Rostocker Mathematisches Kolloquium

Heft 61

Universität Rostock

Institut für Mathematik

2006

Herausgeber: Universität Rostock
Mathematisch – Naturwissenschaftliche
Fakultät
Institut für Mathematik

Wissenschaftlicher Beirat: Hans-Dietrich Gronau
Friedrich Liese
Peter Takáč
Günther Wildenhain

Schriftleitung: Wolfgang Peters

Herstellung der Druckvorlage: Susann Dittmer

Zitat–Kurztitel: Rostock. Math. Kolloq. **61** (2006)

ISSN 0138-3248

BEZUGSMÖGLICHKEITEN: Universität Rostock
Universitätsbibliothek, Schriftentausch
18051 Rostock
Tel.: +49-381-498 22 81
Fax: +49-381-498 22 68
e-mail: maria.schumacher@ub.uni-rostock.de

Universität Rostock
Institut für Mathematik
18051 Rostock
Tel.: +49-381-498 6551
Fax: +49-381-498 6553
e-mail: romako@uni-rostock.de

DRUCK: Universitätsdruckerei Rostock

Rostock. Math. Kolloq. **61**, 3–11 (2006) Subject Classification (AMS)
39A11,41A60

Lothar Berg

On the Difference Equation $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(\gamma x_n + \beta x_{n-1})$

ABSTRACT. The difference equation in the title is solved by means of functions, which can be represented as composed functions of the exponential function and a function being odd with respect to one or two arbitrary parameters. In the case $\beta = 1/4$, $\gamma = 3/4$ there is given a conjecture concerning a solution of a new type. A second conjecture concerns the existence of asymptotically 3-periodic solutions. Though the difference equation is of second order, we point out singular cases where three initial values can be prescribed.

KEY WORDS. Nonlinear difference equations, odd functions, asymptotic behaviour, 3-periodic solutions, three initial values, conjectures.

Rational difference equations of second order are systematically investigated in the book M. R. S. Kulenović and G. Ladas [6], where also various applications of these equations are pointed out.

Here, we consider the special case

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{\gamma x_n + \beta x_{n-1}} \tag{1}$$

with $\beta^2 \neq \gamma^2$ and integer n. According to the classification in [6] it belongs to the type $(2,2)$ or to the type $(1,1)$. In the case of positive coefficients a detailed stability and semicycle analysis of the positive solutions of (1) is carried out in [6, Chapter 6.9].

In this paper we study arbitrary solutions of (1) with respect to its structure and its asymptotic behaviour. These solutions can have zeros, but not at consecutive points.

It can easily be seen that (1) has the following property:

Proposition 1 *If $x_n = w_n$ is a special solution of (1), then $x_n = \frac{1}{w_n}$ is also a solution of (1).*

Here we allow that a solution equals infinity, but not at consecutive points. This means that the forbidden set of the initial values (x_{-1}, x_0) is only $\{(0,0), (\infty,\infty)\}$, cf. [6, pp. 2.17].

Without loss of generality we use for the coefficients the normalization

$$\beta + \gamma = 1\,, \tag{2}$$

so that we can eliminate γ, and it follows $\beta \neq \frac{1}{2}$. Equation (1) has the equilibrium $\overline{x} = 1$, and about this equilibrium it has the linearized equation

$$y_{n+1} = (2\beta - 1)(y_n - y_{n-1})$$

with the characteristic equation

$$\lambda^2 + (1 - 2\beta)(\lambda - 1) = 0\,. \tag{3}$$

At first we study solutions with one arbitrary parameter, and afterwards with two parameters. Moreover, we consider the exceptional case $\beta = \frac{1}{3}$, we ask for asymptotically 3-periodic solutions, and finally we point out cases with three given initial values.

One parameter. Let $\lambda = z$ be a solution of (3) with $|z| \neq 1$. Then (1) possesses a solution of the form

$$x_n = 1 + \sum_{j=1}^{\infty} c_j a^j z^{nj} \tag{4}$$

with $c_1 = 1$, arbitrary a and $|c_j| \leq M^{j-1}$ for a certain constant M, cf. [2]. Hence, (4) converges for

$$|az^n| < \frac{1}{M}\,, \tag{5}$$

i. e. for sufficiently large n in case of $|z| < 1$, and for sufficiently large $-n$ in case of $|z| > 1$. The series in (4) is simultaneously an asymptotic expansion as $n \to +\infty$ resp. $n \to -\infty$.

Proposition 2 *If x_n is the solution (4) of (1), then $c_2 = \frac{1}{2}$ and, under the condition (5), the solution $\frac{1}{x_n}$ has the expansion (4) with $-a$ instead of a. Moreover, under the sharpening of (5)*

$$|az^n| < \frac{1}{M+1} \tag{6}$$

there exists a function $f_n(a)$ being odd in a such that

$$x_n = \exp(f_n(a))\,. \tag{7}$$

Proof: Under the condition (5) it follows from (4) that

$$\frac{1}{x_n} = 1 - az^n + (1-c_2)a^2z^{2n} + \dots\,. \tag{8}$$

According to Proposition 1 the left-hand side is also a solution of (1), and the right-hand side must have the form (4), i. e. the expansion in (8) is the expansion (4) with $-a$ instead of a.

Let $x_n = F_n(a)$ and therefore $\frac{1}{x_n} = F_n(-a)$. Under the condition (6) it is $|F_n(a) - 1| < 1$, so that $f_n(a) = \ln(F_n(a))$ exists, and this function satisfies $f_n(a) = -f_n(-a)$. From this and

$$f_n(a) = \ln(1 + az^n + c_2a^2z^{2n} + \dots) = az^n + \left(c_2 - \frac{1}{2}\right)a^2z^{2n} + \dots$$

it follows that $c_2 = \frac{1}{2}$ ■

The functions $F_n(a)$ and $f_n(a)$ are holomorphic under the condition (6). Hence, the analytic continuation of (7) remains a solution of (1).

Example 1 In the case $\beta = \frac{1}{4}$ we can choose the solution $\lambda = \frac{1}{2}$ of (3) and obtain by means of the DERIVE system

$$x_n = \exp\left\{\frac{a}{2^n} + \frac{1}{108}\frac{a^3}{2^{3n}} + \frac{19}{71280}\frac{a^5}{2^{5n}} + \frac{68437}{6951510720}\frac{a^7}{2^{7n}} + \dots\right\}$$

and for the coefficient in (4)

$$c_3 = \frac{19}{108}\,,\ c_4 = \frac{11}{216}\,,\ c_5 = \frac{943}{71280}\,,\ c_6 = \frac{4159}{1283040}\,,\ c_7 = \frac{764869}{993072960}\,.$$

Note that (1) in this example corresponds to [6, (6.66)] with $p = 1/3$ and $q = 3$, however, neither [6, (6,67)] nor [6, (6,68)] are satisfied.

Two parameters. Let $\lambda = z$ and $\lambda = s$ be two different solutions of (3), and assume that $\lambda = z^js^\ell$ is no solution of (3) for all non-negative integers j, ℓ with $j + \ell \geq 2$, then (1) has also a solution of the form

$$x_n = 1 + \sum_{1 \leq j+\ell} c_{j\ell}\, a^j z^{nj} b^\ell s^{n\ell} \tag{9}$$

with $c_{10} = c_{01} = 1$, arbitrary a, b, and $|c_{j\ell}| \leq M^{j+\ell-1}$ for a certain constant M, which is convergent for $|z| < 1$, $|s| < 1$ and n large, $|z| > 1$, $|s| > 1$ and $-n$ large, cf. [2] in the real, and [3] in the complex case, and also for $|z| < 1 < |s|$, if $|a|$ and $|b|$ are sufficiently small and

$$\frac{\ln(|a|M)}{-\ln|z|} < n < \frac{-\ln(|b|M)}{\ln s}\,.$$

Instead of a detailed analysis we only mention that with some more effort Proposition 2 can be generalized to these cases, and that also the analytic continuation can be applied.

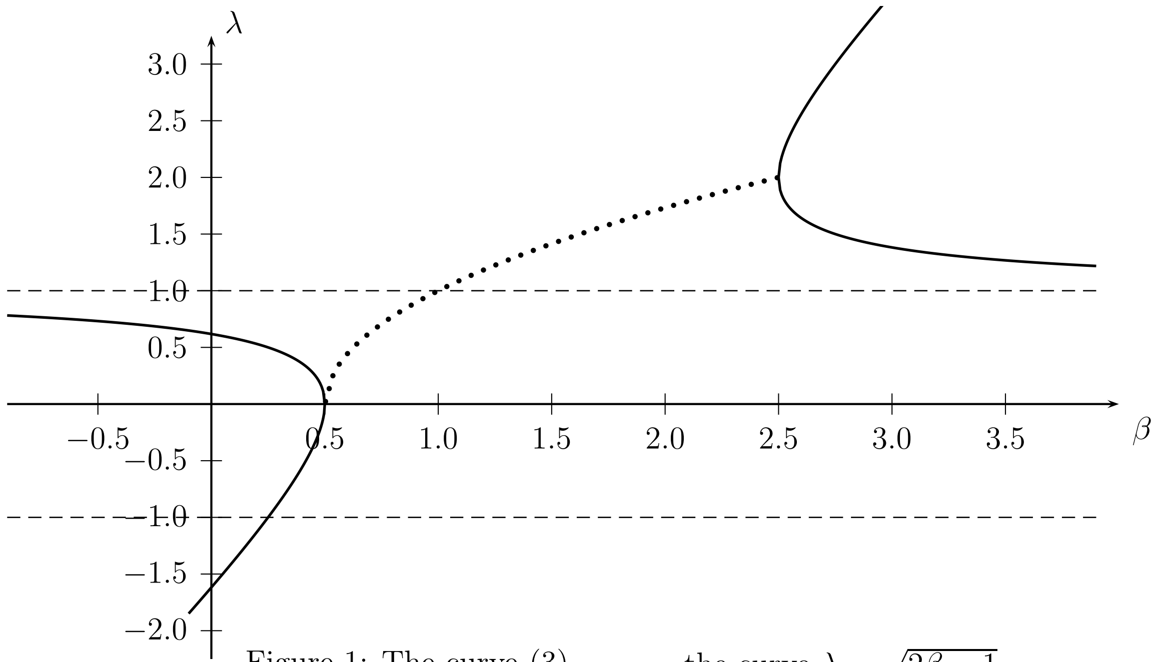


Figure 1: The curve (3) ——— , the curve $\lambda = \sqrt{2\beta - 1}$ $\cdots\cdots$ and the straight lines $\lambda = \pm 1$ - - - -

Figure 2 shows the real branches of the curve (3), and for $\frac{1}{2} < \beta < \frac{5}{2}$ the curve $\lambda = \sqrt{2\beta - 1}$ $(= |z| = |s|)$ concerning the complex branches. For $\frac{1}{4} < \beta < \frac{1}{2}$ there are two real, and for $\frac{1}{2} < \beta < 1$ two complex solutions with $|\lambda| < 1$. For $1 < \beta < \frac{5}{2}$ there are two complex, and for $\frac{5}{2} < \beta$ two real solutions with $|\lambda| > 1$. For $\beta < \frac{1}{4}$ there is one solution with $\frac{1}{2} < \lambda < 1$ and one solution with $\lambda < -1$.

In the two cases, where (1) belongs to the type $(1, 1)$, we have elementary solutions of the form

$$x_n = \exp\{az^n + bs^n\}\,, \tag{10}$$

which visualize Proposition 2, and which can be expanded in the form (9) for all a, b and all n. The first case is $\beta = 0$ with $z = -\frac{1}{2} + \frac{\sqrt{5}}{2}$, $s = -\frac{1}{2} - \frac{\sqrt{5}}{2}$ and therefore $|z| < 1 < |s|$, cf. [6, Chapter 3.3]. The second case is $\beta = 1$ with $z = e^{i\pi/3}$, $s = e^{-i\pi/3}$ and $|z| = |s| = 1$, cf. [6, Chapter 3.2] and [3, Example 4]. Of course, this solution (10) can also be written in the real form

$$x_n = \exp\left\{c\,\cos\frac{n\pi}{3} + d\,\sin\frac{n\pi}{3}\right\}$$

with arbitrary real parameters c, d.

In some of the excluded cases with $z^k s^\ell = 1$ for some integers k, ℓ, there also exist solutions of the form (9), however, with polynomial coefficients, cf. [2].

Example 2 Let us come back to Example 1 with $\beta = \frac{1}{4}$, but concerning the second solution $\lambda = -1$ of (5). In this case equation (1) should be expected to have a 2-periodic solution, however, it turns out that such a solution must be the constant $x_n = 1$.

Hence, we expect that (1) has a solution of the form

$$x_n = 1 + u_n + (-1)^n v_n \tag{11}$$

with functions u_n, v_n tending to zero as $n \to \infty$. In order to find the asymptotic behaviour of these functions, we use the heuristic method from [1], assuming $n = t$ as a continuous variable, replacing u_{n+k} according to Taylor approximately by $u + ku'$, assuming that $u' = o(u)$, and proceeding analogously with v_n. Then (1) can be replaced approximately by

$$(1 + u + u' - (-1)^n(v + v'))(4 + 4u - u' + (-1)^n(2v + v')) = 4 + 4u - 3u' + (-1)^n(3v' - 2v)\,.$$

Comparing coefficients of $(-1)^n$ we obtain

$$\begin{aligned} 6u' + 4u + 4u^2 + 3uu' - u'^2 - 2v^2 - 3vv' - v'^2 &= 0\,, \\ -6v' - 2uv - 3uv' + 3vu' + 2u'v' &= 0\,. \end{aligned}$$

Cancelling all terms of smaller order and dividing by 2, these equations reduce to

$$\begin{aligned} 2u &= v^2\,, \\ -3v' &= uv\,. \end{aligned}$$

Integration yields

$$u = \frac{3}{2t}\,, \quad v = \pm\sqrt{\frac{3}{t}}\,, \tag{12}$$

disregarding the constant of integration. A further analysis shows that we can expect an improvement of (11) with (12) (taking the sign +) in the form

$$x_n = 1 + \frac{3}{2n} + (a\ln n + b)\frac{1}{n^2} + (-1)^n\left(\sqrt{\frac{3}{n}} + (c\ln n + d)\frac{1}{\sqrt{n^3}}\right) \tag{13}$$

up to smaller terms as $n \to \infty$. By means of the DERIVE system we find

$$a = -\frac{3}{8}\,, \quad b = \sqrt{3}d - \frac{9}{8}\,, \quad c = -\frac{\sqrt{3}}{8}\,, \tag{14}$$

where d is an arbitrary constant, cf. [7, Remark 1], and

$$x_{n+1} - \frac{x_n + 3x_{n-1}}{3x_n + x_{n-1}} \sim -\frac{3}{32}\,\frac{\ln^2 n}{n^3}\,.$$

Conjecture 1 *There exists a solution x_n of (1) such that the expansion (13) with (14) is valid up to $o\left(\frac{1}{n^2}\right)$.*

However, similarly as in [8, Conjecture 1] we cannot prove it. Obviously, a solution x_n having the finite asymptotic expansion (13) as $n \to \infty$ is oscillating about the equilibrium 1. From (13) it follows that $\frac{1}{x_n}$ has the asymptotic expansion (13) with $-\sqrt{3}$, $-d$ instead of $\sqrt{3}$, d and that

$$x_n = \exp\left\{(-1)^n\left(\sqrt{\frac{3}{n}} - \frac{1}{8}\left(\sqrt{3}\ln n - 8d + 4\sqrt{3}\right)\frac{1}{\sqrt{n^3}}\right.\right.$$
$$\left.\left. + \frac{3}{80}\left(5\sqrt{3}\ln n - 40d + 18\sqrt{3}\right)\frac{1}{\sqrt{n^5}}\right)\right\}$$

both up to smaller terms as $n \to \infty$, where the argument of the exponential function is an odd function of $\sqrt{n}$.

Asymptotically 3-periodic solutions. Since 2-periodic solutions were already investigated in [6, Section 6.9.1], we ask for 3-periodic solutions. It can easily be seen that

$$\ldots, -1, -1, 1, -1, -1, 1, \ldots \tag{15}$$

is such a solution of (1) for all considered coefficients.

In connection with (15) it makes sense to ask for asymptotically 3-periodic solutions of the form

$$\left.\begin{array}{rcrcl} x_{3n-1} & = & -1 & + & a\lambda^n \\ x_{3n} & = & -1 & + & b\lambda^n \\ x_{3n+1} & = & 1 & + & c\lambda^n \end{array}\right\} \tag{16}$$

up to $O(\lambda^{2n})$, cf. [1, Section 5]. Substituting (16) into (1) we find

$$(1 + c\lambda^n)(\gamma(-1 + b\lambda^n) + \beta(-1 + a\lambda^n)) = \beta(-1 + b\lambda^n) + \gamma(-1 + a\lambda^n)$$
$$(-1 + a\lambda^{n+1})(\gamma(1 + c\lambda^n) + \beta(-1 + b\lambda^n)) = \beta(1 + c\lambda^n) + \gamma(-1 + b\lambda^n)$$
$$(-1 + b\lambda^{n+1})(\gamma(-1 + a\lambda^{n+1}) + \beta(1 + c\lambda^n)) = \beta(-1 + c\lambda^{n+1}) + \gamma(1 + c\lambda^n)$$

again up to $O(\lambda^{2n})$, and, comparing the coefficients of λ^n, it follows

$$\left.\begin{array}{rcrcrcl} \delta a & - & \delta b & - & c & = & 0 \\ \delta\lambda a & + & b & + & c & = & 0 \\ \lambda a & - & \delta\lambda b & + & c & = & 0 \end{array}\right\} \tag{17}$$

using the notation $\delta = \beta - \gamma = 2\beta - 1$ according to (2). The homogeneous system (17) has a non-trivial solution, if its determinant

$$\begin{vmatrix} \delta & -\delta & -1 \\ \delta\lambda & 1 & 1 \\ \lambda & -\delta\lambda & 1 \end{vmatrix} = \delta^2\lambda^2 + (2\delta^2 - \delta + 1)\lambda + \delta$$

vanishes, i.e. with the notation $\eta = \frac{1}{\delta}$, if

$$\lambda^2 + (2 - \eta + \eta^2)\lambda + \eta = 0\,. \tag{18}$$

If the condition (18) is satisfied, then (17) has the solution

$$b = \frac{1+\lambda}{1-\eta}\,a\,, \qquad c = \frac{\eta+\lambda}{\eta(\eta-1)}\,a \tag{19}$$

with arbitrary a.

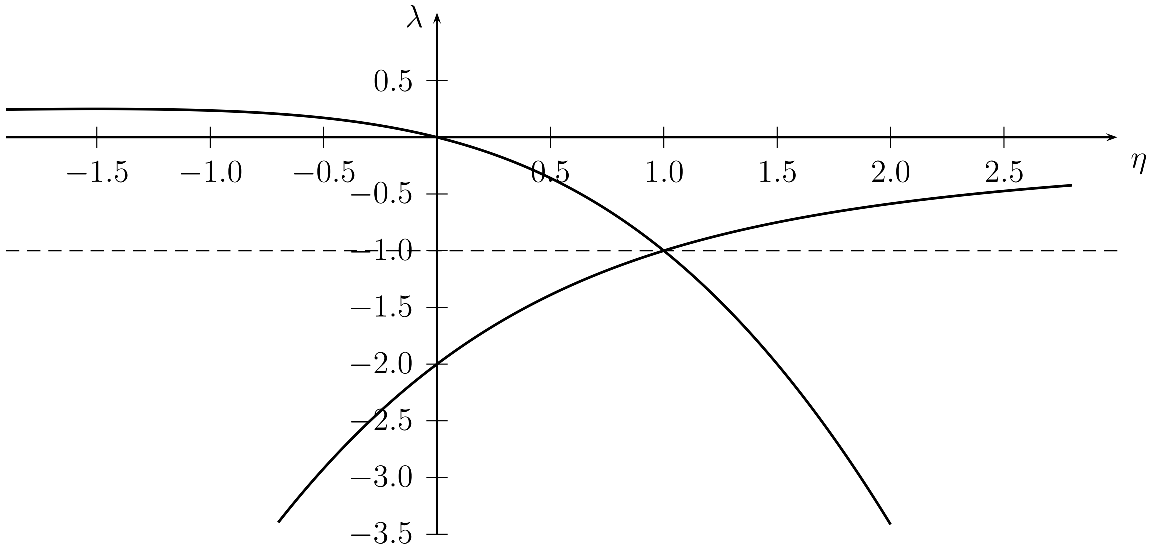


Figure 2: The curve (18) ——— , and the straight line $\lambda = -1$ - - - -

Figure 2 of the curve (18) shows that for all $\eta \neq 1$, i.e. $\beta \neq 1$, there exists one solution of (18) with $|\lambda| < 1$, and another one with $\lambda < -1$. Since for all finite β it is $\eta \neq 0$, it follows that always $\lambda \neq 0$. We expect that for $\beta \neq 1$ there are two solutions of (1) with the asymptotic behaviour (16) for $n \to \infty$ in case of $|\lambda| < 1$ resp. for $n \to -\infty$ in case of $|\lambda| > 1$, and with regard to (4) we make the

Conjecture 2 *For $\beta \neq 1$ there exist two solutions of (1) such that x_{3n-1}, x_{3n}, x_{3n+1} can be expanded into power series in λ^n with the first terms (16) and an arbitrary a. The other parameters are determined by (18), (19) and $\eta = \frac{1}{2\beta-1}$.*

Conjecture 2 comes true in the case $\beta = 0$, where $\eta = -1$. Namely, denoting the 3-periodic solution (15) by ε_n, then (1) has besides of (10) also the solution $\varepsilon_n x_n$, cf. [4, Setion 4.2] as well as [5, p. 175], and the two solutions of (18) are z^3 and s^3 with z and s from (10).

Let us remark that the curve (18) attains its maximum $\lambda = \frac{1}{4}$ at $\eta = -\frac{3}{2}$, where $b = \frac{a}{2}$, $c = -\frac{a}{3}$ and $\beta = \frac{1}{6}$.

Three initial values. Let x_n be a solution of Equation (1) for non-negative integers n. In order to continue this solution to negative n, it is appropriate to write (1) in the form

$$x_{n-1} = x_n \frac{\gamma x_{n+1} - \beta}{\gamma - \beta x_{n+1}}\,, \tag{20}$$

which is singular for $x_{n+1} = \frac{\gamma}{\beta}$. In the case that the initial values (x_{-1}, x_0) are neither $\left(0, \frac{\gamma}{\beta}\right)$ nor $\left(\infty, \frac{\beta}{\gamma}\right)$, the value x_{-2} is uniquely determined by means of (20). Otherwise, this value x_{-2} remains indetermined and can be prescribed arbitrarily, disregarding the countably many cases where the continuation by means of (20) satisfies $x_{n-1} \in \{0, \infty\}$ for some negative n. In particular, we have to avoid the case $x_{-2} = x_{-1}$ in order to avoid the (shifted) forbidden set.

Analogously, if the initial values (x_{-1}, x_0) are given in such a way that for a negative integer m the pair (x_{m-1}, x_m) is either $\left(0, \frac{\gamma}{\beta}\right)$ or $\left(\infty, \frac{\beta}{\gamma}\right)$, then we can choose x_{m-2} as a third arbitrary initial value subject to an analogous restriction as before.

References

[1] **Berg, L. :** *On the asymptotics of nonlinear difference equations.* Z. Anal. Anw. **21**, 1061-1074 (2002)

[2] **Berg, L. :** *Inclusion theorems for non-linear difference equations with applications.* J. Diffenrence Eq. Appl. **10**, 399-408 (2004)
Corrections in same j.: **11**, 181-182 (2005)

[3] **Berg, L. :** *Oscillating solutions of rational difference equations.* Rostock. Math. Kolloq. **58**, 31-35 (2004)

[4] **Çinar, C., Stević, S.**, and **Yalçinkaya, I. :** *A note on global asymptotic stability of a family of rational equations.* Rostock. Math. Kolloq. **59**, 41-49 (2005)

[5] **Kocic, V. L.**, and **Ladas, G. :** *Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications.* Kluwer 1993

[6] **Kulenović, M. R. S.**, and **Ladas, G. :** *Dynamics of Second Order Rational Difference Equations.* Boca Raton etc. 2002

[7] **Stević, S. :** *On positive solutions of a* $(k+1)$*-th order difference equation.* Appl. Math. Letters, to appear

[8] **Stević, S. :** *Asymptotic behaviour of a class on nonlinear difference equations.* To appear

received: August 9, 2005

revised: September 5, 2005

Author:

Lothar Berg
Universität Rostock
Institut für Mathematik
18051 Rostock
Germany

e-mail: lothar.berg@uni-rostock.de

Rostock. Math. Kolloq. **61**, 13–20 (2006)

Subject Classification (AMS)
39A11,41A60

Lothar Berg

Nonlinear Difference Equations with Periodic Solutions

ABSTRACT. For ten nonlinear difference equations with only p-periodic solutions it is shown that the characteristic polynomials of the corresponding linearized equations about the equilibria have only zeros which are p-th roots of unity. An analogous result is shown concerning two systems of such equations. Five counterexamples show that the reverse is not true. Some remarks are made concerning equations with asymptotically p-periodic solutions.

KEY WORDS. Nonlinear difference equations, systems of such equations, p-periodic solutions, asymptotically p-periodic solutions

In their new book [5] the authors E. A. Grove and G. Ladas present a series of examples of nonlinear difference equations

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k})\,, \tag{1}$$

with fixed $k \in \mathbb{N}_0$ and variable $n \in \mathbb{N}_0$, such that all solutions are periodic with the same (not necessarily prime) period p. On p. 25 they put the following two questions:

"*What is it that makes every solution of a difference equation periodic with the same period?*"

"*Is there an easily verifiable test that we can apply to determine whether or not this is true?*"

We deal with these questions under the following conditions:

(i) Assume that $f : G^{k+1} \mapsto G$ for some non-empty complex open set G, and let (1) have an equilibrium $\overline{x} \in G$ defined by

$$\overline{x} = f(\overline{x}, \overline{x}, \dots, \overline{x})\,. \tag{2}$$

(ii) Assume that $f(u_0, u_1, \dots, u_k)$ is holomorphic in some neighbourhood of $(\overline{x}, \overline{x}, \dots, \overline{x})$.

Then there exists the linearized equation of (1) about the equilibrium $\overline{x}$

$$z_{n+1} = f_0 z_n + f_1 z_{n-1} + \dots + f_k z_{n-k} \tag{3}$$

with

$$f_j = \frac{\partial f}{\partial u_j}(\overline{x}, \overline{x}, \dots, \overline{x})$$

and with the corresponding characteristic polynomial

$$\lambda^{k+1} - f_0\lambda^k - \dots - f_{k-1}\lambda - f_k\,. \tag{4}$$

We call a solution *admissible* if the initial values belong to G.

Conjecture *If all admissible solutions of the difference equation* (1) *are p-periodic, and if the conditions* (*i*)-(*ii*) *are satisfied concerning all equilibria $\overline{x} \in G$ then:*

(iii) All zeros λ of (4) *are simple p-th roots of unity (*simple *means that all zeros have multiplicity* 1*).*

Note that the book [5] contains also examples (1) with only periodic solutions, where f is not differentiable at the positive equilibrium. Such examples contain the maximum function, cf. [5, p. 27] or the function $|\cdot|$. We cannot prove this conjecture, but we check assertion (iii) for eight examples of [5] and two further ones, all with periodic solutions only, and we give five counterexamples with nonperiodic solutions, where (iii) is fulfilled nevertheless. In particular, we discuss the case $\lambda = 1$ in (iii). It follows an analogous check concerning two systems of [6], and we make some remarks concerning equations with asymptotically periodic solutions.

Single equations. We begin the following seven cases of [5, Section 2.2]:

(1)	$\overline{x}$	(4)	p
$x_{n+1} = \dfrac{1}{x_n}$	± 1	$\lambda + 1$	2
$x_{n+1} = \dfrac{1}{x_n x_{n-1}}$	$\sqrt[3]{1}$	$\lambda^2 + \lambda + 1$	3
$x_{n+1} = \dfrac{1}{x_{n-1}}$	± 1	$\lambda^2 + 1$	4
$x_{n+1} = \dfrac{1 + x_n}{x_{n-1}}$	$\dfrac{1}{2}(1 \pm \sqrt{5})$	$\lambda^2 - \frac{1}{\overline{x}}\lambda + 1$	5
$x_{n+1} = \dfrac{x_n}{x_{n-1}}$	1	$\lambda^2 - \lambda + 1$	6
$x_{n+1} = \dfrac{1 + x_n + x_{n-1}}{x_{n-2}}$	$1 \pm \sqrt{2}$	$\lambda^3 - \dfrac{1}{\overline{x}}(\lambda^2 + \lambda) + 1$	8
$x_{n+1} = \dfrac{x_n x_{n-2}}{x_{n-1} x_{n-3}}$	1	$\lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1$	10

Here, the validity of assertion (iii) follows immediately from

$$\begin{aligned}
\lambda^2 - 1 &= (\lambda+1)(\lambda-1) \\
\lambda^3 - 1 &= (\lambda^2+\lambda+1)(\lambda-1) \\
\lambda^4 - 1 &= (\lambda^2+1)(\lambda^2-1) \\
\lambda^5 - 1 &= \left(\lambda^2 - \frac{1}{\overline{x}}\lambda + 1\right)(\lambda^2+\overline{x}\lambda+1)(\lambda-1) \\
\lambda^6 - 1 &= (\lambda^2-\lambda+1)(\lambda^2+\lambda+1)(\lambda^2-1) \\
\lambda^8 - 1 &= \left(\lambda^3 - \frac{1}{\overline{x}}(\lambda^2+\lambda)+1\right)\left(\lambda^3 + \frac{1}{\overline{x}}(\lambda^2-\lambda)-1\right)(\lambda^2+1) \\
\lambda^{10} - 1 &= (\lambda^4-\lambda^3+\lambda^2-\lambda+1)(\lambda^4+\lambda^3+\lambda^2+\lambda+1)(\lambda^2-1)\,.
\end{aligned}$$

This assertion also comes true for the *equation* [5, (2.66)]

$$x_{n+1} = \frac{x_n + x_{n-1} + \cdots + x_{n-k}}{x_n x_{n-1} \ldots x_{n-k} - 1} \tag{5}$$

with $k+1$ equilibria $\overline{x} = (k+2)^{\frac{1}{k+1}}$, the further equilibrium $\overline{x} = 0$, the characteristic polynomial $\lambda^{k+1} + \lambda^k + \cdots + 1$, which is a factor of $\lambda^{k+2} - 1$, and the period $p = k+2$. But it does not come true for equation [5, (2.26)] $x_{n+1} = |x_n| - x_{n-1}$ with only 9-periodic solutions, since its equilibrium is $\overline{x} = 0$ and the corresponding linearized equation does not exist.

A *funny example* for an equation with only 3-periodic solutions is

$$x_{n+1} = \frac{5(x_n + x_{n-1}) - 4x_n x_{n-1} - 3}{4(x_n + x_{n-1}) - 5} \tag{6}$$

with the unique equilibrium $\overline{x} = 1$ and the corresponding characteristic polynomial $\lambda^2 + \lambda + 1$.

Recall that a p-periodic sequence x_n can be represented as discrete Fourier series

$$x_n = \sum_{m=0}^{p-1} b_m z^{nm} \tag{7}$$

with $z = \exp\left\{\dfrac{2\pi i}{p}\right\}$ and the inversion

$$b_m = \frac{1}{p}\sum_{k=0}^{p-1} x_k z^{-mk}$$

where we want to point out that the last formula with $p = 5$ was misprinted in [1, p. 1073].

Counterexamples. Next we show by means of two symmetric examples with nonperiodic solutions that nevertheless assertion (iii) comes true at least for one equilibrium. Here we call an equation *symmetric*, if it is uniquely solvable with respect to x_{n-k}, and if this solution reads

$$x_{n-k} = f(x_{n-k+1}, \dots, x_n, x_{n+1})$$

with the same function f as before. It can easily be proved:

Lemma 1 *If equation (1) is symmetric and all solutions of it are p-periodic with $p \geq k+3$, then the solutions x_n with equal initial values $x_0 = x_{-1} = \cdots = x_{-k}$ have the property*

$$x_j = x_{p-k-j} \quad (j = 1, \dots, p-k-1)\,. \tag{8}$$

A third example with nonperiodic solutions concerns the case where (iii) is satisfied completely.

