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On the improper derivatives of Tagaki's continuous nowhere differentiable function

ABSTRACT. This note is a completion of [5] where it was investigated among other things the improper derivatives of Takagi's continuous nowhere differentiable function T. We determine all points x for which T has the one-sided improper derivatives $T'_+(x) = \infty$ and $T'_-(x) = \infty$.

KEY WORDS. Takagi's continuous nowhere differentiable function, improper derivatives

1 Introduction

In 1903, T. Takagi [6] discovered an example of a continuous, nowhere differentiable function that was simpler than a well-known example of K. Weierstrass. Takagi's function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\Delta\,(2^n x)}{2^n} \qquad (x \in \mathbb{R}) \tag{1.1}$$

where $\Delta(y) = \operatorname{dist}(y, \mathbb{Z})$ is a periodic function with period 1. The Takagi function was rediscovered independently by others, e.g. Knopp in 1918, Van der Waerden in 1930 and Hildebrandt in 1933, cf. [3].

It is known that T does not have a finite one-sided derivative anywhere. But at each dyadic rational point $x = \frac{m}{2^n}$ there exist the right-hand improper derivative

$$T'_+(x) = \lim_{h \to +0} \frac{T(x+h) - T(x)}{h} = +\infty$$

and left-hand improper derivative

$$T'_-(x) = \lim_{h \to -0} \frac{T(x+h) - T(x)}{h} = -\infty,$$

cf. [5]. Begle and Ayres [2] have investigated non-dyadic points $x \neq \frac{m}{2^n}$ for which the Takagi function (with the notation Hildebrandt function) does have an improper derivative $T'(x) = +\infty$ or $T'(x) = -\infty$. For given x let I_n and O_n represent the number of $1'$s and $0'$s among the first n terms in the dyadic expansion of x, and $D_n = O_n - I_n$. The claim of Begle and Ayres reads: If $\lim D_n = +\infty$ then $T'(x) = +\infty$ and if $\lim D_n = -\infty$ then $T'(x) = -\infty$. But this cannot be true since in [5] is a counterexample, cf. Example 7.2.

The purpose of this paper is to determine all non-dyadic points $x \neq \frac{m}{2^n}$ for which the improper derivatives do exist. We consider the right-hand and left-hand improper derivatives separately. In view of the symmetry $T(1-x) = T(x)$ it holds $T'_+(x) = \pm\infty$ if and only if $T'_-(1-x) = \mp\infty$. Therefore we only investigate the case $+\infty$. The main results of this note is that for non-dyadic x with the representation

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{a_n}} \tag{1.2}$$

where $1 \leq a_1 < a_2 < \ldots$ are integers, we have:

(i) $T'_+(x) = \infty \iff D_n \to \infty \quad (n \to \infty)$

and

(ii) $T'_-(x) = \infty \iff 2^{D_{a_n}} \frac{d_n}{2^{d_n}} \to \infty \quad (n \to \infty)$

where $d_n = a_{n+1} - a_n$, (Proposition 3.1, Proposition 4.5 and Remark 4.6).
Since $d_n 2^{-d_n}$ is bounded and $D_n \to \infty$ implies $D_{a_n} \to \infty$, from (i) and (ii) it follows

(iii) $T'(x) = \infty \iff 2^{D_{a_n}} \frac{d_n}{2^{d_n}} \to \infty \quad (n \to \infty)$.

Remark 1.1 It is remarkable that if $T'_-(x) = \infty$ then also $T'_+(x) = \infty$ but not conversely. In Example 7.2 from [5] it was considered a point (1.2) where $a_{n+1} \geq 4a_n$. Here $T'_+(x) = \infty$ since $D_n \to \infty$, but in [5] it was shown that $T'_-(x) = \infty$ does not be valid. Hence, the condition in (ii) cannot be satisfied.

Remark 1.2 The condition in (iii) is satisfied if and only if $D_{a_n} \to \infty$ and if e.g. d_n is bounded, but the condition also may be satisfied if $d_n \to \infty$.

Example 1.3 Take the point (1.2) with $a_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. Then $d_n = n + 1$, $D_{a_n} = a_n - 2n = \frac{n(n-3)}{2}$ and

$$2^{D_{a_n}} \frac{d_n}{2^{d_n}} = 2^{n(n-3)/2} \frac{n+1}{2^{n+1}} = 2^{(n^2-5n-2)/2}(n+1) \to \infty$$

as $n \to \infty$. So by (iii) we have $T'(x) = \infty$.

Remark 1.4 Let us mention that in (iii) the term D_{a_n} cannot be replaced by D_n. This shows the Example 1.3 since in view of $d_{a_n} = \frac{n(n+1)}{2} + 1 = \frac{n^2+n+2}{2}$ we have for $k = a_n$

$$2^{D_k} \frac{d_k}{2^{d_k}} = 2^{n(n-3)/2} \frac{\frac{n^2+n+2}{2}}{2^{(n^2+n+2)/2}} = \frac{n^2+n+2}{2^{2n+2}} \to 0$$

though $T'(x) = \infty$.

2 Relations for Takagi's function

In order to determine the improper derivatives we need some relations for the Takagi function. It is known that T satisfies for $0 \le x \le 1$ the following system of functional equations

$$T\left(\frac{x}{2}\right) = \frac{x}{2} + \frac{1}{2}T(x), \qquad T\left(\frac{1+x}{2}\right) = \frac{1-x}{2} + \frac{1}{2}T(x), \tag{2.1}$$

cf. e.g. [4], [5]. Moreover, for $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$ and $x \in [0, 1]$, the Takagi function T satisfies the equations

$$T\left(\frac{k+x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell}x + \frac{1}{2^\ell}T(x) \tag{2.2}$$

and

$$T\left(\frac{k-x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{2s(k-1) - \ell}{2^\ell}x + \frac{1}{2^\ell}T(x) \tag{2.3}$$

where $s(k)$ denotes the binary sum-of-digit function which is the number of ones in the binary representation of k, cf. [5, Proposition 2.1].

Note that for given x with the dyadic expansion

$$x = 0, \xi_1, \xi_2 \ldots \tag{2.4}$$

we have for the difference $D_n = O_n - I_n$ of the number of $0'$s and $1'$ in the first n terms of (2.4)

$$D_n = \sum_{\nu=1}^{n} (-1)^{\xi_\nu}.$$

Besides of (2.4) we consider $y = 0, \eta_1\eta_2 \ldots$ with $\eta_n \in \{0, 1\}$. It is known that if x and y are different points in $[0, 1]$ with $\xi_\nu = \eta_\nu$ for $\nu \le n \in \mathbb{N}$ then

$$\frac{T(x) - T(y)}{x - y} = D_n + \frac{T(x_n) - T(y_n)}{x_n - y_n}, \tag{2.5}$$

where $x_n = 0, \xi_{n+1}\xi_{n+2} \ldots$ and $y_n = 0, \eta_{n+1}\eta_{n+2} \ldots$, cf. [5, Formula (5.3)]. Let us mention that the index in Formula (5.3) is not correct.

The following estimate is already known for $0 < x \le \frac{1}{2}$ from [5, Lemma 3.1].

Lemma 2.1 *For $0 < x \le 1$ the Takagi function T satisfies the estimate*

$$\log_2 \frac{1}{x} \le \frac{1}{x}T(x) \le \log_2 \frac{1}{x} + c \tag{2.6}$$

with a positive constant $c < \frac{2}{3}$.

Proof: Since (2.6) is true for $0 < x \le \frac{1}{2}$ we can assume that $\frac{1}{2} < x \le 1$. By the first relation in (2.1) we have $T(x) = 2T(\frac{x}{2}) - x$ and hence $\frac{1}{x}T(x) = \frac{2}{x}T(\frac{x}{2}) - 1$. In view of $\log_2 \frac{2}{x} = 1 + \log_2 \frac{1}{x}$ and $\frac{x}{2} \le \frac{1}{2}$ it follows that (2.6) is also true for $\frac{1}{2} < x \le 1$. Thus, the lemma is proved. $\square$

3 Right-hand improper derivatives

First we investigate the existence of the right-hand improper derivative.

Proposition 3.1 *The Takagi function T has at the non-dyadic point x the right-hand improper derivative $T'_+(x) = \infty$ if and only if $D_n \to \infty$ as $n \to \infty$.*

Proof: Since x is a non-dyadic point the expansion (2.4) contains infinitely ones and zeros. Let y have the dyadic representation $y = \eta_0, \eta_1\eta_2 \ldots$ where $\eta_\nu = \xi_\nu$ for $\nu \le n$ and $\eta_{n+1} = 1$, $\xi_{n+1} = 0$ so that $x < y < x + 2^{1-n}$. We investigate the term

$$\frac{T(y) - T(x)}{y - x}$$

as $n \to \infty$.

1. Assume $T'_+(x) = \infty$. If we choose $\eta_\nu = 1 - \xi_\nu$ for $\nu > n+1$, then $y_n = 1 - x_n$ and by (2.5) we have

$$\frac{T(y) - T(x)}{y - x} = D_n. \tag{3.1}$$

Since $x < y \le x + \frac{1}{2^n}$ it follows that $T'_+(x) = \infty$ implies $D_n \to \infty$.

2. Suppose $D_n \to \infty$. By (2.5) we have

$$\frac{T(y) - T(x)}{y - x} = D_n + \frac{T(y_n) - T(x_n)}{y_n - x_n}$$

where $x_n = 0, 0\xi_{n+2} \ldots$ and $y_n = 0, 1\eta_{n+2} \ldots$ so that $0 < x_n < \frac{1}{2}$ and $\frac{1}{2} \le y_n \le 1$. We consider two cases:

2.1 In case $\frac{7}{8} < y_n \le 1$ we have $y_n - x_n > \frac{1}{8}$ and

$$\frac{T(y_n) - T(x_n)}{y_n - x_n} > \frac{-\frac{2}{3}}{\frac{1}{8}} = -\frac{16}{3}.$$

2.2 In case $\frac{1}{2} \le y_n \le \frac{7}{8}$ we put $y_n = \frac{1+t}{2}$ with $0 \le t \le \frac{1}{4}$. By (2.1) and $T(t) \ge 2t$ for $0 \le t \le \frac{1}{4}$

$$T(y_n) = T\left(\frac{1+t}{2}\right) = \frac{1-t}{2} + \frac{1}{2}T(t) \ge \frac{1+t}{2}$$

and

$$\frac{T(y_n) - T(x_n)}{y_n - x_n} \ge \frac{1 + t - 2T(x_n)}{1 + t - 2x_n}.$$

For the derivative of the function

$$f(t) = \frac{1 + t - 2T(x_n)}{1 + t - 2x_n}$$

we have

$$f'(t) = \frac{(1+t-2x_n) - (1+t-2T(x_n))}{(1+t-2x_n)^2} = \frac{2T(x_n)-2x_n}{(1+t-2x_n)^2} \geq 0\,.$$

Hence, for $0 \leq t \leq \frac{1}{4}$ the function $f(t)$ is increasing and

$$\frac{T(y_n)-T(x_n)}{y_n - x_n} \geq f(0) = \frac{T(\frac{1}{2})-T(x_n)}{\frac{1}{2}-x_n}.$$

With $h = 1-2x_n$, i.e. $x_n = \frac{1-h}{2}$, we find in view of the symmetry of T with respect to $\frac{1}{2}$ that

$$\frac{T(\frac{1}{2})-T(x_n)}{\frac{1}{2}-x_n} = \frac{\frac{1}{2}-T(\frac{1+h}{2})}{h/2} = \frac{\frac{1}{2}-\frac{1-h}{2}-\frac{1}{2}T(h)}{h/2} = 1 - \frac{T(h)}{h}$$

where we have used the second equation in (2.1). By Lemma 2.1

$$\frac{T(h)}{h} \leq \log_2 \frac{1}{h} + c$$

with $c < \frac{2}{3}$. Note that $h = 1-2x_n = 0, \overline{\xi}_{n+2}\overline{\xi}_{n+3}\ldots$ with $\overline{\xi}_\nu = 1-\xi_\nu$. If $\xi_{n+\nu} = 1$ for $\nu = 2, 3, \ldots, m$ and $\xi_{n+m+1} = 0$ then $m \geq 2$, $h \geq 1/2^m$ and $\log_2 \frac{1}{h} \leq m$. Note that $m = I_{n+m} - I_n$ and $O_{n+m} - O_n = 1$ since $\xi_{n+1} = 0$. Hence, $D_{n+m} - D_n = 1 - m$ and we get

$$D_n + \frac{T(y_n)-T(x_n)}{y_n-x_n} \geq D_n + 1 - m - c = D_{n+m} - c.$$

Both cases 2.1 and 2.2 together yield

$$\frac{T(y)-T(x)}{y-x} \geq \inf_{k \geq n} D_k + O(1)$$

which implies $T'_+(x) = \infty$ since $D_n \to \infty$. □

4 Left-hand improper derivatives

The determining of the conditions for the existence of the left-hand improper derivative $T'_-(x) = \infty$ is more complicated. We need some lemmas.

Lemma 4.1 *Assume that $x = \frac{k+r}{2^m}$ and $y = \frac{k-h}{2^m}$ where k is an odd integer and $0 < r < 1$, $0 \leq h \leq 1$. Then we have*

$$\frac{T(x)-T(y)}{x-y} = D_m + \frac{2h}{r+h} + \frac{T(r)-T(h)}{r+h}. \tag{4.1}$$

Proof: According to equation (2.2) we have

$$T(x) = T\left(\frac{k+r}{2^m}\right) = T\left(\frac{k}{2^m}\right) + \frac{m-2s(k)}{2^m}r + \frac{1}{2^m}T(r)$$

and by equation (2.3)

$$T(y) = T\left(\frac{k-h}{2^m}\right) = T\left(\frac{k}{2^m}\right) + \frac{2s(k-1)-m}{2^m}h + \frac{1}{2^m}T(h).$$

Since k is an odd integer, we have $s(k-1) = s(k) - 1$. It follows

$$T(x) - T(y) = \frac{m-2s(k)}{2^m}(r+h) + \frac{2h}{2^n} + \frac{T(r)-T(h)}{2^m}$$

and in view of $x - y = (r+h)/2^m$ and $D_m = m - 2s(k)$ it follows (4.1). □

Assume that x is a non-dyadic point with the representation (1.2) so that

$$x = \frac{k_n + r_n}{2^{a_n}}, \qquad k_n = 2^{a_n}\sum_{\nu=1}^{n}\frac{1}{2^{a_\nu}}, \qquad r_n = 2^{a_n}\sum_{\nu=n+1}^{\infty}\frac{1}{2^{a_\nu}} \tag{4.2}$$

and that

$$y = \frac{k_n - h_n}{2^{a_n}}, \qquad 0 \le h_n \le 1. \tag{4.3}$$

Note that $r_n > 0$ since x is a non-dyadic point. Put $d_n = a_{n+1} - a_n$ then we have $d_n \ge 1$ and

$$r_n = \frac{1}{2^{d_n}}\sum_{\nu=n+1}^{\infty}\frac{1}{2^{a_\nu - a_{n+1}}} \le \frac{2}{2^{d_n}}$$

and therefore

$$d_n - 1 \le \log_2\frac{1}{r_n} < d_n. \tag{4.4}$$

Lemma 4.2 *If $h_n > 0$ then we put $h_n = 2^t r_n > 0$ and it holds*

$$\frac{T(x)-T(y)}{x-y} = D_{a_n} - d_n + \frac{t2^t + 2d_n}{1+2^t} + O(1) \tag{4.5}$$

for $t \le \log_2\frac{1}{r_n}$.

Proof: Because of $r_n > 0$ and $0 < h_n \le 1$, cf. (4.3), we can write $h_n = 2^t r_n$ with $t \le \log_2\frac{1}{r_n}$. By Lemma 4.1 with $m = a_n$

$$\frac{T(x)-T(y)}{x-y} = D_{a_n} - \frac{2h_n}{r_n+h_n} + \frac{T(r_n)-T(h_n)}{r_n+h_n}.$$

Moreover the term $2h_n/(r_n+h_n)$ is bounded and the last term can be written in the form

$$\frac{T(r_n)-T(h_n)}{r_n+h_n} = \frac{r_n}{r_n+h_n}\frac{T(r_n)}{r_n} - \frac{h_n}{r_n+h_n}\frac{T(h_n)}{h_n}.$$

By Lemma 2.1 and (4.4) we get

$$d_n - 1 \le \frac{1}{r_n} T(r_n) < d_n + c$$

with a constant $c < \frac{2}{3}$, i.e.

$$\frac{1}{r_n} T(r_n) = d_n + \varepsilon_n$$

with $|\varepsilon_n| \le 1$. For $h_n = 2^t r_n$ we have

$$\log_2 \frac{1}{h_n} = \log_2 \frac{1}{r_n} - t$$

and as before

$$\frac{1}{h_n} T(h_n) = d_n - t + \delta_n$$

with $|\delta_n| \le 2$. So with $h_n = 2^t r_n$ we get

$$\begin{aligned}
\frac{T(r_n) - T(h_n)}{r_n + h_n} &= \frac{1}{1+2^t} \frac{T(r_n)}{r_n} - \frac{2^t}{1+2^t} \frac{T(h_n)}{h_n} \\
&= \frac{1}{1+2^t}(d_n + \varepsilon_n) - \frac{2^t}{1+2^t}(d_n - t + \delta_n) \\
&= -d_n + \frac{t2^t + 2d_n}{1+2^t} + \frac{1}{1+e^t}\varepsilon_n - \frac{2^t}{1+2^t}\delta_n
\end{aligned}$$

which yields (4.5). $\square$

In view of (4.5) we want to estimate the minimum of the function

$$f_n(t) = \frac{t2^t + 2d_n}{1+2^t} \qquad (t \in \mathbb{R}). \tag{4.6}$$

Lemma 4.3 *For positive integer d the function $f(t) = (t2^t + 2d)/(1+2^t)$ attains its minimum exactly at one point $t_* = t_*(d)$ where $t_*(d) < d - 1$. It holds*

$$f(t_*) = \log_2 d + O(1). \tag{4.7}$$

Proof: 1. Note that $f(t) \to 2d$ as $t \to -\infty$ and $f(t) \to +\infty$ as $t \to +\infty$. Moreover, for the derivative

$$f'(t) = \frac{(2^t + t2^t \log 2)(1+2^t) - (t2^t + 2d)2^t \log 2}{(1+2^t)^2}$$

we have $f'(t) = 0$ if and only if

$$g(t) = 1 + 2^t + t \log 2 - 2d \log 2$$

vanishes. Now $g(t)$ is strictly increasing with $g(t) \to -\infty$ as $t \to -\infty$ and $g(t) \to +\infty$ as $t \to +\infty$ so that there is exactly one real number $t_* = t_*(d)$ with $g(t_*) = 0$.

In order to show that $t_* < d-1$ we prove the inequality

$$g(d-1) = 1 + 2^{d-1} + (d-1)\log 2 - 2d\log 2 > 0$$

which is true for $d = 1$. Moreover $g(d) - g(d-1) = 2^{d-1} - \log 2 \geq 1 - \log 2 > 0$ so that indeed $g(d-1) > 0$ for all $d \geq 1$. Consequently, $t_* < d-1$.

2. In order to show (4.7) we put $2^{t_*} = \tau_* d$ with suitable $\tau_* = \tau_*(d)$. Then we have

$$g(t_*) = 1 + \tau_* d + \log_2(\tau_* d)\log 2 - 2d\log 2 = 0$$

so that τ_* is a zero of the function

$$h(\tau, d) = 1 + \tau d + \log(\tau d) - 2d\log 2.$$

We show that $a < \tau_* < 2$ where $a = 2\log 2 - 1$. Note that $0 < a < 1$ and hence $h(a,1) = 1 + a + \log a - 2\log 2 = \log a < 0$. Moreover

$$h(a, d+1) - h(a,d) = a + \log(d+1) - \log d - 2\log 2 \leq a - 2\log 2 = -1$$

so that $h(a,d) < 0$ for all $d \geq 1$. On the other hand

$$h(2,d) = 1 + 2d + \log 2 + \log d - 4\log 2 \geq 3 - 3\log 2 > 0$$

and it follows $a < \tau_* < 2$ since $h(\tau, d)$ is strictly increasing with respect to τ.

Finally, with $t_* = \log_2(\tau_* d)$ we get

$$\begin{aligned} f(t_*) &= \frac{(\log_2 \tau_* + \log_2 d)\tau_* d + 2d}{1 + \tau_* d} \\ &= \log_2 d + \frac{\tau_* d(\log_2 \tau_* - 1)}{1 + \tau_* d} + \frac{2d}{1 + \tau_* d} \end{aligned}$$

where in view of $a < \tau_* < 2$ it holds

$$\frac{\tau_* d(\log_2 \tau_* - 1)}{1 + \tau_* d} \sim \log_2 \tau_* - 1, \qquad \frac{2d}{1 + \tau_* d} \sim \frac{2}{\tau_*}$$

as $d \to \infty$. This implies (4.7). □

Corollary 4.4 *The function (4.6) attains its minimum exactly at one point t_n where $t_n < d_n - 1$ and it holds $f_n(t_n) = \log_2 d_n + O(1)$, i.e.*

$$\frac{t2^t + 2d_n}{1 + 2^t} \geq \log_2 d_n + O(1).$$

Proposition 4.5 *The Takagi function has at the non-dyadic point x with the representation (1.2) the left-side improper derivative $T'_-(x) = \infty$ if and only if*

$$D_{a_n} - d_n + \log_2 d_n \to \infty \tag{4.8}$$

as $n \to \infty$.

Proof: First we assume that x has the expansion (2.4). For given positive integer m let be $y = 0, \eta_1\eta_2\ldots$ a number with $\eta_\nu = \xi_\nu$ for $\nu < m$, $\eta_m = 0$, $\xi_m = 1$ so that $x - 2^{1-m} \le y < x$. Again, we investigate the term

$$\frac{T(x) - T(y)}{x - y}$$

as $m \to \infty$. Note that

$$x = \frac{k + r}{2^m}, \qquad k = 2^m \sum_{\nu=1}^{m} \frac{\xi_\nu}{2^\nu}, \qquad r = 2^m \sum_{\nu=m+1}^{\infty} \frac{\xi_\nu}{2^\nu}$$

where $0 < r < 1$ since x is not dyadic rational. In view of

$$y \le \sum_{\nu=1}^{m-1} \frac{\xi_\nu}{2^\nu} + \sum_{\nu=m+1}^{\infty} \frac{1}{2^\nu} = \frac{k-1}{2^m} + \frac{1}{2^m} = \frac{k}{2^m} \qquad y \ge \sum_{\nu=1}^{m-1} \frac{\xi_\nu}{2^\nu} = \frac{k-1}{2^m}$$

we have $y = (k-h)/2^m$ with $0 \le h \le 1$. Let $a_n \le m < a_{n+1}$ then we get the representations (4.2) and (4.3) where $k_n = k/2^{m-a_n}$ is an odd integer, $r_n = r/2^{m-a_n}$, $h_n = h/2^{m-a_n}$, and $m \to \infty$ if and only if $n \to \infty$.
In case $h_n = 0$ we get by Lemma 4.1

$$\frac{T(x) - T(y)}{x - y} = D_{a_n} + \frac{T(r_n)}{r_n} > D_{a_n}. \tag{4.9}$$

In case $h_n > 0$ we put $h_n = 2^{t_n} r_n$ with t_n from Corollary 4.4 which is only possible if $2^{t_n} \le 1$. But $t_n < d_n - 1$ and in view of $d_n - 1 < \log_2 \frac{1}{r_n}$, cf. (4.4), in fact $2^{t_n} r_n < 2^{d_n - 1} r_n < 1$. By Lemma 4.2 and Corollary 4.4 we have

$$\frac{T(x) - T(y)}{x - y} \ge D_{a_n} - d_n + \log_2 d_n + O(1) \tag{4.10}$$

where we have equality if we choose y such that $h_n = 2^{t_n} r_n$ in (4.3). From (4.9) we see that (4.10) is also valid in case $h_n = 0$ since $-d_n + \log_2 d_n < 0$.

Now it is easy to finish the proof. If (4.8) is satisfied then by (4.10) we obtain $T'_-(x) = \infty$. Conversely, if (4.8) fails then there is a strictly increasing sequence $\{n'\}$ of integers so that $D_{a_{n'}} - d_{n'} + \log_2 d_{n'} \to K < \infty$ as $n' \to \infty$. We use (4.2), (4.3) both with n' instead of n, where we put $h_{n'} = 2^{t_{n'}} r_{n'}$. Then by (4.10)

$$\frac{T(x) - T(y)}{x - y} = D_{n'} - d_{n'} + \log_2 d_{n'} + O(1)$$

so that

$$\liminf_{y\to x-}\frac{T(x)-T(y)}{x-y}<\infty.$$

Thus, the proposition is proved. □

Remark 4.6 The condition (4.8) can also be written as

$$2^{D_{a_n}}\frac{d_n}{2^{d_n}}\to\infty \tag{4.11}$$

as in (ii) of the Introduction.

Remark 4.7 Note that $D_{a_n}=a_n-2n\to\infty$ is equivalent to $D_n\to\infty$. It is enough to show that $D_{a_n}\to\infty$ implies $D_n\to\infty$. We assume that $a_n\le m<a_{n+1}$ then $O_m=m-n\ge a_n-n$, $I_m=n$ so that $D_m=O_m-I_m\ge a_n-2n=D_{a_n}\to\infty$. So (4.11) is satisfied if $D_n\to\infty$ and d_n is bounded. It follows that $T'(x)=\infty$ if $D_n\to\infty$ and if the number of consecutive zeros in the dyadic representation of x is bounded, cf. [5, Proposition 5.3].

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Supplement. K. Kawamura and P. C. Allaart also have found the conditions for the existence of the improper derivatives of Takagi's function, cf. [1].

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Green's matrix of the Stokes system in a convex polyhedron

ABSTRACT. The paper deals with the Dirichlet problem for the stationary Stokes system in a convex three-dimensional polyhedron. The author proves Hölder estimates for the elements of Green's matrix and their derivatives.

KEY WORDS. Stokes system, Green's matrix

1 Introduction

The present paper is concerned with the Green matrix $G(x,\xi) = \big(G_{i,j}(x,\xi)\big)_{i,j=1}^4$ of the boundary value problem

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \ \text{ in } \Omega, \tag{1}$$

$$u = 0 \ \text{ on } \partial\Omega, \tag{2}$$

where Ω is a convex polyhedron in $\mathbb{R}^3$. It is well-known that the elements of the Green matrix satisfy the estimate

$$\big|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi)\big| \le c\,|x-\xi|^{-1-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|} \tag{3}$$

for arbitrary multi-indices α and β if the boundary of the domain Ω is smooth (of class C^∞). For nonsmooth domains this result fails. If the domain Ω is of polyhedral type, then the derivatives of the elements of the Green matrix can be estimated by a function which depends not only on $|x-\xi|$ but also on the distances of x and ξ from the vertices and edges of the domain. Such estimates are given in papers of Maz'ya and Plamenevskiĭ [5], Maz'ya and Rossmann [6], Rossmann [8] (see also the monograph by Maz'ya and Rossmann [7]). Using these estimates, it was shown in [8] and [7, Section 11.5] that (3) is satisfied for $|\alpha| \le 1-\delta_{i,4}$ and $|\beta| \le 1-\delta_{j,4}$ if Ω is a convex polyhedron. The goal of the present paper is to prove that the functions $G_{i,j}(x,\xi)$ and their derivatives satisfy even a Hölder estimate

$$\begin{aligned} & \frac{\left|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi) - \partial_x^\alpha \partial_\xi^\beta G_{i,j}(y,\xi)\right|}{|x-y|^\sigma} \\ & \le c\left(|x-\xi|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|} + |y-\xi|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|}\right) \end{aligned} \tag{4}$$

for $|\alpha| \le 1-\delta_{i,4}$ and $|\beta| \le 1-\delta_{j,4}$. Here σ is a sufficiently small positive number depending on the domain Ω. For $i \ne 4$, the estimate (4) was proved in [8] (see also [7, Section 11.5]). However, the proof given in [8] does not work in the case $i = 4$. We modify here the proof of the paper [8] and obtain the estimate (4) for $i = 4$. As a consequence of (4), also the estimate

$$\begin{aligned} & \frac{\left|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi) - \partial_x^\alpha \partial_\eta^\beta G_{i,j}(x,\eta)\right|}{|\xi-\eta|^\sigma} \\ & \le c\left(|x-\xi|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|} + |x-\eta|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|}\right) \end{aligned} \tag{5}$$

holds for $|\alpha| \le 1-\delta_{i,4}$, $|\beta| \le 1-\delta_{j,4}$.

Analogous results were obtained for the Green function of the Laplace equation and some other second order equations and systems including the Lamé system. In papers by Grüter, Widman [2] and Fromm [1] it was shown that the Green function $\mathcal{G}(x,\xi)$ of the Laplace equation satisfies the estimate

$$\left|\partial_x^\alpha \partial_\xi^\beta \mathcal{G}(x,\xi)\right| \le c\,|x-\xi|^{-1-|\alpha|-|\beta|}$$

for $|\alpha|, |\beta| \le 1$ if Ω is an arbitrary (not necessarily polyhedral) convex domain. For a wider class of differential equations, we refer also to the paper by Kozlov [4]. Hölder estimates for the derivatives $\partial_x^\alpha \partial_\xi^\beta \mathcal{G}(x,\xi)$ of orders $|\alpha| \le 1$ and $|\beta| \le 1$ were proved in [2] for domains with $C^{1,\sigma}$ boundary and by Guzman, Leykekhman, Rossmann and Schatz [3] for convex domains of polyhedral type. In [7, Subsection 5.1.5], one can find these estimates for a class of second order differential equations and systems in convex polyhedral domains. Note that the Hölder estimates do not hold for general convex domains (see the counter-example in [2]).

