follows with $f(y) \in f[B]$. But $f(y)_N \in N(\{f(y)\}) \cap \mathrm{GRL}(X)$, according to 2.7. Now, for $F \in f\mathcal{A}$ we will show that $cl_N(F) \in f(y)_N$. $F \in f\mathcal{A}$ implies $F = f[A]$ for some $A \in \mathcal{A}$. We claim $\{f[A]\} \in N(\{f(y)\})$. By hypothesis $f\mathcal{A} \in N(\{f(y)\})$, hence $\{f[A]\} << f\mathcal{A}$, which shows $\{f[A]\} \in N(\{f(y)\})$, and at last $f\mathcal{A} \in N_P(f[B])$ results.

□

**Remark 2.35** The following diagram illustrates the relationship between important former mentioned categories:




## 3 Topological extensions and related superclan spaces

Taking into account example (v), we will now consider the problem for finding a one-to-one corresponding between certain topological extensions and their related supernear spaces. It turns out that there exists an interesting one between pointed supernear spaces and some strict topological extensions.

**Lemma 3.1** *For a topological extension* $(e, \mathcal{B}^X, Y), (X, \mathcal{B}^X, N_e)$ *is a pointed supernear space such that* $cl_{N_e} = cl_X$.

**Proof:** First, we will show the equality of the closure operators. So, let $A \in \underline{P}X$ and $x \in cl_X(A)$. Then by $(\text{tx}_1)$ $e(x) \in cl_Y(e[A])$ hence $\{A\} \in N_e(\{x\})$, and $x \in cl_{N_e}(A)$ follows. Conversely, let $x \in cl_{N_e}(A)$, then $\{A\} \in N_e(\{x\})$. Consequently there exists $y \in e[\{x\}] = \{e(x)\}$ with $y \in cl_Y(e[A])$. Hence $y = e(x)$, and as a consequence of $(\text{tx}_1)$ we get $x \in e^{-1}[cl_Y(e[A])] \subset cl_X(A)$, which was to be proven. Secondly, it is easy to check the axioms $(\text{sn}_1)$ to $(\text{sn}_6)$.

to $(\text{sn}_7)$: Let be $\{cl_{N_e}(F) : F \in \rho\} \in N_e(B)$ for $\rho \subset \underline{P}X, B \in \mathcal{B}^X$ and without restriction $B \neq \varnothing$, then there exists $y \in e[B]$ with $y \in \cap\{cl_Y(e[A]) : A \in \{cl_{N_e}(F) : F \in \rho\}\}$. For $F \in \rho$ we get $y \in cl_Y(e[cl_{N_e}[F]]) = cl_Y(e[cl_X(F)])$ according to the first approved equality. Consequently, $y \in cl_Y(cl_Y(e[F])) \subset cl_Y(e[F])$ results, which shows $\rho \in N_e(B)$, according to $(\text{tx}_1)$. By definition $N_e$ is automatically pointed.

$\square$

**Theorem 3.2** *Let $F$ : TEXT $\longrightarrow$ PT-SN be defined by:*

(a) *For a TEXT-object $(e, \mathcal{B}^X, Y)$ we put $F(e, \mathcal{B}^X, Y) := (X, \mathcal{B}^X, N_e)$;*

(b) *for a TEXT-morphism $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ we put $F(f, g) := f$. Then $F$ : TEXT $\longrightarrow$ PT-SN is a functor.*

**Proof:** With respect to 3.1 we already know that $F(e, \mathcal{B}^X, Y)$ is an object of PT-SN. Let $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ be a TEXT-morphism such that $F(e, \mathcal{B}^X, Y) = (X, \mathcal{B}^X, N_e)$ and $F(e', \mathcal{B}^{X'}, Y') = (X', \mathcal{B}^{X'}, N_{e'})$. It has to be shown that $f : (X, \mathcal{B}^X, N_e) \longrightarrow (X', \mathcal{B}^{X'}, N_{e'})$ preserves B-near collections for each $B \in \mathcal{B}^X$. Without loss of generality, let be $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\rho \in N_e(B)$, hence there exists $y \in e[B]$ such that $y \in \cap\{cl_Y(e[F]) : F \in \rho\}$. Our goal is to verify that $f\rho \in N_{e'}(f[B])$. By hypothesis we have $g(y) \in g[e[B]] = e'[f[B]]$. On the other hand let $D \in f\rho$. We have to verify that $g(y) \in cl_{Y'}(e'[D])$. As $D = f[F]$ for some $F \in \rho, y \in cl_Y(e[F])$. Consequently, $g(y) \in g(cl_Y(e[F])) \subset cl_{Y'}(g(e[F]]) = cl_{Y'}(e'(f[F])) = cl_{Y'}(e'[D])$, which results in $f\rho \in N_{e'}(f[B])$ according to the definitions in 1.4. Then the remainder is clear. □

