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Symmetric Proximal Hypertopology

Dedicated to our friend Professor Dr. Harry Poppe on his 70^th^ birthday.

ABSTRACT. In 1988 a new hypertopology, called **proximal (finite) hypertopology** was discovered. It involves the use of a proximity in the upper part but leaves the lower part the same as the lower Vietoris topology. In 1966, the lower Vietoris topology, which involves finitely many open sets, led to another lower topology involving locally finite families of open sets. In this paper, we change the lower hypertopology using a proximity and thus get a "symmetric" proximal hypertopology, which includes the earlier 'finite' topologies.

KEY WORDS AND PHRASES. Proximities, hyperspace, lower proximal hypertopology, upper proximal hypertopology, symmetric proximal hypertopology, Bombay hypertopology.

1 INTRODUCTION

Let (X, d) be a metric space and let $\mathrm{CL}(X)$ denote the family of all nonempty closed subsets of X. In 1914 Hausdorff defined a metric d_H on $\mathrm{CL}(X)$, which is now known as the **Hausdorff metric** ([10]). In 1922 Vietoris defined, for a T_1 space X, a topology $\tau(\mathrm{V})$, now called the **Vietoris or finite topology** on $\mathrm{CL}(X)$ ([24, 25] and [15]).

The Hausdorff metric topology was first generalized to uniform spaces by Bourbaki and later (1966) to the Wijsman topology ([26]), wherein the convergence of distance functionals is *pointwise* rather than uniform as in the Hausdorff case. A base for the Vietoris topology consists of two parts, the lower which involves the intersection of closed sets with finitely many open sets and the upper part involving closed sets contained in one open set.

In 1962 Fell in [9] changed the upper part by considering the complements of compact sets and this was further generalized by Poppe (see [19] and [20]) in 1966 to complements of members of Δ, a subfamily of $\mathrm{CL}(X)$. Also in 1966 Marjanovic in [14] altered the lower part to consist of intersection of closed sets with members of locally finite families of open sets.

In 1988 the upper part of τ(V) was generalized to consist of closed sets that are proximally contained in an open set while the lower part could be finite or locally finite as before ([7]). It was found that even the well known Hausdorff metric topology is essentially a kind of locally finite proximal topology, thus showing the importance of proximities in hyperspaces. Moreover, recently it was shown that, with the use of two proximities in the upper part, all known hypertopologies could be subsumed under one **Bombay topology** ([6], see also [17]). In the present paper we radically change the lower part by using as a base, families of closed sets that are *proximally near* finitely many open sets. This is the first time that proximities are used in the lower part. The use of proximities in both upper and lower parts yield symmetric proximal hypertopologies and we believe that they will play an important role in the literature. In fact, since a topological space X has an infinite spectrum of proximities compatible with its topology it is possible to have an unusually broad range of symmetrical proximal hypertopologies because our constructs are based on proximal relations between families of closed sets and finite collections of open subsets instead of the usual Boolean operations on open and closed subsets of X, namely unions, intersections, set inclusions.

In what follows (X, τ), (or X), always denotes a T_1 topological space. A binary relation δ on the power set of X is a *generalized proximity* iff

(i) $A\ \delta\ B$ implies $B\ \delta\ A$;

(ii) $A\ \delta\ (B \cup C)$ implies $A\ \delta\ B$ or $A\ \delta\ C$;

(iii) $A\ \delta\ B$ implies $A \neq \emptyset$, $B \neq \emptyset$;

(iv) $A \cap B \neq \emptyset$ implies $A\ \delta\ B$.

A generalized proximity δ is a *LO-proximity* iff it satisfies

(LO) $A\ \delta\ B$ and $b\ \delta\ C$ for every $b \in B$ together imply $A\ \delta\ C$.

Moreover, a LO-proximity δ is a *LR-proximity* iff it satisfies

(R) $x\ \underline{\delta}\ A$ (where $\underline{\delta}$ means the negation of δ) implies there exists $E \subset X$ such that $x\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ A$.

A generalized proximity δ is an *EF-proximity* iff it satisfies

(EF) $A\ \underline{\delta}\ B$ implies there exists $E \subset X$ such that $A\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ B$.

Note that each EF-proximity is a LR-proximity.

Whenever δ is a LO-proximity, $\tau(\delta)$ denotes the topology on X induced by the Kuratowski closure operator $A \to A^{\delta} = \{x \in X : x\ \delta\ A\}$. The proximity δ is declared compatible with respect to the topology τ iff $\tau = \tau(\delta)$ (see [8], [18] or [27]).

A T_1 topological space X admits always a compatible LO-proximity. A topological space X has a compatible LR- (respectively, EF-) proximity iff it is T_3 (respectively, Tychonoff).

If $A\ \delta\ B$, then we say A is δ-near to B; if $A\ \underline{\delta}\ B$ we say A is δ-far from B. A is declared δ-strongly included in B, written $A \ll_{\delta} B$, iff $A\ \underline{\delta}\ B^c$. $A\ \underline{\ll}_{\delta}\ B$ stands for its negation, i.e. $A\ \delta\ B^c$.

We assume, in general, that every compatible proximity δ on X is LO or even LR. These assumptions simplify the results and allow us to display readable statements and makes the theory transparent. In fact, it is a useful fact that in a LO-proximity δ two sets are δ-far iff their closures are δ-far (see [18] or [3]). Moreover, if δ is a compatible LR-proximity, then

(∗) for each $x \notin$ A, with $A = \mathrm{cl}\, A$, there is an open neighbourhood W of x such that $\mathrm{cl}\, W\ \underline{\delta}\ A$.

We use the following notation :

$\mathsf{N}(x)$ denotes the filter of open neighbourhoods of $x \in X$;

$\mathrm{CL}(X)$ is the family of all non-empty closed subsets of X;

$\mathsf{K}(X)$ is the family of all non-empty compact subsets of X.

We set $\Delta \subset \mathrm{CL}(X)$ and assume, without any loss of generality, that it contains all singletons and finite unions of its members.

In the sequel δ, η will denote compatible proximities on X.

The most popular and well studied proximity is the Wallman or fine LO-proximity δ_0 (or η_0) given by

$$A\ \delta_0\ B \Leftrightarrow \mathrm{cl}\, A \cap \mathrm{cl}\, B \neq \emptyset\,.$$

We note that δ_0 is a LR-proximity iff X is regular (see [11, Lemma 2]). Moreover, δ_0 is an EF-proximity iff X is normal (Urysohn Lemma).

Another useful proximity is the discrete proximity η^* given by

$$A\ \eta^*\ B \Leftrightarrow A \cap B \neq \emptyset\,.$$

We note that η^* is not a compatible proximity, unless (X,τ) is discrete.

We point out that in this paper η^* is the only proximity that might be non compatible.

For any open set E in X, we use the notation

$$E_\delta^+ = \{F \in \mathrm{CL}(X) : F \ll_\delta E \text{ or equivalently } F \,\underline{\delta}\, E^c\} \; .$$
$$E^+ = E_{\delta 0}^+ = \{F \in \mathrm{CL}(X) : F \ll_{\delta 0} E \text{ or equivalently } F \subset E\} \; .$$
$$E_\eta^- = \{F \in \mathrm{CL}(X) : F \,\eta\, E\} \; .$$
$$E^- = E_{\eta^*}^- = \{F \in \mathrm{CL}(X) : F \,\eta^*\, E \text{ or equivalently } F \cap E \neq \emptyset\} \; .$$

Now we have the material necessary to define the basic proximal hypertopologies.

(1.1) The lower $\boldsymbol{\eta}$-proximal topology $\boldsymbol{\sigma(\eta^-)}$ is generated by $\{E_\eta^- : E \in \tau\}$.

(1.2) The upper $\boldsymbol{\delta - \Delta}$-proximal topology $\boldsymbol{\sigma(\delta^+, \Delta)}$ is generated by $\{E_\delta^+ : E^c \in \Delta\}$.
We omit Δ and write it as $\sigma(\delta^+)$ if $\Delta = \mathrm{CL}(X)$.

We note that the upper Vietoris topology $\tau(\mathsf{V}^+)$ equals $\sigma(\delta_0^+)$ and the lower Vietoris topology $\tau(\mathsf{V}^-)$ can be written as $\sigma(\eta^{*-})$. These show that proximal topologies are generalizations of the classical upper and lower Vietoris topologies.

Moreover, we have:

The **Vietoris** topology $\tau(\mathsf{V}) = \sigma(\eta^{*-}) \vee \sigma(\delta_0^+) = \tau(\mathsf{V}^-) \vee \tau(\mathsf{V}^+)$.

The **$\boldsymbol{\delta}$-proximal topology** $\sigma(\delta) = \sigma(\eta^{*-}) \vee \sigma(\delta^+) = \tau(\mathsf{V}^-) \vee \sigma(\delta^+)$.

The following is a *new* hypertopology, which generalizes the above as well as the Δ-topologies introduced by Poppe ([19], [20]).

(1.3) The $\boldsymbol{\eta - \delta - \Delta}$-symmetric-proximal topology $\pi(\eta, \delta; \Delta) = \sigma(\eta^-) \vee \sigma(\delta^+; \Delta)$.

For references on proximities, we refer to [8], [18] and [27]. For LO-proximities see [18] and [16]. For LR-proximities see [11], [12] and also [3] and [4].
For references on hyperspaces up to 1993, we refer to [1], except when a specific reference is needed.

2 Three Important Lower Proximal Topologies

Let (X, d) be a metric space. The associated metric proximity η is defined by $A \ \eta \ B$ iff $\inf\{d(a,b) : a \in A, b \in B\} = 0$. Note that a metric proximity η is EF and compatible.

First we observe that, unlike the lower Vietoris topology, $\sigma(\eta^-)$ need not be admissible as the following example shows. The reader is referred to the Appendix (section 7) where the admissibility of the symmetrical proximal topology is investigated in details.

Example 2.1 Let $X = [-1, 1]$ with the metric proximity η. Let $A = \{0\}$, $A_n = \{\frac{1}{n}\}$, for all $n \in \mathbb{N}$. Then $\frac{1}{n}$ converges to 0 in X, but A_n does not converge to A in the topology $\sigma(\eta^-)$. Hence the map $i : X \to \mathrm{CL}(X)$, where $i(x) = \{x\}$, is not an embedding.

We recall that if η and η' are proximities on X, then η is declared *coarser* than η' (or equivalently η' is *finer* than η), written $\eta \leq \eta'$, iff $A \ \underline{\eta} \ B$ implies $A \ \underline{\eta'} \ B$.

We now begin to study lower proximal topologies corresponding to proximities η, η_0, η^*. Observe that for any compatible proximity η we have always $\eta \leq \eta_0 \leq \eta^*$ (since η_0 is the finest compatible LO-proximity and η^* is the discrete proximity). In the case of a metric space, we will take η to be the metric proximity and, in the case of a T_3 topological space, we will take η to be a compatible LR-proximity.

Theorem 2.2 *Let (X, τ) be a T_3 topological space and η a compatible LR-proximity. The following inclusions occur:*

(a) $\tau(V^-) = \sigma(\eta^{*-}) \subset \sigma(\eta^-)$,

(b) $\tau(V^-) = \sigma(\eta^{*-}) \subset \sigma(\eta_0^-)$,

i.e. the finest proximity η^* induces a coarser hypertopology.

Proof: We show (a). Suppose that the net $\{A_\lambda\}$ of closed sets converges to a closed set A in the topology $\sigma(\eta^-)$. If $A \ \eta^* \ U$, where $U \in \tau$, then there is an $x \in A \cap U$ and a $V \in \mathsf{N}(x)$, with $x \in V \subset \mathrm{cl}\, V \subset U$ and $V \ \underline{\eta} \ U^c$ (use $(*)$ in Section 1). We claim that eventually A_λ intersects U. For if not, then frequently $A_\lambda \subset U^c$ and so frequently $A_\lambda \ \underline{\eta} \ V$; a contradiction.

The case (b) is similar.

Remark 2.3 In Section 6, namely Theorem 6.7, we characterize proximities for which the above inclusions hold. Furthermore, we will look for transparent conditions under which a finer proximity induces a coarser hypertopology, a natural phenomenon.

This example shows that the assumption of regularity of the base space X cannot be dropped in Theorem 2.2.

Example 2.4 Let $X = [0, +\infty)$ with the topology τ consisting of the usual open sets together with all sets of the form $U = [0, \varepsilon) \backslash B$ where $\varepsilon > 0$ and $B \subset A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then X is Hausdorff and not regular since $V = [0, \varepsilon') \backslash A \subset U$ for each $0 < \varepsilon' < \varepsilon$, but $\mathrm{cl}\, V \not\subset U$.

Now, set $F = \{0\}$ and for all $n \in \mathbb{N}$, $F_n = \{\frac{1}{n}\}$. Then $\{F_n\}$ does not converge to F in $\sigma(\eta^{*^-})$ (note that $0 \in [0, \varepsilon) \backslash A$ and $\frac{1}{n} \notin [0, \varepsilon) \backslash A$ for all $n \in \mathbb{N}$). However, if $\{0\}\ \eta\ W$, W open, then eventually $\{\frac{1}{n}\}\ \eta\ W$, where η is any compatible LO-proximity. So

$$\sigma(\eta^{*^-}) \not\subset \sigma(\eta^-)\,.$$

Now, we give an example to show that the inclusions in 2.2 (a), (b) are strict except in pathological situations.

Example 2.5 Let $X = [0, 2]$, $A = [0, 1]$, and for each natural number n set $A_n = [0, 1 - \frac{1}{n}]$. Then the sequence $\{A_n\}$ converges to A with respect to the $\sigma(\eta^{*^-})$ topology, but it converges to A neither with respect to the $\sigma(\eta^-)$ topology nor with respect to the $\sigma(\eta_0^-)$ topology.

We note that the space involved is one of the "best" possible spaces and the sets involved are also compact.

In a UC metric space, i.e. one in which the metric proximity $\eta = \eta_0$, $\sigma(\eta_0^-) = \sigma(\eta^-)$.

We now give an example to show that $\sigma(\eta_0^-) \neq \sigma(\eta^-)$.

Example 2.6 Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{M} = \{n - \frac{1}{n} : n \in \mathbb{N}\}$, $X = \mathbb{N} \cup \mathbb{M}$ as subspace of the real line. X is not a UC space. Set $A = \mathbb{N}$ and $A_n = \{m \in \mathbb{N} : m < n\}$, for all $n \in \mathbb{N}$.

Then A_n converges to A in the topology $\sigma(\eta_0^-)$. However, $A\ \eta\ \mathbb{M}$ but $A_n\ \underline{\eta}\ \mathbb{M}$ for each $n \in \mathbb{N}$. So, the sequence $\{A_n\}$ does not converge to A in the topology $\sigma(\eta^-)$.

Note that in the above example $\sigma(\eta_0^-) \subset \sigma(\eta^-)$. We show below that this natural inclusion holds in a locally compact space, too.

Theorem 2.7 *Let (X, τ) be a locally compact Hausdorff space. If η and η_0 are respectively a compatible LO-proximity and the Wallman proximity on X, then:*

$$\sigma(\eta_0^-) \subset \sigma(\eta^-)\,.$$

Proof: Let V be an open set and let $A \in V^-_{\eta 0}$. Then there is a $z \in A \cap \mathrm{cl}\, V$. Let U be a compact neighbourhood of z and set $W = U \cap V$. Note that $\mathrm{cl}\, W$ is compact and that a closed set is η-near a compact set iff it is η_0-near it. Hence $A \in W^-_\eta \subset V^-_{\eta 0}$.

Remark 2.8 If the involved proximities η, η_0 are LR and the net of closed sets A_λ is eventually locally finite and converges to A in $\sigma(\eta^-)$, then it also converges in $\sigma(\eta_0^-)$. Hence, the same inclusion (i.e. $\sigma(\eta_0^-) \subset \sigma(\eta^-)$) as in the previous result holds.

We next consider first and second countability of $\sigma(\eta^-)$.
Observe, that the first and second countability of $\sigma(\eta^{*-})$, i.e. the lower Vietoris topology $\tau(V^-)$, are well known results. $(\mathrm{CL}(X), \tau(V^-))$ is first countable if and only if X is first countable and each closed subset of X is separable (cf. Theorem 1.2 in [13]); $(\mathrm{CL}(X), \tau(V^-))$ is second countable if and only if X is second countable (cf. Proposition 1.11 in [13]).
Thus, we study the first and the second countability of $\sigma(\eta^-)$ when η is different from η^*.
The following definitions have a key role.

Definitions 2.9 (see [2]) *Let (X, τ) be a T_1 topological space, η a compatible LO-proximity and $A \in CL(X)$.*

A family N_A of open sets of X is an ***external proximal local base at A with respect to η*** *(or, briefly a* ***η-external proximal local base at A****) if for any U open subset of X such that $A\,\eta\,U$, there exists $V \in \mathsf{N}_A$ satisfying $A\,\eta\,V$ and $cl V \subset cl U$.*

The ***external proximal character of A with respect to η*** *(or, briefly the* ***η-external proximal character of A****) is defined as the smallest (infinite) cardinal number of the form $|\mathsf{N}_A|$, where N_A is a η-external proximal local base at A, and it is denoted by $E\chi(A, \eta)$.*

The ***external proximal character of CL(X) with respect to η*** *(or, briefly the* ***η-external proximal character****) is defined as the supremum of all number $E\chi(A, \eta)$, where $A \in CL(X)$. It is denoted by $E\chi(CL(X), \eta)$.*

Remark 2.10 If X is a T_3 topological space, η is a compatible LR-proximity and the η-external proximal character $\mathrm{E}\chi(\mathrm{CL}(X), \eta)$ is countable, then X is separable .

Theorem 2.11 *Let (X, τ) be a T_3 topological space with a compatible LR-proximity η. The following are equivalent:*

(a) *$(CL(X),\, \sigma(\eta^-))$ is first countable;*

(b) *the η-external proximal character $E\chi(CL(X), \eta)$ is countable.*

Proof: (a)⇒(b). It suffices to show that for each $A \in \mathrm{CL}(X)$ there exists a countable family N_A of open sets of X which is a η-external proximal local base at A. First note a useful fact:

$$\text{For open sets } V, U, \operatorname{cl} V \subset \operatorname{cl} U \;\Leftrightarrow\; V_\eta^- \subset U_\eta^- . \tag{\#}$$

Now, let $A \in \mathrm{CL}(X)$ and Z a countable subbase $\sigma(\eta^-)$-neighbourhood system of A. Set $\mathsf{N}_A = \{V : V$ an open set with $V^-_\eta \in \mathrm{Z}\}$. Clearly, N_A is a countable family of open sets with the property that $A\,\eta\,U$, U open, implies there is a $V \in \mathsf{N}_A$ satisfying $A\,\eta\,V$ and $V^-_\eta \subset U^-_\eta$. Thus the result follows from (#).

(b)⇒(a). It is obvious.

Definitions 2.12 (see [2]) *Let (X,τ) be a T_1 topological space with a compatible LO-proximity η.*

*A family N of open sets of X is an **external proximal base with respect to η** (or, briefly a **η-external proximal base**) if for any closed subset A of X and any open subset U of X with $A\,\eta\,U$, there exists $V \in \mathsf{N}$ satisfying $A\,\eta\,V$ and $clV \subset clU$.*

*The **external proximal weight of $CL(X)$ with respect to η** (or, briefly the **η-external proximal weight of $CL(X)$**) is the smallest (infinite) cardinality of its η-external proximal bases and it is denoted by $EW(CL(X),\eta)$.*

Theorem 2.14 *Let (X,τ) be a T_3 topological space with a compatible LR-proximity η. The following are equivalent:*

(a) *$(CL(X),\sigma(\eta^-))$ is second countable;*

(b) *the η-external proximal weight $EW(CL(X),\eta)$ is countable.*

3 Six Hyperspace Topologies

The three lower topologies $\sigma(\eta^{*^-})$, $\sigma(\eta^-)$, $\sigma(\eta_0^-)$ combined with two upper ones $\sigma(\delta^+;\Delta)$, $\sigma(\delta_0^+;\Delta)$ yield six distinct hypertopologies of which the first two are already well known as we remarked before.

$$\pi(\eta^*,\delta_0;\Delta) = \sigma(\eta^{*^-}) \vee \sigma(\delta_0^+;\Delta) = \tau(\Delta)\,, \tag{1}$$

the Δ-topology which equals the Vietoris topology when $\Delta = \mathrm{CL}(X)$ and equals the Fell topology when $\Delta = \mathsf{K}(X)$.

$$\pi(\eta^*,\delta;\Delta) = \sigma(\eta^{*^-}) \vee \sigma(\delta^+;\Delta) = \sigma(\delta;\Delta)\,, \tag{2}$$

the proximal-Δ-topology which equals the proximal topology when $\Delta = \mathrm{CL}(X)$ and equals the Fell topology when $\Delta = \mathsf{K}(X)$ and δ is a LR-proximity.

$$\pi(\eta,\delta_0;\Delta) = \sigma(\eta^-) \vee \sigma(\delta_0^+;\Delta)\,. \tag{3}$$

$$\pi(\eta,\delta;\Delta) = \sigma(\eta^-) \vee \sigma(\delta^+;\Delta)\,. \tag{4}$$

$$\pi(\eta_0,\delta_0;\Delta) = \sigma(\eta_0^-) \vee \sigma(\delta_0^+;\Delta)\,. \tag{5}$$

$$\pi(\eta_0,\delta;\Delta) = \sigma(\eta_0^-) \vee \sigma(\delta^+;\Delta)\,. \tag{6}$$

Theorem 3.1 *The following relationships hold when X is a T_3 topological space, η_0 is the Wallman proximity and η is a LR-proximity. Moreover, for simplicity we consider $\Delta = CL(X)$.*

(a) $\sigma(\delta) \subset \tau(V) \subset \pi(\eta_0, \delta_0)$.

(b) $\sigma(\delta) \subset \tau(V) \subset \pi(\eta, \delta_0)$.

(c) $\sigma(\delta) \subset \pi(\eta_0, \delta) \subset \pi(\eta_0, \delta_0)$.

(d) $\sigma(\delta) \subset \pi(\eta, \delta) \subset \pi(\eta, \delta_0)$.

(e) $\sigma(\delta) \subset \tau(V) \subset \pi(\eta_0, \delta_0) \subset \pi(\eta, \delta_0)$ *when X is locally compact.*

(f) $\sigma(\delta) \subset \pi(\eta_0, \delta) \subset \pi(\eta_0, \delta_0) \subset \pi(\eta, \delta_0)$ *when X is locally compact.*

Let (X, τ) be a T_3 topological space with a compatible LR-proximity δ. Then X is called a **PC space** if $\delta = \delta_0$. In such a space, every continuous function on X to an arbitrary proximity space is proximally continuous. In case X is a metric space with the metric proximity δ, then X is PC if and only if X is UC (i.e. continuous functions on X are uniformly continuous).

Theorem 3.2 *Let (X, τ) be a T_3 topological space. The following are equivalent:*

(a) *X is PC;*

(b) $\sigma(\delta) = \tau(V)$*;*

(c) $\pi(\eta_0, \delta) = \pi(\eta_0, \delta_0)$*;*

(d) $\pi(\eta, \delta_0) \subset \pi(\eta, \delta)$.

4 Properties of $(\mathbf{CL}(\mathbf{X}), \pi(\eta, \delta))$

We now study some properties of the topological space $(\mathrm{CL}(X), \pi(\eta, \delta))$.
In general, the map $x \to \{x\}$ from (X, τ) into $(\mathrm{CL}(X), \sigma(\eta^-))$ fails to be an embedding (see Example 2.1). As a result, the topology $\pi(\eta, \delta)$ in general is not admissible. The admissibility of $\pi(\eta, \delta)$ is studied in Appendix (see section 7).
From 2.2 we know that if the space X is regular and the involved proximities are LR, then $(\mathrm{CL}(X), \pi(\eta, \delta))$ is Hausdorff. We begin with some lemmas.

The following is a generalization of the well known result: sets of the form $< U_k^- >= \{E : E \cap U_k \neq \emptyset$ and $E \subset U = \cup U_k\}$ form a base for the Vietoris topology, where $\{U, U_k\}$ is a finite family of open sets.

We recall that given two proximity δ and η on X, $\delta \leq \eta$ iff $A \underline{\delta} B$ implies $A \underline{\eta} B$.

Lemma 4.1 *Let (X,τ) be a T_1 topological space and δ, η compatible proximities. If $\delta \leq \eta$, then all sets of the form*

$$< U_k^-, U^+ >_{\eta,\delta} = \{E : E \;\eta\; U_k, E \ll_\delta U \text{ and } \bigcup U_k \subset U\}$$

form a base for the $\pi(\eta,\delta)$ topology.

Proof: If $A \in \bigcap\{V_{k,\eta}^- : 1 \leq k \leq n\} \cap U_\delta^+$, then $A \underline{\delta} U^c$ and $A \;\eta\; V_k$ for each $k \in \{1, \ldots, n\}$. Thus, we may replace each V_k by $V_k \cap U$. In fact, from $\delta \leq \eta$ we have $A \underline{\eta} U^c$ and thus $A \;\eta\; V_k$ iff $A \;\eta\; (V_k \cap U)$.

Remark 4.2 (i) If $\delta \leq \eta$ and D is a dense subset of X, then the family of all finite subsets of D is dense in $(\mathrm{CL}(X), \pi(\eta,\delta;\Delta))$.

(ii) Note that in the above Lemma and in (i) we need only $\delta \leq \eta$, restricted to the pairs (E,W) where E is closed and W is open.

The following is a generalization of the result: $< \mathrm{cl}\, U_k^- >$, $1 \leq k \leq n$, is closed in $\tau(\mathrm{V})$.

First, we need a definition.

Definition 4.3 *Let (X,τ) be a T_1 topological space and δ,η compatible proximities. The hypertopologies $\pi(\delta,\eta)$ and $\pi(\eta,\delta)$ are called **conjugate**.*

Lemma 4.4 *Let (X,τ) be a T_1 topological space with compatible proximities δ, η. If $P =< clU_k^-, clU >_{\eta,\delta}$, $1 \leq k \leq n$, $U = \cup U_k$ then P is closed with respect to its conjugate $\pi(\delta,\eta)$.*

Proof: $A \notin P$ if and only if $A \underline{\ll}_\delta \mathrm{cl}\, U$ for $A \underline{\eta} \mathrm{cl}\, U_k$ for some k which in turn is equivalent to $A \;\delta\; [\mathrm{cl}\, U]^c$ or $A \ll_\eta [\mathrm{cl}\, U_k]^c$.

Remark 4.5 Obviously, if $\eta \leq \delta$ when restricted to the pairs (E,W) where E is closed and W is open, then P is closed also in $\pi(\eta,\delta)$.

We now consider the uniformizability of $(\mathrm{CL}(X), \pi(\eta,\delta))$.

We assume that (X,τ) is Tychonoff, δ is a compatible EF-proximity and η a LO-proximity. Let $\boldsymbol{\mathcal{W}}$ be the unique totally bounded uniformity compatible with δ (see [8] or [18]). It is known that the Hausdorff-Bourbaki or H-B uniformity $\boldsymbol{\mathcal{W}}_{\mathrm{H}}$, which has as a base all sets of the form $W_{\mathrm{H}} = \{(A, B) : A \subset W(B)$ and $B \subset W(A)\}$ induces the proximal finite topology $\sigma(\delta) = \sigma(\eta^{*^-}) \vee \sigma(\delta^+)$ (cf. [7]).

By 2.2 we know that $\sigma(\eta^{*^-}) \subset \sigma(\eta^-)$. So, in order to get $\sigma(\eta^-)$ we have to augment a typical entourage $W_{\mathrm{H}} \in \boldsymbol{\mathcal{W}}_{\mathrm{H}}$ by adding sets of the type

$$P_{\{U_{\mathrm{k}}\}} = \{(A, B) \in \mathrm{CL}(X) \times \mathrm{CL}(X) : A\ \eta\ U_{\mathrm{k}} \text{ and } B\ \eta\ U_{\mathrm{k}}\}$$

for a finite family of open sets $\{U_{\mathrm{k}}\}$.

Then, by a routine argument we have.

Theorem 4.6 *If δ is a compatible EF-proximity, η a LO-proximity on a Tychonoff space (X, τ) and $\boldsymbol{\mathcal{W}}$ the unique totally bounded uniformity compatible with δ, then the family*

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{W}}_{\mathrm{H}} \cup \{W_{\mathrm{H}} \cup P_{\{U_{\mathrm{k}}\}} : W_{\mathrm{H}} \in \boldsymbol{\mathcal{W}}_{\mathrm{H}}, \{U_{\mathrm{k}}\} \text{ finite family of open sets }\}$$

defines a compatible uniformity on $(CL(X), \pi(\eta, \delta))$.

We next study the first and second countability as well as the metrizability of $\pi(\eta, \delta; \Delta)$, where (X, τ) is a T_3 topological space, η, δ compatible LR-proximities on X. Observe that the first and second countability as well as the metrizability of $\pi(\eta, \delta; \Delta)$ when $\eta = \eta^*$, i.e. $\pi(\eta^*, \delta; \Delta) = \sigma(\delta; \Delta)$, have been studied by Di Maio and Hola in [5].
So, we attack the case $\eta \neq \eta^*$.

Theorem 4.7 *Let (X, τ) be a T_3 topological space, η, δ compatible LR-proximities on X with $\delta \leq \eta$ (cf. Lemma 4.1). The following are equivalent:*

(a) *$(CL(X), \pi(\eta, \delta; \Delta))$ is first countable;*

(b) *$(CL(X), \sigma(\eta^-))$ and $(CL(X), \sigma(\delta^+; \Delta))$ are both first countable.*

Proof: (b)⇒(a). Since $\pi(\eta, \delta; \Delta) = \sigma(\eta^-) \vee \sigma(\delta^+; \Delta)$, this implication is clear.
(a)⇒(b). Let $(\mathrm{CL}(X), \pi(\eta, \delta; \Delta))$ be first countable and take $A \in \mathrm{CL}(X)$.
Let $\mathsf{Z} = \{\mathsf{L} =< S_j^-, V^+ >_{\eta,\delta}$, with $A \ll_\delta V, V^c \in \Delta, S_j$ open, $A\ \eta\ S_j$, $\bigcup\{S_j : j \in J\} \subset V$ and J finite$\}$ be a countable local base of A with respect to the topology $\pi(\eta, \delta)$.
We claim that the family $\mathsf{Z}^+ = \{V_\delta^+ : V_\delta^+$ occurs in some $\mathsf{L} \in \mathsf{Z}\} \cup \{\mathrm{CL}(X)\}$ forms a local base of A with respect to the topology $\sigma(\delta^+; \Delta)$. Indeed, if there is no open subset U with $A \ll_\delta U$, $U^c \in \Delta$, then $\mathrm{CL}(X)$ is the only open set in $\sigma(\delta^+; \Delta)$ containing A. If there is U with $A \ll_\delta U$, $U^c \in \Delta$, then U_δ^+ is a $\pi(\eta, \delta)$-nbhd. of A. Hence, there exists $\mathsf{L} \in \mathsf{Z}$ with $\mathsf{L} \subseteq U_\delta^+$. Note, that L cannot be of the form $\mathsf{L} =< S_j^- >_\eta = \bigcap\{(S_j)_\eta^- : j \in J\}$, otherwise by setting $F = A \cup U^c$, we have $F \in \mathsf{L}$, but $F \notin U_\delta^+$, a contradiction. Thus, L has the form $< S_j^-, V^+ >_{\eta,\delta}$. We claim that $V_\delta^+ \subset U_\delta^+$. Assume not and let $E \in V_\delta^+ \setminus U_\delta^+$. Set $F = E \cup A$,

we have $F \in \mathsf{L} \setminus U_\delta^+$, a contradiction.
We now show, that there is a countable local base of A with respect to the topology $\sigma(\eta^-)$. Without any loss of generality we may suppose that in the expression of every element from Z the family of index set J is non-empty, in fact

$$\{V_\delta^+ : A \ll_\delta V, V^c \in \Delta\} = \{V_\delta^+ : A \ll_\delta V, V^c \in \Delta\} \cap \{V_\eta^-\}$$

By Lemma 4.1 if $\mathsf{L} \in \mathsf{Z}$, then $\mathsf{L} =< S_j^-, V^+ >_{\eta,\delta}= V_\delta^+ \cap \{(S_j)_\eta^- : j \in J\}$ where $A \underline{\delta} V^c, V^c \in \Delta, \bigcup\{S_j : j \in J\} \subset V, A\ \eta\ S_j, S_j \in \tau$ for each $j \in J$ and J finite.
Set $\mathsf{Z}^- = \{(S_j)_\eta^- : (S_j)_\eta^-$ occurs in some $\mathsf{L} \in \mathsf{Z}\}$. We claim that the family Z^- forms a local subbase of A with respect to the topology $\sigma(\eta^-)$. Take U open with $A\ \eta\ U$. Then U_η^- is a $\pi(\eta, \delta)$-nbhd. of A. Hence, there exists $\mathsf{L} \in \mathsf{Z}$ with $\mathsf{L} \subset U_\eta^-$. $\mathsf{L} =< S_j^-, V^+ >_{\eta,\delta}= V_\delta^+ \cap \{(S_j)_\eta^- : j \in J\}$. We claim that there exists a $j \in J$ such that $(S_j)_\eta^- \subset U_\eta^-$. It suffices to show that there exists a $j \in J$ such that $S_j \subset U$. assume not and for each $j \in J$ let $x_j \in S_j \setminus U$. The set $F = \{x_j :\ j \in J\} \in \mathsf{L} \setminus U_\eta^-$, a contradiction.