As *first example* we choose the symmetric equation

$$x_{n+1}x_{n-1} = 2(1-\sqrt{2}) + 2x_n \tag{9}$$

with the equilibria $\overline{x}_1 = \sqrt{2}$, $\overline{x}_2 = 2-\sqrt{2}$. The linearized equation

$$\overline{x}(x_{n+1} + x_{n-1}) - 2x_n = 0$$

has the characteristic polynomial

$$\lambda^2 - \frac{2}{\overline{x}}\lambda + 1\,, \tag{10}$$

and for the first equilibrium its zeros are simple 8th roots of unity in view of

$$\lambda^8 - 1 = (\lambda^2 - \sqrt{2}x + 1)(\lambda^2 + \sqrt{2}x + 1)(\lambda^4 - 1)\,.$$

For the initial values $x_{-1} = x_0 = 1$ (cf. [5, p. 26]) we find

$$x_1 = 4 - 2\sqrt{2}\,, \quad x_2 = 10 - 6\sqrt{2}\,, \quad x_3 = 4 - \frac{3}{2}\sqrt{2}\,, \quad x_4 = \frac{1}{14}(20 + 5\sqrt{2})\,,$$

so that $x_3 \neq x_4$, i.e. (8) with $j = 3$, $p = 8$, $k = 1$ is not fulfilled. According to Lemma 1 Equation (9) cannot have only 8-periodic solutions. Here, the zeros of (10) concerning the second equilibrium are $\lambda = 1 + \frac{1}{2}\sqrt{2} \pm \sqrt{\frac{1}{2} + \sqrt{2}}$, and these real numbers cannot be roots of unity.

As *second example* we choose

$$x_{n+1}x_{n-1} = 2 - x_n \tag{11}$$

with the equilibria $\overline{x}_1 = 1$ and $\overline{x}_2 = -2$. The linearized equation

$$\overline{x}(x_{n+1} + x_{n-1}) + x_n = 0$$

has the characteristic polynomial

$$\lambda^2 + \frac{1}{\overline{x}}\lambda + 1\,, \tag{12}$$

and for the first equilibrium its zeros are simple third roots of unity. For the initial values $x_{-1} = x_0 = 3$ we find $x_1 = -\frac{1}{3}$, $x_2 = \frac{7}{9}$, so that this solution is not 3-periodic. Here, the zeros of (12) concerning the second equilibrium are $\lambda = \frac{1}{4}(1 \pm i\sqrt{15})$, and according to $\lambda^3 = -\frac{1}{32}(22 \pm 6i\sqrt{15})$ no third roots of unity.

The *third example* reads

$$x_{n+1} = 3 - \frac{1}{2}\left(3x_n + \frac{x_{n-1}^2}{x_n}\right) \tag{13}$$

with the single equilibrium $\overline{x} = 1$ and the single characteristic polynomial (12). For the initial values $x_{-1} = x_0 = 2$ we find $x_1 = -1$, $x_2 = \frac{13}{2}$, so that this solution is not 3-periodic.

The case $\boldsymbol{\lambda = 1}$. In the foregoing examples all zeros of (4) are different from 1. But the case $\lambda = 1$ is likewise possible. By differentiation of (2) with respect to $\overline{x}$ we easily see:

Lemma 2 *Let* (2) *be satisfied for all $\overline{x} \in G$, and let (ii) be satisfied in G. Then the characteristic polynomial* (4) *has the zero $\lambda = 1$.*

This case can appear by linear homogeneous difference equations with constant coefficients. A nonlinear *example* is

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}} \tag{14}$$

with arbitrary $\overline{x}$, the single characteristic polynomial $\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda - 1)(\lambda^2 + 1)$, and the 4-periodic solutions

$$a, b, c, \frac{ac}{b}, a, \ldots$$

with arbitrary nonvanishing constants a, b, c.

A *fourth counterexample* is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{x_{n-1}^2}{x_n}\right) \tag{15}$$

with arbitrary $\overline{x}$, the single characteristic polynomial $\lambda^2 - 1$ with the roots ± 1, but $x_2 = x_0$ if and only if $x_0 = x_1$. Hence, disregarding the constant solutions, no further solution is 2-periodic.

As *last counterexample* we consider the equation

$$x_n x_{n-k} = 2x_n - 1 \tag{16}$$

with $k \in \mathbb{N}$, the single equilibrium $\overline{x} = 1$, the linearized equation

$$x_n = x_{n-k}\,,$$

and the single characteristic polynomial $\lambda^k - 1$. All zeros are simple k-th roots of unity, but the solutions of (16) are not k-periodic (disregarding $x_n = 1$). In this case the first hypothesis of Lemma 2 is not fulfilled.

Systems of equations. The foregoing conjecture can also be transferred to systems of nonlinear difference equations. We shall show this for the two systems from [6]

$$x_{n+2}^{(m)}\,x_n^{(m+2)} = 1 + x_{n+1}^{(m+1)} \tag{17}$$

and

$$x_{n+3}^{(m)}\,x_n^{(m+3)} = 1 + x_{n+1}^{(m+2)} + x_{n+2}^{(m+1)} \tag{18}$$

with variable $m, n \in \mathbb{Z}$, where the solutions in both cases shall be k-periodic in m with a fixed $k \in \mathbb{N}$. For $k = 1$ the systems reduce to special cases from before.

In [6] it was proved: *Every admissible solution of* (17) *is p-periodic with $p = 5k$ when $5 \nmid k$ and $p = k$ else. Every admissible solution of* (18) *is p-periodic with $p = 8q$ when $k = 2^j q$ $(0 \le j \le 2)$, $2 \nmid q$ and $p = k$ else.*

In both cases we shall show that *every zero of the corresponding characteristic polynomials about the common equilibria is a p-th root of unity.*

We begin with (17). The common equilibria are the solutions of

$$\overline{x}^2 = 1 + \overline{x}\,. \tag{19}$$

Fixing one of it, the corresponding linearized system reads

$$\overline{x}\left(x_{n+2}^{(m)} + x_n^{(m+2)}\right) = x_{n+1}^{(m+1)}\,,$$

where as before $x_n^{(m)}$ is k-periodic in m. With the ansatz $x_n^{(m)} = \xi_m \lambda^n$ we get the cyclic system

$$\overline{x}\lambda^2\xi_m - \lambda\xi_{m+1} + \overline{x}\xi_{m+2} = 0$$

with k-periodic ξ_m. The matrix is the circulant matrix

$$\mathrm{Circ}\,(\overline{x}\lambda^2, -\lambda, \overline{x}, 0, \ldots, 0)$$

where the eigenvalues are the discrete Fourier transform (7) of the first line

$$\overline{x}\lambda^2 - \lambda\varepsilon^m + \overline{x}\varepsilon^{2m}\,, \tag{20}$$

$m = 0, \ldots, k-1$, with $\varepsilon = \exp\left\{\frac{2\pi i}{k}\right\}$, cf. [4]. The characteristic polynomial is the determinant of the circulant matrix, and its zeros are the zeros of (20), i.e. the solutions of

$$\overline{x}\lambda^2 = \lambda\varepsilon^m - \overline{x}\varepsilon^{2m}\,.$$

From this it follows by means of (19) that

$$\overline{x}\lambda^4 = -\lambda\overline{x}\varepsilon^{3m} + \varepsilon^{4m}$$

and $\lambda^5 = \varepsilon^{5m}$, i.e. $\lambda^{5q} = 1$ and therefore $p = 5q$ when either $k = q$ and $5 \nmid q$ or when $k = 5q$ with an integer q. The result is independent of the chosen equilibrium.

In the case (18) the common equilibria are the solutions of

$$\overline{x}^2 = 1 + 2\overline{x}\,. \tag{21}$$

Fixing one of it, the corresponding linearized system reads

$$\overline{x}\left(x_{n+3}^{(m)} + x_n^{(m+3)}\right) = x_{n+1}^{(m+2)} + x_{n+2}^{(m+1)}\,,$$

and the ansatz $x_n^{(m)} = \xi_m\lambda^n$ with k-periodic ξ_m yields the cyclic system

$$\overline{x}\lambda^3\xi_m - \lambda^2\xi_{m+1} - \lambda\xi_{m+2} + \overline{x}\xi_{m+3} = 0$$

with the circulant matrix

$$\mathrm{Circ}\,(\overline{x}\lambda^3, -\lambda^2, -\lambda, \overline{x}, 0, \ldots, 0)\,.$$

Again, the eigenvalues must vanish, so that

$$\overline{x}\lambda^3 - \lambda^2\varepsilon^m - \lambda\varepsilon^{2m} + \overline{x}\varepsilon^{3m} = 0\,, \tag{22}$$

$m = 0, \ldots, k-1$, with the same ε as before. The left-hand side of (22) can be factorized as

$$(\overline{x}(\lambda^2 - \lambda\varepsilon^m + \varepsilon^{2m}) - \lambda\varepsilon^m)(\lambda + \varepsilon^m)\,,$$

so that one solution of (22) is $\lambda = -\varepsilon^m$. For the zeros of the other factor it follows by means of (21) that $\lambda^4 = -\varepsilon^{4m}$, hence in both cases $\lambda^8 = \varepsilon^{8m}$. This implies $\lambda^{8q} = 1$ and therefore $p = 8q$, when either $k = 2^j q$ $(0 \le j \le 2)$ and q odd or when $k = 8q$ with an integer q. The result is independent of the chosen equilibrium.

Asymptotically periodic solutions. In [5, p. 61] there is contained a further question: "*What is it that makes all the solutions of a difference equation be eventually periodic with the same period?*"

Under the assumptions (i) and (ii) the last assertion comes true, *if some* (*but not all*) *zeros of the characteristic polynomial* (4) *about an equilibrium are simple* p*-th roots of unity, all other zeros* λ_i *satisfy* $0 < |\lambda_i| < 1$*, and if the general solution of* (1) *can be represented in the form* (7)*, where the coefficients are convergent power series in* λ_i^n *for large* n (with polynomial coefficients in case of need), cf. the special cases in [1, (7.12)] and [2, Propositions 3.3 and 3.4]. But the situation can be more complicated, cf. [3, Example 2], in particular in the case $\lambda = 1$ for one zero of (4), cf. [1, (1.7)], i.e. [5, (5.2) with $\alpha = \beta = 0$ and $\gamma = A = 1$].

References

[1] **Berg, L. :** *On the asymptotics of nonlinear difference equations.* Z. Anal. Anw. **21**, 1061-1074 (2002)

[2] **Berg, L. :** *Inclusion theorems for non-linear difference equations with applications.* J. Difference Eq. Appl. **10**, 399-408 (2004)
Corrections in same j.: **11**, 181-182 (2005)

[3] **Berg, L. :** *On the difference equation* $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(\gamma x_n + \beta x_{n-1})$*.* Rostock. Math. Kolloq. **61**, 3-11 (2005)

[4] **Davis, P. J. :** *Circulant Matrices.* AMS Chelsea Publ. 1994

[5] **Grove, E. A.**, and **Ladas, G. :** *Periodicities in Nonlinear Difference Equations.* Chapman & Hall 2005

[6] **Iričanin, B.**, and **Stević, S. :** *Some systems of nonlinear difference equations of order k with periodic solutions.* Dyn. Cont. Discr. Impuls. Systems. To appear

received: September 15, 2005

revised: October 14, 2005

Author:

Lothar Berg
Universität Rostock
Institut für Mathematik
18051 Rostock
Germany

e-mail: lothar.berg@uni-rostock.de

Rostock. Math. Kolloq. **61**, 21–30 (2006)

Subject Classification (AMS)
39A10

Stevo Stević

A Note on Periodic Character of a Higher Order Difference Equation

ABSTRACT. In this note we prove that every positive solution of the difference equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}}, \quad n = 0, 1, \ldots$$

where $k, m \in \mathbf{N}$ are so that $k < m$, and $2m = k(L+1)$ for some $L \in \mathbf{N}$, converges to a k-periodic solution. A similar result is proved for a corresponding symmetric system of difference equations. We also consider the systems of difference equations whose all solutions are periodic with the same period. It is generalized and solved Open Problem 2.9.1 in M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations. With open problems and conjectures.* Chapman and Hall/CRC, 2002.

KEY WORDS. k-periodic solution, difference equation, positive solution, system of difference equations

1 Introduction

In this note we study the difference equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}}, \quad n = 0, 1, \ldots \tag{1}$$

where $k, m \in \mathbf{N}$ are so that $k < m$, and $2m = k(L+1)$ for some $L \in \mathbf{N}$ and initial conditions $x_{-m}, \ldots, x_{-2}, x_{-1}$ are positive real numbers.

In [5, Theorem 4.1] was proved that every positive solution of the difference equation $x_n = 1 + \frac{x_{n-2}}{x_{n-3}}$ converges to a period two solution. This motivated us to generalize the result in [10]. In [6] we prove the following result:

Theorem A *Let I be an open interval of the real line, $\varphi : I^k \to I$ be a continuous function which is nondecreasing in each variable and increasing in the first one and*

$\varphi(x, x, \dots, x) \le x$, *for every* $x \in I$. *If* (a_n) *is a bounded sequence which satisfies the inequality*

$$a_{n+k} \le \varphi(a_{n+k-1}, a_{n+k-2}, \dots, a_n) \quad \textit{for} \quad n \in \mathbf{N} \cup \{0\},$$

then it converges.

Other useful globally convergence results can be found, for example, in [7, 8].

It is easy to prove that every positive solution of Eq. (1) is bounded, moreover, in [9] we prove that if $p \ge 1$ and $m, k \in \mathbf{N}$, then every positive solution of the difference equation

$$x_n = p + \frac{x_{n-k}}{x_{n-m}}, \tag{2}$$

is bounded. By a slight modification of the proof of Theorem 3 in [10] it follows that if $p > 1$, then every positive solution of Eq. (2) converges, see, also [9]. Unlike the case $p > 1$, Theorem 4.1 in [5] shows that positive solutions of equation $x_n = 1 + \frac{x_{n-2}}{x_{n-3}}$ need not converge. Hence, Eq. (2) is more interesting in the case $p = 1$. The case $p \in (0, 1)$ was considered in paper [3]. Our aim is to generalize the main results in [2], [5] and [10]. In Section 2 we generalize the main result in [10] developing the main idea from the same paper. In Section 3 we show that the main result in [2] is an easy consequence of known results, also a generalization of the result is given. Section 4 is devoted to the systems of difference equations which all solutions are periodic with the same period. In the section we generalize and solve Open Problem 2.9.1 from [4].

2 Asymptotic periodicity of solutions of Eq. (1)

In this section we consider the positive solutions of Eq. (1). We prove the following result:

Theorem 1 *Let* $k, m \in \mathbf{N}$ *be such that* $k < m$ *and* $2m = k(L+1)$ *for some* $L \in \mathbf{N}$. *Then every positive solution of Eq.* (1) *converges to a not necessarily prime* k*-periodic solution of Eq.* (1). *If* L *is odd, then every positive solution of Eq.* (1) *converges to the equilibrium* $x^* = 2$.

Proof: We have

$$
\begin{aligned}
x_n &= 1+\frac{x_{n-k}}{1+\dfrac{x_{n-m-k}}{x_{n-2m}}} = 1+\frac{x_{n-k}}{1+\dfrac{1}{x_{n-2m}}\left(1+\dfrac{x_{n-m-2k}}{x_{n-2m-k}}\right)} \\
&= 1+\frac{x_{n-k}}{1+\dfrac{1}{x_{n-2m}}+\dfrac{1}{x_{n-2m}x_{n-2m-k}}\left(1+\dfrac{x_{n-m-3k}}{x_{n-2m-2k}}\right)} \\
&= \cdots \\
&= 1+\frac{x_{n-k}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{x_{n-2m-jk}}+\dfrac{x_{n-m-lk}}{\prod\limits_{j=0}^{l-1}x_{n-2m-jk}}}\,, \qquad (3)
\end{aligned}
$$

for every $n \geq lk+m-k$.

Let $l, t \in \mathbf{N}$ are chosen such that $t < l$ and $l-t = L$. Since $2m = k(L+1)$ we have that

$$
[n-m-lk]+m-k = n-2m-tk\,. \qquad (4)
$$

For such chosen l and t it follows from (1) that

$$
\frac{x_{n-m-lk}}{x_{n-2m-tk}} = \frac{1}{x_{n-2m-(t-1)k}-1}\,, \quad \text{for} \quad n \geq lk\,. \qquad (5)
$$

From (3) and (5) it follows that

$$
x_n = 1+\frac{x_{n-k}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{x_{n-2m-jk}}+\dfrac{(x_{n-2m-(t-1)k}-1)^{-1}}{\prod\limits_{j=0,j\neq t}^{l-1}x_{n-2m-jk}}}\,,
$$

that is

$$
x_n = 1+\frac{x_{n-k}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{x_{n-k(L+1+j)}}+\dfrac{(x_{n-k(L+t)}-1)^{-1}}{\prod\limits_{j=0,j\neq t}^{l-1}x_{n-k(L+1+j)}}}\,, \qquad (6)
$$

Using the changes $y_m^{(i)} = x_{km+i}$, $i = 0, 1, \ldots, k-1$, Eq. (6) separates into the following k equations

$$
y_m^{(i)} = 1+\frac{y_{m-1}^{(i)}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{y_{m-(L+1+j)}^{(i)}}+\dfrac{1}{\left(y_{m-(L+t)}^{(i)}-1\right)}\dfrac{1}{\prod\limits_{j=0,j\neq t}^{l-1}y_{m-(L+1+j)}^{(i)}}}\,, \qquad (7)
$$

$i = 0, 1, \ldots, k-1$.

Eq. (7) can be written in the following form

$$y_m^{(i)} = F\big(y_{m-1}^{(i)}, \ldots, y_{m-(L+l)}^{(i)}\big), \quad i = 0, 1, \ldots, k-1\,.$$

Since

$$F(x, \ldots, x) = 1 + \frac{x}{\displaystyle\sum_{i=0}^{l-1} \frac{1}{x^i} + \frac{1}{x^{l-1}(x-1)}} = x, \qquad \text{for} \quad 0 \neq x \neq 1\,,$$

and since F is nondecreasing in each variable and increasing in the first one, we see that all conditions in Theorem A are satisfied on the interval $(1, \infty)$, which implies that the sequence $y_m^{(i)}$, that is, x_{km+i} converges to y_i^*, for each $i = 0, 1, \ldots, k-1$. It is clear that $(y_0^*, \ldots, y_{k-1}^*)$ is a k-cycle of Eq. (1), from which the first statement follows.

Assume now that L is odd. Then $L = 2s+1$ for some $s \in \mathbf{N} \cup \{0\}$. Hence Eq. (1) can be written as follows

$$x_n = 1 + \frac{x_{n-k}}{x_{n-(s+1)k}}\,. \tag{8}$$

Using the changes $y_m^{(i)} = x_{km+i}$, $i = 0, 1, \ldots, k-1$, Eq. (8) can be separated into the following k equations

$$y_m^{(i)} = 1 + \frac{y_{m-1}^{(i)}}{y_{m-(s+1)}^{(i)}}, \quad i \in \{0, 1, \ldots, k-1\}\,. \tag{9}$$

Each of equations in (9) is a special case of Eq. (1) with $k = 1$, $m = s+1$ and $L = 2m-1$. According to the first part of the theorem it follows that every positive solution of each of equations in (9) converges to a periodic solution of period one, that is, to $y^* = 2$, from which it follows that every positive solution of Eq. (1) in this case, converges to the equilibrium $x^* = 2$, as desired. $\square$

For $k = 1$ we obtain the following global stability result.

Corollary 1 *Let $m \in \mathbf{N}$. Then every positive solution of the difference equation*

$$x_n = 1 + \frac{x_{n-1}}{x_{n-m}}, \quad n = 0, 1, \ldots$$

converges to the positive equilibrium $x^* = 2$.

3 A symmetric system of difference equations

In this section we consider the following symmetric system of difference equations

$$x_n = 1 + \frac{x_{n-k}}{y_{n-m}} \quad \text{and} \quad y_n = 1 + \frac{y_{n-k}}{x_{n-m}}, \quad n = 0, 1, \dots , \tag{10}$$

which corresponds to Eq. (1). A little surprising fact is that the method in Theorem 1 can be used also in studying of system (10). As a by-product we obtain a very short proof of the main result in [2]. The main result in this section is the following:

Theorem 2 *Let $k, m \in \mathbf{N}$ be such that $k < m$ and $2m = k(L+1)$ for some $L \in \mathbf{N}$. Then every positive solution of system (10) converges to a k-periodic solution of the system.*

Proof: We have

$$\begin{aligned} x_n &= 1 + \frac{x_{n-k}}{1 + \dfrac{y_{n-m-k}}{x_{n-2m}}} \\ &= 1 + \frac{x_{n-k}}{1 + \dfrac{1}{x_{n-2m}}\left(1 + \dfrac{y_{n-m-2k}}{x_{n-2m-k}}\right)} \\ &= 1 + \frac{x_{n-k}}{1 + \dfrac{1}{x_{n-2m}} + \dfrac{1}{x_{n-2m}x_{n-2m-k}}\left(1 + \dfrac{y_{n-m-3k}}{x_{n-2m-2k}}\right)} \\ &= \cdots \\ &= 1 + \frac{x_{n-k}}{1 + \displaystyle\sum_{i=0}^{l-2}\prod_{j=0}^{i}\frac{1}{x_{n-2m-jk}} + \dfrac{y_{n-m-lk}}{\prod_{j=0}^{l-1} x_{n-2m-jk}}}, \end{aligned} \tag{11}$$

for every $n \geq lk + m - k$.

Let $l, t \in \mathbf{N}$ are chosen such that $t < l$ and $l - t = L$. Since $2m = k(L+1)$ we have that

$$\frac{y_{n-m-lk}}{x_{n-2m-tk}} = \frac{1}{x_{n-2m-(t-1)k} - 1}, \quad \text{for} \quad n \geq lk. \tag{12}$$

From (11) and (12) it follows (6). As in the proof of Theorem 1 we have that x_n converges to a k-cycle, say $(x_0^*, \dots, x_{k-1}^*)$.

Similarly, it can be proved that y_n satisfies Eq. (6) and that it converges to a k-cycle, say $(y_0^*, \dots, y_{k-1}^*)$. It is easy to see that $(x_0^*, \dots, x_{k-1}^*), (y_0^*, \dots, y_{k-1}^*)$, is a k-periodic solution of system (10), from which the result follows. $\square$

Remark 1 Note that unlike the scalar Eq. (1) with $k = 3$ and $m = 6$, the corresponding system

$$x_n = 1 + \frac{x_{n-3}}{y_{n-6}} \quad \text{and} \quad y_n = 1 + \frac{y_{n-3}}{x_{n-6}}, \quad n = 0, 1, \dots , \tag{13}$$

has prime three-periodic solutions of the form

$$(x_n) = (a, b, c, a, b, c, \dots), \qquad (y_n) = \left(\frac{a}{a-1}, \frac{b}{b-1}, \frac{c}{c-1}, \dots\right).$$

4 Periodic solutions of Eq. (1)

In this section we find a subclass of Eq. (1) which have periodic solutions. Before we formulate and prove the main result of this section say that $GCD(m, k)$ denotes the greatest common divisor of integers m and k.

Theorem 3 *Let $m = 2^i m_1$ where m_1 is odd, and $2^{i+1} \mid k$. Then Eq. (1) has infinitely many periodic solutions with period $2GCD(m, k)$.*

Proof: . First note that $k = 2^{i+1}k_1$, for some $k_1 \in \mathbf{N}$. Then m and k can be written in the following forms

$$m = 2^i GCD(m_1, k_1)m_2 = GCD(m, k)m_2$$

and

$$k = 2^{i+1}GCD(m_1, k_1)k_2 = 2GCD(m, k)k_2 .$$

Hence Eq. (1) can be written

$$x_n = 1 + \frac{x_{n-2GCD(m,k)k_2}}{x_{n-GCD(m,k)m_2}} . \tag{14}$$

Since every natural number n can be written in the following form $n+1 = GCD(m, k)l + r$, where $l \in \mathbf{N} \cup \{0\}$ and $r = 0, 1, \dots, GCD(m, k) - 1$, it follows that Eq. (14) is separated into $GCD(m, k)$ independent equations of the form

$$x_l^{(i)} = 1 + \frac{x_{l-2k_2}^{(i)}}{x_{l-m_2}^{(i)}} , \tag{15}$$

$i \in \{0, 1, \dots, GCD(m, k) - 1\}$.

Now note that m_2 is odd. This means that the numbers l and $l - 2k_2$ have the same parity, but $l - m_2$ has different one. Hence, each equation in (15) has a 2-periodic solution

$$\phi, \psi, \dots, \phi, \psi, \dots$$

with

$$\phi = 1 + \frac{\phi}{\psi} \quad \text{and} \quad \psi = 1 + \frac{\psi}{\phi},$$

which is equivalent to $\phi + \psi = \psi\phi$. It means that each equation in (15) has infinitely many periodic solutions with period two of the form

$$\phi, \frac{\phi}{\phi - 1}, \ldots, \phi, \frac{\phi}{\phi - 1}, \ldots .$$

From all above mentioned the result follows, that is, Eq. (1) has infinitely many periodic solutions with period $2GCD(m, k)$. □

For the readers interested in this topic we leave the following interesting open problem:

Open Problem 1 *Investigate the behavior of the positive solutions of system (10) when $k, m \in \mathbf{N}$ are so that $k \neq m$, and $2m \neq k(L + 1)$ for every $L \in \mathbf{N}$.*

In view of Theorem 1 and Theorem 3 we also believe that the following conjecture holds:

Conjecture 1 *Assume that $k, m \in \mathbf{N}$ such that $k < m$ and 2^i is the largest power of 2 which divides m. Show that the following statements are true:*

(a) *If $2^{i+1} \nmid k$, then every positive solution of Eq. (1) converges to the equilibrium $x^* = 2$.*

(b) *If $2^{i+1} | k$, then every positive solution of Eq. (1) converges to a $2GCD(m, k)$-periodic solution.*

5 On systems which have only periodic solutions

In [4, p. 43] the authors claim that for a linear equation, every solution is periodic with period $p \geq 2$, if and only if every root of the characteristic equation is a pth root of unity. However, this is only true if we add the condition that all these roots are simple and the equation is homogeneous, or the right-hand side constant and no resonance case. Motivated by this observation they posed the following open problem.

Open Problem 2.9.1 *Assume that $f \in C^1[(0, \infty)^2, (0, \infty)]$ is such that every positive solution of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, \ldots \tag{16}$$

is periodic with period $p \geq 2$.

Is it true that the linearized equation about a positive equilibrium of Eq. (16) has the property that every one of its solutions is also periodic with the same period p?

In [1] L. Berg shows that for ten nonlinear difference equations, whose all solutions are periodic with the same period p, all solutions of the corresponding linearized equations are periodic with the same period. This Berg's paper motivated us to believe that not only Open Problem 2.9.1 is true but that a more general result holds.

In order to solve Open Problem 2.9.1 we need a useful result contained in the following lemma. Before formulating it we say that for a mapping $f : X \to X$, $(f^{[p]})_{p\in\mathbf{N}\cup\{0\}}$ denotes the sequence of iterates of f, that is, $f^{[0]} = I$, the identity function on X, $f^{[1]} = f$ and generally $f^{[p+1]} = f \circ f^{[p]}$ for any $p \in \mathbf{N}$.

Lemma 1 *Let $I \subset \mathbf{R}$ be an interval. Consider the system of difference equations*

$$\vec{x}_{n+1} = f(\vec{x}_n)\,, \tag{17}$$

where $f \in C^1[I^k, I^k]$, and x^* is an equilibrium of Eq. (17). If all solutions of Eq. (17) are periodic with period p, then Jacobi's matrix $Df(x^*)$ is diagonalizable and all its eigenvalues are pth roots of unity (here Df denotes Jacobi's matrix of the function f).

Proof: Since all solutions of Eq. (17) are periodic with period p, then we have

$$\vec{x}_p = f^{[p]}(\vec{x}_0) = \vec{x}_0\,, \tag{18}$$

for every $\vec{x}_0 \in I^k$. Differentiating (18) we have that

$$D(f^{[p]}(\vec{x})) = D\vec{x} = Id\,, \tag{19}$$

for every $\vec{x} \in I^k$, where Id denotes the identity operator on $\mathbf{R}^n$.

Now note that every stationary point x^* of system (17) is a fixed point of the equation $f(\vec{x}) = \vec{x}$.

Using this fact, chain rule and taking $\vec{x} = x^*$ in (19), we have that

$$[Df(x^*)]^p = Id\,, \tag{20}$$

that is, pth power of Jacobi's matrix $[Df(x^*)]$ is equal to Id. Using Jordan's decomposition of the matrix and (20), it follows that the matrix $[Df(x^*)]$ is diagonalizable and that all roots of the characteristic polynomial of the matrix are pth roots of unity, as desired. □

Notice that the linearized system of (17) at x^* is

$$\vec{y}_{n+1} = [Df(x^*)]\vec{y}_n.$$

From all above mentioned it follows that the characteristic polynomial of the matrix in the corresponding linearized equation about an equilibrium has only zeros which are pth roots of unity.

As a corollary of Lemma 1 we obtain the next result which among other things solves Open Problem 2.9.1.

Corollary 2 *Let $I \subset \mathbf{R}$ be an interval. Consider the difference equation*

$$x_{n+1} = f(x_n, \dots, x_{n-k+1}), \tag{21}$$

where $f \in C^1[I^k, I]$, and x^* is an equilibrium of Eq. (21). If all solutions of Eq. (21) are periodic with period p, then the zeros of the characteristic polynomial of the linearized equation

$$y_{n+1} = \frac{\partial f}{\partial x_1}(x^*)y_n + \cdots + \frac{\partial f}{\partial x_k}(x^*)y_{n-k+1} \tag{22}$$

about the equilibrium x^* of Eq. (21) are simple pth roots of unity and consequently all solutions of Eq. (22) are periodic with period p.

Proof: By standard transformation Eq. (21) can be written as a $k \times k$ system of difference equations of first order. The corresponding linearized system have the following matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{\partial f}{\partial x_k}(x^*) & \frac{\partial f}{\partial x_{k-1}}(x^*) & \cdot & \cdots & \frac{\partial f}{\partial x_1}(x^*) \end{bmatrix}. \tag{23}$$

By Lemma 1 the characteristic polynomial

$$\lambda^k - \frac{\partial f}{\partial x_1}(x^*)\lambda^{k-1} - \cdots - \frac{\partial f}{\partial x_k}(x^*) = 0 \tag{24}$$

of the system has only zeros which are pth roots of unity.

By a well known result (see [11, No. 9.67 point 4]) if the polynomial (24) has multiple zeros, then matrix (23) cannot be diagonalizable. By Lemma 1 matrix (23) is diagonalizable, which is a contradiction. Hence all zeros of polynomials (24) are simple pth roots of unity, which implies that all solutions of Eq. (22) are periodic with period p, as claimed. □

Acknowledgement. I would like to express my sincere thanks to Professor Lothar Berg for his helpful suggestions and comments during the preparation of this paper.