## 4 Pointed supernear spaces and strict topological extensions

In the previous paragraph we have found a functor from TEXT to PT-SN. Now, we are going to introduce a related one from PT-SN to STREXT.

**Lemma 4.1** *Let $(X, \mathcal{B}^X, N)$ be a supernear space. We put $X^C := \{\mathcal{C} \subset \underline{P}X : \mathcal{C}$ is B-clan in $N$ for some $B \in \mathcal{B}^X\}$, and for each $A^C \subset X^C$ we set: $cl_{X^C}(A^C) := \{\mathcal{C} \in X^C : \triangle A^C \subset \mathcal{C}\}$, where $\triangle A^C := \{F \subset X : \forall \mathcal{C} \in A^C F \in \mathcal{C}\}$, so that by convention $\triangle A^C = \underline{P}X$ if $A^C = \varnothing$. Then $cl_{X^C}$ is a topological closure operator on $X^C$.*

**Proof:** First, we note that for any $\mathcal{C} \in X^C, \mathcal{C} \notin cl_{X^C}(\varnothing)$, because $\varnothing \notin \mathcal{C}$ according to 2.6 and (sn$_2$) respectively. Now, let $A_1^C \subset A_2^C$. Then $\triangle A_2^C \subset \triangle A_1^C$ which yields $cl_{X^C}(A_1^C) \subset cl_{X^C}(A_2^C)$. Further, let $A_1^C$ and $A_2^C$ be subsets of $X^C$. Let $\mathcal{C}$ be an elements of $X^C$ and suppose $\mathcal{C} \notin cl_{X^C}(A_1^C) \cup cl_{X^C}(A_2^C)$. Then we have $\triangle A_1^C \not\subset \mathcal{C}$ and $\triangle A_2^C \not\subset \mathcal{C}$. Choose $F_1 \in \triangle A_1^C$ with $F_1 \notin \mathcal{C}$ and $F_2 \in \triangle A_2^C$ with $F_2 \notin \mathcal{C}$, hence $F_1 \cup F_2 \notin \mathcal{C}$, according to 2.6. On the other hand, we have $F_1 \cup F_2 \in \triangle(A_1^C \cup A_2^C)$, and consequently $\mathcal{C} \notin cl_{X^C}(A_1^C \cup A_2^C)$ results. Now, let $\mathcal{C}$ be the element of $cl_{X^C}(cl_{X^C}(A^C))$ and suppose $\mathcal{C} \notin cl_{X^C}(A^C)$. Choose $F \in \triangle A^C$ $F \notin \mathcal{C}$. By hypothesis we have $\triangle cl_{X^C}(A^C) \subset \mathcal{C}$, hence $F \notin \triangle cl_{X^C}(A^C)$. Choose $\mathcal{D} \in cl_{X^C}(A^C)$ $F \notin \mathcal{D}$. Then $\triangle A^C \subset \mathcal{D}$, hence $F \in \mathcal{D}$, which leads us to a contradiction! □

**Theorem 4.2** *For supernear spaces $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ let $f : X \longrightarrow Y$ be a sn-map. Define a function $f^C : X^C \longrightarrow Y^C$ by setting for each $\mathcal{C} \in X^C$: $f^C(\mathcal{C}) := \{D \subset Y : f^{-1}[cl_M(D)] \in \mathcal{C}\}$. Then the following statements are valid:*

(i) *$f^C : (X^C, cl_{X^C}) \longrightarrow (Y^C, cl_{Y^C})$ is a continuous map;*