Definitions 4.8 (see [2]) *Let (X, τ) be a T_1 topological space, δ a compatible LO-proximity, A a closed subset of X and $\Delta \subset CL(X)$ a ring.*

A family L_A of open neighbourhoods of A is a ***local proximal base at A with respect to δ*** *(or briefly a* ***δ-local proximal base at A****) if for any open subset U of X with $U^c \in \Delta$ and $A \ll_\delta U$ there exists $W \in \mathsf{L}_A$ such that $A \ll_\delta W$, $W^c \in \Delta$ and $W \subset U$.*

The ***δ-proximal character of A*** *is defined as the smallest (infinite) cardinal number of the form $|\mathsf{L}_A|$, where L_A is a δ-local proximal base at A and it is denoted by $\chi(A, \delta)$.*

The ***δ-proximal character of CL(X)*** *is defined as the supremum of all number $\chi(A, \delta)$, where $A \in CL(X)$, and it is denoted by $\chi(CL(X), \delta)$.*

By Theorems 4.7 and 2.11 we have.

Theorem 4.9 *Let (X, τ) be a T_3 topological space, η, δ compatible LR-proximities on X with $\delta \leq \eta$. The following are equivalent:*

(a) *$(CL(X), \pi(\eta, \delta; \Delta))$ is first countable;*

(b) *the η-external proximal character $E\chi(CL(X), \eta)$ and the δ-proximal character $\chi(CL(X), \delta)$ are both countable.*

For the second countability we have a similar result.

Theorem 4.10 *Let (X, τ) be a T_3 topological space, η, δ compatible LR-proximities on X with $\delta \leq \eta$. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta; \Delta))$ *is second countable;*

(b) $(CL(X), \sigma(\eta^-))$ *and* $(CL(X), \sigma(\delta^+; \Delta))$ *are both second countable.*

Proof: We omit the proof that is similar to that in Theorem 4.7.

Definitions 4.11 *Let* (X, τ) *be a* T_1 *topological space,* δ *a compatible LO-proximity and* $\Delta \subset CL(X)$ *a ring.*

A family $\mathcal{B}$ *of open sets of* X *is a **δ-proximal base with respect to Δ** if whenever* $A \ll_\delta V$ *with* $V^c \in \Delta$*, there exists* $W \in \mathcal{B}$ *such that* $A \ll_\delta W$*,* $W^c \in \Delta$ *and* $W \subset V$*.*

*The **δ-proximal weight of $CL(X)$ with respect to Δ** (or, briefly the **δ-proximal weight with respect to Δ**) is the smallest (infinite) cardinality of its* δ*-proximal base with respect to* Δ *and is denoted by* $W(CL(X); \delta, \Delta)$*.*

Theorem 4.12 *Let* (X, τ) *be a* T_3 *topological space,* η*,* δ *compatible LR-proximities on* X *with* $\delta \leq \eta$*. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta; \Delta))$ *is second countable;*

(b) *the* η*-external proximal weight* $EW(CL(X), \eta)$ *and the* δ*-proximal weight* $W(CL(X); \delta, \Delta)$ *with respect to* Δ *are both countable.*

Theorem 4.13 *Let* (X, τ) *be a Tychonoff space,* δ *a compatible EF-proximity,* η *a compatible LR-proximity and* $\delta \leq \eta$*. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta))$ *is metrizable;*

(b) $(CL(X), \pi(\eta, \delta))$ *is second countable and uniformizable.*

Proof: (b)$\Rightarrow$(a). It follows from the Urysohn Metrization Theorem.
(a)$\Rightarrow$(b). Observe that since CL(X) is first countable, X is separable (use Theorem 4.7 and Remark 2.10). Thus, $(\mathrm{CL}(X), \pi(\eta, \delta))$ is second countable (see (i) in Remarks 4.2).

Corollary 4.14 *Let* (X, τ) *be a Tychonoff space,* δ *a compatible EF-proximity and* η *a compatible LR-proximity on* X*. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta))$ *is metrizable;*

(b) $(CL(X), \sigma(\eta^-))$ *and* $(CL(X), \sigma(\delta^+))$ *are second countable.*

5 Subspace Hypertopologies

It is well known that in the lower Vietoris topology $\tau(\mathrm{V}^-)$ subspace topologies behave nicely, i.e. if $A \in \mathrm{CL}(X)$ then

$$(\mathrm{CL}(A), \tau(\mathrm{V}_A^-)) = (\mathrm{CL}(X), \tau(\mathrm{V}^-)) \cap \mathrm{CL}(A)\,.$$

We now give some examples to show that in case of lower proximal topology $\sigma(\eta^-)$ analogous result is not true and the two topologies are not even comparable.

Example 5.1 Let $X = [0,2] \times [-1,1]$, $Q = [0,2] \times [0,1]$, T the closed triangle with vertices at $(0,-1)$, $(1,0)$, $(2,-1)$.

Let $A = \mathrm{T} \cup [0,2] \times \{0\}$, η the metric proximity on X and $V = (0,2) \times (0,1)$. Let $H = V_\eta^- \cap \mathrm{CL}(A)$ which is an open set in $(\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A)$. Then $H = \{\mathrm{CL}([0,2] \times \{0\})\} \cup \{B \in \mathrm{CL}(A) : (1,0) \in B\}$ and is not open in $(\mathrm{CL}(A), \sigma(\eta_A^-))$, where η_A denotes the induced proximity on A. So, $(\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A) \not\subset (\mathrm{CL}(A), \sigma(\eta_A^-))$.

Next examples show that even the reverse inclusion in general does not occur.

Examples 5.2 1. This is an example of a Hausdorff non-regular space X, having a closed subset A such that $(\mathrm{CL}(A), \sigma(\eta_A^-)) \not\subset (\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A)$.

The space X is the "Irrational Slope Topology" (Example 75 on Page 93 [22]). Let $X = \{(x,y) : y \geq 0, x, y \in Q\}$, θ a fixed irrational number and X endowed with the irrational slope topology τ generated on X by neighbourhoods of the form

$$N_\varepsilon((x,y)) = \{(x,y)\} \cup B_\varepsilon(x + y/\theta) \cup B_\varepsilon(x - y/\theta)$$

where $B_\varepsilon(\zeta) = \{r \in Q : |r - \zeta| < \varepsilon\}$, Q being the rationals on the x-axis. Let η_0 be the Wallman proximity. The set $A = \{(x,y) : y > 0, x, y \in Q\}$ is a closed discrete set. Let $\{(x,1) : x \in Q\} = O$ which is clopen in A. Then there is no open neighbourhood H in X such that $[(H_{\eta 0}^- \cap \mathrm{CL}(X)] \subset O^-$ and the claim.

2. This is a less pathological example. Let $X = l_2$ be the space of square summable sequences of real numbers with the usual norm, θ the origin and $\{e_n : n \in \mathbb{N}\}$ the standard basis of unit vectors. Let X be equipped with the Alexandroff proximity η_1 (i.e. $E\ \eta_1\ F$ iff $\mathrm{cl}\, E \cap \mathrm{cl}\, F \neq \emptyset$ or both $\mathrm{cl}\, E, \mathrm{cl}\, F$ are not compact). Since X is not locally compact, η_1 is not an EF-proximity.

   Let $A = \{\theta\} \cup \{e_n : n \in \mathbb{N}\}$. Then $\{\theta\}$ is clopen , $\{\theta\}\eta_1\{\theta\}$.

   Note that $F' \in \mathrm{CL}(A)$ and $F'\eta_1\{\theta\}$ iff $\theta \in F'$.

   Clearly, $A' = (A - \{\theta\}) \in \mathrm{CL}(A)$ and $A'\underline{\eta}_1\{\theta\}$. However, for each open set V in X, $V\eta_1 A'$, showing thereby that $\{\theta\}_{\eta 1}^-$ is not a member of $\sigma(\eta_1^-) \cap \mathrm{CL}(A)$.

   Hence, $(\mathrm{CL}(A), \sigma(\eta_A^-)) \not\subset (\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A)$.

6 COMPARISONS

Now we compare two lower proximal topologies. If η is a LO-proximity on X, then for $A \subset X$, we use the notation $\eta(A) = \{E \subset X : E\ \eta\ A\}$ (cf. [23]).

Lemma 6.1 *Let γ and η be compatible LO-proximities on a T_1 topological space (X, τ). The following are equivalent:*

(a) $W_\gamma^- \subset V_\eta^-$ *on* $CL(X)$*;*

(b) $\gamma(W) \subset \eta(V)$.

Theorem 6.2 *Let γ and η be compatible LO-proximities on a T_1 topological space (X, τ). The following are equivalent:*

(a) $\sigma(\eta^-) \subset \sigma(\gamma^-)$*;*

(b) *for each $F \in CL(X)$ and $U \in \tau$ with $F\ \eta\ U$ there exists $V \in \tau$ such that $F \in \gamma(V) \subset \eta(U)$.*

Proof: (a)$\Rightarrow$(b). Let $F \in \mathrm{CL}(X)$ and $U \in \tau$ with $F\ \eta\ U$. Then $F \in U_\eta^-$.

By assumption there exists $\mathbf{V} =< \mathrm{V}_1^-, \mathrm{V}_2^-, \ldots, \mathrm{V}_n^- >_\gamma \in \sigma(\gamma^-)$ such that $F \in \mathbf{V} \subset U_\eta^-$. Clearly, $F \in \gamma(V_i)$ for each $i \in \{1, 2, \ldots, n\}$. We claim that there exist $i^* \in \{1, 2, \ldots, n\}$ such that $\gamma(V_{i^*}) \subset \eta(U)$. Assume not. Then for each $i \in \{1, 2, \ldots, n\}$ there exists $T_i \in \mathrm{CL}(X)$ such that $T_i\ \gamma\ V_i$ and $T_i\ \underline{\eta}\ U$. Set $T = \bigcup\{T_i : i \in \{1, 2, \ldots, n\}\}$. Then $T \in \mathrm{CL}(X)$, $T\ \gamma\ V_i$ for each $i \in \{1, 2, \ldots, n\}$ and $T\ \underline{\eta}\ U$. This show that $T \in \mathbf{V} \not\subset U_\eta^-$; a contradiction.

(b)$\Rightarrow$(a) It is obvious.

Definition 6.3 *Let (X, τ) be a T_1 topological space and η a LO-proximity.*

(X, τ) is ***nearly regular*** *iff whenever $x \in U$ with $U \in \tau$ there exists $V \in \tau$ such that $x \in cl V \subset U$.*

η is ***nearly regular (n-R*** *for short) iff it satisfies*

(n-R) $x\ \underline{\eta}\ A$ implies there exists $E \subset X$ such that $x\ \eta\ E$ and $E^c\ \underline{\eta}\ A$.

Remarks 6.4 (a) It is easy to verify that each LR-proximity is also an n-R proximity. The converse in general does not occur as (a) in the remark (6.6) shows.

(b) If η is a compatible (n-R)-proximity, then for each $x \in U$ and $U \in \tau$, there is a $V \in \tau$ with $x \in \mathrm{cl}\, V$ and $V\ \underline{\eta}\ U^c$.

(c) If (X, τ) is a T_3 topological space, then (X, τ) is nearly regular but the converse is not true in general as the next examples show.

Examples 6.5 (1) Let $X = \mathbb{R}$ with the topology τ consisting of the usual open sets together with sets of the form $U = (-\varepsilon, \varepsilon)\backslash B$, $\varepsilon > 0$, $B \subset \{1/n : n \in \mathbb{N}\} = A$. Then X is **nearly regular** but not regular since $V = (-\varepsilon', \varepsilon')\backslash B \subset U$ for $0 < \varepsilon' < \varepsilon$, but $\operatorname{cl} V \not\subset U$. Take $W = (-\varepsilon', 0)$ for $0 < \varepsilon' < \varepsilon$, then $0 \in \operatorname{cl} W \subset (-\varepsilon, \varepsilon)\backslash A$.

(2) The space X, of example 2.4, is Hausdorff but not nearly regular.

Remarks 6.6 (a) If in Examples 6.5 we endow X with the proximity η_0, then in the case (1) η_0 is an n-R proximity but not a LR-proximity; whereas in the case (2) η_0 is not n-R.

(b) It is easy to show that if η is a compatible n-R-proximity on X, then X is nearly-regular. Thus by the above result (a) we can state the following.

A topological space (X, τ) admits a compatible n-R proximity η if and only if the base space X is nearly regular.

(c) The above examples show that the nearly regular property is not hereditary. On the other hand it is easy to show that it is open hereditary.

Next Theorem characterizes those proximities η for which the corresponding lower η topologies $\sigma(\eta^-)$ are finer than the lower Vietoris topology $\tau(V^-) = \sigma(\eta^{*-})$.

Theorem 6.7 *Let (X, τ) be a T_1 topological space, η a compatible LO-proximity and η^* the discrete proximity. The following are equivalent:*

(a) $\sigma(\eta^{*-}) \subset \sigma(\eta^-)$*;*

(b) η *is an n-R proximity.*

Proof: (a)⇒(b) Let $x \in X$ and $U \in \tau$ with $x \in U$. The result follows from proof (a)⇒(b) of Theorem 6.2 when $F = \{x\}$.

(b)⇒(a) Let $\mathbf{U} = \mathrm{U}^-_{\eta^*}$ be a subbasic element of $\sigma(\eta^{*-})$ and $F \in \mathbf{U}$. Let $x \in U \cap F$. By assumption there exists a $V \in \tau$ such that $x \in \operatorname{cl} V$ and $V \underline{\eta}\, U^c$. Set $\mathbf{V} = \mathrm{V}^-_{\eta}$. Then $F \in \mathbf{V} \in \sigma(\eta^-)$ and $\mathbf{V} \subset \mathbf{U} = \mathrm{U}^-_{\eta^*}$.

Theorem 6.8 *Let (X, d) be a metric space, η the associated metric proximity and η_0 the Wallman proximity. The following are equivalent:*

(a) *X is UC;*

(b) $\eta = \eta_0$;

(c) $\sigma(\eta^-) = \sigma(\eta_0^-)$;

(d) $\sigma(\eta^-) \subseteq \sigma(\eta_0^-)$;

(e) *for each* $F \in CL(X)$ *and* $U \in \tau$ *with* $F\, \eta\, U$ *there is* $V \in \tau$ *with* $F \in \eta_0(V) \subset \eta(U)$.

Proof: (a)$\Leftrightarrow$(b), (b)$\Leftrightarrow$(c) and (c)$\Rightarrow$(d) are obvious.

(c)$\Leftrightarrow$(e). It follows by the above Theorem 6.2.

(d)$\Rightarrow$(a). It suffices to show that (not a)$\Rightarrow$(not d). So, suppose X is not UC. Then there exists a pair of sequences $\{x_n\}$, $\{y_n\}$ in X without cluster points, which are parallel (i.e. $\lim_{n\to\infty} d(x_n, y_n) = 0$) (see [21] or [1] on Page 54). Let $A = \{x_n : n \in \mathbb{N}\}$, for each $n \in \mathbb{N}$ set $A_n = \{x_m : m \leq n\}$ and $U = \cup\{\mathrm{S}(y_n, \varepsilon_n) : n \in \mathbb{N}\}$ where $\varepsilon_n = \frac{1}{4} d(x_n, y_n)$.

Clearly , $A\, \eta\, U$ but $A_n\, \underline{\eta}\, U$ for each $n \in \mathbb{N}$ showing that $\{A_n : n \in \mathbb{N}\}$ does not converge to A with respect to the $\sigma(\eta^-)$ topology. On the other hand $\{A_n : n \in \mathbb{N}\}$ converges to A with respect to $\sigma(\eta_0^-)$.

Now, we study comparisons between symmetric proximal topologies.

Theorem 6.9 *Let α, γ, δ and η be compatible LO-proximities on a T_1 topological space (X, τ) with $\alpha \leq \gamma$ and $\delta \leq \eta$, and Δ and Λ cobases. The following are equivalent:*

(a) $\pi(\eta, \delta; \Delta) \subset \pi(\gamma, \alpha; \Lambda)$;

(b) (1) *for each* $F \in CL(X)$ *and* $U \in \tau$ *with* $F\, \eta\, U$ *there are* $W \in \tau$ *and* $L \in \Lambda$ *such that* $F \in [\gamma(W) \backslash \alpha(L)] \subset \eta(U)$;

(2) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$ *with* $B \ll_\delta W$, *there exists* $M \in \Lambda$ *such that* $M \ll_\alpha W$ *and* $\delta(B) \subset \alpha(M)$.

Proof: (a)$\Rightarrow$(b) We start by showing (1). So, let $F \in \mathrm{CL}(X)$ and $U \in \tau$ with $F \eta U$. Then U_η^- is a $\pi(\eta, \delta; \Delta)$ neighbourhood of F.

By assumption there is a $\pi(\gamma, \alpha; \Lambda)$ neighbourhood $\boldsymbol{W}$ of F such that $\boldsymbol{W} \subset U_\eta^-$.

$\boldsymbol{W} =< W_1^-, W_2^-, \ldots, W_n^-, W^+ >_{\gamma,\alpha}$, $W_i \in \tau$ for each $i \in \{1, 2, \ldots, n\}$,
$\bigcup\{W_i : i \in \{1, 2, \ldots, n\}\} \subset W$ and $W^c \in \Lambda$.

Set $L = W^c$. By construction $F\, \underline{\alpha}\, L$ as well as $F\, \gamma\, W_i$ for each $i \in \{1, 2, \ldots, n\}$. We claim that there exists an $i^* \in \{1, 2, \ldots, n\}$ such that

$$[\gamma(W_{i^*}) \backslash \alpha(L)] \subset \eta(U)\,.$$

Assume not. Then for each $i \in \{1, 2, \ldots, n\}$ there exists $T_i \in \mathrm{CL}(X)$ with $T_i \in [\gamma(W_i) \backslash \alpha(L)]$ but $T_i \notin \eta(U)$, i.e. $T_i \ \gamma \ W_i$ as well as $T_i \ll_\alpha W = L^c$ and $T_i \ \underline{\eta} \ U$.

Set $T = \bigcup\{T_i : i \in \{1, 2, \ldots, n\}\}$. By construction $T \in \mathrm{CL}(X)$,
$T \in \boldsymbol{W} =< W_1^-, W_2^-, \ldots, W_n^-, W^+ >_{\gamma,\alpha}$ and $T \notin U_\eta^-$ which contradicts $\boldsymbol{W} \subset U_\eta^-$.

Now we show (2). So, let $B \in \Delta$ and $W \in \tau$, $W \neq X$ with $B \ll_\delta W$.

Set $A = W^c$. Then $A \in \mathrm{CL}(X)$ and $A \in (B^c)_\delta^+ \in \pi(\eta, \delta; \Delta)$. Thus there exists a $\pi(\gamma, \alpha; \Lambda)$-neighbourhood

$$\boldsymbol{O} =< O_1^-, O_2^-, \ldots, O_n^-, O^+ >_{\gamma,\alpha} \text{ such that } A \in \boldsymbol{O} \subset (B^c)_\delta^+ .$$

Note that $\bigcup\{O_i : i \in \{1, 2, \ldots, n\}\} \subset O$ and $O^c \in \Lambda$. Set $M = O^c$. Since $A \in \boldsymbol{O}$, then $M \ll_\alpha A^c = W$. We claim $\delta(B) \subset \alpha(M)$. Assume not. Then there exists $F \in \mathrm{CL}(X)$ such that $F\delta B$ but $F\underline{\alpha}M$. Set $E = A \cup F \in \mathrm{CL}(X)$. Then $E \in \boldsymbol{O}$ but $E \notin (B^c)_\delta^+$, which contradicts $\boldsymbol{O} \subset (B^c)_\delta^+$.

(b)$\Rightarrow$(a) Let $F \in \boldsymbol{U} =< U_1^-, U_2^-, \ldots, U_n^-, U^+ >_{\eta,\delta}$ be a $\pi(\eta, \delta; \Delta)$-neighbourhood of F. Then $F\eta U_i$ for each $i \in \{1, 2, \ldots, n\}$ as well as $F \ll_\alpha U$, with $U_i, U \in \tau, \bigcup\{U_i : i \in \{1, 2, \ldots, n\}\} \subset U$ and $B = U^c \in \Delta$.

By (1) for each $i \in \{1, 2, \ldots, n\}$ there are $W_i \in \tau$ and $L_i \in \Lambda$ with $F \in [\gamma(W_i) \backslash \alpha(L_i)] \subset \eta(U_i)$.

By (2), there exists $M \in \Lambda$ with $M \ll_\alpha F^c$ and $\delta(B) \subset \alpha(M)$.

Set $N = \bigcup\{L_i : i \in \{1, 2, \ldots, n\}\} \cup M \in \Lambda$, $O = N^c$ and for each $i \in \{1, 2, \ldots, n\}$ $O_i = W_i \backslash N$.

Note that $F\underline{\alpha}N$ together with $\alpha \leq \gamma$ imply $F\underline{\gamma}N$. $W_i = O_i \cup (W_i \cap N)$. But $F\gamma W_i$ together with $F\underline{\gamma}N$ imply $F\gamma O_i$ as well as $O_i \neq \emptyset$. Moreover $F\underline{\alpha}N$ implies $F \ll_\alpha O = N^c$. Therefore

$$F \in \boldsymbol{O} =< O_1^-, O_2^-, \ldots, O_n^-, O^+ >_{\gamma,\alpha} \in \pi(\gamma, \alpha; \Lambda) .$$

We claim

$$\boldsymbol{O} =< O_1^-, O_2^-, \ldots, O_n^-, O^+ >_{\gamma,\alpha} \subset \boldsymbol{U} =< U_1^-, U_2^-, \ldots, U_n^-, U^+ >_{\eta,\delta} .$$

Assume not. Then there exists $E \in \boldsymbol{O}$, but $E \notin \boldsymbol{U}$.

Hence either ($\diamond$) $E \ \underline{\eta} \ U_i$ for some i or ($\diamond\diamond$) $E \ \delta \ U^c$.

If ($\diamond$) occurs, then since $E\gamma O_i$, $O_i \subset W_i$, $E \ll_\alpha O = N^c$ and $L_i \subset N$ we have $E \in [\gamma(W_i) \backslash \alpha(L_i)] \not\subset \eta(U_i)$, which contradicts (1).

If ($\diamond\diamond$) occurs, then since $E\delta B = U^c$, $E \ll_\alpha O = N^c$ and $M \subset N$ we have $E \in \delta(B) \backslash \alpha(M)$, i.e. $\delta(B) \not\subset \alpha(M)$, which contradicts (2).

7 APPENDIX (ADMISSIBILITY)

It is a well known fact, that if (X,τ) is a T_1 topological space, then the lower Vietoris topology $\tau(V^-)$ is an admissible topology, i.e. the map $i : (X,\tau) \to (\mathrm{CL}(X), \tau(V^-))$, defined by $i(x) = \{x\}$, is an embedding. On the other hand (as observed in Example 2.1), if the involved proximity η is different from the discrete proximity η^*, then the map $i : (X,\tau) \to (\mathrm{CL}(X), \sigma(\eta^-)$ is, in general, not even continuous. So, we start to study the behaviour of $i : (X,\tau) \to (\mathrm{CL}(X), \sigma(\eta^-)$, when $\eta \neq \eta^*$. First we give the following Lemma.

Lemma 7.1 *Let (X,τ) be a T_1 topological space, $U \in \tau$ with $clU \neq X$ and $V = (clU)^c$. If $z \in clU \cap clV$, then there exists a net (z_λ) τ-converging to z such that for all λ either*

(i) $z_\lambda \in U$ and $z_\lambda \neq z$, or

(ii) $z_\lambda \in V$ and $z_\lambda \neq z$

Proof: Let $\mathsf{N}(z)$ be the filter of open neighbourhoods of z. For each $I \in \mathsf{N}(z)$, select $w_I \in I \cap V$ and $y_I \in I \cap U$. Then, the net (w_I) is τ-converging to z and $(w_I) \subset V$ as well as the net (y_I) is τ-converging to z and $(y_I) \subset U$.
We claim that for all $I \in \mathsf{N}(z)$ either $w_I \neq z$ or $y_I \neq z$.
Assume not. Then there exist I and $J \in \mathsf{N}(z)$ such that $y_I = z$ and $w_J = z$. As a result $z \in U \cap V \subset \mathrm{cl}\, U \cap V = \emptyset$, a contradiction.

Recall, that a Hausdorff space X is called **extremally disconnected** if for every open set $U \subset X$ the closure $\mathrm{cl}\, U$ of U is open in X (see [8] on page 368).

Proposition 7.2 *Let (X,τ) be a Hausdorff space with a compatible LO-proximity η. The following are equivalent:*

(a) X is extremally disconnected;

(b) the map $i : (X,\tau) \to (CL(X), \sigma(\eta^-))$, defined by $i(x) = \{x\}$, is continuous.

Proof: (a)⇒(b). Let $x \in X$ and (x_λ) a net τ-converging to x. Let $V \subset X$ with V open and $\{x\}\eta V$. Since $\{x\}\eta V$, then $x \in \mathrm{cl}\, V$. By assumption $\mathrm{cl}\, V$ is an open subset of X and the net (x_λ) τ-converges to x. Thus, eventually $x_\lambda \in \mathrm{cl}\, V$.
(b)⇒(a). By contradiction, suppose (a) fails. Then there exists open set $U \subset X$ such that closure $\mathrm{cl}\, U$ is not open in X. Then $\mathrm{cl}\, U \neq X$. Set $V = (\mathrm{cl}\, U)^c$. V is non-empty and open in X.

We claim that $\mathrm{cl}\,U \cap \mathrm{cl}\,V \neq \emptyset$. Assume not, i.e. $\mathrm{cl}\,U \cap \mathrm{cl}\,V = \emptyset$. Then, $\mathrm{cl}\,U \subset (\mathrm{cl}\,V)^c \subset V^c = \mathrm{cl}\,U$. Thus, $\mathrm{cl}\,U = (\mathrm{cl}\,V)^c$, i.e. $\mathrm{cl}\,U$ is open, a contradiction. Let $z \in \mathrm{cl}\,U \cap \mathrm{cl}\,V$. From Lemma 7.1, there exists a net (z_λ) τ-converging to z such that for all λ either (i) $z_\lambda \in U$ and $z_\lambda \neq z$, or (ii) $z_\lambda \in V$ and $z_\lambda \neq z$. In both cases, there exists an open subset W such that $z \in \mathrm{cl}, W$ and $z_\lambda \notin \mathrm{cl}, W$ for all λ (in fact, if (i) holds, then set $W = V$, otherwise set $W = U$). Thus the net (z_λ) τ-converging to z and the open subset W witness that the map $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\eta^-))$ fails to be continuous.

Now, we investigate when the map $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\eta^-))$ is open.

Proposition 7.3 *Let (X, τ) be a T_1-topological space with a compatible LO-proximity η. The following are equivalent:*

(a) (X, τ) is nearly regular (cf. Definition 6.3);

(b) the map $i : (X, \tau) \to (CL(X), \sigma(\eta^-))$ is open.

Proof: Left to the reader.

Note that if (X, τ) is a T_1-topological space with a compatible LO-proximity δ, then the map $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\delta^+; \Delta))$ is always continuous with respect the upper proximal Δ topology $\sigma(\delta^+; \Delta)$. So, we have:

Proposition 7.4 *Let (X, τ) be a T_1-topological space, δ a compatible LO-proximity and $\Delta \subset CL(X)$ a cobase. The following are equivalent:*

(a) the map $i : (X, \tau) \to (CL(X), \sigma(\delta^+; \Delta))$, defined by $i(x) = \{x\}$, is an embedding;

(b) the map $i : (X, \tau) \to (CL(X), \sigma(\delta^+; \Delta))$, defined by $i(x) = \{x\}$ is an open map;

(c) whenever $U \in \tau$ and $x \in U$, there exists a $B \in \Delta$ such that $x \in B^c \subset U$.

Finally, we have the following result concerning with the admissibility of the entire symmetric proximal Δ topology $\pi(\eta, \delta; \Delta)$. Obviously, we investigate just the significant case $\eta \neq \eta^*$ (the standard proximal Δ topology $\sigma(\delta; \Delta) = \pi(\eta^*, \delta; \Delta)$ is always admissible).

Proposition 7.5 *Let (X, τ) be a Hausdorff space, η, δ compatible LO-proximities and $\Delta \subset CL(X)$ a cobase. The following are equivalent:*

(a) *the map* $i : (X, \tau) \to (CL(X), \pi(\eta, \delta; \Delta))$ *is an embedding;*

(b) $i : (X, \tau) \to (CL(X), \sigma(\eta^-))$ *is continuous and either* $i : (X, \tau) \to (CL(X), \sigma(\eta^-))$ *or* $i : (X, \tau) \to (CL(X), \sigma(\delta^+; \Delta))$ *is open.*

Theorem 7.6 *Let* (X, τ) *be a Hausdorff space,* η, δ *compatible LO-proximities and* $\Delta \subset CL(X)$ *a cobase. The following are equivalent:*

(a) *the map* $i : (X, \tau) \to (CL(X), \pi(\eta, \delta; \Delta))$ *is an embedding;*

(b) X *is extremally disconnected and either* X *is also nearly regular, or whenever* $U \in \tau$ *and* $x \in U$*, there exists a* $B \in \Delta$ *such that* $x \in B^c \subset U$*.*

We raise the following question.

Question 7.1 There exists an extremally disconnected space X which turns out to be nearly regular, but not regular?

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A Recolouring Problem on Undirected Graphs

ABSTRACT. We consider an algorithm on a graph $G = (V, E)$ with a 2-colouring of V, that is motivated from the computer-aided text-recognition. Every vertex changes simultaneously its colour if more than a certain proportion c of its neighbours have the other colour. It is shown, that by iterating this algorithm the colouring becomes either constant or 2-periodic. For $c = \dfrac{1}{2}$ the presented theorem is a special case of a known result [1], but here developed independently with another motivation and a new proof.

There are algorithms for the computer-aided text recognition, that search in a given pixel pattern for characteristic properties of characters. But often this search becomes very difficult because of certain dirt effect like single white pixels in a large black area. So it seems plausible that we can increase the efficiency of such algorithms by first weeding out such effects. For example, we can change simultaneously the colour of every pixel if in a properly defined neighbourhood the proportion of pixels with the opposite colour exceeds a certain number c with $0 < c < 1$. It is easy to see that iterating this recolouring finally runs into a period, so the question for the length of such a period naturally arises. A similar question in the more general situation of an arbitrary finite number of colours is motivated in [1] by a model of a society, where the pixels correspond to the members of the society, whose opinions are influenced by their neighbours. It is shown there that the period is 1 or 2 for this model.

To reformulate our problem in an explicit graph theoretical context let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \dots, v_r\}$. We colour the vertices with two colours, say black and white. Now we can change the colouring by the following algorithm: For i from 1 to r, if v_i is black and has more white than black neighbours, it changes its colour to white, and if v_i is white and has more black than white neighbours, it changes its colour to black. It is a well-known problem in mathematical contests, to show that one always gets a constant colouring by iterating this algorithm. Here every vertex v changes its colour if there are more than $\frac{1}{2}\deg(v)$ vertices of the other colour in its neighbourhood. But what happens, if we vary the fraction of opposite-coloured neighbours that is necessary for changing the colour?