References

[1] **Berg, L. :** *Nonlinear difference equations with periodic solutions.* Rostock. Math. Kolloq. **61**, 13-20 (2005)

[2] **Camouzis, E.**, and **Papaschinopoulos, G. :** *Global asymptotic behavior of positive solutions on the system of rational difference equations* $x_{n+1} = 1 + x_n/y_{n-m}$, $y_{n+1} = 1 + y_n/x_{n-m}$,. Appl. Math. Lett. **17**, 733-737 (2004)

[3] **Karakostas, G.**, and **Stević, S. :** *On a difference equation with min-max response.* Int. J. Math. Math. Sci. **55**, 2915-2926 (2004)

[4] **Kulenović, M. R. S.**, and **Ladas, G. :** *Dynamics of Second Order Rational Difference Equations. With open problems and conjectures.* Chapman and Hall/CRC, 2002

[5] **Patula, W. T.**, and **Voulov, H. D. :** *On the oscillation and periodic character of a third order rational difference equation.* Proc. Amer. Math. Soc. **131** (3), 905-909 (2003)

[6] **Stević, S. :** *A generalization of the Copson's theorem concerning sequences which satisfy a linear inequality.* Indian J. Math. **43** (3), 277-282 (2001)

[7] **Stević, S. :** *A global convergence results with applications to periodic solutions.* Indian J. Pure Appl. Math. **33** (1), 45-53 (2002)

[8] **Stević, S. :** *A global convergence result.* Indian J. Math. **44** (3), 361-368 (2002)

[9] **Stević, S. :** *On the recursive sequence* $x_{n+1} = \frac{A}{\prod_{i=0}^{k} x_{n-i}} + \frac{1}{\prod_{j=k+2}^{2(k+1)} x_{n-j}}$,. Taiwanese J. Math. **7** (2), 249-259 (2003)

[10] **Stević, S. :** *A note on periodic character of a difference equation.* J. Differ. Equations Appl. **10** (10), 929-932 (2004)

[11] **Voevodin, V. V.**, and **Kuznetsov, Yu. A. :** *Matritsy i Vichisleniya.* Moskva 1984 (in Russian)

received: September 15, 2005

Author:

Stevo Stević
Mathematical Institute of the Serbian Academy of Science
Knez Mihailova 35/I,
11000 Beograd,
Serbia

e-mail: sstevic@ptt.yu; sstevo@matf.bg.ac.yu

Rostock. Math. Kolloq. **61**, 31–48 (2006)

Subject Classification (AMS)
35J20,35Q25

Shaolong Xie[1], Weiguo Rui, Xiaochun Hong

The Compactons and Generalized Kink Waves to a generalized CAMASSA-Holm Equation[2]

ABSTRACT. In this paper, the bifurcation method of planar systems and simulation method of differential equations are employed to investigate the bounded travelling waves of a generalized Camassa-Holm equation. The bounded travelling waves defined on finite core regions are found and their integral or implicit expressions are obtained. Their planar simulation graphs show that they possess the properties of compactons or generalized kink waves.

KEY WORDS. Camassa-Holm equation; compactons; generalized kink waves.

1 Introduction

In recent years the so-called Camassa-Holm [1] equation has caught a great deal of attention. It is a nonlinear dispersive wave equation that takes the form

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \, . \tag{1.1}$$

When $k > 0$ this equation models the propagation of unidirectional shallow water waves on a flat bottom, and $u(t, x)$ represents the fluid velocity at time t in the horizontal direction x [1,2]. The Camassa-Holm equation possesses a bi-Hamiltonian structure [1,3] and is completely integrable [1,4,5]. Moreover, when $k = 0$ it has an infinite number of solitary wave solutions, called peakons due to the discontinuity of their first derivatives at the wave peak, interacting like solitons:

$$u(x, t) = c \ \exp(-|x - ct|) \, . \tag{1.2}$$

[1]Corresponding author: E-mail address: xieshlong@163.com

[2]This research was supported by Natural Science Foundation of China (10261008).

Liu and Qian [6] investigated the peakons of the following generalized Camassa-Holm equation

$$u_t + 2ku_x - u_{xxt} + au^m u_x = 2u_x u_{xx} + uu_{xxx} \, . \tag{1.3}$$

with $a > 0, k \in R, m \in N$ and the integral taken as zero. In the case of $m = 1, 2, 3$ and $k \neq 0$, they gave the explicit expressions for the peakons. The concept of compacton: soliton with compact support, or strict localization of solitary waves, appeared in the work of Rosenau and Hyman [7] where a genuinely nonlinear dispersive equation $K(n, n)$ defined by

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0 \, , \tag{1.4}$$

was subjected to experimental and analytical studies. They found certain solitary wave solutions which vanish identically outside a finite core region. These solutions have been called compactons. Several studies have been conducted in [8]-[12]. The aim was to examine if other nonlinear dispersive equation may generate compacton structures.

In fact, When $a = 3$ and $m = 2$, the Eq. (1.3) has another kind of bounded travelling waves which possess some properties of kink waves. We call them generalized kink waves. Therefore, in this paper, we shall consider the compactons and generalized kink waves of the Eq. (1.3) when $a = 3$ and $m = 2$. In the conditions of $a = 3$ and $m = 2$, the Eq. (1.3) can be rewritten as:

$$u_t + 2ku_x - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx} \, , \tag{1.5}$$

where the constant $k \in R$ is given.

The rest of this paper is organized as follows. In Section 2, we firstly derive travelling wave equation and travelling wave system. Then we study the bifurcations of phase portrait of the travelling wave system. In Section 3, using the information of phase portrait, we make the numerical simulation for bounded integral curves of travelling wave equation. In Section 4, we obtain the integral representations of compactons and the implicit or integral representations of the generalized kink waves from the bifurcations of phase portrait and the bounded integral curves. Finally, a short conclusion is given in Section 5.

2 Travelling Waves System and its Bifurcation Phase Portrait

In this section we derive travelling wave system and study its bifurcation phase portrait. Substituting $u(x, t) = \phi(\xi)$ with $\xi = x - ct$ into (1.5), we have

$$-c\phi' + 2k\phi' + c\phi''' + 3\phi^2\phi' = 2\phi'\phi'' + \phi\phi''' \, , \tag{2.1}$$

where c is the wave speed. Integrating it once gives

$$(\phi - c)\phi'' = \phi^3 + (2k - c)\phi - \frac{1}{2}(\phi')^2\,, \tag{2.2}$$

where the integral constant is taken as 0. Letting $\phi' = y$, we obtain a planar system

$$\frac{d\phi}{d\xi} = y\,, \qquad \frac{dy}{d\xi} = \frac{\phi^3 + (2k - c)\phi - \frac{1}{2}y^2}{\phi - c}\,, \tag{2.3}$$

which is called travelling wave system. Our aim is to study the phase portrait of system (2.3). But system (2.3) has a singular line $\phi = c$ which is inconvenient to our study. So we make the transformation $d\xi = (\phi - c)d\tau$. Thus system (2.3) becomes Hamiltonian system

$$\frac{d\phi}{d\tau} = (\phi - c)y, \qquad \frac{dy}{d\tau} = \phi^3 + (2k - c)\phi - \frac{1}{2}y^2\,, \tag{2.4}$$

Thus system (2.3) and (2.4) have the same first integral

$$H(\phi, y) = (\phi - c)y^2 - \frac{1}{2}\phi^4 + (c - 2k)\phi^2 = h\,. \tag{2.5}$$

Therefore both systems (2.3) and (2.4) have same topological phase portraits except the straight line $\phi = c$.

Now we consider the singular points of system (2.4) and their properties. Let

$$y^0_\pm = \pm\sqrt{2(c^2 - c + 2k)c} \quad \text{for} \quad 2(c^2 - c + 2k)c > 0\,, \tag{2.6}$$

$$\phi^0_\pm = \pm\sqrt{c - 2k} \quad \text{for} \quad 2k < c\,, \tag{2.7}$$

$$\phi^*_\pm = \pm\sqrt{-c^2 + 2c - 4k} \quad \text{for} \quad -c^2 + 2c - 4k \geq 0\,, \tag{2.8}$$

$$\phi^1_\pm = \pm\sqrt{2(c - 2k)} \quad \text{for} \quad 2k \leq c\,, \tag{2.9}$$

$$k_1(c) = \frac{c}{2}\,, \tag{2.10}$$

$$k_2(c) = \frac{2c - c^2}{4} \quad \text{for} \quad 0 < c\,, \tag{2.11}$$

$$k_3(c) = \frac{c - c^2}{2}\,, \tag{2.12}$$

Thus, the $k = k_i(c)$ $(i = 1, 2, 3)$ have a unique intersection point $(0, 0)$, and

$$k_3(c) < k_2(c) < k_1(c) \quad \text{for} \quad 0 < c\,, \tag{2.13}$$

and

$$k_3(c) < k_1(c) \quad \text{for} \quad 0 < c\,. \tag{2.14}$$

By the theory of planar dynamical system and (2.4)-(2.14), we derive the following proposition for the equilibrium points of the system (2.4):

Proposition 2.1 1). *When $c<0$ and $k<k_3(c)$ or $0<c$ and $k_3(c)<k$, the (c,y_-^0) and (c,y_+^0) are two singular points of the system (2.4). They are saddle points and $H(c,y_-^0)=H(c,y_+^0)$.*

2). *When $0<c$ and $k_1(c)\le k$, the system (2.4) has three singular points $(0,0),(c,y_-^0)$ and (c,y_+^0). The (0, 0) is a center point.*

3). *When $c=0$ and $0\le k$, the system (2.4) has only one singular point $(0,0)$ and this point is a degenerate saddle point.*

4). *When $c<0$ and $k_1(c)\le k$, the system (2.4) has only one singular point $(0,0)$ and this point is a saddle point.*

5). *When $c<0$ and $k_3(c)<k<k_1(c)$, the system (2.4) has three singular points $(0,0),(\phi_-^0,0)$ and $(\phi_+^0,0)$ and $c<\phi_-^0<0<\phi_+^0$. The $(0,0)$ is a center point, $(\phi_-^0,0)$ and $(\phi_+^0,0)$ are saddle points and $H(\phi_-^0,0)=H(\phi_+^0,0)$.*

6). *When $c<0$ and $k=k_3(c)$, the system (2.4) has three singular points $(0,0),(c,0)$ and $(-c,0)$. The $(0,0)$ is a center point, $(c,0)$ is a degenerate saddle point, $(-c,0)$ is a saddle point and $H(c,0)=H(-c,0)$.*

7). *When $c<0$ and $k<k_3(c)$, the system (2.4) has five singular points $(0,0)$, $(\phi_-^0,0)$, $(\phi_+^0,0)$, (c,y_-^0) and (c,y_+^0), and $\phi_-^0<c<0<\phi_+^0$. The $(0,0)$ and $(\phi_-^0,0)$ are center points, $(\phi_+^0,0)$ is a saddle point.*

8). *When $c=0$ and $k<0$, the system (2.4) has three singular points $(0,0),(\phi_-^0,0)$ and $(\phi_+^0,0)$, and $\phi_-^0<0<\phi_+^0$. The $(0,0)$ is a degenerate saddle point, $(\phi_-^0,0)$ is center point and $(\phi_+^0,0)$ is a saddle point.*

9). *When $c>0$ and $k<k_3(c)$, the system (2.4) has three singular points $(0,0),(\phi_-^0,0)$ and $(\phi_+^0,0)$ and $\phi_-^0<0<c<\phi_+^0$. The $(0,0)$ and $(\phi_+^0,0)$ are saddle points, $(\phi_-^0,0)$ is center point.*

10). *When $c>0$ and $k=k_3(c)$, the system (2.4) has three singular points $(0,0),(-c,0)$ and $(c,0)$. The $(0,0)$ is a saddle point, $(c,0)$ is a degenerate saddle point and $(-c,0)$ is a center point.*

11). *When $c>0$ and $k_3(c)<k<k_2(c)$, the system (2.4) has five singular points $(0,0)$, $(\phi_-^0,0)$, $(\phi_+^0,0)$, (c,y_-^0) and (c,y_+^0), and $\phi_-^0<0<\phi_+^0<c$. The $(0,0)$ is a saddle point, $(\phi_-^0,0)$ and $(\phi_+^0,0)$ are center points.*

12). *When* $c > 0$ *and* $k = k_2(c)$, *the system* (2.4) *has five singular points* $(0, 0)$, $(\phi_-^0, 0)$, $(\phi_+^0, 0)$, (c, y_-^0) *and* (c, y_+^0), *and* $\phi_-^0 < 0 < \phi_+^0 < c$. *The* $(0, 0)$ *is a saddle point,* $(\phi_-^0, 0)$ *and* $(\phi_+^0, 0)$ *are center points, and* $H(0, 0) = H(c, y_-^0) = H(0, y_+^0)$.

13). *When* $c > 0$ *and* $k_2(c) < k < k_1(c)$, *the system* (2.4) *has five singular points* $(0, 0)$, $(\phi_-^0, 0)$, $(\phi_+^0, 0)$, (c, y_-^0) *and* (c, y_+^0), *and* $\phi_-^0 < 0 < \phi_+^0 < c$. *The* $(0, 0)$ *is a saddle point,* $(\phi_-^0, 0)$ *and* $(\phi_+^0, 0)$ *are center points.*

Proof: It is easy to see that all of the singular points of (2.4) are only distributed on ϕ-axis or the line $\phi = c$. Firstly we consider system (2.4) on the line $\phi = c$. From (2.6), on the line $\phi = c$, (2.4) has two singular points (c, y_-^0) and (c, y_+^0) when $c < 0$ and $k < k_3(c)$ or $0 < c$ and $k_3(c) < k$, has one singular point $(c, 0)$ when $k = k_3(c)$, and has not singular point when $c < 0$ and $k > k_3(c)$ or $0 < c$ and $k_3(c) > k$. Assume that $\lambda(\phi, y)$ is an eigenvalue of the linearized system of (2.4) at point (ϕ, y). Then we have

$$\lambda^2(c, y_-^0) = \lambda^2(c, y_+^0) = 2c(c^2 - c + 2k) > 0\,, \tag{2.15}$$

for $c < 0$ and $k < k_3(c)$ or $0 < c$ and $k_3(c) < k$, and

$$\lambda^2(c, 0) = 0, \quad \text{for} \quad k = k_3(c)\,. \tag{2.16}$$

Now we consider system (2.4) on ϕ-axis. Let

$$f(\phi) = \phi^3 + (2k - c)\phi\,, \tag{2.17}$$

then the $(\phi_0, 0)$ is singular point of system (2.4) if and only if $f(\phi_0) = 0$. It is easy to see that we obtain the following facts:

(1^0) When $k_1(c) \le k$, the system (2.4) has one zero point $(0, 0)$. Thus the $(0, 0)$ is singular point of system (2.4) on ϕ-axis. From (2.7) and (2.17) we have $f'(0) > 0$ and $f'(0) = 0$ when $k_1(c) < k$ and $k = k_1(c)$ respectively.

(2^0) When $k_1(c) > k$, the system (2.4) has three zero points $(\phi_-^0, 0)$, $(0, 0)$ and $(\phi_+^0, 0)$. Thus the $(\phi_-^0, 0)$, $(0, 0)$ and $(\phi_+^0, 0)$ are singular points of system (2.4) on ϕ-axis. From (2.7) and (2.17) we have $f'(\phi_-^0) > 0$, $f'(0) < 0$ and $f'(\phi_+^0) > 0$.

On the other hand, we have

$$\lambda^2(\phi_-^0, c) = f'(\phi_-^0)(\phi_-^0 - c)\,, \tag{2.18}$$

$$\lambda^2(0, 0) = -cf'(0)\,, \tag{2.19}$$

$$\lambda^2(\phi_+^0, c) = f'(\phi_+^0)(\phi_+^0 - c)\,, \tag{2.20}$$

From (2.5) and (2.15) - (2.20) the proof is completed.

According to the above analysis, we draw the bifurcation phase portrait of (2.3) and (2.4), shown in Fig. 1.

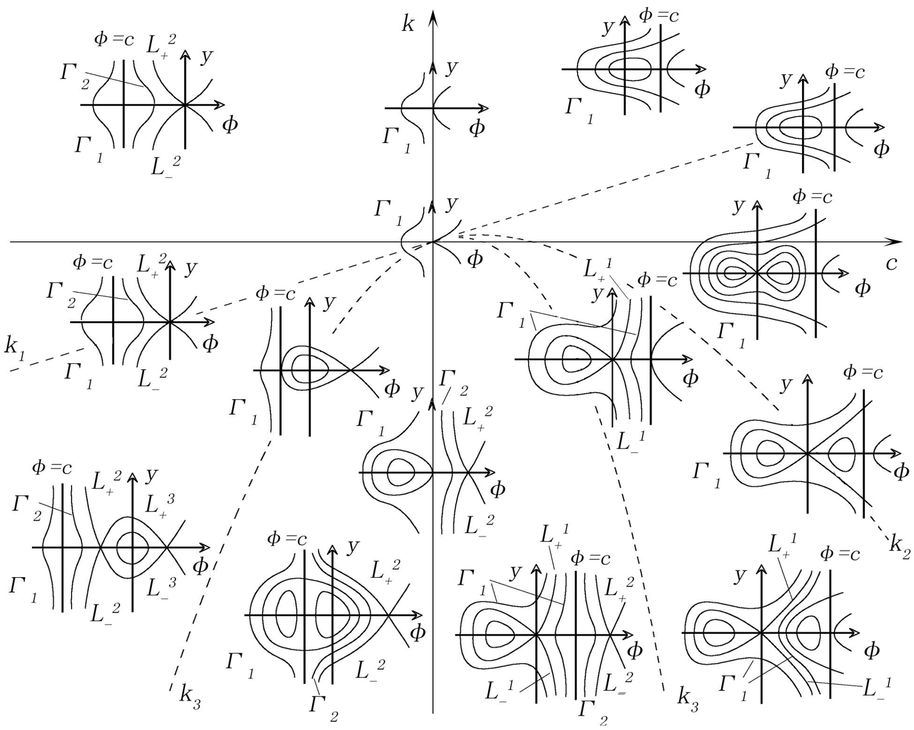


Fig. 1 The bifurcation phase portrait of systems (2.3) and (2.4)

3 Numerical Simulations of Bounded Integral Curves of Travelling Wave Equation

From the derivation in Sec. 2 we see that the bounded travelling waves of Eq. (1.5) correspond to the bounded integral curves of Eq. (2.2), and the bounded integral curves of Eq. (2.2) correspond to the orbits of systems (2.3) in which $\phi = \phi(\xi)$ is bounded. Therefore we can simulate the bounded integral curves of Eq. (2.2) by using the information of the phase portrait of systems (2.3).

From Fig. 1 it is seen that $\phi = \phi(\xi)$ is bounded in the following orbits of system (2.3):

(1). The homoclinic orbits, (2). The periodic orbits, (3). The orbits Γ_1 and Γ_2, (4). The heteroclinic orbits $L^1_\pm$, $L^2_\pm$and $L^3_\pm$.

When (i). $c > 0$ and $k < k_3(c)$, (ii). $c < 0$ and $k_3(c) < k < k_1(c)$, according to the above analysis we will simulate the bounded integral curves of Eq. (2.2) by using the mathematical software *Maple*. In the other case we can use a similar argument. We assume that $(\phi_0, 0)$ is the initial point of an orbit of system (2.3) in the following cases.

Case 1. $c > 0$ and $k < k_3(c)$. For this case, system (2.3) has an orbit Γ_1 on which ϕ is bounded when $\phi_0 < \phi_-^1$ or $0 < \phi_0 < c$, has a homoclinic orbit when $\phi_0 = \phi_-^1$, has a periodic orbit when $\phi_-^1 < \phi_0 < \phi_-^0$, two heteroclinic orbits $L_\pm^1$ on which ϕ are bounded when $\phi_0 = 0$, has an orbit Γ_2 on which is bounded when $c < \phi_0 < \phi_+^0$, and has two heteroclinic orbits $L_\pm^2$ which lie on the left side of the line $\phi = \phi_+^0$ on which ϕ are bounded when $\phi_0 = \phi_+^0$. For example, choosing $c = 2$ and $k = -4$, we have $\phi_-^1 = -4.472135955$ and $\phi_\pm^0 = \pm 3.16227766$.

(i). We respectively take $\phi_0 = -4.48, -4.472135955, -4.4721, 0.01, 3.1622$ and 3.1622776, letting $\phi(0) = \phi_0$ and $\phi'(0) = 0$, we simulate the integral curves of Eq. (2.2) as (a), (b), (c), (f) , (g) and (h) in Fig. 2.

(ii). The two heteroclinic orbits $L_\pm^1$ respectively have expressions

$$y_1^\pm(\phi) = \pm\sqrt{\frac{\phi^4 + 2(2k-c)\phi^2}{2(\phi - c)}}, \text{ for } 0 \le \phi < c\,. \tag{3.1}$$

If $0 < \phi_1^0 < c$, then from the first equation of system (2.3) we have $\frac{d\phi}{d\xi}|_{\xi=\xi_0} = y_1^\pm(\phi_1^0)$ at $\phi = \phi_1^0$. For example, when $c = 2$ and $k = -4$, taking $\phi_1^0 = 0.2$, we have $y_1^\pm(\phi_1^0) = \pm 0.4709328804$. Letting $\phi(0) = 0.2$ and $\phi'(0) = \pm 0.4709328804$, respectively we simulate the integral curves of Eq. (2.2) as (d) and (e) in Fig. 2.

(iii). The two heteroclinic orbits $L_\pm^2$ respectively have expressions

$$y_2^\pm(\phi_2^0) = \pm\sqrt{\frac{\phi^4 + 2(2k-c)\phi^2 + 2h(\phi_+^0)}{2(\phi - c)}}, \text{ for } c < \phi \le \phi_+^0\,, \tag{3.2}$$

$$h(\phi_+^0) = -\frac{1}{2}(\phi_+^0)^4 + (c - 2k)(\phi_+^0)^2\,. \tag{3.3}$$

If $c < \phi_2^0 \le \phi_+^0$, then from the first equation of system (2.3) we have $\frac{d\phi}{d\xi}|_{\xi=\xi_0} = y_2^\pm(\phi_2^0)$ at $\phi = \phi_2^0$. For example, when $c = 2$ and $k = -4$, taking $\phi_2^0 = 3$, we have $y_2^\pm(\phi_2^0) = \pm 0.7071067812$. Letting $\phi(0) = 3$ and $\phi'(0) = \pm 0.7071067812$, respectively we simulate the integral curves of Eq. (2.2) as (i) and (j) in Fig. 2.

Remark 1 Under the conditions of Case 1 the following facts can be seen from Fig.2:

(1) The integral curve is only defined on $[-\xi_0^1, \xi_0^1]$ or $[-\xi_0^2, \xi_0^2]$ and it is of peak form [see

(a), (f), (g) and (h) in Fig. 2] when $\phi_0 < \phi_-^1$ or $0 < \phi_0 < c$ or $c < \phi_0 < \phi_+^0$, where

$$\xi_0^1 = \int_{\phi_0}^{c} \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \quad \text{for} \quad \phi_0 < \phi_-^1 \quad \text{or} \quad 0 < \phi_0 < c\,, \tag{3.4}$$

$$\xi_0^2 = \int_{c}^{\phi_0} \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \quad \text{for} \quad c < \phi_0 < \phi_0^+\,, \tag{3.5}$$

$$\alpha = -\phi_0^2 - 4k + 2c\,. \tag{3.6}$$

The point $(0, \phi_0)$ is the peak of the integral curve $\phi = \phi(\xi)$ which tends to c when $|\xi|$ tends to ξ_0, where $\xi_0 = \xi_0^1$ or ξ_0^2. Following Rosenau and Hyman [7] we call a compacton. For example, when $c = 2$ and $k = -4$, we respectively take $\phi_0 = 0.01$ and 3.1622776, from (3.4) and (3.5) we obtain $\xi_0^1 = 2.417690442$ and $\xi_0^2 = 4.1086580$ which is identical with the simulation [see Figs. 2 (f) and (h)].

(2) When $\phi_0 = 0$, Eq. (2.2) has two bounded integral curves $\phi_1(\xi)$ and $\phi_2(\xi)$ [see Figs. 2 (d) and (e)]. $\phi_1(\xi)$ is defined on $(-\infty, \xi_1]$ and tends to c when ξ tends to ξ_1, to 0 when ξ tends to $-\infty$. $\phi_2(\xi)$ is defined on $[-\xi_1, +\infty)$ and tends to 0 when ξ tends to $+\infty$, to c when ξ tends to $-\xi_1$, where

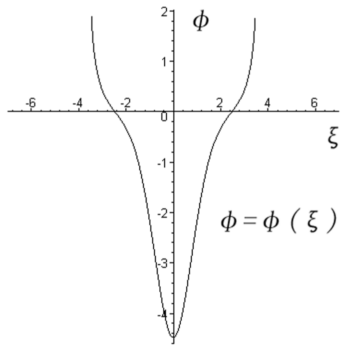
$$\xi_1 = \int_{\phi_1^0}^{c} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2-2(c-2k)}}ds, \quad \text{for} \quad 0 < \phi_1^0 < c\,. \tag{3.7}$$

For example, for the above $c = 2, k = -4$, taking $\phi_1^0 = 0.2$, from (3.7) we obtain $\xi_1 = 0.7904027914$ which is identical with the simulation [see Figs. 2 (d) and (e)].

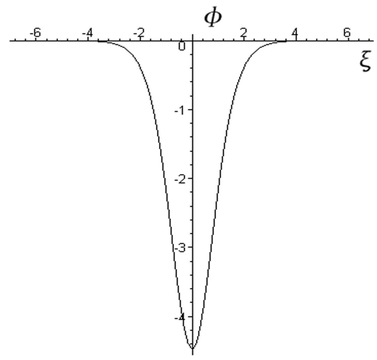
(3) When $\phi_0 = \phi_+^0$, Eq. (2.2) has two bounded integral curves $\phi_3(\xi)$ and $\phi_4(\xi)$ [see Figs. 2 (i) and (j)]. $\phi_3(\xi)$ is defined on $[-\xi_2, +\infty)$ and tends to c when ξ tends to $-\xi_2$, to ϕ_+^0 when ξ tends to $+\infty$. $\phi_4(\xi)$ is defined on $(-\infty, \xi_2]$ and tends to ϕ_+^0 when ξ tends to $-\infty$, to c when ξ tends to ξ_2, where

$$\xi_2 = \int_{c}^{\phi_2^0} \frac{\sqrt{2(s-c)}}{(\phi_+^0 - s)(\phi_+^0 + s)}ds, \quad \text{for} \quad c < \phi_2^0 < \phi_+^0\,. \tag{3.8}$$

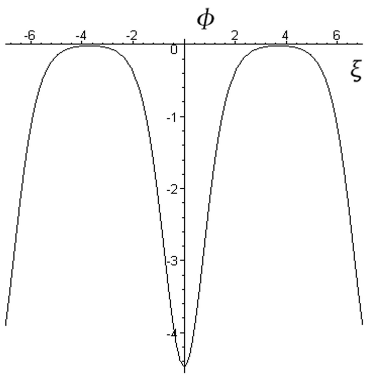
For example, for the above $c = 2, k = -4$, taking $\phi_2^0 = 3$, from (3.8) we obtain $\xi_2 = 0.3697389765$ which is identical with the simulation [see Figs. 2 (i) and (j)].


ϕ
ξ
$\phi = \phi\ (\ \xi\ )$

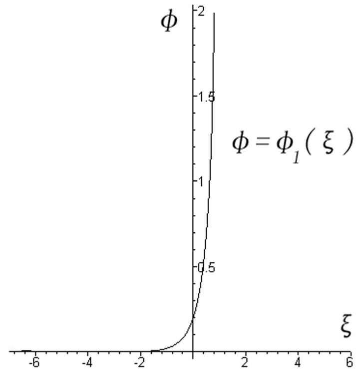
(a)


ϕ
ξ

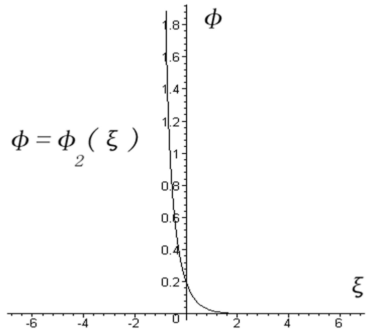
(b)


ϕ
ξ

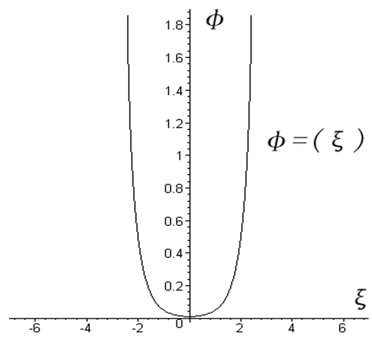
(c)


ϕ
$\phi = \phi_1(\ \xi\ )$
ξ

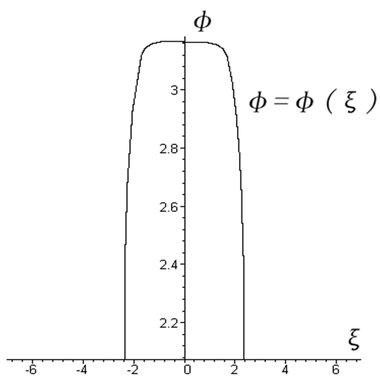
(d)


ϕ
$\phi = \phi_2(\ \xi\ )$
ξ

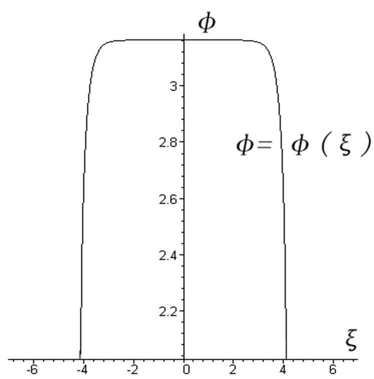
(e)


ϕ
$\phi = (\ \xi\ )$
ξ

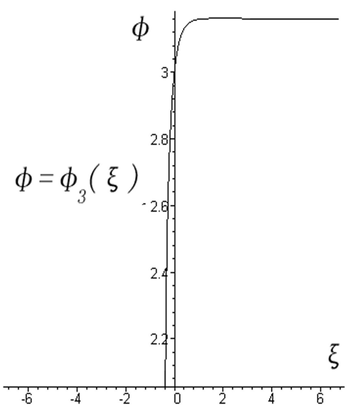
(f)


ϕ
$\phi = \phi\ (\ \xi\ )$
ξ

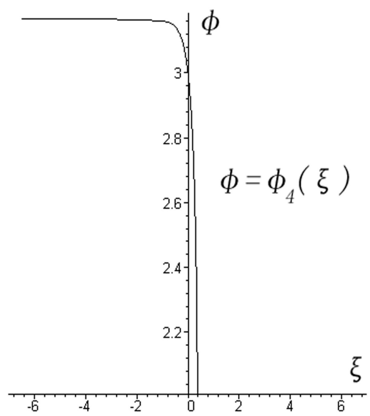
(g)


ϕ
$\phi = \ \phi\ (\ \xi\ )$
ξ

(h)



(i)



(j)

Fig. 2 The simulation of the integral curves of Eq. (2.2) when $c = 2$ and $k = -4$.

(a) $\phi(0) = -4.48$ and $\phi(0)' = 0$, (b) $\phi(0) = -4.472135955$ and $\phi(0)' = 0$, (c) $\phi(0) = -4.4721$ and $\phi(0)' = 0$, (d) $\phi(0) = 0.2$ and $\phi(0)' = 0.4709328804$, (e) $\phi(0) = 0.2$ and $\phi(0)' = -0.4709328804$, (f) $\phi(0) = 0.01$ and $\phi(0)' = 0$, (g) $\phi(0) = 3.1622$ and $\phi(0)' = 0$, (h) $\phi(0) = 3.1622776$ and $\phi(0)' = 0$, (i) $\phi(0) = 3$ and $\phi(0)' = 0.7071067812$, (j) $\phi(0) = 3$ and $\phi(0)' = -0.7071067812$.

Case 2. $c < 0$ and $k_3(c) < k < k_1(c)$. For this case, system (2.3) has an orbit Γ_1 on which ϕ is bounded when $\phi_0 < c$, has an orbit Γ_2 on which ϕ is bounded when $c < \phi_0 < \phi_-^0$ and four heteroclinic orbits $L_\pm^2$ and $L_\pm^3$ are bounded when $\phi_0 = \phi_-^0$, has a periodic orbit when $\phi_-^0 < \phi_0 < 0$. For example, choosing $c = -2$ and $k = -2$, we have $\phi_\pm^0 = \pm 1.414213562$.

(i) We respectively take $\phi_0 = -1.4$ and -1.4133, letting $\phi(0) = \phi_0$ and $\phi'(0) = 0$, the simulation integral curves of Eq. (2.2) are (a) and (b) in Fig. 3.

(ii) The two heteroclinic orbits $L_\pm^3$ respectively have expressions

$$y_3^\pm = \pm\sqrt{\frac{\phi^4 + 2(2k-c)\phi^2 + 2h(\phi_-^0)}{2(\phi - c)}}, \text{ for } \phi_-^0 \le \phi \le \phi_+^0\,, \tag{3.9}$$

where

$$h(\phi_-^0) = -\frac{1}{2}(\phi_-^0)^4 + (c-2k)(\phi_-^0)^2\,. \tag{3.10}$$

If $\phi_-^0 \le \phi_3^0 \le \phi_+^0$, then from the first equation of system (2.3) we have $\frac{d\phi}{d\xi}|_{\xi=\xi_0} = y_3^\pm(\phi_3^0)$ at $\phi = \phi_3^0$. For example, taking $\phi_3^0 = 0$, we have $y_3^\pm(\phi_3^0) = \pm 1$. Letting $\phi(0) = 0$ and $\phi'(0) = \pm 1$ respectively, we simulate the integral curves of Eq. (2.2) as (c) and (d) in Fig. 3.