2 The Green matrix for the Stokes system

Let Ω be a bounded polyhedron in $\mathbb{R}^3$, the boundary $\partial\Omega$ of which consists of the plane faces Γ_j, $j = 1, \dots, N$, the edges M_k, $k = 1, \dots, l$, and the vertices $x^{(1)}, \dots, x^{(d)}$. Throughout this paper, we assume that Ω is convex. As is known, the boundary value problem (1), (2) is solvable in $W^{1,2}(\Omega)^3 \times L_2(\Omega)$ for arbitrary $f \in W^{-1,2}(\Omega)^3$ and $g \in L_2(\Omega)$ satisfying the condition

$$\int_\Omega g(x)\,dx = 0.$$

The solution (u,p) is unique up to vectors $(0,c)$, where c is a constant. Let ϕ be an infinitely differentiable function in Ω which vanishes in a neighborhood of the edges such that

$$\int_\Omega \phi(x)\, dx = 1.$$

The matrix

$$G(x,\xi) = \big(G_{i,j}(x,\xi)\big)_{i,j=1}^4$$

is called *Green's matrix* for the problem (1), (2) if the vector functions $\vec{G}_j = (G_{1,j}, G_{2,j}, G_{3,j})^t$ and the function $G_{4,j}$ are solutions of the problem

$$\begin{aligned}
&-\Delta_x \vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)\,(\delta_{1,j},\delta_{2,j},\delta_{3,j})^t \quad \text{ for } x,\xi\in\Omega,\\
&-\nabla_x\cdot\vec{G}_j(x,\xi) = \big(\delta(x-\xi) - \phi(x)\big)\,\delta_{4,j} \quad \text{ for } x,\xi\in\Omega,\\
&\vec{G}_j(x,\xi) = 0 \ \text{ for } x\in\partial\Omega,\ \xi\in\Omega
\end{aligned}$$

and $G_{4,j}$ satisfies the condition

$$\int_\Omega G_{4,j}(x,\xi)\,\phi(x)\,dx = 0 \text{ for } \xi\in\Omega,\ j=1,2,3,4.$$

As was shown in [5] (see also [6, Theorem 4.5]), there exists a uniquely determined Green matrix $G(x,\xi)$ such that the vector functions $x\to\zeta(x,\xi)\,\big(\vec{G}_j(x,\xi), G_{4,j}(x,\xi)\big)$ belong to the space $\overset{\circ}{W}{}^{1,2}(\Omega)^3\times L_2(\Omega)$ for each $\xi\in\Omega$ and for every infinitely differentiable function $\zeta(\cdot,\xi)$ equal to zero in a neighborhood of the point $x=\xi$. Note that

$$G_{i,j}(x,\xi) = G_{j,i}(\xi,x) \quad \text{ for } x,\xi\in\Omega,\ i,j=1,2,3,4. \tag{6}$$

Remark 1 It is also possible (and perhaps even more natural) to define the columns $(\vec{G}_j, G_{4,j})$ of the Green matrix as the unique solutions of the problem

$$\begin{aligned}
&-\Delta_x \vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)\,(\delta_{1,j},\delta_{2,j},\delta_{3,j})^t \quad \text{ for } x,\xi\in\Omega,\\
&-\nabla_x\cdot\vec{G}_j(x,\xi) = \big(\delta(x-\xi) - (\mathrm{mes}(\Omega))^{-1}\big)\,\delta_{4,j} \quad \text{ for } x,\xi\in\Omega,\\
&\vec{G}_j(x,\xi) = 0 \ \text{ for } x\in\partial\Omega,\ \xi\in\Omega, \quad \int_\Omega G_{4,j}(x,\xi)\,dx = 0 \text{ for } \xi\in\Omega,
\end{aligned}$$

$j=1,2,3,4$. However, then the derivatives of the vector $\vec{G}_4(\cdot,\xi)$ with respect to the variable x cannot be continuous on the edges for any $\xi\in\Omega$. The reason is that from the boundary condition $\vec{G}_4(\cdot,\xi) = 0$ on $\partial\Omega$ it follows that $\nabla_x\cdot\vec{G}_4(\cdot,\xi) = 0$ on the edges M_k. This contradicts the equation $\nabla_x\cdot\vec{G}_4(\cdot,\xi)|_{M_k} = (\mathrm{mes}(\Omega))^{-1}$ which follows from the Stokes system. In particular, then the functions $G_{1,4}, G_{2,4}, G_{3,4}$ cannot satisfy the Hölder estimate (4) for $|\alpha|=1$, $\beta=0$.

3 Point estimates for the elements of Green's matrix

For every $\nu = 1, \dots, d$, let I_ν denote the set of all indices k such that the vertex $x^{(\nu)}$ is an endpoint of the edge M_k. Furthermore, let $\mathcal{U}_\nu$ and $\mathcal{V}_\nu$ be convex neighborhoods of the vertex $x^{(\nu)}$. We assume that

$$\mathcal{U}_\nu \subset \mathcal{V}_\nu \quad \text{and} \quad \bigcup_{\nu=1}^{d} \mathcal{U}_\nu \supset \overline{\Omega}.$$

Moreover, we suppose that there exists a positive number ε_0 such that

$$\operatorname{dist}(\mathcal{U}_\nu, \Omega\backslash\mathcal{V}_\nu) > \varepsilon_0 \quad \text{and} \quad \operatorname{dist}\Big(\mathcal{V}_\nu\,, \bigcup_{k\notin I_\nu} M_k\Big) > \varepsilon_0$$

for $\nu = 1, \dots, d$. In the sequel, Λ_ν and μ' are certain real numbers which depend on the domain Ω,

$$1 < \Lambda_\nu \le 2, \quad 1 < \mu' \le 2.$$

More precisely, we define $\mu' = \min(2, \pi\theta_1^{-1}, \dots, \pi\theta_l^{-1})$, where θ_k denotes the inner angle at the edge M_k. For every vertex $x^{(\nu)}$, we denote by λ_ν the greatest real number such that the strip $1 < \operatorname{Re}\lambda < \lambda_\nu$ is free of eigenvalues of the operator pencil $\mathfrak{A}_\nu(\lambda)$ introduced in [8, Section 3] (see also [7, Subsection 11.1.2]). Then $\Lambda_\nu = \min(\lambda_\nu, 2)$.

The distance of the point x from the vertex $x^{(\nu)}$ is denoted by $\rho_\nu(x)$, the distance from the edge M_k by $r_k(x)$. Furthermore, let

$$r(x) = \min\big(r_1(x), \dots, r_l(x)\big).$$

We will use in this paper the following estimates of Green's matrix which are proved in [7, 8]. First we consider the case, where x and ξ lie in the neighborhood $\mathcal{V}_\nu$ of the same vertex $x^{(\nu)}$.

Lemma 1 1) *Let $x, \xi \in \Omega \cap \mathcal{V}_\nu$ and $\rho_\nu(\xi) < \rho_\nu(x)/2$. Then*

$$\big|\partial_x^\alpha \partial_\xi^\beta G_{4,j}(x,\xi)\big| \le c\,\rho_\nu(x)^{-2-\Lambda_\nu-|\alpha|+\varepsilon}\,\rho_\nu(\xi)^{\Lambda_\nu-|\beta|-\varepsilon}\,\Big(\frac{r(x)}{\rho_\nu(x)}\Big)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}\,\Big(\frac{r(\xi)}{\rho_\nu(\xi)}\Big)^{\mu'-|\beta|-\varepsilon}$$

for $j \ne 4$ and

$$\big|\partial_x^\alpha G_{4,4}(x,\xi)\big| \le c\,\rho_\nu(x)^{-3-|\alpha|}\,\Big(\frac{r(x)}{\rho_\nu(x)}\Big)^{\min(0,\mu'-1-|\alpha|-\varepsilon)},$$

where ε is an arbitrarily small positive number.

2) *Let $x, \xi \in \Omega \cap \mathcal{V}_\nu$ and $\rho_\nu(\xi) > 2\rho_\nu(x)$. Then*

$$\begin{aligned}\big|\partial_x^\alpha \partial_\xi^\beta G_{4,j}(x,\xi)\big| \ \le\ & c\,\rho_\nu(\xi)^{-2-|\alpha|-|\beta|}\,\Big(\frac{\rho_\nu(x)}{\rho_\nu(\xi)}\Big)^{\min(0,\Lambda_\nu-1-|\alpha|-\varepsilon)} \\ & \times\Big(\frac{r(x)}{\rho_\nu(x)}\Big)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}\,\Big(\frac{r(\xi)}{\rho_\nu(\xi)}\Big)^{\mu'-|\beta|-\varepsilon}\end{aligned}$$

for $j \neq 4$ *and*

$$\left|\partial_x^\alpha G_{4,4}(x,\xi)\right| \le c\,\rho_\nu(\xi)^{-3-|\alpha|}\left(\frac{\rho_\nu(x)}{\rho_\nu(\xi)}\right)^{\min(0,\Lambda_\nu-1-|\alpha|-\varepsilon)}\left(\frac{r(x)}{\rho_\nu(x)}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}$$

Lemma 2 *Let* $x,\xi \in \Omega\cap\mathcal{V}_\nu$ *and* $\rho_\nu(x)/2 < \rho_\nu(\xi) < 2\rho_\nu(x)$. *If* $|x-\xi| > \min(r(x),r(\xi))$, *then*

$$\left|\partial_x^\alpha\partial_\xi^\beta G_{4,j}(x,\xi)\right| \;\le\; c\,|x-\xi|^{-2-\delta_{j,4}-|\alpha|-|\beta|}\left(\frac{r(x)}{|x-\xi|}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}$$

for $|\beta|\le 1-\delta_{j,4}$. *If* $|x-\xi| < \min(r(x),r(\xi))$, *then*

$$\left|\partial_x^\alpha\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,|x-\xi|^{-2-\delta_{j,4}-|\alpha|-|\beta|}$$

for all multi-indices α *and* β.

In the next lemma, we consider the case, where x and ξ lie in neighborhoods of different vertices. Then by [8, Theorem 4.3], the following estimates hold.

Lemma 3 *Suppose that* $\mu\neq\nu$, $x\in\Omega\cap\mathcal{U}_\mu$, $\xi\in\Omega\cap\mathcal{U}_\nu$, $\xi\notin\mathcal{V}_\mu$. *Then*

$$\left|\partial_x^\alpha\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\mu(x)^{\min(0,\Lambda_\mu-1-|\alpha|-\varepsilon)}\,\rho_\nu(\xi)^{\Lambda_\nu-|\beta|-\varepsilon}\left(\frac{r(x)}{\rho_\mu(x)}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}$$

for $j\neq 4$, $|\beta|\le 1$ *and*

$$\left|\partial_x^\alpha G_{4,4}(x,\xi)\right| \le c\,\rho_\mu(x)^{\min(0,\Lambda_\mu-1-|\alpha|-\varepsilon)}\left(\frac{r(x)}{\rho_\mu(x)}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}.$$

We also need some sharper estimates for the derivatives $\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)$, where $\rho=\rho(x)=|x-x^{(\nu)}|$. If we apply Lemmas 1–3, we obtain upper bounds for these derivatives, where the factors

$$\frac{r(x)}{\rho_\nu(x)} \quad\text{and}\quad \frac{r(x)}{|x-\xi|}$$

appear with the negative exponent $\mu'-2-\varepsilon$. Since the derivative ∂_ρ is tangent on the faces Γ_j adjacent to the vertex $x^{(\nu)}$, this exponent can be replaced by zero (cf. [6, Remark 4.2] and [7, Remark 10.4.6]). In particular, the following assertions hold.

Lemma 4 1) *Suppose that* $x,\xi\in\Omega\cap\mathcal{V}_\nu$ *and* $|\beta|\le 1-\delta_{j,4}$. *Then*

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\nu(x)^{-3-\Lambda_\nu+\varepsilon}\,\rho_\nu(\xi)^{\Lambda_\nu-|\beta|-\varepsilon} \quad \textit{for } \rho_\nu(\xi)<\rho_\nu(x)/2,\ j\neq 4,$$

$$\left|\partial_\rho G_{4,4}(x,\xi)\right| \le c\,\rho_\nu(x)^{-4} \quad \textit{for } \rho_\nu(\xi)<\rho_\nu(x)/2,$$

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\nu(x)^{\Lambda_\nu-2-\varepsilon}\,\rho_\nu(\xi)^{-1-\Lambda_\nu-\delta_{j,4}-|\beta|+\varepsilon} \quad \textit{for } \rho_\nu(\xi)>2\rho_\nu(x),$$

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,|x-\xi|^{-3-\delta_{j,4}-|\beta|} \quad \textit{for } \rho_\nu(x)/2<\rho_\nu(\xi)<2\rho_\nu(x).$$

2) *If* $\mu\neq\nu$, $x\in\Omega\cap\mathcal{U}_\mu$, $\xi\in\Omega\cap\mathcal{U}_\nu$, $\xi\notin\mathcal{V}_\mu$, *then*

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\mu(x)^{\Lambda_\mu-2-\varepsilon}$$

for $|\beta|\le 1-\delta_{j,4}$.

4 Hölder estimates

Our goal is to prove that

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,\big(|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} + |y-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}\big) \tag{7}$$

for sufficiently small $\sigma>0$, where c is a constant independent of x and ξ.

Lemma 5 *Let m be an arbitrary positive number, and let $0<\sigma<1$. Then the estimate (7) is satisfied for $|\beta|\le 1-\delta_{j,4}$, $x,y,\xi\in\Omega$, $|x-\xi|<m|x-y|$.*

Proof: By (3),

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)\big|}{|x-y|^\sigma} \le c\,\frac{|x-\xi|^{-2-\delta_{j,4}-|\beta|}}{|x-y|^\sigma} \le c\,m^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

for $|x-\xi|<m|x-y|$. Analogously,

$$\frac{\big|\partial_\xi^\beta G_{i,j}(x,\xi)\big|}{|x-y|^\sigma} \le c\,(m+1)^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

since $|y-\xi|<(m+1)|x-y|$. This proves the lemma. □

The last lemma allows us to restrict ourselves to the case $|x-y|<\delta|x-\xi|$, where δ is an arbitrary fixed positive number. We assume in the sequel that σ is a positive number satisfying the inequalities

$$\sigma<\mu'-1 \quad\text{and}\quad \sigma<\Lambda_\nu-1 \ \text{ for } \nu=1,\dots,d \tag{8}$$

and show that

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \tag{9}$$

for $x\ne y$, $|x-y|<\delta|x-\xi|$, $\beta|\le 1-\delta_{j,4}$. Here δ is a sufficiently small positive number. We may assume without loss of generality that x and y lie in the neighborhood $\mathcal{U}_1$ of the vertex $x^{(1)}$ and that $x^{(1)}$ coincides with the origin. In the subsequent three lemmas, we assume moreover that there exists an index $k\in I_1=\{j : x^{(1)}\in\overline{M}_j\}$ such that

$$r_k(x)=\min_{j\in I_1} r_j(x) \quad\text{and}\quad r_k(y)=\min_{j\in I_1} r_j(y), \tag{10}$$

Lemma 6 *Suppose that $\xi\in\Omega$ and that x,y are points in $\Omega\cap\mathcal{U}_1$ satisfying the conditions (10) and $|x-y|<\delta|x-\xi|$, where δ is a sufficiently small positive number. Furthermore, we assume that σ satisfies the inequalities (8) and that there exists a real number $t\in(0,1)$ such that $y-x^*=t(x-x^*)$, where x^* denotes the nearest point to x on the edge M_k. Then the estimate (9) is satisfied for $|\beta|\le 1-\delta_{j,4}$.*

Proof: Obviously,

$$\begin{aligned}\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big| &= \Big|\int_t^1 \frac{d}{d\tau}\partial_\xi^\beta G_{4,j}(x^*+\tau(x-x^*),\xi)\,d\tau\Big| \\ &\le r_k(x)\int_t^1 \big|(\nabla_x\partial_\xi^\beta G_{4,j})(x^*+\tau(x-x^*),\xi)\big|\,d\tau. \qquad (11)\end{aligned}$$

Since x^* is the nearest point to x on the set $\bigcup\limits_{j\in I_1} M_j$ and the polyhedron Ω is convex, there exists a positive constant c_0 such that

$$c_0\,|x| < |x^*| < \big|x^*+\tau(x-x^*)\big| < |x| \quad \text{for } 0<\tau<1.$$

If $\xi\in\mathcal{V}_1$, $|\xi|<|x|/2$, $j\neq 4$ and $|\beta|\le 1$, then (11) together with Lemma 1 yields

$$\begin{aligned}&\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big| \\ &\quad\le c\,r_k(x)\,|x|^{-1-\Lambda_1-\mu'+2\varepsilon}\,|\xi|^{\Lambda_1-|\beta|-\varepsilon}\int_t^1\big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\,d\tau \\ &\quad\le c'\,r_k(x)^{\mu'-1-\varepsilon}\,|x|^{-1-\mu'-|\beta|+\varepsilon}\,(1-t^{\mu'-1-\varepsilon}).\end{aligned}$$

and analogously

$$\big|G_{4,4}(x,\xi)-G_{4,4}(y,\xi)\big| \le c\,r_k(x)^{\mu'-1-\varepsilon}\,|x|^{-2-\mu'+\varepsilon}\,(1-t^{\mu'-1-\varepsilon}).$$

Suppose that $0<\sigma\le\mu'-1-\varepsilon$. Then

$$\frac{1-t^{\mu'-1-\varepsilon}}{(1-t)^\sigma} \le \frac{1-t^{\mu'-1-\varepsilon}}{(1-t)^{\mu'-1-\varepsilon}} \le 1 \qquad (12)$$

and, consequently,

$$\begin{aligned}&\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} = \frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|(1-t)r_k(x)|^\sigma} \\ &\quad\le c\,r_k(x)^{\mu'-1-\varepsilon-\sigma}\,|x|^{-1-\delta_{j,4}-\mu'-|\beta|+\varepsilon} \le c\,|x|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}\end{aligned}$$

for $\xi\in\mathcal{V}_1$ and $|\xi|<|x|/2$, $j\neq 4$, $|\beta|\le 1-\delta_{j,4}$.

Suppose now that $\xi\in\mathcal{V}_1$ and $|\xi|>2|x|$. Then (11) and Lemma 1 imply

$$\begin{aligned}&\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big| \\ &\quad\le c\,r_k(x)\,|\xi|^{-3-\delta_{j,4}-|\beta|}\Big(\frac{|x|}{|\xi|}\Big)^{\Lambda_1-2-\varepsilon}|x|^{2-\mu'+\varepsilon}\int_t^1\big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\,d\tau \\ &\quad\le c'\,r_k(x)^{\mu'-1-\varepsilon}\,|x|^{\Lambda_1-\mu'}\,|\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}\,(1-t^{\mu'-1-\varepsilon})\end{aligned}$$

for $|\beta| \le 1-\delta_{j,4}$. If $0<\sigma<\min(\mu'-1-\varepsilon, \Lambda_1-1-\varepsilon)$, then it follows from the last inequality and (12) that

$$\begin{aligned}\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} &= \frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|(1-t)r_k(x)|^\sigma} \\ &\le c\, r_k(x)^{\mu'-1-\varepsilon-\sigma}\,|x|^{\Lambda_1-\mu'}|\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon} \\ &\le c'\,|\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\,2^{2+\sigma+\delta_{j,4}+|\beta|}\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}\end{aligned}$$

for $|\beta|\le 1-\delta_{j,4}$.

We consider the case $\xi\in\mathcal{V}_1$, $\rho_\nu(x)/2<\rho_\nu(\xi)<2\rho_\nu(x)$. If $|x-\xi|>\min(r_k(x),r_k(\xi))$, then by (11) and Lemma 2,

$$\begin{aligned}&\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big| \\ &\qquad\le c\, r_k(x)\,|x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon}\int_t^1 \big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\,d\tau \\ &\qquad\le c'\, r_k(x)^{\mu'-1-\varepsilon}\,|x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon}\,(1-t^{\mu'-1-\varepsilon})\end{aligned}$$

for $|\beta|\le 1-\delta_{j,4}$. Thus for $0<\sigma\le\mu'-1-\varepsilon$, the estimate

$$\begin{aligned}\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} &\le c\, r_k(x)^{\mu'-1-\varepsilon-\sigma}\,|x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon} \\ \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}&\end{aligned}$$

holds. If $\rho_\nu(x)/2<\rho_\nu(\xi)<2\rho_\nu(x)$ and $|x-\xi|<\min(r_k(x),r_k(\xi))$, then

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le |x-y|^{1-\sigma}\,\big|(\nabla_x\partial_\xi^\beta G_{4,j}(P,\xi)\big|,$$

where P is a point on the line from x to y. Therefore, by Lemma 2

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c|x-y|^{1-\sigma}\,|P-\xi|^{-3-\delta_{j,4}-|\beta|} \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}.$$

Finally, we consider the case $\xi\notin\mathcal{V}_1$. In this case, we have $|x-\xi|>\varepsilon_0$, where ε_0 is the positive number introduced in Section 3. Furthermore, (11) and Lemma 3 imply

$$\begin{aligned}\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big| &\le c\, r_k(x)\,|x|^{\Lambda_1-\mu'}\int_t^1 \big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\,d\tau \\ \le c'\, r_k(x)^{\mu'-1-\varepsilon}\,|x|^{\Lambda_1-\mu'}\,(1-t^{\mu'-1-\varepsilon})&\end{aligned}$$

for $|\beta|\le 1-\delta_{j,4}$. If $\sigma\le\min(\mu'-1-\varepsilon,\Lambda_1-1-\varepsilon)$, we conclude that

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, r_k(x)^{\mu'-1-\varepsilon-\sigma}\,|x|^{\Lambda_1-\mu'} \le c'\,|x|^{\Lambda_1-1-\varepsilon-\sigma} \le C$$

for $|\beta| \le 1 - \delta_{j,4}$. The proof of the lemma is complete. $\Box$

Next, we prove the estimate (9) for the case, where x and y lie in a plane perpendicular to the edge M_k and have the same distance from M_k.

Lemma 7 *Suppose that $x, y \in \Omega \cap \mathcal{U}_1$ and that $x^* \in M_k$ is the nearest point on the set $\bigcup\limits_{j\in I_1} M_j$ both to x and y. Furthermore, we assume that $r_k(x) = r_k(y)$ and $|x-y| < \delta|x-\xi|$, where δ is a sufficiently small positive number. Then the inequality (9) holds for $|\beta| \le 1-\delta_{j,4}$. Here σ is an arbitrary positive number satisfying (8).*

Proof: 1) Suppose first that $\xi \in \mathcal{V}_1$ and $\rho_1(\xi) < \rho_1(x)/2$. Then by Lemma 1,

$$\begin{aligned} &\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le \big|\nabla_x \partial_\xi^\beta G_{4,j}(P,\xi)\big|\ |x-y| \\ &\le c\,|x-y|\,\rho_1(P)^{-3-\Lambda_1+\varepsilon}\,\rho_1(\xi)^{\Lambda_1-|\beta|-\varepsilon}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-2-\varepsilon}\end{aligned}$$

for $j \ne 4$, $|\beta| \le 1$, where P is a point on the straight line between x and y. From the inequality $|x-P| < |x-y| < \delta|x-\xi| < 3\delta\,\rho_1(x)/2$ it follows that

$$(2-3\delta)\,\rho_1(x) < 2\rho_1(P) < (2+3\delta)\,\rho_1(x).$$

Furthermore,

$$r_k(P) \ge (2\tan(\theta_k/2))^{-1}\,|x-y|, \tag{13}$$

where θ_k denotes the angle at the edge M_k. Thus,

$$\begin{aligned} &\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,\rho_1(x)^{-1-\Lambda_1-\mu'+2\varepsilon}\,\rho_1(\xi)^{\Lambda_1-|\beta|-\varepsilon} \\ &\le c'\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,\rho_1(x)^{-1-\mu'-|\beta|+\varepsilon}\end{aligned}$$

for $j \ne 4$, $|\beta| \le 1$. Setting $\varepsilon = \mu' - 1 - \sigma$ and using the inequality $3\rho_1(x) > 2|x-\xi|$, we obtain (9). Analogously, we obtain

$$\frac{\big|G_{4,4}(x,\xi) - G_{4,4}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,\rho_1(x)^{-2-\mu'+\varepsilon} \le c'\,|x-\xi|^{-3-\sigma}$$

for $\varepsilon = \mu' - 1 - \sigma$.

2) We consider the case $\xi \in \mathcal{V}_1$, $\rho_1(x)/2 < \rho_1(\xi) < 2\rho_1(x)$. There exists a point P on the line between x and y such that

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le \big|\nabla_x \partial_\xi^\beta G_{4,j}(P,\xi)\big|\ |x-y|$$

If $|P-\xi| > \min(r_k(P), r_k(\xi))$, then Lemma 2 implies

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le c\,|x-y|\,|P-\xi|^{-3-\delta_{j,4}-|\beta|}\Big(\frac{r_k(P)}{|P-\xi|}\Big)^{\mu'-2-\varepsilon}$$

for $|\beta| \le 1-\delta_{j,4}$. Using the inequalities (13) and $|P-\xi| > |x-\xi| - |x-P| > (1-\delta)\,|x-\xi|$, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,|x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon}.$$

For $\varepsilon = \mu'-1-\sigma$, the inequality (9) holds. Analogously,

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{1-\sigma}\,|P-\xi|^{-3-\delta_{j,4}-|\beta|} \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

for $|P-\xi| < \min(r_k(P), r_k(\xi))$.

3) Suppose that $\xi \in \mathcal{V}_1$ and $\rho_1(\xi) > 2\rho_1(x)$. Then, for an arbitrary point P on the line between x and y, we have

$$\rho_1(P) < \rho_1(x) + |x-y| < (1+\delta)\rho_1(x) + \delta\rho_1(\xi) < \frac{1+3\delta}{2}\,\rho_1(\xi). \tag{14}$$

Thus by Lemma 1,

$$\begin{aligned}\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| &\le c\,|x-y|\,\rho_1(P)^{\Lambda_1-2-\varepsilon}\,\rho_1(\xi)^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-2-\varepsilon}\\ &= c\,|x-y|^\sigma\,\rho_1(P)^{\Lambda_1-1-\varepsilon-\sigma}\,\rho_1(\xi)^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}\Big(\frac{|x-y|}{r_k(P)}\Big)^{1-\sigma}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-1-\varepsilon-\sigma}\end{aligned}$$

for $|\beta| \le 1-\delta_{j,4}$. Using the inequalities (13), (14) and $r_k(P) < \rho_1(P)$, we obtain

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le c\,|x-y|^\sigma\,\rho_1(\xi)^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\,|x-y|^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

if $\varepsilon < \min(\Lambda_1-1-\sigma, \mu'-1-\varepsilon)$.

4) Finally, we consider the case $\xi \in \Omega\backslash\mathcal{V}_1$. Then by Lemma 3,

$$\begin{aligned}\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| &\le c\,|x-y|\,\rho_1(P)^{\Lambda_1-2-\varepsilon}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-2-\varepsilon}\\ &= c\,|x-y|^\sigma\,\rho_1(P)^{\Lambda_1-1-\varepsilon-\sigma}\Big(\frac{|x-y|}{r_k(P)}\Big)^{1-\sigma}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-1-\varepsilon-\sigma},\end{aligned}$$

where again P is a point on the line from x to y. Since all factors on the right-hand side have an upper bound independent of x, y and ξ if $\varepsilon < \min(\Lambda_1-1-\sigma, \mu'-1-\varepsilon)$, we get

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le c\,|x-y|^\sigma \le c'\,|x-y|^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}.$$

The proof of the lemma is complete. □

In the next lemma, we assume that x and y lie on the same ray starting from the vertex $x^{(1)}$.