Thus the new condition for colour-changing is that there are more than $c\deg(v)$ vertices of the other colour in the neighbourhood for some c, $0 < c < 1$. Furthermore we want to recolour the vertices not one after another, but all simultaneously. The standard-solution of the mentioned contest problem gives a hint, how to tackle this problem, namely by the search for an integer-valued, bounded and monotonous function of the number of steps.

We may assume that G has no isolated vertices, because if there were any, they would never change their colour. Every 2-colouring of the vertices is given by a function $V \to \{-1, 1\}$. Then the series of recolouring steps corresponds to a series of functions $(f_n : V \to \{-1, 1\})_{n\in\mathbb{N}}$. To describe the recolouring steps, we define the series of functions:

$$g_n : E \to \{-1, 1\}, \quad \{v, w\} \mapsto f_n(v)f_n(w)$$

So the number of neighbours of v, that are coloured in a different way than v before the n-th recolouring step is $\frac{1}{2}\left(\deg(v) - \sum_{e\in E: v\in e} g_n(e)\right)$, and v changes its colour iff $\frac{1}{2\deg(v)}\left(\deg(v) - \sum_{e\in E: v\in e} g_n(e)\right) > c$. With

$$h_n : V \to \mathbb{Z}, \quad v \mapsto \sum_{e\in E: v\in e} g_n(e) - (1 - 2c)\deg(v)$$

this is equivalent to $h_n(v) < 0$. Next we introduce another global parameter, for which we will see, that it grows monotonously with n:

$$s_n = \sum_{v\in V} |h_n(v)|$$

Lemma *For every $n \in \mathbb{N}$ we define $V_n^- := \{v \in V \mid h_n(v) < 0\}$, the set of vertices, that change their colours in the n-th step. Furthermore we set $V_n^+ := V \setminus V_n^-$. Then, for all $n \in \mathbb{N}$,*

$$s_{n+1} = s_n + 2\left(\sum_{v\in V_n^-\cap V_{n+1}^+} |h_{n+1}(v)| + \sum_{v\in V_n^+\cap V_{n+1}^-} |h_{n+1}(v)|\right).$$

Proof: We choose $n \in \mathbb{N}$. For $i \in \{1, 2, \ldots, r\}$ let k_i^- and k_i^+ denote the numbers of edges e, that are incident with v_i, and fulfill $g_n(e) = 1 = -g_{n+1}(e)$ and $g_n(e) = -1 = -g_{n+1}(e)$, respectively.
Obviously, $h_{n+1}(v_i) = h_n(v_i) + 2\left(k_i^+ - k_i^-\right)$ for all i, and hence

$$|h_n(v_i)| = \begin{cases} |h_{n+1}(v_i)| - 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^+ \cap V_{n+1}^+ \\ |h_{n+1}(v_i)| + 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^- \cap V_{n+1}^- \\ -|h_{n+1}(v_i)| + 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^- \cap V_{n+1}^+ \\ -|h_{n+1}(v_i)| - 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^+ \cap V_{n+1}^-. \end{cases}$$

Summing up these equations yields

$$\begin{aligned} s_n = & s_{n+1} - 2\left(\sum_{v\in V_n^- \cap V_{n+1}^+} |h_{n+1}(v)| + \sum_{v\in V_n^+ \cap V_{n+1}^-} |h_{n+1}(v)|\right) \\ & + 2\left(\sum_{i:v_i\in V_n^-} (k_i^+ - k_i^-) - \sum_{i:v_i\in V_n^+} (k_i^+ - k_i^-)\right). \end{aligned} \tag{1}$$

For every $v \in V_n^+$ and every $e = \{v, w\} \in E$ we have $g_n(e) = 1 = -g_{n+1}(e)$ iff $w \in V_n^-$. Therefore $\sum_{i:v_i\in V_n^+} k_i^- = \sum_{i:v_i\in V_n^-} k_i^-$ and analogously $\sum_{i:v_i\in V_n^+} k_i^+ = \sum_{i:v_i\in V_n^-} k_i^+$. So the last two sums in (1) cancel each other and the claim follows. □

Now we have an upper bound for the series (s_n) by

$$s_n \leq \sum_{v\in V} \deg(v) + \sum_{v\in V} |1-2c| \deg(v) = 4\alpha|E|$$

with $\alpha = 1 - c$ if $c \leq \frac{1}{2}$ and $\alpha = c$ if $c > \frac{1}{2}$. For every $v \in V_{n+1}^-$ we have

$$\begin{aligned} &|h_{n+1}(v)| \geq 1 && \text{if } (1-2c)\deg(v) \in \mathbb{Z}, \\ &|h_{n+1}(v)| \geq |1-2c|\deg(v) - \lfloor |1-2c|\deg(v) \rfloor && \text{if } (1-2c)\deg(v) \notin \mathbb{Z}. \end{aligned}$$

So there is a constant $\beta > 0$ with $|h_{n+1}(v)| \geq \beta$ for all $v \in V_{n+1}^-$ and for all $n \in \mathbb{N}$. It follows

$$\sum_{n\in\mathbb{N}} \left|V_n^+ \cap V_{n+1}^-\right| \leq \frac{4\alpha|E| - s_0}{2\beta}. \tag{2}$$

Now we consider any $v \in V$. Let $n_0, n_1, n_2, \ldots$ denote the numbers with $v \in V_{n_i}^- \cap V_{n_i+1}^+$ in increasing order. Then either $v \in V_0^-$ or $v \in V_m^+ \cap V_{m+1}^-$ for an $m < n_0$, and, for all $k \in \mathbb{N}$, $v \in V_m^+ \cap V_{m+1}^-$ for an m with $n_k < m < n_{k+1}$. So with the characteristic functions $\chi_A : V \to \{0,1\}$, $\chi_A(v) = 1$ for $v \in A$ and $\chi_A(v) = 0$ for $v \notin A$ we have

$$\sum_{n\in\mathbb{N}} \chi_{V_n^- \cap V_{n+1}^+}(v) \leq \chi_{V_0^-}(v) + \sum_{n\in\mathbb{N}} \chi_{V_n^+ \cap V_{n+1}^-}(v).$$

Summing up over V yields

$$\sum_{v\in V}\sum_{n\in\mathbb{N}} \chi_{V_n^- \cap V_{n+1}^+}(v) \leq \sum_{v\in V} \chi_{V_0^-}(v) + \sum_{v\in V}\sum_{n\in\mathbb{N}} \chi_{V_n^+ \cap V_{n+1}^-}(v),$$

$$\sum_{n\in\mathbb{N}}\sum_{v\in V} \chi_{V_n^- \cap V_{n+1}^+}(v) \leq \sum_{v\in V} \chi_{V_0^-}(v) + \sum_{n\in\mathbb{N}}\sum_{v\in V} \chi_{V_n^+ \cap V_{n+1}^-}(v).$$

$$\sum_{n\in\mathbb{N}} \left|V_n^- \cap V_{n+1}^+\right| \leq \left|V_0^-\right| + \sum_{n\in\mathbb{N}} \left|V_n^+ \cap V_{n+1}^-\right| \leq |V_0^-| + \frac{4\alpha|E| - s_0}{2\beta},$$

and hence with (2)

$$\sum_{n\in\mathbb{N}} \left(\left|V_n^- \cap V_{n+1}^+\right| + \left|V_n^+ \cap V_{n+1}^-\right|\right) \leq |V_0^-| + \frac{4\alpha|E| - s_0}{\beta}.$$

Consequently there is an $n_0 \leq |V_0^-| + \frac{4\alpha|E|-s_0}{\beta}$ with $(V_{n_0}^+ \cap V_{n_0+1}^-) \cup (V_{n_0}^- \cap V_{n_0+1}^+) = \emptyset$. From this follows:

Theorem *Let $G = (V, E)$ be a graph with a 2-colouring of V and $0 < c < 1$. In every time step every vertex v changes its colour iff more than $c\deg(v)$ vertices in the neighbourhood of v are coloured in a different way than v. Finally this algorithm runs into a period of length* 1 *or* 2*.*

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Oscillating Solutions of Rational Difference Equations

ABSTRACT. For a special rational difference equation of order two oscillating series solution are constructed. An example is given where Bessel functions arise as coefficients.

KEY WORDS. Rational difference equation, oscillating solutions, periodic solutions, Bessel functions

A detailed investigation of the rational difference equation

$$x_{n+2} = \frac{\alpha + \beta x_{n+1} + \gamma x_n}{A + Bx_{n+1} + Cx_n} \quad (n \in \mathbb{N}_0) \tag{1}$$

with non-negative parameters ($A+B+C > 0$) is contained in the book Kulenović and Ladas [3]. Under the conditions

$$A + B + C = \alpha + \beta + \gamma = 1 \tag{2}$$

it has the positive equilibrium $\tilde{x} = 1$, and the corresponding linearized equation has the characteristic equation $D(s) = 0$ with

$$D(s) = s^2 + (B - \beta)s + C - \gamma\,. \tag{3}$$

In the case that the zeros of (3) are real, series solutions of (1) were constructed in [2]. Here, we deal with the case

$$C > \gamma + \frac{1}{4}(\beta - B)^2 \tag{4}$$

where the zeros

$$z = \frac{1}{2}\left(\beta - B + i\sqrt{4(C-\gamma) - (\beta - B)^2}\right) \tag{5}$$

and $\overline{z}$ of (3) are complex, and we construct series solutions which are oscillating. In the following we use the notation

$$g_{jk}(r) = \frac{1 - r^{j+1}}{1 - r}\frac{1 - r^{k+1}}{1 - r} - 1 - r^{j+k} \tag{6}$$

with $j,k \in \mathbb{N}_0$. Moreover, we put $r = |z|$ where (2), (4) and (5) imply that $0 < r = \sqrt{C-\gamma} \le 1$. Some calculations were carried out by means of the DERIVE system.

Proposition 1 *Under the assumptions* (2), (4) *and* $r < 1$ *the difference equation* (1) *has the solution*

$$x_n = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk} a^j z^{nj} \overline{a}^k \overline{z}^{nk} \tag{7}$$

with $c_{00} = c_{10} = c_{01} = 1$, *an arbitrary complex* a, *and*

$$c_{jk} = -\frac{1}{D(z^j \overline{z}^k)} {\sum_{\mu=0}^{j}}' {\sum_{\nu=0}^{k}}' c_{\mu\nu} c_{j-\mu,k-\nu} z^{\mu} \overline{z}^{\nu} (B z^j \overline{z}^k + C z^{\mu} \overline{z}^{\nu}) \tag{8}$$

for $j+k \ge 2$, *where the primes at the sums shall indicate that the pairs* $(0,0)$ *and* (j,k) *are excluded for* (μ,ν). *The series* (7) *converges for*

$$\lambda |a| r^n < 1 \tag{9}$$

where

$$\lambda = \sup_{j+k\ge 2} \frac{1}{|D(z^j \overline{z}^k)|} \left(B r^{j+k} g_{jk}(r) + C g_{jk}(r^2)\right) . \tag{10}$$

Proof: Writing (1) in the form

$$x_{n+2}(A + Bx_{n+1} + Cx_n) = \alpha + \beta x_{n+1} + \gamma x_n$$

and replacing x_n by means of (7) with $c_{00} = c_{10} = c_{01} = 1$, we obtain by comparing coefficients that the coefficients c_{jk} can be determined recursively by (8), whereas a remains arbitrary.

In order to prove the convergence condition (9) we show that

$$|c_{jk}| \le \lambda^{j+k-1} \tag{11}$$

for $j+k \ge 1$. This estimate is valid in the case $j+k = 1$. Assuming that $|c_{\mu\nu}| \le \lambda^{\mu+\nu-1}$ is valid for $0 \le \mu \le j$, $0 \le \nu \le k$ but $1 \le \mu+\nu < j+k$, then (8) implies the estimate

$$|c_{jk}| \le \frac{1}{|D(z^j \overline{z}^k)|} \left(B r^{j+k} g_{jk}(r) + C g_{jk}(r^2)\right) \lambda^{j+k-2} ,$$

and (11) is proved by induction in view of (10) □

The coefficients of (7) satisfy $c_{jk} = \overline{c}_{jk}$. Writing $z = re^{i\varphi}$, $c_{jk} a^j \overline{a}^k = \varrho_{jk} e^{i\vartheta_{jk}}$ and using $c_{jk} a^j z^{nj} \overline{a}^k \overline{z}^{nk} + c_{kj} a^k z^{nk} \overline{a}^j \overline{z}^{nj} = 2\varrho_{jk} r^{(j+k)n} \cos(n\varphi(j-k) + \vartheta_{jk})$, we see that the solution (7) oscillates around the equilibrium 1 when $a \ne 0$.

The estimate (9) implies that (7) converges at least for

$$n > \frac{\ln(\lambda|a|)}{\ln \frac{1}{r}} .$$

Proposition 2 *The supremum (10) allows the estimate*

$$\lambda \leq \frac{2(Br+C)}{(1-r)^2} \,. \tag{12}$$

Proof: We use the abbreviations $x = r^j$, $y = r^k$. In view of $D(s) = (z-s)(\overline{z}-s)$ we have

$$\left|D\left(z^j \overline{z}^k\right)\right| \geq (r-xy)^2 \,,$$

so that (12) is valid if we show that both

$$0 \leq \frac{2r}{(1-r)^2} - \frac{xy}{(r-xy)^2}\left(\frac{(1-rx)(1-ry)}{(1-r)^2} - 1 - xy\right) \tag{13}$$

and

$$0 \leq \frac{2}{(1-r)^2} - \frac{1}{(r-xy)^2}\left(\frac{(1-r^2x^2)(1-r^2y^2)}{(1-r^2)^2} - 1 - x^2y^2\right) \,. \tag{14}$$

The right-hand side of (13) can be written as

$$ry(1-x)(r-x) + rx(1-y)(r-y) + r(r-y)(r-xy) + (r-x)(r^2-xy^2) \tag{15}$$

divided by the positive denominator $(1-r)^2(r-xy)^2$, and the right-hand side of (14) as

$$r^2(x-y)^2 + 3(r^2-xy)^2 + 4r(r^2-xy)(1-xy) \tag{16}$$

divided by the positive denominator $(1-r^2)^2(r-xy)^2$. In view of $xy \leq r^2 < 1$ the expression (16) is always non-negative. For both $x \leq r$ and $y \leq r$ also the expression (15) is non-negative. In the case $x = 1$ and $y \leq r^2$ the expression (15) can be written as

$$r(r-y)^2 + (r-y)(r^2-y)$$

so that it is also non-negative and, in view of the symmetry of (14), also the case $x \leq r^2$ and $y = 1$ is settled □

The right-hand sides of (13) and (14) vanish for $x = y = r$.

Example 3 Pielou's equation

$$x_{n+2} = \frac{2x_{n+1}}{1+x_n} \,,$$

cf. [3, Theorem 4.4.1 (b)], is a special case of (1), (2) with the non-vanishing coefficients $A = C = \frac{1}{2}$, $\beta = 1$. Hence, $z = \frac{1}{2}(1+i)$ with $r = \frac{1}{\sqrt{2}}$, and

$$c_{20} = \frac{1}{5}(2-i)\,, \quad c_{11} = 0\,, \quad c_{30} = \frac{1}{15}(1-2i)\,, \quad c_{21} = \frac{1}{5}(1+2i)\,.$$

The estimate (12) seems to be rather bad because it only yields $\lambda \le 6 + 4\sqrt{2}$.

The case $r > 1$ is impossible in view of (2). If $r = 1$ without z being a root of unity, then the coefficients (8) exist, but the convergence of (7) is an open problem. If z is a root of unity, then the general solution of (1) is periodic and we do not need the expansion (7). A special example is Lyness' equation with $C = 1$ and $\alpha = \beta^2$ having 5-periodic solutions, cf. [3, p. 71].

A further one is

Example 4 with $C = \beta = 1$, i.e.

$$x_{n+2} = \frac{x_{n+1}}{x_n} \tag{17}$$

and 6-periodic solutions, cf. [3, p. 48]. The general positive solution of (17) reads

$$x_n = \exp(az^n + \overline{a}\,\overline{z}^n) \tag{18}$$

with $z = e^{\frac{i\pi}{3}}$ and an arbitrary complex constant a. In this case the corresponding expansion (7) has the coefficients $c_{jk} = \frac{1}{j!k!}$ and it can be written in a finite form. In order to show this we introduce the notation $a = \varrho e^{i\vartheta}$ and write it first as

$$x_n = \sum_{\ell=-\infty}^{+\infty} \mathrm{I}_\ell(2\varrho) \exp\left[i\left(\frac{\pi n}{3} + \vartheta\right)\ell\right] \tag{19}$$

with the Bessel functions

$$\mathrm{I}_\ell(2\varrho) = \sum_{k=0}^{\infty} \frac{1}{k!(k+\ell)!}\varrho^{\ell+2k}\,.$$

Setting $\ell = 6\mu + \nu$, expression (19) turns over into the finite Fourier sum

$$x_n = \sum_{\nu=0}^{5} C_\nu(\varrho,\vartheta) \exp\left[i\left(\frac{\pi n}{3} + \vartheta\right)\nu\right] \tag{20}$$

with

$$C_\nu(\varrho,\vartheta) = \sum_{\mu=-\infty}^{+\infty} \mathrm{I}_{6\mu+\nu}(2\varrho)\exp(6i\vartheta\mu)\,. \tag{21}$$

The series (21) converges in view of

$$\mathrm{I}_\ell(2\varrho) = \mathrm{I}_{-\ell}(2\varrho) \sim \frac{\varrho^\ell}{\ell!}$$

as $\ell \to \infty$. The coefficients in (20) can be simplified using the discrete Fourier transform as in [1, p. 1073].

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An Integral Expression and Some Inequalities of Mathieu Type Series

ABSTRACT. Let $r > 0$ and $a = \{a_k > 0, k \in \mathbb{N}\}$ such that the series $g(x) = \sum_{k=1}^{\infty} e^{-a_k x}$ converges for $x > 0$, then the Mathieu type series $\sum_{k=1}^{\infty} \frac{a_k}{\left(a_k^2+r^2\right)^2} = \frac{1}{2r} \int_0^{\infty} x g(x) \sin(rx) \,\mathrm{d}x$.

If $a = \{a_k > 0, k \in \mathbb{N}\}$ is an arithmetic sequence, then some inequalities of Mathieu type series $\sum_{k=1}^{\infty} \frac{a_k}{\left(a_k^2+r^2\right)^2}$ are obtained for $r > 0$.

KEY WORDS AND PHRASES. Mathieu type series, integral expression, Laplace transform, inequality

1 Introduction

In 1890, Mathieu defined $S(r)$ in [14] as

$$S(r) = \sum_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^2}, \quad r > 0, \tag{1}$$

and conjectured that $S(r) < \frac{1}{r^2}$. We call formula (1) Mathieu's series.

In [3, 13], Berg and Makai proved

$$\frac{1}{r^2 + \frac{1}{2}} < S(r) < \frac{1}{r^2}. \tag{2}$$

H. Alzer, J. L. Brenner and O. G. Ruehr in [2] obtained

$$\frac{1}{r^2 + \frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2 + \frac{1}{6}}, \tag{3}$$

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where ζ denotes the zeta function and the number $\zeta(3)$ is the best possible.

The integral form of Mathieu's series (1) was given in [7, 8] by

$$S(r) = \frac{1}{r}\int_0^\infty \frac{x}{e^x-1}\sin(rx)\,\mathrm{d}x. \tag{4}$$

Recently, the following results were obtained in [16, 18]:

(1) Let Φ_1 and Φ_2 be two integrable functions such that $\frac{x}{e^x-1}-\Phi_1(x)$ and $\Phi_2(x)-\frac{x}{e^x-1}$ are increasing. Then, for any positive number r, we have

$$\frac{1}{r}\int_0^\infty \Phi_2(x)\sin(rx)\,\mathrm{d}x \le S(r) \le \frac{1}{r}\int_0^\infty \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{5}$$

(2) For positive number $r>0$, we have

$$S(r) \le \frac{(1+4r^2)\left(e^{-\pi/r}-e^{-\pi/(2r)}\right)-4\left(1+r^2\right)}{\left(e^{-\pi/r}-1\right)\left(1+r^2\right)\left(1+4r^2\right)}. \tag{6}$$

(3) For positive number $r>0$, we have

$$S(r) < \frac{1}{r}\int_0^{\pi/r} \frac{x}{e^x-1}\sin(rx)\,\mathrm{d}x < \frac{1+\exp(-\frac{\pi}{2r})}{r^2+\frac{1}{4}}. \tag{7}$$

Remark 1 For $0<r<0.83273\cdots$, inequality (6) is better than the right hand side inequality in (3). If $r>1.57482\cdots$, inequality (7) is better than (6). When $r<1.574816\cdots$, inequality (7) is not better than (6). When $0<r<0.734821\cdots$, inequality (7) is better than the corresponding one in (3).

In [11, 16, 18], the following open problem was proposed by B.-N. Guo and F. Qi respectively: Let

$$S(r,t,\alpha) = \sum_{n=1}^\infty \frac{2n^{\alpha/2}}{(n^\alpha+r^2)^{t+1}} \tag{8}$$

for $t>0$, $r>0$ and $\alpha>0$. Can one obtain an integral expression of $S(r,t,\alpha)$? Give some sharp inequalities for the series $S(r,t,\alpha)$.

In [20], the open problem stated above was considered and an integral expression of $S(r,t,2)$ was obtained: Let $\alpha>0$ and $p\in\mathbb{N}$, then

$$\begin{aligned} S(\alpha,p,2) = \sum_{n=1}^\infty \frac{2n}{(n^2+\alpha^2)^{p+1}} &= \frac{2}{(2\alpha)^p p!}\int_0^\infty \frac{t^p\cos(\frac{p\pi}{2}-\alpha t)}{e^t-1}\,\mathrm{d}t \\ &\quad - 2\sum_{k=1}^p \frac{(k-1)(2\alpha)^{k-2p-1}}{k!(p-k+1)}\binom{-(p+1)}{p-k}\int_0^\infty \frac{t^k\cos[\frac{\pi}{2}(2p-k+1)-\alpha t]}{e^t-1}\,\mathrm{d}t. \end{aligned} \tag{9}$$

Using the quadrature formulas, some new inequalities of Mathieu series (1) were established in [9]. By the help of Laplace transform, the open problem mentioned above was partially solved, for example, among other things, an integral expression for $S\left(r, \frac{1}{2}, 2\right)$ was given as follows:

$$S\left(r, \frac{1}{2}, 2\right) = \frac{2}{r} \int_0^\infty \frac{t J_0(rt)}{e^t - 1} \,\mathrm{d}t, \tag{10}$$

where J_0 is Bessel function of order zero.

There has been a rich literature on the study of Mathieu's series, for example, [5, 6, 13, 19, 21, 22, 23], also see [4, 12, 15].

In this paper, we are about to investigate the following Mathieu type series

$$S(r, a) = \sum_{k=1}^{\infty} \frac{a_k}{(a_k^2 + r^2)^2}, \tag{11}$$

where $a = \{a_k > 0, k \in \mathbb{N}\}$ is a sequence satisfying $\lim_{k\to\infty} a_k = \infty$, and obtain an integral expression and some inequalities of $S(r, a)$ under some suitable conditions.

2 An integral expression of Mathieu type series (11)

Using Laplace transform of $x\sin(rx)$ we can immediately establish an integral expression of Mathieu type series (11).

Theorem 1 *Let $r > 0$ and $a = \{a_k > 0, k \in \mathbb{N}\}$ be a sequence such that the series*

$$g(x) \triangleq \sum_{k=1}^{\infty} e^{-a_k x} \tag{12}$$

converges for $x > 0$ and $xg(x)$ is Lebesgue integrable in $[0, \infty)$. Then we have

$$S(r, a) = \frac{1}{2r} \int_0^\infty x g(x) \sin(rx) \,\mathrm{d}x. \tag{13}$$

Proof: In [1] and [10, p. 559], Laplace transform of $t\sin(\alpha t)$ is given by

$$\int_0^\infty t \sin(\alpha t) e^{-st} \,\mathrm{d}t = \frac{2\alpha s}{(s^2 + \alpha^2)^2}, \tag{14}$$

where $s = \sigma + i\omega$ is a complex variable, α a complex number, and $\sigma > |\operatorname{Im}\alpha|$.

Applying (14) to the case $\alpha = r > 0$ and $s = a_k$, summing up, and interchanging between integral and summation produces

$$\begin{aligned} \int_0^\infty x g(x) \sin(rx) \,\mathrm{d}x &= \sum_{k=1}^{\infty} \int_0^\infty x \sin(rx) e^{-a_k x} \,\mathrm{d}x \\ &= 2r \sum_{k=1}^{\infty} \frac{a_k}{(a_k^2 + r^2)^2} = 2r S(r, a) \end{aligned} \tag{15}$$

according to Lebesgue's dominanted convergence theorem. The proof is complete. □

Remark 2 If $a_k = k$ in (13), then we can easily obtain the formula (4) in [8].

Corollary 1 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ be a sequence with $a_k = kd - c$ and $d > 0$. Then for any positive real number $r > 0$, we have*

$$S(r, a) = \frac{1}{2r} \int_0^\infty \frac{x e^{cx}}{e^{dx} - 1} \sin(rx)\, \mathrm{d}x. \tag{16}$$

Proof: Since $a = \{a_k, k \in \mathbb{N}\}$ is an arithmetic sequence with difference $d > 0$, then $\{e^{-a_k x}\}_{k=1}^\infty$ is a positive geometric sequence with constant ratio $e^{-dx} < 1$ for $x > 0$, thus

$$g(x) = \sum_{k=1}^\infty e^{-a_k x} = e^{cx} \sum_{k=1}^\infty e^{-kdx} = \frac{e^{cx}}{e^{dx} - 1}. \tag{17}$$

Then formula (16) follows from combination of (13) and (17) in view of $d > c$. □

Remark 3 In fact, in Corollary 1 and the following Theorem 2, Theorem 3 and Theorem 4, it suffices to consider the case $d = 1$, since from this one the general case arises by replacing c and r by $\frac{c}{d}$ and $\frac{r}{d}$, respectively, and dividing $S(r, a)$ by d^3.

3 Some inequalities of Mathieu type series (16)

The following result was obtained in [16, 18].

Lemma 1 ([16, 18]) *For a given positive number T, let $\phi(x)$ be an integrable function such that $\phi(x) = -\phi(x + T)$ and $\phi(x) \geq 0$ for $x \in [0, T]$, and let $f(x)$ and $g(x)$ be two integrable functions on $[0, 2T]$ such that*

$$f(x) - g(x) \geq f(x + T) - g(x + T) \tag{18}$$

on $[0, T]$. Then

$$\int_0^{2T} \phi(x) f(x)\, \mathrm{d}x \geq \int_0^{2T} \phi(x) g(x)\, \mathrm{d}x. \tag{19}$$

Now we give a general estimate of Mathieu type series (16) as follows.

Theorem 2 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ such that $a_k = kd - c$ and $d > 0$. If Φ_1 and Φ_2 are two integrable functions such that $\frac{xe^{cx}}{e^{dx}-1} - \Phi_1(x)$ and $\Phi_2(x) - \frac{xe^{cx}}{e^{dx}-1}$ are increasing, then for $r > 0$,*

$$\frac{1}{2r} \int_0^\infty \Phi_2(x) \sin(rx)\, \mathrm{d}x \leq S(r, a) \leq \frac{1}{2r} \int_0^\infty \Phi_1(x) \sin(rx)\, \mathrm{d}x. \tag{20}$$

Proof: The function $\phi(x) = \sin(rx)$ has a period $\frac{2\pi}{r}$, and $\phi(x) = -\phi\left(x + \frac{\pi}{r}\right)$.

Since $f(x) = \frac{xe^{cx}}{e^{dx}-1} - \Phi_1(x)$ is increasing, for any $\alpha > 0$, we have $f(x+\alpha) \geq f(x)$. Therefore, from Lemma 1, we obtain

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{xe^{cx}}{e^{dx}-1} \sin(rx)\,\mathrm{d}x \leq \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{21}$$

Then, from formula (16), we have

$$\begin{aligned} S(r,a) &= \frac{1}{2r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{xe^{cx}}{e^{dx}-1}\sin(rx)\,\mathrm{d}x \\ &\leq \frac{1}{2r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x)\sin(rx)\,\mathrm{d}x \\ &= \frac{1}{2r}\int_0^{\infty} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \end{aligned} \tag{22}$$

The right hand side of inequality (20) follows.

Similar arguments yield the left hand side of inequality (20). □

Lemma 2 *For $x > 0$, we have*

$$\frac{1}{e^x} < \frac{x}{e^x-1} < \frac{1}{e^{x/2}}. \tag{23}$$

Proof: This follows from standard argument of calculus. □

Theorem 3 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ satisfying $a_k = kd - c$ and $d > 0$. If $d > 2c$, then for $r > 0$, we have*

$$\begin{aligned} &\frac{1}{d}\left\{\frac{1+e^{-\pi(d-c)/r}}{2\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)} - \frac{2\left[e^{-\pi(d-2c)/r}+e^{-\pi(d-2c)/(2r)}\right]}{\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]}\right\} \\ &\leq S(r,a) \\ &\leq \frac{1}{d}\left\{\frac{2\left[1+e^{-\pi(d-2c)/(2r)}\right]}{\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]} - \frac{e^{-2\pi(d-c)/r}+e^{-\pi(d-c)/r}}{2\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)}\right\}. \end{aligned} \tag{24}$$

Proof: For $r > 0$, using (16), by direct calculation, we have

$$S(r,a) = \frac{1}{2r}\sum_{k=0}^{\infty}\left[\int_{2k\pi/r}^{(2k+1)\pi/r} + \int_{(2k+1)\pi/r}^{(2k+2)\pi/r}\right]\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x. \tag{25}$$

The inequality (23) gives us

$$\begin{aligned}\frac{r\left(1+e^{-\pi(d-c)/r}\right)}{d\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)}&=\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{de^{(d-c)x}}\,\mathrm{d}x\\&\le\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x\\\le\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{de^{(\frac{d}{2}-c)x}}\,\mathrm{d}x&=\frac{4r\left[1+e^{-\pi(d-2c)/(2r)}\right]}{d\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]}\end{aligned}\tag{26}$$

and

$$\begin{aligned}-\frac{4r\left[e^{-\pi(d-2c)/r}+e^{-\pi(d-2c)/(2r)}\right]}{d\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]}&=\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{de^{(\frac{d}{2}-c)x}}\,\mathrm{d}x\\&\le\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x\\\le\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{de^{(d-c)x}}\,\mathrm{d}x&=-\frac{r\left(e^{-2\pi(d-c)/r}+e^{-\pi(d-c)/r}\right)}{d\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)}.\end{aligned}\tag{27}$$

Substituting (26) and (27) into (25) yields (24). The proof is complete. □

Theorem 4 *Let $a=\{a_k>0,k\in\mathbb{N}\}$ be a sequence such that $a_k=kd-c$ and $d>0$. If $d>2c$, then for any positive number $r>0$, we have*

$$S(r,a)<\frac{1}{2r}\int_0^{\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x<\frac{2\left[1+e^{\pi(2c-d)/(2r)}\right]}{d[(2c-d)^2+4r^2]}.\tag{28}$$

Proof: It is easy to see that

$$\int_0^{\infty}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x=\sum_{k=0}^{\infty}\int_{k\pi/r}^{(k+1)\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x,\tag{29}$$

and

$$\frac{xe^{cx}}{e^{dx}-1}=\frac{xe^{(c-\frac{d}{2})x}}{2\sinh\left(\frac{dx}{2}\right)}.\tag{30}$$

Since the functions $\frac{\sinh x}{x}$ and $e^{(\frac{d}{2}-c)x}$ are both increasing with $x>0$ for $d>2c$, then the function $\frac{xe^{cx}}{e^{dx}-1}$ is decreasing with $x>0$. Furthermore, $\lim_{x\to\infty}\frac{xe^{cx}}{e^{dx}-1}=0$.

Therefore, the series in (29) is an alternating series whose moduli of the terms are decreasing to zero. As well known, such a series in (29) is always less than its first term $\int_0^{\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x$. Hence

$$\int_0^{\infty}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x<\int_0^{\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x.\tag{31}$$

Using inequality (23), we have

$$\int_0^{\pi/r} \frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x < \int_0^{\pi/r} \frac{\sin(rx)}{de^{(\frac{d}{2}-c)x}}\,\mathrm{d}x = \frac{4r\left[1+e^{\pi(2c-d)/(2r)}\right]}{d[(2c-d)^2+4r^2]}. \tag{32}$$

Inequality (28) follows from combination of (31) and (32) with (16). □

Remark 4 If taking $a_k = k$ for $k \in \mathbb{N}$ or equivalently $d = 1$ and $c = 0$ in (20), (24) and (28), inequalities (5), (6) and (7) are deduced.