(iii) When $\phi_0 = \phi_-^0$, $L_\pm^2$ lie on the left side of the line $\phi = \phi_-^0$, the simulation integral curve of Eq. (2.2) is similar to Figs. 2 (i) - (j), when $\phi_0 < c$, to Fig. 2 (a) or (f), when $c < \phi_0 < \phi_-^0$, to Fig. 2 (g) or (h).

Remark 2 The simulation in Fig. 3 imply that under of case 2, the integral curve $\phi = \phi_5(\xi)$ and $\phi = \phi_6(\xi)$ are defined on $(-\infty, +\infty)$, $\phi_5(\xi)$ tends to ϕ_-^0 when ξ tends to $-\infty$ or tends to ϕ_+^0 when ξ tends to $+\infty$ and $\phi_6(\xi)$ tends to ϕ_+^0 when ξ tends to $-\infty$ or tends to ϕ_-^0 when ξ tends to $+\infty$.

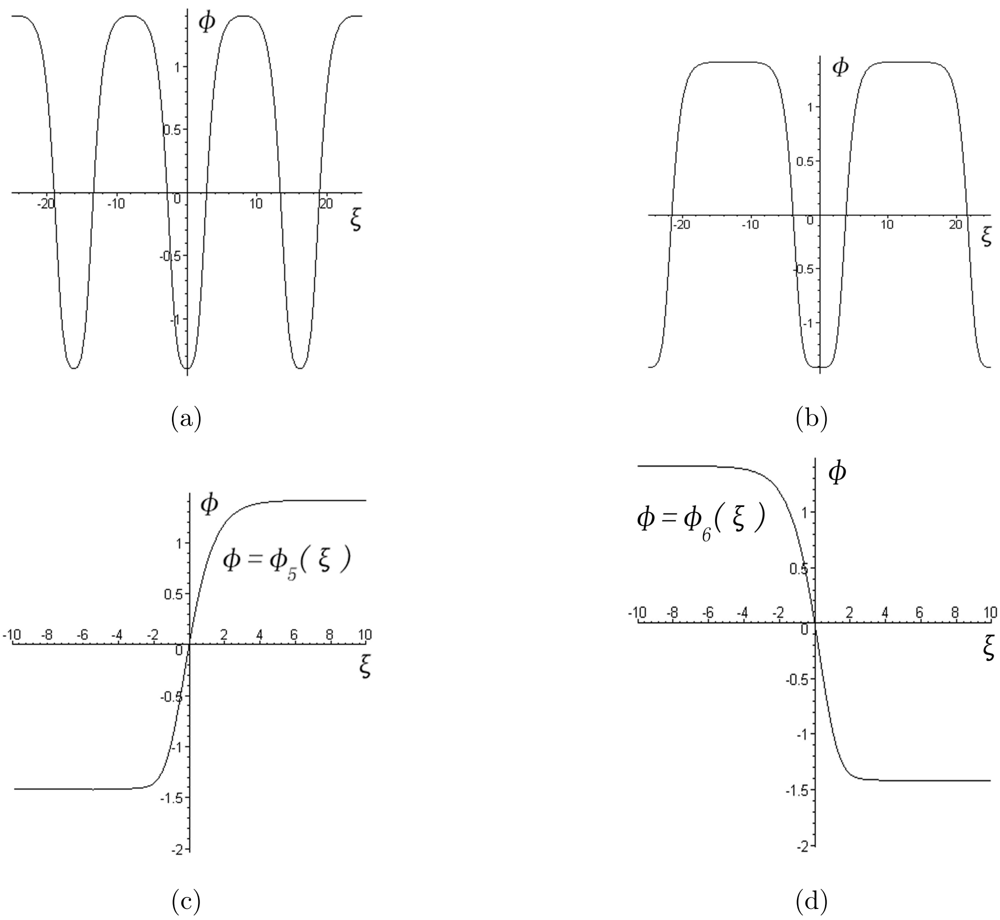


Fig. 3 The simulation of the integral curves of Eq. (2.2) when $c = -2$ and $k = -2$.

(a) $\phi(0) = -1.4$ and $\phi'(0) = 0$, (b) $\phi(0) = -1.4133$ and $\phi'(0) = 0$, (c) $\phi^(0) = 0$ and $\phi'(0) = 1$, (d) $\phi^(0) = 0$ and $\phi'(0) = -1$.

4 The Expressions of Compactons and Generalized Kink Waves

In this section, we derive the expressions of compactons and generalized kink waves by using the information obtained from above sections.

4.1 Integral Expressions of Compactons

For given c and k, we give hypotheses as follows:

(H1) $c < 0$, $k < k_3(c)$ and ϕ_0 satisfies $\phi_0 < \phi_-^*$.

(H2) $c = 0$, $k < 0$ and ϕ_0 satisfies $\phi_0 < \phi_-^1$.

(H3) $c > 0$, $k \le k_3(c)$ and ϕ_0 satisfies $\phi_0 < \phi_-^1$ *or* $0 < \phi_0 < c$.

(H4) $c > 0$, $k_3(c) < k < k_2(c)$ and ϕ_0 satisfies $\phi_0 < \phi_-^1$ *or* $0 < \phi_0 < \phi_+^*$.

(H5) $c \ge 0$, $k_2(c) \le k$ and ϕ_0 satisfies $\phi_0 < -c$.

(H6) $c < 0$, $k \ge k_3(c)$ and ϕ_0 satisfies $\phi_0 < c$.

(H7) $c < 0$, $k < k_3(c)$ and ϕ_0 satisfies $-c < \phi_0 < \phi_+^0$.

(H8) $c \ge 0$, $k < k_3(c)$ and ϕ_0 satisfies $c < \phi_0 < \phi_+^0$.

(H9) $c < 0$, $k \ge k_1(c)$ and ϕ_0 satisfies $c < \phi_0 < 0$.

(H10) $c < 0$, $k_3(c) < k < k_1(c)$ and ϕ_0 satisfies $c < \phi_0 < \phi_-^0$.

Proposition 4.1 (i) *If one of hypotheses (H1) – (H6) holds, then Eq. (1.5) has a concave compacton $u = \phi(\xi)$ which satisfies integral equation*

$$\xi_0 - |\xi| = \int_\phi^c \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \ \ for\ |\xi| \le \xi_0\,, \tag{4.1}$$

where

$$\xi_0 = \int_{\phi_0}^c \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds\,. \tag{4.2}$$

(ii) *If one of hypotheses (H7) – (H10) holds, then Eq. (1.5) has a convex compacton $u = \phi(\xi)$ which satisfies integral equation*

$$\xi_0 - |\xi| = \int_c^\phi \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \ \ for\ |\xi| \le \xi_0\,, \tag{4.3}$$

where

$$\xi_0 = \int_c^{\phi_0} \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds\,. \tag{4.4}$$

Proof: From Fig. 1 it is seen that the unique orbit Γ_1 or Γ_2 of system (2.3) passes the point $(\phi_0, 0)$ when one of above hypotheses holds. From (2.5) the Γ_1 and Γ_2 have expression

$$2(\phi - c)y^2(\phi) = (\phi^2 - \phi_0^2)(\phi^2 - \alpha)\,. \tag{4.5}$$

Substituting $y = \frac{d\phi}{d\xi}$ into (4.5), we have

$$\pm\sqrt{\frac{2(\phi - c)}{(\phi^2 - \phi_0^2)(\phi^2 - \alpha)}}d\phi = d\xi\,. \tag{4.6}$$

Thus along Γ_1 and Γ_2 respectively integrate (4.6), the (4.1) and (4.3) are obtained.

4.2 Implicit or Integral Expressions of Generalized Kink Waves

For given c and k, we give hypotheses as follows:

(H11) $c > 0, k < k_2(c)$ and ϕ_1^0 satisfies $0 < \phi_1^0 < c < \phi_+^0$.

(H12) $k < k_3(c)$ and ϕ_2^0 satisfies $\phi_-^0 < c < \phi_2^0 < \phi_+^0$.

(H13) $c < 0, k \geq k_1(c)$ and ϕ_2^0 satisfies $c < \phi_2^0 < 0$.

(H14) $c < 0, k_3(c) < k < k_1(c)$ and ϕ_2^0 satisfies $c < \phi_2^0 < \phi_-^0$.

(H15) $c < 0, k_3(c) < k < k_1(c)$ and ϕ_3^0 satisfies $c < \phi_-^0 < \phi_3^0 < \phi_+^0$.

Proposition 4.2 (i) *If hypothesis (H11) holds, then Eq. (1.5) has two generalized kink waves $u = \phi_1(\xi)$ and $u = \phi_2(\xi)$ which satisfy integral equation*

$$\int_{\phi_1^0}^{\phi_1} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k-c)}}ds = \xi,\ \textit{for}\ -\infty < \xi < \xi_1 \tag{4.7}$$

and

$$\int_{\phi_1^0}^{\phi_2} -\frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k-c)}}ds = \xi,\ \textit{for}\ -\xi_1 < \xi < +\infty \tag{4.8}$$

where

$$\xi_1 = \int_c^{\phi_1^0} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k-c)}}ds\,. \tag{4.9}$$

(ii) *If hypothesis (H12) holds, then Eq. (1.5) has two generalized kink waves $u = \phi_3(\xi)$ and $u = \phi_4(\xi)$ which respectively satisfy equation*

$$\frac{\sqrt{\phi_+^0 - c}}{2} \ln\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_3 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_3 - c}}\right) - \sqrt{\phi_+^0 + c}\,\arctan\frac{\sqrt{\phi_3 - c}}{\sqrt{\phi_+^0 + c}} = \frac{\phi_+^0}{\sqrt{2}}(\xi + \xi_2)\,, \tag{4.10}$$

for $-\xi_2 < \xi < +\infty$. And

$$\frac{\sqrt{\phi_+^0 - c}}{2} \ln\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_4 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_4 - c}}\right) - \sqrt{\phi_+^0 + c}\,\arctan\frac{\sqrt{\phi_4 - c}}{\sqrt{\phi_+^0 + c}} = \frac{\phi_+^0}{\sqrt{2}}(-\xi + \xi_2)\,, \tag{4.11}$$

for $-\infty < \xi < \xi_2$. Where

$$\xi_2 = \frac{\sqrt{2}}{\phi_+^0}\left[\frac{\sqrt{\phi_+^0 - c}}{2} \ln\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_2^0 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_2^0 - c}}\right) - \sqrt{\phi_+^0 + c}\,\arctan\frac{\sqrt{\phi_2^0 - c}}{\sqrt{\phi_+^0 + c}}\right]. \tag{4.12}$$

(iii) *If hypothesis (H13) holds, then Eq. (1.5) has two generalized kink waves $u = \phi_3(\xi)$ and $u = \phi_4(\xi)$ which respectively satisfy integral equation*

$$\int_{\phi_2^0}^{\phi_3} -\frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k-c)}}ds = \xi, \ \text{for} \ -\xi_2 < \xi < +\infty \tag{4.13}$$

and

$$\int_{\phi_2^0}^{\phi_4} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k-c)}}ds = \xi, \ \text{for} \ -\infty < \xi < \xi_2 \tag{4.14}$$

where

$$\xi_2 = \int_{c}^{\phi_2^0} -\frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k-c)}}ds\,. \tag{4.15}$$

(iv) *If hypothesis (H14) holds, then Eq. (1.5) has two generalized kink waves $u = \phi_3(\xi)$ and $u = \phi_4(\xi)$ which satisfy equation*

$$\left(\frac{\sqrt{\phi_-^0 - c} - \sqrt{\phi_3 - c}}{\sqrt{\phi_-^0 - c} + \sqrt{\phi_3 - c}}\right)^{\sqrt{\phi_-^0 - c}} \left(\frac{\sqrt{-\phi_-^0 - c} + \sqrt{\phi_3 - c}}{\sqrt{-\phi_-^0 - c} - \sqrt{\phi_3 - c}}\right)^{\sqrt{-\phi_-^0 - c}} = \beta_1 e^{(\sqrt{2}\phi_-^0 \xi)}\,, \tag{4.16}$$

for $-\xi_2 < \xi < +\infty$, *and*

$$\left(\frac{\sqrt{\phi_-^0 - c} - \sqrt{\phi_4 - c}}{\sqrt{\phi_-^0 - c} + \sqrt{\phi_4 - c}}\right)^{\sqrt{\phi_-^0 - c}} \left(\frac{\sqrt{-\phi_-^0 - c} + \sqrt{\phi_4 - c}}{\sqrt{-\phi_-^0 - c} - \sqrt{\phi_4 - c}}\right)^{\sqrt{-\phi_-^0 - c}} = \beta_1 e^{-\sqrt{2}\phi_-^0 \xi}, \tag{4.17}$$

for $-\infty < \xi < \xi_2$, *where*

$$\beta_1 = \left(\frac{\sqrt{\phi_-^0 - c} - \sqrt{\phi_2^0 - c}}{\sqrt{\phi_-^0 - c} + \sqrt{\phi_2^0 - c}}\right)^{\sqrt{\phi_-^0 - c}} \left(\frac{\sqrt{-\phi_-^0 - c} + \sqrt{\phi_2^0 - c}}{\sqrt{-\phi_-^0 - c} - \sqrt{\phi_2^0 - c}}\right)^{\sqrt{-\phi_-^0 - c}}, \tag{4.18}$$

and

$$\xi_2 = \ln \beta_1 . \tag{4.19}$$

(v) *If hypotheses (H15) holds, then Eq. (1.5) has two generalized kink waves* $u = \phi_5(\xi)$ *and* $u = \phi_6(\xi)$ *which satisfies equation*

$$\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_5 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_5 - c}}\right)^{\sqrt{\phi_+^0 - c}} \left(\frac{\sqrt{\phi_5 - c} - \sqrt{-\phi_+^0 - c}}{\sqrt{\phi_5 - c} + \sqrt{-\phi_+^0 - c}}\right)^{\sqrt{-\phi_+^0 - c}} = \beta_2 e^{(\sqrt{2}\phi_+^0 \xi)}, \tag{4.20}$$

for $-\infty < \xi < +\infty$ *and*

$$\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_6 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_6 - c}}\right)^{\sqrt{\phi_+^0 - c}} \left(\frac{\sqrt{\phi_6 - c} - \sqrt{-\phi_+^0 - c}}{\sqrt{\phi_6 - c} + \sqrt{-\phi_+^0 - c}}\right)^{\sqrt{-\phi_+^0 - c}} = \beta_2 e^{-(\sqrt{2}\phi_+^0 \xi)}, \tag{4.21}$$

for $-\infty < \xi < +\infty$. *Where*

$$\beta_2 = \left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_3^0 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_3^0 - c}}\right)^{\sqrt{\phi_+^0 - c}} \left(\frac{\sqrt{\phi_3^0 - c} - \sqrt{-\phi_+^0 - c}}{\sqrt{\phi_3^0 - c} + \sqrt{-\phi_+^0 - c}}\right)^{\sqrt{-\phi_+^0 - c}}. \tag{4.22}$$

Proof: Here we only proof (ii), in the other cases one can use a similar arguments. If hypotheses (H12) holds, then there are two heteroclinic orbits L_+^2 and L_-^2 of system (2.3) passes the point $(\phi_+^0, 0)$, From (2.5) they have expressions

$$2(\phi - c)y^2(\phi) = [(\phi_+^0)^2 - \phi^2]^2, \text{ for } \phi_-^0 < c < \phi < \phi_+^0 . \tag{4.23}$$

Substituting $y = \frac{d\phi}{d\xi}$ into (4.23) and letting $\phi(0) = \phi_2^0$, we have

$$\frac{\sqrt{2(\phi - c)}}{(\phi_+^0)^2 - \phi^2} d\phi = d\xi, \ -\xi_2 < \xi < +\infty \text{ and } \phi_-^0 < c < \phi < \phi_+^0 , \tag{4.24}$$

and

$$-\frac{\sqrt{2(\phi-c)}}{(\phi_+^0)^2-\phi^2}d\phi = d\xi,\ -\infty<\xi<\xi_2 \text{ and } \phi_-^0<c<\phi<\phi_+^0\,. \tag{4.25}$$

Integrating (4.24) and (4.25) along L_+^2 and L_-^2 respectively, we have

$$\int_{\phi_2^0}^{\phi_3}\frac{\sqrt{2(s-c)}}{(\phi_+^0)^2-s^2}ds = \int_0^{\xi}ds,\ -\xi_2<\xi<+\infty \text{ and } \phi_-^0<c<\phi<\phi_+^0\,, \tag{4.26}$$

and

$$-\int_{\phi_2^0}^{\phi_4}\frac{\sqrt{2(s-c)}}{(\phi_+^0)^2-s^2}ds = \int_0^{\xi}ds,\ -\infty<\xi<\xi_2 \text{ and } \phi_-^0<c<\phi<\phi_+^0\,. \tag{4.27}$$

From (4.26) and (4.27) we obtain (4.10) and (4.11).

5 Conclusion

In this paper, we have employed both the bifurcation method of planar dynamical systems and numerical simulation method of differential equations to investigate the bounded traveling waves of a generalized Camassa-Holm equation. We have found another kind of bounded traveling waves which have the properties of compactons or generalized kink waves. Their planar graphs are simulated (see Figs. 2 (a), (f), (g) and (h) for compactons; Figs. 2 (d), (e), (i) and (j) and Figs. 3 (c) and (d) for generalized kink waves). Their integral or implicit representations are obtained (see Proposition 4.1 for compactons; Proposition 4.2 for generalized kink waves).

References

[1] **Camassa, R.**, and **Holm, D. D. :** *An integrable shallow water equation with peaked solitons.* Phys Rev Lett, **71** (11), 1661-4 (1993)

[2] **Johnson, R. S.**, and **Camassa-Holm :** *Korteweg-de Vries and related models for waves.* J. Fluid Mech., 455, 63-82 (2002)

[3] **Fuchssteiner, B.**, and **Fokas, A. S. :** *Symplectic structuctures, their Böcklund transformations and hereditary symmetries.* Phys. D., **4** (1), 47-66 (1981/82)

[4] **Beals, R., Sattinger, D. H.**, and **Szmigielski, J. :** *Acoustic scattering and the extended Korteweg-de Vries hierarchy.* Adv. Math., **140** (2), 190-206 (1998)

[5] **Constantin, A. :** *On the scattering problem for the Camassa - Holm equation.* R. Soc. Lond. Proc. Ser. A, **457** (2008), 953-970 (2001)

[6] **Zhengrong, L.**, and **Tifei, Q. :** *Peakons and their bifurcation in a generalized Camassa-Holm equation.* International Journal of Bifurcation and Choas, **11** (3), 781-792(2001)

[7] **Rosenau**, and **Hyman, J. M. :** *Compactons: Solitons with finite wavelength.* Phys. Rev. Lett. **70**, 564-567 (1993)

[8] **Zhengrong, L.**, and **Can, Ch. :** *Compactons in a general compressible hyperelastic rod.* Chaos, Solitions and Fractals **22**, 627-640 (2004)

[9] **Rosenau, P. :** *Compact and noncompact dispersive patterns.* Phys. Lett. **A275**, 193-203 (2000)

[10] **Wazwaz, A. M. :** *Compact dispersive structures for variants of the $k(n, n)$ and the KP equations.* Chaos, Solitions and Fractals **13**, 1053-1062 (2002b)

[11] **Liu, Z. R., Lin, Q. M.**, and **Li, Q. X. :** *Integral approach to compacton solutions of Boussinesq-like $B(m, n)$ equations with fully nonlinear dispersion.* Chaos, Solitions and Fractals **19**, 1071-1081 (2004b)

[12] **Liu, Z. R., Li, Q. X.**, and **Lin, Q. M. :** *New bounded traveling waves to Camassa-Holm equation.* International Journal of Bifurcation and Chaos, Vol. **14**, No. 10, 3541-3556 (2004)

received: September 15, 2005

Authors:

Shaolong Xie
Department of Mathematics
of Yuxi Normal College
Yuxi
Yunnan, 653100
P.R. China
e-mail: xieshlong@163.com

Xiaochun Hong
Primary Education Department
of Qujing Normal College
Qujing
Yunnan, 655000
P.R. China
email:xchhong@sina.com

Weiguo Rui
Department of Mathematics
of Honghe University
Mengzi
Yunnan, 661100
P.R. China
e-mail: weiguorhhu@yahoo.com.cn

Rostock. Math. Kolloq. **61**, 49–55 (2006)

Subject Classification (AMS)
05A16

Sadek Bouroubi, Moncef Abbas

New identities for Bell's polynomials New approaches

ABSTRACT. In this work we suggest a new approach to the determination of new identities for Bell's polynomials, based on the Lagrange inversion formula, and the binomial sequences. This approach allows the easy recovery of known identities and deduction of some new identities including these polynomials.

KEY WORDS. Bell's polynomials, Bell's numbers, Lagrange inversion formula, binomial sequences.

1 Introduction

Using a proof by recurrence, Salim KHELIFA and Yves CHERRUAULT gave the following identity on Bell's polynomials [3].

Theorem 1 *For n and $k \in \mathbb{N}^*$,$k \leq n$, it holds*

$$B_{n,k}\left(1^0, 2^1, 3^2, \ldots\right) = \binom{n-1}{k-1} n^{n-k}. \tag{1}$$

This identity allowed the authors to demonstrate a new theorem of convergence for the Adomian decomposition method [4], but the proof is excessively long (7 pages) and consequently it requires another proof. Here we propose two new identities, with shorter proofs. The first uses the Lagrange inversion formula, having as an immediate consequence the identity (1), and the second uses binomial sequences, begetting new identities, and the known identities in the literature.

Definition 2 *The Bell polynomials are the polynomials $B_{n,k}(x_1, x_2, \ldots)$ in an infinite number of variables $x_1, x_2, \ldots$, defined by (see [2], p. 133)*

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots \cdot \tag{2}$$

Their exact expression is (see [2], p. 134)

$$B_{n,k}(x_1, x_2, \ldots) = \sum_{\pi(n)} \frac{n!}{k_1! k_2! \cdots (1!)^{k_1} (2!)^{k_2} \cdots} x_1^{k_1} x_2^{k_2} \ldots,$$

where $\pi(n)$ denotes a partition of n, with $k_1 + 2k_2 + \ldots = n$; k_i is of course, the number of parts of size i. Also $k_1 + k_2 + \ldots = k$ is the number of parts in the partition.

2 Main results

2.1 Method based on the Lagrange inversion formula

Let f be an analytic function about the origin such that $f(0) \neq 0$ and for n and $m \in \mathbb{N}$ let

$$f_n(m) = \begin{cases} D^n [f(w)]^m \big|_{w=0} & if\, n \geq 1 \\ (f(0))^m & if\, n = 0 \end{cases}$$

where D is the differential operator $\dfrac{\mathrm{d}}{\mathrm{d}\, w}$.

Theorem 3 *For n and $k \in \mathbb{N}^*$, $k \leq n$, it holds*

$$B_{n,k}(f_0(1), f_1(2), f_2(3), \ldots) = \binom{n-1}{k-1} f_{n-k}(n) \ .$$

Proof: For $z \in \mathbb{C}$, let us consider the equation of the unknown $w \in \mathbb{C}$,

$$w - zf(w) = 0 \, .$$

This equation admits a unique solution $w = g(z)$ around the origin (see [1], p. 234) and for any analytic function F around the origin we have by Lagrange inversion formula

$$F(g(z)) = F(0) + \sum_{n \geq 1} D^{n-1} \left\{ F'(w) [f(w)]^n \right\} \Big|_{w=0} \frac{z^n}{n!} \, . \tag{3}$$

If we choose $F(w) = w$, we get from (3)

$$\begin{aligned} g(z) &= \sum_{n \geq 1} D^{n-1} [f(w)]^n \big|_{w=0} \frac{z^n}{n!} \\ &= \sum_{n \geq 1} f_{n-1}(n) \frac{z^n}{n!} \, . \end{aligned}$$

Thus from (2) we have

$$\begin{aligned}\frac{1}{k!}\left(g\left(z\right)\right)^{k} &= \frac{1}{k!}\left(\sum_{n\geq 1} f_{n-1}(n)\frac{z^{n}}{n!}\right)^{k} \\ &= \sum_{n\geq k} B_{n,k}\left(f_{0}(1), f_{1}(2), f_{2}(3), \ldots\right)\frac{z^{n}}{n!}.\end{aligned}$$

On the other hand, if we choose $F\left(w\right)=\frac{w^{k}}{k!}$, we get by (3)

$$\begin{aligned}\frac{1}{k!}\left(g\left(z\right)\right)^{k} &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{w^{k-1}\left[f\left(w\right)\right]^{n}\right\}\Big|_{w=0}\frac{z^{n}}{n!} \\ &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{w^{k-1}\sum_{j\geq 0} D^{j}\left[f\left(w\right)\right]^{n}\Big|_{w=0}\frac{w^{j}}{j!}\right\}\Bigg|_{w=0}\frac{z^{n}}{n!} \\ &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{\sum_{j\geq 0}\frac{f_{j}\left(n\right)}{j!}w^{j+k-1}\right\}\Bigg|_{w=0}\frac{z^{n}}{n!} \\ &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{\sum_{j\geq k-1}\frac{f_{j-k+1}\left(n\right)}{(j-k+1)!}w^{j}\right\}\Bigg|_{w=0}\frac{z^{n}}{n!} \\ &= \frac{1}{(k-1)!}\sum_{n\geq k}\frac{(n-1)!}{(n-k)!}f_{n-k}\left(n\right)\frac{z^{n}}{n!} \\ &= \sum_{n\geq k}\binom{n-1}{k-1}f_{n-k}\left(n\right)\frac{z^{n}}{n!}.\end{aligned}$$

□

Corollary 4 *Let $a\in\mathbb{R}$. We have for all n and $k\in\mathbb{N}^{*}$, $k\leq n$,*

$$B_{n,k}\left(\left(1a\right)^{0},\left(2a\right)^{1},\left(3a\right)^{2},\ldots\right)=\binom{n-1}{k-1}\left(an\right)^{n-k}.$$

Proof: We have just to apply Theorem 3 by putting $f\left(w\right)=e^{aw}$, that gives

$$f_{n}\left(m\right)=\begin{cases}\left(am\right)^{n} & \text{if } n\geq 1\\ 1 & \text{if } n=0\end{cases}$$

□

Remark 5

1) If we choose $a=1$ we find the identity (1).

2) It is obvious that the identity of Corollary 4 is not the only consequence of Theorem 3, because it depends on the choice of f. If, for instance, we choose the function $f(w) = 1 + aw$, we get

$$f_n(m) = \begin{cases} a^n [m]_n & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

where $[m]_n = m(m-1)\cdots(m-n+1)$. Thus we have

$$B_{n,k}\left(1!a^0, 2!a^1, 3!a^2, \ldots\right) = a^{n-k}\binom{n-1}{k-1}\frac{n!}{k!}.$$

If we choose :

△ $a = 1$, we recover the known identity

$$B_{n,k}(1!, 2!, 3!, \ldots) = \binom{n-1}{k-1}\frac{n!}{k!}.$$

△ $a = 0$, we get

$$B_{n,k}(1, 0, 0, \ldots) = 0, \quad \textit{except} \;\; B_{n,n} = 1.$$

2.2 Method based on binomial sequences

A sequence of definite functions $(\varphi_n(x))_n$ on a subset I of $\mathbb{R}$ is called binomial if,

$$\varphi_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}\varphi_k(x)\,\varphi_{n-k}(y), \forall x, y \in I.$$

Theorem 6 *Let $(\varphi_n(x))_n$ be a binomial sequence defined on I, $\mathbb{N} \subseteq I \subseteq \mathbb{R}$, with $\varphi_0 \neq 0$. Then for all n and $k \in \mathbb{N}^*$, $k \leq n$, we have*

$$B_{n,k}(\varphi_0(1), 2\varphi_1(1), 3\varphi_2(1), \ldots) = \binom{n}{k}\varphi_{n-k}(k).$$

Proof: Let by Φ_x denote the exponential generating function associated to the sequence $(\varphi_n(x))_n$, i.e.

$$\Phi_x(t) = \sum_{n\geq 0}\varphi_n(x)\frac{t^n}{n!}.$$

(*We suppose, of course, that the radius of convergence satisfies $R > 0$.*)

The sequence $(\varphi_n(x))_n$ is binomial, then we have, from Cauchy product

$$\Phi_{x+y}(t) = \Phi_x(t)\cdot\Phi_y(t), \forall x, y \in I.$$

Hence

$$\Phi_k(t) = (\Phi_1(t))^k, \forall k \in \mathbb{N}^*. \tag{4}$$

It comes then, on the one hand

$$\begin{aligned}\frac{1}{k!}\left(t\Phi_1\left(t\right)\right)^k &= \frac{1}{k!}\left(\sum_{n\geq 0}\varphi_n\left(1\right)\frac{t^{n+1}}{n!}\right)^k \\ &= \frac{1}{k!}\left(\sum_{n\geq 1}n\varphi_{n-1}\left(1\right)\frac{t^n}{n!}\right)^k \\ &= \sum_{n\geq k}B_{n,k}\left(\varphi_0\left(1\right),2\varphi_1\left(1\right),3\varphi_2\left(1\right),\ldots\right)\frac{t^n}{n!}\cdot\end{aligned}$$

On the other hand by (4), we have

$$\begin{aligned}\frac{1}{k!}\left(t\Phi_1\left(t\right)\right)^k &= \frac{1}{k!}t^k\Phi_k\left(t\right) \\ &= \frac{1}{k!}\sum_{n\geq 0}\varphi_n\left(k\right)\frac{t^{n+k}}{n!} \\ &= \sum_{n\geq k}\binom{n}{k}\varphi_{n-k}\left(k\right)\frac{t^n}{n!}\cdot\end{aligned}$$

□

Application

Let $S\left(n,k\right)$ denote the Stirling number of the second kind, and put

$$B_n\left(x\right) = \sum_{k=0}^{n}S\left(n,k\right)x^k\,.$$

The sequence $\left(B_n\left(x\right)\right)_n$ is defined in $\mathbb{R}$, where $B_0\left(x\right)\equiv 1$ and $B_n\left(1\right) = B_n$, the Bell numbers.

Corollary 7 *We have*

$$B_{n,k}\left(B_0,2B_1,3B_2,\ldots\right) = \binom{n}{k}\sum_{j=0}^{n-k}S\left(n-k,j\right)k^j\cdot$$

Proof: It is well known and easily verified that

$$\sum_{n=0}^{+\infty}B_n\left(x\right)\frac{z^n}{n!} = \exp\left(x\left(e^z-1\right)\right)\cdot \tag{5}$$

In fact, from (5) it follows that

$$\sum_{n=0}^{\infty}B_n\left(x+y\right)\frac{z^n}{n!} = \exp\left(\left(x+y\right)\left(e^z-1\right)\right) = \sum_{n=0}^{+\infty}B_n\left(x\right)\frac{z^n}{n!}\sum_{n=0}^{+\infty}B_n\left(y\right)\frac{z^n}{n!}\cdot$$

Therefore

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) \cdot$$

Thus the sequence $(B_n(x))_n$ is binomial and the result is proved by means of Theorem 6. □

Remark 8 Corollary 7 is not the only consequence of Theorem 6. It all depends on the choice of binomial sequence. If we choose for example the binomial sequence defined on $\mathbb{R}$ by $\varphi_n(x) = x^n$, we recover the known identity

$$B_{n,k}(1,2,3,\ldots) = \binom{n}{k} k^{n-k}.$$

References

[1] **Evgrafov, M.**, and **Coll :** *Recueil de problèmes sur la théorie des fonctions analytiques.* Traduction française, Editions Mir. 1974

[2] **Comtet, L. :** *Advanced combinatorics.* D. Reidel, 1974

[3] **Khelifa, S.**, and **Cherruault, Y. :** *Nouvelle identité pour les polynômes de Bell.* Maghreb Mathematical Review. Vol.9, n°1 and 2, June and December 2000, 115-123

[4] **Khelifa, S.**, and **Cherruault, Y. :** *New results on Adomian method.* Kybernetes. Vol. 29; n° 3, (2000) 332-354

[5] **Khelifa, S. :** *Equations aux dérivées partielles et méthode décompositionnelle d'Adomian.* Thèse de Doctorat Es Sciences en Mathématiques, U.S.T.H.B., Alger 2002

received: March 26, 2003

Authors:

Sadek Bouroubi
U.S.T.H.B.
Faculty of Mathematics
Department of Operational Research
B.P. 32 16111 El-Alia
Bab-Ezzouar
Algiers
Algeria

e-mail: bouroubis@yahoo.fr

Moncef Abbas
U.S.T.H.B.
Faculty of Mathematics
Department of Operational Research
B.P. 32 16111 El-Alia
Bab-Ezzouar
Algiers
Algeria

Rostock. Math. Kolloq. **61**, 57–71 (2006)

Subject Classification (AMS)
37K40, 37K50

Weiguo Rui[1], Yao Long and Bin He

Periodic wave solutions and solitary cusp wave solutions for a higher order wave equation of KdV type[2]

ABSTRACT. This paper is the continuation of Ref. [1]. Both the bifurcation theory of planar dynamical system and elliptic function integral method are applied to study a higher order wave equation of KdV type. And the parametric space is redivided when the integral constant $g \neq 0$. Many explicit and implicit solutions of periodic wave and solitary cusp wave are obtained.