Lemma 8 *Suppose that x, y are points in $\Omega \cap \mathcal{U}_1$ satisfying the condition* (10). *If $x^{(1)}$ is the origin, $y = sx$ and $|x-y| < \delta|x-\xi|$, then* (9) *holds for $|\beta| \le 1-\delta_{j,4}$, where c is independent of x and ξ.*

Proof: Suppose first that $\xi \in \Omega \cap \mathcal{V}_1$. If $|\xi| < 2|x|$, then

$$|(s-1)x| = |x-y| < \delta\, |x-\xi| < 3\delta |x|$$

and, consequently, $|s-1| < 3\delta$. Let $\rho = |x|$. Using the equality $x \cdot \nabla_x = \rho \partial_\rho$, we obtain

$$\begin{aligned}
&\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} = \frac{1}{|(s-1)x|^\sigma} \Big| \int_s^1 \frac{d}{d\tau}\, \partial_\xi^\beta G_{4,j}(\tau x,\xi)\, d\tau \Big| \\
&= \frac{1}{|(s-1)x|^\sigma} \Big| \int_s^1 x \cdot \big(\nabla_x\, \partial_\xi^\beta G_{4,j}\big)(\tau x,\xi)\, d\tau \Big| \\
&= \frac{1}{|(s-1)x|^\sigma} \Big| \int_s^1 \tau^{-1}\, (\rho\partial_\rho\, \partial_\xi^\beta G_{4,j})(\tau x,\xi)\, d\tau \Big|.
\end{aligned} \tag{15}$$

We consider the case $|\xi| < |x|/2$. Then Lemma 4 yields

$$\begin{aligned}
&\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} = |s-1|^{1-\sigma}\, |x|^{-\sigma}\, \tau^{-1}\, (\rho\partial_\rho\, \partial_\xi^\beta G_{4,j})(\tau x,\xi) \\
&\le c\, |s-1|^{1-\sigma}\, |x|^{-\sigma}\, \tau^{-1}\, |\tau x|^{-2-\Lambda_1+\varepsilon}\, |\xi|^{\Lambda_1-|\beta|-\varepsilon}
\end{aligned}$$

for $j \ne 4$, $|\beta| \le 1$, where τ is a real number between s and 1. Since $|\tau-1| < |s-1| < 3\delta$ and $|x-\xi| < 3|x|/2$, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, |x|^{-2-\sigma-|\beta|} \le c'\, |x-\xi|^{-2-\sigma-|\beta|}$$

for $j \ne 4$, $|\beta| \le 1$. Furthermore,

$$\frac{\big|G_{4,4}(x,\xi) - G_{4,4}(y,\xi)\big|}{|x-y|^\sigma} \le c\, |s-1|^{1-\sigma}\, |x|^{-\sigma}\, \tau^{-1}\, |\tau x|^{-3} \le c'\, |x|^{-3-\sigma} \le c''\, |x-\xi|^{-3-\sigma}.$$

Analogously, in the case $\xi \in \mathcal{V}_1$, $|x|/2 < |\xi| < 2|x|$, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, |(s-1)x|^{1-\sigma}\, |\tau x - \xi|^{-3-\delta_{j,4}-|\beta|},$$

where again τ is a number between s and 1. Using the inequalities $|(s-1)x| = |x-y| < \delta|x-\xi|$ and

$$|\tau x - \xi| \ge |x-\xi| - |(\tau-1)x| \ge |x-\xi| - |(s-1)x| > (1-\delta)\, |x-\xi|,$$

we get (9). Now let $\xi \in \Omega \cap \mathcal{V}_1$, $|\xi| > 2|x|$. Then, by Lemma 4, (12) and (15), we have

$$\begin{aligned}\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} &\le \frac{c}{|(s-1)x|^\sigma} \Big| \int_s^1 \tau^{-1}\, |\tau x|^{\Lambda_1 - 1 - \varepsilon}\, |\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}\, d\tau \Big| \\ &\le c' \, \frac{|s^{\Lambda_1-1-\varepsilon} - 1|}{|s-1|^\sigma}\, |x|^{\Lambda_1-1-\varepsilon-\sigma}\, |\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}.\end{aligned}$$

Setting $\varepsilon = \Lambda_1 - 1 - \sigma$ and using the inequality $|s^\sigma - 1| \le |s-1|^\sigma$ for $s > 0$, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, |\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\, |x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}.$$

It remains to consider the case $\xi \in \Omega \backslash \mathcal{V}_1$. Then Lemma 4, (12) and (15) imply

$$\begin{aligned}&\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le \frac{c}{|(s-1)x|^\sigma} \Big| \int_s^1 \tau^{-1}\, |\tau x|^{\Lambda_1-1-\varepsilon}\, d\tau \Big| \\ &\le c'\, |x|^{\Lambda_1-1-\varepsilon-\sigma}\, \frac{|s^{\Lambda_1-1-\varepsilon} - 1|}{|s-1|^\sigma}.\end{aligned}$$

Setting $\varepsilon = \Lambda_1 - 1 - \sigma$ and using the inequality $|x - \xi| > \varepsilon_0$, we obtain (9). The proof of the lemma is complete. □

Now we can prove the main result of the paper

Theorem 9 *Suppose that σ is a positive number satisfying the condition (8). Then the elements $G_{4,j}(x,\xi)$ of Green's matrix satisfy the inequality (7) for $|\beta| \le 1 - \delta_{j,4}$.*

Proof: For $|x-\xi| < m|x-y|$ the inequality (7) is already shown (see Lemma 5). We consider the case $|x-y| < \delta|x-\xi|$, where δ is a given sufficiently small positive number. Since then $|x-y| < \delta\, \mathrm{diam}(\Omega)$, we may assume in this case that x and y lie in the neighborhood $\mathcal{U}_1$ of the same vertex $x^{(1)}$ and that this vertex coincides with the origin. Let I_1 be the set of all indices j such that $x^{(1)}$ is an endpoint of the edge M_j. Suppose first that that there exists an index $k \in I_1$ such that

$$r_k(x) = \min_{j \in I_1} r_j(x) \quad \text{and} \quad r_k(y) = \min_{j\in I_1} r_j(y). \tag{16}$$

By x^* and y^* we denote the nearest points to x and y on the edge M_k. Without loss of generality, we may assume that

$$\frac{r_k(x)}{|x|} > \frac{r_k(y)}{|y|}. \tag{17}$$

We define

$$s = \Big(\frac{|x|^2 - r_k(x)^2}{|y|^2 - r_k(y)^2} \Big)^{1/2}, \quad t = s\, \frac{r_k(y)}{r_k(x)} \quad \text{and} \quad z = x^* + t(x - x^*).$$

Then x^* is also the nearest point to sy on M_k. From (17) it follows that $t < 1$. Furthermore, there exists a constant c_0 depending only on the domain Ω such that

$$|x - sy| < c_0 \, |x - y|. \tag{18}$$

To see this, we consider the line ℓ_y through the origin and the point y. Then $|x - y| > \text{dist}(x, \ell_y)$, while $|x - sy|$ is the distance of x from the intersection of ℓ_y with the plane perpendicular to M_k through the point x^*. Since M_k is the nearest edge to y and Ω is convex, the angle between ℓ_y and the last plane is greater than a certain angle $\alpha_0 > 0$. Thus, $|x - sy| < (\sin \alpha_0)^{-1} \, \text{dist}(x, \ell_y)$ which proves (18). Since sy and z have the same distance $t\, r_k(x) = s\, r_k(y)$ from the point x^* and z lies on the straight line from x to x^*, it follows that

$$|x - z| \le |x - sy| < c_0 \, |x - y| \quad \text{and} \quad |z - sy| < 2c_0 \, |x - y|.$$

Moreover,

$$|y - sy| \le |x - sy| + |x - y| < (c_0 + 1) \, |x - y|.$$

We assume in the following that $c_0\delta$ is sufficiently small. Applying Lemma 6, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(z,\xi)\big|}{|x-y|^\sigma} \le c \, \frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(z,\xi)\big|}{|x-z|^\sigma} \le c' \, |x - \xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

for $|\beta| \le 1 - \delta_{j,4}$. Analogously, Lemmas 7 and 8 imply

$$\begin{aligned} \frac{\big|\partial_\xi^\beta G_{4,j}(z,\xi) - \partial_\xi^\beta G_{4,j}(sy,\xi)\big|}{|x-y|^\sigma} &\le c\, |z - \xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c' \, |x - \xi|^{-2-\sigma-\delta_{j,4}-|\beta|}, \\ \frac{\big|\partial_\xi^\beta G_{4,j}(sy,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} &\le c\, |y - \xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c' \, |x - \xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \end{aligned}$$

for $|\beta| \le 1 - \delta_{j,4}$. This proves the inequality (9) for the case where the nearest points to x and y on the set $\bigcup_{j \in I_1} M_j$ lie on the same edge M_k. If

$$r_k(x) = \min_{j \in I_1} r_j(x) \quad \text{and} \quad r_l(y) = \min_{j \in I_1} r_j(y), \tag{19}$$

where $k, l \in I_1$ and $k \ne l$, then one can find a set of points $z_1, \dots, z_k$ on the straight line from $x = z_1$ to $y = z_k$, where for every pair $(i, i+1)$ there exits an index $n(i) \in I_1$ such that

$$r_{n(i)}(z_i) = \min_{j \in I_1} r_j(z_i) \quad \text{and} \quad r_{n(i)}(z_{i+1}) = \min_{j \in I_1} r_j(z_{i+1}) \quad \text{for } i = 1, \dots, k-1.$$

Obviously $(1-\delta)|x-\xi| < |z_i - \xi| < (1+\delta)|x - \xi|$ if $|x - y| < \delta |x - \xi|$. Thus, the inequalities

$$\frac{\big|\partial_\xi^\beta G_{4,j}(z_i,\xi) - \partial_\xi^\beta G_{4,j}(z_{i+1},\xi)\big|}{|z_i - z_{i+1}|^\sigma} \le c\, |z_i - \xi|^{-2-\sigma-\delta_{j,4}-|\beta|}, \quad i = 1, \dots, k-1,$$

imply (9). The proof of the theorem is complete. $\square$

Using the analogous result for the elements $G_{i,j}(x,\xi)$, $i \ne 4$, in [8], we conclude that the estimate (4) is valid for $i, j = 1, 2, 3, 4$, $|\alpha| \le 1 - \delta_{i,4}$, $|\beta| \le 1 - \delta_{j,4}$. The estimate (5) can be deduced directly from (4) and (6).

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Systems of Schrödinger Equations in the Whole Space

ABSTRACT. We present in this paper results for the sign of the weak solutions of some elliptic systems defined in $\mathbb{R}^N$ involving Schrödinger operators with indefinite weight functions and with potentials which tend to infinity at infinity.

KEY WORDS. Schrödinger operators, indefinite weight, principal eigenvalue, positivity and negativity, maximum and antimaximum principles, existence of solutions

1 Introduction

1.1 The problem settings

We study the elliptic system:

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + g_i(x, u_1, \dots, u_n) \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n, \tag{1.1}$$

for $i = 1, \dots, n$. We consider the following hypothesis for each $i = 1, \dots, n$:

($\mathbf{H_q^1}$) $q_i \in L^2_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N)$, $p > \frac{N}{2}$, such that $\lim_{|x|\to\infty} q_i(x) = \infty$ and $q_i \geq cst > 0$.

We will later specify the form and the hypotheses on each weight m_i and on each function g_i and we denote by μ_i real parameters for $i = 1, \dots, n$. The variational space is denoted by $V_{q_1}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N)$, where for each $i = 1, \dots, n$, $V_{q_i}(\mathbb{R}^N)$ is the completion of $D(\mathbb{R}^N)$, the set of $\mathcal{C}^\infty$ functions with compact supports, with respect to the norm

$$\|u\|^2_{q_i} = \int_{\mathbb{R}^N} [|\nabla u|^2 + q_i u^2]. \tag{1.2}$$

We recall that the embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.

The aim of this paper is to study the sign of the solutions of (1.1). This extends earlier results already obtained for the Laplacian operator in a bounded domain (see [16, 18]), for equations or systems involving Schrödinger operators $-\Delta + q_i$ in $\mathbb{R}^N$ with positive weights (see [9–11]).

Our paper is organized as follows: In section 1.2 we recall some results for the scalar case, for the existence of principal eigenvalues in the case of indefinite weights. We also recall extensions of the maximum and antimaximum principles called ground state positivity and negativity (see [3, 4]). We study systems of the form (1.1) in Section 2. In Section 2.1 we give results for the maximum principle in the case of cooperative systems (2.1) by considering the positive principal eigenvalue and the negative principal eigenvalue of each operator $-\Delta + q_i$ associated with the indefinite weight m_i. Note that our results are more restrictive than those usually obtained when the weights m_i are positive (see [11, 16, 18]). In Section 2.2, first we give a result concerning the existence (and also Courant-Fischer formula) of a global positive eigenvalue $\Lambda_{1,M}$ for the cooperative system (2.8). Note that we can compare $\Lambda_{1,M}$ to each principal eigenvalue of $-\Delta + q_i$ associated with m_i. Then we obtain a maximum principal result for (2.8). Finally, in Section 2.3, for the two-by-two system (2.17), we present some results for the sign of the solutions. We decouple the system (2.17) in order to apply the results of the ground state positivity or negativity for each equation. Note that even if our conditions are restrictive, there are few results for the antimaximum principle for such systems (see [2]). Besides note that, to our knowledge, even the antimaximum principle, for the operator $-\Delta + q$ associated with an indefinite weight function m defined in the whole space, is not achieved yet (whereas it is well known for the Laplacian operator $-\Delta$ on a bounded domain in the case of an indefinite weight function, see [20], and for the Schrödinger operator $-\Delta + q$ in $\mathbb{R}^N$ but without any weight, see [3, 4]). In Appendix A, we give a brief recall of the proof of the antimaximum principle for the scalar case in the case of a positive and bounded weight m.

1.2 Review of results for the scalar case

1.2.1 The Schrödinger operator

We begin this section studying the Schrödinger operator $-\Delta + q$ associated with the weight m. We will assume throughout the paper that q is a potential which satisfies $(\mathbf{H_q^1})$. The weight m will assume one of the following hypotheses:

$(\mathbf{H_m^1})$ There exist two positive reals α and β such that $0 < \alpha \le m \le \beta$ in $\mathbb{R}^N$.

$(\mathbf{H_m^{*1}})$ $0 < m \le cst$ in $\mathbb{R}^N$.

$(\mathbf{H_m^2})$ $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ $(N \ge 3)$, $m \ge 0$, $meas\{x \in \mathbb{R}^N, m(x) > 0\} \ne 0$.

$(\mathbf{H_m'^1})$ $m \in L^\infty(\mathbb{R}^N)$, m is positive in an open subset $\Omega_m^+ = \{x \in \mathbb{R}^N, m(x) > 0\}$ with non zero measure and m is negative in an open subset $\Omega_m^- = \{x \in \mathbb{R}^N, m(x) < 0\}$ with non zero measure.

$(\mathbf{H_m'^2})$ $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ $(N \geq 3)$, $meas(\Omega_m^+) > 0$, $meas(\Omega_m^-) > 0$.

For a positive weight m, we have:

Theorem 1.1 (cf. [12, Theorems 2.1,2.2]) *Assume that q satisfies $(\mathbf{H_q^1})$ and m satisfies $(\mathbf{H_m^1})$ or $(\mathbf{H_m^{*1}})$ or $(\mathbf{H_m^2})$. Then there exists a unique principal eigenvalue $\lambda_{1,q,m}$ which is simple and associated with a positive eigenfunction $\phi_{1,q,m}$ and:*

$$(-\Delta + q)\phi_{1,q,m} = \lambda_{1,q,m}\ m\ \phi_{1,q,m} \ \ in\ \mathbb{R}^N; \quad \lambda_{1,q,m} > 0; \quad \phi_{1,q,m} > 0. \tag{1.3}$$

$$\lambda_{1,q,m} = \inf\{\frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q(\mathbb{R}^N)\ \ s.\ t.\ \int_{\mathbb{R}^N} m\phi^2 > 0\}. \tag{1.4}$$

For a weight m which changes sign in $\mathbb{R}^N$, we have:

Theorem 1.2 (cf. [12, Theorem 3.1]) *Assume that q satisfies $(\mathbf{H_q^1})$ and m satisfies $(\mathbf{H_m'^1})$ or $(\mathbf{H_m'^2})$. Then the operator $-\Delta + q$ associated with the weight m has a unique positive principal eigenvalue $\lambda_{1,q,m}$ associated with a positive eigenfunction $\phi_{1,q,m}$ and $(\lambda_{1,q,m}, \phi_{1,q,m})$ satisfy (1.3) and (1.4). Moreover the operator $-\Delta + q$ associated with the weight m has a unique negative principal eigenvalue $\tilde{\lambda}_{1,q,m}$ associated with a positive eigenfunction $\tilde{\phi}_{1,q,m}$ and $(\tilde{\lambda}_{1,q,m}, \tilde{\phi}_{1,q,m})$ satisfy*

$$(-\Delta + q)\tilde{\phi}_{1,q,m} = \tilde{\lambda}_{1,q,m}\ m\ \tilde{\phi}_{1,q,m} \ \ in\ \mathbb{R}^N; \quad \tilde{\lambda}_{1,q,m} < 0; \quad \tilde{\phi}_{1,q,m} > 0. \tag{1.5}$$

$$\tilde{\lambda}_{1,q,m} = \sup\{\frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q(\mathbb{R}^N)\ \ s.\ t.\ \int_{\mathbb{R}^N} m\phi^2 < 0\}. \tag{1.6}$$

We have: $\tilde{\lambda}_{1,q,m} = -\lambda_{1,q,-m}$.

1.2.2 Maximum principle for the scalar case

We consider the following equation in a variational sense

$$(-\Delta + q)u = \mu m u + f \text{ in } \mathbb{R}^N \tag{1.7}$$

where μ is a real parameter and $f \in L^2(\mathbb{R}^N)$. First we recall the classical weak maximum principle for (1.7) in the case of a positive weight m.

Theorem 1.3 (cf. [12, Theorem 2.3]) *Assume that q satisfies $(\mathbf{H_q^1})$, m satisfies $(\mathbf{H_m^1})$ or $(\mathbf{H_m^{*1}})$ or $(\mathbf{H_m^2})$, $f \geq 0$ and u is a solution of the equation (1.7). If $\mu < \lambda_{1,q,m}$, then $u \geq 0$.*

Now we consider the equation (1.7) in the case of an indefinite weight m.

Theorem 1.4 (cf. [12, Theorem 3.2] *Assume that q satisfies* $(\mathbf{H_q^1})$, *m satisfies* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$, *$\mu \in \mathbb{R}$, $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ and u is a solution of the equation* (1.7). *If $\tilde{\lambda}_{1,q,m} < \mu < \lambda_{1,q,m}$, then $u \geq 0$.*

1.2.3 Ground state positivity or negativity for the scalar case

We recall here a result of ground state positivity or negativity for the Schrödinger operator $-\Delta + q$ associated with a strictly positive and bounded weight m in $\mathbb{R}^N$ (see [1]). We will add in this section the following hypothesis upon the potentiel q.

$$(\mathbf{H_q^2}) \begin{cases} \text{(i)} & q \text{ is radially symmetric.} \\ \text{(ii)} & \text{There exists a constant } c_1 > 0 \text{ and a positive real } R_0 \text{ such that} \\ & c_1 Q(r) \leq q(r) \text{ for } R_0 \leq r \text{ with } Q \text{ an auxiliary function which satisfies} \\ & Q \text{ is positive and locally absolutely continuous }, Q'(r) \geq 0, \\ & \int_{R_0}^{+\infty} Q(r)^{-\beta} dr < +\infty \text{ with } 0 < \beta < \frac{1}{2}. \end{cases}$$

Definition 1.1 i) *A function $u \in L^2(\mathbb{R}^N)$ satisfies the ground state positivity if there exists a constant $c > 0$ such that $u \geq c\phi_{1q,m}$ almost everywhere in $\mathbb{R}^N$.*

ii) *A function $u \in L^2(\mathbb{R}^N)$ satisfies the ground state negativity if there exists a constant $c > 0$ such that $u \leq -c\phi_{1q,m}$ almost everywhere in $\mathbb{R}^N$.*

These notions are similar to the maximum and antimaximum principles in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, which have been established by [13], [22], [23] (for a function $f \in L^p(\Omega)$, $p > N$). But for the Schrödinger operator defined in the whole space, the hypothesis $f \in L^p(\Omega)$, $p > N$, is no longer sufficient and we need to take a smaller space for f, namely, a stronger ordered Banach space introduced in [4]

$$X_{q,m} = \{u \in L^2(\mathbb{R}^N), \frac{u}{\phi_{1,q,m}} \in L^\infty(\mathbb{R}^N)\}$$

endowed with the ordered norm $\|u\|_{X_{q,m}} = \inf\{C \in \mathbb{R}, |u| \leq C\phi_{1,q,m} \text{ a. e. in } \mathbb{R}^N\}$. We denote by S^{N-1} the unit sphere in $\mathbb{R}^N$ centered at the origin and by σ the surface measure on S^{N-1}. For any $s > 0$, we introduce the Banach space $X_{q,m}^{s,2}$ of all functions $f \in L^2_{loc}(\mathbb{R}^N)$ having the following properties:

$$[(-\Delta_S)^{s/2} f](r, .) \in L^2(S^{N-1}) \text{ for all } r > 0,$$

where Δ_S denotes the Laplace-Beltrami operator on the sphere S^{N-1}, and there is a constant $C \geq 0$ such that

$$\frac{1}{\sigma(S^{N-1})} (\int_{S^{N-1}} |f(r, x')|^2 d\sigma(x') + \int_{S^{N-1}} |[(-\Delta_S)^{s/2} f](r, x')|^2 d\sigma(x')) \leq [C\phi_{1,q,m}(r)]^2$$

for almost every $r > 0$. The smallest such constant C defined the norm $\|f\|_{X^{s,2}_{q,m}}$ in $X^{s,2}_{q,m}$. Notice that, for $f(x) = f(|x|)$, we have $f \in X^{s,2}_{q,m}$ if and only if $f \in X_{q,m}$ together with the norms $\|f\|_{X^{s,2}_{q,m}} = \|f\|_{X_{q,m}}$. We recall from [1] the following result (which extends, for a Schrödinger equation with weight, former results in [4]):

Theorem 1.5 (see [1] Theorem 2.1) *Assume that the potential q is radially symmetric and satisfies* $(\mathbf{H^1_q})$, $(\mathbf{H^2_q})$ *and the weight m satisfies* $(\mathbf{H^1_m})$. *Assume that $u \in D(-\Delta + q)$ is one solution of* (1.7), *$\mu \in \mathbb{R}$, $f \geq 0$ a.e. in $\mathbb{R}^N$ with $f > 0$ in some set of positive Lebesgue measure.*

(i) *For every $\mu \in (-\infty, \lambda_{1,q,m})$, there exists a constant $C(f, \mu) > 0$ such that: $u \geq C(f, \mu)\phi_{1,q,m}$ in $\mathbb{R}^N$. Moreover, if the weight m is radially symmetric and if $f \in X^{s,2}_{q,m}$, then there exists a positive number $\delta(f)$ (depending upon f) such that, for every $\mu \in (\lambda_{1,q,m} - \delta(f), \lambda_{1,q,m})$, $C(f, \mu) = \frac{\int_{\mathbb{R}^N} f\phi_{1,q,m}}{\lambda_{1,q,m}-\mu} + \Gamma(\mu, f)$ with $\lim_{\mu \to \lambda_{1,q,m}} \Gamma(\mu, f) = \Gamma < +\infty$. And furthermore, if $f \in X_{q,m}$, then there exists a constant $C'(\mu, f, m) > 0$ such that:*

$$C(f, \mu)\phi_{1,q,m} \leq u \leq \frac{C'(\mu, f, m)}{\lambda_{1,q,m} - \mu}\phi_{1,q,m} \text{ in } \mathbb{R}^N.$$

(ii) *Assume that the weight m is radially symmetric and that $f \in X^{s,2}_{q,m}$. Then there exists a positive number $\delta'(f)$ (depending upon f) such that, for every $\mu \in (\lambda_{1,q,m}, \lambda_{1,q,m} + \delta'(f))$, $u \leq -C''(f, \mu)\phi_{1,q,m}$ in $\mathbb{R}^N$ with $C''(f, \mu) = \frac{\int_{\mathbb{R}^N} f\phi_{1,q,m}}{\mu-\lambda_{1,q,m}} - \Gamma'(\mu, f)$ and with $\lim_{\mu \to \lambda_{1,q,m}} \Gamma'(\mu, f) = \Gamma' < +\infty$.*

For the proof, see Appendix A.

As for the case of a positive weight, we can obtain a result on ground state positivity but not on ground state negativity (because our proof for the antimaximum principle in Theorem 1.5 (ii) needs to consider a weight m such that $\|u\|_m = \sqrt{\int_{\mathbb{R}^N} mu^2}$ defines a norm in $L^2(\mathbb{R}^N)$ equivalent to the usual norm).

Theorem 1.6 *Assume that the potential q is radially symmetric and satisfies* $(\mathbf{H^1_q})$, $(\mathbf{H^2_q})$ *and the weight m satisfies* $(\mathbf{H'^1_m})$ *or* $(\mathbf{H'^2_m})$. *Furthermore if m satisfies* $(\mathbf{H'^2_m})$, *assume also that $m^+ \in L^\infty(\mathbb{R}^N)$ and that $|m(x)| \leq cstQ(|x|)^{1/2-\beta}$ for all $x \in \mathbb{R}^N$ (with Q the auxiliary function associated with q which satisfies* $(\mathbf{H^2_q})$*). Assume that $u \in D(-\Delta + q)$ is one solution of* (1.7), *$\mu \in \mathbb{R}$, $f \geq 0$ a.e. in $\mathbb{R}^N$ with $f > 0$ in some set of positive Lebesgue measure. Then for every μ such that $\tilde{\lambda}_{1,q,m} < \mu < \lambda_{1,q,m}$, there exists a constant $C(f, \mu) > 0$ such that: $u \geq C(f, \mu)\phi_{1,q,m}$ in $\mathbb{R}^N$.*

Proof: Assume that $\tilde{\lambda}_{1,q,m} < \mu < \lambda_{1,q,m}$ and $(-\Delta + q)u = \mu mu + f$ in $\mathbb{R}^N$. Note that $u \geq 0$ by the maximum principle (Theorem 1.4). Let $\alpha > 0$ be a positive real such that

$\alpha+(\mu-\lambda_{1,q,m})m>0$ in $\mathbb{R}^N$ (which is possible for α sufficiently large since either m is bounded (case $(\mathbf{H'^1_m})$) or $m^+\in L^\infty(\mathbb{R}^N)$ (case $(\mathbf{H'^2_m})$). Therefore u satisfies $(-\Delta+q-\lambda_{1,q,m}m)u=-\alpha u+g$ in $\mathbb{R}^N$ with $g=(\alpha+(\mu-\lambda_{1,q,m})m)u+f\geq 0$ in $\mathbb{R}^N$. Moreover 0 is the principal eigenvalue of the operator $-\Delta+q-\lambda_{1,q,m}m$ in $\mathbb{R}^N$. Thus, since $-\alpha<0$ we can apply the Theorem 2.1 in [4] to obtain that $u\geq C\phi_{1,q,m}$ with C a positive constant which only depends of μ and f. $\square$

2 Results for systems

2.1 Results for linear systems

In this section, we consider (1.1) in the form:

$$(-\Delta+q_i)u_i=\mu_i m_i u_i+\sum_{j=1;j\neq i}^{n}a_{ij}u_j+f_i \text{ in } \mathbb{R}^N, \quad i=1,\dots,n \tag{2.1}$$

where each of the potentials q_i satisfy $(\mathbf{H^1_q})$ and each of the weights m_i satisfy one of the hypotheses among $(\mathbf{H^1_m})$, $(\mathbf{H^{*1}_m})$, $(\mathbf{H'^1_m})$, $(\mathbf{H^2_m})$, $(\mathbf{H'^2_m})$. We consider the hypotheses:

(H3) For all $i,j=1,\cdots,n,\ a_{ij}\in L^\infty(\mathbb{R}^N)$ and $a_{ij}\geq 0$ if $i\neq j$.

(H4) For all $i=1,\cdots,n,\ f_i\in L^2(\mathbb{R}^N)$.

(H5) For all $i,j=1,\cdots,n,\ i\neq j$, there exists a positive constant K_{ij} such that $a_{ij}\leq K_{ij}\sqrt{|m_i m_j|}$.

Note that if each of the weights m_i satisfy $(\mathbf{H^1_m})$, then **(H5)** is automatically satisfied. Note also that in the particular case where $m_i=1$ for each i, we can take $K_{ij}=\|a_{ij}\|_{L^\infty(\mathbb{R}^N)}$. We denote by

$$\lambda_i:=\lambda_{1,q_i,m_i} \text{ and } \phi_i:=\phi_{1,q_i,m_i} \tag{2.2}$$

the eigenpair for the operator $-\Delta+q_i$ associated with the weight m_i in $\mathbb{R}^N$. We denote by $L=(l_{ij})$ and $P=(p_{ij})$ the $n\times n$-matrices given as follows

$$l_{ii}:=\lambda_i-\mu_i \text{ and } l_{ij}=-K_{ij}\ (i\neq j) \tag{2.3}$$

$$p_{ii}:=1-|\mu_i|C_i\|m_i\| \text{ and } p_{ij}=-K_{ij}\sqrt{C_iC_j\|m_i\|\|m_j\|}\ (i\neq j) \tag{2.4}$$

where $\|m_i\|$ denotes either $\|m_i\|_{L^\infty(\mathbb{R}^N)}$ if m_i satisfies $(\mathbf{H'^1_m})$ or $\|m_i\|_{L^{N/2}(\mathbb{R}^N)}$ if m_i satisfies $(\mathbf{H'^2_m})$ and where $C_i=\max(1,\frac{1}{\inf q_i})\tilde{C}_0$ with either $\tilde{C}_0=1$ if m_i satisfies $(\mathbf{H'^1_m})$ or $\tilde{C}_0$ is the square of the Sobolev constant for the embedding of $H^1(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ if m_i satisfies $(\mathbf{H_m'^2})$. Note that (see (1.2))

$$\int_{\mathbb{R}^N} m_i u_i^2 \le C_i \|m_i\| \, \|u_i\|_{q_i}^2 \text{ for all } u_i \in V_{q_i}(\mathbb{R}^N). \tag{2.5}$$

For positive weights m_i, we recall the maximum principle (see [11, Theorem 2.1] in the case of weights m_i which satisfy $(\mathbf{H_m^1})$).

Theorem 2.1 *Assume that each of the potentials q_i satisfy $(\mathbf{H_q^1})$ and each of the weights m_i satisfy $(\mathbf{H_m^{*1}})$ or $(\mathbf{H_m^2})$. Assume also that* **(H3)**-**(H5)** *are satisfied and that the matrix L, defined by (2.3), is a non singular M-matrix.*

(i) *Then the cooperative system (2.1) satisfies the maximum principle (i. e. for any $f = (f_1, \cdots, f_n) \ge 0$, then $u_i \ge 0$ for all i, with $u = (u_1, \cdots, u_n)$ solution of (2.1)).*

(ii) *Assume here that each of the weights m_i satisfy $(\mathbf{H_m^{*1}})$. Then the cooperative system (2.1) satisfies the ground state positivity (i. e. for any $f = (f_1, \cdots, f_n) \ge 0$, $f_i \ne 0$ then there exists a positive constant C such that $u_i \ge C\phi_i$ for all i, with ϕ_i defined by (2.2)).*

Proof:

(i) Assume that for all $i = 1, \cdots, n$, $f_i \ge 0$. Let $u = (u_1, \cdots, u_n)$ be a solution of the system (2.1) and define $u_i^- = \max(0, -u_i)$. Multiplying by u_i^- and integrating over $\mathbb{R}^N$, using **(H5)** we get:

$$0 \le \|u_i^-\|_{q_i}^2 \le \mu_i \int_{\mathbb{R}^N} m_i (u_i^-)^2 + \sum_{j=1; j\ne i}^{n} K_{ij} \left(\int_{\mathbb{R}^N} m_i (u_i^-)^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} m_j (u_j^-)^2 \right)^{1/2}. \tag{2.6}$$

Let X the vector be defined by ${}^tX = (x_1, \cdots, x_n)$ with $x_i = \left(\int_{\mathbb{R}^N} m_i (u_i^-)^2\right)^{1/2}$. From the characterization of λ_i and from (2.6), we have:

$$(\lambda_i - \mu_i) \int_{\mathbb{R}^N} m_i (u_i^-)^2 - \sum_{j=1; j\ne i}^{n} K_{ij} \left(\int_{\mathbb{R}^N} m_i (u_i^-)^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} m_j (u_j^-)^2 \right)^{1/2} \le 0. \tag{2.7}$$

We denote by $(LX)_i = (\lambda_i - \mu_i) x_i - \sum_{j=1; j\ne i}^{n} K_{ij} x_j$. From (2.7) note that $(LX)_i \le 0$ for each i and so $LX \le 0$. Since L is a non singular M-matrix (see [6]), we can deduce that $X \le 0$ and thus $X = 0$, i. e. $x_i = 0$ for each i. So from (2.6) we get for each i : $\|u_i^-\|_{q_i} = 0$ i. e. $u_i \ge 0$.

(ii) We combine the maximum principle for the system (2.1) with the ground sate positivity for an equation. Indeed, from (i) we know that $u_i \ge 0$ for all i and so $g_i := \sum_{j=1; j\ne i}^{n} a_{ij} u_j + f_i \ge 0$, $g_i > 0$ in a set of non zero measure. Therefore, since $\mu_i < \lambda_i$, we get that there exists a positive constant C_i such that $u_i \ge C_i \phi_i$.

□

Proceeding as for Theorem 2.1, we obtain the following maximum principle for indefinite weights.

Theorem 2.2 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and each of the weights m_i satisfy* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$. *Assume also that* **(H3)**-**(H5)** *are satisfied.*

(i) *If the matrix P, defined by (2.4), is a non singular M-matrix, then the cooperative system (2.1) satisfies the maximum principle.*

(ii) *Assume also that, in the case of each of the weights m_i satisfy* $(\mathbf{H_m'^2})$, $m_i^+ \in L^\infty(\mathbb{R}^N)$ *and* $|m_i(x)| \leq cstQ_i(|x|)^{1/2-\beta}$ *for* $x \in \mathbb{R}^N$ *(with Q_i the auxiliary function associated with the potential q_i which satisfies* $(\mathbf{H_q^2})$*). If the matrix P is a non singular M-matrix, then the cooperative system (2.1) satisfies the ground state positivity.*

Note that, as for one equation, the condition "P is a non singular M-matrix" is a stronger hypothesis than the condition "L is a non singular M-matrix." Indeed, note that the hypothesis $1 - |\mu_i|C_i\|m_i\| > 0$ is stronger than the hypothesis $\tilde{\lambda}_{1,q_i,m_i} < \mu_i < \lambda_{1,q_i,m_i}$ (see (1.3)-(1.6),(2.5)).