By exploiting a technique presented by E. Makai in [13], we obtain the following inequalities of Mathieu type series (11).

Theorem 5 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ with $a_k = k - c$. If $r > 0$ satisfies $r^2 + c^2 > c$, then*

$$\frac{1}{2r^2 + 2\left(c-\frac{1}{2}\right)^2 + \frac{1}{2}} < S(r,a) < \frac{1}{2r^2 + 2\left(c-\frac{1}{2}\right)^2 - \frac{1}{2}}. \tag{33}$$

Proof: By standard argument, we obtain

$$\begin{aligned}
&\frac{1}{\left[(k-c)-\frac{1}{2}\right]^2 + r^2 - \frac{1}{4}} - \frac{1}{\left[(k-c)+\frac{1}{2}\right]^2 + r^2 - \frac{1}{4}} \\
&= \frac{2(k-c)}{\left[(k-c)^2 + r^2 - (k-c)\right]\left[(k-c)^2 + r^2 + (k-c)\right]} \\
&> \frac{2(k-c)}{\left[(k-c)^2 + r^2\right]^2 - (k-c)^2} \\
&> \frac{2(k-c)}{\left[(k-c)^2 + r^2\right]^2} \\
&> \frac{2(k-c)}{\left[(k-c)^2 + r^2\right]^2 + r^2 + \frac{1}{4}} \\
&= \frac{2(k-c)}{\left\{\left[(k-c)-\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}\right\}\left\{\left[(k-c)+\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}\right\}} \\
&= \frac{1}{\left[(k-c)-\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}} - \frac{1}{\left[(k-c)+\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}},
\end{aligned} \tag{34}$$

summing up for $k = 1, 2, \ldots$ yields inequalities in (33). □

Remark 5 If letting $c = 0$, inequality (2) is deduced from (33).

Inequalities (24), (28) and (33) for every case do not include each other. This can be verified by using the well known software Mathematica [24].

It is also worthwhile to note that inequality

$$\frac{1}{c^2 + \frac{1}{2}} < \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha + c^2)^2} < \frac{1}{c^2} \tag{35}$$

obtained in [16, 18] and mentioned in [17] is a wrong result.

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Gerhard Preuss

Hyperräume – von den Ideen Hausdorff's bis in die Gegenwart

Herrn Professor Harry Poppe anläßlich seines 70. Geburtstages gewidmet

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Ist **X** ein Raum (z.B. ein metrischer Raum, ein topologischer Raum, ein uniformer Raum oder ein semiuniformer Konvergenzraum), so ist ein **Hyperraum** von **X** (Bezeichnung: $H(\mathbf{X})$) ein Raum, dessen Punkte geeignete Teilmengen von **X** sind und in den **X** eingebettet werden kann (evtl. unter zusätzlichen Bedingungen). Man sagt, daß die Einbettung von **X** in $H(\mathbf{X})$ irgend eine Eigenschaft E bewahrt (bzw. reflektiert), wenn $H(\mathbf{X})$ (bzw. **X**) die Eigenschaft E besitzt, falls **X** (bzw. $H(\mathbf{X})$) die Eigenschaft E besitzt.

Hyperräume metrischer Räume sind erstmals 1914 von F. Hausdorff [6, S. 290ff] betrachtet worden:

Ist (X, d) ein metrischer Raum, so wird auf

$$\mathcal{F}(X) = \{E \subset X : E \text{ nicht-leer, abgeschlossen und beschränkt}\}$$

eine Metrik d_H definiert durch

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

d_H heißt **Hausdorff-Metrik**. Sind a, b Elemente von X, so gilt

$$d_H(\{a\}, \{b\}) = d(a, b)\,,$$

d.h. $i : (X, d) \to (\mathcal{F}(X), d_H)$, definiert durch $i(x) = \{x\}$ für alle $x \in X$, ist eine metrische Einbettung.

Häufig wird für die Hausdorff-Metrik folgende äquivalente Formulierung benutzt:

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset \mathcal{U}_d(B, \varepsilon) \text{ und } B \subset \mathcal{U}_d(A, \varepsilon)\}\,,$$

wobei $\mathcal{U}_d(C, \delta) = \{x \in X : d(x, c) < \delta \text{ für irgendein } c \in C\}$, falls $\delta > 0$ und $C \subset X$, d.h. $\mathcal{U}_d(C, \delta)$ ist die Vereinigung aller offenen δ-Kugeln um c für jedes $c \in C$.

Der **Satz von Hahn** (1932) besagt, daß *für jeden vollständigen metrischen Raum* (X, d) *der Hyperraum* $(\mathcal{F}(X), d_H)$ *vollständig* ist (s. [5]).
Bereits 1922 hat L. Vietoris [14] Hyperräume topologischer Räume studiert: Für jeden topologischen Raum $(X, \mathcal{X})$ bezeichne $\mathcal{A}(X)$ die Menge aller nicht-leeren, abgeschlossenen Teilmengen von X . Dann ist

$$\mathcal{B} = \{\langle U_1, \ldots, U_k\rangle : (U_i)_{i\in\{1,\ldots,k\}} \text{ endliche Folge offener Teilmengen von } (X, \mathcal{X})\}$$

Basis einer Topologie $\mathcal{A}(\mathcal{X})$ auf $\mathcal{A}(X)$ (d.h. jede in $(\mathcal{A}(X), \mathcal{A}(\mathcal{X}))$ offene Menge ist Vereinigung von Elementen aus $\mathcal{B}$), wobei

$$\langle U_1, \ldots, U_k\rangle = \{B \in \mathcal{A}(X) : B \subset \bigcup_{i=1}^{n} U_i \text{ und } B \cap U_i \neq \emptyset \text{ für jedes } i \in \{1, \ldots, k\}\}$$

$(\mathcal{A}(X), \mathcal{A}(\mathcal{X}))$ heißt **Vietoris'scher Hyperraum** von $(X, \mathcal{X})$. Falls $(X, \mathcal{X})$ T_1-Raum ist (d.h. die einpunktigen Teilmengen von $(X, \mathcal{X})$ sind abgeschlossen), ist
$i : (X, \mathcal{X}) \to (\mathcal{A}, (X), \mathcal{A}(\mathcal{X}))$, definiert durch $i(x) = \{x\}$, eine topologische Einbettung, d.h. $(X, \mathcal{X})$ ist Unterraum seines Hyperraumes.
Der Zusammenhang zwischen dem Hausdorff'schen und dem Vietoris'schen Ansatz ist gegeben durch folgendes

Lemma *Sei* (X, d) *ein metrischer Raum und*

$$\mathcal{K}(X) = \{K \subset X : K \textit{ nicht-leer und kompakt}\}.$$

Dann stimmt die von der Hausdorff-Metrik auf $\mathcal{K}(X)$ *induzierte Topologie mit der Vietoris-Topologie auf* $\mathcal{K}(X)$ *überein.*

Für den Fall, daß $(X, \mathcal{X})$ ein T_1-Raum ist, hat E. Michael [8] 1951 einige topologische Invarianten angegeben, die von der Einbettung von $(X, \mathcal{X})$ in den Vietoris'schen Hyperraum $(\mathcal{A}(X), \mathcal{A}, (\mathcal{X}))$ bewahrt und reflektiert werden:

1) quasikompakt (bzw. kompakt),

2) lokal kompakt

3) separabel.

Ist (X, d) ein kompakter metrischer Raum, so stimmen $\mathcal{F}(X)$, $\mathcal{A}(X)$ und $\mathcal{K}(X)$ überein und aufgrund obigen Lemmas ist, falls $\mathcal{X}_d$ die von d induzierte Topologie bezeichnet, $(\mathcal{A}(X), \mathcal{A}(\mathcal{X}_d))$ ein kompakter topologischer Raum, der durch die Hausdorff-Metrik metrisiert werden kann.

Bekanntlich heißt ein lokal zusammenhängendes Kontinuum ein *Peano-Raum*, wobei ein Kontinuum wie üblich ein kompakter, zusammenhängender metrischer Raum ist.
Unter der Voraussetzung, daß X ein Kontinuum ist, haben 1923 L. Vietoris [15] ("$\Leftarrow$") und T. Wazewski [16] ("$\Rightarrow$") gezeigt:

$$\mathcal{A}(X) \text{ Peano-Raum } \Leftrightarrow X \text{ Peano-Raum}$$

(Der Satz von Hahn/Mazurkiewicz besagt, daß ein Hausdorff-Raum genau dann ein Peano-Raum ist, wenn er stetiges Bild des Einheitsintervalls $[0, 1]$ ist.)
Ebenfalls unter der Voraussetzung, daß X ein Kontinuum ist, haben 1974 D. W. Curtis und R. M. Schori [4] folgendes tiefliegendes Resultat erzielt (durch Anwendung der Methoden der unendlich-dimensionalen Topologie):

> *$\mathcal{A}(X)$ ist homöomorph zum Hilbert-Quader genau dann, wenn X ein Peano-Raum mit mehr als einem Punkt ist.*

(Der Hilbert-Quader ist das Produkt von abzählbar vielen Kopien des Einheitsintervalls $[0, 1]$).
Uniforme Konzepte, die für metrische Räume einen Sinn ergeben, wie etwa Vollständigkeit, können in topologischen Räumen nicht erklärt werden. Deshalb wurden 1937 von A. Weil [17] uniforme Räume als Verallgemeinerung metrischer Räume eingeführt. 1940 definierte N. Bourbaki [2, p. 97, ex. 7)] Hyperräume uniformer Räume wie folgt:
Ist $(X, \mathcal{V})$ ein separierter uniformer Raum (d.h. der zugrundeliegende topologische Raum ist Hausdorff'sch), bezeichnet $\mathcal{A}$ die Menge der nicht-leeren abgeschlossenen Teilmengen von X und setzt man für jedes $V \in \mathcal{V}$

$$H(V) = \{(A, B) \in \mathcal{A} \times \mathcal{A} : A \subset V[B] \text{ und } B \subset V[A]\},$$

so ist $\{H(V) : V \in \mathcal{V}\}$ Basis einer Uniformität $H(\mathcal{V})$ für $\mathcal{A}$. Es gilt:

(1) $i : (X, \mathcal{V}) \to (\mathcal{A}, H(\mathcal{V}))$, definiert durch $i(x) = \{x\}$, ist eine uniforme Einbettung,

(2) $(\mathcal{A}, H(\mathcal{V}))$ ist separiert.

$(\mathcal{A}, H(\mathcal{V}))$ heißt der **uniforme Hyperraum** von $(X, \mathcal{V})$.
Ist $(X, \mathcal{V})$ ein metrisierbarer uniformer Raum (d.h. es gibt eine Metrik auf X , die o.B.d.A. als beschränkt angenommen werden kann, so daß

$$V \in \mathcal{V} \;\Leftrightarrow\; \text{Es existiert } \varepsilon > 0 \text{ mit } \{(x, y) \in X \times X : d(x, y) < \varepsilon\} \subset V),$$

so ist $(\mathcal{A}, H(\mathcal{V}))$ metrisierbar mit Hilfe der Hausdorff-Metrik d_H. Falls $(X, \mathcal{V})$ außerdem vollständig ist, ist aufgrund des Satzes von Hahn auch $(\mathcal{A}, H(\mathcal{V}))$ vollständig, allerdings:

$(\mathcal{A}, H(\mathcal{V}))$ ist i.a. selbst dann **nicht** vollständig, wenn $(X, \mathcal{V})$ ein vollständiger uniformer Raum ist (vgl. dazu J. Isbell [7]).
Im vergangenen Jahrhundert sind viele Versuche unternommen worden, topologische und (oder) uniforme Räume zu verallgemeinern. Ein besonders nützliches Konzept stellen die semiuniformen Konvergenzräume dar, die sowohl topologische als auch uniforme Aspekte in voller Allgemeinheit berücksichtigen (vgl. hierzu G. Preuß [12]).

[Zur Erinnerung: Ein *semiuniformer Konvergenzraum* ist ein Paar $(X, \mathcal{J}_X)$, wobei X eine Menge und $\mathcal{J}_X$ eine Menge von Filtern auf $X \times X$ ist derart, daß gelten:

$UC1)$ $\dot{x} \times \dot{x} = \{M \subset X \times X : (x,x) \in M\} \in \mathcal{J}_X$ für alle $x \in X$

$UC2)$ $\mathcal{G} \in \mathcal{J}_X$, falls $\mathcal{F} \in \mathcal{J}_X$ und $\mathcal{F} \subset \mathcal{G}$

$UC3)$ $\mathcal{F} \in \mathcal{J}_X$ impliziert $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\} \in \mathcal{J}_X$,
wobei $F^{-1} = \{(y,x) : (x,y) \in F\}$]

Im folgenden sei $\mathcal{A}$ eine Menge von nicht-leeren Teilmengen eines semiuniformen Konvergenzraumes $(X, \mathcal{J}_X)$ derart, daß $\{x\} \in \mathcal{A}$ für jedes $x \in X$. Wird eine injektive Abbildung $i : X \to \mathcal{A}$ definiert durch $i(x) = \{x\}$ für jedes $x \in X$, so setze man $i[X] = X'$ und nehme o.B.d.A. an, daß $X = X'$ ist, d.h. i ist eine Inklusionsabbildung. Auf $\mathcal{A}$ werde eine semiuniforme Konvergenzstruktur $\mathcal{J}^f_{\mathcal{A}}$ definiert durch

$$\begin{aligned} &\mathcal{J}^f_{\mathcal{A}} = \{\mathcal{H} \in F(\mathcal{A} \times \mathcal{A}) : (i \times i)^{-1}(\mathcal{A}) \text{ existiert und gehört zu } \mathcal{J}_X \\ &\text{oder } (i \times i)^{-1}(\mathcal{A}) \text{ existiert nicht}\}\,, \end{aligned}$$

wobei $F(\mathcal{A} \times \mathcal{A})$ die Menge aller Filter auf $\mathcal{A} \times \mathcal{A}$ bezeichnet. Dann ist $(X, \mathcal{J}_X)$ ein Unterraum von $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ und $\mathcal{J}^f_{\mathcal{A}}$ ist die gröbste semiuniforme Konvergenzstruktur auf $\mathcal{A}$ mit dieser Eigenschaft. $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ heißt **finaler Hyperraum** von $(X, \mathcal{J}_X)$.

Satz [13, 1.8] *Für jeden semiuniformen Konvergenzraum $(X, \mathcal{J}_X)$ ist der finale Hyperraum $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ vollständig, falls $\mathcal{A} \setminus X$ nicht leer ist, und enthält X als dichte Teilmenge, d.h. er ist eine Vervollständigung von $(X, \mathcal{J}_X)$.*

Ist $(X, \mathcal{V})$ ein separierter uniformer Raum sowie $(X, [\mathcal{V}])$ sein entsprechender semiuniformer Konvergenzraum, d.h. $[\mathcal{V}] = \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset \mathcal{V}\}$, und besteht $\mathcal{A}$ aus allen nicht-leeren abgeschlossenen Teilmengen von X, so ist $(\mathcal{A}, [H(\mathcal{V})])$ ein separierter semiuniformer Konvergenzraum, der $(X, [\mathcal{V}])$ als Unterraum enthält. Folglich ist $[H(\mathcal{V})]$ feiner als $\mathcal{J}^f_{\mathcal{A}}$, d.h. $[H(\mathcal{V})] \subset \mathcal{J}^f_{\mathcal{A}}$. Der finale Hyperraum $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ braucht jedoch nicht uniform zu sein, wie folgendes *Beispiel* zeigt: Sei $\mathcal{V}$ die diskrete Uniformität auf $\{0,1\}$. Dann besteht die Menge $\mathcal{A}$

aller abgeschlossenen nicht-leeren Teilmengen von $(\{0,1\},\mathcal{V})$ aus genau drei Elementen. Es gibt jedoch keine gröbste Uniformität auf $\mathcal{A}$, die $\mathcal{V}$ induziert (vgl. [12, 3.2.7.②]).
Die Einbettung eines semiuniformen Konvergenzraumes in seinen finalen Hyperraum bewahrt und reflektiert u.a. die Eigenschaften "semiuniform *" und "präkompakt", und sie bewahrt "kompakt", "zusammenhängend" und "uniform zusammenhängend" (vgl. [13]).
Seit längerem werden auch Zusammenhänge zwischen Hyperräumen und Funktionenräumen studiert (vgl. hierzu S. A. Naimpally [10]). Ein jüngeres Resultat von T. Mizokami ([9]) besagt, daß für Hausdorff-Räume X, Y die Menge $C(X,Y)$ der stetigen Abbildungen zwischen X und Y, versehen mit der kompakt-offenen Topologie, abgeschlossen eingebettet werden kann in $C(\mathcal{K}(X),\mathcal{K}(Y))$, versehen mit der punktweisen Konvergenz, wobei $\mathcal{K}(X)$ (bzw. $\mathcal{K}(Y)$) die Menge der nicht-leeren kompakten Teilmengen von X (bzw. Y) ist, versehen mit der Vietoris-Topologie. Durch Ausweitung der Ideen von Mizokami gelingt es schließlich R. Bartsch [1], einem Schüler von H. Poppe, im Jahre 2002 Sätze vom Ascoli-Typ zu beweisen, d.h. Kompaktheitskriterien in Funktionenräumen zu entwickeln. In diesem Zusammenhang muß auch das Buch "Compactness in General Function Spaces" von H. Poppe [11] erwähnt werden, das internationale Beachtung gefunden hat und in das eigene Ergebnisse von H. Poppe zum Ascoli-Satz eingeflossen sind.

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Laure Cardoulis

Existence of Solutions for an Elliptic Equation Involving a Schrödinger Operator with Weight in all of the Space

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ABSTRACT. In this paper, we obtain some results about the existence of solutions for the following elliptic semilinear equation $(-\Delta + q)u = \lambda mu + f(x,u)$ in $\mathbb{R}^N$ where q is a positive potential satisfying $\lim_{|x|\to+\infty} q(x) = +\infty$ and m is a bounded positive weight.

1 Introduction

In this paper, we study the existence of solutions for the elliptic semilinear equation:

$$(-\Delta + q)u = \lambda mu + f(x,u) \text{ in } \mathbb{R}^N \tag{1}$$

where the following hypotheses are satisfied:

(h1) $q \in L^2_{loc}(\mathbb{R}^N)$ such that $\lim_{|x|\to+\infty} q(x) = +\infty$ and $q \geq \text{const} > 0$.

(h2) $m \in L^\infty(\mathbb{R}^N)$ such that $\exists m_1 \in \mathbb{R}^{*+}$, $\exists m_2 \in \mathbb{R}^{*+}$, $\forall x \in \mathbb{R}^N$, $0 < m_1 \leq m(x) \leq m_2$.

We will specify later the hypothesis on f. We denote by λ a real parameter.
The variational space is denoted by $V_q(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N),\ (-\Delta + q)u \in L^2(\mathbb{R}^N)\}$ which is the completed of $\mathcal{D}(\mathbb{R}^N)$ for the norm $\|u\|_q = \sqrt{\int_{\mathbb{R}^N} |\nabla u|^2 + qu^2}$.
Recall (see [1] for example) that the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.
We denote by $\|u\|_m = \sqrt{\int_{\mathbb{R}^N} mu^2}$ for all $u \in L^2(\mathbb{R}^N)$. According to the hypothesis $(h2)$, $\|.\|_m$ is a norm in $L^2(\mathbb{R}^N)$ equivalent to the usual norm. We denote by M the operator of multiplication by m in $L^2(\mathbb{R}^N)$. The operator $(-\Delta + q)^{-1}M : (L^2(\mathbb{R}^N), \|.\|_m) \to (L^2(\mathbb{R}^N), \|.\|_m)$ is positive self-adjoint and compact. So its spectrum is discrete and consists of a positive sequence $\mu_1 \geq \mu_2 \geq ...\mu_n \to 0$ when $n \to +\infty$. We denote by $\lambda_1 = \frac{1}{\mu_1}$ and u_1 the corresponding eigenfunction which satisfy $(-\Delta + q)u_1 = \lambda_1 mu_1$ in $\mathbb{R}^N$ and $\|u_1\|_m = 1$. (We know that

λ_1 is simple and $u_1 > 0$ (see [2, Th2.2]).) By the Courant-Fischer formulas, λ_1 is given by:

$$\lambda_1 = \inf\{\frac{\int_{\mathbb{R}^N} |\nabla\phi|^2 + q\phi^2}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in \mathcal{D}(\mathbb{R}^{\mathcal{N}})\}.$$

We recall now some results already obtained for the existence of solutions in the linear cases or semilinear cases.
Using the Lax-Milgram theorem and the above characterization of λ_1, we obtain the following result:

Theorem 1.1 (see [3],[4]) *We consider the linear case (i.e. $f(x,u) = f(x)$.) Assume that the hypotheses $(h1)$ and $(h2)$ are satisfied and that $f \in L^2(\mathbb{R}^N)$. If $\lambda < \lambda_1$, then the equation (1) has a unique solution $u_\lambda \in V_q(\mathbb{R}^N)$. Moreover, the Maximum Principle is satisfied i.e.: if $f \geq 0$ and $\lambda < \lambda_1$ then $u_\lambda \geq 0$.*
If $\lambda = \lambda_1$ (which is the case of the Fredholm Alternative), then the equation (1) admits a solution iff $\int_{\mathbb{R}^N} fu_1 = 0$.

Using a method of sub- and supersolutions and a Schauder Fixed Point Theorem (see [3]) or an approximation method (see [4]), we get the following results in the semilinear case:

Theorem 1.2

1. *(see [3]). Assume that the hypotheses $(h1)$ and $(h2)$ are satisfied. Assume also that f is Lipschitz in u uniformly in x and that:*
   $\exists\theta \in L^2(\mathbb{R}^N), \theta > 0, \forall u \geq 0,\ 0 \leq f(x,u) \leq su + \theta.$
   If $\lambda < \lambda_1$, the equation (1) has at least a positive solution.

2. *(see [4]). Assume that the hypothesis $(h1)$ is satisfied, $N \geq 3$ and $0 \leq m \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$. Assume also that f is Lipschitz in u uniformly in x and that: $\exists\theta \in L^2(\mathbb{R}^N)$, $\forall u \in L^2(\mathbb{R}^N),\ |f(x,u)| \leq \theta$.*
   If $\lambda < \lambda_1$, then the equation (1) has at least a solution.

Finally, for the linear case (i.e. $f(x,u) = f(x)$), assuming $N = 2$, m a radial weight and q a radial potential with some strong properties of growth at infinity (not recalled here) (see [5]), we obtain the following result for the Antimaximum Principle:

Theorem 1.3 (see [5]) *Assume that the hypotheses $(h1)$ and $(h2)$ are satisfied.*
We denote by $X^{1,2} = \{f \in L^2_{loc}(\mathbb{R}^2), \frac{\partial f}{\partial\theta}(r,.) \in L^2(-\pi,\pi)$ for all $r > 0,$ and $\exists C \geq 0,$
$\|f(r,\theta)\| + (\frac{1}{2\pi}\int_{-\pi}^{\pi} |\frac{\partial f}{\partial\theta}(r,\theta)|^2 d\theta)^{\frac{1}{2}} \leq Cu_1(r)$ for all $r \geq 0$ and $\theta \in]-\pi,\pi].\}$
Assume that $f \geq 0$ in $\mathbb{R}^2$, $f > 0$ in a subset with a non zero Lebesgue measure and $f \in X^{1,2}$.
Let u be a solution of the equation (1).
Then $\exists\delta(f) > 0, \forall\lambda \in (\lambda_1, \lambda_1 + \delta(f)), \exists c(\lambda,f) > 0,\ u \leq -c(\lambda,f)u_1$.

In this paper, we sudy the existence of solutions for the equation (1) in the case $\lambda > \lambda_1$, λ near λ_1.
For the linear case (i.e. $f(x,u) = f(x)$), if $\lambda \in (\lambda_1, \lambda_2)$, $\lambda_2 = \frac{1}{\mu_2}$ where μ_2 is the second eigenvalue of $(-\Delta + q)^{-1}M$, then there are obviously existence and uniqueness of a solution for the equation (1).
In the second section, following a bifurcation method developped in [6], we get the following result:

Theorem 1.4 *Assume that the hypotheses* $(h1)$ *and* $(h2)$ *are satisfied. Assume also that* $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *(defined by* $f(x,y)$*) satisfies the following hypothesis* $(h3)$*:*

i) $f(x,0) = 0$.

ii) f *is Frechet differentiable with respect to the second variable* y *and its derivative* $f'_y(x,.)$ *is continuous and bounded, uniformly in* x.

iii) $f'_y(x,0) = 0$.

Then there exists for λ *sufficiently near* λ_1 *a nontrivial solution for the equation* (1)*.*

Finally, in the third section, following a method developped in [7] for the p-Laplacian in a bounded domain of $\mathbb{R}^N$, we get results for the case where $f(x,u) = f(x)|u(x)|^{\gamma-2}u(x)$. Before stating the results, we need some notations. We define for $C \in \mathbb{R}^{*+}$ the set $X_{q,C} = \{u \in V_q(\mathbb{R}^N),\ u_1 \le u \le C \quad a.e.\}$.
Let $F(u) := \int_{\mathbb{R}^N} f|u|^\gamma$ for all $u \in V_q(\mathbb{R}^N)$.
Let $\lambda^* = \sup_{u\in V_q(\mathbb{R}^N), u\ge 0}\{\inf_{\phi\in V_q(\mathbb{R}^N)}\{\frac{\int_{\mathbb{R}^N}\nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} mu\phi},\ F'(u)(\phi) \ge 0,\ \phi \ge 0\}\}$ and
$\lambda^{**} = \sup_{u\in X_{q,C}}\{\inf_{\phi\in V_q(\mathbb{R}^N)}\{\frac{\int_{\mathbb{R}^N}\nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} mu\phi},\ F'(u)(\phi) \ge 0,\ \phi \ge 0\}\}$.
(Note that $\lambda^{**} \le \lambda^*$.)
We consider also hypotheses of the following forms:

(h4) $\lambda_1 < \lambda^{**} \le \lambda^* < +\infty$.

(h5) $f \in L^\infty(\mathbb{R}^N)$.

(h6) The sets $\Omega^+ = \{x \in \mathbb{R}^N, f(x) > 0\}$ and $\Omega^- = \{x \in \mathbb{R}^N, f(x) < 0\}$ have non zero measures.

(h7) $f \ge -\frac{\epsilon u_1 m}{l^{\gamma-2}C^{\gamma-1}}$.

Theorem 1.5 *Assume that the hypotheses* $(h1)$ *and* $(h2)$ *are satisfied,* $N = 3, 4$ *so that* $\gamma = 2^* = \frac{2N}{N-2} \in \mathbb{N}^*$.

1. *If in addition the hypotheses* $(h4)$ *and* $(h5)$ *are satisfied, and if* $\lambda > \lambda^*$, *then the equation* (1) *has no positive solution.*

2. *Assume additionally that the hypotheses* $(h4)-(h7)$ *are satisfied, where the numbers* $l \geq 1$, $\epsilon > 0$, ϵ *involved in (h7) are small enough such that* $\lambda_1 \leq \epsilon\gamma l^{\gamma-2}$ *and* $\epsilon < \frac{\lambda_1}{\gamma}$. *If there holds* $\lambda_1 + \epsilon l^{\gamma-2} < \lambda < \lambda^{**}$ *with the same numbers* ϵ, l *as in (h7), then the equation* (1) *has at least a positive solution.*

2 A bifurcation result

In this section, we follow a method developped in [6].
We obtain some results of the existence of solutions for the semilinear equation

$$(-\Delta + q)u = \lambda m u + f(x,u) \text{ in } \mathbb{R}^N \tag{1}$$

by considering bifurcating solutions from the zero solution. We suppose that the hypotheses $(h1), (h2), (h3)$ are satisfied in all this section. We denote by $< .,. >_q$ the inner product in $V_q(\mathbb{R}^N)$. We define the operator $T : \mathbb{R} \times V_q(\mathbb{R}^N) \to V_q(\mathbb{R}^N)$ by: $\forall \phi \in V_q(\mathbb{R}^N)$

$$< T(\lambda, u), \phi >_q = \int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi - \lambda \int_{\mathbb{R}^N} mu\phi - \int_{\mathbb{R}^N} f(x, u(x))\phi(x)dx.$$

Lemma 2.1 *The operator* T *is well defined.*

Proof: Let $u \in V_q(\mathbb{R}^N)$. We introduce
$F(\phi) = \int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi - \lambda \int_{\mathbb{R}^N} mu\phi - \int_{\mathbb{R}^N} f(x,u(x))\phi(x)dx$ for all $\phi \in V_q(\mathbb{R}^N)$.
Since m is bounded, f is Lipschitz in u uniformly in x and $f(x,0) = 0$, we deduce that: $\forall \phi \in V_q(\mathbb{R}^N)$, $|F(\phi)| \leq \text{const} \cdot ||u||_q ||\phi||_q$. The operator F is linear and continuous. By the Riesz Theorem, we can well define the operator T. □

Lemma 2.2 *The operator* T *is continuous, Frechet differentiable with continuous derivatives given by:* $\forall \phi \in V_q(\mathbb{R}^N)$, $\forall \psi \in V_q(\mathbb{R}^N)$,

$$< T'_u(\lambda, u)\phi, \psi >_q = \int_{\mathbb{R}^N} \nabla\phi.\nabla\psi + q\phi\psi - \lambda \int_{\mathbb{R}^N} m\phi\psi - \int_{\mathbb{R}^N} f'_y(x, u(x))\phi(x)\psi(x)dx.$$

$$< T'_\lambda(\lambda, u), \phi >_q = -\int_{\mathbb{R}^N} mu\phi \;\; ; \;\; < T''_{\lambda u}(\lambda, u)\phi, \psi >_q = -\int_{\mathbb{R}^N} m\phi\psi.$$

Proof: We do not give here the details of the proof which is technical but simple. Since m is bounded and f is Lipschitz in u uniformly in x, we obtain the continuity of T and T'_λ. By using the hypothesis that $f'_y(x, .)$ is bounded uniformly in x and using the Lebesgue Dominated Convergence Theorem, we get the continuity of T'_u. □

Remarks $T'_u(\lambda_1, 0)$ is a continuous self-adjoint operator (by $(h3)$); the kernel $N(T'_u(\lambda_1, 0))$ is generated by u_1. So $dim\, N(T'_u(\lambda_1, 0)) = 1 = dim\, R(T'_u(\lambda_1, 0))$. Moreover $T''_{\lambda u}(\lambda_1, 0)u_1 \notin R(T'_u(\lambda_1, 0))$.
Indeed, denote by $< u_1 >$ the sub-space of $V_q(\mathbb{R}^N)$ generated by u_1. Since $T'_u(\lambda_1, 0)$ is a self-adjoint operator, the range $R(T'_u(\lambda_1, 0))$ of $T'_u(\lambda_1, 0)$ is the orthogonal of $< u_1 >$. But $< T''_{\lambda u}(\lambda_1, 0)u_1, u_1 >_q = -\int_{\mathbb{R}^N} m u_1^2 < 0$.
So $T''_{\lambda u}(\lambda_1, 0)u_1 \notin R(T'_u(\lambda_1, 0))$.
We can now apply the Theorem 1.7 in [8] to obtain a local bifurcation result.

Theorem 2.1 *Assume that the hypotheses $(h1) - (h3)$ are satisfied. Then there exist a number $\epsilon_0 > 0$, and two continuous functions $\eta : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ and $\psi : (-\epsilon_0, \epsilon_0) \to < u_1 >^\perp$ such that: $\eta(0) = \lambda_1$, $\psi(0) = 0$ and all non trivial solutions of $T(\lambda, u) = 0$ in a small neighbourhood of $(\lambda_1, 0)$ have the form $(\lambda_\epsilon, u_\epsilon) = (\eta(\epsilon), \epsilon u_1 + \epsilon\psi(\epsilon))$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$.*

Remark $T(\lambda, u) = 0$ iff u is solution of the equation (1). So near λ_1 (including the cases where $\lambda > \lambda_1$), the equation (1) admits non trivial solutions.

Adding another hypothesis on f, we are going to study now the sign of u_ϵ for $\epsilon \in (-\epsilon_0, \epsilon_0)$. First, we study the asymptotic behaviour of each solution of the equation (1).