KEY WORDS. higher order wave equation of KdV type, solitary cusp wave solution, periodic wave solution, elliptic function integral method.

1 Introduction

In this paper, we will seek periodic wave and solitary cusp wave solutions for the following higher order wave equation of KdV type (see [1, 2]):

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x + \alpha\beta(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}) = 0\,, \tag{1.1}$$

where $\rho_i (i = 1, 2, 3)$ are free parameters and α, β are positive real constants which characterize, respectively, the long wavelength and short amplitude of the waves. Just as Tzirtzilakis, E. [2] said, the equation (1.1) is a water wave equation of KdV type which is more physically and practically meaningful. By the local coordinate transformation

$$u = v - \alpha\rho_1 v^2 - \beta\left(3\rho_1 + \frac{7}{4}\rho_2 - \frac{1}{2}\rho_3\right) v_{xx}\,, \tag{1.2}$$

Eq. (1.1) can be transformed into the following simple equation, see [3, 4]:

$$v_t - \frac{3}{2}\beta\rho_2 v_{xxt} + \beta(1 - \frac{3}{2}\rho_2) v_{xxx} + \alpha v v_x - \frac{1}{2}\alpha\beta\rho_2(v v_{xxx} + 2 v_x v_{xx}) = 0\,, \tag{1.3}$$

[1]Corresponding author: E-mail address: weiguorhhu@yahoo.com.cn

[2]This research was supported by Natural Science Foundation of Honghe University (XJZ10401) and the National Natural Science Foundation of China (10261008).

where $\rho_2 \neq 0$. In Ref. [1], we obtained two explicit parametric representations of periodic solutions of equation (1.3) when integral constant $g = 0$. In this case, we also proved the existence of all travelling wave solutions. However, when $g \neq 0$, the bifurcation of travelling solutions had not been studied. In fact, when the integral constant $g \neq 0$, the dynamical behaviors of the equation (1.3) are better than the case of $g = 0$. Therefore, we shall use bifurcation method of planar dynamical system [5]-[8] and elliptic function integral method [9, 10] to investigate the explicit and implicit travelling wave solutions of (1.3) when $g \neq 0$.

Let $v(x,t) = \psi(x-ct) = \psi(\xi)$, where c is the wave speed, then the equation (1.3) becomes the following ordinary differential equation

$$\frac{1}{2}\alpha(\psi^2)_\xi - c\psi_\xi + \left(\frac{3}{2}c\beta\rho_2 + \beta(1-\frac{3}{2}\rho_2)\right)\psi_{\xi\xi\xi} - \frac{1}{2}\alpha\beta\rho_2\left(\psi\psi_{\xi\xi} + \frac{1}{2}\psi_\xi^2\right)_\xi = 0\,, \tag{1.4}$$

Integrating once with respect to ξ, we obtain the following wave equation of (1.3)

$$\beta(3c\rho_2 + 2 - 3\rho_2 - \alpha\rho_2\psi)\psi_{\xi\xi} - \frac{1}{2}\alpha\beta\rho_2\psi_\xi^2 + \alpha\psi^2 - 2c\psi + g = 0\,, \tag{1.5}$$

where g is the integral constant and $g \neq 0$.

Clearly, (1.5) is equivalent to the following two-dimensional systems:

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\frac{1}{2}\alpha\beta\rho_2 y^2 - \alpha\psi^2 + 2c\psi - g}{\beta(3\rho_2(c-1) + 2 - \alpha\rho_2\psi)}\,. \tag{1.6}$$

System (1.6) is a planar dynamical system defined by the 5-parameter space $(\alpha, \beta, c, \rho_2, g)$. Because the phase orbits defined by the vector field of (1.6) determine all travelling wave solutions, we will investigate bifurcations of phase portraits of the system, when the parameters vary. Since (1.3) is a physical model where only the bounded travelling waves are meaningful, so we only consider their bounded travelling wave solutions.

Suppose that $\psi(x-ct) = \psi(\xi)$ is a continuous solution of (1.6) for $\xi \in (-\infty, \infty)$ and $\lim\limits_{\xi\to\infty}\psi(\xi) = a$, $\lim\limits_{\xi\to-\infty}\psi(\xi) = b$. It is well known that (i) $\psi(x,t)$ is called a solitary wave solution if $a = b$; (ii) $\psi(x,t)$ is called a kink or anti-kink solution if $a \neq b$. Usually, a solitary wave solution of (1.3) corresponds to a homoclinic orbit of (1.6); a kink (or anti-kink) wave solution of (1.3) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of (1.6). Similarly, a periodic orbit of (1.3) corresponds to a periodically travelling wave solution of (1.6). Therefore, we must find all periodic annuli, heterclinic and homoclinic orbits of (1.6) in order to investigate the bifurcations of periodic waves and solitary cusp waves of (1.3). Thus, the bifurcation theory of dynamical systems and some computational method of travelling wave solutions are very important and useful, see [5]-[11].

We notice that the right-hand side of the second equation in (1.6) is not continuous when $\psi = \psi_s = \frac{\eta}{\alpha\rho_2}$, where $\eta = 3\rho_2(c-1) + 2$. In other words, on such straight line $\psi = \psi_s$ in the phase plane (ψ, y), the function $\psi_{\xi\xi}$ is not defined. It implies that the smooth system (1.3) sometimes has non-smooth travelling wave solutions. The similar phenomenon has been considered before, see [1, 7, 8, 10].

In Section 2, we discuss bifurcations of phase portraits of (1.6), where explicit parametric conditions will be derived. In Section 3, we derive the explicit parameter representations of the smooth periodic wave and non-smooth solitary cusp wave solutions of (1.3). In Sections 4, we derive the implicit parameter representations of the smooth periodic wave solutions.

2 Bifurcations of phase portraits of system (1.6)

Because the function $\psi_{\xi\xi}$ is not defined on the singular straight line $\psi = \frac{\eta}{\alpha\rho_2}$, we make a transformation $d\zeta = 2\beta(\eta - \alpha\rho_2\psi)d\xi$, $\eta = 3\rho_2(c-1)+2$. Then the system (1.6) becomes the following system:

$$\frac{d\psi}{d\zeta} = 2\beta(\eta - \alpha\rho_2\psi)y, \quad \frac{dy}{d\zeta} = -(-\alpha\beta\rho_2 y^2 + 2\alpha\psi^2 - 4c\psi + 2g)\,. \tag{2.1}$$

It is easy to see that (1.6) and (2.1) have the same first integral

$$H(\psi, y) = \beta(\eta - \alpha\rho_2\psi)y^2 + \frac{2}{3}\alpha\psi^3 - 2c\psi^2 + 2g\psi = h\,, \tag{2.2}$$

where h is integral constant.

By system (2.1), we define the $\psi = \psi_s = \frac{\eta}{\alpha\rho_2}$ is a singular straight line L and write

$$f(\psi) = \alpha\psi^2 - 2c\psi + g, \quad \Delta = c^2 - \alpha g, \quad \psi_{1,2} = \frac{c \pm \sqrt{\Delta}}{\alpha}, \quad Y_\pm = \pm\sqrt{\frac{2f(\psi_s)}{\alpha\beta\rho_2}}\,. \tag{2.3}$$

Thus, we obtain the following conclusion for equilibrium points of system (2.1):

(1) when $\Delta > 0$, (2.1) has two equilibrium points at $A_{1,2}(\psi_{1,2}, 0)$ in the ψ-axis;

(2) when $\Delta = 0$ and $c \neq 0$, (2.1) has only one equilibrium point at $A_0(\frac{c}{\alpha}, 0)$ in the ψ-axis;

(3) When $\rho_2 f(\psi_s) > 0$, there exist two equilibrium points of (2.1) at $S_\pm(\psi_s, Y_\pm)$ in L;

(4) When $f(\psi_s) = 0$, there exist only one equilibrium point of (2.1) at $S_0(\psi_s, 0)$ which is the intersection point of the line L and the ψ-axis.

Let $M(\psi_i, y_j)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point, (ψ_i, y_j). Then we have $Trace(M(\psi_{1,2}, 0)) = 0$ and

$$J(\psi_{1,2}, 0) = detM(\psi_{1,2}, 0) = -8\beta[\rho_2\sqrt{\Delta} \pm (3\rho_2 - 2\rho_2 c - 2)]\sqrt{\Delta}, \; J\left(\frac{c}{\alpha}, 0\right) = 0\,. \tag{2.4}$$

$$J(\psi_s, Y_\pm) = detM(\psi_s, Y_\pm) = -4\alpha^2\beta^2\rho_2^2 Y_\pm^2 < 0, \quad J(\psi_s, 0) = 0\,. \tag{2.5}$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(\psi_i, 0)) = 0$ then it is a center point; if $J > 0$ and $(Trace(M(\psi_i, 0)))^2 - 4J(\psi_i, 0) > 0$ then it is a node; if $J = 0$ and the index of the equilibrium point is zero then it is a cusp; if $J = 0$ and the index of the equilibrium point is'nt zero then it is a high order singular point.

Notice that for $H(\psi, y) = h$ defined by (2.2), we have

$$h_{1,2} = H(\psi_{1,2}, 0) = -\frac{2(c \pm \sqrt{\Delta})[(2c^2 \pm c\sqrt{\Delta}) - (\Delta + 3\alpha g)]}{3\alpha^2}\,, \tag{2.6}$$

$$h_0 = H(\frac{c}{\alpha}, 0) = -\frac{2c(2c^2 - 3\alpha g)}{3\alpha^2}\,, \tag{2.7}$$

$$h_s = H(\psi_s, Y_\pm) = \frac{2\eta(\eta^2 - 3c\rho_2\eta + 3g\alpha\rho_2^2)}{3\alpha^2\rho_2^3}\,, \tag{2.8}$$

$$h_{s0} = H(\phi_s, 0) = \frac{2\eta^2(3c\rho_2 - 2\eta)}{3\alpha^2\rho_2^3}\,. \tag{2.9}$$

From $\Delta = 0$, we have

$$(\Gamma_1): \qquad g = g_1(c) = \frac{c^2}{\alpha}\,. \tag{2.10}$$

For a fixed ρ_2, the case of $h_1 = h_s$ or $h_2 = h_s$ imply

$$(\Gamma_2): \qquad g = g_2(c) = \frac{(12\rho_2^2 - 8\rho_2)c - 9\rho_2^2 + 12\rho_2 - 4}{4\alpha\rho_2^2}\,, \tag{2.11}$$

and

$$(\Gamma_3): \qquad g = g_3(c) = -\frac{3\rho_2^2c^2 - 12\rho_2^2c + 8\rho_2c + 9\rho_2^2 - 12\rho_2 + 4}{\alpha\rho_2^2}\,. \tag{2.12}$$

It is easy to see that $\psi_s = \psi_1 (\psi_s = \psi_2)$ corresponds to $J(\psi_1, 0) = 0 (J(\psi_2, 0) = 0)$ when the parameter $(c, g) \in \Gamma_3$. In this case, $f(\psi_s) = 0$ corresponds to $\rho_2\sqrt{\Delta} \pm (3\rho_2 - 2\rho_2 c - 2) = 0$ when the parameter $(c, g) \in \Gamma_3$.

Write

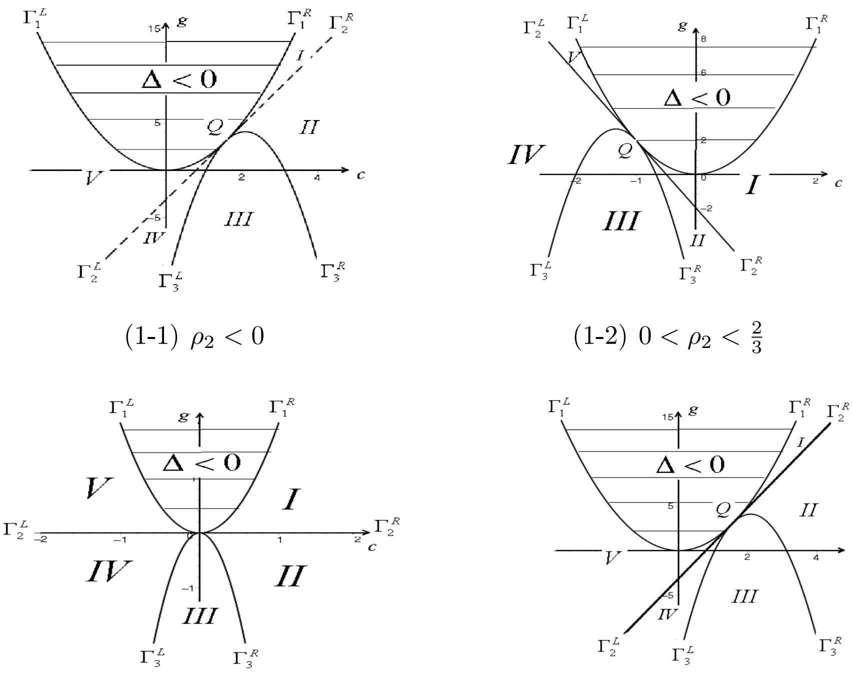
$$\Gamma_1 \bigcap \Gamma_2 \bigcap \Gamma_3 = Q\,, \tag{2.13}$$

where $Q\left(\frac{3\rho_2-2}{2\rho_2}, \frac{(3\rho_2-2)^2}{4\alpha\rho_2^2}\right)$ is intersection point of $\Gamma_1,\ \Gamma_2,\ \Gamma_3$.

Here, we express the part of $c > \frac{3\rho_2-2}{2\rho_2}$ on curve Γ with Γ^R; We express the part of $c < \frac{3\rho_2-2}{2\rho_2}$ on curve Γ with Γ^L; Similarly, we express the part of $g > 0$ at the regional $I - V$ with

$I^+ - V^+$; we express the part of $g < 0$ at the regional $I - V$ with $I^- - V^-$; we express the part of $g > 0$ on curve Γ with Γ^+; We express the part of $g < 0$ on curve Γ with Γ^-.

Thus, the bifurcation curves Γ_1^R, Γ_2^R, Γ_3^R, Γ_1^L, Γ_2^L, Γ_3^L which are defined by (2.10), (2.11), (2.12) divided the plane (c, g) into six regions, i.e. $(I) - (V)$ and the region of $\Delta < 0$, shown in Fig. 1.



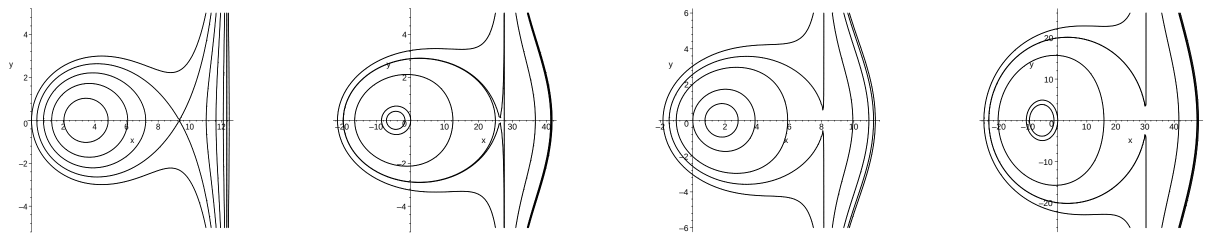
(1-1) $\rho_2 < 0$ (1-2) $0 < \rho_2 < \frac{2}{3}$

(1-3) $\rho_2 = \frac{2}{3}$ (1-4) $\rho_2 > \frac{2}{3}$

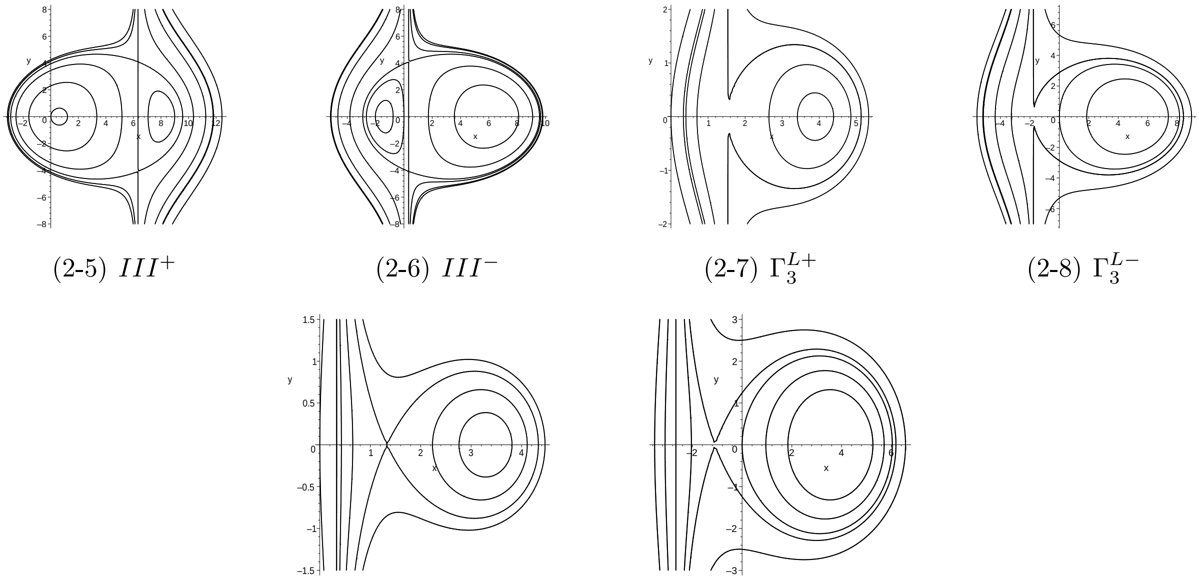
Fig. 1 The bifurcation curves and the six regions of (2.1)

For a fixed h, the level curve $H(\phi, y) = h$ defined by (2.2) determines a set of invariant curves of (2.1), which contains different branches of curves. As h vary, it defines different families of orbits of (2.1), with different dynamical behaviors.

Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-1), we obtain the following different phase portraits



(2-1) I, Γ_2^R, II^+ (2-2) II^- (2-3) Γ_3^{R+} (2-4) Γ_3^{R-}

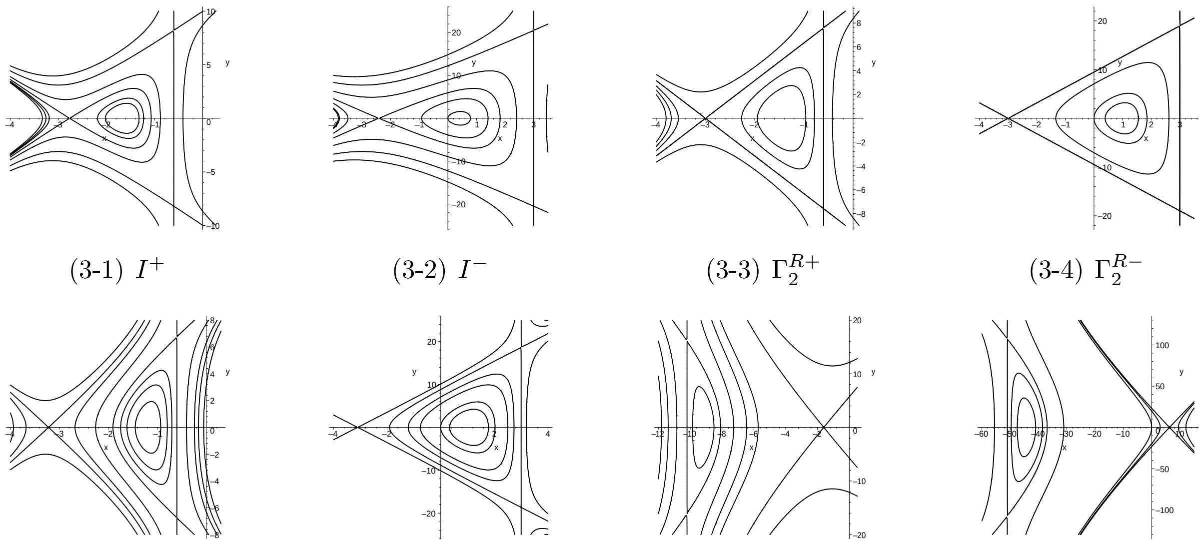


(2-5) III^+ (2-6) III^- (2-7) Γ_3^{L+} (2-8) Γ_3^{L-}

(2-9) IV^+, Γ_2^{L+}, V^+ (2-10) IV^-, Γ_2^{L-}, V^-

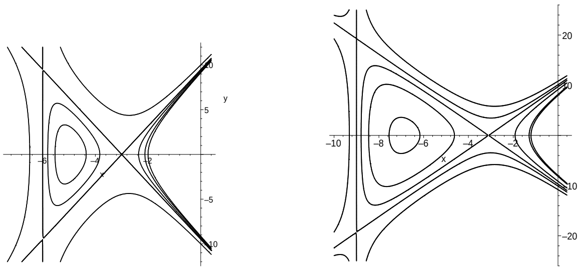
Fig. 2 The phase portraits of (2.1) for $\rho_2 < 0,\ g \neq 0$

Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-2), we obtain the following different phase portraits



(3-1) I^+ (3-2) I^- (3-3) Γ_2^{R+} (3-4) Γ_2^{R-}

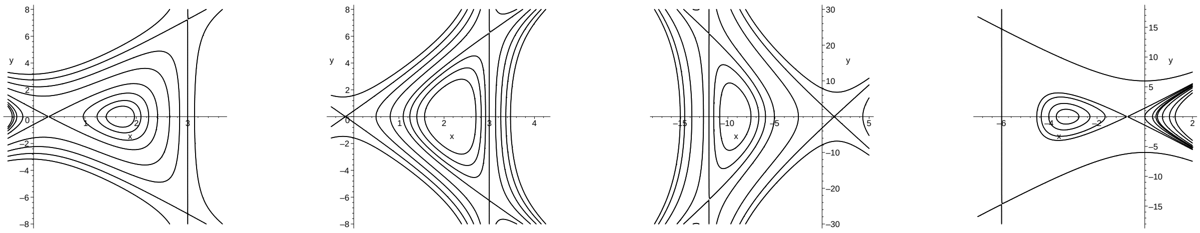
(3-5) II^+ (3-6) II^- (3-7) IV^+ (3-8) IV^-



(3-9) Γ_2^L (3-10) V

Fig. 3 The phase portraits of (2.1) for $0 < \rho_2 < \frac{2}{3}$, $g \neq 0$.

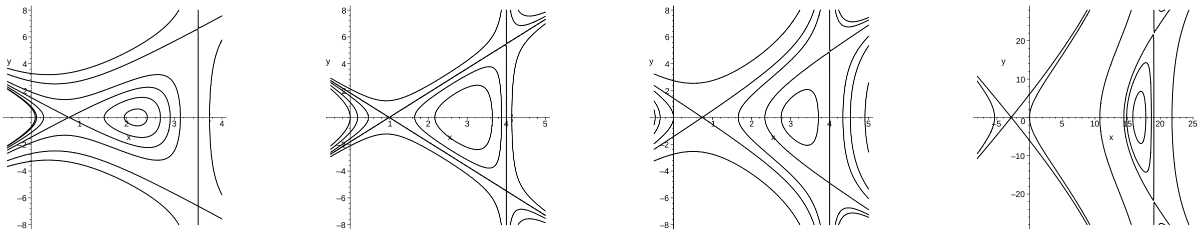
Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-3), we obtain the following different phase portraits



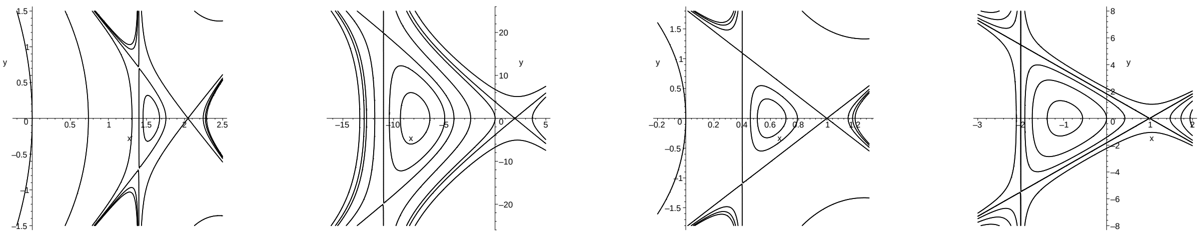
(4-1) I (4-2) II (4-3) IV (4-4) V

Fig. 4 The phase portraits of (2.1) for $\rho_2 = \frac{2}{3}$, $g \neq 0$.

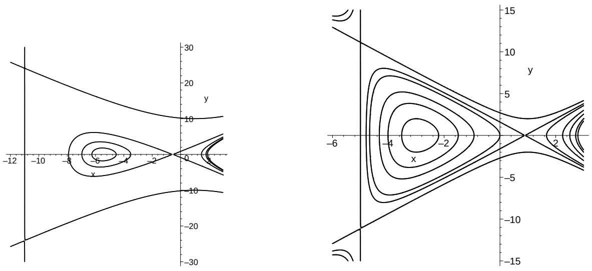
Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-4), we obtain the following different phase portraits



(5-1) I (5-2) Γ_2^R (5-3) II^+ (5-4) II^-



(5-5) IV^+ (5-6) IV^- (5-7) Γ_2^{L+} (5-8) Γ_2^{L-}



(5-9) V^+ (5-10) V^-

Fig. 5 The phase portraits of (2.1) for $\rho_2 > \frac{2}{3},\ g \neq 0$.

Note: When $\Delta < 0$ and $\rho_2 > 0, (c, g) \in \Gamma_1, \Gamma_3, III$, system (2.1) has not closed orbit. Here we omit their phase portraits.

3 Explicit expressions of periodic wave solutions and solitary cusp wave solutions of (1.3)

According to the analysis in the section 2, we derive the explicit expressions of periodic wave solutions and solitary cusp wave solutions of (1.3). See the computational process and results below.

3.1 Suppose that $\rho_2 < 0,\ (c, g) \in \Gamma_3^R$ i.e. $\rho_2 < 0,\ c > \frac{3\rho_2 - 2}{2\rho_2},\ g = g_3(c)$. In this case, we get $\psi_1 = \psi_s,\ h_1 = h_{s0} = \frac{2\eta^2(3c\rho_2 - 2\eta)}{3\alpha^2\rho_2^3}$. When $h = h_{s0}$, system (2.1) has a periodic orbit to the point $S_0(\psi_s, 0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-3), (2-4). Substituting $h = h_{s0}$ into (2.2) yields the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{(\psi_s - \psi)(\psi - \psi_0)}\,, \tag{3.1}$$

where $\psi_s = \frac{\eta}{\alpha\rho_2} = \frac{3\rho_2(c-1)+2}{\alpha\rho_2}$, $\psi_0 = \psi(0) = \frac{-4+6\rho_2-3\rho_2 c}{\alpha\rho_2}$ and $\psi_0 < \psi_s$.

Substituting (3.1) into the first equation of (2.1) yields the following equation

$$\pm\frac{d\psi}{\sqrt{(\psi_s - \psi)(\psi - \psi_0)}} = \sqrt{\frac{2}{3\beta(-\rho_2)}}d\xi\,. \tag{3.2}$$

Integrating (3.2) along this periodic orbit yields

$$\int_{\psi_s}^{\psi}\frac{d\psi}{\sqrt{(\psi_s - \psi)(\psi - \psi_0)}} = \sqrt{\frac{2}{3\beta(-\rho_2)}}\int_0^{\xi} d\xi, \qquad \xi > 0 \tag{3.3}$$

and

$$-\int_{\psi}^{\psi_s} \frac{d\psi}{\sqrt{(\psi_s-\psi)(\psi-\psi_0)}} = \sqrt{\frac{2}{3\beta(-\rho_2)}} \int_{\xi}^{0} d\xi, \qquad \xi \le 0 \tag{3.4}$$

By (3.3) and (3.4), we obtain a smooth periodic wave solution of (1.3):

$$v(x-ct) = \psi(x-ct) = \frac{1}{2}\left[(\psi_s+\psi_0) + (\psi_s-\psi_0)\cos\omega(x-ct)\right], \tag{3.5}$$

where $\omega = \sqrt{\frac{2}{3\beta(-\rho_2)}}$.

3.2 Suppose that $\rho_2 < 0$, $(c,g) \in \Gamma_3^L$ i.e. $\rho_2 < 0$, $c < \frac{3\rho_2-2}{2\rho_2}$, $g = g_3(c)$. In this case, we get $\psi_2 = \psi_s$, $h_2 = h_{s0} = \frac{2\eta^2(3c\rho_2-2\eta)}{3\alpha^2\rho_2^3}$. When $h = h_{s0}$, system (2.1) has a periodic orbit to the point $S_0(\psi_s, 0)$ and around the center point $A_2(\psi_1, 0)$, see Fig. 2 (2-7), (2-8). Substituting $h = h_{s0}$ into (2.2) yields the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{(\psi_0-\psi)(\psi-\psi_s)}\,, \tag{3.6}$$

where ψ_s, ψ_0 are given above and $\psi_0 > \psi_s$.

Similarly, substituting (3.6) into the first equation of (2.1) to integrate along this orbit, we obtain a smooth periodic wave solution of (1.3):

$$v(x-ct) = \psi(x-ct) = \frac{1}{2}\left[(\psi_s+\psi_0) - (\psi_s-\psi_0)\cos\omega(x-ct)\right], \tag{3.7}$$

where $\omega = \sqrt{\frac{2}{3\beta(-\rho_2)}}$.