(ii) *the equality $f^C \circ e_X = e_Y \circ f$ holds, where $e_X : X \longrightarrow X^C$ denotes that function which assigns the $\{x\}$-clan $x_N$ to each $x \in X$.*

**Proof:** First, let $\mathcal{C} \in X^C$, we must show that $f^C(\mathcal{C}) \in Y^C$. $f^C(\mathcal{C}) \in \mathrm{GRL}(Y)$, since $\mathcal{C} \in \mathrm{GRL}(X)$ and $f^{-1}$ respectively $cl_M$ are compatible with finite union. By hypothesis $\mathcal{C} \in N(B)$ for some $B \in \mathcal{B}^X$, hence $f\mathcal{C} \in N(f[B])$, because $f$ is sn-map. Now, we will show that $\{cl_M(D) : D \in f^C(\mathcal{C})\} << f\mathcal{C}$. $cl_M(D)$ for some $D \in f^C(\mathcal{C})$ implies $f^{-1}[cl_M(D)] \in \mathcal{C}$, hence $cl_M(D) \supset f[f^{-1}[cl_M(D)]] \in f\mathcal{C}$. According to ($\mathrm{sn}_7$), $f^C(\mathcal{C}) \in M(f[B])$ follows. $f[B] \in f^C(\mathcal{C})$, since $f^{-1}[cl_M(f[B])] \supset f^{-1}[f[cl_N(B)]] \supset B \in \mathcal{C}$ by hypothesis.

At last, let be $D \in f^C(\mathcal{C})$ and $D \subset cl_M(F)$, we have to verify $F \in f^C(\mathcal{C})$. By supposition $f^{-1}[cl_M(D)] \in \mathcal{C}$. $f^{-1}[cl_M(D)] \subset cl_N(f^{-1}[cl_M(F)])$, because $x \in f^{-1}[cl_M(D)]$ implies $f(x) \in cl_M(D)$; but $cl_M(D) \subset cl_M(cl_M(F)) \subset cl_M(F)$, hence $f(x) \in cl_M(F)$. Consequently, $x \in f^{-1}[cl_M(F)] \subset cl_N(f^{-1}[cl_M(F)])$ results. Since $\mathcal{C}$ satisfies ($\mathrm{cla}_2$), $f^{-1}[cl_M(F)] \in \mathcal{C}$ is valid, which shows $F \in f^C(\mathcal{C})$.

to (i): Let $A^C \subset X^C, \mathcal{C} \in cl_{X^C}(A^C)$ and suppose $f^C(\mathcal{C}) \notin cl_{Y^C}(f^C[A^C])$. Then $\triangle f^C[A^C] \not\subset f^C(\mathcal{C})$, hence $D \notin f^C(\mathcal{C})$ for some $D \in \triangle f^C[A^C]$, which means $f^{-1}[cl_M(D)] \notin \mathcal{C}$.

But $\triangle A^C \subset \mathcal{C}$ implies $f^{-1}[cl_M(D)] \notin \mathcal{D}$ for some $\mathcal{D} \in A^C$. Therefore $D \notin f^C(\mathcal{D})$, which leads us to a contradiction, because $D \in \triangle f^C[A^C]$.

to (ii): Let $x$ be an element of $X$. We will prove that the equality $f^C(e_X(x)) = e_Y(f(x))$ is valid. To this end let $T \in e_Y(f(x))$, hence $f(x) \in cl_M(T)$, and consequently $x \in f^{-1}[cl_M(T)]$ follows, which shows $f^{-1}[cl_M(T)] \in x_N = e_X(x)$. Thus, $T \in f^C(e_X(x))$ which proves the inclusion $e_Y(f(x)) \subset f^C(e_X(x))$.

Consequently, since $e_Y(f(x))$ is maximal in $M(\{f(x)\}) \setminus \{\varnothing\}$ (see 2.7 and note also that $\{cl_M(D) : D \in f^C(e_X(x))\} << fx_N \in M(\{f(x)\})$, since by hypothesis $f$ is sn-map) we obtain the desired equality.