Lemma 2.3 *Assume that the hypothesis $(h1) - (h3)$ are satisfied. Let u be a solution of the equation (1). Then $\lim_{|x| \to +\infty} u(x) = 0$.*

Proof: We have in a weak sense: $(-\Delta + q)u = \lambda m u + f(x, u) = [\lambda m + \frac{f(x,u)}{u}]u$ in $\mathbb{R}^N$. By $(h3)$, $\exists K > 0$, $|f(x, u)| \leq K|u|$. Using $(h2)$ we obtain that $\lambda m + \frac{f(x,u)}{u} \in L^\infty(\mathbb{R}^N)$. This implies by Theorem 4.1.3 in [3] combining with Theorem 8.17 in [9] that $\lim_{|x| \to +\infty} u(x) = 0$. □

Theorem 2.2 *Assume that the hypotheses $(h1) - (h3)$ are satisfied. Assume also that the following hypothesis $(h'3)$ is satisfied where:*
$(h'3)$ $\exists R > 0$, $\exists \epsilon^* > 0$, $\forall x \in \mathbb{R}^N$, $\forall y \in \mathbb{R}^{*-}$, $|x| > R$ *and* $|\lambda - \lambda_1| < \epsilon^* \Rightarrow \lambda m(x)y + f(x, y) > 0$.
Then $u_\epsilon \geq 0$ for ϵ small enough.

Proof:

i) Recall that $\lim_{|x| \to +\infty} u_\epsilon(x) = 0$.

ii) Let $0 < \epsilon < \epsilon_0$. We have: $\forall x$, $u_\epsilon(x) = \epsilon u_1(x) + \epsilon\psi(\epsilon)(x)$. Since $u_1 > 0$ and $\psi(\epsilon) \to 0$ when $\epsilon \to 0$, we deduce that:$\exists \epsilon_1 > 0$, $0 < \epsilon < \epsilon_1 \Rightarrow \forall x \in B(0, R)$, $u_\epsilon(x) > 0$.
We suppose that: $\exists x_0 \in \mathbb{R}^N$, $u_\epsilon(x_0) < 0$. Since $\lim_{|x| \to +\infty} u_\epsilon(x) = 0$, we deduce that

there exists $x_1 \in \mathbb{R}^N$, $|x_1| > R$ such that u_ϵ has a negative minimum in x_1.
If $(-\Delta + q)(u_\epsilon)(x_1) > 0$, then there exists a bounded domain Ω, containing x_1 such that $\forall x \in \Omega$, $(-\Delta + q)(u_\epsilon)(x) \geq 0$.
By the Maximum Principle (see Corollary 3.2 in [9]), we have: $\inf_\Omega u_\epsilon = u_\epsilon(x_1) \geq \inf_{\partial\Omega} u_\epsilon^- \geq 0$ where $u_\epsilon^- = \max\{0, -u_\epsilon\}$. Since $u_\epsilon(x_1) < 0$, we get a contradiction. Therefore $(-\Delta + q)(u_\epsilon)(x_1) \leq 0$. Using $(h'3)$, we have also: $(-\Delta + q)(u_\epsilon)(x_1) = \lambda m(x_1)u_\epsilon(x_1) + f(x_1, u_\epsilon(x_1)) > 0$.
So we get again a contradiction. Therefore $u_\epsilon \leq 0$.

□

We sudy now the global nature of the continuum of solutions obtained by bifurcation from the $(\lambda_1, 0)$ solution. Using Theorems 1.3 and 1.40 in [10], we obtain the following result:

Theorem 2.3 *There exists a continuum $\mathcal{C}$ of non trivial solutions for the equation (1) obtained by bifurcation from the $(\lambda_1, 0)$ solution, which is either unbounded or contains a point $(\lambda, 0)$ where $\lambda \neq \lambda_1$ is the inverse of an eigenvalue of the operator L. (L is defined by $< Lu, \phi >_q = \int_{\mathbb{R}^N} mu\phi$.) Since $\lambda_1$is simple, $\mathcal{C}$ has two connected subsets $\mathcal{C}^+$ and $\mathcal{C}^-$ which satisfy also the above alternatives.*

Proof:

i) We define an operator S by setting $S(\lambda, u) = u - T(\lambda, u)$ i.e. $\forall \phi \in V_q(\mathbb{R}^N)$,

$$< S(\lambda, u), \phi >_q = \int_{\mathbb{R}^N} [\lambda mu\phi + f(x, u)\phi].$$

So u is a solution of the equation (1) iff $u = S(\lambda, u)$. We write $S(\lambda, u) = \lambda Lu + H(\lambda, u)$ where $< Lu, \phi >_q = \int_{\mathbb{R}^N} mu\phi$ and $< H(\lambda, u), \phi >_q = \int_{\mathbb{R}^N} f(x, u)\phi$.

ii) For applying the results in [10], we must prove that $S : \mathbb{R} \times V_q(\mathbb{R}^N) \to V_q(\mathbb{R}^N)$ is continuous and compact, that $L : V_q(\mathbb{R}^N) \to V_q(\mathbb{R}^N)$ is linear and compact, that $H(\lambda, u) = O(\|u\|)$ for u near 0 uniformly on bounded intervals of λ and that $\frac{1}{\lambda_1}$ is a simple eigenvalue of L (which is true because it's a simple eigenvalue of $(-\Delta + q)^{-1}M$.)

iii) We show here that S is continuous and compact. S is continuous since T is continuous. Let $((\lambda_n, u_n))_n$ be a bounded sequence in $\mathbb{R} \times V_q(\mathbb{R}^N)$. Since the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $((\lambda_n, u_n))_n$ in $\mathbb{R} \times L^2(\mathbb{R}^N)$.
We have: $\forall \phi \in V_q(\mathbb{R}^N)$,

$< S(\lambda_n, u_n) - S(\lambda_p, u_p), \phi >_q = \lambda_n \int_{\mathbb{R}^N} m u_n \phi - \lambda_p \int_{\mathbb{R}^N} m u_p \phi + \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_p)]\phi$.
So $\|S(\lambda_n, u_n) - S(\lambda_p, u_p)\|_q^2 = (\lambda_n - \lambda_p) \int_{\mathbb{R}^N} m u_n [S(\lambda_n, u_n) - S(\lambda_p, u_p)]$

$$+\lambda_p \int_{\mathbb{R}^N} m(u_n - u_p)[S(\lambda_n, u_n) - S(\lambda_p, u_p)] + \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_p)][S(\lambda_n, u_n) - S(\lambda_p, u_p)].$$

By $(h2)$ and $(h3)$ we deduce that $(S(\lambda_n, u_n))_n$ is a Cauchy sequence and therefore a convergent sequence. So S is compact.

iv) We show here that L is linear and compact. L is obviously linear and continuous.
Let $(u_n)_n$ be a bounded sequence in $V_q(\mathbb{R}^N)$. Since the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $(u_n)_n$ in $L^2(\mathbb{R}^N)$.
We have: $\|Lu_n - Lu_p\|_q^2 = \int_{\mathbb{R}^N} m(u_n - u_p)[Lu_n - Lu_p]$.
By the Cauchy-Schwartz inequality, we get: $\|Lu_n - Lu_p\|_q \leq cst\|u_n - u_p\|_{L^2(\mathbb{R}^N)}$.
Therefore $(Lu_n)_n$ is a Cauchy sequence and so L is compact.

v) Finally note that $H(\lambda, u)$ is independant of λ. We denote it $H(u)$. We have: $\|H(u)\|_q^2 = \int_{\mathbb{R}^N} f(x, u)H(u) \leq cst\|u\|_q \|H(u)\|_q$.
So $H(u) = O(\|u\|)$.

□

3 Existence of positive solutions

We follow here a method developped in [7] for the p-Laplacian in a bounded domain.
Our results are more restrictive than in [7] because of the unboundedness of our domain.
We consider the equation

$$(-\Delta + q)u = \lambda m u + f|u|^{\gamma-2}u \text{ in } \mathbb{R}^N \tag{1}$$

for which the hypotheses $(h1)$ and $(h2)$ are satisfied, and $N = 3, 4$ so that $\gamma = 2^* = \frac{2N}{N-2} \in \mathbb{N}^*$.
Our aim is to study the existence of positive solutions for the equation (1) where $\lambda > \lambda_1$.
We define for $C \in \mathbb{R}^{*+}$, $C \geq u_1$, the set $X_{q,C} = \{u \in V_q(\mathbb{R}^N),\ u_1 \leq u \leq C \quad a.e.\}$
Let $F(u) := \int_{\mathbb{R}^N} f|u|^\gamma$ and $H_\lambda(u) := \int_{\mathbb{R}^N} |\nabla u|^2 + qu^2 - \lambda \int_{\mathbb{R}^N} m u^2$ for all $u \in V_q(\mathbb{R}^N)$.
Let $\lambda^* = \sup_{u \in V_q(\mathbb{R}^N), u \geq 0} \{\inf_{\phi \in V_q(\mathbb{R}^N)} \{\frac{\int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} m u \phi},\ F'(u)(\phi) \geq 0,\ \phi \geq 0\}\}$ and
$\lambda^{**} = \sup_{u \in X_{q,C}} \{\inf_{\phi \in V_q(\mathbb{R}^N)} \{\frac{\int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} m u \phi},\ F'(u)(\phi) \geq 0,\ \phi \geq 0\}\}$.
(Note that $\lambda^{**} \leq \lambda^*$.)
Let $l \geq 1$, $\epsilon > 0$, ϵ be small enough such that $\lambda_1 \leq \epsilon\gamma l^{\gamma-2}$ and $\epsilon < \frac{\lambda_1}{\gamma}$.

Remark There holds $\lambda_1 \leq \lambda^*$. On the contrary, if $\lambda_1 > \lambda^*$, then by the characterization of λ_1 we have $H_{\lambda_1}(u_1) = 0$. By the definition of λ^*, $\exists \phi \in V_q(\mathbb{R}^N)$, $\phi \geq 0$, $F'(u_1)(\phi) \geq 0$, $\frac{\int_{\mathbb{R}^N} \nabla u_1.\nabla\phi + qu_1\phi}{\int_{\mathbb{R}^N} mu_1\phi} \leq \lambda^* < \lambda_1$.
So $H'_{\lambda_1}(u_1)(\phi) < 0$.
We have: $\forall \eta \in \mathbb{R}^{*+}$, $H_{\lambda_1}(u_1 + \eta\phi) = H_{\lambda_1}(u_1) + \eta H'_{\lambda_1}(u_1)(\phi) + \|\eta\phi\| h(\eta\phi)$ with $h(\eta\phi) \to 0$ when $\eta \to 0$. Therefore, for η small enough, we have $H_{\lambda_1}(u_1 + \eta\phi) < 0$ and this contradicts the definition of λ_1.

Theorem 3.1 *Assume that the hypotheses $(h1) - (h7)$ are satisfied, $N = 3, 4$ and $\gamma = 2^* = 2N/(N-2)$.*

a) *If $\lambda > \lambda^*$, then the equation (1) has no positive solution.*

b) *If $\lambda_1 + \epsilon l^{\gamma-2} < \lambda < \lambda^{**}$, then the equation (1) has at least a positive solution.*

Proof:

i) By $(h7)$ we have: $f \geq -\frac{\epsilon u_1 m}{l^{\gamma-2}C^{\gamma-1}} \geq -\frac{\lambda_1 m}{\gamma l^{\gamma-2}C^{\gamma-2}} \geq -\frac{\epsilon m}{u_1^{\gamma-2}}$.

ii) Since $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ with continuous imbedding, we deduce that $V_q(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ with continuous imbedding.

Note that $\forall \phi \in V_q(\mathbb{R}^N),\ F'(u)(\phi) = \gamma \int_{\mathbb{R}^N} f|u|^{\gamma-2}u\phi$ and
$H'_\lambda(u)(\phi) = 2\int_{\mathbb{R}^N}[\nabla u.\nabla\phi + qu\phi - \lambda mu\phi]$.
Note also that u is a solution of the equation (1) iff $\forall \phi \in V_q(\mathbb{R}^N),\ H'_\lambda(u)(\phi) = \frac{2}{\gamma}F'(u)(\phi)$.
Moreover, if $t \in \mathbb{R}^{*+}$, $F'(tu)(\phi) = t^{\gamma-1}F'(u)(\phi)$ and $H'_\lambda(tu)(\phi) = tH'_\lambda(u)(\phi)$.

Assume here that $\lambda > \lambda^*$.
So: $\forall u \in V_q(\mathbb{R}^N)$, $u \geq 0$, $\exists \phi \geq 0$, $F'(u)(\phi) \geq 0$ and $H'_\lambda(u)(\phi) < 0$. Therefore the equation (1) has no positive solution.

Assume now that $\lambda_1 + \epsilon l^{\gamma-2} < \lambda < \lambda^{**}$.
We are going to prove that the equation (1) admits at least a positive solution by using the sub and supper solutions method and a Schauder Fixed Point Theorem.

a) Note by the definition of λ^{**} that:
$\exists u^* \in X_{q,C}$, $\forall \phi \geq 0$, $F'(u^*)(\phi) \geq 0 \Rightarrow H'_\lambda(u^*)(\phi) > 0$. (e)
We suppose that $\forall 0 < t \leq l$, $\exists \psi_t \geq 0$, $H'_\lambda(tu^*)(\psi_t) < \frac{2}{\gamma}F'(tu^*)(\psi_t)$.

If $\forall t$, $\forall \psi_t$, $F'(tu^*)(\psi_t) \geq 0$, then:

Let $\phi \geq 0$ such that $F'(u^*)(\phi) < 0$.

So $\forall t > 0$, $H'_\lambda(tu^*)(\phi) \geq \frac{2}{\gamma}F'(tu^*)(\phi)$ i.e. $\forall t > 0$, $t^{\gamma-2}\int_{\mathbb{R}^N} f(u^*)^{\gamma-1}\phi \leq \int_{\mathbb{R}^N}[\nabla u^*.\nabla\phi + qu^*\phi - \lambda m u^*\phi]$.

When $t \to 0$, we get: $0 \leq H'_\lambda(u^*)(\phi)$.

So $F'(u^*)(\phi) < 0 \Rightarrow H'_\lambda(u^*)(\phi) \geq 0$.

Using the property (e), we get: $\forall\phi \geq 0$, $H'_\lambda(u^*)(\phi) \geq 0$.

In particular, for $\phi = u_1$, we obtain: $\lambda_1\int_{\mathbb{R}^N} mu^*u_1 \geq \lambda\int_{\mathbb{R}^N} mu^*u_1 > 0$.

Since $\lambda_1 < \lambda$, we get a contradiction.

If $\forall t$, $\forall\psi_t$, $F'(tu^*)(\psi_t) \leq 0$, then:

Let $\phi \geq 0$ such that $F'(tu^*)(\phi) > 0$. We have $H'_\lambda(tu^*)(\phi) \geq \frac{2}{\gamma}F'(tu^*)(\phi) > 0$.

So $\forall t$, $\int_{\mathbb{R}^N}[\nabla u^*.\nabla\phi + qu^*\phi - \lambda mu^*\phi] \geq t^{\gamma-2}\int_{\mathbb{R}^N} f(u^*)^{\gamma-1}\phi > 0$ and this is impossible for t large enough (because we can take a bigger l.)

Then we have: $\exists\phi \geq 0$, $\exists\psi \geq 0$, $H'_\lambda(u^*)(\phi) < \frac{2}{\gamma}t^{\gamma-2}F'(u^*)(\phi) < 0$ and $0 < H'_\lambda(u^*)(\psi) < \frac{2}{\gamma}t^{\gamma-2}F'(u^*)(\psi)$ (for at least one t).

Since $F'(u^*)$ is a continuous function, $\exists\alpha \in (0,1)$, $F'(u^*)(\alpha\phi + (1-\alpha)\psi) = 0$.

Therefore we deduce that $H'_\lambda(u^*)(\alpha\phi + (1-\alpha)\psi) > 0$.

But: $\frac{\alpha\gamma}{2t^{\gamma-2}}H'_\lambda(u^*)(\phi) < \alpha F'(u^*)(\phi) = -(1-\alpha)F'(u^*)(\psi) < -\frac{(1-\alpha)\gamma}{2t^{\gamma-2}}H'_\lambda(u^*)(\psi)$.

So $\frac{\gamma}{2t^{\gamma-2}}[\alpha H'_\lambda(u^*)(\phi) + (1-\alpha)H'_\lambda(u^*)(\psi)] < 0$ and we get a contradiction.

Therefore $\exists t \in (o, l]$, $\forall\phi \geq 0$, $H'_\lambda(tu^*)(\phi) \geq \frac{2}{\gamma}F'(tu^*)(\phi)$ i.e. tu^* is a supper solution of the equation (1). Note that $tu^* \geq su_1$ if $0 < s \leq t$. Let $s > 0$ such that $\frac{1}{s} \leq l^{\gamma-3}$. This is possible because we can choose l sufficiently big such that $\frac{1}{l^{\gamma-3}} \leq t \leq l$.

b) We show now that su_1 is a sub solution of the equation (1).

We have: $\frac{\lambda_1-\lambda}{s^{\gamma-2}} < -\epsilon$ (since $l \geq s$) and $f \geq -\frac{\epsilon m}{u_1^{\gamma-2}}$.

So: $fu_1^{\gamma-1} > \frac{\lambda_1-\lambda}{s^{\gamma-2}}mu_1$ and therefore su_1 is a sub solution of the equation (1).

c) Let $\sigma = [su_1, tu^*]$ and the operator T be defined by $T(u) = v$ with v solution of $(-\Delta + q)v = \lambda mu + f|u|^{\gamma-2}u$ in $\mathbb{R}^N$.

We want to prove that $T(\sigma) \subset \sigma$ and that T is a continuous compact operator.

Let $u \in \sigma$ and $T(u) = v$.

We have, in a weak sense: $(-\Delta + q)(v - su_1) = \lambda mu + fu^{\gamma-1} - \lambda_1 msu_1$.

By $(h7)$, $f \geq -\frac{\epsilon u_1 m}{l^{\gamma-2}C^{\gamma-1}}$.

So, since $u > 0$, we have: $\lambda mu + fu^{\gamma-1} - \lambda_1 msu_1 \geq -\frac{\epsilon u_1 m}{l^{\gamma-2}C^{\gamma-1}}u^{\gamma-1} + \lambda mu - \lambda_1 msu_1$.

Moreover $u \in \sigma$ so $u^{\gamma-1} \leq l^{\gamma-1}C^{\gamma-1}$

and $\lambda mu + fu^{\gamma-1} - \lambda_1 msu_1 \geq m[\lambda u - (\lambda_1 + \frac{\epsilon l}{s})su_1] > 0$.

Therefore, since $u \geq su_1$ and $\lambda > \lambda_1 + \epsilon l^{\gamma-2} \geq \lambda_1 + \epsilon \frac{l}{s}$, we obtain that: $(-\Delta + q)(v - su_1) \geq 0$.
By the Maximum Principle, we deduce that $v \geq su_1$.

Moreover we have: $\forall \phi \geq 0, < (-\Delta + q)(tu^* - v), \phi >_{L^2(\mathbb{R}^N)} \geq \int_{\mathbb{R}^N} [\lambda m(tu^* - u) + f((tu^*)^{\gamma-1} - u^{\gamma-1})]\phi$.
By $(h7)$, since $t \leq l$ and $\lambda_1 < \lambda$ we have:
$f \geq -\frac{\lambda_1 m}{\gamma C^{\gamma-2} l^{\gamma-2}} \geq -\frac{\lambda_1 m}{\gamma C^{\gamma-2} t^{\gamma-2}} \geq -\frac{\lambda m}{\gamma C^{\gamma-2} t^{\gamma-2}}$.
But $\lambda m(tu^* - u) + f((tu^*)^{\gamma-1} - u^{\gamma-1}) \geq 0$ iff $f \geq -\frac{\lambda m}{\sum_{i=0}^{\gamma-2} (tu^*)^i u^{\gamma-2-i}}$.
Since $\sum_{i=0}^{\gamma-2} (tu^*)^i u^{\gamma-2-i} \leq \gamma C^{\gamma-2} t^{\gamma-2}$, we get $f \geq -\frac{\lambda m}{\sum_{i=0}^{\gamma-2} (tu^*)^i u^{\gamma-2-i}}$.
Therefore, by the Maximum Principle, we obtain $(-\Delta + q)(tu^* - v) \geq 0$ and so $v \leq tu^*$.

d) Let $(u_n)_n$ be a convergent sequence in σ, with limit u for the norm $\|.\|_q$. Let $T(u_n) = v_n$ and $T(u) = v$.
We have: $\forall n$,
$\|v_n - v\|_q^2 \leq cst\|u_n - u\|_q \|v_n - v\|_q + \|f\|_\infty \int_{\mathbb{R}^N} |u_n^{\gamma-1} - u^{\gamma-1}||v_n - v|$.
Since $u_n, u \in \sigma$, $|u_n^{\gamma-1} - u^{\gamma-1}| \leq cst|u_n - u|$ we obtain that:
$\|v_n - v\|_q \leq cst\|u_n - u\|_q$ and so T is a continuous operator. We finish this proof by showing that T is compact. Let now $(u_n)_n$ be a bounded sequence in σ for the norm $\|.\|_q$. Since the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $(u_n)_n$, in $L^2(\mathbb{R}^N)$. Let $T(u_n) = v_n$.
We have: $\forall n, p$
$\|v_n - v_p\|_q^2 = \lambda \int_{\mathbb{R}^N} m(u_n - u_p)(v_n - v_p) + \int_{\mathbb{R}^N} f(u_n^{\gamma-1} - u_p^{\gamma-1})(v_n - v_p)$.
Since $|u_n^{\gamma-1} - u_p^{\gamma-1}| \leq cst|u_n - u_p|$ we obtain that:
$\|v_n - v_p\|_q \leq cst\|u_n - u_p\|_{L^2(\mathbb{R}^N)}$.
We can deduce that $(v_n)_n$ is a Cauchy sequence and so T is a compact operator.

$\square$

To finish, we obtain some results assuring the validity of the hypothesis $(h4)$. First, we need the following lemma: (we still follow a method developped in [7]).

Lemma 3.1 $\forall u \in V_q(\mathbb{R}^N)$, $u > 0, \forall \phi \in V_q(\mathbb{R}^N)$, $\phi \geq 0$,
$H'_\lambda(u)((\frac{\phi}{u})^{\gamma-1}\phi) - H'_\lambda(\phi)((\frac{\phi}{u})^{\gamma-1}u) \leq 0$.

Proof: We denote by $A = H'_\lambda(u)((\frac{\phi}{u})^{\gamma-1}\phi) - H'_\lambda(\phi)((\frac{\phi}{u})^{\gamma-1}u)$.
We have: $A = 2\int_{\mathbb{R}^N} [\nabla u.\nabla((\frac{\phi}{u})^{\gamma-1}\phi) - \nabla\phi.\nabla((\frac{\phi}{u})^{\gamma-1}u)]$.
$A = 2\int_{\mathbb{R}^N} [\phi\nabla u.\nabla((\frac{\phi}{u})^{\gamma-1}) - u\nabla\phi.\nabla((\frac{\phi}{u})^{\gamma-1})]$.

Since $\nabla((\frac{\phi}{u})^{\gamma-1}) = (\gamma-1)(\frac{\phi}{u})^{\gamma-2}[\frac{1}{u}\nabla\phi - \frac{\phi}{u^2}\nabla u]$, we get:
$A = 2(\gamma-1)\int_{\mathbb{R}^N}(\frac{\phi}{u})^{\gamma-2}[2\frac{\phi}{u}\nabla u.\nabla\phi - (\frac{\phi}{u})^2|\nabla u|^2 - |\nabla\phi|^2] \leq 0.$ □

So we get the last theorem:

Theorem 3.2 *Assume that the hypotheses* $(h1),(h2),(h5)$ *are satisfied,* $N == 3,4$ *and* $\gamma = 2^*$.

i) *If* $\Omega^+ = \{x \in \mathbb{R}^N,\ f(x) > 0\}$ *is a nonempty, bounded domain of* $\mathbb{R}^N$ *with a smooth frontier* $\partial\Omega^+$, *then* $\lambda^* < +\infty$.

ii) *If* $F(u_1) \geq 0$, *then* $\lambda^* = \lambda_1 < +\infty$.

iii) *Moreover* $\lambda_1 < \lambda^*$ *iff* $F(u_1) < 0$.

Proof:

i) Consider the following equation $(-\Delta+q)u = \lambda mu$ defined in Ω^+ with Dirichlet condition on $\partial\Omega^+$. We denote by λ_{1+} the first eigenvalue (which is simple and positive) and by ϕ_1 the first eigenfunction associated i.e:
$(-\Delta+q)\phi_1 = \lambda_{1+}m\phi_1$ in Ω^+, $\phi_1 > 0$ in Ω^+, $\phi_1 = 0$ on $\partial\Omega^+$.
Since $supp\phi_1 \subset \Omega^+$, by the above lemma, we get:
$\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}})$, $H'_{\lambda_{1+}}(u)((\frac{\phi_1}{u})^{\gamma-1}\phi_1) \leq 0$
i.e. $\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}})$, $u \geq 0$

$$\frac{\int_{\mathbb{R}^N}[\nabla u.\nabla((\frac{\phi_1}{u})^{\gamma-1}\phi_1) + qu(\frac{\phi_1}{u})^{\gamma-1}\phi_1]}{\int_{\mathbb{R}^N} mu(\frac{\phi_1}{u})^{\gamma-1}\phi_1} \leq \lambda_{1+} < +\infty.$$

Moreover, $F'(u)((\frac{\phi_1}{u})^{\gamma-1}\phi_1) = \gamma\int_{\Omega^+} f\phi_1^\gamma \geq 0$.
So $\lambda^* \leq \lambda_{1+} < +\infty$.

ii) As remarked before, there holds always $\lambda^* \geq \lambda_1$. We need to show that $\lambda^* \leq \lambda_1$, under the condition that $F(u_1) \geq 0$. We use again the above lemma.
We have $H'_{\lambda_1}(u_1)((\frac{u_1}{u})^{\gamma-1}u) = 0$ so
$\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}})$, $H'_{\lambda_1}(u)((\frac{u_1}{u})^{\gamma-1}u_1) \leq 0$.
Therefore, $\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}})$, $u \geq 0$

$$\frac{\int_{\mathbb{R}^N}[\nabla u.\nabla((\frac{u_1}{u})^{\gamma-1}u) + qu(\frac{u_1}{u})^{\gamma-1}u_1]}{\int_{\mathbb{R}^N} mu(\frac{u_1}{u})^{\gamma-1}u_1} \leq \lambda_1 < +\infty.$$

Since $F'(u)((\frac{u_1}{u})^{\gamma-1}u_1) = \gamma F(u_1) \geq 0$ we get that $\lambda^* \leq \lambda_1$ and therefore $\lambda^* = \lambda_1$.

iii)

a) Moreover, if $\lambda_1 < \lambda^*$, then, by *ii)* we obtain $F(u_1) < 0$.

b) Assume now that $F(u_1) < 0$.

1. We denote by $\lambda^- = \inf_{\phi \in V_q(\mathbb{R}^N),\, \phi \geq 0,\, F(\phi) \geq 0,} \frac{\int_{\mathbb{R}^N} [|\nabla \phi|^2 + q|\phi|^2]}{\int_{\mathbb{R}^N} m|\phi|^2}$.
   We are going to prove that $\lambda_1 < \lambda^-$ then that $\lambda^- \leq \lambda^*$.
   Let $W = \{\phi \in V_q(\mathbb{R}^N),\, \phi \geq 0,\, F(\phi) \geq 0\}$. Since $W \subset V_q(\mathbb{R}^N)$, we have $\lambda_1 \leq \lambda^-$. Since $u_1 \notin W$, then $\lambda_1 < \lambda^-$.
   We have to prove now that $\lambda^- \leq \lambda^*$.
2. First we prove that $\exists u^- \in V_q(\mathbb{R}^N)$, $u^- \geq 0$, $F(u^-) \geq 0$,
   $\lambda^- = \frac{\int_{\mathbb{R}^N} [|\nabla u^-|^2 + q|u^-|^2]}{\int_{\mathbb{R}^N} m|u^-|^2}$.
   On the contrary, we suppose that
   $\forall u \in V_q(\mathbb{R}^N)$, $u \geq 0$, $F(u) \geq 0 \Rightarrow \lambda^- < \frac{\int_{\mathbb{R}^N} [|\nabla u|^2 + q|u|^2]}{\int_{\mathbb{R}^N} m|u|^2}$.
   Let $v \geq 0$ such that $F(v) > 0$. Then $H_{\lambda^-}(v) > 0$.
   Since $\lambda_1 < \lambda^-$, we have $H_{\lambda^-}(u_1) < 0$ and so $H_{\lambda^-}(\eta u_1) < 0$ for all $\eta > 0$.
   Since the function H_{λ^-} is continuous, we get:
   $\exists \alpha \in (0,1)$, $H_{\lambda^-}(\alpha \eta u_1 + (1-\alpha)v) = 0$.
   Then $F(\alpha \eta u_1 + (1-\alpha)v) < 0$.
   Since $F((1-\alpha)v) > 0$, there exists $\eta > 0$ small enough such that $F(\alpha \eta u_1 + (1-\alpha)v) > 0$.
   So we get a contradiction and therefore we can deduce the existence of u^-.
3. Finally, we have to prove that $\lambda^- \leq \lambda^*$.
   On the contrary, we suppose that $\lambda^- > \lambda^*$.
   So $\exists \phi \in V_q(\mathbb{R}^N)$, $\phi \geq 0$, $F'(u^-)(\phi) \geq 0$, $\frac{\int_{\mathbb{R}^N} [\nabla u^- . \nabla \phi + q u^- \phi]}{\int_{\mathbb{R}^N} m u^- \phi} < \lambda^-$
   i.e. $H'_{\lambda^-}(u^-)(\phi) < 0$.
   Since $F(u^-) \geq 0$ and $F'(u^-)(\phi) \geq 0$, then $F(u^- + \eta\phi) \geq 0$ for $\eta > 0$ small enough.
   Moreover, since $H'_{\lambda^-}(u^-)(\phi) < 0$ and $H_{\lambda^-}(u^-) = 0$, we can choose $\eta > 0$ small enough such that $H_{\lambda^-}(u^- + \eta\phi) < 0$.
   So we obtain that: $\frac{\int_{\mathbb{R}^N} [|\nabla(u^- + \eta\phi)|^2 + q(u^- + \eta\phi)^2]}{\int_{\mathbb{R}^N} m(u^- + \eta\phi)^2} < \lambda^-$ and this contradicts the definition of λ^-.
   Therefore $\lambda^- \leq \lambda^*$.

$\square$

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Coincidence Points for Hybrid Mappings

1 Introduction

There have been several extensions of known fixed point theorems in which a mapping takes each point of a metric space into a closed (resp. closed and bounded) subset of the same (cf. [3, 4, 5, 7, 10, 11]). Hybrid fixed point theory for nonlinear mappings is relatively a recent development within the ambit of fixed point theory of point to set mappings (multivalued mappings) with a wide range of applications (see, for instance, [2, 8, 12, 13, 14, 15, 16]). Recently, in an attempt to improve /generalize certain results of Naidu, Sastry and Prasad [11] and Kaneko [4] and others, Chang [1] obtained some fixed point theorems for a hybrid of multivalued and singlevalued mappings.

However, his main theorem (see Theorem A below) admits a counter example. Our main purpose in this paper is to present a correct version of this result which, in turn, generalizes several known results in this direction.

Let (X, d) be a metric space. We shall use the following notations and definitions:

$$\begin{aligned}
CL(X) &= \{A : A \text{ is a nonempty closed subset of } X\},\\
CB(X) &= \{A : A \text{ is a nonempty closed and bounded subset of } X\},\\
N(\epsilon, A) &= \{x \in X : d(x,a) < \epsilon \text{ for some } a \in A,\ \epsilon > 0\},\ A \in CL(X),\\
E_{A,B} &= \{\epsilon > 0 : A \subset N(\epsilon, B),\ B \subset N(\epsilon, A)\},\ A, B \in CL(X),\\
H(A,B) &= \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \phi \\ \infty & \text{if } E_{A,B} = \phi\,, \end{cases}\\
D(x, A) &= \inf\{d(x,a) : a \in A\}
\end{aligned}$$

for each $A, B \in CL(X)$, and for each $x \in X$.