3.3 Suppose that (1) $0 < \rho_2 < \frac{2}{3}$, $(c,g) \in \Gamma_2^R$ i.e. $0 < \rho_2 < \frac{2}{3}$, $c > \frac{3\rho_2-2}{2\rho_2}$, $g = g_2(c)$; (2) $\rho_2 > \frac{2}{3}$, $(c,g) \in \Gamma_2^R$ i.e. $\rho_2 > \frac{2}{3}$, $c > \frac{3\rho_2-2}{2\rho_2}$, $g = g_2(c)$. In these two cases, we get $\psi_2 = \frac{3\rho_2-2}{2\alpha\rho_2} < \psi_s$ and $h_2 = h_s = \frac{2\eta(\eta^2-3c\rho_2\eta+3g\alpha\rho_2^2)}{3\alpha^2\rho_2^3}$. When $h = h_2 = h_s$, system (2.1) has two heterclinic orbits connect three saddle points $A_2(\psi_2, 0)$ and $S_\pm(\psi_s, Y_\pm)$, see Fig. 3 (3-3), (3-4) and Fig. 5 (5-2). Substituting $h = h_s$ into (2.2) yields the following algebraic equations for these two heterclinic orbits

$$y = \pm\frac{2\alpha\rho_2\psi - 3\rho_2 + 2}{\alpha\rho_2\sqrt{6\beta\rho_2}} = \pm\frac{2}{\sqrt{6\beta\rho_2}}(\psi-\psi_2)\,. \tag{3.8}$$

Similarly, substituting (3.8) into the first equation of (2.1) to integrate along these two orbits, we obtain a non-smooth solitary cusp wave solution of peak type of (1.3):

$$v(x-ct) = \psi(x-ct) = \psi_2 + (\psi_s-\psi_2)\exp\left(-\frac{2|x-ct|}{\sqrt{6\beta\rho_2}}\right). \tag{3.9}$$

3.4 Suppose that (1) $0 < \rho_2 < \frac{2}{3}$, $(c,g) \in \Gamma_2^L$ i.e. $0 < \rho_2 < \frac{2}{3}$, $c < \frac{3\rho_2-2}{2\rho_2}$, $g = g_2(c)$; (2) $\rho_2 > \frac{2}{3}$, $(c,g) \in \Gamma_2^L$ i.e. $\rho_2 > \frac{2}{3}$, $c < \frac{3\rho_2-2}{2\rho_2}$, $g = g_2(c)$. In these two cases, we have $\psi_1 = \frac{1}{\alpha}(\frac{3}{2} - \frac{2}{\rho_2}) > \psi_s$ and $h_1 = h_s = \frac{2\eta(\eta^2-3c\rho_2\eta+3g\alpha\rho_2^2)}{3\alpha^2\rho_2^3}$. When $h = h_1 = h_s$, system (2.1) has two heterclinic orbits connect three saddle points $A_1(\psi_2, 0)$ and $S_\pm(\psi_s, Y_\pm)$, see Fig. 3 (3-9) and Fig. 5 (5-7), (5-8). Substituting $h = h_s$ into (2.2) yields the following algebraic equations for these two heterclinic orbits

$$y = \pm\frac{-2\alpha\rho_2\psi + 3\rho_2 - 2}{\alpha\rho_2\sqrt{6\beta\rho_2}} = \pm\frac{2}{\sqrt{6\beta\rho_2}}(\psi_1 - \psi)\,. \tag{3.10}$$

Similarly, substituting (3.10) into the first equation of (2.1) to integrate along these two orbits, we obtain a non-smooth solitary cusp wave solution of valley type of (1.3):

$$v(x-ct) = \psi(x-ct) = \psi_1 - (\psi_1 - \psi_s)\exp\left(-\frac{2|x-ct|}{\sqrt{6\beta\rho_2}}\right)\,. \tag{3.11}$$

4 Implicit expressions of periodic wave solutions which is defined by $H(\psi, y) = 0$

By the phase portraits of (2-2)-(2-6), (2-8), (2-10), (3-2), (3-4), (3-6), (5-8) and (5-10) in Fig. 2-Fig. 5, it is easy to know that there is a periodic annuli through the point $O(0,0)$. This periodic annuli is defined by $H(\psi, y) = 0$. By using the elliptic function integral method, see [9, 10] and their references, we derive the implicit expressions of periodic wave solutions of (1.3). See the below computational process and results. Here, we only consider the case of $\rho_2 < 0$, see Fig. 2. The other cases are similar to $\rho_2 < 0$, see Fig. 3-Fig. 5.

4.1.1 Suppose that $\rho_2 < 0$, $(c,g) \in \Gamma_3^{R+}$ i.e. $\rho_2 < 0$, $c > \frac{3\rho_2-2}{2\rho_2}$, $g = g_3(c) > 0$. In this case, there is $\psi_1 = \psi_s = \frac{3\rho_2(c-1)+2}{\alpha\rho_2}$. And, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-3). From $H(\psi, y) = 0$, we obtain the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_M - \psi)(\psi_m - \psi)(\psi - 0)}{\psi_s - \psi}}\,, \tag{4.1}$$

where $\psi_m = \frac{3\rho_2 c + \sqrt{3(5\rho_2 c - 6\rho_2 + 4)(3\rho_2 c - 6\rho_2 + 4)}}{2\alpha\rho_2}$, $\psi_M = \frac{3\rho_2 c - \sqrt{3(5\rho_2 c - 6\rho_2 + 4)(3\rho_2 c - 6\rho_2 + 4)}}{2\alpha\rho_2}$ and $0 < \psi < \psi_m < \psi_s < \psi_M$.

Substituting (4.1) into the first equation of (2.1) yields

$$\pm\sqrt{\frac{\psi_s - \psi}{(\psi_M - \psi)(\psi_m - \psi)(\psi - 0)}}d\psi = \sqrt{\frac{2}{3\beta(-\rho_2)}}d\xi\,. \tag{4.2}$$

Integrating (4.2) along this periodic orbit, we get

$$\int_{\psi}^{\psi_m} \sqrt{\frac{\psi_s - \psi}{(\psi_M - \psi)(\psi_m - \psi)(\psi - 0)}} d\psi = \sqrt{\frac{2}{3\beta(-\rho_2)}} \int_{\xi}^{0} d\xi, \quad \xi > 0 \tag{4.3}$$

and

$$-\int_{\psi_m}^{\psi} \sqrt{\frac{\psi_s - \psi}{(\psi_M - \psi)(\psi_m - \psi)(\psi - 0)}} d\psi = \sqrt{\frac{2}{3\beta(-\rho_2)}} \int_{0}^{\xi} d\xi, \quad \xi \leq 0 \tag{4.4}$$

By (4.2) and (4.3), we obtain

$$\int_{\psi}^{\psi_m} \sqrt{\frac{\psi_s - \psi}{(\psi_M - \psi)(\psi_m - \psi)(\psi - 0)}} d\psi = \sqrt{\frac{2}{3\beta(-\rho_2)}} |\xi| \,. \tag{4.5}$$

By using the elliptic integral formulas citelon12, we obtain

$$\int_{\psi}^{\psi_m} \sqrt{\frac{\psi_s - \psi}{(\psi_M - \psi)(\psi_m - \psi)(\psi - 0)}} d\psi = (\psi_s - \psi_m) e_0 \int_{0}^{u_0} \frac{du}{1 - \alpha_0^2 sn^2 u} \,, \tag{4.6}$$

where $e_0 = \frac{2}{\sqrt{\psi_s(\psi_M - \psi_m)}}$, $u_0 = sn^{-1}\left(\sqrt{\frac{\psi_s(\psi_m - \psi)}{\psi_m(\psi_s - \psi)}}, k_0\right)$, $k_0 = \sqrt{\frac{\psi_m(\psi_M - \psi_s)}{\psi_s(\psi_M - \psi_m)}}$, $k_0^2 < \alpha_0^2 = \frac{\psi_m}{\psi_s} < 1$.
And

$$\int_{0}^{u_0} \frac{du}{1 - \alpha_0^2 sn^2 u} = \Pi(u_0, \alpha_0^2) \,. \tag{4.7}$$

By (4.5), (4.6) and (4.7), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1} \sqrt{\frac{\psi_s(\psi_m - \psi)}{\psi_m(\psi_s - \psi)}}, k_0), \alpha_0^2 \right) = \frac{1}{(\psi_s - \psi_m)} \sqrt{\frac{\psi_s(\psi_M - \psi_m)}{6\beta(-\rho_2)}} |\xi| \,, \tag{4.8}$$

where $sn^{-1}(* \,,\, *)$ is the inverse function of $sn(* \,,\, *)$ which is the Jacobian elliptic function, $\Pi(* \,,\, *)$ is Legendre's incomplete elliptic integral of the third kind.

4.1.2 Suppose that $\rho_2 < 0$, $(c, g) \in \Gamma_3^{R-}$ i.e. $\rho_2 < 0$, $c > \frac{3\rho_2 - 2}{2\rho_2}$, $g = g_3(c) < 0$. In this case, there is $\psi_1 = \psi_s = \frac{3\rho_2(c-1)+2}{\alpha\rho_2}$. And, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-4). From $H(\psi, y) = 0$, we get the following algebraic equations for this periodic orbit

$$y = \pm \sqrt{\frac{2}{3\beta(-\rho_2)}} \sqrt{\frac{(\psi_M - \psi)(0 - \psi)(\psi - \psi_m)}{\psi_s - \psi}} \,, \tag{4.9}$$

where $\psi_m,\ \psi_M$ are given above and $\psi_m < \psi < 0 < \psi_s < \psi_M$.

Corresponding to (4.9), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1}\sqrt{\frac{(\psi_s-\psi_m)\psi}{\psi_m(\psi_s-\psi)}},k_1),\alpha_1^2\right)=\frac{1}{\psi_s}\sqrt{\frac{\psi_M(\psi_s-\psi_m)}{6\beta(-\rho_2)}}|\xi|\,, \tag{4.10}$$

where the computational process is similar to (4.2)-(4.8) and $k_1^2 = \frac{-\psi_m(\psi_M-\psi_s)}{\psi_M(\psi_s-\psi_m)}$, $\alpha_1^2 = \frac{-\psi_m}{\psi_s-\psi_m} < 1$.

4.1.3 Suppose that $\rho_2 < 0,\ (c,g)\in\Gamma_3^{L-}$ i.e. $\rho_2<0,\ c<\frac{3\rho_2-2}{2\rho_2},\ g=g_3(c)<0$. In this case, there is $\psi_2=\psi_s=\frac{3\rho_2(c-1)+2}{\alpha\rho_2}$. And, when $h=0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_1(\psi_1,0)$, see Fig. 2 (2-8). From $H(\psi,y)=0$, we get the following algebraic equations for this periodic orbit

$$y=\pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_M-\psi)(\psi-0)(\psi-\psi_m)}{\psi-\psi_s}}\,, \tag{4.11}$$

where $\psi_m,\ \psi_M$ are given above and $\psi_m < \psi_s < 0 < \psi < \psi_M$.

Substituting (4.11) into the first equation of (2.1) yields

$$\pm\sqrt{\frac{\psi_s-\psi}{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}d\xi\,. \tag{4.12}$$

Integrating (4.12) along this periodic orbit, we get

$$\int_{\psi}^{\psi_M}\sqrt{\frac{\psi-\psi_s}{(\psi_M-\psi)(\psi-0)(\psi-\psi_m)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}|\xi|\,, \tag{4.13}$$

By using the elliptic integral formulas [11], we obtain

$$\int_{\psi}^{\psi_M}\sqrt{\frac{\psi-\psi_s}{(\psi_M-\psi)(\psi-0)(\psi-\psi_m)}}d\psi=(\psi_M-\psi_s)e_2\int_0^{u_2}\frac{dn^2udu}{1-\alpha_2^2sn^2u}\,, \tag{4.14}$$

where $e_2=\frac{2}{\sqrt{-\psi_m(\psi_M-\psi_s)}}$, $u_2=sn^{-1}\left(\sqrt{\frac{-\psi_m(\psi_M-\psi)}{\psi_M(\psi-\psi_m)}},k_2\right)$, $k_2^2=\frac{\psi_M(\psi_s-\psi_m)}{-\psi_m(\psi_M-\psi_s)}$, $\alpha_2^2=\frac{\psi_M}{\psi_m}<0$.
And

$$\int_0^{u_2}\frac{dn^2udu}{1-\alpha_2^2sn^2u}=\frac{1}{\alpha_2^2}[k_2^2u_2+(\alpha_2^2-k_2^2)\Pi(u_2,\alpha_2^2)]\,. \tag{4.15}$$

By (4.13), (4.14) and (4.15), we obtain a smooth periodic wave solution of (1.3):

$$k_2^2sn^{-1}(\sqrt{\frac{-\psi_m(\psi_M-\psi)}{\psi_M(\psi-\psi_m)}},k_2)+(\alpha_2^2-k_2^2)\Pi\left((sn^{-1}\sqrt{\frac{-\psi_m(\psi_M-\psi)}{\psi_M(\psi-\psi_m)}},k_2),\alpha_2^2\right)=\Omega_1|\xi|\,, \tag{4.16}$$

where $\Omega_1 = \alpha_2^2\sqrt{\frac{-\psi_m}{6\beta(-\rho_2)(\psi_M-\psi_s)}}$.

4.2.1 Suppose that $\rho_2 < 0$, $(c,g) \in III^+$ i.e. $\rho_2 < 0$, $0 < g < g_3(c)$. In this case, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-5). From $H(\psi, y) = 0$, we get the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_G-\psi)(\psi_l-\psi)(\psi-0)}{\psi_s-\psi}}\,, \tag{4.17}$$

where $\psi_l = \frac{3c-\sqrt{9c^3-12\alpha g}}{2\alpha}$, $\psi_G = \frac{3c+\sqrt{9c^3-12\alpha g}}{2\alpha}$ and $0 < \psi < \psi_l < \psi_s < \psi_G$.

Corresponding to (4.17), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1}\sqrt{\frac{\psi_s(\psi_l-\psi)}{\psi_l(\psi_s-\psi)}}, k_3), \alpha_3^2\right) = \frac{1}{(\psi_s-\psi_l)}\sqrt{\frac{\psi_s(\psi_G-\psi_l)}{6\beta(-\rho_2)}}|\xi|\,, \tag{4.18}$$

where $k_3^2 = \sqrt{\frac{\psi_l(\psi_G-\psi_s)}{\psi_s(\psi_G-\psi_l)}}$, $k_3^2 = \sqrt{\frac{\psi_l(\psi_G-\psi_s)}{\psi_s(\psi_G-\psi_l)}}$, $k_3^2 < \alpha_3^2 = \frac{\psi_l}{\psi_s} < 1$.

4.2.2 Suppose that (1) $\rho_2 < 0$, $(c,g) \in III^-$ i.e. $\rho_2 < 0$, $g < g_3(c) < 0$; (2) $\rho_2 < 0$, $(c,g) \in II^-$ i.e. $\rho_2 < 0$, $g_3(c) < g < 0$. In these two cases, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-6), (2-2). From $H(\psi, y) = 0$, we get the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_G-\psi)(0-\psi)(\psi-\psi_l)}{\psi_s-\psi}}\,, \tag{4.19}$$

where ψ_l, ψ_G are given above and $\psi_l < \psi < 0 < \psi_s < \psi_G$.

Corresponding to (4.21), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1}\sqrt{\frac{(\psi_s-\psi_l)\psi}{\psi_l(\psi_s-\psi)}}, k_4), \alpha_4^2\right) = \frac{1}{\psi_s}\sqrt{\frac{\psi_G(\psi_s-\psi_l)}{6\beta(-\rho_2)}}|\xi|\,, \tag{4.20}$$

where $k_4^2 = \frac{-\psi_l(\psi_G-\psi_s)}{\psi_G(\psi_s-\psi_l)}$, $\alpha_4^2 = \frac{-\psi_l}{\psi_s-\psi_l} < 1$.

4.2.3 Suppose that $\rho_2 < 0$, $(c,g) \in IV^-, \Gamma_2^{L-}, V^-$ i.e. $\rho_2 < 0$, $c < \frac{3\rho_2-2}{2\rho_2}$, $g_3(c) < g < 0$. In this case, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_1(\psi_1, 0)$, see Fig. 2 (2-10). From $H(\psi, y) = 0$, we get the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_G-\psi)(\psi-0)(\psi-\psi_l)}{\psi-\psi_s}}\,, \tag{4.21}$$

where $\psi_l,\ \psi_G$ are given above and $\psi_l < \psi_s < 0 < \psi < \psi_G$.

Corresponding to (4.21), we obtain a smooth periodic wave solution of (1.3):

$$k_5^2 sn^{-1}(\sqrt{\frac{-\psi_l(\psi_G-\psi)}{\psi_G(\psi-\psi_l)}}, k_5) + (\alpha_5^2 - k^2)\Pi\left((sn^{-1}\sqrt{\frac{-\psi_l(\psi_G-\psi)}{\psi_G(\psi-\psi_l)}}, k_5), \alpha_5^2\right) = \Omega_2|\xi|\,, \tag{4.22}$$

where $\Omega_2 = \alpha_5^2\sqrt{\frac{-\psi_l}{6\beta(-\rho_2)(\psi_G-\psi_s)}}$, $k_5^2 = \frac{\psi_G(\psi_s-\psi_l)}{-\psi_l(\psi_G-\psi_s)}$, $\alpha_5^2 = \frac{\psi_G}{\psi_m} < 0$.

References

[1] **Long, Y., Rui, W. G.**, and **He, B. :** *Travelling wave solutions for a higher order wave equation of KdV type (I).* Chaos, Solitons and Fractals **23**, 469-475 (2005)

[2] **Tzirtzilakis, E., Marinakis, V., Apokis, C.**, and **Bountis, T. :** *Soliton-like solutions of higher order wave equations of the Korteweg-de-Vries Type.* J. Math. Phys. **43**, 12: 6151-61 (2002)

[3] **Fokas, A. S. :** *On a class of physically important integral equations.* Physica D **87**, 145-150 (1995)

[4] **Tzirtzilakis, E., Xenos, M., Marinakis, V.**, and **Bountis, T. :** *Interactions and stability of solitary waves in shallow water.* Chaos, Solitons and Fractals, **14**, 87-95 (2002)

[5] **Chow, S. N.**, and **Hale, J. K. :** *Method of Bifurcation Theory.* Springer-Verlag, New York, 1981

[6] **Guckenheimer, J.**, and **Holmes, P. J. :** *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* New York: Springer-Verlag 1983

[7] **Li, J. B.**, and **Liu, Z. R. :** *Smooth and non-smooth travelling waves in a nonlinearly dispersive equation.* Appl. Math. Modelling **25**, 41-56 (2000)

[8] **Li, J. B.**, and **Liu, Z. R. :** *Travelling wave solutions for a class of nonlinear dispersive equations.* Chin. Ann. of Math. **23B**, 397-418 (2002)

[9] **Liu, Z. R., Li, Q. X.**, and **Lin, Q. M. :** *New bounded traveling waves os Camassa-Holm equation.* Journal of Bifurcation and Chaos, Vol.14, No. 10, 3541-56 (2004)

[10] **Liu, Z. R.**, and **Chen, C. :** *Compactons in a general compressible hyperelastic rod.* Chaos, Solitons and Fractals **22**, 627-640, 2004

[11] **Liu, Z. R., Lin, Q. M.**, and **Li, Q. X. :** *Integral approach to compacton solutons of Boussinesq-like* $B(m, n)$ *equations with fully nonlinear dispersion.* Chaos, Solitions and Fractals **19**, 1071-1081, 2004b

[12] **Byrd, P. F.**, and **Friedman, M. D. :** *Handbook of Elliptic Integrals for Engineers and Scientists [M].* Springer, NY, 1971

received: Septembre 27, 2005

Authors:

Weiguo Rui
Department of Mathematics
of Honghe University
Mengzi
Yunnan, 661100
P.R. China

e-mail: weiguorhhu@yahoo.com.cn

Yao Long
Department of Mathematics
of Honghe University
Mengzi
Yunnan, 661100
P.R. China

Bin He
Department of Mathematics
of Honghe University
Mengzi
Yunnan, 661100
P.R. China

Rostock. Math. Kolloq. **61**, 73–83 (2006)

Subject Classification (AMS)
39A10

Yixiang Hu, Xianyi Li[1]

Dynamics of a Nonlinear Difference Equation[2]

ABSTRACT. In this paper the dynamics for a third-order rational difference equation is considered. The rule for the trajectory structure of solutions of this equation is clearly described out. The successive lengths of positive and negative semicycles of nontrivial solutions of this equation are found to occur periodically with prime period 7. And the rule is $3^+, 2^-, 1^+, 1^-$ in a period. By utilizing the rule, the positive equilibrium point of the equation is verified to be globally asymptotically stable.

KEY WORDS. rational difference equation, semicycle, cycle length, global asymptotic stability.

1 Introduction and Preliminaries

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. For this, see, for example, [1, 2] and the papers in the journal of " Advances in Difference Equations änd the references cited therein. Furthermore, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently, M. R. S. Kulenović et al [3], Tim Nesemann [4] and Yang et.al [6, 7] investigated the global asymptotic stability of some second or higher order rational difference equations.

---

[1]Corresponding author; Present address (Until September 2006): Département de Mathematique, Université de Lille 1, Cité Scientifique, 59655,Villeneuve d'Ascq, France, xli@math.univ-lille1.fr.

[2]This work is supported by NNSF of China (grant: 10371040), Mathematical Tianyuan Foundation of China (grant: TY10026002-01-05-03), Excellent Youth Project of Educational Committee of Hunan Province (grant: 04B056), the Foundation for " New Century '121' Talents in Hunan Province" and the Foundation for " Chief Professor of Mathematical Discipline in Hunan Province".

From the known work, one can see that it is extremely difficult to understand thoroughly the trajectory structure of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [1]-[10], especially [1, 2] for examples to illustrate this.

In this paper we consider the following third - order rational difference equation

$$x_{n+1} = \frac{x_{n-1} + x_{n-2} + a}{x_{n-1}x_{n-2} + 1 + a}, \quad n = 0, 1, 2, \cdots, \tag{1.1}$$

where $a \in [0, \infty)$ and the initial values $x_{-2}, x_{-1}, x_0 \in (0, \ \infty)$,

Mainly, by analyzing the rule for the length of semi-cycle to occur successively, we describe clearly out the rule for the trajectory structure of its solutions and further derive the global asymptotic stability of positive equilibrium of equation (1.1). Whereas, it is extremely difficult to use those methods in the known literature, such as [1]-[7], to obtain the rule of trajectory structure of solutions of equation (1.1).

It is easy to see that the positive equilibrium $\bar{x}$ of equation (1.1) satisfies

$$\bar{x} = \frac{2\bar{x} + a}{\bar{x}^2 + 1 + a},$$

from which one can see that equation (1.1) has a unique positive equilibrium $\bar{x} = 1$.

Here, for convenience of readers, we give some corresponding definitions, also review some results which will be useful in our investigation of the behavior of solutions of Eq. (1.1). Let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function. Then for every group of initial conditions $x_{-2}, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_{n-1}, \ x_{n-2}), \quad n = 0, 1, 2, \cdots, \tag{E}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

A point $\bar{x}$ is called an equilibrium point of Eq. (E) if $\bar{x} = f(\bar{x}, \ \bar{x})$. That is, $x_n = \bar{x}$, for $n \geq 0$, is a solution of Eq. (E), or, equivalently, $\bar{x}$ is a fixed point of f.

Definition 1.1 *Let $\bar{x}$ be an equilibrium point of Eq. (E).*

(a) *The equilibrium $\bar{x}$ is called stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$ for all ≥ 1.*

(b) *The equilibrium $\bar{x}$ is called locally asymptotically stable if it is stable and if there exists $\gamma > 0$ such that if $x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.*

(c) *The equilibrium $\bar{x}$ is called a global attractor if*

$$\lim_{n\to\infty} x_n = \bar{x} \quad \textit{for any} \quad x_{-2}, x_{-1},\ x_0 \in I.$$

(d) *The equilibrium $\bar{x}$ is called globally asymptotically stable if it is stable and is a global attractor.*

(e) *The equilibrium $\bar{x}$ is called unstable if it is not stable.*

(f) *The equilibrium $\bar{x}$ is called a repeller if there exists $\gamma > 0$ such that for $x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, there exists $N \geq -2$ such that $|x_N - \bar{x}| \geq \gamma$.*

Clearly, a repeller is an unstable equilibrium.

Let

$$p = \frac{\partial f(\bar{x},\ \bar{x})}{\partial u} \qquad \text{and} \qquad q = \frac{\partial f(\bar{x},\ \bar{x})}{\partial v},$$

where $f(u,\ v)$ is the function in Eq. (E) and $\bar{x}$ is an equilibrium of the equation. Then the equation

$$y_{n+1} = p y_{n-1} + q y_{n-2}, \qquad n = 0, 1, \cdots$$

is called the linearized equation associated with Eq. (E) about the equilibrium point $\bar{x}$.

Definition 1.2 *A positive semicycle of a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$, all greater than or equal to the equilibrium $\bar{x}$, with $l \geq -2$ and $m \leq \infty$ such that*

$$\textit{either} \quad l = -2 \quad \textit{or} \quad l > -2 \quad \textit{and} \quad x_{l-1} < \bar{x}$$

and

$$\textit{either} \quad m = \infty \quad \textit{or} \quad m < \infty \quad \textit{and} \quad x_{m+1} < \bar{x}.$$

A negative semicycle of a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$, all less than $\bar{x}$, with $l \geq -2$ and $m \leq \infty$ such that

$$\textit{either} \quad l = -2 \quad \textit{or} \quad l > -2 \quad \textit{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\textit{either} \quad m = \infty \quad \textit{or} \quad m < \infty \quad \textit{and} \quad x_{m+1} \geq \bar{x}.$$

The length of a semicycle is the number of the total terms contained in it.

Definition 1.3 *A solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) is said to be eventually trivial if x_n is eventually equal to $\bar{x} = 1$; Otherwise, the solution is said to be nontrivial.*

A solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) is said to be eventually positive(negative) if x_n is eventually great (less) than $\bar{x} = 1$;

For the other concepts in this paper, see [1, 2].

2 Main Results and Their Proofs

In this section we will formulate our main results in this paper, that is, with respect to the nontrivial solutions, oscillation and non-oscillation and global asymptotic stability for equation (1.1).

2.1 Nontrivial solution

Theorem 2.1 *A positive solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) is eventually trivial if and only if*

$$(x_{-2}-1)(x_{-1}-1)(x_0-1)=0\,. \tag{2.1}$$

Proof: Sufficiency. Assume that (2.1) holds. Then it follows from equation (1.1) that the following conclusions hold.

i) If $x_{-2}=1$, then $x_n=1$ for $n\geq 3$;

ii) If $x_{-1}=1$, then $x_n=1$ for $n\geq 1$;

iii) If $x_0=1$, then $x_n=1$ for $n\geq 2$.

Necessity. Conversely, assume that

$$(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0\,. \tag{2.2}$$

Then one can show that

$$x_n\neq 1 \quad \text{for any} \quad n\geq 1.$$

Assume the contrary that for some $N\geq 1$,

$$x_N=1 \quad \text{and that} \quad x_n\neq 1 \quad \text{for} \quad -2\leq n\leq N-1\,. \tag{2.3}$$

Clearly,

$$1=x_N=\frac{x_{N-2}+x_{N-3}+a}{x_{N-2}x_{N-3}+1+a}\,,$$

which implies $(x_{N-2}-1)(x_{N-3}-1)=0$, which contradicts (2.3).

Remark 2.2 Theorem 2.1 actually demonstrates that a positive solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) is eventually nontrivial if and only if $(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0$. Therefore, if a solution $\{x_n\}_{n=-2}^{\infty}$ is nontrivial, then $x_n\neq 1$ for $n\geq -2$.

Next we consider some properties of nontrivial solutions of equation (1.1).

2.2 Oscillation and Non-oscillation

Before stating the oscillation and non-oscillation of solutions, we need the following key lemma.

Lemma 2.3 *For any nontrivial positive solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1), the following conclusions are true:*

(a) $(x_{n+1}-1)(x_{n-1}-1)(x_{n-2}-1)<0$ *for* $n\geq 0$*;*

(b) $(x_{n+1}-x_{n-1})(x_{n-1}-1)<0$ *for* $n\geq 0$*;*

(c) $(x_{n+1}-x_{n-2})(x_{n-2}-1)<0$ *for* $n\geq 0$.

Proof: In view of equation (1.1), we can see that

$$x_{n+1}-1=-\frac{(x_{n-1}-1)(x_{n-2}-1)}{x_{n-1}x_{n-2}+1+a},\quad n=0,1,2,\cdots$$

and

$$x_{n+1}-x_{n-1}=\frac{(1-x_{n-1})[a+x_{n-2}(1+x_{n-1})]}{x_{n-1}x_{n-2}+1+a},\quad n=0,1,2,\cdots,$$

from which inequalities (a) and (b) follow. The proof for inequality (c) is similar to that of inequality (b). So the proof is complete.

Theorem 2.4 *There exist non-oscillatory solutions of equation (1.1), which must be eventually negative. There don't exist eventually positive non-oscillatory solutions of equation (1.1).*

Proof: Consider a solution of equation (1.1) with $x_{-2}<1$, $x_{-1}<1$ and $x_0<1$. We then know from Lemma 2.3 (a) that $x_n<1$ for $n\geq -2$. So, this solution is just a non-oscillatory solution and furthermore eventually negative.

Suppose that there exist eventually positive non-oscillatory solutions of equation (1.1). Then, there exists a positive integer N such that $x_n>1$ for $n\geq N$. Thereout, for $n\geq N+2$, $(x_{n+1}-1)(x_{n-1}-1)(x_{n-2}-1)>0$. This contradicts Lemma 2.3 (a). So, There don't exist eventually positive non-oscillatory solutions of equation (1.1), as desired.

We now analyze the rule for trajectory structure of strictly oscillatory solutions of equation (1.1).

Theorem 2.5 *Let $\{x_n\}_{n=-2}^{\infty}$ be a strictly oscillatory solution of equation (1.1). Then the rule for the lengths of positive and negative semi-cycles of this solution to successively occur is* $\cdots,3^+,2^-,\ 1^+,1^-,3^+,\ 2^-,1^+,1^-,3^+,\ 2^-,1^+,1^-,3^+,\ 2^-,1^+,1^-,\ 3^+,2^-,1^+,1^-,3^+,2^-,1^+,1^-,\cdots$.

Proof: By Lemma 2.3 (a), one can see that the length of a negative semi-cycle is not larger than 2, whereas, the length of a positive semi-cycle is at most 3. Based on the strictly oscillatory character of the solution, we see, for some integer $p \geq 0$, one of the following two cases must occur:
Case 1: $x_{p-2} > 1, x_{p-1} < 1, x_p > 1$;
Case 2: $x_{p-2} > 1, x_{p-1} < 1, x_p < 1$.

If Case 1 occurs, it follows from Lemma 2.3 (a) that $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} < 1$, $x_{p+4} < 1, x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} > 1$, $x_{p+8} > 1, x_{p+9} > 1$, $x_{p+10} < 1$, $x_{p+11} < 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} > 1$, $x_{p+15} > 1$, $x_{p+16} > 1$, $x_{p+17} < 1$, $x_{p+18} < 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} > 1$, $x_{p+22} > 1$, $x_{p+23} > 1$, $x_{p+24} < 1$, $x_{p+25} < 1$, $x_{p+26} > 1$, $x_{p+27} < 1$, $x_{p+28} > 1$, $x_{p+29} > 1$, $x_{p+30} > 1$, $x_{p+31} < 1$, $x_{p+32} < 1$, $x_{p+33} > 1$, $x_{p+34} < 1$, $x_{p+35} > 1$, $x_{p+36} > 1$, $x_{p+37} > 1$, $x_{p+38} < 1$, $x_{p+39} < 1$, $x_{p+40} > 1$, $x_{p+41} > 1$, $\cdots$, which means that the rule for the lengths of positive and negative semi-cycles of the solution of equation (1.1) to successively occur is $\cdots, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, \cdots$.

If Case 2 happens, then Lemma 2.3 (a) tells us that $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} > 1$, $x_{p+4} > 1$, $x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} < 1$, $x_{p+8} > 1, x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} > 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} < 1$, $x_{p+22} > 1$, $x_{p+23} < 1$, $x_{p+24} > 1$, $x_{p+25} > 1$, $x_{p+26} > 1$, $x_{p+27} < 1$, $x_{p+28} < 1$, $x_{p+29} > 1$, $x_{p+30} < 1$, $x_{p+31} > 1$, $x_{p+32} > 1$, $x_{p+33} > 1$, $x_{p+34} < 1$, $x_{p+35} < 1$, $x_{p+36} > 1$, $x_{p+37} < 1$, $x_{p+38} > 1$, $x_{p+39} > 1$, $x_{p+40} > 1$, $x_{p+41} < 1$, $x_{p+42} < 1$, $x_{p+43} < 1$, $x_{p+44} > 1$, $\cdots$. This shows the rule for the numbers of terms of positive and negative semicycles of the solution of equation (1.1) to successively occur still is $\cdots 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, \cdots$.

Hence, the proof is complete.

Remark 2.6 It is well known to all that the two cases in the proof of Theorem 2.5 are caused by the perturbation of the initial around the equilibrium point. So, the theorem 2.5 actually indicates that the perturbation of the initial values may lead to the variation of the trajectory structure rule for the solutions of equation (1.1).

2.3 Global Asymptotic Stability

First, we consider the local asymptotic stability for unique positive equilibrium point $\bar{x}$ of equation (1.1). We have the following results.

Theorem 2.7 *Then the positive equilibrium of equation (1.1) is locally asymptotically stable.*

Proof: The linearized equation of equation (1.1) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+1} = 0 \times y_n + 0 \times y_{n-1} + 0 \times y_{n-2}, \quad n = 0, 1, \cdots .$$

By virtue of [2, Remark 1.3.7], $\bar{x}$ is locally asymptotically stable. The proof is complete.