□

**Theorem 4.3** *Let $G : SN \longrightarrow STREXT$ be defined as follows:*

(a) *For any supernear space $(X, \mathcal{B}^X, N)$ we put $G(X, \mathcal{B}^X, N) := (e_X, \mathcal{B}^X, X^C)$ with $X := (X, cl_N)$ and $X^C := (X^C, cl_{X^C})$;*

(b) *for any sn-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$ we put: $G(f) := (f, f^C)$.*

*Then $G : SN \longrightarrow STREXT$ is a functor.*

**Proof:** With respect to ($\text{sn}_7$) $cl_N$ is topological, and by 4.1 this also holds for $cl_{X^C}$. Therefore we get topological spaces with $\underline{B}$-set $\mathcal{B}^X$, and $e_X : X \longrightarrow X^C$ is a map according to 4.2. Now, we have to verify that $(e_X, \mathcal{B}^X, X^C)$ satisfies the axioms ($\text{tx}_1$) to ($\text{tx}_3$).

to ($\text{tx}_1$): Let $A$ be a subset of $X$ and suppose $x \in cl_N(A)$. Since $\triangle e_X[A] = \{T \subset X : A \subset cl_N(T)\}$ we get $e_X(x) \in cl_{X^C}(e_X[A])$, hence $x \in e_X^{-1}[cl_{X^C}[e_X[A]]]$ follows. Conversely, let $x$ be an element of $e_X^{-1}[cl_{X^C}(e_X[A])]$, then by definition we have $e_X(x) \in cl_{X^C}(e_X[A])$, and consequently the statement $\triangle e_X[A] \subset e_X(x)$ results. In applying the above mentioned equation we get $A \in e_X(x)$, which means $x \in cl_N(A)$.

to ($\text{tx}_2$): Let $\mathcal{C} \in X^C$ and suppose $\mathcal{C} \notin cl_{X^C}(e_X[X])$. By definition we get $\triangle e_X[X] \not\subset \mathcal{C}$, so that there exists a set $F \in \triangle e_X[X] F \notin \mathcal{C}$.

Consequently, the inclusion $X \subset cl_N(F)$ holds. By hypothesis $\mathcal{C}$ is B-clan for some $B \in \mathcal{B}^X$, hence $B \in \mathcal{C}$ according to ($\text{cla}_1$), and $B \subset X \subset cl_N(F)$ follows, which imply $F \in \mathcal{C}$ according to ($\text{cla}_2$). But this is a contradiction, hence $\mathcal{C} \in cl_{X^C}(e_X[X])$ holds.

to ($\text{tx}_3$): Let $\mathcal{C} \in X^C$ and let $A^C$ be closed in $X^C$ with $\mathcal{C} \notin A^C$. Then $\mathcal{C} \notin cl_{X^C}(A^C)$ and so $\triangle A^C \not\subset \mathcal{C}$. There exists $F \in \triangle A^C$ such that $F \notin \mathcal{C}$. Now, for each $\mathcal{D} \in A^C$ we have $F \in \mathcal{D}$, which implies $\triangle e_X[F] \subset \mathcal{D}$, and so at last $\mathcal{D} \in cl_{X^C}(e_X[F])$ results. On the other hand since $F \notin \mathcal{C}$ we have $\triangle e_X[F] \not\subset \mathcal{C}$, and so $\mathcal{C} \notin cl_{X^C}(e_X[F])$.

□

Now it is interesting to see, how the composite functor $F \circ G$ works on the category PT-SN.

**Theorem 4.4** *Let $G$ : PT-SN $\longrightarrow$ TEXT and $F$ : TEXT $\longrightarrow$ PT-SN be the functors given in theorem 3.2 and 4.3. For each object $(X, \mathcal{B}^X, N)$ of PT-SN let $t_(X, \mathcal{B}^X, N)$ denote the identity map $t_(X, \mathcal{B}^X, N) := id_X : F(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$. Then $t : \mathcal{F} \circ G \longrightarrow 1_{PT\text{-}SN}$ is natural equivalence from $F \circ G$ to the identity functor $1_{PT\text{-}SN}$, i.e. $id_X : F(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$ is in <u>both</u> directions a sn-map for each object $(X, \mathcal{B}^X, N)$, and the following diagram commutes for each sn-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$:*

$$\begin{tikzcd}
F(G(X, \mathcal{B}^X, N)) \arrow[r, "id_X"] \arrow[d, "F(G(f))"'] & (X, \mathcal{B}^X, N) \arrow[d, "f"] \\
F(G(Y, \mathcal{B}^Y, M))) \arrow[r, "id_Y"'] & (Y, \mathcal{B}^Y, M).
\end{tikzcd}$$