H is called the generalized Hausdorff metric for $CL(X)$ induced by d. If $H(A, B)$ is defined for $A, B \in CB(X)$, then H is called the Hausdorff metric induced by d (cf. Nadler [6]).

Definition 1 ([4]) *Mappings $S : X \to CB(X)$ and $I : X \to X$ are called compatible if $ISx \in CB(X)$ for all $x \in X$ and $H(SIx_n, ISx_n) \to 0$, as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $Sx_n \to M \in CB(X)$ and $Ix_n \to t \in M$ as $n \to \infty$.*

Following Singh and Mishra [16] (see also [3], [4] and [9]), we introduce the notion of R-sequentially commuting mappings for a hybrid pair of single-valued and multi-valued maps.

Definition 2 *Let K be a nonempty subset of a metric space X and $I : K \to X$ and $S : K \to CL(X)$ be respectively single-valued and multi-valued mappings. Then I and S will be called R-sequentially commuting on K if for a given sequence $\{x_n\} \subset K$ with $\lim_n Ix_n \in K$, there exists $R > 0$ such that*

$$\lim_n D(Iy, SIx_n) \le R \lim_n D(Ix_n, Sx_n) \tag{$*$}$$

for each $y \in K \cap \lim_n Sx_n$.

If $x_n = x (x \in K)$ for all $n \in \mathbb{N}$ (naturals), $Ix \in K$ and ($*$) holds for some $R > 0$, then I and S have been defined to be pointwise R-weakly commuting at $x \in K$ (see [16, Def. 1]). If it holds for all $x \in K$, then I and S are called R-weakly commuting on K. Further, if $R = 1$, we get the definition of weak commutativity of I and S on K due to Hadzic and Gajec [3]. If $I, S : X \to X$, then as mentioned in [16], we recover the definitions of pointwise R-weak commutativity and R-commutativity of single-valued self-maps due to Pant [9] and all the remarks as given in [16] apply.

We now introduce the following.

Definition 3 *Maps $I : K \to X$ and $S : K \to CL(X)$ are to be called sequentially commuting (or s-commuting) at a point $x \in K$ if*

$$I(\lim_n Sx_n) \subset SIx \tag{$**$}$$

whenever there exists a sequence $\{x_n\} \subset K$ such that $\lim_n Ix_n = x \in \lim_n Sx_n \in CL(X)$.

If $x_n = x$ for all $n \in \mathbb{N}$, then the maps I and S will be said to be weakly s-commuting at a point $x \in K$.

The following example shows that s-commutativity of I and S is indeed more general than their R-sequential commutativity (and hence their pointwise R-commutativity and compatibility).

Example 1 Let $X = [0, \infty)$ with the usual metric d and define $I : X \to X$ and $S : X \to CL(X)$ by

$$Ix = \begin{cases} 0, & \text{if } x \in [0,1] \\ x, & \text{if } x \in (1, \infty), \end{cases} \qquad Sx = [x, \infty).$$

Then for the sequence $\{x_n\} \subset X$ defined by $x_n = 1 + \frac{1}{n}$, we have $1 = \lim_n Ix_n \in [1, \infty) = \lim_n Sx_n \in CL(X)$ and $I(\lim_n Sx_n) = \{0\} \cup (1, \infty) \subset [0, \infty) = SI1$. Therefore, I and S are s-commuting but $(*)$ is not satisfied for $y = 1 \in [1, \infty) = \lim_n Sx_n$.

Definition 4 ([1]) *Let $\mathbb{R}^+$ denote the set of all non-negative real numbers, and let $A \subset \mathbb{R}^+$. A function $\varphi : A \to \mathbb{R}^+$ is upper semicontinuous from the right if $\lim_{x \to u+} \sup \varphi(x) \leq \varphi(u)$ for all $u \in A$.*

A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy (Φ)-conditions if:

(i) *φ is upper semi-continuous from the right on $(0, \infty)$ with $\varphi(t) < t$ for all $t > 0$, and*

(ii) *there exists a real number $s > 0$ such that φ is non-decreasing on $(0, s]$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t \in (0, s]$, where φ^n denotes the composition of φ with itself n times and $\varphi^0(t) = t$.*

Let Γ denote the set of all functions which satisfy the (Φ)-condition.

The following lemmas will be useful in proving our main results.

Lemma 1 *Let (X, d) be a metric space and $I, J : X \to X$ and $S, T : X \to CL(X)$ be such that $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and for all $x, y \in X$,*

$$H(Sx, Ty) \leq \varphi(aL(x, y) + (1 - a)N(x, y)), \tag{1}$$

where $a \in [0, 1]$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous from the right on $(0, \infty)$ with $\varphi(t) < t$ for all $t > 0$, and

$$\begin{aligned} L(x, y) &= \max\{d(Ix, Jy),\ D(Ix, Sx),\ D(Jy, Ty),\ \frac{1}{2}[D(Ix, Ty) + D(Jy, Sx)]\}, \\ N(x, y) &= [\max\{d^2(Ix, Jy),\ D(Ix, Sx)\, D(Jy, Ty),\ D(Ix, Ty)\, D(Jy, Sx), \\ &\qquad \frac{1}{2} D(Ix, Sx)\, D(Jy, Sx),\ \frac{1}{2} D(Jy, Ty)\, D(Ix, Ty)\}]^{1/2}. \end{aligned}$$

Then $\inf_{x \in X} D(Ix, Sx) = 0 = \inf_{x \in X} D(Jx, Tx)$.

Proof: Due to symmetry, we may suppose that

$$\inf_{x \in X} D(Ix, Sx) = \inf_{x \in X} D(Jx, Tx) = \delta.$$

If $\delta > 0$, then $\varphi(\delta) < \delta$. Since φ is upper semi-continuous from the right, there exists $\epsilon > 0$ such that $\varphi(t) < \delta$ for all $t \in [\delta, \delta + \epsilon)$. Pick $x_0 \in X$ such that $D(Ix_0, Sx_0) < \delta + \epsilon$. By $S(X) \subset J(X)$, there exists $x_1 \in X$ such that $Jx_1 \in Sx_0$ and $d(Ix_0, Jx_1) < \delta + \epsilon$.

Consider

$$\delta \le D(Jx_1, Tx_1) \le H(Sx_0, Tx_1) \le \varphi(aL(x_0, x_1) + (1-a)N(x_0, x_1)),$$

where

$$\begin{aligned} L(x_0, x_1) &= \max\{d(Ix_0, Jx_1),\ D(Ix_0, Sx_0),\ D(Jx_1, Tx_1),\ \frac{1}{2}[D(Ix_0, Tx_1) + D(Jx_1, Sx_0)]\} \\ &= \max\{d(Ix_0, Jx_1),\ D(Jx_1, Tx_1)\} \end{aligned}$$

and

$$\begin{aligned} N(x_0, x_1) &= [\max\{d^2(Ix_0, Jx_1),\ D(Ix_0, Sx_0)\, D(Jx_1, Tx_1),\ D(Ix_0, Tx_1)\, D(Jx_1, Sx_0), \\ &\qquad \frac{1}{2}D(Ix_0, Sx_0)\, D(Jx_1, Sx_0),\ \frac{1}{2}D(Jx_1, Tx_1)\, D(Ix_0, Tx_1)\}]^{1/2} \\ &\le [\max\{d^2(Ix_0, Jx_1),\ d(Ix_0, Jx_1)D(Jx_1, Tx_1)\}]^{1/2} \\ &\le [\max\{d^2(Ix_0, Jx_1),\ d(Ix_0, Jx_1)D(Jx_1, Tx_1),\ D^2(Jx_1, Tx_1)\}]^{1/2} \\ &\le [\max\{d^2(Ix_0, Jx_1),\ D^2(Jx_1, Tx_1)\}]^{1/2} \\ &= \max\{d(Ix_0, Jx_1),\ D(Jx_1, Tx_1)\}. \end{aligned}$$

Hence,

$$\delta \le D(Jx_1, Tx_1) \le \varphi(\max\{d(Ix_0, Jx_1),\ D(Jx_1, Tx_1)\}),$$

which is a contradiction, since $\varphi(d(Ix_0, Jx_1)) < \delta$ and $\varphi(D(Jx_1, Tx_1)) < D(Jx_1, Tx_1)$ proving that $\delta = 0$.

Lemma 2 *Let X, I, J, S, T and φ be as defined Lemma 1 such that the inequality (1) holds. If $Ix \in Sx$ for some $x \in X$, then there exists a $y \in X$ such that $Ix = Jy$ and $Jy \in Ty$.*

Proof: Suppose $Ix \in Sx$. Since $S(X) \subset J(X)$, we may choose a $y \in X$ such that $Jy = Ix \in Sx$. By (1), we have

$$D(Jy, Ty) \le H(Sx, Ty) \le \varphi(aL(x, y) + (1-a)N(x, y)),$$

where

$$\begin{aligned} L(x, y) &= \max\{d(Ix, Jy),\ D(Ix, Sx),\ D(Ix, Sx),\ D(Jy, Ty),\ \frac{1}{2}[D(Ix, Ty) + D(Jy, Sx)]\} \\ &= D(Jy, Ty), \end{aligned}$$

and

$$\begin{aligned} N(x, y) &= [\max\{d^2(Ix, Jy),\ D(Ix, Sx)\, D(Jy, Ty),\ D(Jx, Ty)\, D(Jy, Sx), \\ &\qquad \frac{1}{2}D(Ix, Sx)\, D(Jy, Sx),\ \frac{1}{2}D(Jy, Ty)\, D(Ix, Ty)\}]^{1/2} \\ &= (1/\sqrt{2})D(Jy, Ty). \end{aligned}$$

Hence

$$D(Jy, Ty) \leq \varphi([a + (1-a)/\sqrt{2})]D(Jy, Ty)) < D(Jy, Ty)\,,$$

a contradiction, and so $D(Jy, Ty) = 0$, i.e., $Jy \in Ty$.

Remark 1 If the assumptions of Lemma 2 hold, then setting $x_{2n} = x$ and $x_{2n-1} = y$ for all $n \in \mathbb{N}$ and $z = Ix$ we observe that $Ix_{2n} \to z$, $Jx_{2n-1} \to z$, $D(Ix_{2n}, Sx_{2n}) \to 0$ and $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$.

Lemma 3 ([11]) *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function such that*

(i) $\varphi(t+) < t$ *for all* $t > 0$ *and* $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ *for all* $t > 0$.

Then there exists a strictly increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(ii) $\varphi(t) < \psi(t)$ *for all* $t > 0$ *and* $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ *for all* $t > 0$.

Lemma 4 ([1]) *If $\varphi \in \Gamma$, then there exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that:*

(i) *ψ is upper semi-continuous from the right with $\varphi(t) \leq \psi(t) < t$ for all $t > 0$,*

(ii) *ψ is strictly increasing with $\varphi(t) < \psi(t)$ for $t \in (0, s]$, $s > 0$ and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for $t \in (0, s]$.*

2 Main Results

The following theorem is the main result of Chang [1, Theorem 1].

Theorem A *Let (X, d) be a complete metric space, let I, J be two functions from X into X, and let $S, T : X \to CB(X)$ be two set-valued functions with $SX \subset JX$ and $TX \subset IX$. If there exists $\varphi \in \Gamma$ such that for all x, y in X,*

$$H(Sx, Ty) \leq \varphi\left(\max\left\{d(Ix, Jy),\ D(Ix, Sx),\ D(Jy, Ty),\ \frac{1}{2}[D(Ix, Ty) + D(Jy, Sx)]\right\}\right), \tag{C}$$

then there exists a sequence $\{x_n\}$ in X such that $Ix_{2n} \to z$ and $Jx_{2n-1} \to z$ for some z in X and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. Moreover, if $Iz = z$ and T and J are compatible, then $z \in Sz$ and $Jz \in Tz$. That is, z is a common fixed point of I and S, and z is a coincidence point of J and T.

The following example shows that Theorem A in its present form is incorrect.

Example 2 Let $X = [0,1]$ with absolute value metric d and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\varphi(t) = t^2$ for $t \in [0,1)$ and $\varphi(t) = 1/2$ for $t \geq 1$. Define $I = J : X \to X$ and $S = T : X \to CB(X)$ by $Ix = 1 - x$, $x \in X$ and $Sx = \{0, 1/3, 2/3, 1\}$ for all $x \in X$. Then for each $x, y \in X$ and $\varphi \in \Gamma$, we have

$$\begin{aligned} H(Sx,Ty) &= 0 \\ &\leq \psi\left(\max\left\{d(Ix,Jy),\ D(Ix,Sx),\ D(Jy,Ty),\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\right\}\right) \end{aligned}$$

and for the sequence $\{x_n\} \subset X$ defined by $x_n = 1/n$ for all $n \in \mathbb{N}$, we have $Sx_n, Tx_n \to \{0, 1/3, 2/3, 1\} = M$, $Ix_n, Jx_n = 1 - 1/n \to 1 \in M \subset X$, $D(Ix_{2n}, Sx_{2n}) \to 0$ and $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. Also, $z = 1/2 \in X$ is such that $Iz = z$ and for $\{x_n\}$ as defined above we have $\lim_n H(TJx_n, JTx_n) = 0$, that is, T and J are compatible. Thus, all the conditions of Theorem A are satisfied. Evidently, $z \notin Sz$, $Jz \notin TZ$, that is, $z = 1/2$ is neither a common fixed point of I and S nor it is a coincidence point of J and T.

Before we present a corrected version of Theorem A, we have the following:

Theorem 1 *Let (X,d) be a complete metric space, and let $I, J : X \to X$, $S, T : X \to CL(X)$. Let A be a nonempty subset of X such that $I(A)$ and $J(A)$ are closed subsets of X, and $Tx \subseteq I(A)$ and $Sx \subseteq J(A)$ for all $x \in A$ and there exists a $\varphi \in \Gamma$ such that for all $x, y \in X$, (1) holds. Then*

(i) $F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi$,

(ii) $G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi$,

(iii) $F = G$ *if* $A = X$.

Proof: Let ψ be the function satisfying the conclusion of Lemma 4. By (1), we have for any $x, y \in X$, and $Ix \in Ty$,

$$\begin{aligned} D(Ix,Sx) &\leq H(Ty,Sx) \\ &\leq \varphi(aL(x,y) + (1-a)N(x,y)), \end{aligned}$$

where

$$\begin{aligned} L(x,y) &= \max\{d(Ix,Jy),\ D(Ix,Sx),\ D(Jy,Ty),\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\} \\ &\leq \max\{d(Ix,Jy),\ D(Ix,Sx),\ [d(Jy,Ix)+D(Ix,Ty)], \\ &\qquad \frac{1}{2}[D(Ix,Ty)+d(Ix,Jy)+D(Ix,Sx)]\} \\ &= \max\{d(Ix,Jy),\ D(Ix,Sx),\ \frac{1}{2}[d(Ix,Jy)+D(Ix,Sx)]\} \\ &= \max\{d(Ix,Jy),\ D(Ix,Sx)\} \end{aligned}$$

and

$$\begin{aligned}
N(x,y) &= [\max\{d^2(Ix,Jy),\ D(Ix,Sx)D(Jy,Ty),\ D(Ix,Ty)D(Jy,Sx),\\
&\qquad \frac{1}{2}D(Ix,Sx)D(Jy,Sx),\ \frac{1}{2}D(Jy,Ty)D(Ix,Ty)\}]^{1/2}\\
&= \max\{d^2(Ix,Jy),\ D(Ix,Sx)D(Jy,Ty),\ \frac{1}{2}(Ix,Sx)D(Jy,Sx)\}^{1/2}\\
&\le [\max\{d^2(Ix,Jy),\ d(Ix,Jy)D(Ix,Sx)\,,\\
&\qquad \frac{1}{2}[d(Ix,Jy)+D(Ix,Sx)]D(Ix,Sx)\}]^{1/2}\\
&\le [\max\{d^2(Ix,Jy),\ d(Ix,Jy)D(Ix,Sx),\ D^2(Ix,Sx)\}]^{1/2}\\
&= [\max\{d^2(Ix,Jy),\ D^2(Ix,Sx)\}]^{1/2}\\
&= \max\{d(Ix,Jy),\ D(Ix,Sx)\}\,.
\end{aligned}$$

Since $D(Ix,Sx) \le \varphi(aD(Ix,Sx)+(1-a)D(Ix,Sx))$ is inadmissible for any $a \in [0,1]$, $D(Ix,Sx) \le \varphi(aD(Ix,Sx)+(1-a)d(Ix,Jy))$ is inadmissible for $a=1$ and $D(Ix,Sx) \le \varphi(ad(Ix,Jy)+(1-a)D(Ix,Sx))$ is inadmissible for $a=0$, it follows that

$$\begin{aligned}
D(Ix,Sx) &\le (ad(Ix,Jy)+(1-a)d(Ix,Jy))\\
&= \varphi(d(Ix,Jy))\,.
\end{aligned}$$

Similarly we can show that

$$D(Jy,Ty) \le \varphi(d(Ix,Jy)) \text{ if } Jy \in Sx\,.$$

Pick $x_0 \in A$ such that $D(Ix_0,Sx_0) < s$. Since $Sx_0 \subseteq J(A)$, there exists $x_1 \in A$ such that $Jx_1 \in Sx_0$. Then we have

$$\begin{aligned}
D(Jx_1,Tx_1) &\le H(Sx_0,Tx_1)\\
&\le \varphi(aL(x_0,x_1)+(1-a)N(x_0,x_1))\\
&\le \psi(aL(x_0,x_1)+(1-a)N(x_0,x_1))\,.
\end{aligned}$$

Since $Tx_1 \subseteq I(A)$, we may choose $x_2 \in A$ such that $Ix_2 \in Tx_2$ and

$$d(Jx_1,Ix_2) \le \psi(aL(x_0,x_1)+(1-a)N(x_0,x_1))\,.$$

Therefore

$$\begin{aligned}
D(Ix_2,Sx_2) &\le H(Sx_2,Tx_1)\\
&\le \varphi(aL(x_2,x_1)+(1-a)N(x_2,x_1))\\
&< \psi(aL(x_2,x_1)+(1-a)N(x_2,x_1))\,.
\end{aligned}$$

Hence we can choose $x_3 \in A$ such that $Jx_3 \in Sx_2$ and $d(Ix_2, Jx_3) \leq \psi(aL(x_2, x_1) + (1-a)N(x_2, x_1))$. Proceeding in this way, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in A such that $Jx_{2n+1} \in Sx_{2n}$, $Ix_{2n+2} \in Tx_{2n+1}$ $(n = 0, 1, 2, \dots)$ and

$$
\begin{aligned}
d(Ix_{2n}, Jx_{2n+1}) &\leq \psi(aL(x_{2n}, x_{2n-1}) + (1-a)N(x_{2n}, x_{2n-1}))\,,\\
d(Jx_{2n-1}, Ix_{2n}) &\leq \psi(aL(x_{2n-2}, x_{2n-1}) + (1-a)N(x_{2n-2}, x_{2n-1}))
\end{aligned}
$$

for all $n \in \mathbb{N}$ (naturals). By the construction of $\{x_n\}$ we have

$$
\begin{aligned}
L(x_{2n}, x_{2n-1}) &\leq \max\{d(Ix_{2n}, Jx_{2n-1}),\ d(Ix_{2n}, Jx_{2n+1})\}\,,\\
N(x_{2n}, x_{2n-1}) &\leq \max\{d(Ix_{2n}, Jx_{2n-1}),\ d(Ix_{2n}, Jx_{2n+1})\}\,,\\
L(x_{2n-2}, x_{2n-1}) &\leq \max\{d(Ix_{2n-2}, Jx_{2n-1}),\ d(Ix_{2n}, Jx_{2n+1})\} \text{ and}\\
N(x_{2n-2}, x_{2n-1}) &\leq \max\{d(Ix_{2n-2}, Jx_{2n-1}),\ d(Ix_{2n-2}, Jx_{2n+1})\} \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Since ψ is strictly increasing on $(0, s]$ and $\psi(t) < t$ for $t > 0$, we have

$$
\begin{aligned}
d(Ix_{2n}, Jx_{2n+1}) &\leq \psi(ad(Ix_{2n}, Jx_{2n-1}) + (1-a)d(Ix_{2n}, Jx_{2n-1}))\\
&= \psi(d(Ix_{2n}, Jx_{2n-1}))\\
&\leq d(Jx_{2n-1}, Ix_{2n})\psi(ad(Ix_{2n-2}, Jx_{2n-1}) + (1-a)d(Ix_{2n-2}, J_{2n-1}))\\
&= \psi(d(Ix_{2n-2}, Jx_{2n-1})) \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
d(Ix_{2n}, Jx_{2n+1}) &\leq \psi^{2n}(d(Ix_0, Jx_1)) \text{ and}\\
d(Jx_{2n-1}, Ix_{2n}) &\leq \psi^{2n-1}(d(Ix_0, Jx_1)) \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Set $y_{2n} = Ix_{2n}$ and $y_{2n+1} = Jx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$
d(y_n, y_{n+1}) \leq \psi^n(d(y_0, y_1)) \text{ for all } n \in \mathbb{N}.
$$

Since $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for $t \in (o, s]$ and $d(y_0, y_1) = d(Ix_0, Jx_1) < s$, it follows that $\sum_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent. Hence by the completeness of X, $\{y_n\}$ converges to z for some $z \in X$. Since $\{y_{2n}\}$ is a sequence in $I(A)$ converging to z and $I(A)$ is closed, it follows that $z \in I(A)$. So there exists a $w \in X$ such that $Iw = z$. Now by (1), we have

$$
\begin{aligned}
D(Ix_{2n}, Sw) &\leq H(Sw, Tx_{2n-1})\\
&\leq \varphi(aL(w, x_{2n-1}) + (1-a)N(w, x_{2n-1})) \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Making $n \to \infty$ in the above inequality, we obtain

$$
D(z, Sw) \leq \varphi(aD(z, Sw)_+ + (1-a)D(z, Sw)_+) = \varphi(D(z, Sw)_+)\,.
$$

By the definition of φ, we have $\varphi(t_+) < t$ for all $t \in (0, \infty)$, it follows that $D(z, Sw) = 0$. Hence $Iw \in Sw$ and so

$$F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi\,.$$

Similarly

$$G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi\,.$$

We now suppose that $Sx \subseteq J(X)$ and $Tx \subseteq I(X)$ for all $x \in X$. Pick $u \in X$ such that $Iu \in Su$. Then since $Su \subseteq J(X)$, there exists a $v \in X$ such that $Jv = Iu$. By the inequality (1), we have

$$\begin{aligned} D(Jv, Tv) &\leq H(Su, Tv) \\ &\leq \varphi(aD(Jv, Tv) + (1-a)D(Jv, Tv)) \\ &< D(Jv, Tv)\,. \end{aligned}$$

Hence $Jv \in Tv$. It follows that $F \subseteq G$. Similarly we can prove that $G \subseteq F$. Hence $F = G$. Further, suppose that $I(X)$ and $J(X)$ are closed. Choose a sequence $\{u_n\}$ in X such that $Iu_n \in Su_n$ for all $n \in \mathbb{N}$ and $\{Iu_n\}$ is convergent. Since $I(X)$ is closed, it follows that $\lim_n Iu_n = Iu$ for some $u \in X$. Since $Iu_n \in Su_n \subseteq J(X)$ for all $n \in \mathbb{N}$ and $J(X)$ is closed, it follows that $Iu \in J(X)$. So there exists a $v \in X$ such that $Iu = Jv$. Again by (1), we have

$$\begin{aligned} D(Iu_n, Tv) &\leq H(Su_n, Tv) \\ &\leq \varphi(aL(u_n, v) + (1-a)N(u_n, v))\,. \end{aligned}$$

Making $n \to \infty$ in the above inequality, we obtain

$$D(Jv, Tv) \leq \varphi(aD(Jv, Tv)_+ + [(1-a)/\sqrt{2}]D(Jv, Tv)_+)\,.$$

Hence $Jv \in Tv$. By (1), we have

$$\begin{aligned} D(Iu, Su) &\leq H(Su, Tv) \\ &\leq \varphi(aD(Iu, Su) + [(1-a)/\sqrt{2}]D(Iu, Su))\,. \end{aligned}$$

Hence $Iu \in Su$. It follows that G is closed. #

Remark 2 Theorem 1 of Naidu [7] and Theorem 9 of Sastry, Naidu and Prasad [11] follow as direct corollaries of Theorem 1.

Remark 3 For $a = 1$, Example 10 of Sastry, Naidu and Prasad [11] shows that Theorem 1 fails if $\frac{1}{2}[D(Ix, Ty) + D(Jy, Sx)]$ is replaced by $\max\{D(Ix, Ty), D(Jy, Sx)\}$ even if $S = T$, $I = J = i_d$ (the identity mapping on X) and φ is continuous on $\mathbb{R}^+$.

Remark 4 If (1) is assumed to be valid only for those $x, y \in X$ for which $Ix \neq Jy$, $Ix \notin Sx$ and $Jy \notin Ty$ instead of all $x, y \in X$, then we conclude from Theorem 1 that: either $F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi$ or $G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi$.

The following theorem presents a correct version of Theorem A.

Theorem 2 *Let (X, d) be a complete metric space, and let $I, J : X \to X$, $S, T : X \to CL(X)$ be such that $S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$. If there exists a $\varphi \in \Gamma$ such that for all $x, y \in X$, (1) holds, then there is a sequence $\{x_n\}$ in X such that $Ix_{2n} \to z$ and $Jx_{2n-1} \to z$ for some $z \in X$ and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. Moreover,*

(i) *if $Iz \in Sz$ and $d(Iz, z) \leq D(z, Sx)$ for all $x \in X$, then $z \in Sz$, and if $d(Iz, z) \leq D(z, Tx)$ for all $x \in X, J$ and T are weakly s-commuting, then $Jz \in Tz$.*

(ii) *if $Jz \in Tz$ and $d(Jz, z) \leq D(z, Tx)$ for all $x \in X$, then $z \in Tz$; and if $d(Jz, z) \leq D(z, Sx)$ for all $x \in X, I$ and S are weakly s-commuting, then $Iz \in Sz$.*

(iii) *if $Iz = z$ and J and T are weakly s-commuting, then $z \in Sz$ and $Jz \in Tz$.*

(iv) *if $Jz = z$ and I and S are weakly s-commuting, then $z \in Tz$ and $Iz \in Sz$.*

Proof: By replacing A with X throughout in the proof of Theorem 1, we can construct a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that $Jx_{2n+1} \in Sx_{2n}$, $Ix_{2n+2} \in Tx_{2n+1}$ $(n = 0, 1, 2, \dots)$ and the sequences $\{Ix_{2n}\}$, $\{Jx_{2n-1}\}$ are Cauchy sequences which converge to the same limit $z \in X$ and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. It then follows that $D(z, Sx_{2n}) \to 0$ and $D(z, Tx_{2n-1}) \to 0$ as $n \to \infty$.

(i) Suppose $Iz \in Sz$, since $d(Iz, z) \leq D(z, Sz)$ and J and T are weakly s-commuting. Choose $m \in \mathbb{N}$ such that

$$\sup\{d(Ix_{2n}, z),\ d(Jx_{2n-1}, z),\ D(z, Sx_{2n}),\ D(z, Tx_{2n-1}) : n \geq m\} < s\,.$$

Then for $n \geq m$ we have

$$\begin{aligned} D(z, Sz) &\leq d(z, Ix_{2n}) + D(Ix_{2n}, Sz) \\ &\leq d(z, Ix_{2n}) + H(Sz, Tx_{2n-1}) \\ &\leq d(z, Ix_{2n}) + \varphi(aL(z, x_{2n-1}) + (1-a)N(z, x_{2n-1}))\,, \end{aligned} \tag{2}$$

where

$$\begin{aligned} L(z,x_{2n-1}) &= \max\{d(Iz,Jx_{2n-1}),\ D(Iz,Sz),\ D(Jx_{2n-1},Tx_{2n-1}), \\ &\qquad \frac{1}{2}[D(Iz,Tx_{2n-1})+D(Jx_{2n-1},Sz)]\} \\ &\le \max\{d(Iz,Jx_{2n-1}),\ 0,\ d(x_{2n-1},Tx_{2n-1}), \\ &\qquad \frac{1}{2}[d(Iz,z)+D(z,Tx_{2n-1}),\ d(Jx_{2n-1},z)+D(z,Tx_{2n-1})]\} \\ &\qquad \to \max\{d(Iz,z),\ 0,\ 0,\ \frac{1}{2}d(Iz,z)\} \text{ as } n\to\infty, \end{aligned}$$

i.e.

$$\lim_n L(z,x_{2n-1}) \le D(z,Sz);$$

and

$$N(z,x_{2n-1}) \le [\max\{d^2(Iz,z),\ 0,\ 0,\ 0,\ 0\}]^{1/2} \text{ as } n\to\infty$$

i.e.

$$\lim_n N(z,x_{2n-1}) \le D(z,Sz).$$

Hence making $n\to\infty$ in (2), we obtain

$$D(z,Sz) \le 0+\varphi(aD(z,Sz)+(1-a)D(z,Sz)),$$

that is, $D(z,Sz)=0$ and so $z\in Sz$. Choose $z'\in X$ such that $Jz'=z$, then

$$\begin{aligned} D(z,Tz') &\le H(Sz,Tz') \\ &\le \varphi(aL(z,z')+(1-a)N(z,z')), \end{aligned} \tag{3}$$

where

$$\begin{aligned} L(z,z') &= \max\{d(Iz,Jz'),\ D(Iz,Sz), D(Jz',Tz'), \\ &\qquad \frac{1}{2}[D(Iz,Tz')+D(Jz',Sz)]\} \\ &\le \max\{d(Iz,z),\ D(Iz,Sz),\ D(z,Tz'), \\ &\qquad \frac{1}{2}[d(Iz,z)+D(z,Tz')+D(z,Sz)]\} \\ &= \max\{d(Iz,z),\ D(z,Tz')\} \le D(z,Tz') \end{aligned}$$

and

$$N(z,z') \le [\max\{d^2(Iz,z),0,0,0,\ \frac{1}{2}D(z,Tz')[d(Iz,z)+d(z,Tz')]\}]^{1/2}$$
$$\le D(z,Tz')\,.$$

Hence by (3)

$$D(z,Tz') \le \varphi(D(z,Tz'))\,,$$

and so $D(z,Tz')=0$; i.e.,$Jz'=z\in Tz'$.

Since J and T are weakly s-commuting and $Jz'\in Tz'$, we have

$$JJz' \in JTz' \subset TJz'\,,$$

which implies that $Jz\in Tz$.

(ii) The proof is analogous to the proof of (i) due to symmetry.

(iii) Suppose $Iz=z$ and J and T are weakly s-commuting. Choose m as in (i), then for all $n\ge m$

$$\begin{aligned} D(z,Sz) &\le d(z,Ix_{2n})+D(Ix_{2n},Sz) \\ &\le d(z,Ix_{2n})+H(Sz,Tx_{2n-1}) \\ &\le d(z,Ix_{2n})+\varphi(aL(z,x_{2n-1})+(1-a)N(z,x_{2n-1}))\,, \end{aligned} \tag{4}$$

where

$$L(z,x_{2n-1}) \to \max\{0,\ D(z,Sz),\ 0,\ \frac{1}{2}D(z,Sz)\} \text{ as } n\to\infty\,,$$

i.e.,

$$\lim_n L(z,x_{2n-1}) = D(z,Sz)$$

and

$$N(z,x_{2n-1}) \to [\max\{0,0,0,\ \frac{1}{2}D^2(z,Sz),0\}]^{1/2} \text{ as } n\to\infty$$

i.e.,

$$\lim_n N(z,x_{2n-1}) = D(z,Sz)\,.$$

Making $n\to\infty$ in (4), we obtain

$$D(z,Sz) \le 0+\varphi(aD(z,Sz)+[(1-a)/\sqrt{2}]D(z,Sz))$$
$$< D(z,Sz)\,,$$

which implies $D(z,Sz)=0$ and so $z\in Sz$. Choose $z'\in X$ such that $Jz'=z$, then

$$D(z,Tz') \le H(Sz,Tz')$$
$$\le \varphi(aL(z,z')+(1-a)N(z,z'))\,,$$

where

$$L(z,z') = \max\{d(Iz,Jz'),\ D(Iz,Sz),\ D(Jz',Tz'),\ \frac{1}{2}[D(Iz,Tz')+D(Jz,Sz)]\}$$
$$= D(z,Tz')$$

and

$$N(z,z') = [\max\{d^2(Iz,Jz'), D(Iz,Sz)D(Jz',Tz'),\ D(Iz,Tz')D(Jz',Sz),$$
$$\frac{1}{2}D(Iz,Sz)D(Jz',Sz),\ \frac{1}{2}D(Jz',Tz')D(Iz,Tz')\}]^{1/2}$$
$$= (1/\sqrt{2})D(z,Tz')\,.$$

Hence

$$D(z,Tz') \le \varphi(aD(z,Tz') + [(1-a)/\sqrt{2}]D(z,Tz'))$$
$$< D(z,Tz')\,.$$

It follows that $D(z,Tz') = 0$ and so $Jz' = z \in Tz'$. Since J and T are weakly s-commuting $Jz' \in Tz'$, we have $JJz' \in JTz'$. Hence $Jz \in Tz$.