We now are in a position to study the global asymptotic stability of positive equilibrium point $\bar{x}$.

Theorem 2.8 *The positive equilibrium point of equation (1.1) is globally asymptotically stable.*

Proof: We must prove that the positive equilibrium point $\bar{x}$ of equation (1.1) is both locally asymptotically stable and globally attractive. Theorem 2.7 has shown the local asymptotic stability of $\bar{x}$. Hence, it remains to verify its global attractivity. That is, it suffices to prove that every solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) converges to $\bar{x}$ as $n \to \infty$, i.e., to prove

$$\lim_{n\to\infty} x_n = \bar{x} = 1 . \tag{2.4}$$

We can divide the solutions into two kinds of types.

i) Trivial solutions;

ii) Nontrivial solutions.

If the solution is a trivial solution, then it is obvious for (2.4) to hold because $x_n = 1$ eventually.

If the solution is a nontrivial solution, then we can further divide the solution into two cases.

a) Non-oscillatory solution;

b) Oscillatory solution.

If case a) happens, then it follows from Theorem 2.4 that the solution is actually an eventually negative one. Accordingly, there exists a positive integer N such that $x_n < 1$ for $n \geq N$. From Lemma 2.3 (b), we know that two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of the solution $\{x_n\}_{n=-2}^{\infty}$ are increasing and have upper bound 1. So, the limits $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exist and are finite, denoted by L and M, respectively. It is clear from equation (1.1) that

$$x_{2n+1} = \frac{x_{2n-1} + x_{2n-2} + a}{x_{2n-1}x_{2n-2} + 1 + a} \qquad \text{and} \qquad x_{2n+2} = \frac{x_{2n} + x_{2n-1} + a}{x_{2n}x_{2n-1} + 1 + a}.$$

Taking limits on both sides of the above equalities produces

$$M = \frac{M + L + a}{LM + 1 + a} \qquad \text{and} \qquad L = \frac{L + M + a}{LM + 1 + a}.$$

Solving these two equations, we get $L = M = 1$. This manifests that (2.4) is valid for non-oscillatory solutions.

Thus, it suffices to prove that (2.4) holds for the solution to be oscillatory, i.e., case b) occurs.

Consider now $\{x_n\}$ to be strictly oscillatory about the positive equilibrium point $\bar{x}$ of equation (1.1). By virtue of Theorem 2.5, we know that the rule for the lengths of positive and negative semi-cycles which occur successively is $\cdots, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, \cdots$.

For simplicity, for some nonnegative integer p, we denote by $\{x_p, x_{p+1}, x_{p+2}\}^+$ the terms of a positive semi-cycle of length three, followed by $\{x_{p+3}, x_{p+4}\}^-$ a negative semi-cycle with length two, then a positive semi-cycle $\{x_{p+5}\}^+$ and a negative semi-cycle $\{x_{p+6}\}^-$, and so on. Namely, the rule for the positive and negative semi-cycles of the solution to occur successively can be periodically expressed as follows:
$\{x_{p+7n}, x_{p+7n+1}, x_{p+7n+2}\}^+$, $\{x_{p+7n+3}, x_{p+7n+4}\}^-$, $\{x_{p+7n+5}\}^+$, $\{x_{p+7n+6}\}^-$, $n = 0, 1, \cdots$.

From Lemma 2.3 (b) and (c) the following results can be derived straightforward:

(i) $x_{p+7n+7} < x_{p+7n+5} < x_{p+7n+2} < x_{p+7n}$;

(ii) $x_{p+7n+6} > \max\{x_{p+7n+4},\ x_{p+7n+3}\}$.

From i), one can see that $\{x_{p+7n}\}$ is monotonically decreasing and has lower bound 1. So, the limit $\lim_{n\to\infty} x_{p+7n}$ exists and is finite, denoted by L. Moreover, it follows from i) that

$$\lim_{n\to\infty} x_{p+7n+5} = \lim_{n\to\infty} x_{p+7n+2} = \lim_{n\to\infty} x_{p+7n} = L\,. \tag{2.5}$$

According to the taking values of variable in positive and negative semi-cycles and equation (1.1), we also have

$$x_{p+7n+3} = \frac{x_{p+7n+1} + x_{p+7n} + a}{x_{p+7n+1}x_{p+7n} + 1 + a} > \frac{1}{x_{p+7n+1}}\,,$$

and

$$x_{p+7n+4} = \frac{x_{p+7n+2} + x_{p+7n+1} + a}{x_{p+7n+2}x_{p+7n+1} + 1 + a} > \frac{1}{x_{p+7n+2}}\,.$$

So, we further obtain

$$x_{p+7n+7} = \frac{x_{p+7n+5} + x_{p+7n+4} + a}{x_{p+7n+5}x_{p+7n+4} + 1 + a} < \frac{1}{x_{p+7n+4}} < x_{p+7n+2} \tag{2.6}$$

and

$$x_{p+7n+8} = \frac{x_{p+7n+6} + x_{p+7n+5} + a}{x_{p+7n+6}x_{p+7n+5} + 1 + a} < \frac{1}{x_{p+7n+6}} < \frac{1}{x_{p+7n+3}} < x_{p+7n+1}\,. \tag{2.7}$$

we see by (2.5) and (2.6) that $\lim_{n\to\infty} x_{p+7n+4} = 1/L$.

(2.7) indicates that $\{x_{p+7n+1}\}$ is monotonically decreasing and has lower bound 1. So, the limit $\lim_{n\to\infty} x_{p+7n+1}$ exists and is finite, denoted by M. Furthermore, it is clear from (2.7) that

$$\lim_{n\to\infty} x_{p+7n+6} = \lim_{n\to\infty} x_{p+7n+3} = \frac{1}{M}. \tag{2.8}$$

Now it's sufficient for us to verify that $L = M = 1$. To this end, noting

$$x_{p+7n+6} = \frac{x_{p+7n+4} + x_{p+7n+3} + a}{x_{p+7n+4}x_{p+7n+3} + 1 + a},$$

taking the limits on both sides of the above equality, we obtain $\frac{1}{M} = \frac{1/L+1/M+a}{1/L\times 1/M+1+a}$. Solving this equation, we can derive $M = 1$.

By taking the limits on both sides of

$$x_{p+7n+5} = \frac{x_{p+7n+3} + x_{p+7n+2} + a}{x_{p+7n+3}x_{p+7n+2} + 1 + a},$$

we have $L = \lim_{n\to\infty} x_{p+7n+5} = 1$.

Up to this, we have shown $\lim_{n\to\infty} x_{p+7n+k} = 1, k = 0, 1, 2\cdots, 6$, which indicates $\lim_{n\to\infty} x_n = 1$. So, the proof for Theorem 2.8 is complete.

2.4 Rule of Trajectory Structure

Finally, we can sum the general rule for the trajectory structure of solutions of equation (1.1) as follows.

Theorem 2.9 *The rule for the trajectory structure of any solution of equation (1.1) is as follows.*

I). *The solution is either eventually trivial or;*

II). *The solution is eventually nontrivial and further either*

II-1). *The solution is eventually negative non-oscillatory or;*

II-2). *The solution is strictly oscillatory and moreover, the successive lengths for positive and negative semi-cycles occur periodically with prime period 7 and in a period the rule is* $3^+, 2^-, 1^+, 1^-$.

The positive equilibrium point of equation (1.1) is a global attractor of all its solutions.

It follows from the results stated previously that Theorem 2.9 is true.

References

[1] **Agarwal, R. P. :** *Difference Equations and Inequalities.* Marcel Dekker, New York, 1992(1st edition), 2000(2nd edition)

[2] **Kocic, V. L.**, and **Ladas, G. :** *Global behavior of nonlinear difference equations of higher order with applications.* Kluwer Academic Publishers, Dordrecht, 1993

[3] **Kulenović, M. R. S., Ladas, G., Martins, L. F.**, and **Rodrigues, I. W. :** *The Dynamics of* $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}$*: Facts and Conjectures.* Computers Math. Appl., 45: 1087-1099, 2003

[4] **Nesemann, T. :** *Positive nonlinear difference equations: some results and apllications.* Nonlinear Analysis, 47, 4707-4717, 2001

[5] **Ladas, G. :** *Progress report on* $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$. J. Difference Equa. Appl., 1(2), 211-215, 1995

[6] **Yang, X., Lai, H., Evans, D. J.**, and **Megson, G. M. :** *Global asymptotic stability in a rational recursive sequence.* Appl. Math. Comput., 158(3), 703-716, 2004

[7] **Yang, X., Cao, J.**, and **Megson, G. M. :** *Global asymptotic stability in a class of Putnam-type equations.* Nonlinear Anal., 64(1), 42-50, 2006

[8] **Li, X.**, and **Zhu, D. :** *Global asymptotic stability of a nonlinear recursive sequence.* Appl. Math. Letters, 17(7), 833-838, 2004

[9] **Li, X.**, and **Zhu, D. :** *Global asymptotic stability in a rational equation.* J. Difference Equa. Appl., 9 (9), 833-839, 2003

[10] **Li, X.**, and **Zhu, D. :** *Global asymptotic stability for two recursive difference equations.* Appl. Math. Comput., 150 (2), 481-492, 2004

received: April 3, 2006

Authors:

Yixiang Hu
School of Mathematics and Physics
Nanhua University,
Hengyang, Hunan 421001,
P. R. China

Xianyi Li
School of Mathematics and Physics
Nanhua University,
Hengyang, Hunan 421001,
P. R. China
e-mail: xianyili6590@163.com

Rostock. Math. Kolloq. **61**, 85–94 (2006)

Subject Classification (AMS)
47H09, 47H10, 47H17

Xue Zhiqun

Ishikawa Iterative Process with Errors for Generalized Lipschitz Φ-Accretive Mappings in Uniformly Smooth Banach Spaces[1]

ABSTRACT. Let E be a uniformly smooth real Banach space and $T : E \to E$ be generalized Lipschitz Φ-accretive mapping with $\Phi(r) \to +\infty$ as $r \to +\infty$. Let$\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be six real sequences in $[0,1]$ satisfying the following conditions: (i)$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, (ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b'_n = \lim\limits_{n\to\infty} c'_n = 0$, (iii)$\sum\limits_{n=0}^{\infty} b_n = \infty$, (iv)$c_n = o(b_n)$. For arbitrary $x_0 \in E$, define the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ by (ISE): $y_n = a'_n x_n + b'_n S x_n + c'_n v_n, x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, n \geq 0$. where $S : E \to E$ is defined by $Sx = f + x - Tx, f \in E, \forall x \in E$. Assume that the equation $Tx = f$ has solution and $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are arbitrary two bounded sequences in E. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$. A related result deals with approximation of fixed point of generalized Lipschitz Φ-pseudocontractive mapping.

KEY WORDS. Ishikawa iterative process with errors; generalized Lipschitz; Φ-accretive mapping; Φ- pseudocontractive mapping; uniformly smooth Banach space.

1 Introduction

Let E be real Banach space and E^* be the dual space on E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* :< x, f >= ||x|| \cdot ||f|| = ||f||^2\} \tag{1.1}$$

for all $x \in E$, where $< \cdot, \cdot >$ denotes the generalized duality pairing. It is well known that if E is an uniformly smooth Banach space, then J is single-valued and such that $J(-x) = -J(x), J(tx) = tJ(x)$ for all $x \in E$ and $t \geq 0$; and J is uniformly continuous on any bounded subset of E. In the sequel we shall denote single-valued normalized duality

[1]Project supported by the National Science Foundation of China and Shijiazhuang Railway College Sciences Foundation.

mapping by j. By means of the normalized duality mapping J. In the following we give some concepts.

Definition 1.1 *Let E be real Banach space, and $T : E \supset D(T) \to E$ be a mapping with domain $D(T)$ and range $R(T)$. A mapping T is said to be strongly accretive if for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that*

$$< Tx - Ty, j(x-y) > \geq k\|x-y\|^2 \tag{1.2}$$

for some constant $k \in (0,1)$. A mapping T is called Φ-strongly accretive if for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< Tx - Ty, j(x-y) > \geq \Phi(\|x-y\|)\|x-y\| \tag{1.3}$$

The mapping T is called Φ-accretive if, there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$, and for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$< Tx - Ty, j(x-y) > \geq \Phi(\|x-y\|) \tag{1.4}$$

Recently, Zhou [6] proved the following result: Let X be real uniformly smooth Banach space. Assume that $A : X \to X$ is Lipschitz Φ-strongly accretive mapping with $\Phi(r) \to +\infty$ as $r \to +\infty$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $(0,1)$ satisfying the conditions:

(i) $0 < \alpha_n \leq \dfrac{1}{4(1+L_1)^2}, n \geq 0,$ *where $L_1 = 1 + L, L \geq 1$ is Lipschitz constant of A;*

(ii) $b(\alpha_n), \beta_n \to 0$ *as* $n \to \infty$;

(iii) $\sum\limits_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty.$

Assume that $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are two sequences in X satisfying condition: $\|u_n\| = o(\alpha_n)$, $\|v_n\| \to 0$ as $n \to \infty$, and $\|v_n\| \leq 1, \forall n \geq 0$. Define $S : E \to E$ by $Sx = f + x - Tx, f \in X, \forall x \in X$. Then the Ishikawa iterative process $\{x_n\}_{n=0}^{\infty}$ $x_0 \in X$ by

$$\begin{cases} x_0 \in X, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, n \geq 0. \end{cases}$$

converges strongly to the unique solution of the equation $Tx = f$. one question arises naturally: If T neither is Lipschitzian nor has the bounded range , whether or not the Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by converges strongly to the unique solution of the equation $Tx = f$. It is our purpose in this paper to solve the above part question by proving the *following much more general result: If E is an uniformly smooth real Banach space. Assume that $T : E \to E$ is Φ-accretive mapping, and T neither is Lipschizian nor has the bounded range, then the Ishikawa iteration sequence with errors generated by converges strongly to the unique solution of the equation $Tx = f$. For this, we need to give the following concept and Lemma.*

Definition 1.2 *A mapping $T : E \to E$ is called a generalized Lipschitz mapping, if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L(1 + \|Tx - Ty\|)$, $\forall x, y \in E$. Clear, every Lipschitz mapping is generalized Lipschitz mapping. However, generalized Lipschitz mapping must not be Lipschitz. See the following example.*

Example Let $E = (-\infty, +\infty)$ and $T : E \to E$ be

$$Tx = \begin{cases} x - 1, & \text{if } \mathrm{x} \in (-\infty, 0), \\ x - \sqrt{1 - (x+1)^2}, & \text{if } \mathrm{x} \in [-1, 0), \\ x + \sqrt{1 - (x-1)^2}, & \text{if } \mathrm{x} \in [0, 1], \\ x + 1, & \text{if } \mathrm{x} \in (1, +\infty). \end{cases}$$

Lemma 1.1 ([4]) *Let E be a real Banach space, then for all $x, y \in E$, there exists $j(x+y) \in J(x+y)$ such that $||x+y||^2 \leq ||x||^2 + 2 < y, j(x+y) >$.*

2 Main Results

Now we prove the main the results of this paper, In the sequel, we always assume that E is a uniformly smooth real Banach space.

Theorem 2.1 *Assume that $T : E \to E$ is generalized Lipschitz Φ-accretive mapping with $\Phi(r) \to +\infty$ as $r \to +\infty$. Let$\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$;

(ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b'_n = \lim\limits_{n\to\infty} c'_n = 0$;

(iii) $\sum\limits_{n=0}^{\infty} b_n = \infty$;

(iv) $c_n = o(b_n)$.

For arbitrary $x_0 \in E$, define the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ by (ISE):

$$\begin{cases} y_n = a'_n x_n + b'_n S x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \end{cases} \tag{2.1}$$

where $S : E \to E$ is defined by $Sx = f + x - Tx, \forall x \in E$. Assume that the equation $Tx = f$ has solution and $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are arbitrary two bounded sequences in E. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.

Proof: Let q be the solution of the equation $Tx = f$, then q is the unique solution. Since T is generalized Lipschitz Φ-accretive, then there exists $L_0 > 0$ such that $\|Tx-Ty\| \le L_0(1+\|x-y\|)$ and $< Tx-Ty, J(x-y) >\ge \Phi(\|x-y\|)$, for all $x, y \in E$, i.e., $\|Sx-Sy\| \le L(1+\|x-y\|)$, $< Sx - Sy, J(x - y) >\le \|x - y\|^2 - \Phi(\|x - y\|)$, where $L = 1 + L_0$. Especially, for $\forall x \in E$, $< Sx - Sq, J(x - q) >\le \|x - q\|^2 - \Phi(\|x - q\|)$. Observe that (ISE) equivalent form

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Sx_n + V_n + c_n'(q - x_n) \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n + U_n + c_n(q - x_n) \end{cases} \tag{2.2}$$

where $V_n = c_n'(v_n - q)$, $U_n = c_n(u_n - q)$, $\beta_n = b_n'$, $\alpha_n = b_n$. Then $\|V_n\| \to 0$ as $n \to \infty$, $\|U_n\| = o(b_n)$. From form (2.2), we obtain that

$$\begin{aligned} \|y_n - q\| &= \|(1-\beta_n)(x_n - q) + \beta_n(Sx_n - Sq) + V_n + c_n'(q - x_n)\| \\ &\le (1 - \beta_n + \beta_n L + c_n')\|x_n - q\| + \beta_n L + \|V_n\|, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)(x_n - q) + \alpha_n(Sy_n - Sq) + U_n + c_n(q - x_n)\| \\ &\le (1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c_n') + c_n)\|x_n - q\| \\ &+ \alpha_n(L + \beta_n L^2 + L\|V_n\|) + \|U_n\|. \end{aligned} \tag{2.4}$$

Furthermore, we have the following estimates

$$\begin{aligned} 2c_n\|x_n - q\| \cdot \|x_{n+1} - q\| &\le 2c_n(1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c_n') + c_n)\|x_n - q\|^2 \\ &+ 2c_n(\alpha_n(L + \beta_n L^2 + L\|v_n\|) + \|U_n\|)\|x_n - q\| \\ &\le R_n\|x_n - q\|^2 + P_n, \end{aligned} \tag{2.5}$$

where $R_n = 2c_n(1-\alpha_n+\alpha_n L(1-\beta_n+\beta_n L+c_n')+c_n)+c_n(\alpha_n(L+\beta_n L^2+L\|V_n\|)+\|U_n\|)$, $P_n = c_n(\alpha_n(L + \beta_n L^2 + L\|V_n\|) + \|U_n\|)$. And have

$$\begin{aligned} &2\left(\|V_n\| + c_n'\|x_n - q\|\right)\|y_n - q\| \\ &\le 2(\|V_n\| + c_n'\|x_n - q\|)((1 - \beta_n + \beta_n L + c_n')\|x_n - q\| + \beta_n L + \|V_n\|) \\ &\le 2c_n'((1 - \beta_n + \beta_n L + c_n')\|x_n - q\|^2 + 2\|V_n\|(\beta_n L + \|V_n\|)) \\ &+ 2(\|V_n\|(1 - \beta_n + \beta_n L + c_n') + c_n'(\beta_n L + \|v_n\|)\|x_n - q\| \\ &\le 2c_n'((1 - \beta_n + \beta_n L + c_n')\|x_n - q\|^2 + 2\|V_n\|(\beta_n L + \|V_n\|)) \\ &+ (\|V_n\|(1 - \beta_n + \beta_n L + c_n') + c_n'(\beta_n L + \|V_n\|)(1 + \|x_n - q\|^2) \\ &\le ((2c_n' + \|V_n\|)(1 - \beta_n + \beta_n L + c_n') + c_n'(\beta_n L + \|V_n\|))\|x_n - q\|^2) \\ &+ (\|V_n\|(1 - \beta_n + \beta_n L + c_n') + (c_n' + 2\|V_n\|)(\beta_n L + \|V_n\|) \\ &= G_n\|x_n - q\|^2 + H_n \end{aligned} \tag{2.6}$$

where $G_n = (2c'_n + \|V_n\|)(1 - \beta_n + \beta_n L + c'_n) + c'_n(\beta_n L + \|V_n\|)$, $H_n = (\|V_n\|(1 - \beta_n + \beta_n L + c'_n) + (c'_n + 2\|V_n\|)(\beta_n L + \|V_n\|)$. Set $A_n = \left\| J\left(\frac{x_{n+1}-q}{1+\|x_n-q\|}\right) - J\left(\frac{y_n-q}{1+\|x_n-q\|}\right)\right\|$, $D_n = \left\| J\left(\frac{y_n-q}{1+\|x_n-q\|}\right) - J\left(\frac{x_n-q}{1+\|x_n-q\|}\right)\right\|$, then $A_n \to 0$, $D_n \to 0$ as $n \to \infty$. Indeed $\left\{\frac{y_n-q}{1+\|x_n-q\|}\right\}_{n=0}^{\infty}$, $\left\{\frac{x_n-q}{1+\|x_n-q\|}\right\}_{n=0}^{\infty}$, and $\left\{\frac{x_{n+1}-q}{1+\|x_n-q\|}\right\}_{n=0}^{\infty}$ are all bounded, and $\left\|\frac{x_{n+1}-q}{1+\|x_n-q\|} - \frac{y_n-q}{1+\|x_n-q\|}\right\| \to 0$, $\left\|\frac{y_n-q}{1+\|x_n-q\|} - \frac{x_n-q}{1+\|x_n-q\|}\right\| \to 0$ as $n \to \infty$. Applying uniformly continuity of J on any bounded subset, hence $A_n \to 0$, $D_n \to 0$ as $n \to \infty$. Using Lemma 1.1 and (2.4), (2.5), we may obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q) + \alpha_n(Sy_n - Sq) + U_n + c_n(q - x_n)\|^2 \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n < Sy_n - Sq, J(x_{n+1} - q) > \\
&+ 2 < U_n, J(x_{n+1} - q) > + 2c_n < q - x_n, J(x_{n+1} - q) > \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n < Sy_n - Sq, J(y_n - q) > \\
&+ 2\alpha_n < Sy_n - Sq, J(x_{n+1} - q) - J(y_n - q) > \\
&+ 2\|U_n\| \cdot \|x_{n+1} - q\| + 2c_n\|x_n - q\| \cdot \|x_{n+1} - q\| \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(\|y_n - q\|^2 - \Phi(\|y_n - q\|)) \qquad (2.7) \\
&+ 2\alpha_n < Sy_n - Sq, J\left(\frac{x_{n+1} - q}{1 + \|x_n - q\|}\right) - J\left(\frac{y_n - q}{1 + \|x_n - q\|}\right) > (1 + \|x_n - q\|) \\
&+ 2\|U_n\|(1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c'_n) + c_n)\|x_n - q\| \\
&+ 2\|U_n\|(\alpha_n L + \alpha_n\beta_n L^2 + \alpha_n L\|V_n\| + \|U_n\|) + R_n\|x_n - q\|^2 + P_n \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(\|y_n - q\|^2 - \Phi(\|y_n - q\|)) \\
&+ 2\alpha_n A_n L(1 + \|y_n - q\|)(1 + \|x_n - q\|) + E_n + P_n + R_n\|x_n - q\|^2 \\
&+ 2\|U_n\|(1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L))\|x_n - q\|
\end{aligned}$$

where $E_n = 2\|U_n\|(\alpha_n L + \alpha_n\beta_n L^2 + \alpha_n L\|V_n\| + \|U_n\|)$. Furthermore,

$$\begin{aligned}
&2\|U_n\|(1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c'_n) + c_n)\|x_n - q\| \\
&\quad \le \|U_n\|((1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c'_n) + c_n)^2 + \|x_n - q\|^2) \qquad (2.8) \\
&\quad \le \|U_n\|M_1 + \|U_n\|\|x_n - q\|^2,
\end{aligned}$$

$$\begin{aligned}
&2\alpha_n A_n L(1 + \|y_n - q\|)(1 + \|x_n - q\|) \\
&\quad \le 2\alpha_n A_n L((1 - \beta_n + \beta_n L + c'_n))\|x_n - q\| + 1 + \beta_n L + \|v_n\|)(1 + \|x_n - q\|) \\
&\quad \le 4\alpha_n A_n L(1 + \beta_n L + c'_n)(1 + \|x_n - q\|^2) \\
&\quad = F_n\|x_n - q\|^2 + F_n
\end{aligned} \qquad (2.9)$$

where $M_1 = \sup\{(1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c'_n) + c_n)^2\}$. $F_n = 4\alpha_n A_n L(1 + \beta_n L + c'_n)$. Substituting (2.8) and (2.9) in (2.7), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 \le{}& ((1-\alpha_n)^2 + F_n + R_n + \|U_n\|)\|x_n - q\|^2 + E_n + F_n \\
&+ P_n + \|U_n\|M_1 + 2\alpha_n(\|y_n - q\|^2 - \Phi(\|y_n - q\|)).
\end{aligned} \qquad (2.10)$$

Again using Lemma 1.1 and (2.6), we obtain

$$
\begin{aligned}
\|y_n-q\|^2 &\le (1-\beta_n)^2\|x_n-q\|^2+2\beta_n < Sx_n-Sq, J(y_n-q) > \\
&+2(\|V_n\|+c_n'\|x_n-q\|)\|y_n-q\| \\
&\le (1-\beta_n)^2\|x_n-q\|^2+2\beta_n < Sx_n-Sq, J(y_n-q)-J(x_n-q) > \\
&+2\beta_n < Sx_n-Sq, J(x_n-q) > +G_n\|x_n-q\|^2+H_n \\
&\le (1+\beta_n^2+G_n)\|x_n-q\|^2-2\beta_n\Phi(\|x_n-q\|)+H_n \\
&+2\beta_n < Sx_n-Sq, J\left(\frac{y_n-q}{1+\|x_n-q\|}\right)-J\left(\frac{x_n-q}{1+\|x_n-q\|}\right) > (1+\|x_n-q\|) \\
&\le (1+\beta_n^2+G_n)\|x_n-q\|^2-2\beta_n\Phi(\|x_n-q\|)+H_n \\
&+2\beta\|Sx_n-Sq\|D_n(1+\|x_n-q\| \\
&\le (1+\beta_n^2+G_n+4\beta_nD_nL)\|x_n-q\|^2 \\
&+H_n+4\beta_nD_nL-2\beta_n\Phi(\|x_n-q\|)\,.
\end{aligned}
\tag{2.11}
$$

Substituting (2.11) in (2.10), get

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\le (1+\alpha_n^2+F_n+R_n+\|U_n\|+2\alpha_n(\beta_n^2+G_n+4\beta_nD_nL)) \\
&\times\|x_n-q\|^2+E_n+F_n+P_n+\|U_n\|M_1+2\alpha_nH_n \\
&+8\alpha_n\beta_nD_nL-2\alpha_n\Phi(\|y_n-q\|)) \\
&-4\alpha_n\beta_n\Phi(\|x_n-q\|)\|x_n-q\|) \\
&\le \|x_n-q\|^2+I_n\|x_n-q\|^2+2\alpha_n(O_n-\Phi(\|y_n-q\|))
\end{aligned}
\tag{2.12}
$$

where $I_n=\alpha_n^2+F_n+R_n+\|U_n\|+2\alpha_n(\beta_n^2+G_n+4\beta_nD_nL)$, $O_n=(E_n+F_n+P_n+\|U_n\|M_1+2\alpha_nH_n+8\alpha_n\beta_nD_nL)/2\alpha_n$. Base on definition of S, for any $\forall x\in E$, $< Sx-Sq-x+q, J(x-q) >\le -\Phi(\|x-q\|)$. Thus, $\Phi(\|x-q\|)\le\|x-Sx\|$. Any choose $x_0\in E$ such that $\|x_0-Sx_0\|\ne 0$, i.e,$x_0\ne q$. If $x_0=q$, then we are done. Suppose this is not the case, then have $\|x_0-q\|\le\Phi^{-1}(\|x_0-Sx_0\|)$. Since $\alpha_n,\beta_n\to 0$ $(n\to\infty)$, so that $I_n=o(\alpha_n), O_n=o(\alpha_n), \|U_n\|=o(\alpha_n)$ and $\|V_n\|\to 0 (n\to\infty)$, there exists positive integer N such that $\alpha_n<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{4(1+L+L^2)\Phi^{-1}(\|x_0-Sx_0\|)+2L^2+4L}$, $\beta_n<\frac{2\Phi^{-1}(\|x_0-Sx_0\|)}{3(1+L)\Phi^{-1}(\|x_0-Sx_0\|)+3L}$, $\|U_n\|<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{2}$, $\|V_n\|<\min\left\{1,\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{6}\right\}$, $1-\beta_n-\beta_nL-c_n'>\frac{3}{4}$, $\beta_nL+\|V_n\|<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{6}$, $\frac{I_n(2\Phi^{-1}(\|x_0-Sx_0\|))^2}{2\alpha_n}+O_n<\frac{\Phi\left(\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{2}\right)}{2}$. For all $n\ge N$. Suppose $\|x_N-q\|\le 2\Phi^{-1}(\|x_0-Sx_0\|)$ holds, we prove $\|x_{N+1}-q\|\le 2\Phi^{-1}(\|x_0-Sx_0\|)$. Assume that this is not true, then $\|x_{N+1}-q\|>2\Phi^{-1}(\|x_0-Sx_0\|)$. From (2.2) we may get $(1-\alpha_N)\|x_N-q\|\ge\|x_{N+1}-q\|-\alpha_N\|x_N-Sy_N\|-\|U_N\|-c_N\|x_N-q\|$ (N is enough big, $1-\alpha_N+c_N<1$).