For positive weights, we now recall the following result for the existence of solutions for the system (2.1) (see [11, Theorem 2.2 and Theorem 2.3] in the case of weights m_i which satisfy $(\mathbf{H_m^1})$).

Theorem 2.3 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and each of the weights m_i satisfy* $(\mathbf{H_m^{*1}})$ *or* $(\mathbf{H_m^2})$. *Assume also that* **(H3)**-**(H5)** *are satisfied. If the matrix L is a non singular M-matrix, then the system (2.1) has a unique solution $u = (u_1, \cdots, u_n) \in V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$.*

For indefinite weights m_i, existence and uniqueness of a solution is stated as follows and is an application of the Lax-Milgram Theorem (see [11]).

Theorem 2.4 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and each of the weights m_i satisfy* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$. *Assume also that* **(H3)**-**(H5)** *are satisfied. If the matrix P is a non singular M-matrix, then the system (2.1) has a unique solution $u = (u_1, \cdots, u_n) \in V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$.*

2.2 Existence of a global principal eigenvalue for a system

In this section, we consider the eigenvalue problem for the following system

$$(-\Delta + q_i)u_i = \lambda \left(m_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij} u_j \right) \text{ in } \mathbb{R}^N, \; i = 1, \cdots, n, \tag{2.8}$$

where each of the potentials q_i satisfy $(\mathbf{H^1_q})$ and each of the weights m_i satisfy one of the hypotheses among $(\mathbf{H^1_m})$, $(\mathbf{H^{*1}_m})$, $(\mathbf{H'^1_m})$. We denote by M is the $n \times n$-matrix given by $M = (m_{ij})$ with $m_{ii} := m_i$. We will consider the following hypotheses:

(H8) For all $i \neq j$, $m_{ij} \in L^\infty(\mathbb{R}^N)$ and $m_{ij} > 0$.

(H9) M is a symmetric matrix.

(H10) $\Omega := \cap_{i=1}^n \Omega_i^+$ is an open subset of $\mathbb{R}^N$ with non zero measure and with $\Omega_i^+ := \{x \in \mathbb{R}^N, m_i(x) > 0\}$.

We add another hypothesis upon the potentials q_i which assures that any weak solution $u_i \in V_{q_i}(\mathbb{R}^N)$ of the equation $(-\Delta + q_i)u_i = f_i$ in $\mathbb{R}^N$, with $f_i \in L^2(\mathbb{R}^N)$, belongs to the strong domain $D(-\Delta + q_i) \subset L^2(\mathbb{R}^N)$. It is the following hypothesis. For all $i = 1, \cdots, n$,

$(\mathbf{H^3_q})$ For all $x \in \mathbb{R}^N$ and all $h \in \mathbb{R}^N$, $h \neq 0$, $|\frac{q_i(x+h)-q_i(x)}{h}| \leq cst\ \sqrt{q_i(x)}$.

Note that for example, the potential $q(x) = 1 + |x|$ satisfies $(\mathbf{H^3_q})$.

Lemma 2.1 *Assume that the potential q satisfy $(\mathbf{H^1_q})$ and $(\mathbf{H^3_q})$. Let u be a weak solution of $(-\Delta + q)u = f$ in $\mathbb{R}^N$ with $f \in L^2(\mathbb{R}^N)$. Then $u \in H^2(\mathbb{R}^N)$, $qu \in L^2(\mathbb{R}^N)$ and therefore $u \in D(-\Delta + q)$.*

The proof of Lemma 2.1 is based on the methods of translations (see Appendix B). For strictly positive and bounded weights m_i, proceeding as for one equation (see [12, Theorem 2.1]), we can prove the existence of a positive eigenvalue associated with a positive eigenfunction for (2.8). Therefore, we extend here to Schrödinger operators defined in the whole space, some results of [21] and [8] for elliptic operators defined in a bounded domain.

Theorem 2.5 *Assume that each of the potentials q_i satisfy $(\mathbf{H^1_q})$ and each of the weights m_i satisfy $(\mathbf{H^{*1}_m})$. Assume also that (H8) is satisfied. Then there exists a unique principal eigenvalue $\Lambda_{1,M} > 0$ associated with a positive eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M}) \in V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ for the system (2.8). Moreover if (H9) and $(\mathbf{H^3_q})$ are satisfied then*

$$\Lambda_{1,M} = \inf \left\{ \frac{\sum_{i=1}^n \|u_i\|_{q_i}^2}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j},\ u = (u_1, \cdots, u_n) \in V \right.$$

$$\left. \textit{such that } \sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j > 0 \right\}. \tag{2.9}$$

Note that the condition $\sum_{i=1}^{n}\int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^{n}\int_{\mathbb{R}^N} m_{ij}u_iu_j > 0$ is automatically satisfied if M is a definite positive matrix (i. e. for all $X \neq 0$, ${}^tXMX > 0$).

Proof: We denote by M the operator of multiplication by the matrix M in $(L^2(\mathbb{R}^N))^n$ and we consider the operator

$$L^{-1}M : ((L^2(\mathbb{R}^N))^n, \|.\|_{(L^2(\mathbb{R}^N))^n}) \to ((L^2(\mathbb{R}^N))^n, \|.\|_{(L^2(\mathbb{R}^N))^n}).$$

The operator $L^{-1}M$ is compact and strongly positive in the sense of quasi-interior points in $(L^2(\mathbb{R}^N))^n$, in the sense of Daners and Koch-Medina [15]. This implies that $L^{-1}M$ is irreducible and we apply the version of the Krein-Rutman Theorem given in [15, Theorem 12.3] to deduce that $r(L^{-1}M)$, the spectral radius of $L^{-1}M$, is a strictly positive and simple eigenvalue associated with an eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M})$ which is a quasi-interior point of $(L^2(\mathbb{R}^N))^n$, that is $\phi_{i,M} > 0$ in $\mathbb{R}^N$ for all i. Of course $\Lambda_{1,M} = \frac{1}{r(L^{-1}M)} > 0$ and $r(L^{-1}M)$ is the only one eigenvalue of $L^{-1}M$ associated with a positive eigenfunction.

We recall that $V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ and the inner product in V is defined by $< u, v >_V = \sum_{i=1}^{n} < u_i, v_i >_{q_i}$ for all $u = (u_1, \cdots, u_n) \in V$ and $v = (v_1, \cdots, v_n) \in V$. We set the bilinear form

$$\beta(u, v) = \sum_{i=1}^{n}\int_{\mathbb{R}^N} m_i u_i v_i + \sum_{i,j=1;i\neq j}^{n}\int_{\mathbb{R}^N} m_{ij}u_jv_i \text{ for all } u \in V \text{ and } v \in V.$$

From hypotheses **(H8)** and **(H9)**, β is a bilinear, symmetric and continous form. From the Riesz Theorem, we get the existence of a continuous operator $T : V \to V$, $T = (T_1, \cdots, T_n)$, such that $\beta(u, v) =< Tu, v >_V$ for all $u \in V$ and $v \in V$ (see [17] for the Lax-Milgram Theorem). We can easily prove that the operator T is compact.

Moreover, since the matrix M is assumed to be symmetric, the operator T is selfadjoint. So the largest eigenvalue of T is given by:

$$\mu_{1,M} = \sup_{u\in V, u\neq 0}\frac{< Tu, u >_V}{< u, u >_V} = \sup_{u\in V, u\neq 0}\frac{\sum_{i=1}^{n}\int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j=1;i\neq j}^{n}\int_{\mathbb{R}^N} m_{ij}u_ju_i}{\sum_{i=1}^{n}\int_{\mathbb{R}^N}[|\nabla u_i|^2 + q_iu_i^2]}.$$

Choosing $u = (u_1, \cdots, u_n) \in V$ such that *supp* $u_i \subset \{x \in \mathbb{R}^N, m_i(x) > 0\}$ for one i and $u_j = 0$ if $j \neq i$, we get that $\mu_{1,M} > 0$.

Now, we prove that $\Lambda_{1,M} = \frac{1}{\mu_{1,M}}$. We have $L^{-1}M\Phi_{1,M} = r(L^{-1}M)\Phi_{1,M}$ or equivalently $L\Phi_{1,M} = \Lambda_{1,M}M\Phi_{1,M}$. Therefore for all $i = 1, \cdots, n$:

$$(-\Delta + q_i)\phi_{i,M} = \Lambda_{1,M}(m_i\phi_{i,M} + \sum_{j=1;j\neq i} m_{ij}\phi_{j,M}) \text{ in } \mathbb{R}^N.$$

Thus for all $v = (v_1, \cdots, v_n) \in V$, we have:

$$\sum_{i=1}^{n}\int_{\mathbb{R}^N}[\nabla\phi_{i,M}.\nabla v_i + q_i\phi_{i,M}v_i] = \Lambda_{1,M}\sum_{i=1}^{n}\left(\int_{\mathbb{R}^N} m_i\phi_{i,M}v_i + \sum_{j=1;j\neq i}^{n}\int_{\mathbb{R}^n} m_{ij}\phi_{j,M}v_i\right).$$

For $v_i = \phi_{i,M}$, we get:

$$\frac{1}{\Lambda_{1,M}} = \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i \phi_{i,M}^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\phi_{j,M}\phi_{i,M}}{\sum_{i=1}^n \int_{\mathbb{R}^N}[|\nabla \phi_{i,M}|^2 + q_i \phi_{i,M}^2]} \leq \mu_{1,M}. \tag{2.10}$$

Moreover, since $\mu_{1,M}$ is an eigenvalue of the operator T defined above, let $\psi = (\psi_1, \cdots, \psi_n)$ be an eigenfunction associated with $\mu_{1,M}$. Since $T\psi = \mu_{1,M}\psi$, we have for all $v \in V$: $\mu_{1,M} < \psi, v >_V = < T\psi, v >_V = \beta(\psi, v)$ and so

$$\mu_{1,M} \sum_{i=1}^n \int_{\mathbb{R}^N} [\nabla \psi_i . \nabla v_i + q_i \psi_i v_i] = \sum_{i=1}^n \int_{\mathbb{R}^N} m_i \psi_i v_i + \sum_{i,j=1;j\neq i}^n \int_{\mathbb{R}^n} m_{ij}\psi_j v_i.$$

For $v = (0, \cdots, 0, v_i, 0, \cdots, 0) \in V$, we get:

$$\int_{\mathbb{R}^N} [\nabla \psi_i . \nabla v_i + q_i \psi_i v_i] = \frac{1}{\mu_{1,M}} \left(\int_{\mathbb{R}^N} m_i \psi_i v_i + \sum_{j=1;j\neq i}^n \int_{\mathbb{R}^n} m_{ij}\psi_j v_i \right).$$

Therefore, using Lemma 2.1, we have $L\psi = \frac{1}{\mu_{1,M}} M\psi$ or equivalently $L^{-1}M\psi = \mu_{1,M}\psi$. Thus $\mu_{1,M}$ is an eigenvalue of the operator $L^{-1}M$ and

$$0 < \mu_{1,M} \leq r(L^{-1}M) = \frac{1}{\Lambda_{1,M}}. \tag{2.11}$$

From (2.10) and (2.11), we deduce that $\mu_{1,M} = \frac{1}{\Lambda_{1,M}}$ and $\Lambda_{1,M}$ satisfies (2.9). □

Now, for indefinite bounded weights m_i, proceeding as for one equation (see [12, Theorem 3.1]), we prove the existence and the uniqueness of a principal positive eigenvalue for (2.8). This is the following result.

Theorem 2.6 *Assume that each of the potentials q_i satisfy* **($\mathbf{H_q^1}$)** *and* **($\mathbf{H_q^3}$)** *and each of the weights m_i satisfy* **($\mathbf{H_m'^1}$)**. *Assume also that* **(H8)**-**(H10)** *are satisfied. Then there exists a unique principal eigenvalue $\Lambda_{1,M} > 0$ associated with a positive eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M}) \in V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$, $\phi_{i,M} > 0$ and $\Lambda_{1,M}$ satisfies* (2.9).

Proof: We follow a method developed in [19] (for one equation in a bounded domain). Let $\Omega_i^+ = \{x \in \mathbb{R}^N, m_i(x) > 0\}$, *meas* $(\Omega_i^+) > 0$, $\Omega_i^- = \{x \in \mathbb{R}^N, m_i(x) < 0\}$, *meas* $(\Omega_i^-) > 0$, and $\Omega_i^0 = \{x \in \mathbb{R}^N, m_i(x) = 0\}$. Let $(u_1, \cdots, u_n)$ be a solution of (2.8). We have for all i :

$$(-\Delta + q_i)u_i + \lambda m_i^- u_i = \lambda(m_i^+ u_i + \sum_{j=1;j\neq i}^n m_{ij}u_j) \text{ in } \mathbb{R}^N. \tag{2.12}$$

For given $\lambda > 0$, we rewrite (2.12) as an eigenvalue problem with parameter $\sigma(\lambda)$. For all i,

$$(-\Delta + q_i)u_i + \lambda(m_i^- + 1_i)u_i = \sigma(\lambda) \left((m_i^+ + 1_i)u_i + \sum_{j=1;j\neq i}^n m_{ij}u_j \right) \text{ in } \mathbb{R}^N \tag{2.13}$$

where 1_i denotes the characteristic function of $\Omega_i^0 \cup \Omega_i^-$. We denote by $Q_i := q_i + \lambda(m_i^- + 1_i)$ and $\rho_i := m_i^+ + 1_i$. Then (2.13) is equivalent to

$$(-\Delta + Q_i)u_i = \sigma(\lambda)(\rho_i u_i + \sum_{j=1; j\neq i}^{n} m_{ij}u_j) \text{ in } \mathbb{R}^N. \tag{2.14}$$

Note that the weight $\rho_i > 0$ in $\mathbb{R}^N$, $\rho_i \in L^\infty(\mathbb{R}^N)$ since $m_i \in L^\infty(\mathbb{R}^N)$ and Q_i satisfies $(\mathbf{H_q^1})$ since $\lambda > 0$. From Theorem 2.5, we deduce that the system (2.14) has a unique principal eigenvalue $\sigma(\lambda)$ associated with a principal eigenfunction $\Phi_\lambda = (\phi_{1,\lambda}, \cdots, \phi_{n,\lambda})$, $\phi_{i,\lambda} > 0$. Moreover, since $D(-\Delta + Q_i) = D(-\Delta) \cap D(Q_i)$, from (2.9), we get:

$$\sigma(\lambda) = \inf\left\{ \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\psi_i|^2 + q_i\psi_i^2] + \lambda\sum_{i=1}^n \int_{\mathbb{R}^N}(m_i^- + 1_i)\psi_i^2}{\sum_{i=1}^n \int_{\mathbb{R}^N}(m_i^+ + 1_i)\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j}, \psi = (\psi_1, \cdots, \psi_n) \in V \right.$$
$$\left. \text{such that } \sum_{i=1}^n \int_{\mathbb{R}^N}(m_i^+ + 1_i)\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j > 0 \right\}. \tag{2.15}$$

Therefore, $\sigma(\lambda) < \Lambda_{1,Q,N}^+$ the principal eigenvalue of the operator L_Q associated with the matrix $N = (n_{ij})$ where $L_Q = diag\ (-\Delta + Q_i)$, $n_{ii} = \rho_i$ and $n_{ij} = m_{ij}$ in $\Omega = \cap_{i=1}^n \Omega_i^+$ with Dirichlet boundary condition. Note that $\sigma : \lambda \mapsto \sigma(\lambda)$ is increasing and continuous and that $\sigma(0) > 0$. Therefore for all $\lambda > 0$, we have $0 < \sigma(0) < \sigma(\lambda) < \Lambda_{1,Q,N}^+$ and $\Lambda_{1,Q,N}^+$ is in fact independant of λ. Thus we deduce that there exists $0 < \tilde{\lambda} < \Lambda_{1,Q,N}^+$ such that $\sigma(\tilde{\lambda}) = \tilde{\lambda}$. Proceeding as in [19], we can show that $\tilde{\lambda}$ is unique.
Now, we verify that $\tilde{\lambda}$ satisfies (2.9). Let us denote by

$$\Lambda_{1,M} = \inf\left\{ \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\psi_i|^2 + q_i\psi_i^2]}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j}, \psi = (\psi_1, \cdots, \psi_n) \in V \right.$$
$$\left. \text{such that } \sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j > 0 \right\}.$$

Since

$$\tilde{\lambda} = \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\phi_{i,\tilde{\lambda}}|^2 + q_i\phi_{i,\tilde{\lambda}}^2] + \tilde{\lambda}\sum_{i=1}^n \int_{\mathbb{R}^N}(m_i^- + 1_i)\phi_{i,\tilde{\lambda}}^2}{\sum_{i=1}^n \int_{\mathbb{R}^N}(m_i^+ + 1_i)\phi_{i,\tilde{\lambda}}^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\phi_{i,\tilde{\lambda}}\phi_{j,\tilde{\lambda}}},$$

we have $\tilde{\lambda} \geq \Lambda_{1,M}$.
Moreover let $\psi = (\psi_1, \cdots, \psi_n) \in V$ be such that $\sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j > 0$. From (2.15), since $\tilde{\lambda} = \sigma(\tilde{\lambda})$, we get $\tilde{\lambda} \leq \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\psi_i|^2 + q_i\psi_i^2]}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j}$. Thus $\tilde{\lambda} \leq \Lambda_{1,M}$. $\square$

Note that for all $i = 1, \cdots, n$, $\Lambda_{1,M} < \lambda_i$.
Indeed, from (1.4) and (2.9), we have $\Lambda_{1,M} \leq \lambda_i$. Suppose that $\Lambda_{1,M} = \lambda_i$. Then

$$(-\Delta + q_i)(\phi_{i,M} - \phi_i) = \lambda_i m_i(\phi_{i,M} - \phi_i) + \lambda_i \sum_{j=1; j\neq i}^{n} m_{ij}\phi_{j,M} \text{ in } \mathbb{R}^N,$$

where ϕ_i (resp. $\phi_{i,M}$) is defined by (2.2) (resp. Theorem 2.6). Multiplying by ϕ_i and integrating over $\mathbb{R}^N$, we obtain (since $\lambda_i > 0$), $\int_{\mathbb{R}^N} \sum_{j=1;j\neq i}^n m_{ij}\phi_{j,M}\phi_i = 0$. Since $m_{ij} > 0$, $\phi_{i,M} > 0$ and $\phi_i > 0$ we get a contradiction.

Now, we consider the following system

$$(-\Delta + q_i)u_i = \lambda \left(m_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij} u_j \right) + f_i \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n. \tag{2.16}$$

We give a maximum principle result.

Theorem 2.7 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and* $(\mathbf{H_q^3})$ *and each of the weights m_i satisfy* $(\mathbf{H_m^{*1}})$ *or* $(\mathbf{H_m'^1})$*. Assume also that* (**H8**)-(**H9**) *are satisfied. Furthermore if the weights m_i satisfy* $(\mathbf{H_m'^1})$*, assume also that* (**H10**) *is satisfied.Assume that $f_i \in L^2(\mathbb{R}^N)$ for all i. If $0 \leq \lambda < \Lambda_{1,M}$, then the system* (2.16) *satisfies the maximum principle: if $f = (f_1, \cdots, f_n) \geq 0$, then $u_i \geq 0$ for all i with $u = (u_1, \cdots, u_n)$ solution of* (2.16).

Note that we have the same condition $0 \leq \lambda < \Lambda_{1,M}$ as in [21, Proposition 2.2].

Proof: Multiplying (2.16) by u_i^-, integrating over $\mathbb{R}^N$, since $\lambda \geq 0$ and $f_i \geq 0$, we have:

$$0 \leq \sum_{i=1}^n \|u_i^-\|_{q_i}^2 \leq \lambda(\sum_{i=1}^n \int_{\mathbb{R}^N} m_i(u_i^-)^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}u_i^- u_j^-) := \lambda C(u^-) = \lambda C(u_1^-, \cdots, u_n^-).$$

If $C(u^-) > 0$, then $\Lambda_{1,M} \leq \frac{\sum_{i=1}^n \|u_i^-\|_{q_i}^2}{C(u^-)} \leq \lambda$ and we get a contradiction with the hypothesis $\lambda < \Lambda_{1,M}$. Thus $C(u^-) = 0$. Then $\sum_{i=1}^n \|u_i^-\|_{q_i}^2 = 0$ and therefore $u_i \geq 0$ for all i. $\square$

We can state a result for the existence of solutions for the system (2.16) as follows.

Theorem 2.8 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and* $(\mathbf{H_q^3})$ *and each of the weights m_i satisfy* $(\mathbf{H_m^{*1}})$ *or* $(\mathbf{H_m'^1})$*. Assume also that* (**H8**)-(**H9**) *are satisfied. Furthermore if the weights m_i satisfy* $(\mathbf{H_m'^1})$*, assume also that* (**H10**) *is satisfied. Assume that $f_i \in L^2(\mathbb{R}^N)$ for all i. If $0 \leq \lambda < \Lambda_{1,M}$, then the system* (2.16) *has a unique solution $u = (u_1, \cdots, u_n) \in V$.*

Proof: We introduce a bilinear continuous form l and we apply the Lax-Milgram Theorem. Let $l : (V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N))^2 \to \mathbb{R}$ be defined by

$$l(u, v) = \sum_{i=1}^n \int_{\mathbb{R}^N} [\nabla u_i.\nabla v_i + q_i u_i v_i - \lambda m_i u_i v_i - \lambda \sum_{j=1;j\neq i}^n m_{ij} u_j v_i].$$

First note that from (2.9) we have: $\Lambda_{1,M} C(u) \leq \sum_{i=1}^n \|u_i\|_{q_i}^2$ for all $u = (u_1, \cdots, u_n) \in V$. Therefore, since $\lambda \geq 0$, we get: $l(u, u) \geq \frac{\Lambda_{1,M} - \lambda}{\Lambda_{1,M}} \sum_{i=1}^n \|u_i\|_{q_i}^2$ and so l is coercive. By the Lax-Milgram Theorem, we get the existence and the uniqueness of a weak solution for the system (2.16). $\square$

2.3 Study of the signs of the solutions for a 2×2 system

We consider in this section the following system (for $N \geq 2$):

$$\begin{cases} (-\Delta + q)u = \lambda u + au + bv + f \text{ in } \mathbb{R}^N \\ (-\Delta + q)v = \lambda v + cu + dv + g \text{ in } \mathbb{R}^N. \end{cases} \tag{2.17}$$

The real λ is a real parameter and the potential q is radially symmetric and satisfies $(\mathbf{H_q^1})$ and $(\mathbf{H_q^2})$. The aim of this section is to present some results concerning positivity or negativity of the solutions of the system (2.17). We can find results for the antimaximum principle for a system of two equations with constant coefficients in [2]; the ideas, there, are to decouple the system, and then to apply the results of the antimaximum principle for each equation. We will follow this method in this section.

We denote by $M(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ the coupling matrix of the coefficients of the system (2.17). Following [14], we introduce S an invertible 2×2 matrix of constants such that S diagonalises $M(x)$ for all x. In [14], it is proved that such a choice is possible only if either (case I) $b(x)$ and $c(x)$ are both multiples of $a(x) - d(x)$ or (case II) $a(x) = d(x)$ for all x and $b(x)$ and $c(x)$ are positive multiples of each other. We define the functions u^* and v^* by

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = S^{-1} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} f^* \\ g^* \end{pmatrix} = S^{-1} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{2.18}$$

and since S is a constant matrix, we obtain from (2.17)

$$\begin{pmatrix} -\Delta + q & 0 \\ 0 & -\Delta + q \end{pmatrix} \begin{pmatrix} u^* \\ v^* \end{pmatrix} = \lambda \begin{pmatrix} u^* \\ v^* \end{pmatrix} + S^{-1} M(x) S \begin{pmatrix} u^* \\ v^* \end{pmatrix} + \begin{pmatrix} f^* \\ g^* \end{pmatrix}. \tag{2.19}$$

We suppose that the coefficients a, b, c, d of the system satisfy the following hypothesis:

$$\textbf{(H11)} \begin{cases} \text{(i)} & a, b, c, d \in L^\infty(\mathbb{R}^N). \\ \text{(ii)} & \text{either } b \text{ and } c \text{ are positive multiples of } a - d \text{ (case I)} \\ & \text{or } a = d \text{ and } b \text{ and } c \text{ are positive multiples of each other (case II)} \\ \text{(iii)} & a, b, c, d \text{ are radially symmetric functions.} \end{cases}$$

Note that the hypothesis **(H11)**(iii) upon the coefficients of the matrix M of the system (2.17) assures that the weights of each equation (after decoupling (2.17)) are radially symmetric. Here we consider the case I and we rewrite the matrix $M(x)$ under the following form:

$$M(x) = \begin{pmatrix} a(x) & b^*(a(x) - d(x)) \\ c^*(a(x) - d(x)) & d(x) \end{pmatrix} \quad \text{(case I)} \tag{2.20}$$

where $a \neq d$ and b^* and c^* are constants such that $1 + 4b^*c^* > 0$.

Moreover we assume that the following hypothesis is satisfied:

(H12) $f, g \in L^2(\mathbb{R}^N)$.

Then we define the two following constants $\rho_1 = \frac{1+\sqrt{1+4b^*c^*}}{2}$, $\rho_2 = \frac{1-\sqrt{1+4b^*c^*}}{2}$ and we choose $S = \begin{pmatrix} -b^* & -b^* \\ \rho_1 & \rho_2 \end{pmatrix}$. Thus we have $u = -b^*(u^* + v^*)$ and $v = \rho_1 u^* + \rho_2 v^*$. Now, if we define the functions

$$\mu_1(x) := \frac{1}{\rho_1 - \rho_2}[\rho_1 d(x) - \rho_2 a(x) + 2\rho_1\rho_2(a(x) - d(x))] \tag{2.21}$$

$$\mu_2(x) := \frac{1}{\rho_1 - \rho_2}[\rho_1 a(x) - \rho_2 d(x) - 2\rho_1\rho_2(a(x) - d(x))], \tag{2.22}$$

then we can write the decoupled system (see (2.17)-(2.22)) as

$$\begin{cases} (-\Delta + q)u^* = \lambda u^* + \mu_1 u^* + \frac{1}{b^*(\rho_1-\rho_2)}[\rho_2 f + b^* g] \text{ in } \mathbb{R}^N \\ (-\Delta + q)v^* = \lambda v^* + \mu_2 v^* - \frac{1}{b^*(\rho_1-\rho_2)}[\rho_1 f + b^* g] \text{ in } \mathbb{R}^N. \end{cases}$$

Theorem 2.9 *Assume that the potential q satisfies $(\mathbf{H_q^1})$-$(\mathbf{H_q^2})$ and that the hypotheses* **(H11)**-**(H12)** *are satisfied. Assume also that the matrix M has the form (2.20) with $b^*c^* < 0$ and $1 + 4b^*c^* > 0$. Let μ_1 and μ_2 functions be defined as in (2.21) and (2.22). Assume that μ_1 and μ_2 are functions such that $\lambda + \mu_1 \geq cst > 0$ and $\lambda + \mu_2 \geq cst > 0$. Define $f^* = \frac{1}{b^*(\rho_1-\rho_2)}[\rho_2 f + b^* g]$ and $g^* = -\frac{1}{b^*(\rho_1-\rho_2)}[\rho_1 f + b^* g]$.*

1. *Assume that $\lambda_{1,q,\lambda+\mu_1} - \delta(f^*) < 1 < \lambda_{1,q,\lambda+\mu_1}$, $\lambda_{1,q,\lambda+\mu_2} - \delta(g^*) < 1 < \lambda_{1,q,\lambda+\mu_2}$, $0 < f^* \in X^{s,2}_{q,\lambda+\mu_1}$ and $0 < g^* \in X^{s,2}_{q,\lambda+\mu_2}$, with $\delta(f^*)$, $\delta(g^*)$ defined in Theorem 1.5*
   Then u has the same sign as $-b^*$ and $v > 0$.

2. *Assume that $\lambda_{1,q,\lambda+\mu_1} < 1 < \lambda_{1,q,\lambda+\mu_1} + \delta'(f^*)$, $\lambda_{1,q,\lambda+\mu_2} < 1 < \lambda_{1,q,\lambda+\mu_2} + \delta'(g^*)$, $0 < f^* \in X^{s,2}_{q,\lambda+\mu_1}$ and $0 < g^* \in X^{s,2}_{q,\lambda+\mu_2}$.*
   Then $v < 0$ and u has the same sign as b^*.

Note that the above results are just consequences of the diagonalization of the coupling matrix M and applications of Theorem 1.5. We can also obtain similar results in the case II. Note that for λ sufficiently large, since each function μ_i is bounded, we have $\lambda + \mu_i \geq cst > 0$. Moreover if $b^* > 0$ e.g., choosing $g > 0$ and f such that $-\frac{b^* g}{\rho_2} < f < -\frac{b^* g}{\rho_1}$, we have $f^* > 0$ and $g^* > 0$.

A Appendix: Ground state positivity and negativity

We only give a sketch of the proof in $\mathbb{R}^2$. We recall that the space $X_{q,m}$ is defined by $X_{q,m} = \{u \in L^2(\mathbb{R}^2), \frac{u}{\phi_{1,q,m}} \in L^\infty(\mathbb{R}^2)\}$ and the space $X^{1,2}_{q,m}$ is defined by

$X^{1,2}_{q,m} = \{f : \mathbb{R}^2 \to \mathbb{R},\ \frac{\partial f}{\partial \theta}(r,.) \in L^2(-\pi,\pi)$ for all $r \geq 0$ and there exists a constant $C \geq 0$ such that $|f(r,\theta)| + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\frac{\partial f}{\partial \theta}(r,\theta)|^2\, d\theta\right)^{1/2} \leq C\phi_{1,q,m}(r)$ a.e.$\}$.

Note that the ground state positivity is a simple application of the weak maximum principle combined with [4, Theorem 2.1]. Note also that if $f \in X_{q,m}$ and $f \geq 0$ then there exists a positive constant $C(f)$ such that $0 \leq f \leq C(f)\phi_{1,q,m}$. Choosing $C'(f,m) = \frac{C(f)}{m_0(\lambda_{1,q,m}-\mu)}$ with $m_0 = \inf m > 0$, from the weak maximum principle for the scalar case, writing $(-\Delta + q)(C'(f,m)\phi_{1,q,m} - u) = \mu m(C'(f,m)\phi_{1,q,m} - u) + (\lambda_{1,q,m} - \mu)mC'(f,m)\phi_{1,q,m} - f$ in $\mathbb{R}^2$, we obtain that $u \leq C'(f,m)\phi_{1,q,m}$.