**Proof:** The commutativity of the diagram is obvious, because $F(G(f)) = f$.

It remains to prove that in each case $F(G(X, \mathcal{B}^X, N)) \xrightarrow{id_X} (X, \mathcal{B}^X, N) \xrightarrow{id_X} F(G(X, \mathcal{B}^X, N))$ is sn-map for any object $(X, \mathcal{B}^X, N) \in$ PT-SN. To fix the notation, let $N_1$ be such that

$F(G(X,\mathcal{B}^X,N)) = F(e_X,\mathcal{B}^X,X^C) = (X,\mathcal{B}^X,N_1)$. First we show that for each $B \in \mathcal{B}^X \setminus \{\varnothing\}, \rho \in N_1(B)$ implies $\rho \in N(B)$. To this end assume that $\rho \in N_1(B)$, then there exists $\mathcal{C} \in e_X[B]$ such that $\mathcal{C} \in \cap\{cl_{X^C}(e_X[F]) : F \in \rho\}$. We have $\mathcal{C} = e_X(x)$ for some $x \in B$, hence $\mathcal{C} \in N(B)$ according to 2.7 and 4.2, respectively. $\rho \subset \mathcal{C}$, because $F \in \rho$ implies $\mathcal{C} \in cl_{X^C}(e_X[F])$, and in consequence $\triangle e_X[F] \subset \mathcal{C}$ results. Since $F \in \triangle e_X[F]$ we get $F \in \mathcal{C}$, which shows $\rho \in N(B)$, according to $(\mathrm{sn}_1)$. Conversely, let be $B \in \mathcal{B}^X \setminus \{\varnothing\}$ and $\rho \in N(B)$, we have to show that $\rho \in N_1(B)$.

In assuming the above we get $\rho \in N(\{x\})$ for some $x \in B$, since $(X,\mathcal{B}^X,N)$ is pointed. But $x_N = e_X(x) \in e_X[B]$. We have to show that for each $F \in \rho$ the statement $x_N \in cl_{X^C}(e_X[F])$ is valid. So let be $F \in \rho$ and $T \in \triangle e_X[F]$. By hypothesis $F \subset cl_N(T)$ results with $F \in x_N$, hence $x \in cl_N(F)$, and consequently we get $T \in x_N$, which concludes the proof.

Now, in making this part of searching more transparent, we give a short characterization of the subject as follows: □

*Comment 1* Let be given an arbitrary supernear space $(X,\mathcal{B}^X,N)$. Then his property of being pointed can be described in such a way that there exists a topological space $Y$ in which it is densely "embedded", so that non-empty B-near collections are characterized by the fact, that its closure meet in $Y$ by the image of an element of $B$. Hence, we can resume, that pointed supernear spaces can be strictly extended in such a manner!

**Corollary 4.5** *If $(X,\mathcal{B}^X,N)$ is separated, which means $N$ satisfies* (sep)*, e.g.*

(sep) *$x, z \in X$ and $\{\{z\}\} \in N(\{x\})$ imply $x = z$, then $e_X : X \longrightarrow X^C$ is injective! Conversely, for a $T_1$-extension $(e,\mathcal{B}^X,Y)$, where $e$ is a topological embedding, and $Y$ is a $T_1$-space, then $(X,\mathcal{B}^X,N_e)$ is separated!*

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# Letter to the Editors

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**Dear Editors of the Rostocker Mathematisches Kolloquium,**

We would like you know that the paper *A New Criterion of Stability for Stochastic Networks With Two Stations and Two Heterogeneous Servers* by A. Kandouci et al published in Rostock. Math. Kolloq., 62, 3-19 (2007) contains ONLY copied-and-pasted material from our paper published in Queueing System in 1999 and cited as [4] (see RoMaKo 62 (2007) p. 18) in the paper by Kandouci et al.

It should be obvious even to a non-specialist that A. Kandouci et al have just re-published results from our paper under their names.

Yours faithfully, Sergey Foss and Artyom Kovalevskii

**received:** November 25, 2011

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