We also obtain the following inequality:

$$
\begin{aligned}
\|x_N - q\| &\geq \|x_{N+1} - q\| - \alpha_N \|x_N - Sy_N\| - \|U_N\| \\
&\geq 2\Phi^{-1}(\|x_0 - Sx_0\|) \\
&- \alpha_N(2(1 + L + L^2)\Phi^{-1}(\|x_0 - Sx_0\|) + L^2 + 2L) - \|U_N\| \\
&\geq \Phi^{-1}(\|x_0 - Sx_0\|)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_N - q\| &\geq (1 - \beta_N)\|x_N - q\| - \beta_N L\|x_N - q\| - \beta_N L - \|V_N\| - c'_N\|x_N - q\| \\
&= (1 - \beta_N - \beta_N L - c'_N)\|x_N - q\| - \beta_N L - \|V_N\| \\
&\geq \frac{\Phi^{-1}(\|x_0 - Sx_0\|)}{2},
\end{aligned}
$$

so that $\Phi(\|y_N - q\|) \geq \Phi(\frac{\Phi^{-1}(\|x_0 - Sx_0\|)}{2})$. Using (2.12) and above relevant form, we compute as follows:

$$
\begin{aligned}
\|x_{N+1} - q\|^2 &\leq \|x_N - q\|^2 + I_N\|x_N - q\|^2 + 2\alpha_N(O_N - \Phi(\|y_N - q\|)) \\
&\leq \|x_N - q\|^2 + 2\alpha_N(\frac{I_N\|x_N - q\|^2}{2\alpha_N} + O_N - \Phi(\|y_N - q\|)) \\
&\leq \|x_N - q\|^2 - \alpha_N\Phi(\frac{\Phi^{-1}(\|x_0 - Sx_0\|)}{2}) \\
&\leq \|x_N - q\|^2 \leq (2\Phi^{-1}(\|x_0 - Sx_0\|))^2 .
\end{aligned}
$$

contradicting with assumption. By induction, so sequence $\{\|x_n - q\|\}_{n=0}^{\infty}$ is bounded, therefore $\{\|y_n - q\|\}_{n=0}^{\infty}$ is also bounded. Set $W = \sup\{\|x_n - q\|\} + \sup\{\|y_n - q\|\}$, $Q_n = \frac{I_n W^2}{2\alpha_n} + O_n$. Then using (2.12), we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|x_n - q\|^2 + 2\alpha_n\left(\frac{I_n\|x_n - q\|^2}{2\alpha_n} + O_n - \Phi(\|y_n - q\|)\right) \\
&\leq \|x_n - q\|^2 + 2\alpha_n(Q_n - \Phi(\|y_n - q\|)) \\
&= \|x_n - q\|^2 + \alpha_n(2Q_n - \Phi(\|y_n - q\|)) - \alpha_n\Phi(\|y_n - q\|)) .
\end{aligned}
\tag{2.13}
$$

In the following we prove that $\liminf_{n\to\infty} \|y_n - q\| = 0$ holds. If not true. Let $\liminf_{n\to\infty} \|y_n - q\| = 2\delta > 0$. Then, there exists an integer N_1 such that $\|y_n - q\| \geq \delta$, $\forall n \geq N_1$, i.e., $\Phi(\|y_n - q\|) \geq \Phi(\delta)$. Since $Q_n \to 0 (n \to \infty)$, there exists positive integer $N_2 > N_1$ such that $Q_n \leq \Phi(\delta), \forall n \geq N_2$. The implies that $Q_n \leq \Phi(\|y_n - q\|)$. Hence, for all $n \geq N_2$, we obtain that

$$
\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\Phi(\|y_n - q\|)) \leq \|x_n - q\|^2 - \alpha_n\Phi(\delta) .
$$

This implies that

$$
\Phi(\delta)\sum_{n=N_2}^{\infty}\alpha_n \leq \sum_{n=N_2}^{\infty}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) \leq \|x_{N_2} - q\|^2 < \infty
$$

contradicting with condition (iii) of Theorem. So there exists a infty subsequence $\{y_{n_j}-q\}_{j=0}^{\infty}$ of $\{y_n-q\}_{n=0}^{\infty}$. Again from (2.2), get $\|y_{n_j}-q\| \geq (1-\beta_{n_j}-\beta_{n_j}L-c'_{n_j})\|x_{n_j}-q\|-\beta_{n_j}L-\|V_{n_j}\|$, so that $\lim\limits_{j\to\infty}\|x_{n_j}-q\|=0$. Hence, for any $\forall\varepsilon>0$, there exists a positive integer n_{j_0}, such that $\|x_{n_j}-q\|<\varepsilon, \alpha_{n_j}(LW+L)+\|U_{n_j}\|<\frac{\varepsilon}{4}, (\beta_{n_j}+\beta_{n_j}L+c'_{n_j})W+(\beta_{n_j}L+\|V_{n_j}\|)<\frac{\varepsilon}{4}$, for all $n_j>n_{j_0}$. Again choose a positive integer $N_0 \geq n_{j_0}$ such that $Q_n<\Phi(\frac{\varepsilon}{2})\frac{1}{2}$, for all $n>N_0$. Next, we want to prove: for arbitrary $\forall m\geq 1, \|x_{n_j+m}-q\|<\varepsilon, n_j>n_{j_0}$. First, we prove that $\|x_{n_j+1}-q\|<\varepsilon$. If it is not the case, then there exists $n_{j_1}>n_{j_0}$ such that $\|x_{n_{j_1}+1}-q\|<\varepsilon$. Using (2.2) again, we have

$$\begin{aligned}
\|x_{n_{j_1}+1}-q\| &\leq (1-\alpha_{n_{j_1}})\|x_{n_{j_1}}-q\|+\alpha_{n_{j_1}}\|Sy_{n_{j_1}}-Sq\|+\|U_{n_{j_1}}\|+c_{n_{j_1}}\|x_{n_{j_1}}-q\| \\
&\leq (1-\alpha_{n_{j_1}}+c_{n_{j_1}})\|x_{n_{j_1}}-q\|+\alpha_{n_{j_1}}L(1+\|y_{n_{j_1}}-q\|)+\|U_{n_{j_1}}\| \\
&\leq \|x_{n_{j_1}}-q\|+\alpha_{n_{j_1}}L(1+W)+\|U_{n_{j_1}}\| \\
&\leq \|x_{n_{j_1}}-q\|+\frac{\varepsilon}{4}
\end{aligned}$$

lead to $\|x_{n_{j_1}}-q\|>\|x_{n_{j_1}+1}-q\|-\frac{\varepsilon}{4}>\frac{3\varepsilon}{4}$. And we get also

$$\begin{aligned}
\|y_{n_{j_1}}-q\| &\geq (1-\beta_{n_{j_1}})\|x_{n_{j_1}}-q\|-\beta_{n_{j_1}}L(\|x_{n_{j_1}}-q\|+1)-\|V_{n_{j_1}}\|-c'_{n_{j_1}}\|x_{n_{j_1}}-q\| \\
&\geq \|x_{n_{j_1}}-q\|-(\beta_{n_{j_1}}+\beta_{n_{j_1}}L+c'_{n_{j_1}})\|x_{n_{j_1}}-q\|-(\beta_{n_{j_1}}L+\|V_{n_{j_1}}\|) \\
&> \frac{3\varepsilon}{4}-(\beta_{n_{j_1}}+\beta_{n_{j_1}}L+c'_{n_{j_1}})W-(\beta_{n_{j_1}}L+\|V_{n_{j_1}}\|) \\
&> \frac{\varepsilon}{2}.
\end{aligned}$$

Hence $\Phi(\|y_{n_{j_1}}-q\|)>\Phi(\frac{\varepsilon}{2})$. By (2.12), we obtain that

$$\begin{aligned}
\varepsilon^2 &\leq \|x_{n_{j_1}+1}-q\|^2 \\
&\leq \|x_{n_{j_1}}-q\|^2+2\alpha_{n_{j_1}}(Q_{n_{j_1}}-\Phi(\|y_{n_{j_1}}-q\|)) \\
&< \varepsilon^2+2\alpha_{n_{j_1}}(\Phi(\frac{\varepsilon}{2})\frac{1}{2}-\Phi(\frac{\varepsilon}{2})) \\
&= \varepsilon^2-\alpha_{n_{j_1}}\Phi(\frac{\varepsilon}{2})\frac{1}{2} \\
&\leq \varepsilon^2
\end{aligned}$$

contradiction. By induction, we obtain that $\|x_{n_j+m}-q\|<\varepsilon$. This show that $x_n\to q$ as $n\to\infty$. About case $\sum\limits_{n=0}^{\infty}\|U_n\|<\infty$, repeating above-mentioned course, we can get the conclusion. Completing proof of Theorem 2.1. □

Remark 1 Theorem 2.1 contains a good number of the known results as its special cases. In particular, if the mapping T considered here satisfies one of the following assumptions: (i) $T:K\to K$ is a Lipschitzian. (ii) T has the bounded range. Then T satisfied the conditions of Theorem 2.1.

Remark 2 It is well known that T is strongly pseudocontractive (Φ-strongly pseudocontractive, Φ-pseudocontractive)if and only if $(I-T)$ is strongly accertive (Φ-strongly accretive, Φ-accretive), where I denotes the identity operator. In the following we give about the results of Φ-pseudocontractive.

Theorem 2.2 *Let K be nonempty closed convex subset of E and $T : K \to K$ be generalized Lipschitz Φ-pseudocontractive mapping. Assume that $\Phi(r) \to +\infty$ as $r \to +\infty$ and $F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be six real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$;

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = 0$;

(iii) $\sum_{n=0}^{\infty} b_n = \infty$;

(iv) $c_n = o(b_n)$ *or* $\sum_{n=0}^{\infty} c_n < +\infty$.

For arbitrary $x_0 \in K$, define the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ by (ISE):

$$\begin{cases} y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \end{cases} \tag{2.14}$$

Suppose $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are arbitrary two bounded sequences in K. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Proof: Applying Theorem 2.1, we obtain directly conclusion of Theorem 2.2. □

Remark 3 Our two Theorems extend the main known results from Lipschitzian or the boundedness range to more general class of neithe Lipschitzian nor the range boundedness mappings, and also from strongly pseudocontractive (accretive) to Φ- pseudocontractive (accretive).

References

[1] **Browder, F. E. :** *Nonlinear operators and nonlineat equation of evolution in Banach spaces.* Proc, Sympos, Pure. Math. **18** (1976)

[2] **Barbu, V. :** *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff, Leyden 1976

[3] **Bogin, J. :** *On strict pseudo-contractions and a fixed point theorem.* Technion Preprint MT-29, Haifa 1974

[4] **Deimling, K. :** *Nonlinear Functional Analysis.* Springer-Verlag

[5] **Zhou, H. Y.**, and **Jia, Y. T. :** *Approximating of fixed points of strongly pseudocontractive maps without Lipschitz assumption.* Proc. Amer. Math. soc. **125**, 1705-1709, 1997

[6] **Haiyun, Z. :** *Iterative approximation of the fixed point of* Φ*-hemicontractive mapping.* Chinese Annals of Mathematics, 20A:**3**,399-402 (1999)

[7] **Haiyun, Z. :** *Approximating Zeros of* φ*-strongly Acretive Operators by the Ishikawa Iteration Process with Errors.* Acta Mathematica Sinica, (41)**5**, 1091-1100, 1998

[8] **Chang, S. S., Cho, Y. J. :** *Jung, J. S.*. Iterative approximation of fixed points and solutions for strongly accretive and strongly pseudocontractive mappings in Banach spacesJ. Math. Anal. Appl. 224: 149-165, 1998

[9] **Xu, Y. G. :** *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations.* J. Math. Anal. Appl. 224: 91-101, 1998

[10] **Xue, Z. Q., Zhou, H. Y.**, and **Cho, Y. J. :** *Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces.* J. Nonlinear and Convex Analysis 1(3): 313-320, (2003)

received: April 12, 2006

Author:

Xue Zhiqun
Department of Mathematics,
Shijiazhuang Railway College,
Shijiazhuang 050043
China

e-mail: xuezhiqun@126.com

Rostock. Math. Kolloq. **61**, 95–105 (2006)

Subject Classification (AMS)
Primary 05C38, 15A15;
Secondary 05A15, 15A18

Arif Rafiq

Convergence of an iterative scheme due to Agarwal et al.

ABSTRACT. In this paper, we are concerned with the study of an iterative scheme with errors due to Agarwal et al [1] associated with two mappings satisfying certain condition. We approximate the common fixed points of these two mappings by weak and strong convergence of the scheme in a uniformly convex Banach space under a certain condition.

KEY WORDS AND PHRASES. Iterative Scheme with Errors, Common Fixed Point, Condition (AU-N), Condition (AR), Weak and Strong Convergence

1 Introduction

Let C be a nonempty convex subset of a normed space E and $S, T : C \to C$ be two mappings. Xu [15] introduced the following iterative schemes known as Mann iterative scheme with errors and Ishikawa iterative scheme with errors:

(1) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \ n \geq 1, \end{cases} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme [8] if $c_n = 0$, i.e.,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - b_n) x_n + b_n T x_n, \ n \geq 1, \end{cases} \tag{M}$$

where $\{b_n\}$ is a sequence in $[0, 1]$.

(2) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, n \geq 1, \end{cases} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}\{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme [5] if $c_n = 0 = c'_n$, i.e.,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - b_n) x_n + b_n T y_n, \\ y_n = (1 - b'_n) x_n + b'_n T x_n, \ n \geq 1, \end{cases} \tag{I}$$

where $\{b_n\}, \{b'_n\}$ are sequences in $[0,1]$.

A generalization of Mann and Ishikawa iterative schemes [5, 8] was given by Das and Debata [4] and Takahashi and Tamura [13]. This scheme dealt with two mappings:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n S y_n \\ y_n = (1 - b_n)\, x_n + b_n T x_n, \ n \geq 1. \end{cases} \tag{1.3}$$

In [1] Agarwal et al introduced the following scheme for quasi-contractive mappings as follows.

(3) The sequence $\{x_n\}$, in this case, is defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{a_n\}, \{b_n\}, \{c_n\}\{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

A Banach space E is said to satisfy Opial's condition [9] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Next we state the following useful lemmas.

Lemma 1 [11] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2 [14] *Let $\{s_n\}, \{t_n\}$ be two nonnegative sequences satisfying*

$$s_{n+1} \leq s_n + t_n \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n\to\infty} s_n$ exists.

Lemma 3 [2] *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

Nonexpansive mappings since their introduction have been extensively studied by many authors in different frames of work. One is the convergence of iteration schemes constructed through nonexpansive mappings.

Recently Khan et al presented the following results in [6].

Definition 1 *Two mappings $S, T : C \to C$ where C a subset of E, are said to satisfy condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - x^*\| : x^* \in F = F(S) \cap F(T)\}$.*

Lemma 4 *Let E be a normed space and C its nonempty bounded convex subset. Let $S, T : C \to C$ be nonexpansive mappings. Let $\{x_n\}$ be the sequence as defined in (1.4) with $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.*

Lemma 5 *Let E be a uniformly convex Banach space and C its nonempty bounded closed convex subset. Let $S, T : C \to C$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.4) with $0 < \delta \leq b_n$, $b'_n \leq 1 - \delta < 1$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$.*

Lemma 6 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S, T and $\{x_n\}$ be as taken in Lemma 5. If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Lemma 7 *Let E be a uniformly convex Banach space and C, $\{x_n\}$ be as taken in Lemma 5. Let $S, T : C \to C$ be two nonexpansive mappings satisfying condition (A'). If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

The following observations about the results of Khan et al [6] have been made.

1: In [6] the authors claimed that, the iterative scheme (1.4) is new. Infect it is studied by Agarwal et al in [1].

2: Unfortunately, just as in [10], one cannot directly deduce the Mann type convergence theorems for one mapping due to the condition $0 < \delta \leq b'_n \leq 1 - \delta < 1$ (similar to the condition $1 - \beta_n < 1 - \epsilon$, $\epsilon > 0$ in [10]).

3: To say that S and T are nonexpansive (separately) is meaningless, the classical definition of two nonexpansive mappings is stated as follows: Two mappings $S, T : C \to C$ are said to be nonexpansive, if

$$\|Sx - Ty\| \leq \|x - y\|, \tag{AU-N}$$

for all $x, y \in C$. For $S = T$, we get the usual definition of nonexpansive mappings.

4: In [6] the authors stated that, $\{u_n\}, \{v_n\}$ are bounded sequences in C, while they are taking C as bounded. Hence $\{u_n\}, \{v_n\}$ should be arbitrary sequences in C (just as in [3]).

In this paper, we study the iterative scheme given in (1.4) for weak and strong convergence for two mappings satisfying (AU-N) in a uniformly convex Banach space. In order to prove our results, we do not need C to be bounded. We also remove the condition $0 < \delta \leq b'_n \leq 1 - \delta < 1$. Similar results for usual Ishikawa iterations for one mapping can be obtained, and consequently results including of Schu [11] can be recapture.

2 Main Results

In this section, we shall prove the weak and strong convergence of the iteration scheme (1.4) to a common fixed point of two mappings S and T satisfying (AU-N). Let $F(T)$ denote the set of all fixed points of T. We first prove the following lemmas.

Lemma 8 *Let E be a normed space and C its nonempty convex subset. Let $S, T : C \to C$ be two mappings satisfying (AU-N). Let $\{x_n\}$ be the sequence as defined in (1.4) with $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.*

Proof: Assume that

$$M = \max\{\sup_{n\geq 1} \|u_n - x^*\|, \sup_{n\geq 1} \|v_n - x^*\|\},$$

and $F(S) \cap F(T) \neq \phi$. Let $x^* \in F(S) \cap F(T)$. Then

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a_n x_n + b_n S y_n + c_n u_n - x^*\| \\ &= \|a_n(x_n - x^*) + b_n(S y_n - x^*) + c_n(u_n - x^*)\| \\ &\leq a_n \|x_n - x^*\| + b_n \|S y_n - x^*\| + c_n \|u_n - x^*\| \\ &\leq (1 - b_n) \|x_n - x^*\| + b_n \|S y_n - T x^*\| + M c_n \\ &\leq (1 - b_n) \|x_n - x^*\| + b_n \|y_n - x^*\| + M c_n. \end{aligned} \tag{2.1}$$

$$\begin{aligned} \|y_n - x^*\| &= \|a'_n x_n + b'_n T x_n + c'_n v_n - x^*\| \\ &= \|a'_n(x_n - x^*) + b'_n(T x_n - x^*) + c'_n(v_n - x^*)\| \\ &\leq a'_n \|x_n - x^*\| + b'_n \|T x_n - x^*\| + c'_n \|v_n - x^*\| \\ &\leq (1 - b'_n) \|x_n - x^*\| + b'_n \|T x_n - S x^*\| + M c'_n \\ &\leq (1 - b'_n) \|x_n - x^*\| + b'_n \|x_n - x^*\| + M c'_n \\ &= \|x_n - x^*\| + M c'_n. \end{aligned} \tag{2.2}$$

Substituting (2.2) in (2.1) yields

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + M(b_n c'_n + c_n).$$

Using Lemma 2, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for each $x^* \in F(S) \cap F(T)$, and the sequence $\{x_n\}$ is bounded. □

Lemma 9 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S, T : C \to C$ be two mappings satisfying (AU-N) and $\{x_n\}$ be the sequence as defined in (1.4) with $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $\limsup_{n\to\infty} b'_n < 1$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|S x_n - x_n\| = 0 = \lim_{n\to\infty} \|T x_n - x_n\|$.*

Proof: Assume that

$$M_1 = \max\{\sup_{n\geq 1} \|x_n - x^*\|, \sup_{n\geq 1} \|u_n - x^*\|, \sup_{n\geq 1} \|v_n - x^*\|\}.$$

By Lemma 8, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Suppose $\lim_{n\to\infty} \|x_n - x^*\| = c$ for some $c \geq 0$.

Taking limsup on both the sides of (2.2), we have

$$\limsup_{n\to\infty} \|y_n - x^*\| \leq c. \tag{2.3}$$

Next consider

$$\begin{aligned} \|S y_n - x^* + c_n(u_n - x_n)\| &\leq \|S y_n - x^*\| + c_n \|u_n - x_n\| \\ &\leq \|S y_n - T x^*\| + 2M_1 c_n \\ &\leq \|y_n - x^*\| + 2M_1 c_n. \end{aligned}$$

Taking limsup on both the sides in the above inequality and then using (2.3), we get that

$$\limsup_{n\to\infty} \|Sy_n - x^* + c_n(u_n - x_n)\| \leq c.$$

Also

$$\begin{aligned} \|x_n - x^* + c_n(u_n - x_n)\| &\leq \|x_n - x^*\| + c_n \|u_n - x_n\| \\ &\leq \|x_n - x^*\| + 2M_1 c_n, \end{aligned}$$

gives that

$$\limsup_{n\to\infty} \|x_n - x^* + c_n(u_n - x_n)\| \leq c.$$

Further, $\lim_{n\to\infty} \|x_{n+1} - x^*\| = c$ means that

$$\lim_{n\to\infty} \|(1-b_n)(x_n - x^* + c_n(u_n - x_n)) + b_n(Sy_n - x^* + c_n(u_n - x_n))\| = c.$$

Hence applying Lemma 1, we obtain that

$$\lim_{n\to\infty} \|x_n - Sy_n\| = 0. \tag{2.4}$$

Next consider

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - Sy_n\| + \|Sy_n - Tx_n\| \\ &\leq \|x_n - Sy_n\| + \|y_n - x_n\|. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \|y_n - x_n\| &= \|a'_n x_n + b'_n Tx_n + c'_n v_n - x_n\| \\ &\leq b'_n \|x_n - Tx_n\| + c'_n \|v_n - x_n\| \\ &\leq b'_n \|x_n - Tx_n\| + 2M_1 c'_n. \end{aligned} \tag{2.6}$$

Substituting (2.6) in (2.5), we get

$$\|x_n - Tx_n\| \leq \|x_n - Sy_n\| + b'_n \|x_n - Tx_n\| + 2M_1 c'_n,$$

implies

$$\|x_n - Tx_n\| \leq \frac{1}{1-b'_n} \|x_n - Sy_n\| + 2M_1 \frac{c'_n}{1-b'_n},$$

gives us with the help of condition $\limsup_{n\to\infty} b'_n < 1$,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.7}$$

Now observe that

$$\begin{aligned}
\|x_{n+1}-Sx_{n+1}\| &= \|a_nx_n+b_nSy_n+c_nu_n-Sx_{n+1}\| \\
&= \|(1-b_n)x_n+b_nSy_n+c_n(u_n-x_n)-Sx_{n+1}\| \\
&= \|(1-b_n)(x_n-Sx_{n+1})+b_n(Sy_n-Sx_{n+1})+c_n(u_n-x_n)\| \\
&\leq (1-b_n)\|x_n-Sx_{n+1}\|+b_n\|Sy_n-Sx_{n+1}\|+c_n\|u_n-x_n\| \\
&\leq (1-b_n)(\|x_n-x_{n+1}\|+\|x_{n+1}-Sx_{n+1}\|) \\
&\quad +b_n(\|Sy_n-Tx_n\|+\|Tx_n-Sx_{n+1}\|)+2M_1c_n \\
&\leq (1-b_n)(\|x_n-x_{n+1}\|+\|x_{n+1}-Sx_{n+1}\|) \\
&\quad +b_n(\|y_n-x_n\|+\|x_n-x_{n+1}\|)+2M_1c_n \\
&= \|x_n-x_{n+1}\|+(1-b_n)\|x_{n+1}-Sx_{n+1}\|+b_n\|y_n-x_n\| \\
&\quad +2M_1c_n,
\end{aligned}$$

implies

$$\begin{aligned}
\|x_{n+1}-Sx_{n+1}\| &\leq \frac{1}{b_n}\|x_n-x_{n+1}\|+\|y_n-x_n\|+2M_1\frac{c_n}{b_n} \\
&\leq \frac{1}{\delta}\|x_n-x_{n+1}\|+\|y_n-x_n\|+2M_1\frac{c_n}{\delta}. \qquad (2.8)
\end{aligned}$$

Also

$$\begin{aligned}
\|x_n-x_{n+1}\| &= \|x_n-a_nx_n+b_nSy_n+c_nu_n\| \\
&\leq b_n\|x_n-Sy_n\|+c_n\|u_n-x_n\| \\
&\leq (1-\delta)\|x_n-Sy_n\|+2M_1c_n. \qquad (2.9)
\end{aligned}$$

Substituting (2.6) and (2.9) in (2.8) yields

$$\|x_{n+1}-Sx_{n+1}\| \leq \frac{1-\delta}{\delta}\|x_n-Sy_n\|+b_n'\|x_n-Tx_n\|+4M_1\frac{c_n}{\delta}+2M_1c_n',$$

implies

$$\lim_{n\to\infty}\|x_{n+1}-Sx_{n+1}\|=0.$$

Thus

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

Hence

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0=\lim_{n\to\infty}\|Tx_n-x_n\|.$$

This completes the proof of the lemma. □

Theorem 1 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S, T and $\{x_n\}$ be as taken in Lemma 9. If $F(S)\cap F(T)\neq\phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof: Let $x^* \in F(S) \cap F(T)$. Then as proved in Lemma 8, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S) \cap F(T)$. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 9, $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 3, therefore we obtain $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$. Again in the same way, we can prove that $z_2 \in F(S) \cap F(T)$. Next, we prove the uniqueness. For this suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in $F(S) \cap F(T)$. □

The following condition is due to Senter and Dotson [12].

Definition 2 *A mapping $T : C \to C$ where C is a subset of E, is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.*

Senter and Dotson [12] approximated fixed points of a nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [7] and Tan and Xu [14] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same condition (A) which is weaker than the the requirement that T is demicompact.

We modify the condition (A) and (A') for two mappings $S, T : C \to C$ as follows:

Definition 3 *Two mappings $S, T : C \to C$ where C a subset of E, are said to satisfy condition (AR) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ and $\lambda \in [0,1]$ such that $\lambda \|x - Tx\| + (1-\lambda)\|x - Sx\| \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - x^*\| : x^* \in F = F(S) \cap F(T)\}$.*

Note that condition (AR) reduces to condition (A) when $S = T$ and (A') if we take $\lambda = \frac{1}{2}$. We shall use condition (AR) instead of compactness of C to study the strong convergence of $\{x_n\}$ defined in (1.4). It is worth noting that in case of two mappings $S, T : C \to C$ satisfying (AU-N), condition (AR) is weaker than the compactness of C.

Theorem 2 *Let E be a uniformly convex Banach space and C, S, T and $\{x_n\}$ be as taken in Lemma 9. Further let $S, T : C \to C$ be two mappings satisfying condition (AR). If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof: By Lemma 8, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Lemma 9, $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$. Moreover, $\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + M(b_n c'_n + c_n)$ for all $x^* \in F(S) \cap F(T)$. This implies that $d(x_{n+1}, F) \leq d(x_n, F) + (b_n c'_n + c_n)$ gives that $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of Lemma 2. Now by condition (AR), $\lim_{n\to\infty} f(d(x_n, F) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of proof is the same as provided by Tan and Xu [14]. □

Lemma 10 *Let E be a normed space and C its nonempty convex subset. Let $S, T : C \to C$ be two mappings satisfying (AU-N) and $\{x_n\}$ be the sequence as defined in (1.3). If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.*

Lemma 11 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S, T : C \to C$ be two mappings satisfying (AU-N) and $\{x_n\}$ be the sequence as defined in (1.3) with $\{b_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ and $\limsup_{n\to\infty} b'_n < 1$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$.*

Theorem 3 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S, T and $\{x_n\}$ be as taken in Lemma 11. If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Theorem 4 *Let E be a uniformly convex Banach space and C, S, T and $\{x_n\}$ be as taken in Lemma 11. Further let $S, T : C \to C$ be two mappings satisfying condition (AR). If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Lemma 12 *Let E be a normed space and C its nonempty convex subset. Let $T : C \to C$ be a nonexpansive mapping and $\{x_n\}$ be the sequence as defined in (M) with $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If $F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$.*

Lemma 13 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $T : C \to C$ be a nonexpansive mapping and $\{x_n\}$ be the sequence as defined in (M) with $\{b_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. If $F(T) \neq \phi$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Theorem 5 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, T and $\{x_n\}$ be as taken in Lemma 13. If $F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of T.*

Theorem 6 *Let E be a uniformly convex Banach space and C, T and $\{x_n\}$ be as taken in Lemma 13. Furthere let $T : C \to C$ be a nonexpansive mapping satisfying condition (A). If $F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a fixed point of T.*

References

[1] **Agarwal, R. P., Cho, Y. J., Li, J.**, and **Huang, N. J. :** *Stability of iterative procedures with errors approximating common fixed points for a couple of quasi-contractive mappings in q-uniformly smooth Banach spaces.* J. Math. Anal. Appl. **272**, 435-447, (2002)

[2] **Browder, F. E. :** *Nonlinear operators and nolinear equations of evolution in Banach spaces.* "Proc. Symp. Pure Math.", Vol. 18, Proc. Amer. Math. Soc., Providence, RI, 1976

[3] **Chidume, C. E.**, and **Moore, Chika :** *Fixed points iteration for pseudocontractive maps.* Proc. Amer. Math. Soc. **127(4)**, 1163-1170, (1999)

[4] **Das, G.**, and **Debata, J. P. :** *Fixed points of Quasi-nonexpansive mappings.* Indian J. Pure. Appl. Math. **17**, 1263-1269, (1986)

[5] **Ishikawa, S. :** *Fixed points by a new iteration method.* Proc. Amer. Math. Soc. 44, 147-150, (1974)

[6] **Khan, S. H. et al. :** *Weak and strong convergence of a scheme with errors for two nonexpansive mappings.* Nonlinear Anal. **61**, 1295-1301, (2005)

[7] **Maiti, M.**, and **Gosh, M. K. :** *Approximatig fixed points by Ishikawa iterates.* Bull. Austral. Math. Soc. **40**, 113-117, (1989)

[8] **Mann, W. R. :** *Mean value methods in iteration.* Proc. Amer. Math. Soc. 4, 506-510, (1953)

[9] **Opial, Z. :** *Weak convergence of the sequence of successive approximations for nonexpansive mappings.* Bull. Amer. Math. Soc. **73**, 591-597, (1967)

[10] **Rhoades, B. E. :** *Fixed point iterations for certain nonlinear mappings.* J. Math. Anal. Appl. **183**, 118-120, (1994)

[11] **Schu, J. :** *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings.* Bull. Austral. Math. Soc. **43**, 153-159, (1991)

[12] **Senter, H. F.**, and **Dotson, W. G. :** *Approximatig fixed points of nonexpansive mappings.* Proc. Amer. Math. Soc. **44(2)**, 375-380, (1974)

[13] **Takahashi, W.**, and **Tamura, T. :** *Convergence theorems for a pair of nonexpansive mappings.* J. Convex Analysis **5(1)**, 45-58, (1998)

[14] **Tan, K. K.**, and **Xu, H. K. :** *Approximatig fixed points of nonexpansive mappings by the Ishikawa iteration process.* J. Math. Anal. Appl. **178**, 301–308, (1993)

[15] **Xu, Y. :** *Ishikawa and Mann Iteration process with errors for nonlinear strongly accretive operator equations.* J. Math. Anal. Appl. **224**, 91-101, (1998)

received: May 9, 2006

Author:

Arif Rafiq
COMSATS
Institute of Information Technology,
Islamabad,
Pakistan

e-mail: arafiq@comsats.edu.pk

Hints for Authors

Rostock. Math. Kolloq. appears once or twice per year.

Submission
Papers should be sent by e-mail to

romako@uni-rostock.de

or by mail to

Universität Rostock
Institut für Mathematik
Universitätsplatz 1
D-18051 Rostock

All papers will be reviewed. Only original contributions will be considered.

AMS-Subject-Classification
Please add one or two AMS-classification numbers which describe the content of your paper.

Manuscript Format
Papers should be written in German or English. Please send your article as a Latex-file and a pdf-file or a ps-file. The Latex-file should not contain self defined commands.

Authors Adresses
Please add the current complete adresses of all authors including first name / surname / institution / department / street / house number / postal code / place / country / e-mail-address.

Bibliography
Current numbers within the text ([3], [4]; [7, 8, 9]) refer to the corresponding item in the bibliography at the end of the article, which has the headline References. Please follow the structure of the examples:

[3] **Zariski, O.**, and **Samuel, P.:** *Commutative Algebra.* Princeton 1958

[4] **Steinitz, E.:** *Algebraische Theorie der Körper.* J. Reine Angew. Math. **137**, 167-309 (1920)

[8] **Gnedenko, B.W.:** *Über die Arbeiten von C.F. Gauß zur Wahrscheinlichkeitsrechnung.* In: Reichardt, H. (Ed.): C.F. Gauß, Gedenkband anläßlich des 100. Todestages. S. 193-204, Leipzig 1957

Each citation should be written in the original language. Cyrillic letters must be transliterated as it is usual in libraries.

Hinweise für Autoren

Das Rostock. Math. Kolloq. erscheint ein- bis zweimal pro Jahr.

Einreichen
Senden Sie bitte Ihre Arbeiten per e-mail an

romako@uni-rostock.de

oder per Post an

Universität Rostock
Institut für Mathematik
Universitätsplatz 1
D-18051 Rostock

Alle Arbeiten werden begutachtet. Wir berücksichtigen nur Originalarbeiten.

AMS-Subject-Klassifikation
Bitte geben Sie zur inhaltlichen Einordnung der Arbeit ein bis zwei Klassifizierungsnummern (AMS-Subject-Classification) an.

Manuskript
Manuskripte sollen in Deutsch oder Englisch abgefasst sein. Bitte senden Sie uns Ihre Arbeit als als LaTeX- und als PDF–Datei bzw. PS-Datei. In der LaTeX-Datei sollten selbst definierte Befehle vermieden werden.

Adressen der Autoren
Die aktuelle, vollständige Adresse des Autors sollte enthalten: Vornamen Name / Institution / Struktureinheit / Straße Hausnummer / Postleitzahl Ort / Land / e-mail-Adresse.

Literaturzitate
Literaturzitate sind im Text durch laufende Nummern (vgl. [3], [4]; [7, 8, 10]) zu kennzeichnen und am Schluss der Arbeit unter der Zwischenüberschrift **Literatur** zusammenzustellen. Hierbei ist die durch die nachfolgenden Beispiele veranschaulichte Form einzuhalten.

[3] **Zariski, O.**, and **Samuel, P.:** *Commutative Algebra.* Princeton 1958

[4] **Steinitz, E.:** *Algebraische Theorie der Körper.* J. Reine Angew. Math. **137**, 167-309 (1920)

[8] **Gnedenko, B.W.:** *Über die Arbeiten von C.F. Gauß zur Wahrscheinlichkeitsrechnung.* In: Reichardt, H. (Ed.): C.F. Gauß, Gedenkband anläßlich des 100. Todestages. S. 193-204, Leipzig 1957

Die Angaben erfolgen in Originalsprache; bei kyrillischen Buchstaben sollte die (bibliothekarische) Transliteration verwendet werden.