The proof of the ground state negativity is based upon ideas of [20] and [4]. We decompose it in several steps.

Step 1 : We denote by $L_q := -\Delta + q$ and by M the operator of multiplication by m. As in Hess [20] we consider the operator $L_q^{-1}M$ and the same decomposition of $L^2(\mathbb{R}^2) = span(\phi_{1,q,m}) \oplus R(I - \lambda_{1,q,m}L_q^{-1}M)$ where $R(I - \lambda_{1,q,m}L_q^{-1}M)$ is the range of the operator $I - \lambda_{1,q,m}L_q^{-1}M$. But because of the unboundedness of our domain, we cannot study $R(I - \lambda_{1,q,m}L_q^{-1}M)$ as it done in [20] and we adapt to our case an idea developed in [3] which is the following decomposition of $L^2(\mathbb{R}^2) = H_1 \oplus H_2 \oplus H_3$ with

$$\begin{aligned} H_1 =& span(\phi_{1,q,m}) \\ H_2 =& \{f \in L^2(\mathbb{R}^2) \colon f(x) \equiv f(|x|) \text{ with } \int_0^\infty m(r)f(r)\phi_{1,q,m}(r)r\,dr = 0\}; \\ H_3 =& \{f \in L^2(\mathbb{R}^2) \colon \int_{-\pi}^{\pi} f(r,\theta)\,d\theta = 0 \text{ for almost all } r \geq 0\}. \end{aligned}$$

Note that $\|.\|_m$ defined by (1.2) is a norm equivalent to the usual norm in $L^2(\mathbb{R}^2)$ since m satisfies $(\mathbf{H^1_m})$. It is obvious that $L^2(\mathbb{R}^2) = H_1 \oplus H_2 \oplus H_3$ is an orthogonal decomposition. The corresponding orthogonal projections P_1, P_2 and P_3, respectively, take the following forms, for each $f \in L^2(\mathbb{R}^2)$: $P_1 f = \frac{(f,\phi_{1,q,m})_m}{(\phi_{1,q,m},\phi_{1,q,m})_m}\,\phi_{1,q,m}$, $P_2 f = (I - P_1)f^*$ with $f^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)\,d\theta$, $P_3 f = f - f^*$.

Step 2 : Let u be a solution of (1.7), we decompose u and $L_q^{-1}f = g$ in $L^2(\mathbb{R}^2)$ under the following way: $u = \beta_\mu\phi_{1,q,m} + u_2 + u_3$ with $u_2 \in H_2$, $u_3 \in H_3$ and $g = g_1 + g_2 + g_3$. It is easy to check that: $g_1 = (I - \mu L_q^{-1}M)\beta_\mu\phi_{1,q,m}$, $g_2 = (I - \mu L_q^{-1}M)u_2$ and $g_3 = (I - \mu L_q^{-1}M)u_3$. The idea then is to show that the sign of u is given by β_μ and that u_2 and u_3 belong to $X_{q,m}$. For that we need the two following Propositions based on Propositions 3.5 and 3.6 in [3].

Proposition A.1 (see [1, Proposition 3.1]) *Assume that q is a radially symmetric potential which satisfies* $(\mathbf{H^1_q})$-$(\mathbf{H^2_q})$ *and that m is a radially symmetric weight which satisfies*

$(\mathbf{H^1_m})$. *Assume that* $u_2, g_2 \in D(L_q)$, $L_q u_2 - \lambda_{1,q,m} M u_2 = L_q g_2 \in L^2(\mathbb{R}^2)$ *with* g_2 *a radial symmetric function.*

(i) *If* $\int_{\mathbb{R}^2} L_q g_2.\phi_{1,q,m} = 0$ *and* $\int_{\mathbb{R}^2} u_2 m \phi_{1,q,m} = 0$, *then* u_2 *is radial and there exists a constant* $\Gamma > 0$ *(depending exclusively upon the potential* q *and the weight* m*) such that* $|L_q g_2| \leq c\phi_{1,q,m} \Rightarrow |u_2| \leq \Gamma c \phi_{1,q,m}$.

(ii) *If* $\int_{\mathbb{R}^2} m.L_q g_2.\phi_{1,q,m} = 0$ *and* $\int_{\mathbb{R}^2} u_2 m \phi_{1,q,m} = 0$, *then* u_2 *is radial and there exists a constant* $\Gamma > 0$ *(depending exclusively upon the potential* q *and the weight* m*) such that* $|L_q g_2| \leq c\phi_{1,q,m} \Rightarrow |u_2| \leq \Gamma c \phi_{1,q,m}$.

Proposition A.2 (see [1, Proposition 3.2]) *Assume that* q *is a radially symmetric potential which satisfies* $(\mathbf{H^1_q})$-$(\mathbf{H^2_q})$ *and that* m *is a radially symmetric weight which satisfies* $(\mathbf{H^1_m})$. *Assume that* $u_3, g_3 \in D(L_q)$, $L_q u_3 - \lambda_{1,q,m} M u_3 = L_q g_3 \in L^2(\mathbb{R}^2)$ *with* $L_q g_3 \in H_3$ *and* $u_3 \in H_3$. *If* $L_q g_3 \in X^{1,2}_{q,m}$, *then there exists a constant* $\Gamma > 0$ *(depending exclusively upon the potential* q *and the weight* m*) such that* $\|u_3\|_{X^{1,2}_{q,m}} \leq \Gamma \|L_q g_3\|_{X^{1,2}_{q,m}}$.

Step 3 : First note that if $f = L_q g = L_q g_1 + L_q g_2 + L_q g_3$ then $L_q g_1 + L_q g_2$ is obviously radially symmetric and so $L_q g_3 = P_3 f$. Note also that if $f \in X_{q,m}$ then $L_q g_1 \in X_{q,m}$, $L_q g_2 \in X_{q,m}$ and $L_q g_3 \in X_{q,m}$. Indeed $f^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)\,d\theta$ is in $X_{q,m}$ too and $L_q g_3 = P_3 f = f - f^*$ is in $X_{q,m}$. More $L_q g_1$ belongs to $X_{q,m}$ since m is bounded. Then we get $L_q g_2 \in X_{q,m}$.

Now, we study each component of the decomposition of u.
First, we calculate β_μ. Recall that $g_1 = \alpha \phi_{1,q,m}$ with the constant $\alpha = (L_q^{-1} f, \phi_{1,q,m})_m = \frac{1}{\lambda_{1,q,m}} \int_{\mathbb{R}^2} f \phi_{1,q,m}$. Since f is positive, $\alpha > 0$. Therefore, we get $\beta_\mu = \frac{\alpha \lambda_{1,q,m}}{\lambda_{1,q,m} - \mu}$.
Then, we prove that $u_2 \in X_{q,m}$. Writing down the Neumann series for the resolvant $(I - \mu L_q^{-1} M)^{-1}$:

$$u_2 = \sum_n (\mu - \lambda_{1,q,m})^n (M^{-1} L_q - \lambda_{1,q,m} I)^{-n} (I - \lambda_{1,q,m} L_q^{-1} M)^{-1} g_2.$$

Let call $g_2^0 = (I - \mu L_q^{-1} M)^{-1} g_2$ and apply Proposition A.1. Indeed $g_2 \in H_2$ and $L_q g_2$ satisfies:

$$\int_{\mathbb{R}^2} L_q g_2.\phi_{1,q,m} = \int_{\mathbb{R}^2} g_2.L_q \phi_{1,q,m} = \lambda_{1,q,m} \int_{\mathbb{R}^2} m.g_2.\phi_{1,q,m} = 0.$$

We obtain $g_2^0 \in H_2$ and $|g_2^0| \leq \Gamma c \phi_{1,q,m}$.
Then call $g_2^1 = (M^{-1} L_q - \lambda_{1,q,m} I)^{-1} g_2^0$; g_2^1 satisfies the following equation:

$$(I - \lambda_{1,q,m} L_q^{-1} M) g_2^1 = L_q^{-1} M g_2^0.$$

We check that

$$\int_{\mathbb{R}^2} m.L_q^{-1} M g_2^0.\phi_{1,q,m} = 0.$$

Applying again Proposition A.1, we get that $g_2^1 \in H_2$ and $|g_2^1| \leq \Gamma \|Mg_2^0\|_{X_{q,m}} \phi_{1,q,m}$. Using the same method at each step, we deduce that the following sequence:

$$g_2^{n+1} = (M^{-1}L_q - \lambda_{1,q,m} I)^{-1} g_2^n$$

satisfies $|g_2^{n+1}| \leq \Gamma \|Mg_2^n\|_{X_{q,m}} \phi_{1,q,m}$. Finally, we get that, if $|\mu - \lambda_{1,q,m}|$ is small enough,$u_2 \in X_{q,m}$. To conclude, we prove similarly that $u_3 \in X_{q,m}$.

We finish the proof, saying that there exists some λ_0 such that for $\lambda_{1,q,m} < \mu < \lambda_0$

$$u = \frac{\alpha \lambda_{1,q,m}}{\lambda_{1,q,m} - \mu} \phi_{1,q,m} + u_2 + u_3 \leq (\frac{\alpha \lambda_{1,q,m}}{\lambda_{1,q,m} - \mu} + C)\phi_{1,q,m}$$

where the constant C depends only on λ_0. Then the Theorem 1.5 follows immediately.

B Appendix: Proof of Lemma 2.1

We use the methods of translations (see [5], [7, p. 182]). Let u be a weak solution of $(-\Delta + q)u = f$ in $\mathbb{R}^N$.
Let $h \in \mathbb{R}^N$ and define

$$(D_h u)(x) = \frac{u(x+h) - u(x)}{|h|}.$$

Let $v = D_{-h}(D_h u)$, $v \in V_q(\mathbb{R}^N)$. From $\int_{\mathbb{R}^N} [\nabla u . \nabla v + quv] = \int_{\mathbb{R}^N} fv$, we get:

$$\int_{\mathbb{R}^N} |\nabla (D_h u)|^2 + \int_{\mathbb{R}^N} D_h(qu).(D_h u) = \int_{\mathbb{R}^N} f D_{-h}(D_h u).$$

Since $D_h(qu)(x) = q(x+h) D_h u(x) + u(x) D_h q(x)$, we get:

$$\int_{\mathbb{R}^N} |\nabla (D_h u)|^2 + \int_{\mathbb{R}^N} q(x+h) |D_h u(x)|^2 \, dx + \int_{\mathbb{R}^N} u D_h q D_h u = \int_{\mathbb{R}^N} f D_{-h}(D_h u).$$

Using $q \geq cst > 0$, we deduce that there exists a positive constant $C = C(q)$ (depending upon q) such that:

$$\int_{\mathbb{R}^N} |\nabla (D_h u)|^2 + \int_{\mathbb{R}^N} |D_h u|^2 \leq C(q) \int_{\mathbb{R}^N} |f D_{-h}(D_h u)| + \int_{\mathbb{R}^N} |u D_h(q)||D_h u|.$$

Recall from [7, Proposition IX.3] that for all $w \in H^1(\mathbb{R}^N)$, $\|D_{-h} w\|_{L^2(\mathbb{R}^N)} \leq \|\nabla w\|_{L^2(\mathbb{R}^N)}$. Thus, since for all h, $|D_h(q)| \leq cst \sqrt{q}$, we have $|u D_h(q)| \leq cst \; |u| \sqrt{q}$ and there exists a positive constant C such that

$$\|D_h u\|_{H^1(\mathbb{R}^N)} \leq C[\|f\|_{L^2(\mathbb{R}^N)} + \|u\sqrt{q}\|_{L^2(\mathbb{R}^N)}] < \infty.$$

We conclude as in [7] by for all i and for all h,

$$\|D_h \frac{\partial u}{\partial x_i}\|_{L^2(\mathbb{R}^N)} \leq C[\|f\|_{L^2(\mathbb{R}^N)} + \|u\sqrt{q}\|_{L^2(\mathbb{R}^N)}].$$

Using [7, Proposition IX.3] we get that $\frac{\partial u}{\partial x_i} \in H^1(\mathbb{R}^N)$. Therefore $u \in H^2(\mathbb{R}^N)$ and $-\Delta u \in L^2(\mathbb{R}^N)$. Moreover, we have for all $\phi \in D(\mathbb{R}^N)$, $\int_{\mathbb{R}^N}(-\Delta u + qu - f)\phi = 0$ and $-\Delta u + qu - f \in L^1_{loc}(\mathbb{R}^N)$. From [7, Lemma IV.2], we get that $-\Delta u + qu = f$ a. e. in $\mathbb{R}^N$. Thus $qu \in L^2(\mathbb{R}^N)$ and in particular we deduce that $u \in D(-\Delta + q)$.

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Comparison with ground state for solutions of non cooperative systems of Schrödinger operators on $\mathbb{R}^N$

ABSTRACT. We study the sign of solutions of a system $\mathcal{L}U = \lambda U + MU + F$, on the whole space $\mathbb{R}^N$, more precisely, we compare the components of U with the ground state solution. Here $\mathcal{L}$ is a diagonal matrix of Schrödinger operators of the form $Lu := -\Delta u + qu$, F is a vector of functions in $L^2(\mathbb{R}^N)$, and M is a matrix, not necessarily cooperative. When M is a constant matrix, we prove the existence of a real Λ playing the role of principal eigenvalue: if $|\lambda - \Lambda|$ is sufficiently small, U exists and the sign of each entry is fixed. The sign of each entry changes as λ grows and get over Λ. We study the case of a variable M for a 2×2 system.

1 Introduction

In this paper we study systems defined on the whole space $\mathbb{R}^N$ and acting on $(L^2(\mathbb{R}^N))^n$:

$$Lu_i := (-\Delta + q(x))u_i = \lambda u_i + \sum_{j=1}^{n} m_{ij}u_j + f_i, \; 1 \le i \le n \tag{1}$$

which we write:

$$\mathcal{L}U = \lambda U + MU + F, \tag{2}$$

with $U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} L & & 0 \\ & \ddots & \\ 0 & & L \end{pmatrix}$, and M is a $n \times n$ matrix with coefficients m_{ij}.

The potential $q(x)$ is assumed to be a continuous function $q\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$\inf_{\mathbb{R}^N} q > 0 \quad \text{and} \quad q(x) \to +\infty \text{ as } |x| \to \infty. \tag{3}$$

The potential is a “relatively small” perturbation of a radially symmetric potential which is assumed to be monotone increasing (in the radial variable) and growing somewhat faster than $|x|^2$ as $|x| \to \infty$.

For a unique equation

$$(-\Delta + q(x))\, u = \lambda u + f \text{ on } \mathbb{R}^N,$$

where q is a perturbation of a radially symmetric function, under the hypothesis $f \geq 0$, B. Alziary, J. Fleckinger, and P. Takáč consider the eigenvalue λ^*, associated to a function $\varphi^* > 0$. They show that for $|\lambda - \lambda^*|$ sufficiently small, if $\lambda < \lambda^*$ then $u > C\varphi^* > 0$ (fundamental positivity), and if $\lambda > \lambda^*$, and f comparable to φ^*, then $u < -C\varphi^* < 0$ (fundamental negativity).

First we are concerned with the anti-maximum principle for the system when M is a constant matrix. In the case of cooperative systems, there are several results related to the maximum principle. B. Alziary L. Cardoulis, and J. Fleckinger, obtained a maximum principle for cooperative systems, then B. Alziary, J. Fleckinger, and P. Takáč, proved a result of fundamental positivity. For the anti-maximum principle N. Besbas [10, Theorem 4.3.2, p. 40] gave a theorem on the fundamental negativity for a special cooperative problem involving a radial potential q. In the present work, we study general systems (in particular non cooperative systems are allowed) and we obtain a comparison with the ground state, for the spectral parameter λ close to the ground state energy level. In this part, we extend to a $n \times n$ system some results of fundamental positivity or negativity established by B. Alziary,J. Fleckinger and MH. Lécureux [3] for 2×2 systems.

In the second part, we tackle the case of a variable matrix M. Our result concerns 2×2 systems with M restricted to very specific forms.

Organization:

The paper is organized as follows. In Section 2, we introduce some notation. In Section 3 we recall some known results, in Section 4 we state our main results. Finally, in Section 5, we prove them.

2 Notations and hypotheses

2.1 Fundamental positivity, fundamental negativity, notation

It is established that the Schrödinger operator: $L_q \stackrel{\text{def}}{=} -\Delta + q(x)\bullet$ defined on $L^2(\mathbb{R}^N)$ with a positive continuous potential tending to $+\infty$ as $|x| \to \infty$ has a compact inverse and therefore a discrete spectrum. This holds since the variational space V_q is compactly embedded in $L^2(\mathbb{R}^N)$ (see D. E. EDMUNDS AND W. D. EVANS, [14], J. FLECKINGER,[16]) where

$$V_q(\mathbb{R}^N) \stackrel{\text{def}}{=} \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q(x)|u|^2 < \infty \right\}. \tag{1}$$

The smallest eigenvalue is simple and is given by:

$$\lambda^*(q) \ = \inf_{u \in V_q(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q(x)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} \right\}. \tag{2}$$

Eigenfunctions associated to $\lambda^*(q)$ do not change sign and $\lambda^*(q)$ is referred to as the **"principal eigenvalue"**. Denote by φ^* (or $\varphi^*(q)$) the associated eigenfunction which is positive and normalized by $\|\varphi^*\|^2_{L^2(\mathbb{R}^N)} = 1$. The function φ^* is $C^1(\mathbb{R}^N)$, and exponentially decreasing near infinity. Usually, φ^* is called the **"ground state"** or **"principal eigenfunction"**.

As in the paper of B. ALZIARY and P. TAKÁČ [8], we consider the operator $L_q \stackrel{\text{def}}{=} -\Delta + q(x)\bullet$ on a subspace X of $L^2(\mathbb{R}^N)$ defined, by

$$X \stackrel{\text{def}}{=} \{u \in L^2(\mathbb{R}^N) \colon u/\varphi^* \in L^\infty(\mathbb{R}^N)\}. \tag{3}$$

The space X equipped with the norm

$$\|u\|_X \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\mathbb{R}^N} \ (|u|/\varphi^*)$$

is a Banach space.

Notation: We note $u \stackrel{*}{\succ} 0$ and we say that $u \in X$ is fundamentally positive if there exists a real number $c > 0$ such that $u > c\varphi^*$.
Similarly we write $u \stackrel{*}{\prec} 0$ and we say that $u \in X$ is fundamentally negative if there exists a real number $c > 0$ such that $u < -c\varphi^*$.

2.2 Hypotheses on potential

Now we give the precise assumptions on the potential q, which guarantee the compactness of the resolvant $(\lambda I - L)^{-1}$. For a single equation, ALZIARY, FLECKINGER, and TAKÁČ obtain this compactness and so the fundamental positivity and negativity for different classes of potentials [6], [9]. We choose here hypotheses used in [9], but there is no problem for obtaining the same results with the class of potential used in [6].

More precisely, we introduce a class of growth for potentials:

$$\mathcal{C}_Q := \{Q \in \mathcal{C}(\mathbb{R}_+, (0,\infty)) \,/\, \exists r_0 > 0,\ Q' > 0 \, a.e. \text{ on } [r_0, \infty),\ \int_{r_0}^{\infty} Q(r)^{-1/2}\, \mathrm{d}r < \infty\}. \tag{4}$$

We assume that the potential q satisfies Hypothesis (H_q):

Hypothesis (H_q) *The potential q is positive continuous and tends to $+\infty$ as $|x| \to \infty$. Moreover, there exist two functions Q_1 and Q_2 in $\mathcal{C}_Q$ and two positive constants $C_0, r_0 \in (0,\infty)$, such that*

$$Q_1(|x|) \le q(x) \le Q_2(|x|) \le C_0\, Q_1(|x|) \quad \textit{for all } x \in \mathbb{R}^N\,, \tag{5}$$

$$\int_{r_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{r_0}^{s} \exp\Big(- \int_r^s [Q_1(t)^{1/2} + Q_2(t)^{1/2}]\,\mathrm{d}t\Big)\,\mathrm{d}r\,\mathrm{d}s < \infty\,. \tag{6}$$

In their paper, ALZIARY, and TAKÁČ ([9] Corollary 3.3) show that the ground states $\varphi^*(q)$, $\varphi^*(Q_1)$ and $\varphi^*(Q_2)$ are comparable: there exist some constants $0 < \gamma_1 \le \gamma_2 < \infty$ such that $\gamma_1\varphi^*(q) \le \varphi^*(Q_j) \le \gamma_2\varphi^*(q)$ with $j = 1, 2$. We have $X_q = X_{Q1} = X_{Q2}$.

Remark 2.1 The set X does not change if we change q into $q - \widetilde{q}$ where $\widetilde{q}$ is a bounded function such that $q - \widetilde{q} \ge 0$.

2.3 Hypotheses on matrix M and vector F

2.3.1 Case of constant matrix M

⋄ Hypothesis on M

In this case, we suppose the whole spectrum of M real. More precisely:

Hypothesis (H_M): *The whole spectrum of Matrix M is in $\mathbb{R}$. We denote the p real eigenvalues $(\mu_i)_{1\le i\le p}$ of matrix M, by*

$$\mu_1 > \mu_2 \ge \ldots \ge \mu_p.$$

We assume that the largest eigenvalue μ_1 of M is algebraically and geometrically simple.

Remark 2.2 We choose to write eigenvalues μ_i in decreasing order. The Jordan's canonical form allows us to write $M = PTP^{-1}$ with :

$$T = \left(\begin{array}{c|c|c|c} J_1 & & 0 & \\ \hline & J_2 & 0 & \\ \hline & 0 & \ddots & \\ \hline & & & J_p \end{array}\right)$$

where P is a change-of-basis matrix.

Every Jordan's block J_i is a square $k_i \times k_i$ matrix, in the form :

$$J_i = \begin{pmatrix} \mu_i & 1 & 0 & \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & \mu_i \end{pmatrix}$$

By Hypothesis (H_M), the first block is 1×1 : $J_1 = (\mu_1)$.

Notation: Let G be the eigenspace associated with μ_1 ($\dim G = 1$) and H the hyperplan spanned by other column vectors of Matrix P. By hypothesis (H_M), we have $\mathbb{R}^n = G \oplus H$. It is important to notice that, in matrix P, we can choose for the first column, every non null vector of G.

$\diamond$ **Hypothesis on** F

We recall that in the whole space, the anti-maximum principle could be violated for the equation

$$-\Delta u + q(x)u = \lambda u + f$$

if the function f is in $L^2(\mathbb{R}^N) \setminus X$ (cf. [5, Example 4.1, pp. 377–379]). So the fundamental negativity does not hold for $0 \leq f \not\equiv 0$. For results on systems presented in this article, of course we need to consider vector F with all the components f_k in X.
We can decompose $F(x)$ into $F(x) = F_G(x) + F_H(x)$ with $F_G(x) \in G$ and $F_H(x) \in H$.

Hypothesis (H_F): *All components f_i of vector F are in X and let us decompose $F(x) = F_G(x) + F_H(x)$ where $F_G(x) \in G$ and $F_H(x) \in H$. We assume there exists $\Psi \in G$ such that $F(x) = \widetilde{f_1}(x)\Psi + F_H(x)$ with $\widetilde{f_1} \geq 0$ (a.e.), and $F_G = \widetilde{f_1}\Psi \not\equiv 0$.*

Vector Ψ is in G so we have : $M\Psi = \mu_1\Psi$. Its components ψ_i are constant real numbers. In Matrix $P = (p_{ij})$ we choose Ψ for the first column. So $\psi_i = p_{i1}$.

2.3.2 Case of variable M

In this case, M is a 2×2 matrix. We note $M = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$.

Assumptions on Matrix M allow us to diagonalise this matrix with the help of a change-of-basis matrix with real and constant coefficients. These very particular forms of matrix are studied by Cosner and Schaefer [13]. If $a \not\equiv d$, we need b and c proportional to $a - d$; if $a \equiv d$, b is proportional to c and have the same sign. In the first case, where $a \not\equiv d$, we need to have a constant sign for $a - d$. In the second case, we suppose $a \equiv d$.

Hypothesis (H_{Mv1}) *(case $a \not\equiv d, a \geq d$): We assume*:

- $\diamond$ *Functions a and d are continuous, in $L^\infty(\mathbb{R}^N)$, and $a \geq d \geq 0$ with $a \not\equiv d$.*
- $\diamond$ *There exist two real numbers $\widehat{b}$ and $\widehat{c}$ such that $b = \widehat{b}(a-d)$ and $c = \widehat{c}(a-d)$, and $\widehat{D} = 1 + 4\widehat{b}\,\widehat{c} > 0$.*

Note that with hypotheses $a(x) \geq 0$ and $d(x) \geq 0$ we do not loose generality: we can add a positive number to each side to obtain these hypotheses.

In this case, we always use Hypothesis (H_F), but we can write it differently.

Hypothesis (H_{Fv1}) *(case $a \not\equiv d, a \geq d$): We assume $f_1,\ f_2 \in X$,*

$$\widetilde{f_1} = f_1 + \frac{2\widehat{b}}{1+\sqrt{\widehat{D}}} f_2 \geq 0 \quad \text{and} \quad \widetilde{f_1} \not\equiv 0.$$

Hypothesis (H_{Mv2}) *(case $a \equiv d$): We assume:*

- ◇ *The equality $a = d$ and this function is in $L^\infty(\mathbb{R}^N)$. Moreover $\forall x \in \mathbb{R}^N$, $a(x) \geq 0$.*
- ◇ *There exist two positive real numbers $\widehat{b}$ and $\widehat{c}$ such that $b = \epsilon\widehat{b}r$ and $c = \epsilon\widehat{c}r$, where ϵ is ± 1 and $r \in L^\infty(\mathbb{R}^N)$ is a bounded, positive and continuous function.*

Hypothesis (H_F) can now be written:

Hypothesis (H_{Fv2}) *(case $a \equiv d$): We assume $f_1,\ f_2 \in X$,*

$$\sqrt{\widehat{c}} f_1 + \epsilon\sqrt{\widehat{b}} f_2 \geq 0 \text{ and } \sqrt{\widehat{c}} f_1 + \epsilon\sqrt{\widehat{b}} f_2 \not\equiv 0.$$

Remark 2.3 Under Hypotheses (H_{Mv1}) or (H_{Mv2}), M has two real eigenvalues. We denote them by $\nu^+(x) \geq \nu^-(x)$. The two functions ν^+ and ν^- are in $L^\infty(\mathbb{R}^N)$.

3 Known Results

We recall here some results of fundamental positivity and fundamental negativity.

Our proof uses some results in Alziary, Takáč, ([8]) then Alziary, Fleckinger, Takáč,([5]) and Alziary, Takáč, ([9]) for fundamental positivity, in Besbas, ([10]) for fundamental negativity.

For q with superquadratical growth and for $f/\varphi^*(q) \in L^\infty$, they study

$$(-\Delta + q)u = \lambda u + f \tag{1}$$

and they show that there exist positive numbers c and δ (depending on q, f and λ) such that:

$$\lambda < \lambda^*(q) \;\Rightarrow\; u \overset{*}{\succ} 0, \text{ (fundamental positivity)}$$

$$\lambda^*(q) < \lambda < \lambda^*(q) + \delta \;\Rightarrow\; u \overset{*}{\prec} 0, \text{ (fundamental negativity).}$$

Fundamental Positivity

Theorem 3.1 ([8, Theorem 2.1, p. 284])([9, Theorem 3.1, p. 41])
Assume (H_q) is satisfied and $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ a.e. on $\mathbb{R}^N$, $f \not\equiv 0$. For $\lambda < \lambda^*(q)$ there exists a unique solution u to Equation (1) which is positive; and there exists a constant $c > 0$ such that

$$u > c\varphi^*(q) > 0 \quad \textit{(fundamental positivity)}. \tag{2}$$

Moreover, if also $f \leq C\varphi^*(q)$, with some constant $C > 0$, then we have

$$u \leq c'\varphi^*(q) \quad \textit{everywhere, with } c' = \frac{C}{\lambda^*(q) - \lambda}. \tag{3}$$

Corollary 3.2 ([9]): *The constant c defined in (2) tends to ∞ as $\lambda \to \lambda^*(q)$.*

This result plays an important role in the proof of our main Theorems:

Corollary 3.3 *Assume $f \in X$ (not necessarily $f \geq 0$), for $\lambda < \lambda^*(q)$, u exists and we have*

$$|u| \leq \frac{\|f\|_X}{\lambda^*(q) - \lambda}\varphi^*(q). \tag{4}$$

Indeed if we denote by $\mathcal{K}|_X$ the restriction of $\mathcal{K} = (L_q - \lambda I)^{-1}$ to the Banach space X, the operator $\mathcal{K}|_X$ is linear and bounded in X with norm $\leq \dfrac{1}{\lambda^*(q) - \lambda}$ ([9], p. 41).

Fundamental Negativity

It has been shown first in [1] for a radial potential and then in [9].

Theorem 3.4 ([9, Theorem 3.4, p. 42]) *Assume (H_q) is satisfied; let $f \in X$ be such that $f \geq 0$ a.e. on $\mathbb{R}^N$, $f \not\equiv 0$. Then there exists $\delta(f) > 0$ and $c > 0$ such that for all $\lambda \in (\lambda^*(q); \lambda^*(q) + \delta)$,*

$$u \leq -c\varphi^*(q) \quad \textit{(fundamental negativity)}. \tag{5}$$

Remark 3.5 The same holds if we assume only $\int_{\mathbb{R}^N} f\varphi^*(q)\,dx > 0$.

Corollary 3.6 ([10]): *The constant c defined in (5) tends to ∞ as $\lambda \to \lambda^*(q)$.*

Remark 3.7 Besbas ([10]) uses a slightly different space $X^{1,2} \subset X$; it coincides with X for radially symmetric functions.

Remark 3.8 Fundamental negativity improves the antimaximum principle introduced in Clément-Pelletier ([12]).

4 Main Results

4.1 System $n \times n$

This result concerns System (2) where M is a constant matrix:

$$(2) \quad \mathcal{L}U := \begin{pmatrix} (-\Delta + q(x)) & & 0 \\ & \ddots & \\ 0 & & (-\Delta + q(x)) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \lambda U + MU + F,$$

Recall that, by hypothesis (H_q), (H_M) and (H_F), M has only real eigenvalues; its largest eigenvalue μ_1 is simple and there exists Ψ eigenvector of M associated with μ_1, such that $F(x) = \widetilde{f_1}(x)\Psi + F_H(x)$ with $\widetilde{f_1} \geq 0$ (a.e.) Denote (ψ_i) the components of Ψ.