(iv) Due to symmetry, the proof is analogous to the proof of (iii).#

Theorem 3 *Suppose that* $\lim_{t\to+\infty}(t-\varphi(t)) = +\infty$*, there are sequences* $\{x_n\}_{n=0}^{\infty}$ *and* $\{y_n\}_{n=0}^{\infty}$ *in* X *such that* $\{Ix_n, Ix_{n+1}\} \subset Sx_n$ *and* $\{Jy_n, Jy_{n+1}\} \subset Ty_n$ $(n = 0,1,2,\dots)$*, and*

$$H(Sx,Ty) \le \varphi(aL_1(x,y) + (1-a)N_1(x,y)) \tag{5}$$

for all $x, y \in X$ *and* $a \in [0,1]$*, where*

$$L_1(x,y) = \max\{D(Ix,Sx),\ D(Jy,Ty),\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\}$$

and

$$N_1(x,y) = [\max\{D(Ix,Sx)D(Jy,Ty),\ D(Ix,Ty)D(Jy,Sx),$$
$$\frac{1}{2}D(Ix,Sx)D(Jy,Sx),\ \frac{1}{2}D(Jy,Ty)D(Ix,Ty)\}]^{1/2}\,.$$

Then:

(i) *the sequences* $\{Sx_n\}$ *and* $\{Ty_n\}$ *converge in* $CL(X)$ *to the same limit* A *for some* $A \in CL(X)$.

(ii) $F = \{Ix : x \in X \text{ and } Ix \in Sx\} = I(X) \cap A$, *and*
$G = \{Jy : y \in y \in X \text{ and } Jy \in Ty\} = J(X) \cap A$.

(iii) $Sx = A$ *whenever* $Ix \in Sx$ *and* $Ty = A$ *whenever* $Jy \in Ty$.

Proof: For a fixed $n \in \mathbb{N}$, let

$$\beta_n = \sup\{H(Sx_i, Ty_j) : 1 \le i, j \le n\}\,.$$

Let $\delta = \max\{H(Ix_0, Sx_1),\ H(Ty_0, Ty_1)\}$.

For $i, j \in \mathbb{N}$, the inequality (5) yields $H(Sx_i, Ty_j) \le \varphi(aL_1(x_i, y_j)) + (1-a)N_1(x_i, y_j)$, where

$$\begin{aligned} L_1(x_i, y_j) &= \max\{D(Ix_i, Sx_j),\ D(Jy_j, Ty_j),\ \frac{1}{2}[D(Ix_i, Ty_j) + D(Jy_j, Sx_i)]\} \\ &\le \frac{1}{2}[H(Sx_{i-1}, Ty_j) + H(Ty_{j-1}, Sx_i)] \\ &\le \max\{H(Sx_i, Ty_j),\ H(Ty_{j-1}, Sx_i)\} \end{aligned}$$

and

$$\begin{aligned} N_1(x_i, y_j) &\le [H(Sx_{i-1}, Ty_j)H(Ty_{j-1}, Sx_1)]^{1/2} \\ &\le \max\{H(Sx_{i-1}, Ty_j),\ H(Ty_{j-1}, Sx_i)\}\,. \end{aligned}$$

Hence for $i, j \in \mathbb{N}$, we have

$$H(Sx_i, Ty_j) \le \varphi(\max\{H(Sx_{i-1}, Ty_j),\ H(Ty_{j-1}, Sx_i)\})\,. \tag{6}$$

It follows that $\beta_n \le \varphi(\beta_n + \delta)$ for all $n = 1, 2, 3, \ldots$. Hence $(\beta_n + \delta) - \varphi(\beta_n + \delta) \le \delta$ for all $n = 1, 2, 3, \ldots$. Since $t - \varphi(t) \to +\infty$ as $t \to +\infty$, it follows that $\{\beta_n\}$ is bounded. Hence $\sup\{H(Sx_i, Ty_j) : i, j \in \mathbb{N}\}$ is finite.

For $n \in \mathbb{N}$,

$$\text{let } \nu_n = \sup\{H(Sx_i, Ty_j) : i, j \ge n\}\,.$$

Then the inequality (6) yields $\nu_n \le (\nu_{n-1})$ for all $n \in \mathbb{N}$. It follows that $\nu_n \le \varphi^n(\nu_0)$ for all $n \in \mathbb{N}$. Since $\varphi(t+) < t$ for all $t \in (0, \infty)$ and $\varphi(0) = 0$, it follows that $\varphi^n(\nu_0) \to 0$ as $n \to \infty$. So $\{\nu_n\}$ converges to zero. Again for all $i, j \in \mathbb{N}$, we have

$$H(Sx_i, Sx_j) \le H(Sx_i, Ty_i) + H(Ty_i, Sx_j)$$

Thus for all $i, j \ge n$ and using the fact that $\nu_n \to 0$ as $n \to \infty$ we have

$$H(Sx_i, Sx_j) \le z\nu_n \to o \text{ as } i, j \to +\infty\,.$$

It follows that $\{Sx_n\}$ is Cauchy. Since $(CL(X), H)$ is complete, $\{Sx_n\}$ is convergent in $CL(X)$. We can similarly show that $\{Ty_n\}$ is also convergent in $CL(X)$. Since $H(Sx_n, Ty_n) \to 0$ as $n \to \infty$, it follows that the sequences $\{Sx_n\}$ and $\{Ty_n\}$ converge in $CL(X)$ to the same limit A for some $A \in CL(X)$.

Suppose $u \in X$ such that $Iu \in A$. Then from the inequality (5) it follows that, for all $n \in \mathbb{N}$,

$$H(Su, Ty_n) \leq \varphi(aL_1(u, y_n) + (1-a)N_1(u, y_n)), \tag{7}$$

where

$$\begin{aligned} L_1(u, y_n) &\leq \max\{H(A, Su),\ \frac{1}{2}[H(A, Ty_n) + H(Ty_n, Su)]\} \\ &\to H(A, Su)_+ \text{ as } n \to \infty \end{aligned}$$

and

$$\begin{aligned} N_1(u, y_n) &\leq [\max\{H(A, Ty_n)H(Ty_n, Su),\ \frac{1}{2}H(A, Su)H(Ty_n, Su)\}]^{1/2} \\ &\to (1/\sqrt{2})H(A, Su)_+ \text{ as } n \to \infty\,. \end{aligned}$$

Hence passing over to limit as $n \to \infty$ in (7), we obtain

$$\begin{aligned} H(Su, A) &\leq \varphi(aH(Su, A)_+ + [(1-a)/\sqrt{2}]H(Su, A)_+) \\ &\leq \varphi(H(Su, A)_+)\,. \end{aligned}$$

Since $\varphi(t_+) < t$ for all $t \in (0, \infty)$, it follows that $H(Su, A) = 0$. Hence $Su = A$. We now suppose that $v \in A$ such that $Iv \in Sv$. Then from the inequality (5), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} H(Sv, Ty_n) \leq \varphi\big(&a \cdot \max\{D(Iv, Sv),\ D(Jy_n, Ty_n),\ \frac{1}{2}[D(Iv, Ty_n) + D(Jy_n, Sv)]\} \\ &+ (1-a) \cdot [\max\{D(Iv, Sv)D(Jy_n, Ty_n), D(Iv, Ty_n)D(Jy_n, Sv)\,, \\ &\quad \frac{1}{2}D(Iv, Sv)D(Jy, Sv), \frac{1}{2}D(Jy_n, Ty_n)D(Iv, Ty_n)\}]^{1/2}\big) \\ \leq\ & \varphi(a \cdot H(Sv, Ty_n) + (1-a) \cdot H(Sv, Ty_n)) \\ =\ & \varphi(H(Sv, Ty_n))\,. \end{aligned}$$

Passing over to limit as $n \to \infty$ in the above inequality, we obtain $H(Sv, A) \leq \varphi(A(Sv, A)_+)$. Hence $H(Sv, A) = 0$. It follows that $Sv = A$. Thus we have shown that $F = I(X) \cap A$ and $Sx = A$ whenever $Ix \in Sx$. We can similarly show that $G = J(X) \cap A$ and $Ty = A$ whenever $Jy \in Ty$.#

Remark 5 Theorem 3 improves Theorem 2 of Naidu [7].

Corollary 2 *Suppose that* $\lim_{t\to+\infty}[t-\varphi(t)] = +\infty$*,* $S(X)$ *and* $T(X)$ *are closed subsets of* X*,* $Sx \subseteq I(X)$ *and* $Gx \subseteq J(X)$ *for all* $x \in X$ *and the inequality* (5) *holds for all* $x, y \in X$*,* $a \in [0,1]$*. Then:*

(i) $\{Ix : x \in X \text{ and } Ix \in Sx\} = \{Jx : x \in X \text{ and } Jx \in Tx\} = A$ *for some* $A \in CL(X)$*,*

(ii) $Sx = A = Ty$ *for all* $x \in I^{-1}(A)$ *and for all* $y \in J^{-1}(A)$*.*

Proof: The conclusion follows immediately from Theorem 3.#

Theorem 4 *Let* (X,d) *be a complete metric space, and let* $I, J : X \to X$ *and* $S, T : X \to CL(X)$*. Suppose that* $\lim_{t\to+\infty}(t-\varphi(t)) = +\infty$*, there are sequences* $\{x_n\}_{n=0}^{\infty}$ *and* $\{y_n\}_{n=0}$ *in* X *such that* $\{Ix_n, Ix_{n+1}\} \subset Sx_n$ *and* $\{Jy_n, Jy_{n+1}\} \subset Ty_n$ $(n = 0, 1, 2, \dots)$*, and* (5) *holds for all* $x, y \in X$*. If* I*,* J*,* S *and* T *are continuous ,* I*,* S *and* J*,* T *are compatible mappings, then there exists a point* $t \in X$ *such that* $It \in St$ *and* $Jt \in Tt$*, i.e.,* t *is a coincidence point of* I *and* S *and* J *and* T*.*

Proof: Following the proof technique of Theorem 3, we can show that the sequences $\{Sx_n\}$ and $\{Ty_n\}$ converge in $CL(X)$ to the same limit A for some A in $CL(X)$. By (5), for $m \geq n$ $(m, n \in \mathbb{N})$, we have

$$\begin{aligned} d(Ix_n, Jy_m) &\leq D(Ix_n, Sx_n) + D(Jy_m, Sx_n) \\ &\leq D(Ix_n, Sx_n) + H(Sx_n, Ty_m) \\ &\leq D(Ix_n, Sx_n) + \varphi(aL_1(x_n, y_m) + (1-a)N_1(x_n, y_m))\,, \end{aligned} \tag{8}$$

where

$$\begin{aligned} L_1(x_n, y_m) &= \max\{D(Ix_n, Sx_n),\ D(Jy_m, Ty_m),\ \frac{1}{2}[D(Ix_n, Ty_n) + D(Jy_m, Sx_n)]\} \\ &\leq H(Sx_n, Ty_m) \end{aligned}$$

and

$$\begin{aligned} N_1(x_n, y_m) &= [\max\{D(Ix_n, Sx_n)D(Jy_m, Ty_m),\ D(Ix_n, Ty_m)D(Jy_m, Sx_n)\,, \\ &\qquad \frac{1}{2}D(Ix_n, Sx_n)D(Jy_m, Sx_n),\ \frac{1}{2}D(Jy_m, Ty_n)D(Ix_n, Ty_m)\}]^{1/2} \\ &\leq H(Sx_n, Ty_m)\,. \end{aligned}$$

Making $n \to \infty$ in (8), we obtain

$$\lim_n d(Ix_n, Jy_m) \leq 0 + \varphi(0)\,.$$

It follows that $Ix_n, Jy_n \to t$ as $n \to \infty$ for some $t \in X$, since X is complete, $d(Ix_n, Ix_m) \leq d(Ix_n, Jy_m) + d(Jy_m, Ix_n)$ and $d(Jy_n, Jy_m) \leq d(Jy_n, Ix_m) + d(Ix_m, Jy_m)$. Again since $D(t, A) \leq D(t, Sx_n) + H(Sx_n, A) \to 0$ as $n \to \infty$, it follows that $t \in A$. By continuity of I and S, and since S and I are compatible, we have

$$\begin{aligned} D(It, St) &= \lim_n D(It, SIx_n) \leq \lim_n H(IA, SIx_n) \\ &= \lim_n H(ISx_n, SIx_n) = 0 \end{aligned}$$

Hence $It \in St$. Due to symmetry, we can similarly show that $Jt \in Tt$.#

By applying the same arguments as in the proof of Theorem 3, we can easily prove the following theorems:

Theorem 5 *Let (X, d) be a complete metric space, and $I, J : X \to X$ and $S, T : X \to CL(X)$. Suppose that $\lim\limits_{t\to+\infty}(t - \varphi(t)) = +\infty$, there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in X such that $Ix_{n+1} \in Sx_n$ and $Jy_{n+1} \in Ty_n$ $(n = 0, 1, 2, \dots)$ and*

$$H(Sx, Ty) \leq \varphi\left(\frac{a}{2}[D(Ix, Ty) + D(Jy, Sx)] + (1 - a)[D(Ix, Ty)D(Jy, Sx)]^{1/2}\right) \qquad (5')$$

for all $x, y \in X$ and $a \in [0, 1]$. Then the sequences $\{Sx_n\}$, $\{Ty_n\}$ converge in $CL(X)$ to the same limit A for some $A \in CL(X)$, $\{Ix| \in X$ and $Ix \in Sx\} = I(X) \cap A$ and $\{Jy|y \in X$ and $Jy \in Ty\} = J(X) \cap A$. Further, $Sx = A$ whenever $Ix \in Sx$ and $Ty = A$ whenever $Jy \in Ty$.

Theorem 6 *Let (X, d) be a complete metric space, and let $I, J : X \to X$ and $S, T : X \to CL(X)$. Suppose that $\lim\limits_{t\to+\infty}(t - \varphi(t)) = +\infty$, there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$in X such that $Ix_{n+1} \in Sx_n$ and $Jy_{n+1} \in Ty_n$ $(n = 0, 1, 2, \dots)$ and (5′) holds for all $x, y \in X$. If I, J, S and T are continuous, I, S and J, T are compatible mappings. Then there exists a point $t \in X$ such that $It \in St$ and $Jt \in Tt$; i.e., t is a coincidence point of I and S and J and T.*

Remark 6 In view of Example 10 of Sastry, Naidu and Prasad [11], the condition $\lim\limits_{t\to\infty}(t - \varphi(t)) = +\infty$ in Theorems 3-6 cannot be dispensed with even if $\sum\limits_{n=1}^{\infty} \varphi^n(t) < +\infty$ for all $t \in \mathbb{R}^+$ with $S = T$ and $I = J = i_d$, the identity mapping on X.

Remark 7 It is not yet known whether the continuity of all four maps I, J, S and T in Theorems 4 and 6 are necessary or not.

Remark 8 Condition (2) of Naidu [7] is implied by condition (5′) of Theorem 5, and hence Theorem 2 of Naidu [7] is a direct consequence of Theorem 5 .

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Coincidence Points For Multivalued Mappings

ABSTRACT. In this paper we show some coincidence theorems for contractive type multivalued mappings in compact metric spaces, which extend properly the results of Kubiak and Kubiaczyk.

KEY WORDS. Coincidence point, multivalued mappings, compact metric space.

1 Introduction and preliminaries

Let (X, d) be a metric space. For any nonempty subsets A, B of X we define $D(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}, \delta(A, B) = \sup\{d(a, b) : a \in A \text{ and } b \in B\}$ and $H(A, B) = \max\{\sup[D(a, B) : a \in A], \sup[D(A, b) : b \in B]\}$. Let $CL(X) = \{A : A \text{ is a nonempty closed subset of } X\}$ and $CB(X) = \{A : A \text{ is a nonempty bounded closed subset of } X\}$. It is well known that $(CB(X), H)$ is a metric space. Obviously $CB(X) = CL(X)$ if (X, d) is a compact metric space. Let S be a mapping of X into $CL(X)$, f a selfmapping of X. A point $x \in X$ is called a coincidence point of f and S if $fx \in Sx$.

Kubiak [1] and Kubiaczyk [2] proved some fixed point theorems for contractive type multivalued mappings in compact metric spaces. The purpose of this paper is to extend their results to a more general case.

2 Coincidence theorems

Theorem 2.1 *Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Suppose that f and g are selfmappings of X satisfying*

$$\begin{aligned}\delta(Sx, Ty) < \max\Big\{&d(fx, gy), H(fx, Sx), H(gy, Ty),\\ &\frac{1}{2}[D(fx, Ty) + D(gy, Sx)],\\ &H(fx, Sx)H(gy, Ty)/d(fx, gy),\\ &D(fx, Ty)D(gy, Sx)/d(fx, gy)\Big\} \qquad (2.1)\end{aligned}$$

for all $x, y \in X$ with $fx \neq gy$. Let $SX \subseteq gX$ and $TX \subseteq fX$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point u with $Su = \{fu\}$ or $Tu = \{gu\}$.

Proof: We assume without loss of generality that f and S are continuous. It follows that $H(fx, Sx)$ is a continuous function on X. By the compactness of X, there exists a point $u \in X$ such that $H(fu, Su) = \inf\{H(fx, Sx) : x \in X\}$. It is easy to check that there is a point $y \in Su$ with $d(fu, y) = H(fu, Su)$. Since $SX \subseteq gX$, then there exists a point $v \in X$ with $y = gv$. Consequently $d(fu, gv) = H(fu, Su)$ for some $gv \in Su$. Similarly, there are two points $w, x \in X$ such that $d(gv, fw) = H(gv, Tv)$, $d(fw, gx) = H(fw, Sw)$, where $fw \in Tv, gx \in Sw$. We now assert that $H(fu, Su)H(gv, Tv) = 0$. Otherwise $H(fu, Su)H(gv, Tv) > 0$. Using (2.1) we have

$$\begin{aligned}\delta(Su, Tv) &< \max\Big\{d(fu, gv), H(fu, Su), H(gv, Tv),\\ &\qquad \frac{1}{2}[D(fu, Tv) + D(gv, Su)],\\ &\qquad H(fu, Su)H(gv, Tv)/d(fu, gv),\\ &\qquad D(fu, Tv)D(gv, Su)/d(fu, gv)\Big\}\\ &= \max\Big\{H(fu, Su), H(gv, Tv),\\ &\qquad \frac{1}{2}[d(fu, gv) + H(gv, Tv)]\Big\}\\ &= \max\Big\{H(fu, Su), H(gv, Tv)\Big\}\end{aligned}$$

which implies

$$H(gv, Tv) \leq \delta(Su, Tv) < \max\big\{H(fu, Su), H(gv, Tv)\big\} = H(fu, Su)\,. \qquad (2.2)$$

Similarly we can show

$$H(fw, Sw) \leq \delta(Sw, Tv) < \max\big\{H(gv, Tv), H(fw, Sw)\big\} = H(gv, Tv)\,. \qquad (2.3)$$

It follows from (2.2) and (2.3) that

$$H(fw, Sw) < H(gv, Tv) < H(fu, Su) = \inf \left\{ H(fx, Sx) : x \in X \right\}$$

which is a contradiction and hence $H(fu, Su)H(gv, Tv) = 0$, which implies that $Su = \{fu\}$ or $Tv = \{gv\}$. This completes the proof.

If f and g are the identity mapping on X, Theorem 2.1 reduces to the following.

Corollary 2.2 *Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$ satisfying*

$$\begin{aligned} \delta(Sx, Ty) < \max \Big\{ & d(x, y), H(x, Sx), H(y, Ty), \\ & \frac{1}{2}[D(x, Ty) + D(y, Sx)], \\ & H(x, Sx)H(y, Ty)/d(x, y), \\ & D(x, Ty)D(y, Sx)/d(x, y) \Big\} \end{aligned} \tag{2.4}$$

for all $x, y \in X$ with $x \neq y$. If S or T is continuous, then S or T has a fixed point u with $Su = \{u\}$ or $Tu = \{u\}$.

Remark 2.1 Theorem 4 in [1] and Theorem 4 in [2] are special cases of Corollary 2.2. The following example demonstrates that Corollary 2.2 extends properly Theorem 4 in [1] and Theorem 4 in [2].

Example 2.1 Let $X = \{1, 3, 6, 10\}$, d the ordinary distance, and define S and T by $S1 = \{3, 6\}$, $S3 = \{3, 6, 10\}$, $S6 = S10 = T1 = T6 = T10 = \{6\}$ and $T3 = \{10\}$. Then (X, d) is a compact metric space, S and T are continuous mappings of X into $CL(X)$. It is easy to show that S and T satisfy (2.4). But Theorem 4 in [1] and Theorem 4 in [2] are not applicable since

$$\delta(Sx, Ty) < \max \left\{ d(x, y), H(x, Sx), H(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Sx)] \right\}$$

and

$$\begin{aligned} \delta(Sx, Ty) &< a(x, y)d(x, y) + b(x, y)[H(x, Sx) + H(y, Ty)] \\ &\quad + c(x, y)[D(x, Ty) + D(y, Tx)] \end{aligned}$$

are not satisfied for $x = 1$ and $y = 3$, where a, b and c are functions of $X \times X$ into $[0, \infty)$ with $\sup\{a(x, y) + 2b(x, y) + 2c(x, y) : (x, y) \in X \times X\} \leq 1$.

Theorem 2.3 *Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Assume that f and g are selfmappings of X satisfying*

$$\begin{aligned} H(Sx, Ty) < \max\Big\{ & d(fx, gy), D(fx, Sx), D(gy, Ty), \\ & \frac{1}{2}[D(fx, Ty) + D(gy, Sx)], \\ & D(fx, Sx)D(gy, Ty)/d(fx, gy), \\ & D(fx, Ty)D(gy, Sx)/d(fx, gy)\Big\} \end{aligned} \tag{2.5}$$

for all $x, y \in X$ with $fx \neq gy$. Let $SX \subseteq gX$ and $TX \subseteq fX$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point.

Proof: We may assume that f and S are continuous on X. Then $D(fx, Sx)$ is continuous and attains its minimum at some $u \in X$. As in the proof of Theorem 2.1, there exist $v, w,\ x \in X$ such that $d(fu, gv) = D(fu, Su)$, $d(gv, fw) = D(gv, Tv)$ and $d(fw, gx) = D(fw, Sw)$, where $gv \in Su$, $fw \in Tv$, $gx \in Sw$. Assume that $D(fu, Su)D(gv, Tv) > 0$. The same argument as that of the proof of Theorem 2.1 shows that $D(fw, Sw) < D(gv, Tv) < D(fu, Su)$, which contradicts the miniality of $D(fu, Su)$. Hence $D(fu, Su)D(gv, Tv) = 0$. That is, $fu \in Su$ or $gv \in Tv$. This completes the proof.

As an immediate consequence of Theorem 2.3 we have the following.

Corollary 2.4 *Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Suppose that f and g are selfmappings of X satisfying*

$$\begin{aligned} H(Sx, Ty) < \max\Big\{ & d(fx, gy), D(fx, Sx), D(gy, Ty), \\ & \frac{1}{2}[D(fx, Ty) + D(gy, Sx)]\Big\} \end{aligned} \tag{2.6}$$

for all $x, y \in X$ with $fx \neq gy$. Let $SX \subseteq gX$ and $TX \subseteq fX$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point.

Remark 2.2 If f and g are the identity mapping on X, Corollary 2.4 reduces to Theorem 2 in [1] and includes Theorem 3 in [2]. The following example verifies that Corollary 2.4 does indeed generalize Theorem 2 in [1] and Theorem 3 in [2], that not both f, S and g, T of Corollary 2.4 need have a coincidence point and that the coincidence point may not be unique.

Example 2.2 Let $X = \{1, 3, 6\}$ with the usual metric, and define S, T, f and g by $S1 = S3 = T6 = \{1, 3\}$, $S6 = T1 = \{3\}$, $T3 = \{1\}$, $f1 = f6 = 3$, $f3 = 1$, $g1 = g3 = g6 = 6$. It is easy to see that the hypothesis of Corollary 2.4 is satisfied. Clearly f and S have three coincidence points while g and T have none. However, Theorem 2 in [1] and Theorem 3 in [2] are not applicable since

$$\begin{aligned} H(Sx,Ty) &< \max\Big\{d(x,y), D(x,Sx), D(y,Ty), \\ &\qquad \frac{1}{2}[D(x,Ty)+D(y,Sx)]\Big\} \end{aligned}$$

and

$$\begin{aligned} H(Sx,Ty) &< a(x,y)d(x,y)+b(x,y)[D(x,Sx)+D(y,Ty)] \\ &\quad +c(x,y)[D(x,Ty)+D(y,Sx)] \end{aligned}$$

are not satisfied for $x = 1$ and $y = 3$, where a, b and c are functions of $X \times X$ into $[0,\infty)$ with $\sup\{a(x,y)+2b(x,y)+2c(x,y) : (x,y) \in X \times X\} \leq 1$.

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Stability of Mann Iterative Process with Random Errors for the Fixed Point of Strongly-Pseudocontractive Mapping in Arbitrary Banach Spaces[1]

ABSTRACT. Suppose that X is a arbitrary real Banach space and $T : X \to X$ is a Strongly pseudocontractive mapping. It is proved that certain Mann iterative process with random errors for the fixed point of T is stable(almost stable) with respect to T with(without) Lipschitz condition. And, two related results are obtained that deals with stability(or almost stability) of Mann iterative process for solution of nonlinear equations with strongly accretive mapping. Consequently, the corresponding results of Osilike are improved.

KEY WORDS. strongly pseudocontractive mapping, strongly accretive mapping, Mann iterative process with random errors, stable, almost stable

To set the framework, we recall some basic notations as follows.

Let X be a real Banach space and $K \subset X$ a nonempty subset.

(a) A mapping $T : K \to X$ is said to be strongly pseudocontractive if for any $x, y \in K$ we have

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\| \tag{1}$$

for all $r > 0$, where I is the identity mapping on X and the constant $k \in (0,\ 1)$. A mapping $A : K \to X$ is said to be strongly accretive if $I - A$ is strongly pseudocontractive. Hence, the mapping theory for accretive mappings is intimately connected with the fixed point theory for pseudocontractive mappings.

(b) Let $T : X \to X$ be a mapping. For any given $x_0 \in X$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_nTx_n + c_nu_n \qquad (n \geq 0) \tag{2}$$

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is called Mann[1] iteration sequence with random errors, where $u_n \in X (n \geq 0)$ is a bounded and random error term, and $\{a_n\}$ and $\{c_n\}$ are two real sequences in $(0,\ 1)$ satisfying some conditions. By the way, Xu, one of authors introduced another definition of Mann iteration process with random errors on a nonempty convex subset of Banach space in 1998(see, Xu [2]).

(c) Let K be a nonempty convex subset of X and T be a selfmapping of K. Assume that $x_0 \in K$ and $x_{n+1} = f_n(T,\ x_n)$ define an iterative process which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in K. Suppose $F(T) = \{x \in K:\ Tx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point $q \in F(T)$. For any $\{y_n\}_{n=0}^{\infty} \subset K$, let $\varepsilon_n = \|y_{n+1} - f_n(T,\ y_n)\|$. If $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim y_n = q$ then the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is said to be almost T-stable. Furthermore, If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim y_n = q$ then the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is said to be T-stable(see, Zhang [3]).

In recent, some stability results have been established(see [4]-[8]), for example, Osilike[6] showed that the Mann and Ishikawa iterative processes are stable with respect to Lipschitz strongly pseudocontractive mapping T in p-uniformly smooth Banach space. Then, he extended the results to arbitrany real Banach spaces in [7]. Since the consideration of error terms is an important part of any iteration methods and many mappings without Lipschitz condition, therefore, we introduced the Mann iterative process with random errors and to prove that the iterative process is stable(almost stable) with respect to T with(without) Lipschitz condition where T is a strongly pseudocontractive mapping in arbitrary Banach space. And, two related results are obtained that deals with stability(or almost stability) of Mann iterative process for solution of nonlinear equations with Strongly accretive mapping. Consequently, the corresponding results of Osilike are improved.

Now, we prove the following theorems.

Theorem 1 *Suppose that $T : X \to X$ be a Lipschitz strongly pseudocontractive mapping. If q is a fixed point of T and for arbitrary $x_0 \in X$, the Mann iteration sequence with random errors defined by (2) satisfying*

$$0 < a \leq a_n \leq k[2(L^2 + 3L + 3)]^{-1} \qquad \text{and} \qquad \lim_{n\to\infty} c_n = 0 \tag{1.1}$$

where $L > 1$ is Lipschitz constant of T and $a > 0$ is a constant. Then

(1) *$\{x_n\}$ converges strongly to unique fixed point q of T;*

(2) *Let $\{y_n\}$ be any sequence in X. Then y_n converges strongly to q if and only if ε_n converges to 0.*

Proof: Let $\sup\{\|u_n\| : \ n = 0, 1, 2, \cdots\} = M$. Using (2), we have

$$\begin{aligned} x_n - q &= (x_{n+1} - q) - a_n(Tx_n - q) + a_n(x_n - q) - c_n u_n \\ &= (1 + a_n)(x_{n+1} - q) + a_n[(I - T - kI)x_{n+1} - (I - T - kI)q] \\ &\quad + a_n(Tx_{n+1} - Tx_n) - (2 - k)a_n^2(Tx_n - q) - (1 - k)a_n(x_n - q) \\ &\quad + (2 - k)a_n^2(x_n - q) - [1 + (2 - k)a_n]c_n u_n \end{aligned}$$

for all $n \geq 0$. Furthermore,

$$\begin{aligned} \|x_n - q\| &\geq (1 + a_n)\|x_{n+1} - q + \tfrac{a_n}{1+a_n}[(I - T - kI)x_{n+1} - (I - T - kI)q]\| \\ &\quad - a_n\|Tx_{n+1} - Tx_n\| - (2 - k)a_n^2\|Tx_n - q\| - (1 - k)a_n\|x_n - q\| \\ &\quad - (2 - k)a_n^2\|x_n - q\| - [1 + (2 - k)a_n]c_n\|u_n\| \end{aligned} \tag{3}$$

for all $n \geq 0$. by virtue of (1), we have

$$\begin{aligned} \|x_n - q\| &\geq (1 + a_n)\|x_{n+1} - q\| - La_n\|x_{n+1} - x_n\| - (1 - k)a_n\|x_n - q\| \\ &\quad - 2(L + 1)a_n^2\|x_n - q\| - 3Mc_n \\ &\geq (1 + a_n)\|x_{n+1} - q\| - (1 - k)a_n\|x_n - q\| \\ &\quad - (L + 1)(L + 2)a_n^2\|x_n - q\| - (3 + L)Mc_n \end{aligned} \tag{4}$$

for all $n \geq 0$. It follows from (4) and the condition (1.1) that

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - a_n + a_n^2)\|x_n - q\| + (1 - k)a_n\|x_n - q\| \\ &\quad + (L + 1)(L + 2)a_n^2\|x_n - q\| + (3 + L)Mc_n \\ &\leq (1 - ka_n)\|x_n - q\| + (L^2 + 3L + 3)a_n^2\|x_n - q\| + (3 + L)Mc_n \\ &\leq (1 - ka_n/2)\|x_n - q\| + (3 + L)Mc_n \\ &\quad + a_n[a_n(L^2 + 3L + 3) - k/2]\|x_n - q\| \\ &\leq (1 - ka/2)\|x_n - q\| + (3 + L)Mc_n \end{aligned} \tag{5}$$

for all $n \geq 0$. Putting

$$\alpha = 1 - ka/2, \quad t_n = \|x_n - q\| \ \textit{and} \ \ \beta_n = (3 + L)Mc_n \qquad (n \geq 0).$$

Hence, the inequality (5) reduces to

$$t_{n+1} \leq \alpha t_n + \beta_n \qquad (n \geq 0).$$

It follows from the inequality of Q. H. Liu (see Lemma of [9]) that $\lim_{n\to 0}\|x_n - q\| = 0$. I.e., $\{x_n\}$ converges strongly to fixed point q of T. If q' also is a fixed point of T, putting $r = 1$ in (1) we obtain $\|q - q'\| \leq (1 - k)\|q - q'\|$. It implies that $q = q'$.
We now prove part (2). Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. Then

$$\begin{aligned} \|y_{n+1} - q\| &= \|y_{n+1} - (1 - a_n)y_n - a_nTy_n - c_nu_n + (1 - a_n)y_n + a_nTy_n - q + c_nu_n\| \\ &\leq \varepsilon_n + \|(1 - a_n)(y_n - q) + a_n(Ty_n - q) + c_nu_n\| \\ &\leq \|(1 - ka/2)\|y_n - q\| + (3 + L)Mc_n + \varepsilon_n \\ &\leq \alpha\|y_n - q\| + \beta_n + \varepsilon_n \end{aligned}$$

for all $n \geq 0$.
By virtue of the inequality of Q. H. Liu again, we obtain that $y_n \to q$ $(as\ n \to \infty)$. I.e., the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is T-stable.
On the contrary, if $\lim_{n\to\infty} y_n = q$ then

$$\begin{aligned}\varepsilon_n &= \|y_{n+1} - (1-a_n)y_n - a_nTy_n - c_nu_n\| \\ &\leq \|y_{n+1} - q\| + (1-a_n)\|y_n - q\| + La_n\|y_n - q\| + Mc_n \to 0(as\ \ n \to \infty).\end{aligned}$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$. The Proof is completed.