Theorem 4.1 *We assume Hypotheses (H_q), (H_M) and (H_F).*
Let $\Lambda := \lambda^*(q) - \mu_1$. Then there exist two real numbers $\delta > 0$ and $\delta' > 0$, depending on q, M, F, such that

- *If $\lambda \in (\Lambda - \delta; \Lambda)$ then System (2) admits a unique solution $U = (u_i)$. Moreover, for each integer $i \in [1, n]$, $u_i \in X$ and $\psi_i u_i \overset{*}{\succ} 0$.*
- *If $\lambda \in (\Lambda; \Lambda + \delta')$ then System (2) admits a unique solution $U = (u_i)$. Moreover, for each integer $i \in [1, n]$ $u_i \in X$ and $\psi_i u_i \overset{*}{\prec} 0$.*

Remark 4.2 If M is irreducible and cooperative, we know that there exists Ψ with all components strictly positive. We obtain the fundamental positivity below Λ and the fundamental negativity above Λ.

4.2 Variable Matrix M

Here M is a variable 2×2 matrix $M = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$.

The system is:

$$\begin{pmatrix} -\Delta + q(x) & 0 \\ 0 & -\Delta + q(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{1}$$

As we will see in the proof, the two real eigenvalues of M are $\nu^+(x) \geq \nu^-(x)$, and the functions ν^+ and ν^- are continuous, bounded. Let $\nu^+_{max} = \sup\{\nu^+(x),\, x \in \mathbb{R}\}$.
By Remark 2.1, we know that X is the same set for q, for $q^+ = q - \nu^+ + \nu^+_{max}$ and for $q^- = q - \nu^- + \nu^+_{max}$. We denote $\lambda^*(q^+)$ the principal eigenvalue of $-\Delta + q^+$ and $\lambda^*(q^-)$ the principal eigenvalue of $-\Delta + q^-$.

1. First case

Under Hypothesis 2.3.2 (H_{Mv1}), let us set $\widehat{b}$, $\widehat{c}$ the two real numbers such that $b = \widehat{b}(a-d)$ and $c = \widehat{c}(a-d)$.

Theorem 4.3 *(case $a \not\equiv d$) We assume Hypotheses (H_q), (H_{Mv1}) and (H_{Fv1}):*

$$f_1 + \frac{2\widehat{b}}{1+\sqrt{\widehat{D}}} f_2 \geq 0 \ \ a.e., \ \ f_1 + \frac{2\widehat{b}}{1+\sqrt{\widehat{D}}} f_2 \not\equiv 0$$

Let $\Lambda = \lambda^*(q^+) - \nu^+_{max}$. Then there exist two real numbers $\delta > 0$ and $\delta' > 0$, depending on q, M, F, such that

- *If $\Lambda - \delta < \lambda < \Lambda$, then System (1) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\succ} 0 \ \ and \ \ \widehat{c}u_2 \overset{*}{\succ} 0.$$

- *If $\Lambda < \lambda < \Lambda + \delta'$, then System (1) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\prec} 0 \ \ and \ \ \widehat{c}u_2 \overset{*}{\prec} 0.$$

Under Hypothesis 2.3.2 (H_{Mv2}), recall that functions $b = \epsilon\widehat{b}r$ and $c = \epsilon\widehat{c}r$ have the same sign, given by $\epsilon = \pm 1$.

Theorem 4.4 *(case $a \equiv d$) We assume Hypotheses (H_q), (H_{Mv2}) and (H_{Fv2}):*

$$\sqrt{\widehat{b}} f_1 + \epsilon\sqrt{\widehat{c}} f_2 \geq 0 \ \ a.e., \ \ \sqrt{\widehat{b}} f_1 + \epsilon\sqrt{\widehat{c}} f_2 \not\equiv 0.$$

Let $\Lambda = \lambda^*(q^+)$. Then there exist two real numbers $\delta > 0$ and $\delta' > 0$, depending on q, M, F, such that

- *If $\Lambda - \delta < \lambda < \Lambda$, then System (1) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\succ} 0 \ \ and \ \ \epsilon u_2 \overset{*}{\succ} 0.$$

- *If $\Lambda < \lambda < \Lambda + \delta'$, then System (1) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\prec} 0 \ \ and \ \ \epsilon u_2 \overset{*}{\prec} 0.$$

5 Proofs

5.1 Proof of Theorem 4.1

1/ First case: $\lambda < \Lambda = \lambda^*(q) - \mu_1$

First step: change of basis

We use the Jordan's block matrix $T = \left(\begin{array}{c|c|c|c} J_1 & & 0 & \\ \hline & J_2 & 0 & \\ \hline & 0 & \ddots & \\ \hline & & & J_p \end{array}\right)$ associated with matrix M in System (2):

$$\mathcal{L}U := \lambda U + MU + F.$$

There is a matrix P such that $T = P^{-1}MP$. More precisely, by Hypothesis (H_M) and Hypothesis (H_F) we can choose for the first column of change-of-basis matrix P : $\Psi \in G$ such that $F = \widetilde{f_1}\Psi + F_H$ with $\widetilde{f_1} \geq 0$ and $F_H(x) \in H$.

Now let us introduce the following notation:

$$U = P\widetilde{U} \quad \Leftrightarrow \quad \widetilde{U} = \begin{pmatrix} \widetilde{u_1} \\ \vdots \\ \widetilde{u_n} \end{pmatrix} = P^{-1}U \quad \text{and} \quad F = P\widetilde{F} \quad \Leftrightarrow \quad \widetilde{F} = \begin{pmatrix} \widetilde{f_1} \\ \vdots \\ \widetilde{f_n} \end{pmatrix} = P^{-1}F.$$

All potentials are equal, so System (2) becomes

$$\mathcal{L}\widetilde{U} = \lambda\widetilde{U} + T\widetilde{U} + \widetilde{F}. \tag{1}$$

By Hypothesis (H_M) the first equation in System (1) is

$$L\widetilde{u_1} = \lambda\widetilde{u_1} + \mu_1\widetilde{u_1} + \widetilde{f_1}, \tag{2}$$

where, by Hypothesis (H_F), $\widetilde{f_1} \geq 0$ and $\widetilde{f_1} \not\equiv 0$.

Look at the Jordan's block J_i with $2 \leq i \leq p$. The matrix J_i is $k_i \times k_i$. Set $s_i = \sum_{m=1}^{i-1} k_m$ with $k_1 = 1$.

From line $s_i + 1$ to line $s_i + k_i - 1$, we obtain $k_i - 1$ equations:

$$L\widetilde{u_j} = \lambda\widetilde{u_j} + \mu_i\widetilde{u_j} + \widetilde{u}_{j+1} + \widetilde{f_j} \qquad \text{if } s_i + 1 \leq j < s_i + k_i - 1, \tag{3}$$

and the last one:

$$L\widetilde{u_j} = \lambda\widetilde{u_j} + \mu_i\widetilde{u_j} + \widetilde{f_j} \qquad \text{for } j = s_i + k_i = s_{i+1}. \tag{4}$$

Second step: study of the triangular system (1)

In the first line
Using Theorem 3.1, we obtain that $L\widetilde{u_1} = \lambda\widetilde{u_1} + \mu_1\widetilde{u_1} + \widetilde{f_1}$ has a solution, $u_1 \overset{*}{\succ} 0$ (fundamental positivity), and since $\widetilde{f_1} \geq 0$ a.e. on $\mathbb{R}^N$,

$$c(\lambda)\varphi^* \leq \widetilde{u_1}.$$

If $\lambda \to \Lambda$, by Corollary (3.2) $c(\lambda) \to +\infty$.

In other lines we look at every Jordan's block.
In i^{th} block, with $2 \leq i \leq p$, from line $s_i + 1$ to line s_{i+1}.
• Line s_{i+1}: In Equation (4) $L\widetilde{u}_{s_{i+1}} = \lambda\widetilde{u}_{s_{i+1}} + \mu_i\widetilde{u}_{s_{i+1}} + \widetilde{f}_{s_{i+1}}$ by Corollary 3.3 the solution $\widetilde{u}_{s_{i+1}}$ exists and satisfies the inequality

$$|\widetilde{u}_{s_{i+1}}| \leq \frac{\|\widetilde{f}_{s_{i+1}}\|_X}{\lambda^*(q) - \mu_i - \lambda}\varphi^*. \tag{5}$$

By (H_M), $\lambda < \lambda^*(q) - \mu_1 < \lambda^*(q) - \mu_i$. So $|\widetilde{u}_{s_{i+1}}| \leq \dfrac{\|\widetilde{f}_{s_{i+1}}\|_X}{\mu_1 - \mu_i}\varphi^*$.
Hence, for $i > 1$, the function $\widetilde{u}_{s_{i+1}}$ is in X, and $\|\widetilde{u}_{s_{i+1}}\|_X \leq c_{s_{i+1}}$ where the constant $c_{s_{i+1}}$ depends only on F and M.
• From line $s_i + 1$ to line $s_{i+1} - 1$
For $j = s_{i+1} - 1$, we have $L\widetilde{u_j} = \lambda\widetilde{u_j} + \mu_i\widetilde{u_j} + \widetilde{u}_{s_{i+1}} + \widetilde{f_j}$.
Set $\widetilde{g_j} = \widetilde{u}_{s_{i+1}} + \widetilde{f}_{s_{i+1}}$. This function $\widetilde{g_j}$ is in X, and $\|\widetilde{g_j}\|_X \leq l_j$ where the constant l_j depends only on F and M.
Therefore, by Corollary 3.3 we obtain the existence of $\widetilde{u_j}$ and

$$|\widetilde{u_j}| \leq \frac{\|\widetilde{g_j}\|_X}{\lambda^*(q) - \mu_i - \lambda}\varphi^* \leq \frac{l_j}{\mu_1 - \mu_i}\varphi^*.$$

So, for $j = s_{i+1} - 1$, $\widetilde{u_j} \in X$, and $\|\widetilde{u_j}\|_X \leq c_j$ where c_j depends only on F and M.
Step by step, we can use the same argument from line $s_{i+1} - 1$ to line $s_i + 1$. Therefore we obtain, in each block, for each integer j with $s_i + 1 \leq j \leq s_{i+1} - 1$, the existence of the solution $\widetilde{u_j}$ which is in X. Moreover, $\|\widetilde{u_j}\|_X \leq c_j$ where the real c_j depends only on F and M.

To sum up, we have, for $2 \leq j \leq n$,

$$|\widetilde{u_j}| \leq c_j\varphi^*, \tag{6}$$

where the real c_j depends only on F and M,
and for $j = 1$,

$$c(\lambda)\varphi^* \leq \widetilde{u_1}, \tag{7}$$

where $c(\lambda)$ depends on F, M, λ and $c(\lambda) \nearrow +\infty$ when $\lambda \nearrow \Lambda$.

Third step: consequence for the initial system (2)

$U = P\widetilde{U}$ implies for each component $1 \leq i \leq n$:

$$u_i = p_{i1}\widetilde{u_1} + \sum_{j=2}^{n} p_{ij}\widetilde{u_j}.$$

As $\lambda \to \Lambda$, we have $\widetilde{u_1} \geq c(\lambda)\varphi^*(q)$, where $c(\lambda)$ tends to infinity; and by (6), $\sum_{j=2}^{n} p_{ij}\widetilde{u_j}$ is bounded by a constant times φ^*.

Therefore there exists $\delta_i > 0$ such that for $\lambda \in (\Lambda - \delta_i; \Lambda)$ the function

$$u_i = p_{i1}\widetilde{u_1} + \sum_{j=2}^{n} p_{ij}\widetilde{u_j}$$

has the same sign than p_{i1}. More precisely, if $p_{i1} > 0,\ u_i \overset{*}{\succ} 0,$ and if $p_{i1} < 0$ $u_i \overset{*}{\prec} 0$.

But the first eigenvector Ψ is the first column of the change-of-basis matrix $P : \psi_i = p_{i1}$ We obtain, in the case $\Lambda - \delta \leq \lambda < \Lambda$, where $\delta = \min_i \delta_i$,

$$\psi_i u_i \overset{*}{\succ} 0 \text{ (fundamentally positive)}$$

2/ **Second case** $\lambda > \Lambda = \lambda^* - \mu_1$ and $|\lambda - \Lambda|$ small:
there is $\delta_0 > 0$ with $\Lambda < \lambda < \Lambda + \delta_0 < \lambda^* - \mu_2 \leq \ldots \leq \lambda^* - \mu_n$.

First step

We transform System (2) into System (1) exactly as above.

Second step: study of the triangular system (1)

In the first line (2) $\quad L\widetilde{u_1} = \lambda\widetilde{u_1} + \mu_1\widetilde{u_1} + \widetilde{f_1}$,
we can apply the fundamental negativity results (Theorem 3.4): there is $\delta_1(F) > 0$ such that if $\Lambda < \lambda < \Lambda + \delta_1 < \Lambda + \delta_0$, then $\widetilde{u_1} \leq -c(\lambda)\varphi^*(q)$, and by Corollary 3.6: $c(\lambda)$ grows to $+\infty$ when $\lambda \to \Lambda$.

In the other equations, $L\widetilde{u_i} = \lambda\widetilde{u_i} + \mu_k\widetilde{u_i} + \widetilde{f_i}$ we have $\lambda < \lambda^* - \mu_i$. Hence by fundamental positivity and corollary 3.2, as in the case $\lambda < \Lambda$, we have by (6), $\sum_{j=2}^{n} p_{ij}\widetilde{u_j}$ bounded by a constant times φ^*.

Third step: consequence for the initial system (2)

In $u_i = p_{i1}\widetilde{u_1} + \sum_{j=2}^{n} p_{ij}\widetilde{u_j}$, we have $\sum_{j=2}^{n} p_{ij}\widetilde{u_j}$ bounded by a constant times φ^* and $\widetilde{u_1} < -c(\lambda)\varphi^*(q)$ tending to $-\infty$ when λ tends to Λ.
So there is $\delta' > 0$ such that : if $\Lambda < \lambda < \Lambda + \delta'$ we obtain $p_{j1}u_j = \psi_j u_j$ fundamentally negative: $\psi_j u_j \overset{*}{\prec} 0$. $\square$

5.2 Proof of Theorems 4.3 and 4.4

Here we study System (1):

$$\begin{pmatrix} -\Delta + q(x) & 0 \\ 0 & -\Delta + q(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

1/ Proof of Theorem 4.3

First step : study of eigenvalues

By Hypothesis (H_{Mv1}), there exist two real numbers $\widehat{b}$, $\widehat{c}$ such that $b = \widehat{b}(a-d)$ and $c = \widehat{c}(a-d)$. Since $a \geq d$, the two functions b and c never change sign. Moreover $\widehat{D} = 1 + 4\widehat{b}\,\widehat{c}$ is positive.

By calculation we obtain two eigenvalues : $\nu^+(x) = \frac{1}{2}\Big(a(x) + d(x) + (a(x) - d(x))\sqrt{\widehat{D}}\Big)$, and $\nu^-(x) = \frac{1}{2}\Big(a(x) + d(x) - (a(x) - d(x))\sqrt{\widehat{D}}\Big)$.

Since $a \geq d$, $a \not\equiv d$ and $\widehat{D} > 0$, we have $\nu^+ \geq \nu^-$, $\nu^+ \not\equiv \nu^-$. By (H_{Mv1}), the two functions a and d are continuous and bounded, so ν^+ and ν^- are continuous and bounded.
Set $\nu^+_{max} = \sup_x \nu^+(x)$.
By Remark 2.1, the set X is the same for the two potentials $q^+ = q + \nu^+_{max} - \nu^+$ and $q^- = q + \nu^+_{max} - \nu^-$. We have $q^- \geq q^+ > 0$, with $q^- \not\equiv q^+$.

The principal eigenvalue of $L_{q^-} \overset{\text{def}}{=} -\Delta + q^-(x)\bullet$ is

$$\lambda^*(q^-) = \inf_{u \in V_{q^-}(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q^-(x)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} \right\}$$

and we know that

$$\lambda^*(q^-) = \int_{\mathbb{R}^N} |\nabla\varphi^*(q^-)|^2 dx + \int_{\mathbb{R}^N} q^- |\varphi^*(q^-)|^2 dx,$$

where $\varphi^*(q^-)$ is the ground state of $-\Delta + q^-(x)\bullet$, which is positive and normalized by $\|\varphi^*(q^-)\|^2_{L^2(\mathbb{R}^N)} = 1$.
By $\nu^-(x) \leq \nu^+(x)$, $\nu^- \not\equiv \nu^+$, and by continuity we have

$$\int_{\mathbb{R}^N} \left(\nu^+_{max} - \nu^-(x)\right) |\varphi^*(q^-(x))|^2 dx > \int_{\mathbb{R}^N} \left(\nu^+_{max} - \nu^+(x)\right) |\varphi^*(q^-(x))|^2 dx,$$

so

$$\int_{\mathbb{R}^N} q^-(x) |\varphi^*(q^-)|^2 dx > \int_{\mathbb{R}^N} q^+(x) |\varphi^*(q^-)|^2 dx.$$

Therefore

$$\lambda^*(q^-) > \int_{\mathbb{R}^N} |\nabla\varphi^*(q^-)|^2 dx + \int_{\mathbb{R}^N} q^+(x) |\varphi^*(q^-)|^2 dx.$$

We obtain $\varphi^*(q^-) \in V_{q^+}$ and

$$\lambda^*(q^-) > \inf_{u \in V_{q^+}(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q^+(x)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} \right\} = \lambda^*(q^+).$$

Second step: diagonalization of the system (1)

We choose the eigenvectors $v^+ = \begin{pmatrix} \dfrac{1+\sqrt{\widehat{D}}}{2} \\ \widehat{c} \end{pmatrix}$ associated with ν^+ and $v^- = \begin{pmatrix} -\widehat{b} \\ \dfrac{1+\sqrt{\widehat{D}}}{2} \end{pmatrix}$ associated with ν^-.

Let P the matrix with columns vectors v^+ and v^-. The inverse matrix is

$$P^{-1} = \frac{1}{\sqrt{\widehat{D}}} \begin{pmatrix} 1 & \dfrac{2\widehat{b}}{1+\sqrt{\widehat{D}}} \\ \dfrac{-2\widehat{c}}{1+\sqrt{\widehat{D}}} & 1 \end{pmatrix}.$$

As before, we note: $\widetilde{U} = \begin{pmatrix} \widetilde{u_1} \\ \widetilde{u_2} \end{pmatrix} = P^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\begin{pmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.
The components of P and P^{-1} are constants. So, if f_1, $f_2 \in X$, then $\widetilde{f_1}$ and $\widetilde{f_2}$ are also in X.

By this change of basis, System (1)

$$\mathcal{L}U = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

is written in two equations:

$$-\Delta\widetilde{u_1} + q\widetilde{u_1} = \lambda\widetilde{u_1} + \nu^+\widetilde{u_1} + \widetilde{f_1},$$

$$-\Delta\widetilde{u_2} + q\widetilde{u_2} = \lambda\widetilde{u_2} + \nu^-\widetilde{u_2} + \widetilde{f_2}$$

where $\widetilde{f_1} \geq 0$, $\widetilde{f_1} \not\equiv 0$ by Hypothesis $(H_{Fv1}.)$

Set $q^+ = q + \nu^+_{max} - \nu^+$, and $q^- = q + \nu^+_{max} - \nu^-$, we derive

$$-\Delta\widetilde{u_1} + q^+\widetilde{u_1} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_1} + \widetilde{f_1}, \tag{8}$$

$$-\Delta\widetilde{u_2} + q^-\widetilde{u_2} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_2} + \widetilde{f_2}. \tag{9}$$

If $\lambda < \lambda^*(q^-) - \nu^+_{max}$, Equation (9) satisfies the Theorem of Fundamental Positivity, and by Corollary 3.3 we have

$$|\widetilde{u_2}| \leq \left(\lambda^*(q^-) - \nu^+_{max} - \lambda\right)^{-1} C_{\widetilde{f_2}}\varphi^*$$

• If $\lambda < \lambda^*(q^+) - \nu^+_{max} < \lambda^*(q^-) - \nu^+_{max}$,
we obtain

$$|\widetilde{u_2}| \leq \left(\lambda^*(q^-) - \lambda - \nu^+_{max}\right)^{-1} C_{\widetilde{f_2}}\varphi^* \leq \frac{C_{\widetilde{f_2}}}{\lambda^*(q^-) - \lambda^*(q^+)}\varphi^*.$$

Equation (8) satisfies the fundamental positivity result, so we have

$$\widetilde{u_1} \geq C(\lambda, \widetilde{f_1})\varphi^*$$

and $C(\lambda, \widetilde{f_1})$ tends to infinity, when λ tends to $\lambda^*(q^+) - \nu^+_{max}$. Consequently $\widetilde{u_2}$ is bounded, and $\widetilde{u_1}$ tends to infinity.

Now we can derive U from $U = P\widetilde{U}$; we have:

$$u_1 = \frac{1 + \sqrt{\widehat{D}}}{2}\widetilde{u_1} - \widehat{b}\widetilde{u_2}, \tag{10}$$

$$u_2 = \widehat{c}\widetilde{u_1} + \frac{1 + \sqrt{\widehat{D}}}{2}\widetilde{u_2}. \tag{11}$$

So there exists a real number $\delta > 0$, depending on F and M, such that for all
$\lambda^*(q^+) - \nu^+_{max} - \delta < \lambda < \lambda^*(q^+) - \nu^+_{max}$,

$$u_1 \overset{*}{\succ} 0 \text{ and } \widehat{c}u_2 \overset{*}{\succ} 0.$$

• If $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^-) - \nu^+_{max}$
By Theorem 3.4 in Equation (8) there exists δ_1 (depending on F) such that for all λ with

$\lambda^*(q^+) < \lambda + \nu^+_{max} < \lambda^*(q^+) + \delta_1$, $\widetilde{u_1}$ exists and $\widetilde{u_1} \overset{*}{\prec} 0$. We can choose $\delta_1 < \lambda^*(q^-) - \lambda^*(q^+)$, and assume $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^+) - \nu^+_{max} + \delta_1 < \lambda^*(q^-) - \nu^+_{max}$.
In Equation 9, by $\lambda < \lambda^*(q^-) - \nu^+_{max}$ we can apply the Fundamental Positivity Result. So $\widetilde{u_2}$ exists, and

$$|\widetilde{u_2}| \le \left(\lambda^*(q^-) - \lambda - \nu^+_{max}\right)^{-1} C_{\widetilde{f_2}} \varphi^* \le \frac{1}{\lambda^*(q^-) - \lambda^*(q^+) - \delta_1} \varphi^*.$$

We have $\widetilde{u_2}$ bounded by a constant times φ^*, and $\widetilde{u_1} \le -C(\lambda, \widetilde{f_1})\varphi^*$, with $C(\lambda, \widetilde{f_1})$ tending to infinity when λ tends to $\lambda^*(q^+) - \nu^+_{max}$.
Relations (10) and (11) are always true. So there exists a real $0 < \delta \le \delta_1$ such that:
if $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^+) - \nu^+_{max} + \delta < \lambda^*(q^-) - \nu^+_{max}$, we have $u_1 \overset{*}{\prec} 0$ and $\widehat{c}u_2 \overset{*}{\prec} 0$. $\square$

2/ Proof of Theorem 4.4

By Hypothesis (H_{Mv2}), $a = d$ and there exist two real numbers $\widehat{b}$, $\widehat{c}$ such that $b = \epsilon\widehat{b}r$ and $c = \epsilon\widehat{c}r$, with $\epsilon = \pm 1$. The function $r \in L^\infty(\mathbb{R}^N)$ is continuous, positive and bounded.
The matrix $M(x)$ has two eigenvalues, $\nu^+(x) = a(x) + \sqrt{\widehat{b}\,\widehat{c}}\,r(x)$ and $\nu^-(x) = a(x) - \sqrt{\widehat{b}\,\widehat{c}}\,r(x)$. The function r is positive, bounded and continuous so the function $\nu^+ - \nu^- = 2\sqrt{\widehat{b}\,\widehat{c}}\,r(x)$ is positive, bounded and continuous. Let $q^+ = q + \nu^+_{max} - \nu^+$ and $q^- = q + \nu^+_{max} - \nu^-$.
We have, as in the first step of the proof of Theorem 4.3, $\lambda(q^-) > \lambda(q^+)$.
Eigenvectors associated to ν^+ and ν^- are $v^+ = \begin{pmatrix} \sqrt{\widehat{b}} \\ \epsilon\sqrt{\widehat{c}} \end{pmatrix}$ and $v^- = \begin{pmatrix} -\epsilon\sqrt{\widehat{b}} \\ \sqrt{\widehat{c}} \end{pmatrix}$.
With these eigenvectors, we obtain

$$P^{-1} = \begin{pmatrix} \dfrac{1}{2\sqrt{\widehat{b}}} & \dfrac{\epsilon}{2\sqrt{\widehat{c}}} \\ \dfrac{-\epsilon}{2\sqrt{\widehat{b}}} & \dfrac{1}{2\sqrt{\widehat{c}}} \end{pmatrix}.$$

The components of P and P^{-1} are constants.

We always denote $\begin{pmatrix} \widetilde{u_1} \\ \widetilde{u_2} \end{pmatrix} = P^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\begin{pmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. Functions $\widetilde{f_1}$ and $\widetilde{f_2}$ are in X, and by Hypothesis (H_{Fv2}), $\widetilde{f_1} \ge 0$, and $\widetilde{f_1} \not\equiv 0$. We obtain the same equations as above:

$$(8)\quad -\Delta\widetilde{u_1} + q^+\widetilde{u_1} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_1} + \widetilde{f_1},$$
$$(9)\quad -\Delta\widetilde{u_2} + q^-\widetilde{u_2} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_2} + \widetilde{f_2},$$

where $\widetilde{f_1} \ge 0$, $\widetilde{f_1} \not\equiv 0$ by Hypothesis (H_{Fv1}).

The study of the comparison with the ground state is the same as in Theorem 4.3. So $\widetilde{u_2}$ is still bounded in X. For $\widetilde{u_1}$:

- if $\lambda < \lambda^*(q^+) - \nu^+_{max}$, then $\widetilde{u_1} \geq C(\lambda, F)\varphi^*$, where $C(\lambda, F) \to \infty$ when $\lambda \to \lambda^*(q^+) - \nu^+_{max}$,
- if $\lambda > \lambda^*(q^+) - \nu^+_{max}$ and $|\lambda - (\lambda^*(q^+) - \nu^+_{max})|$ small, we have $\widetilde{u_1} \leq -C(\lambda, F)\varphi^*$, where $C(\lambda, F) \to \infty$ when $\lambda \to \lambda^*(q^+) - \nu^+_{max}$.

But now the change of basis gives:

$$u_1 = \sqrt{\widehat{b}}\,\widetilde{u_1} - \epsilon\sqrt{\widehat{b}}\,\widetilde{u_2}, \tag{12}$$

$$u_2 = \epsilon\sqrt{\widehat{c}}\,\widetilde{u_1} + \sqrt{\widehat{c}}\,\widetilde{u_2}. \tag{13}$$

By similar arguments, we obtain
- the existence of δ such that: if $\lambda^*(q^+) - \nu^+_{max} - \delta < \lambda < \lambda^*(q^+) - \nu^+_{max} < \lambda^*(q^-) - \nu^+_{max}$, then $u_1 \overset{*}{\succ} 0$ and $\epsilon u_2 \overset{*}{\succ} 0$,
- the existence of δ' such that: if $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^+) - \nu^+_{max} + \delta' < \lambda^*(q^-) - \nu^+_{max}$, then $u_1 \overset{*}{\prec} 0$ and $\epsilon u_2 \overset{*}{\prec} 0$. ■

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The infimal convolution can be used to easily prove the classical Hahn-Banach theorem

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ABSTRACT. By using a particular case of the infimal convolution, we provide an instructive proof for the dominated Hahn-Banach extension theorem.

Former proofs have only used this convolution rather implicitly.

KEY WORDS. Infimal convolution, Hahn-Banach theorem

1 Hahn-Banach extensions and the infimal convolution

Notation 1.1 Suppose that X *is a vector space over* $\mathbb{R}$ and p *is a positively homogeneous, subadditive function of* X *to* $\mathbb{R}$.

Moreover, assume that V *is a subspace of* X and φ *is a linear function of* V *to* $\mathbb{R}$ such that φ is dominated by p on V in the sense that $\varphi(v) \le p(v)$ *for all* $v \in V$.

Under the above assumptions, the subsequent dominated extension theorem was first proved by Banach in [1, pp. 227–29] with reference to his former paper in 1929. At some later pages, he also mentions the pioneering works of Riesz in 1907, Helly in 1912, and Hahn in 1927. See the reliable historical notes of Saccoman [10].

The term *Hahn-Banach theorem* has been coined to the following theorem of Banach, or an important consequence of it proved earlier by Hahn, after a paper of Bohnenblust and Sobczcyk in 1938 who proved a complex form Hahn's theorem independently of the works of Murray in 1936 and Sukhomlinov in 1938. See the excellent surveys of Buskes [5] and Narici and Beckenstein [9].

Theorem 1.2 *There exists a linear function* f *of* X *to* $\mathbb{R}$ *that extends* φ *and is dominated by* p *on* X.

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This theorem is usually proved with the help of an elementary, but tricky computation and some important, non-direct consequence of the Axiom of Choice such as the well-ordering principle or transfinite induction, and Zorn's lemma or Hausdorff's maximal principle. See Bridges [3, pp. 261–262] for a nice instructive treatment.

In the present note, we shall show that the computational part of the proof can be put into a proper perspective with the help of the $p * \varphi$ particular case of the infimal convolution. The latter notion was already intensively studied by Moreau [8] and Strömberg [11] with several applications. See also [13], [6] and [14] for some further results.

However, up till now, it has only been implicitly used in the proofs of the Hahn-Banach theorems. Unfortunately, the second author in [12] considered the intersection convolution to be a more convenient tool for proving linear extension theorems than the infimal one. Though, he observed that the former one is only a particular case of an obvious extension of the latter one.

In the sequel, in addition to Notation 1.1, we shall only need the following two fundamental definitions.

Definition 1.3 If U is a linear subspace of X containing V, then a linear function ψ of U to $\mathbb{R}$, that extends φ and is dominated by p on U, will be called a *Hahn-Banach extension* of φ to U.

Remark 1.4 By using this definition, the assertion of Theorem 1.2 can be briefly expressed by saying that there exists a Hahn-Banach extension f of φ to X.

Definition 1.5 The function $q = p * \varphi$, defined by

$$q(x) = \inf_{v \in V} \big(p(x-v) + \varphi(v) \big)$$

for all $x \in X$, will be called the *infimal convolution* of p and φ.

Remark 1.6 The above definition can be put a more instructive form by observing that

$$q(x) = \inf \big\{ p(u) + \varphi(v) : \quad u \in X, \quad v \in V, \quad x = u + v \big\}$$

for all $x \in X$. Note that the latter form can be applied to more general situations.

The close relationship that exists between the Hahn-Banach extensions and the infimal convolution can already be nicely clarified by the following

Theorem 1.7 *If ψ is a Hahn-Banach extension of φ to U, then for any $u \in U$ we have*

$$-q(-u) \leq \psi(u) \leq q(u).$$

Proof: For any $v \in V$, we have

$$\psi(u) = \psi(u - v + v) = \psi(u - v) + \psi(v) \leq p(u - v) + \varphi(v).$$

Hence, we can already infer that

$$\psi(u) \leq \inf_{v \in V} \big(p(u - v) + \varphi(v) \big) = q(u).$$

Now, by writing $-u$ in place of u, we can see that

$$-\psi(u) = \psi(-u) \leq q(-u), \quad \text{and thus} \quad -q(-u) \leq \psi(u)$$

also holds.

Now, as an immediate consequence of this theorem, we can also state

Corollary 1.8 *If ψ is as in Theorem 1.7 and q is odd on U, then q is an extension of ψ.*

Proof: In this case, for any $u \in U$, we have

$$q(-u) = -q(u), \quad \text{and hence} \quad -q(-u) = q(u).$$

Therefore, by Theorem 1.7, $\psi(u) = q(u)$ is also true.

Thus, in particular, we can also state

Corollary 1.9 *If U is a subspace of X such that $V \subset U$ and q is odd on U, then there exists at most one Hahn-Banach extension ψ of φ to U.*

2 Further inequalities for the function q

Theorem 2.1 *For any $x \in X$, we have*

$$-p(-x) \leq q(x) \leq p(x).$$

Proof: For any $v \in V$, we have

$$\begin{aligned} 0 = \varphi(0) = \varphi(-v) + \varphi(v) &\leq p(-v) + \varphi(v) \\ &= p(-x + x - v) + \varphi(v) \leq p(-x) + p(x - v) + \varphi(v), \end{aligned}$$

and thus

$$-p(-x) \leq p(x - v) + \varphi(v).$$

Hence, we can already infer that

$$-p(-x) \leq \inf_{v\in V} \big(p(x-v) + \varphi(v)\big) = q(x).$$

Moreover, we can at once see that

$$q(x) = \inf_{v\in V} \big(p(x-v) + \varphi(v)\big) \leq p(x-0) + \varphi(0) = p(x)$$

also holds.

Now, as an immediate consequence of this theorem, we can also state

Corollary 2.2 *q is a real-valued function of X such that $q(0)=0$.*

From Theorem 2.1, by writing $-x$ in place x, we can also immediately get

Corollary 2.3 *For any $x \in X$, we have*

$$-p(-x) \leq -q(-x) \leq p(x).$$

In addition to the above results and Theorem 1.7, it is also worth proving the following

Theorem 2.4 *For any $x \in X$, we have*

$$-q(-x) \;\leq\; q(x).$$

Proof: For any $v, s \in V$ we have

$$\begin{aligned}
&-p(x-v) - \varphi(v) = -p(x-v) - \varphi(v) - \varphi(s) + \varphi(s)\\
&\qquad = -p(x-v) + \varphi(-v-s) + \varphi(s) \leq -p(x-v) + p(-v-s) + \varphi(s)\\
&= -p(x-v) + p(x-v-x-s)) + \varphi(s) \leq -p(x-v) + p(x-v) + p(-x-s) + \varphi(s)\\
&\qquad\qquad = p(-x-s) + \varphi(s).
\end{aligned}$$

Hence, we can infer that

$$-p(x-v) - \varphi(v) \leq \inf_{s\in V} \big(p(-x-s) + \varphi(s)\big) = q(-x).$$

Now, we can already see that

$$-q(-x) \leq p(x-v) + \varphi(v),$$

and thus

$$-q(-x) \leq \inf_{v\in V} \big(p(x-v) + \varphi(v)\big) = q(x).$$

also holds.

This theorem makes the less obvious part of the proof of Theorem 2.1 superfluous. Moreover, it immediately yields the following

Corollary 2.5 *q is a superodd function of X in the sense that $-q(x) \leq q(-x)$ for all $x \in X$.*

Remark 2.6 Later, we shall see that the function q is not, in general, odd. Therefore, in contrast to Corollary 1.8, it cannot usually be a Hahn–Banach extension of φ to X.

Moreover, we shall also see that q is not, in general, even. Therefore, it cannot usually be a seminorm even if p is so. However, due to the linearity of φ, it will turn out to have some better additivity and homogeneity properties than p.

3 Additivity and homogeneity properties of q

Theorem 3.1 *For any $x \in X$ and $v \in V$, we have*

$$q(x+v) = q(x) + \varphi(v).$$

Proof: For any $s \in V$, we have

$$q(x) = \inf_{t \in V} \big(p(x-t) + \varphi(t) \big) \leq p\big(x - (s-v) \big) + \varphi(s-v),$$

and thus

$$q(x) + \varphi(v) \leq p\big(x - (s-v) \big) + \varphi(s-v) + \varphi(v) = p(x+v-s) + \varphi(s).$$

Hence, we can already infer that

$$q(x) + \varphi(v) \leq \inf_{s \in V} \big(p(x+v-s)) + \varphi(s) \big) = q(x+v).$$

Now, we can easily see that

$$q(x+v) = q(x+v) + \varphi(0) = q(x+v) + \varphi(-v) + \varphi(v) \leq q(x) + \varphi(v)$$

also holds.

From this theorem, by using Corollary 2.2, we can immediately derive

Corollary 3.2 *q is an extension of φ.*

Proof: Namely, by Theorem 3.1 and Corollary 2.2, we have

$$q(v) = q(0+v) = q(0) + \varphi(v) = 0 + \varphi(v) = \varphi(v)$$

for all $v \in V$.

Now, as an immediate consequence of Theorem 3.1 and Corollary 3.2, we can also state

Corollary 3.3 *q is an $X \times V$–additive function of X in the sense that*

$$q(x+v) = q(x) + q(v)$$

for all $x \in X$ and $v \in V$.

Concerning the function q, we can also easily prove the following

Theorem 3.4 *q is a subadditive function of X.*

Proof: If $x, y \in X$, then by Definition 1.3 and Corollary 2.2, for any $\varepsilon > 0$ there exist $s, t \in V$ such that

$$p(x-s) + \varphi(s) < q(x) + \varepsilon \qquad \text{and} \qquad p(y-t) + \varphi(t) < q(y) + \varepsilon .$$

Now, we can already see that

$$\begin{aligned} q(x+y) = \inf_{v \in V} \big(p(x+y-v) + \varphi(v) \big) \\ \leq p\big(x+y-(s+t) \big) + \varphi(s+t) \leq p(x-s) + p(y-t) + \varphi(s) + \varphi(t) \\ \leq q(x) + q(y) + 2\varepsilon . \end{aligned}$$

Hence, by letting ε tend to 0, we can infer that

$$q(x+y) \leq q(x) + q(y) .$$

Remark 3.5 This theorem makes the proof of Theorem 2.4 superfluous. Namely, by Theorem 3.4 and [4, Theorem 4.3], the function q is superodd.

Moreover, by the above theorems, we can also at once state that q is $\mathbb{N}$–subhomogeneous and $\{0\} \cup \mathbb{N}^{-1}$–superhomogeneous.

However, the latter facts are of no particular importance for us now since we can also prove the following

Theorem 3.6 *q is a positively homogeneous function of X.*

Proof: For any $x \in X$, $v \in V$ and $\lambda \in \mathbb{R}$, with $\lambda > 0$, we have

$$q(x) = \inf_{s \in V} \big(p(x-s) + \varphi(s) \big) \leq p\big(x - \lambda^{-1} v \big) + \varphi\big(\lambda^{-1} v \big) ,$$

and thus

$$\lambda\, q(x) \leq \lambda\, p\big(x - \lambda^{-1} v \big) + \lambda\, \varphi\big(\lambda^{-1} v \big) = p(\lambda x - v) + \varphi(v) .$$

Hence, we can already infer that

$$\lambda\, q(x) \leq \inf_{v \in V} \big(p(\lambda\, x - v) + \varphi(v) \big) = q\,(\,\lambda\, x\,)\,.$$

Now, we can easily see that

$$q\,(\,\lambda\, x\,) = \lambda\, \lambda^{-1} q\,(\,\lambda\, x\,) \leq \lambda\, q\,(\,\lambda^{-1} \lambda\, x\,) = \lambda\, q(x)$$

also holds.

Now, as a useful consequence of Theorem 3.6 and Corollary 2.5, we can also prove the following

Corollary 3.7 *q is an $\mathbb{R}$–superhomogeneous function of X in the sense that*

$$\lambda\, q\,(x) \leq q\,(\,\lambda\, x\,)$$

for all $\lambda \in \mathbb{R}$ and $x \in X$.

Proof: By Corollary 2.5 and Theorem 3.6, for any $x \in X$ and $\lambda \in \mathbb{R}$, with $\lambda < 0$, we also have

$$\lambda\, q(\,x\,) = (-\lambda\,)\big(-q(x)\big) \leq (-\lambda\,)\, q(-x\,) = q\,\big(\,(-\lambda\,)(-x\,)\big) = q\,(\,\lambda\, x\,)\,.$$

From this corollary, by writing $-\lambda$ and $-x$ in place of λ and x, respectively, we can immediately infer

Corollary 3.8 *For any $\lambda \in \mathbb{R}$ and $x \in X$, we have*

$$-q\,(\,\lambda\, x\,) \leq \lambda\, q\,(-x\,)\,.$$

4 An instructive proof of the Hahn-Banach theorem

We first state the following basic theorem whose proof may be left to the reader.

Theorem 4.1 *If $a \in X$ such that $a \notin V$ and*

$$U = \mathbb{R}\, a + V = \big\{\, \lambda\, a + v \;:\;\; \lambda \in \mathbb{R}\,, \;\; v \in V\, \big\}\,,$$

then

(1) *U is the smallest linear subspace of X such that $a \in U$ and $V \subset U$;*

(2) *for each $u \in U$ there exists a unique pair $(\,\lambda_u,\, v_u) \in \mathbb{R} \times V$ such that $u = \lambda_u a + v_u$;*

(3) *the mappings $u \mapsto \lambda_u$ and $u \mapsto v_u$, where $u \in U$, are linear functions of U to $\mathbb{R}$ and V, respectively, such that $\lambda_v = 0$ and $v_v = v$ for all $v \in V$.*

Now, by using this theorem and our former results on the infimal convolution, we can quite easily prove the following simple, but important particular case of a slight improvement of Theorem 1.2.

Theorem 4.2 *If $a \in X$ such that $a \notin V$, then there exists a linear function ψ of the subspace $U = \mathbb{R}\, a + V$ to $\mathbb{R}$ that extends φ and satisfies*

$$-q(-u) \leq \psi(u) \leq q(u)$$

for all $u \in U$.

Proof: Note that if ψ is as above, then under the notation of Theorem 4.1, for any $u \in U$, we have

$$\psi(u) = \psi(\lambda_u a + v_u) = \lambda_u \psi(a) + \psi(v_u) = \lambda_u \psi(a) + \varphi(v_u).$$

Moreover, we also have

$$-q(-a) \leq \psi(a) \leq q(a).$$

Therefore, to prove the theorem, we may naturally define a function ψ of U to $\mathbb{R}$ such that

$$\psi(u) = \lambda_u q(a) + \varphi(v_u)$$

for all $u \in U$. Now, by using Theorem 4.1, we can easily see that ψ is a linear extension of φ.

Therefore, by Theorem 1.7, we need only show that ψ is dominated by p on U. For this, note that by Corollary 3.7 and Theorems 3.1 and 2.1 we have

$$\psi(u) = \lambda_u q(a) + \varphi(v_u) \leq q(\lambda_u a) + \varphi(v_u) = q(\lambda_u a + v_u) = q(u) \leq p(u)$$

for all $u \in U$.

Remark 4.3 Note that, in the above proof, instead of $q(a)$ we may take any number $b \in \mathbb{R}$ with

$$-q(-a) \leq b \leq q(a).$$

Therefore, the required extension ψ of φ is unique if and only if the function q is odd at the point a.

Now, as a slight improvement of Theorem 1.2, we can also prove the following

Theorem 4.4 *There exists a linear function f of X to $\mathbb{R}$ that extends φ and satisfies*

$$-q(-x) \leq f(x) \leq q(x)$$

for all $x \in X$.

Proof: Denote by Ψ the family of all Hahn–Banach extensions ψ of φ. Then, it is clear Ψ is a nonvoid partially ordered set with the ordinary set inclusion.

Moreover, if Φ is a nonvoid totally ordered subset of Ψ, then it can be easily seen that $\phi = \cup \Phi$ is an upper bound of Φ in Ψ. Thus, by Zorn's lemma, there exists a maximal element f of Ψ.

Now, by Theorem 1.7, it remains only to show that the domain D_f of f is X. For this, note that if for some $a \in X$ we have $a \notin D_f$, then by Theorem 4.2 and 2.1 there exists a Hahn-Banach extension ψ of f to the subspace $U = \mathbb{R}\, a + D_f$. However, this contradicts the maximality of f.

Remark 4.5 Note that if f is as in the above theorem, then by Theorem 2.1 f is, in particular, a Hahn-Banach extension of φ to X.

Now, as a useful consequence of our former results, we can briefly prove the following

Theorem 4.6 *The following assertions are equivalent:*

(1) q *is odd* X*;*

(2) q *is a Hahn-Banach extension of* φ *to* X*;*

(3) *there exists a unique Hahn-Banach extension* f *of* φ *to* X*;*

(4) *there exists at most one Hahn-Banach extension* f *of* φ *to* X*.*

Proof: By Corollary 1.9, it is clear that (1) implies (4). Moreover, from Theorems 4.4 and 2.1, we can see that there exists a Hahn-Banach extension f of φ to X. Therefore, (4) implies (3). Moreover, if (1) holds, then by Corollary 1.8 we necessarily have $f = q$. Therefore, (1) also implies (2).

Now, since the implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (4) trivially hold, we need only show that (4) also implies (1). For this, note that if (1) does not hold, then there exists $a \in X$ such that

$$q(-a) \neq -q(a).$$

Hence, by Corollary 3.2 and Theorem 2.4, we can infer that

$$a \notin V \qquad \text{and} \qquad -q(-a) < q(a).$$

Now, by Remark 4.3 and Theorem 2.1, we can construct two Hahn-Banach extensions ψ_1 and ψ_2 of φ to $U = \mathbb{R}\, a + V$ such that $\psi_1(a) \neq \psi_2(a)$. Moreover, by Theorems 4.4 and 2.1, we can state that there exist some Hahn-Banach extensions f_1 and f_2 of ψ_1 and ψ_2 to X, respectively. Thus, (4) does not also hold. This proves the required implication.

Remark 4.7 Sections 7 and 11 of [9] and [5] respectively, show that the question of the uniqueness of the Hahn–Banach extension has also been intensively studied by several authors. Moreover, some further uniqueness results can also be found on the MathSciNet. However, the above simple convolutional characterization seems to be new.

5 A simple illustrating example to Theorems 4.2 and 2.4

Example 5.1 Take

$$a = (-1, 1) \qquad \text{and} \qquad V = \mathbb{R}\,(1, 1).$$

Moreover, define

$$\varphi(s, s) = s \qquad \text{and} \qquad p(s, t) = \max\{\,|s|,\, |t|\,\}$$

for all $s, t \in \mathbb{R}$.

Then, it is clear that V is a linear subspace of $\mathbb{R}^2$ such that $a \notin V$. Moreover, for any $(s, t) \in \mathbb{R}^2$, by taking

$$\lambda(s, t) = 2^{-1}(t - s) \qquad \text{and} \qquad v(s, t) = 2^{-1}(s + t)(1, 1),$$

we can easily check that

$$(s, t) = \lambda(s, t)\,a + v(s, t).$$

Therefore, $\mathbb{R}^2 = \mathbb{R}\,a + V$. Moreover, we can also at once state that φ is a linear function of V to $\mathbb{R}$ and p is a norm on $\mathbb{R}^2$ such that

$$|\varphi(s, s)| = |s| = p(s, s)$$

for all $s \in \mathbb{R}$. Thus, in particular, φ is dominated by p on V.

Therefore, by Theorem 4.2, there exists a linear function ψ of $\mathbb{R}^2$ to $\mathbb{R}$ that extends φ and satisfies

$$-q(-s, -t) \le \psi(s, t) \le q(s, t)$$

for all $s, t \in \mathbb{R}$, with $q = p * \varphi$. Moreover, by the proof Theorem 4.2, we can take

$$\psi(s, t) = \lambda(s, t)\,q(a) + \varphi\big(v(s, t)\big) = 2^{-1}(t - s)\,q(a) + 2^{-1}(s + t)$$

for all $s, t \in \mathbb{R}$.

Now, by drawing pictures of the functions involved, we can also easily see that

$$\begin{aligned} q(a) &= \inf_{v \in V}\big(p(a - v) + \varphi(v)\big) = \inf_{s \in \mathbb{R}}\big(p((-1, 1) - (s, s)) + \varphi(s, s)\big) \\ &= \inf_{s \in \mathbb{R}}\big(\max\{\,|1 + s|,\, |1 - s|\,\} + s\big) = \inf_{s \in \mathbb{R}}\big(1 + |s| + s\big) = 1. \end{aligned}$$

Therefore,

$$\psi(s,t) = 2^{-1}(t-s) + 2^{-1}(s+t) = t$$

for all $s, t \in \mathbb{R}$.

Remark 5.2 Quite similarly, we can also see that $q(-a) = 1$. Therefore,

$$-q(-a) = -1 < 1 = q(a).$$

Thus, the superodd function q fails to be odd at the point a.

In this respect, it is also worth noticing that, by Corollary 3.2, q is an extension of φ. Thus, it is also not even.

Remark 5.3 Now, by using our former observations, we can also state that if ψ is a Hahn-Banach extension of φ to $\mathbb{R}^2$, then there exists $b \in [-1, 1]$ such that

$$\begin{aligned}\psi(s,t) &= \lambda(s,t)b + \varphi\big(v(s,t)\big) \\ &= 2^{-1}(t-s)b + 2^{-1}(s+t) = 2^{-1}(1-b)s + 2^{-1}(1+b)t\end{aligned}$$

for all $s, t \in \mathbb{R}$. Hence, by taking

$$c = 2^{-1}(1-b),$$

we can already infer that $0 \le c \le 1$ such that

$$\psi(s,t) = cs + (1-c)t$$

for all $s, t \in \mathbb{R}$.

Conversely, we can also note that if ψ is of the above form for some $c \in [0, 1]$, then ψ is a linear extension of φ to $\mathbb{R}^2$ such that

$$\begin{aligned}\psi(s,t) \le |\psi(s,t)| &= |cs + (1-c)t| \le c|s| + (1-c)|t| \\ &\le cp(s,t) + (1-c)p(s,t) = p(s,t)\end{aligned}$$

for all $s, t \in \mathbb{R}$. Thus, we have obtained all the Hahn-Banach extensions of φ to $\mathbb{R}^2$.

Remark 5.4 Now, if $s, t \in \mathbb{R}$ such that $\lambda(s,t) \ge 0$, i.e, $s \le t$, then by using Theorems 3.1 and 3.6 we can also easily see that

$$\begin{aligned}q(s,t) &= q\big(\lambda(s,t)a + v(s,t)\big) \\ &= \lambda(s,t)q(a) + \varphi\big(v(s,t)\big) = 2^{-1}(t-s) + 2^{-1}(s+t) = t.\end{aligned}$$

Hence, because of the symmetry of s and t in the formula

$$q(s,t)=\inf_{r\in\mathbb{R}}\big(p((s,t)-(r,r))+\varphi(r,r)\big)=\inf_{r\in\mathbb{R}}\big(\max\{\,|s-r|,\,|t-r|\,\}+r\big)\,,$$

we can already infer that

$$q(s,t)=\max\{s,t\}$$

for all $s, t\in\mathbb{R}$.

Thus, in particular

$$q(s,s)=\varphi(s,s)\qquad\text{and}\qquad q(s,-s)=|\varphi(s)|$$

and

$$q\big(|s|,|t|\big)=p(s,t)$$

for all $s, t\in\mathbb{R}$.

The value $q(s,t)$ can also be computed directly by observing that

$$\begin{aligned}\max\{\,|s-r|,\,|t-r|\,\}+r&=\big|r-2^{-1}(s+t)\big|+2^{-1}|s-t|+r\\&=\big|r-2^{-1}(s+t)\big|+r-2^{-1}(s+t)+2^{-1}(s+t)+2^{-1}|s-t|=\\&=\big|r-2^{-1}(s+t)\big|+r-2^{-1}(s+t)+\max\{s,t\}\end{aligned}$$

for all $r, s, t\in\mathbb{R}$.

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Functional Equations for Knopp Functions and Digital Sums

ABSTRACT. The well-known Delange formula expressed the usual sum-of-digits function to a basis $q \geq 2$ by means of a continuous, nowhere differentiable function. The aim of this paper is to clarify the actually reason for this phenomenon. For this we show that specific Knopp functions satisfy functional equations which allow to calculate, for any positive integer n, the number of times of digits in the q-ary representation of n which are equal to a fixed $m \in \{1, \ldots, q-1\}$. By linear combination for arbitrary Knopp functions we get functional equations contained certain digital sums. These functional equations imply sum formulas for certain digital sums. Simple examples are the formula of Delange for the usual sum-of-digits function and a formula for the number of zeros

KEY WORDS. Knopp functions, functional equations, digital sums, Fourier expansion.

1 Introduction

Throughout in this paper let q be a fixed integer with $q \geq 2$. For an integer $k \in \mathbb{N}$ we introduce the q-ary representation

$$k = \sum_{j=0}^{\infty} a_j q^j \tag{1.1}$$

with $a_j \in \{0, 1, \ldots, q-1\}$ and $a_j = 0$ for $j > \log_q k$. It is known that the sum

$$S(n) = \sum_{k=1}^{n-1} s(k), \tag{1.2}$$

where $s(k) = a_0 + a_1 + \ldots$, can be represented by the Delange formula [3]

$$\frac{1}{n} S(n) = \frac{q-1}{2} \log_q n + F\left(\log_q n\right) \tag{1.3}$$

where $F(u)$ is a continuous, nowhere differentiable function with the period 1, cf. [9] for $q = 2$. In the case $q = 2$ this function can be expressed by

$$F(u) = -\frac{u}{2} - \frac{1}{2^{u+1}} T(2^u) \qquad (u \leq 0)$$

where T is Takagi's function, cf. [6]. Takagi's function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\Delta\left(2^n x\right)}{2^n} \qquad (0 \le x \le 1) \tag{1.4}$$

where $\Delta(x) = \operatorname{dist}(x, \mathbb{Z})$, and it was introduced in 1903 by T. Takagi [8] as an example of a continuous, nowhere differentiable function.

In this paper we investigate so-called Knopp functions ([7])

$$G(x) = \sum_{\nu=0}^{\infty} \frac{g\left(q^\nu x\right)}{q^\nu} \qquad (x \in \mathbb{R}) \tag{1.5}$$

where the function $g(x)$ is continuous, 1-periodic with $g(0) = g(1) = 0$ and linear in the intervals $[\frac{k}{q}, \frac{k+1}{q}]$, $(k \in \mathbb{Z})$. First we consider $q-1$ specific functions $g_m(x)$ $(m \in \{1, \ldots, q-1\})$ which form a basis for all such $g(x)$, i.e.

$$g(x) = \sum_{m=1}^{q-1} \lambda_m g_m(x) \qquad (x \in \mathbb{R}) \tag{1.6}$$

with suitable coefficients λ_m. By means of certain functional equations for the corresponding Knopp functions

$$G_m(x) = \sum_{\nu=0}^{\infty} \frac{g_m\left(q^\nu x\right)}{q^\nu} \qquad (x \in \mathbb{R}) \tag{1.7}$$

we are able to express the number $s_m(k)$ of exactly those digits of the integer k in the q-ary representation which equal m. We show that

$$\frac{1}{n} \sum_{k=1}^{n-1} s_m(k) = \frac{1}{q} \log_q n + F_m\left(\log_q n\right) \tag{1.8}$$

where $F_m(u)$ is a continuous nowhere differentiable function with period 1 which is connected with G_m by

$$F_m(u) = -\frac{u}{q} - \frac{1}{q^{u+1}} G_m(q^u) \qquad (u \le 0).$$

The coefficients of the Fourier expansion of F_m can be expressed by means of the Hurwitz zeta function $\zeta(s, a)$ which for $\operatorname{Re} s > 1$ is defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{1.9}$$

where a is a fixed real number, $0 < a \le 1$. When $a = 1$ this reduces to the Riemann zeta function, $\zeta(s) = \zeta(s, 1)$, cf. [1], p. 249.

Next for arbitrary numbers $\lambda_1, \dots, \lambda_{p-1}$ we consider the Knopp function G from (1.5) with g from (1.6) and the function

$$s(k) = \sum_{m=1}^{q-1} \lambda_m s_m(k) \qquad (k \in \mathbb{N}_0). \tag{1.10}$$

For the sum (1.2) with $s(k)$ from (1.10) we show that it holds the formula

$$\frac{1}{n} S(n) = \frac{1}{q} S(q) \log_q n + F(\log_q n) \tag{1.11}$$

where F is a 1-periodic continuous nowhere differentiable function. Moreover, we can express the Fourier coefficients of F by means of the zeta function $\zeta(s, a)$. The connection between F in (1.11) and G from (1.5) with g from (1.6) is given by

$$F(u) = -\frac{1}{q} S(q) u - \frac{1}{q^{u+1}} G(q^u) \qquad (u \le 0). \tag{1.12}$$

As application we get formulas for several digital sums. In particular, for $\lambda_m = m$ ($m = 1, \dots, q-1$) we get the formula (1.3) of Delange for the sum-of-digits function and for $\lambda_m = 1$ a formula for the number of all digits which are different from zero. Finally, we also give a formula for the number of zeros.

2 Functional equations for specific Knopp functions

Throughout in this paper let q be a fixed integer with $q \ge 2$. In this paper for $m \in \{1, \dots, q-1\}$ we need the function G_m defined by (1.7) where the generated function g_m is given by

$$g_m(x) = \begin{cases} x & \text{for} \quad 0 \le x \le \frac{m}{q} \ , \\ m - (q-1)x & \text{for} \quad \frac{m}{q} \le x \le \frac{m+1}{q} \ , \\ x - 1 & \text{for} \quad \frac{m+1}{q} \le x \le 1 \ , \end{cases} \tag{2.1}$$

and by $g_m(x+1) = g_m(x)$ for $x \in \mathbb{R}$. This function can also be written as

$$g_m(x) = x - \left[\frac{qx}{m}\right](qx - m) + \left[\frac{qx}{m+1}\right](qx - m - 1) \qquad (0 \le x \le 1). \tag{2.2}$$

In particular, for $k \in \{0, 1, \dots, q\}$ we have

$$g_m\left(\frac{k}{q}\right) = \begin{cases} \frac{k}{q} & \text{for} \quad 1 \le k \le m \ , \\ \frac{k-q}{q} & \text{for} \quad m < k \le q-1 \ . \end{cases} \tag{2.3}$$

Obviously, the function G_m from (1.7) is continuous with $G_m(0) = 0$ and it holds $G_m(x+1) = G_m(x)$ for $x \in \mathbb{R}$. The function G_m satisfies the functional equation

$$G_m\left(\frac{x}{q}\right) = g_m\left(\frac{x}{q}\right) + \frac{1}{q}G_m(x) \qquad (x \in \mathbb{R}). \tag{2.4}$$

The function $s_m(k)$ which counts the digits m in (1.1) is given for $k \in \{0, 1, \ldots, q-1\}$ by

$$s_m(k) = \begin{cases} 1 & \text{for} \quad k = m\ , \\ 0 & \text{for} \quad k \neq m \end{cases} \tag{2.5}$$

and for arbitrary $k \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, q-1\}$ by

$$s_m(qk + r) = s_m(k) + s_m(r). \tag{2.6}$$

Proposition 2.1 *For $m \in \{1, \ldots, q-1\}$ the function G_m from (1.7) satisfies the functional equations*

$$G_m\left(\frac{k+x}{q^\ell}\right) = G_m\left(\frac{k}{q^\ell}\right) + \frac{\ell - qs_m(k)}{q^\ell}x + \frac{1}{q^\ell}G_m(x) \tag{2.7}$$

where $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, q^\ell - 1$, $x \in [0, 1]$. Moreover, for $n = 0, 1, \ldots, q^\ell$ we have

$$G_m\left(\frac{n}{q^\ell}\right) = \frac{n\ell}{q^\ell} - \frac{1}{q^{\ell-1}}\sum_{k=0}^{n-1} s_m(k). \tag{2.8}$$

Proof: Since $g_m(r) = 0$ for $r \in \mathbb{N}_0$ we get from (1.7) that

$$G_m\left(\frac{k}{q^\ell}\right) = \sum_{\nu=0}^{n-1} \frac{g_m(q^\nu \frac{k}{q^\ell})}{q^\nu}$$

and this implies

$$G_m\left(\frac{k+x}{q^\ell}\right) - G_m\left(\frac{k}{q^\ell}\right) = \sum_{\nu=0}^{\ell-1} \frac{g_m(q^\nu \frac{k+x}{q^\ell}) - g_m(q^\nu \frac{k}{q^\ell})}{q^\nu} + \sum_{\nu=\ell}^{\infty} \frac{g_m(q^\nu \frac{k+x}{q^\ell})}{q^\nu}.$$

For $\nu \geq \ell$ we find with $\mu = \nu - \ell \geq 0$ that $g_m(q^\nu \frac{k+x}{q^\ell}) = g_m(q^\mu k + q^\mu x) = g_m(q^\mu x)$ so that the last sum in the last equation is equal to $\frac{1}{q^\ell}G_m(x)$. For $\nu = 0, \ldots, \ell - 1$ there is no integer in the open interval $(q^\nu \frac{k}{q^\ell}, q^\nu \frac{k+1}{q^\ell})$, and hence the both numbers $q^\nu \frac{k+x}{q^\ell}$ and $q^\nu \frac{k}{q^\ell}$ belong to the same interval of the form $[r + \frac{s}{q}, r + \frac{s+1}{q}]$ with $r \in \mathbb{N}_0$ and $s \in \{0, 1, \ldots, q-1\}$. Since g_m is linear in each of these intervals we find that

$$\frac{g_m(q^\nu \frac{k+x}{q^\ell}) - g_m(q^\nu \frac{k}{q^\ell})}{q^\nu} = \varepsilon_\nu \frac{x}{q^\ell}$$

where $\varepsilon_\nu = -(q-1)$ when $q^\nu \frac{k}{q^\ell} \in [r + \frac{m}{q}, r + \frac{m+1}{q}]$ and where $\varepsilon_\nu = +1$ elsewhere in view of (2.1). If k has the representation (1.1) then we write shortly $\frac{k}{q^\ell} = a_\ell, a_{\ell-1} \dots a_0$ with $a_\ell = 0$ since $k < q^\ell$ and then $q^\nu \frac{k}{q^\ell} = a_\ell \dots a_{\ell-\nu}, a_{\ell-\nu-1} \dots a_0$ for $0 \le \nu \le \ell - 1$. Hence $\varepsilon_\nu = -(q-1)$ when $a_{\ell-\nu-1} = m$ which happens for $s_m(k)$ elements, and $\varepsilon_\nu = +1$ when $a_{\ell-\nu-1} \neq m$ which happens for $\ell - s_m(k)$ elements. This implies

$$\sum_{\nu=0}^{\ell-1} \varepsilon_\nu = -(q-1)s_m(k) + \ell - s_m(k) = \ell - q s_m(k)$$

and hence (2.7) is proved. Equation (2.8) follows from (2.7) with $x = 1$ and summation over k in view of $G_m(1) = 0$. □

3 The number of occurrences of a single digit

The equation (2.8) can be considered as sum formula for

$$S_m(n) = \sum_{k=1}^{n-1} s_m(k) \tag{3.1}$$

which is equal to the number of digits m in the q-ary representations of the integers $1, 2, \dots, n-1$. For this sum we have according to (2.8)

$$S_m(n) = \frac{n\ell}{q} - q^{\ell-1} G_m\left(\frac{n}{q^\ell}\right) \tag{3.2}$$

where $n \le q^\ell$ and G_m is given by (1.7). In particular, for $n = q^\ell$ we find from (3.2) in view of $G_m(1) = 0$ that the special sum $S_m(q^\ell) = \ell q^{\ell-1}$ is independent of m.

In order to obtain a representation of $S_m(k)$ $(m \in \{1, \dots, q-1\})$ which does not contain ℓ we introduce the function

$$f_m(x) = -\frac{1}{q}\left\{\frac{1}{x} G_m(x) + \log_q x\right\} \qquad (0 < x \le 1). \tag{3.3}$$

For $0 < x \le 1$ equation (2.4) simplifies to

$$G_m\left(\frac{x}{q}\right) = \frac{x}{q} + \frac{1}{q} G_m(x),$$

and therefore the function f_m has the property

$$f_m\left(\frac{x}{q}\right) = f_m(x) \qquad (0 < x \le 1).$$

Hence, we can extend the function $f_m(x)$ for all $x > 0$ by

$$f_m(qx) = f_m(x) \qquad (x > 0). \tag{3.4}$$

Theorem 3.1 *For the number of digits equal to m ($m \in \{1, \dots, q-1\}$) in the q-ary representation of the integers $1, 2, \dots, n-1$ we have*

$$\frac{1}{n} S_m(n) = \frac{1}{q} \log_q n + f_m(n) \tag{3.5}$$

where f_m is given by (3.3) and (3.4).

Proof: From (3.2) we get

$$\frac{1}{n} S_m(n) = \frac{1}{q} \left\{ \ell - \frac{q^\ell}{n} G_m \left(\frac{n}{q^\ell} \right) \right\}.$$

By means of (3.3) the term in brackets can be written as

$$\ell - \frac{q^\ell}{n} G_m \left(\frac{n}{q^\ell} \right) = \log_q n - \frac{q^\ell}{n} G_m \left(\frac{n}{q^\ell} \right) - \log_q \frac{n}{q^\ell} = \log_q n + q f_m \left(\frac{n}{q^\ell} \right).$$

In view of the property (3.4) we have

$$f_m \left(\frac{n}{q^\ell} \right) = f_m(n)$$

so that the representation (3.5) follows. $\square$

4 Periodic functions and Fourier expansions

According to (3.4) the function

$$F_m(u) = f_m(q^u) \qquad (u \in \mathbb{R}) \tag{4.1}$$

is periodic with period 1 so that in view of (3.3) Theorem 3.1 implies the

Corollary 4.1 *Let m be a fixed integer with $1 \le m \le q-1$. Then for the sum (3.1) we have*

$$\frac{1}{n} S_m(n) = \frac{1}{q} \log_q n + F_m(\log_q n) \tag{4.2}$$

where F_m is a continuous function of period 1 which is given by

$$F_m(u) = -\frac{u}{q} - \frac{1}{q^{u+1}} G_m(q^u) \qquad (u \le 0) \tag{4.3}$$

with G_m from (1.7).

In order to determine the Fourier expansion of the periodic function $F_m(u)$ we need the zeta function $\zeta(s, a)$ defined by (1.9) for $\operatorname{Re} s > 1$ and $0 < a \le 1$. The only singularity of $\zeta(s, a)$ is at the point $s = 1$, cf. [10], p. 265.

Lemma 4.2 *Let be $m \in \{1, \dots, q-1\}$ and $0 < \alpha \le \frac{m}{q}$. Then for the periodic function g_m from (2.1) we have for $\mathrm{Re}\, s > -1$, $s \ne 0, 1$*

$$\int_\alpha^\infty \frac{g_m(x)}{x^{s+2}} dx = \frac{1}{s\alpha^s} + q \frac{\zeta(s, \frac{m+1}{q}) - \zeta(s, \frac{m}{q})}{s(s+1)}. \tag{4.4}$$

Moreover, for the excluded values $s = 0$ and $s = 1$ we have

$$\int_\alpha^\infty \frac{g_m(x)}{x^2} dx = 1 - \log\alpha + q \log \frac{\Gamma(\frac{m+1}{q})}{\Gamma(\frac{m}{q})} \tag{4.5}$$

and

$$\int_\alpha^\infty \frac{g_m(x)}{x^3} dx = \frac{1}{\alpha} + \frac{q}{2} \left\{ \frac{\Gamma'(\frac{m}{q})}{\Gamma(\frac{m}{q})} - \frac{\Gamma'(\frac{m+1}{q})}{\Gamma(\frac{m+1}{q})} \right\}. \tag{4.6}$$

Proof: The integral (4.4), denoted by $I_m(s)$, converges absolutely for $\mathrm{Re}\, s > -1$. In view of (2.2) we have

$$I_m(s) = \int_\alpha^\infty \frac{x - [x]}{x^{s+2}} dx - J_m(s) + J_{m+1}(s)$$

where

$$J_m(s) = \int_\alpha^\infty \frac{1}{x^{s+2}} \left[\frac{(x - [x])q}{m} \right] ((x - [x])q - m)\, dx.$$

For $\mathrm{Re}\, s > 0$ the first integral can be computed by

$$\int_\alpha^\infty \frac{dx}{x^{s+1}} = \frac{1}{s\alpha^s}$$

and

$$\int_\alpha^\infty \frac{[x]}{x^{s+2}} dx = \int_1^\infty \frac{[x]}{x^{s+2}} dx = \frac{1}{s+1} \zeta(s+1),$$

cf. [3] (see also [1], p. 246). Moreover, for $\mathrm{Re}\, s > 1$ we have

$$\begin{aligned} J_m(s) &= \sum_{n=0}^\infty q \int_{n+m/q}^{n+1} \frac{dx}{x^{s+1}} - \sum_{n=0}^\infty (nq+m) \int_{n+m/q}^{n+1} \frac{dx}{x^{s+2}} \\ &= \frac{q}{s} \sum_{n=0}^\infty \left(\frac{1}{(n+\frac{m}{q})^s} - \frac{1}{(n+1)^s} \right) - \frac{1}{s+1} \sum_{n=0}^\infty \left(\frac{nq+m}{(n+\frac{m}{q})^{s+1}} - \frac{nq+m}{(n+1)^{s+1}} \right) \\ &= \frac{1}{s+1} \sum_{n=0}^\infty \frac{nq+m}{(n+1)^{s+1}} + \frac{q\zeta(s, \frac{m}{q})}{s(s+1)} - \frac{q}{s}\zeta(s) \end{aligned}$$

so that

$$J_{m+1}(s) - J_m(s) = \frac{1}{s+1}\zeta(s+1) + q \frac{\zeta(s, \frac{m+1}{q}) - \zeta(s, \frac{m}{q})}{s(s+1)}.$$

Hence,

$$I_m(s) = \frac{1}{s\alpha^s} + q \frac{\zeta(s, \frac{m+1}{q}) - \zeta(s, \frac{m}{q})}{s(s+1)}$$

which proves (4.4) for $\operatorname{Re} s > 1$. Since $\zeta(s,a)$ is analytic for $s \neq 1$ it follows that (4.4) is valid for $\operatorname{Re} s > -1$ excluded $s = 0$ and $s = 1$. In order to determine $I_m(0)$ we let s tend to zero and by means of the rule of de l' Hospital we get

$$\begin{aligned} I_m(0) &= \lim_{s\to 0} I_m(s) \\ &= \log \tfrac{1}{\alpha} + q\zeta'(0, \tfrac{m+1}{q}) - q\zeta'(0, \tfrac{m}{q}) - q\zeta(0, \tfrac{m+1}{q}) + q\zeta(0, \tfrac{m}{q}) \\ &= -\log \alpha + 1 + q \log \Gamma(\tfrac{m+1}{q}) - q \log \Gamma(\tfrac{m}{q}), \end{aligned}$$

since $\zeta(0,a) = \frac{1}{2} - a$ and $\zeta'(0,a) = \log \Gamma(a) - \frac{1}{2}\log(2\pi)$ (cf. [10], p. 271), and so we get (4.5). Finally, in view of

$$\lim_{s\to 1}\left(\zeta(s,a) - \frac{1}{s-1}\right) = -\frac{\Gamma'(a)}{\Gamma(a)}$$

(cf. [10], p. 271), we obtain

$$\lim_{s\to 1}\left(\zeta(s,a) - \zeta(s,b)\right) = \frac{\Gamma'(b)}{\Gamma(b)} - \frac{\Gamma'(a)}{\Gamma(a)}$$

and therefore (4.6). □

Proposition 4.3 *The continuous 1-periodic function $F_m(u)$ has the Fourier expansion*

$$F_m(u) = \sum_{k\in\mathbb{Z}} c_{mk} e^{2k\pi i u} \tag{4.7}$$

with

$$c_{m0} = \log_q \left(\frac{\Gamma(\frac{m}{q})}{\Gamma(\frac{m+1}{q})}\right) - \frac{1}{2q} - \frac{1}{q\log q}, \tag{4.8}$$

$$c_{mk} = \frac{\zeta(s_k, \frac{m}{q}) - \zeta(s_k, \frac{m+1}{q})}{s_k(s_k+1)\log q}, \qquad s_k = \frac{2\pi i k}{\log q}, \quad k \neq 0. \tag{4.9}$$

Proof: In view of the periodicity of $F_m(u)$ we have from (4.3)

$$F_m(u) = \frac{1}{q}(1-u) - \frac{1}{q^u} G_m(q^{u-1}) \qquad (0 \le u \le 1).$$

As in [3], p. 44, for the Fourier coefficients

$$c_{mk} = \int_0^1 F_m(u) e^{-2k\pi i u} du$$

we put $c_{mk} = a_{mk} + b_{mk}$ with

$$a_{mk} = \frac{1}{q}\int_0^1 (1-u) e^{-2k\pi i u} du,$$

i.e. $a_{m0} = \frac{1}{2q}$ and $a_{mk} = \frac{1}{2qk\pi i}$ for $k \neq 0$, and

$$b_{mk} = -\int_0^1 \frac{1}{q^u} G_m(q^{u-1}) e^{-2k\pi i u} du = -\sum_{\nu=0}^{\infty} \int_0^1 \frac{1}{q^{u+\nu}} g_m(q^{u+\nu-1}) e^{-2k\pi i u} du.$$

As in [3] we get by means of the substitution $u = 1 - \nu + \log_q x$ that

$$\int_0^1 \frac{1}{q^{u+\nu}} g_m(q^{u+\nu-1}) e^{-2k\pi i u} du = \frac{1}{q \log q} \int_{q^{\nu-1}}^{q^\nu} \frac{1}{x^2} g_m(x) e^{-2\pi i k \log_q x} dx$$

and hence

$$b_{mk} = -\frac{1}{q \log q} \int_{1/q}^{\infty} \frac{g_m(x)}{x^{2+2k\pi i/\log q}} dx.$$

By Lemma 4.2 with $\alpha = \frac{1}{q}$ we get the assertion. □

5 General Knopp functions

Now we consider the general Knopp function

$$G(x) = \sum_{\nu=0}^{\infty} \frac{g(q^\nu x)}{q^\nu} \qquad (x \in \mathbb{R}). \tag{5.1}$$

where the function g is continuous, 1-periodic with $g(0) = 0$, and linear in each interval $[\frac{k}{q}, \frac{k+1}{q}]$, $(k \in \mathbb{Z})$. Since the functions g_m from (2.1) form a basis for these functions, every g can be written as linear combination

$$g(x) = \sum_{m=1}^{q-1} \lambda_m g_m(x) \qquad (x \in \mathbb{R}) \tag{5.2}$$

with certain coefficients λ_m $(m \in \{1, \dots, q-1\})$. From (2.1) we get

$$g\left(\frac{k}{q}\right) = \left(\frac{k}{q} - 1\right) \sum_{m=1}^{k-1} \lambda_m + \frac{k}{q} \sum_{m=k}^{q-1} \lambda_m \tag{5.3}$$

and it easy to see that

$$\lambda_m = g\left(\frac{1}{q}\right) + g\left(\frac{m}{q}\right) - g\left(\frac{m+1}{q}\right). \tag{5.4}$$

According to (5.2) the Knopp function G from (5.1) can be written as

$$G(x) = \sum_{m=1}^{q-1} \lambda_m G_m(x) \qquad (x \in \mathbb{R}) \tag{5.5}$$

with G_m from (1.7).

Now for $k \in \mathbb{N}_0$ we consider the function

$$s(k) = \sum_{m=1}^{q-1} \lambda_m s_m(k) \tag{5.6}$$

with $s_m(k)$ from (2.5) and (2.6). By (2.5) we have $s(0) = 0$ and $s(m) = \lambda_m$ for $m = 1, \ldots, q-1$, and (2.6) implies

$$s(kq + r) = s(k) + s(r) \tag{5.7}$$

for $k \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, q-1\}$.

Proposition 5.1 *Every function $s(k)$ with the property (5.7) can be written in the form (5.6) with*

$$\lambda_m = s(m) \qquad (m = 1, \ldots, q-1). \tag{5.8}$$

Proof: Assume that $s(k)$ is a given function satisfying (5.7) then for $k \in \mathbb{N}_0$ we put

$$s_0(k) = s(k) - \sum_{m=1}^{q-1} s(m) s_m(k). \tag{5.9}$$

In view of (2.5) it holds $s_0(k) = 0$ for $k = 0, 1, \ldots, q-1$. Moreover, according to (5.7) and (2.6) we have for $k \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, q-1\}$

$$s_0(qk + r) = s_0(k) + s_0(r).$$

It follows $s_0(k) = 0$ for all $k \in \mathbb{N}_0$ so that (5.9) implies the assertion. $\square$

Let

$$S(n) = \sum_{k=1}^{n-1} s(k) \tag{5.10}$$

with $s(k)$ from (5.6), then (5.3) can be written as

$$g\left(\frac{k}{q}\right) = \frac{k}{q} S(q) - S(k) \qquad (k = 0, 1, \ldots, q). \tag{5.11}$$

In particular, $g(0) = g(1) = 0$ and $g(\frac{1}{q}) = \frac{1}{q} S(q)$.

In view of (5.5), (5.6) and (5.10) we get from Proposition 2.1 the

Theorem 5.2 *For $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, q^\ell - 1$, $x \in [0,1]$ the Knopp function G from (5.1) with g from (5.2) satisfies the functional equations*

$$G\left(\frac{k+x}{q^\ell}\right) = G\left(\frac{k}{q^\ell}\right) + \frac{S(q)\ell - qs(k)}{q^\ell}x + \frac{1}{q^\ell}G(x). \tag{5.12}$$

Moreover, for $n = 0, 1, \ldots, q^\ell$ we have

$$G\left(\frac{n}{q^\ell}\right) = \frac{S(q)n\ell - qS(n)}{q^\ell} \tag{5.13}$$

with $S(n)$ from (5.10).

It is known that in case $g(x) \not\equiv 0$ the Knopp function G from (5.1) is nowhere differentiable, cf. [2] and [5]. In [5] it was shown even that in the case $g(x) \not\equiv 0$ the function G from (5.1) does not have anywhere a finite one-sided derivative. We show that this property is a consequence of (5.12) where we need the following simple lemma, cf. [4].

Lemma 5.3 *Let $f : [0,1] \mapsto \mathbb{R}$ have a finite right-hand derivative $f'_+(x_0)$ at the point $x_0 \in [0,1)$. Let further (u_ℓ) and (v_ℓ) be sequences in $[0,1]$ with $x_0 < u_\ell < v_\ell$ for all $\ell \in \mathbb{N}$ and $v_\ell \to x_0$ as $\ell \to \infty$. If there exists a $p > 0$ with $u_\ell - x_0 \le p(v_\ell - u_\ell)$ for all $\ell \in \mathbb{N}$ then*

$$\frac{f(v_\ell) - f(u_\ell)}{v_\ell - u_\ell} \to f'_+(x_0) \qquad (\ell \to \infty).$$

Proposition 5.4 *If $g(x) \not\equiv 0$ then the Knopp function G from (5.1) has nowhere a finite one-sided derivative.*

Proof: Assume, at $x_0 \in [0,1)$ there exists the finite right-hand derivative $G'_+(x_0)$. For $\ell \in \mathbb{N}$ and $k = 0, 1, \ldots, q^\ell - 1$ we put $x_{k,\ell} = k/q^\ell$ and $N_{a,b} = \{k \in \mathbb{N} : a \le k \le b\}$. If $x_{k',\ell} \le x_0 < x_{k'+1,\ell}$ then for every $k \in N_{k'+1,k'+2q-1}$ we put $u_{k,\ell} = x_{k,\ell}$ and $v_{k,\ell} = x_{k+1,\ell}$ so that $x_0 < u_{k,\ell} < v_{k,\ell}$ and $u_{k,\ell} - x_0 \le p(v_{k,\ell} - u_{k,\ell})$ with $p = 2q$. Applying (5.12) with $x = 1$ we get

$$\frac{G(v_{k,\ell}) - G(u_{k,\ell})}{v_{k,\ell} - u_{k,\ell}} - \frac{G(v_{k+1,\ell}) - G(u_{k+1,\ell})}{v_{k+1,\ell} - u_{k+1,\ell}} = \{S(q)\ell - qs(k)\} - \{S(q)\ell - qs(k+1)\}$$

and Lemma 5.3 implies that for $k \in N_{k'+1,k'+2q-1}$ we have

$$s(k+1) - s(k) \to 0 \qquad (\ell \to \infty).$$

The set $N_{k'+1,k'+2q-1}$ contains a section of the form $N_{d,d+q-2}$ with $d = qk_0 \le k' + q$. For $k \in N_{d,d+q-2}$, i.e. $k = qk_0 + r$ with $r = 0, 1, \ldots, q-2$, we have in view of (5.7) and (5.8) that $s(k) = s(qk_0 + r) = s(k_0) + s(r) = s(k_0) + \lambda_r$ with $\lambda_0 = 0$ and hence

$$s(k+1) - s(k) = \lambda_{r+1} - \lambda_r \to 0 \qquad (\ell \to \infty).$$

This implies $\lambda_r = 0$ for all $r = 1, \ldots, q-1$ since $\lambda_0 = 0$. $\square$

6 Digital sums

From Corollary 4.1, Proposition 4.3 and Proposition 5.4 we get for the sum $S(n)$ from (5.10) in view of $\lambda_m = s(m)$ for $m = 1, \ldots, q-1$ and $\lambda_1 + \ldots + \lambda_{q-1} = S(q)$ the main result concerning digital sums.

Theorem 6.1 *For $S(n)$ from (5.10) with $s(k)$ from (5.6) we have the formula*

$$\frac{1}{n} S(n) = \frac{S(q)}{q} \log_q n + F(\log_q n) \tag{6.1}$$

where $F(u) = \lambda_1 F_1(u) + \ldots + \lambda_{q-1} F_{q-1}(u)$ is a continuous, nowhere differentiable function of period 1 which is given by

$$F(u) = -\frac{S(q)u}{q} - \frac{1}{q^{u+1}} G(q^u) \qquad (u \le 0) \tag{6.2}$$

with G from (5.1). The Fourier coefficients of F read

$$c_k = \sum_{m=1}^{q-1} \lambda_m c_{mk} \tag{6.3}$$

with c_{mk} from (4.8), (4.9).

We want to point out this for two examples.

1. The sum-of-digits function. For the sum of digits in the q-ary expansion of the integer k we have $\lambda_m = s(m) = m$ for $m \in \{1, \ldots, q-1\}$. Theorem 6.1 for $\lambda_m = m$ yields the well-known formula (1.3) of Delange where F is a continuous nowhere differentiable function which is given by

$$F(u) = -\frac{q-1}{2} u - \frac{1}{q^{u+1}} G(q^u) \qquad (u \le 0) \tag{6.4}$$

where G is given by (5.1) with g from (5.2). The Fourier coefficients of $F(u)$ are

$$\begin{aligned} c_0 &= \frac{q-1}{2} \log_q(2\pi) - \frac{q+1}{4} - \frac{q-1}{2\log q}, \\ c_k &= -\frac{q-1}{\log q} \frac{\zeta(s_k)}{s_k(s_k+1)}, \qquad s_k = \frac{2k\pi i}{\log q}, \quad k \ne 0 \end{aligned}$$

which follow from (6.3) with $\lambda_m = m$ in view of the relations

$$\prod_{m=1}^{q-1} \left(\frac{\Gamma(\frac{m}{q})}{\Gamma(\frac{m+1}{q})} \right)^m = \prod_{m=1}^{q-1} \Gamma\left(\frac{m}{q}\right) = \frac{(2\pi)^{\frac{q-1}{2}}}{\sqrt{q}}$$

and

$$\sum_{m=1}^{q-1} m\left(\zeta(s,\tfrac{m}{q})-\zeta(s,\tfrac{m+1}{q})\right) = \sum_{m=1}^{q-1}\zeta(s,\tfrac{m}{q})-(q-1)\zeta(s)$$
$$= (q^s-q)\zeta(s).$$

2. The number of digits different from zero. For the number of digits which are different from zero in the q-ary representation of the integer k we have (5.6) with $\lambda_m=1$ for $m\in\{1,\dots,q-1\}$ and the function (5.2) for $0\le x\le 1$ reads

$$g(x)=\sum_{m=1}^{q-1} g_m(x)=\begin{cases}(q-1)x & \text{for } 0\le x\le \frac{1}{q}\,,\\ 1-x & \text{for } \frac{1}{q}<x\le 1\,.\end{cases} \tag{6.5}$$

Theorem 6.1 for $\lambda_m=1$ yields:

Corollary 6.2 *Let $S(n)$ denote the numbers of digits different from zero in the q-ary representations of the integers $1,2,\dots,n-1$. Then it holds*

$$\frac{1}{n}S(n)=\frac{q-1}{q}\log_q n+F(\log_q n) \tag{6.6}$$

where $F(u)$ is a continuous nowhere differentiable function of period 1 which is given by

$$F(u)=-\frac{(q-1)u}{q}-\frac{1}{q^{u+1}}G(q^u) \qquad (u\le 0) \tag{6.7}$$

where G is given by (5.1) with g from (5.2). The Fourier expansion of the periodic function $F(u)$ has the coefficients

$$c_0 = \log_q\Gamma\left(\frac{1}{q}\right)-\frac{q-1}{2q}-\frac{q-1}{q\log q}\,,$$
$$c_k = \frac{\zeta(s_k,\frac{1}{q})-\zeta(s_k)}{s_k(s_k+1)\log q}\,, \qquad s_k=\frac{2k\pi i}{\log q}\,, \quad k\neq 0.$$

7 The number of zeros

In Corollary 4.1 we have given a formula for the number of a fixed digit $m\in\{1,\dots,q-1\}$. Now, we consider the digit $m=0$. In order to determine the number of zeros in the q-ary expansion first we compute the number of all digits. Let $a(k)$ denote the number of all digits in the q-ary expansion of k, i.e. $a(k)=\ell+1$ if $q^\ell\le k<q^{\ell+1}$. We state a formula for the sum

$$A(n)=\sum_{k=1}^{n-1}a(k). \tag{7.1}$$

Proposition 7.1 *For the number of all digits in the q-ary representations of the integers $1, 2, \dots, n-1$ we have*

$$\frac{1}{n}A(n) = \log_q n + \frac{1}{(q-1)n} + H(\log_q n) \tag{7.2}$$

where H is a continuous function of period 1 which is given by

$$H(u) = 1 - u - \frac{1}{q-1}q^{1-u} \qquad (0 \le u < 1). \tag{7.3}$$

Proof: We have $a(k) = 1$ for $k = 1, \dots, q-1$, $a(k) = 2$ for $k = q, \dots, q^2-1$ and so on. Since for $k \ge 1$ the first digit may be $1, \dots, q-1$ and the following digits may be $0, 1, \dots, q-1$ we get for the sum (7.1) the special values $A(q) = q-1$, $A(q^2) = q-1+2q(q-1)$, $A(q^3) = q-1+2q(q-1)+3q^2(q-1)$ and in general

$$A(q^\ell) = (q-1)(1+2q+3q^2+\ldots+\ell q^{\ell-1}).$$

In view of

$$1+2t+3t^2+\ldots+\ell t^{\ell-1} = \frac{(\ell+1)t^\ell(t-1)-(t^{\ell+1}-1)}{(t-1)^2} \qquad (t \ne 1)$$

we get

$$A(q^\ell) = \ell q^\ell - \frac{q^\ell - 1}{q-1}.$$

It follows for $0 \le k \le q^{\ell+1} - q^\ell$ that

$$A(q^\ell + k) = \ell q^\ell - \frac{q^\ell-1}{q-1} + (\ell+1)k$$

i.e.

$$A(q^\ell + k) = \ell(q^\ell + k) - \frac{q^\ell - 1}{q-1} + k.$$

Write $n = q^\ell + k = q^\ell(1+x)$ with $0 \le x < q-1$ we get in view of $\frac{q^\ell}{n} = \frac{1}{1+x}$, $\frac{k}{n} = 1 - \frac{1}{1+x}$ and $\ell = \log_q n + \log_q(\frac{q^\ell}{n}) = \log_q n - \log_q(1+x)$

$$\begin{aligned}
\frac{1}{n}A(n) &= \ell - \frac{q^\ell-1}{n(q-1)} + \frac{k}{n} \\
&= \log_q n + \frac{1}{n(q-1)} + \left\{-\log_q(1+x) - \frac{1}{(q-1)(1+x)} + 1 - \frac{1}{1+x}\right\} \\
&= \log_q n + \frac{1}{n(q-1)} + \left\{1 - \log_q(1+x) - \frac{q}{(q-1)(1+x)}\right\}.
\end{aligned}$$

This yields the assertion since in view of the periodicity of H we have for $n = q^\ell(1+x)$

$$H(\log_q n) = H(\log_q[q^\ell(1+x)]) = H(\log_q(1+x)) = H(u)$$

with $1 + x = q^u$ $(0 \le u < 1)$. □

The following result is a generalization of Theorem 3.2 in [6].

Proposition 7.2 *Let $s_0(k)$ be the number of zeros of k in the q-ary representation of k. Then it holds*

$$\frac{1}{n}\sum_{k=1}^{n-1} s_0(k) = \frac{\log_q n}{q} + \frac{1}{(q-1)n} + F_0(\log_q n) \tag{7.4}$$

where F_0 is a continuous nowhere differentiable function of period 1 which is given by

$$F_0(u) = \frac{1-u}{q} + \frac{1}{q^u} G(q^{u-1}) - \frac{q^{1-u}}{q-1} \qquad (0 \le u < 1) \tag{7.5}$$

where G is given by (5.1) with the 1-periodic function g given by (6.5). The continuous periodic function $F_0(u)$ has the Fourier expansion

$$F_0(u) = \sum_{k\in\mathbb{Z}} c_{0k} e^{2k\pi i u} \tag{7.6}$$

with

$$\begin{aligned} c_{00} &= \frac{2q-1}{2q} - \frac{1}{q\log q} - \log_q \Gamma\left(\frac{1}{q}\right), \\ c_{0k} &= \frac{1-\zeta(s_k)+\zeta(s_k,\frac{1}{q})}{s_k(s_k+1)\log q}, \qquad s_k = \frac{2\pi i k}{\log q}, \quad k \neq 0. \end{aligned}$$

Proof: We have $s_0(n) = a(n) - s(n)$ where $a(n)$ counts the number of all digits of n in the q-ary expansion and $s(n)$ counts the number of all digits different from zero. Hence Proposition 7.1 and Corollary 6.2 imply the assertion. Formulas (6.6) and (7.2) imply (7.4) with $F_0(u) = H(u) - F(u)$. Since the Fourier coefficients c_k of F are known, we have to compute the Fourier coefficients h_k of

$$H(u) = \sum_{k\in\mathbb{Z}} h_k e^{2k\pi i u}.$$

We put $h_k = a_k + b_k$ with

$$a_k = \int_0^1 (1-u)e^{-2k\pi i u}du,$$

i.e. $a_0 = \frac{1}{2}$ and $a_k = \frac{1}{2k\pi i}$ for $k \neq 0$, and

$$b_k = \frac{-q}{q-1}\int_0^1 q^{-u}e^{-2k\pi i u}du.$$

Substitution $x = q^u$ yields that

$$\int_0^1 q^{-u}e^{-2k\pi i u}du = \frac{1}{\log q}\int_1^q \frac{1}{x^2}e^{-2\pi i k \log_q x}dx$$

and hence

$$b_k = -\frac{q}{(q-1)\log q}\int_1^q \frac{dx}{x^{2+2k\pi i/\log q}}.$$

So $b_k = \frac{-1}{\log q + 2k\pi i}$ and by $h_k = a_k + b_k$ we get $h_0 = \frac{1}{2} - \frac{1}{\log q}$ and

$$h_k = \frac{1}{s_k(1+s_k)\log q}\,, \qquad s_k = \frac{2k\pi i}{\log q}, \quad k \neq 0.$$

This completes the proof. □

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