From this Theorem we can prove

Corollary 1 *Suppose that $A : X \to X$ be a Lipschitz strongly accretive mapping. Let x^* be a solution of $Ax = f$ where f is any given and $Sx = f + x - Ax\ \ \forall\ x \in X$. For arbitrary $x_0 \in X$, if Mann iteration sequence with random errors defined by*

$$x_{n+1} = (1-a_n)x_n + a_nSx_n + c_nu_n \qquad (n \geq 0) \tag{6}$$

satisfying

$$0 < a \leq a_n \leq k[2(L_*^2 + 3L_* + 3)]^{-1} \quad \text{and} \quad \lim_{n\to\infty} c_n = 0$$

where $L_* > 1$ is Lipschitz constant of S. Then

(1) *$\{x_n\}$ converges strongly to unique solution of $Ax = f$;*

(2) *It is S-stable to approximate the solution of $Ax = f$ by (6) (Mann iteration sequence with random errors).*

In fact, from $Sx = f + x - Ax$, it is easy to see that x^* is unique solution of $Ax = f$ if and only if x^* is unique fixed point of S. Since S is a Lipschitz strongly pseudocontractive mapping, by virtue of theorem 1, we know the conclusions of corollary 1 are true.

Theorem 2 *Suppose that $T : X \to X$ be an uniformly continuous strongly pseudocontractive mapping with bounded range. If q is a fixed point of T and for arbitrary $x_0 \in X$, the Mann iteration sequence with random errors defined by (2) satisfying*

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \sum_{n=0}^{\infty} a_n^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} c_n < \infty,$$

then

(1) *$\{x_n\}$ converges strongly to unique fixed point of T;*

(2) *Let $\{y_n\}$ be any sequence in X. Then $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that y_n converges strongly to q;*

(3) *y_n converges strongly to q implies that $\lim_{n\to\infty} \varepsilon_n = 0$.*

Proof: Putting

$$\begin{aligned} c &= \sup\{\|Tx - q\| : x \in X\} + \|x_0 - q\| \\ d &= \sup\{\|u_n\| : n \geq 0\}. \end{aligned}$$

For any $n \geq 0$, using induction, we obtain

$$\|x_n - q\| \leq c + d\sum_{i=0}^{n-1} c_i \leq c + d\sum_{i=0}^{+\infty} c_i.$$

Hence, we set

$$M = c + d\sum_{i=0}^{+\infty} c_i.$$

Since $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|a_n(Tx_n - x_n) + c_n u_n\| = 0$, therefore,

$$e_n := \|Tx_{n+1} - Tx_n\| \to 0 (as \quad n \to \infty)$$

by the uniform continuity of T. From (3) and using (1), we have

$$\begin{aligned} \|x_n - q\| &\geq (1 + a_n)\|x_{n+1} - q + \tfrac{a_n}{1+a_n}[(I - T - kI)x_{n+1} - (I - T - kI)q]\| \\ &\quad -a_n e_n - (2-k)Ma_n^2 - (1-k)a_n\|x_n - q\| \\ &\quad -(2-k)a_n^2\|x_n - q\| - [1 + (2-k)a_n]c_n\|u_n\| \\ &\geq (1 + a_n)\|x_{n+1} - q\| - (1-k)a_n\|x_n - q\| \\ &\quad -a_n e_n - (2-k)a_n^2\|x_n - q\| - 2Ma_n^2 - 3Mc_n \end{aligned} \tag{7}$$

for all $n \geq 0$. It follows from (7) that

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - a_n + a_n^2)\|x_n - q\| + (1-k)a_n\|x_n - q\| \\ &\quad +a_n e_n + (2-k)a_n^2\|x_n - q\| + 2Ma_n^2 + 3Mc_n \\ &\leq (1 - ka_n)\|x_n - q\| + a_n e_n + 5Ma_n^2 + 3Mc_n \end{aligned} \tag{8}$$

for all $n \geq 0$. Putting

$$\alpha_n = ka_n, \quad t_n = \|x_n - q\|, \quad a_n e_n = O(\alpha_n) \quad and \quad \beta_n = 5Ma_n^2 + 3Mc_n \qquad (n \geq 0).$$

Hence, the inequality (8) reduces to

$$t_{n+1} \leq (1 - \alpha_n)t_n + O(\alpha_n) + \beta_n \qquad (n \geq 0).$$

It follows from the inequality of L. S. Liu(see Lemma 2 of [10]) that $\lim_{n\to 0}\|x_n - q\| = 0$. So, $\{x_n\}$ converges strongly to unique fixed point q of T.
We now prove part (2) and (3). Suppose $\sum_{n=0}^{\infty}\varepsilon_n \leq \infty$. Observe

$$\begin{aligned}\|y_{n+1} - q\| &\leq \varepsilon_n + \|(1-a_n)(y_n - q) + a_n(Ty_n - q) + c_n u_n\| \\ &\leq (1-\alpha_n)\|y_n - q\| + O(\alpha_n) + \beta_n + \varepsilon_n\end{aligned}$$

for all $n \geq 0$.
By virtue of the inequality of L. S. Liu again, we obtain that $y_n \to q$ $(as\ n \to \infty)$. I.e., the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is almost T-stable.
On the contrary, if $\lim_{n\to\infty} y_n = q$ then

$$\begin{aligned}\varepsilon_n &= \|y_{n+1} - (1-a_n)y_n - a_n T y_n - c_n u_n\| \\ &\leq \|y_{n+1} - q\| + (1-\alpha_n)\|y_n - q\| + O(\alpha_n) + \beta_n \to 0(as\ \ n \to \infty).\end{aligned}$$

This implies that $\lim_{n\to\infty}\varepsilon_n = 0$. The proof is completed.

From Theorem 2 we can prove

Corollary 2 *Suppose that $A : X \to X$ is an uniformly continuous strongly accretive mapping and the range of $I - A$ is bounded. Let x^* be a solution of $Ax = f$ where f is any given. for arbitrary $x_0 \in X$, if Mann iteration sequence with random errors defined by (6) satisfying*

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \sum_{n=0}^{\infty} a_n^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} c_n < \infty,$$

then

(1) *$\{x_n\}$ converges strongly to unique solution of $Ax = f$;*

(2) *It is almost S-stable to approximate the solution of $Ax = f$ by (6) (Mann iteration sequence with random errors).*

In fact, from $Sx = f + x - Ax$, it is easy to see that x^* is unique solution of $Ax = f$ if and only if x^* is unique fixed point of S. Since S is an uniformly continuous strongly pseudocontractive mapping, by virtue of theorem 2, we know the conclusions of corollary 2 are true.

Remark The iterative parameters $\{\alpha_n\}$ and $\{c_n\}$ do not depend on any geometric structure of space X and on any property of the mappings, but, the selection of the parameters is deal with the convergence rate of the iterative sequence. In Theorem 2 and Corollary 2, a prototype of iteration parameters is

$$a_n = \frac{1}{n+1} \quad \text{and} \quad c_n = \frac{1}{(n+1)^2} \qquad \forall\ \ n \geq 0.$$

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Recursions for the Solution of an Integral-Functional Equation

ABSTRACT. In this paper, we continue our considerations in [1, 2, 3] about a homogeneous integral-functional equation with a parameter $a > 1$. Here we assume that $a \geq 2$, disregarding some explicitly mentioned cases where a can be smaller than 2. We derive new recursions which allow to calculate the solution and its derivatives effectively, and which contain formulas of R. Schnabl [8] and W. Volk [10] as special cases for $a = 2$.

KEY WORDS. Integral-functional equation, generating functions, Cantor sets, relations containing polynomials, recursions, directed graphs.

1 Introduction

There exists a long history concerning compactly supported C^∞-functions, which are solutions of differential-functional equations, cf. [7], [3] and the literature quoted there. By integration these equations can be transformed into integral-functional equations. Here, we deal with the special equation

$$\phi(t) = b \int_{at-a+1}^{at} \phi(\tau)d\tau \qquad \left(b = \frac{a}{a-1}\right) \tag{1.1}$$

with the real variable t and a parameter $a > 1$. Applications of (1.1) to probability problems were given by G.J. Wirsching for $a = 3$ in [11], and for $a \geq \frac{3}{2}$ in [12].

In this paper, we continue our considerations in [1, 2, 3] concerning the solutions of (1.1) under the assumption $a \geq 2$ disregarding some explicitly mentioned cases where a can be smaller than 2, in particular, in Section 8. We derive new recursions which allow to calculate the solution and its derivatives effectively, and which contain formulas of R. Schnabl [8] and W. Volk [10] as special cases for $a = 2$. For convenience of the reader we first list those results from [1, 2, 3] which are needed later on.

For $a > 1$ equation (1.1) has a C^∞-solution with the support $[0,1]$, which is uniquely determined by means of the normalization

$$\int_0^1 \phi(t)dt = 1 \ . \tag{1.2}$$

In particular, it is $\phi(0) = \phi(1) = 0$. The solution of (1.1)-(1.2) is symmetric with respect to the point $\frac{1}{2}$, monotone at both sides of $\frac{1}{2}$ and it is strictly positive for $t \in (0,1)$. The Laplace transform Φ of the solution ϕ of (1.1)-(1.2) is an entire function satisfying $\Phi(0) = 1$ and the functional equation

$$\Phi(z) = \frac{1 - e^{-z/b}}{z/b}\Phi\left(\frac{z}{a}\right) . \tag{1.3}$$

It has the Taylor series

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!} z^n \qquad (z \in \mathbb{C})$$

where the coefficients are rational functions with respect to a and, starting with $\rho_0(a) = 1$ for $n \geq 1$, they can be determined by means of the recursion

$$\rho_n(a) = \frac{1}{(n+1)(a^n - 1)} \sum_{\nu=0}^{n-1} \binom{n+1}{\nu} \rho_\nu(a)(1-a)^{n-\nu} \ . \tag{1.4}$$

Moreover, for fixed n, the functions $(-1)^n \rho_n(a)$ are increasing for $a \geq 1$ and it holds

$$\frac{1}{2^n} \leq (-1)^n \rho_n(a) \leq \frac{1}{n+1}, \tag{1.5}$$

cf. [1, (2.14)].

For $a > 2$, the solution ϕ of (1.1)-(1.2) is a polynomial on each component of an open Cantor set with Lebesgue measure 1. These polynomials can be expressed by means of the polynomials

$$\psi_n(t) = \sum_{\nu=0}^{n} \binom{n}{\nu} \rho_{n-\nu}(a) t^\nu, \tag{1.6}$$

which have the special values

$$\psi_n(0) = \rho_n(a), \qquad \psi_n(1) = (-1)^n \rho_n(a), \tag{1.7}$$

and which have the generating function

$$e^{tz}\Phi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \psi_n(t) z^n \tag{1.8}$$

so that they are Appell polynomials, cf. [8], [1]. Note that in [1] we have used the abbreviation ψ_n for the polynomials (1.6) with $\frac{1}{a}$ instead of a. In [3] we have modified the polynomials (1.6) by

$$f_n(t) = c_n \psi_n(t) \tag{1.9}$$

where c_n is given by

$$c_n = \frac{b^{n+1}}{n!\, a^{\frac{n(n+1)}{2}}} = \frac{1}{n!\, a^{\frac{(n+1)(n-2)}{2}}(a-1)^{n+1}} \,. \tag{1.10}$$

These polynomials can be calculated recursively by

$$f_n(t) = \frac{b}{na^n}\left(t - \frac{1}{2}\right) f_{n-1}(t) + \frac{1}{n}\sum_{\nu=2}^{n} \frac{1}{\nu!} B_\nu \frac{a^{\frac{1}{2}\nu(\nu+1-2n)}}{a^\nu - 1} f_{n-\nu}(t) \quad (n \geq 1), \tag{1.11}$$

starting with $f_0(t) = b$ and using the Bernoulli numbers

$$B_0 = 1\,, \quad B_1 = -\frac{1}{2}\,, \quad B_2 = \frac{1}{6}\,, \quad B_3 = 0\,, \quad B_4 = -\frac{1}{30}\,, \quad \ldots\,.$$

They satisfy the relations

$$f_n(t) - f_n(t-a+1) = f_{n-1}\left(\frac{t}{a}\right) \tag{1.12}$$

and

$$f_n(t) = (-1)^n f_n(1-t) \tag{1.13}$$

with $n \in \mathbb{N}_0$ and $f_{-1} = 0$. The simplest connection between the solution ϕ of (1.1)-(1.2) and polynomials f_n $(n \geq -1)$ is valid for $a \geq 2$, and reads

$$\phi\left(\frac{\tau}{a^{n+1}}\right) = f_n(\tau) \qquad (1 \leq \tau \leq a-1). \tag{1.14}$$

In particular for $n = 0$, ϕ attains its maximum $\phi(t) = b$ for $\frac{1}{a} \leq t \leq 1 - \frac{1}{a}$. In order to state more complicated connections between ϕ and f_n we need an auxiliary sequence $\gamma_k = \gamma_k(a)$ defined as follows: If $k \in \mathbb{N}$ has the dyadic representation $k = d_p \ldots d_1 d_0$ with $d_p = 1$ and $d_\nu \in \{0, 1\}$ then

$$\gamma_k = (a-1)\sum_{\nu=0}^{p} d_\nu a^\nu. \tag{1.15}$$

The sequence γ_k $(k \in \mathbb{N}_0)$ can also be defined by

$$\gamma_{2k} = a\gamma_k\,, \quad \gamma_{2k+1} = a\gamma_k + a - 1\,, \quad k = 0, 1, 2, \ldots\,, \tag{1.16}$$

so that in particular $\gamma_0 = 0$ and $\gamma_1 = a - 1$. For $p \in \mathbb{N}_0$ these numbers satisfy the relations

$$\gamma_{2^p} = (a-1)a^p, \qquad \gamma_{2^p-1} = a^p - 1, \qquad \gamma_{2^p-2} = a^p - a \quad (p \neq 0), \tag{1.17}$$

$$\gamma_{2k+1} = \gamma_{2k} + \gamma_1, \qquad \gamma_{2^\sigma\kappa} = a^\sigma \gamma_\kappa \qquad (\sigma, \kappa \in \mathbb{N}_0), \tag{1.18}$$

$$\gamma_k + \gamma_u + 1 = a^{p+1} \qquad \text{if} \qquad k + u + 1 = 2^{p+1} \tag{1.19}$$

and the inequality

$$\gamma_{k+1} \geq \gamma_k + \gamma_1 \qquad (k \in \mathbb{N}_0,\ a \geq 2). \tag{1.20}$$

For integers a also the numbers γ_k are integers. In particular, for $a = 2$, we have $\gamma_k = k$. Moreover, we need the sign sequence $\varepsilon_k = (-1)^{\nu(k)}$, where $\nu(k)$ denotes the number of "1s" in the dyadic representation of k, i.e. $\nu(k)$ is the binary sum-of-digits function (cf. [4]).

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ε_k	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1

Table 1: The first numbers ε_k

In the case $a \geq 2$ we define the following closed intervals

$$\overline{G}_{kn} = \left[\frac{\gamma_{2k}+1}{a^{n+1}}, \frac{\gamma_{2k+1}}{a^{n+1}}\right], \qquad F_{kn} = \left[\frac{\gamma_k}{a^n}, \frac{\gamma_k + 1}{a^n}\right] \tag{1.21}$$

with $\overline{G}_{kn} \subset F_{kn}$, since

$$F_{kn} = F_{2k,n+1} \cup \overline{G}_{kn} \cup F_{2k+1,n+1} \tag{1.22}$$

$k = 0, 1, \dots, 2^n - 1$, $n \in \mathbb{N}_0$ (for $a = 2$ the intervals $\overline{G}_{kn}$ degenerate to a single point). In the intervals $\overline{G}_{kn}$, the solution ϕ of (1.1)-(1.2) has the representation

$$\phi(t) = \sum_{\nu=0}^{2k} \varepsilon_\nu f_n(a^{n+1}t - \gamma_\nu) \quad (t \in \overline{G}_{kn}) \tag{1.23}$$

for $k = 0, 1, \dots, 2^n - 1$, $n \in \mathbb{N}_0$. Moreover, for $t \in F_{kn}$, i.e. $t = \frac{\gamma_k + \tau}{a^n}$ with $0 \leq \tau \leq 1$ we have the *main formula*

$$\phi\left(\frac{\gamma_k + \tau}{a^n}\right) - \varepsilon_k \phi\left(\frac{\tau}{a^n}\right) = \sum_{\nu=0}^{k-1} \varepsilon_\nu f_{n-1}(\gamma_k + \tau - \gamma_\nu). \tag{1.24}$$

Another relation is

$$\phi\left(\frac{\gamma_k + \tau}{a^{n+1}}\right) + \phi\left(\frac{\gamma_\ell + \tau}{a^{n+1}}\right) = f_{n-p}\left(\frac{\gamma_k + \tau}{a^p}\right) \quad (0 \leq \tau \leq a,\ a \geq 2,\ n \geq p) \tag{1.25}$$

where k is even and $k = 2^p + \ell$ $(0 \leq \ell < 2^p, p \in \mathbb{N})$.

Remark 1.1 Formula (1.25) is also valid for arbitrary $k \in \mathbb{N}$ ($p \in \mathbb{N}_0$) when $0 \le \tau \le 1$. If $k = 2^\sigma \kappa$ and $\ell = 2^\sigma \lambda$ with $\sigma \in \mathbb{N}$ and integers κ, λ then $\kappa = 2^{p-\sigma} + \lambda$ and, according to (1.18), it holds

$$\frac{\gamma_k + \tau}{a^{n+1}} = \frac{\gamma_\kappa + \frac{\tau}{a^\sigma}}{a^{n-\sigma+1}}$$

and an analogous formula with ℓ and λ instead of k and τ, respectively. Writing (1.25) for $0 \le \tau \le 1$ with κ, λ, $n-\sigma$, $p-\sigma$ and $\frac{\tau}{a^\sigma}$ instead of k, ℓ, n, p and τ, respectively, we see that (1.25) is even valid for $0 \le \tau \le a^\sigma$ (i.e. at least for $0 \le \tau \le a$ when k is even). This assertion is already contained in [2, Proposition 6.1], however without proof.

Finally, we quote a relation which is valid even for $a \ge \frac{3}{2}$, namely

$$\phi\left(\frac{\tau}{a^{n+1}}\right) + (-1)^n \phi\left(\frac{1-\tau}{a^{n+1}}\right) = f_n(\tau) \qquad (2-a \le \tau \le a-1,\, n \ge -1), \tag{1.26}$$

and the relation valid for $a > 1$

$$\sum_{\nu=-\infty}^{+\infty} \phi\left(t - \frac{\nu}{b}\right) = b \qquad (t \in \mathbb{R}). \tag{1.27}$$

2 Polynomial relations

First we state two sets of new formulas for the polynomials f_n.

Proposition 2.1 *The polynomials f_n ($n \in \mathbb{N}_0$) satisfy the addition theorem*

$$f_n(at + (1-a)s) = a^n \sum_{\nu=0}^{n} \frac{(-1)^\nu}{\nu!} B_\nu(s) a^{\frac{\nu}{2}(\nu-1-2n)} f_{n-\nu}(t) \qquad (s, t \in \mathbb{R}) \tag{2.1}$$

where $B_\nu(s)$ are the Bernoulli polynomials, and the multiplication theorem

$$f_n(t) = a^{-\frac{n(n+1)}{2}} \sum_{\nu=0}^{n} \frac{(-1)^{n-\nu}}{(n+1-\nu)!} a^{\frac{\nu(\nu-1)}{2}} f_\nu(at) \qquad (t \in \mathbb{R}). \tag{2.2}$$

Proof: Equation (1.3) can be written in the form

$$e^{(at+(1-a)s)\frac{z}{a}} \Phi\left(\frac{z}{a}\right) = \frac{-\frac{z}{b} e^{\frac{-sz}{b}}}{e^{-\frac{z}{b}} - 1} e^{tz} \Phi(z) \qquad (s, t \in \mathbb{R}),$$

since $b = \frac{a}{a-1}$. Expanding both sides into power series with respect to z, using (1.8) and the generating function of the Bernoulli polynomials, and comparing the coefficients we obtain the formula

$$\psi_n(at + (1-a)s) = \sum_{\nu=0}^{n} \binom{n}{\nu} (1-a)^\nu B_\nu(s) a^{n-\nu} \psi_{n-\nu}(t) \qquad (n \in \mathbb{N}_0).$$

In view of (1.9) and

$$\frac{c_n}{c_{n-\nu}} = \frac{(n-\nu)!}{n!}\frac{a^{\frac{\nu}{2}(\nu+1-2n)}}{(a-1)^\nu}$$

the last equation turns over into (2.1).

From (1.3) and (1.8) we obtain analogously

$$\psi_n(t) = n!\sum_{\nu=0}^{n}\frac{\psi_\nu(at)}{a^\nu\nu!}\frac{1}{(n+1-\nu)!}\left(-\frac{1}{b}\right)^{n-\nu},$$

and in view of (1.9) and

$$\frac{c_n}{c_\nu} = \frac{\nu!}{n!}b^{n-\nu}\frac{a^{\frac{\nu(\nu+1)}{2}}}{a^{\frac{n(n+1)}{2}}} \tag{2.3}$$

we obtain (2.2) ■

Formula (2.1) is equivalent for $a = \frac{1}{2}$ and $s = t$ to [8, (C)], and for $s = 0$ to a formula in [2, p.1012].

Relation (2.2) is a generalization of (1.4), because for $t = 0$ it can be transferred into (1.4), using (1.9) and the first relation of (1.7). Though formula (2.2) is not a usual recursion, it is possible to calculate $f_n(t)$ recursively by means of it if we additionally use from (1.6) and (1.9) that the polynomial must have the main term c_nt^n. Relation (2.2) can be considered as the inversion of (2.1) with $s = 0$ and vice versa.

3 Special recursions for the solutions

Formula [10, (1.14)] from W. Volk can be generalized to the case $a \geq 2$, which shall be the general assumption in the Sections 3 – 7.

Proposition 3.1 *For $n \geq 2$ we have the recursion*

$$\phi\left(\frac{1}{a^n}\right) = \frac{1}{1-a^{1-n}}\sum_{\nu=2}^{n}\frac{1}{\nu!}a^{\frac{1}{2}\nu(\nu+1-2n)}\phi\left(\frac{1}{a^{n+1-\nu}}\right) \tag{3.1}$$

with the initial value $\phi(\frac{1}{a}) = b$.

Proof: According to (1.14) with $\tau = 1$ we have $\phi(\frac{1}{a^{n+1}}) = f_n(1)$ so that (1.9) and (1.7) yield

$$\phi\left(\frac{1}{a^{n+1}}\right) = (-1)^n c_n\rho_n(a) \tag{3.2}$$

and in particular $\phi(\frac{1}{a}) = b$. Substituting (3.2) into (1.4) we get

$$\phi\left(\frac{1}{a^{n+1}}\right) = \frac{1}{(n+1)(a^n-1)} \sum_{\nu=0}^{n-1} \binom{n+1}{\nu} \frac{c_n}{c_\nu}(a-1)^{n-\nu}\phi\left(\frac{1}{a^{\nu+1}}\right),$$

and in view of (2.3) we obtain the equation

$$\phi\left(\frac{1}{a^{n+1}}\right) = \frac{a^n}{a^n-1} \sum_{\mu=2}^{n+1} \frac{1}{\mu!} a^{\frac{1}{2}\mu(\mu-1-2n)}\phi\left(\frac{1}{a^{n+2-\mu}}\right),$$

which turns over into (3.1) replacing n by $n-1$ ■

Formula [10, (1.14)] is the special case $a = 2$. Besides of (3.1) we also can state recursions for $\phi(\frac{\tau}{a^n})$. Inserting (1.14) into (1.11) with $t = \tau$ and into (2.1) with $t = s = \tau$, respectively, we immediately obtain

Corollary 3.2 *For $1 \leq \tau \leq a-1$ and $n \geq 1$ we have the recursion formulas*

$$\phi\left(\frac{\tau}{a^{n+1}}\right) = \frac{(\tau-\frac{1}{2})}{n(a-1)a^{n-1}}\phi\left(\frac{\tau}{a^n}\right) + \frac{1}{n}\sum_{\nu=2}^{n} \frac{1}{\nu!} B_\nu \frac{a^{\frac{\nu}{2}(\nu+1-2n)}}{a^\nu - 1}\phi\left(\frac{\tau}{a^{n-\nu}}\right) \tag{3.3}$$

and

$$\phi\left(\frac{\tau}{a^{n+1}}\right) = \frac{a^n}{1-a^n}\sum_{\nu=1}^{n} \frac{(-1)^\nu}{\nu!} B_\nu(\tau) a^{\frac{\nu}{2}(\nu-1-2n)}\phi\left(\frac{\tau}{a^{n+1-\nu}}\right), \tag{3.4}$$

both with the initial value $\phi(\frac{\tau}{a}) = b$.

Equations (3.3) and (3.4), both for $\tau = 1$, lead to new recursions for $\phi(\frac{1}{a^n})$ which are different from (3.1). Moreover, for $\tau = a-1$ both equations yield recursions for $\phi(\frac{\gamma_1}{a^n})$ which are the initial values for more general recursions yielding $\phi(\frac{\gamma_k}{a^n})$. In order to state such recursions we apply Taylor's formula and hence we need the derivatives of higher order of the solutions. Moreover, we have to extend the interval of validity of the main formula (1.24).

4 The domain of validity of the main formula

We preserve the assumption $a \geq 2$ and show that formula (1.24) with $n \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, 2^n - 1\}$ has in fact a greater interval of validity when $a > 2$.

Proposition 4.1 *The main formula (1.24) for the solution ϕ of (1.1)-(1.2) is valid even for $2 - a \leq \tau \leq a-1$.*

Proof: Since (1.24) is trivial for $k = 0$ we assume that $k \geq 1$ and therefore also $n \geq 1$. For convenience we introduce the notation $\overline{G}_{2^m,m} = [1, 1 + \frac{a-2}{a^{m+1}}]$ where $m \in \mathbb{N}_0$. Then according to (1.21) every F_{kn} has two $\overline{G}_{\ell m}$ with $m \leq n-1$ as neighbouring intervals. Since $t = \frac{\gamma_k+\tau}{a^n} \in F_{kn}$ for $0 \leq \tau \leq 1$ and $|\overline{G}_{\ell,n-1}| = \frac{a-2}{a^n}$, we see that t lies in intervals $\overline{G}_{\ell m}$ both for $2-a \leq \tau \leq 0$ and for $1 \leq \tau \leq a-1$. Hence, in both cases $\phi(t)$ is a polynomial. But in both cases also $\phi(\frac{\tau}{a^n})$ is a polynomial, namely 0 and $f_{n-1}(\tau)$, respectively, cf. (1.14). This implies that the left-hand side of (1.24) is a polynomial spline for $2-a \leq \tau \leq a-1$. But it is also a C^∞-function, i.e. it must be a unique polynomial ■

The interval $2-a \leq \tau \leq a-1$ is optimal if k is odd, cf. (1.22). The case that $k = 2^\sigma \kappa$ is even can be reduced to the odd case as in Remark 1.1 using $\gamma_k = a^\sigma \gamma_\kappa$.

As consequence of Proposition 4.1, formula (1.26) can be generalized in the case $a \geq 2$ as follows:

Proposition 4.2 *For $n \in \mathbb{N}_0$, $k \in \{0, 1, \ldots, 2^n - 1\}$ and $2-a \leq \tau \leq a-1$, the solution ϕ of (1.1)-(1.2) has the property*

$$\phi\left(\frac{\gamma_k+\tau}{a^n}\right) + (-1)^{n-1}\phi\left(\frac{\gamma_k+1-\tau}{a^n}\right) = P(\tau) \tag{4.1}$$

where P is the polynomial

$$P(\tau) = \varepsilon_k f_{n-1}(\tau) + \sum_{\nu=0}^{k-1} \varepsilon_\nu \left[f_{n-1}(\gamma_k+\tau-\gamma_\nu) + f_{n-1}(\gamma_\nu+\tau-\gamma_k)\right]. \tag{4.2}$$

Proof: The inequality $2-a \leq \tau \leq a-1$ implies $2-a \leq 1-\tau \leq a-1$. Hence, according to Proposition 4.1 besides of (1.24) we also have

$$\phi\left(\frac{\gamma_k+1-\tau}{a^n}\right) - \varepsilon_k\phi\left(\frac{1-\tau}{a^n}\right) = \sum_{\nu=0}^{k-1} \varepsilon_\nu f_{n-1}(\gamma_k+1-\tau-\gamma_\nu). \tag{4.3}$$

Multiplying the last equation with $(-1)^{n-1}$, using (1.13) and (1.26) with $n-1$ instead of n, we obtain the assertion by adding (1.24) ■

5 On the derivatives of higher order

As before it shall be $a \geq 2$. Besides of the intervals (1.21) we need the open intervals

$$\overset{\circ}{F}_{kn} = \left(\frac{\gamma_k}{a^n}, \frac{\gamma_k+1}{a^n}\right) \qquad (k = 0, 1, \ldots, 2^n - 1,\ n \in \mathbb{N}_0), \tag{5.1}$$

with the decomposition

$$\overset{\circ}{F}_{kn} = \overset{\circ}{F}_{2k,n+1} \cup \overline{G}_{kn} \cup \overset{\circ}{F}_{2k+1,n+1} \tag{5.2}$$

where the three sets on the right-hand side are disjoint. As in [3] we introduce the set

$$M = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} \overline{G}_{kn}$$

and its complement $CM = (0,1) \setminus M$ which can also be represented as

$$CM = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} \overset{\circ}{F}_{kn}\ . \tag{5.3}$$

For t in one of the intervals $\overset{\circ}{F}_{\ell m}$ it holds

$$\phi^{(m)}(t) = \varepsilon_\ell a^{\frac{m(m+1)}{2}} b^m \phi(a^m t - \gamma_\ell) \quad (t \in \overset{\circ}{F}_{\ell m}) \tag{5.4}$$

and otherwise we have $\phi^{(m)}(t) = 0$, cf. [2]. This means that for fixed $t \in (0,1)$ and $m \in \mathbb{N}_0$ we have $\phi^{(m)}(t) \neq 0$ if and only if there is an index ℓ satisfying

$$0 < a^m t - \gamma_\ell < 1, \tag{5.5}$$

i.e. $t \in \overset{\circ}{F}_{\ell m}$. Note that there exists at most one number $\ell = \ell_m$ with (5.5), since the intervals $F_{\ell m}$ $(\ell = 0, 1, \ldots, 2^m - 1)$ are pairwise disjoint. Next, we modify [3, Definition 2.2]:

Definition 5.1 *For given $t \in (0,1)$ we define a sequence $\delta_m = \delta_m(a,t)$ $(m \in \mathbb{N}_0)$ by $\delta_m = 1$ if (5.5) is satisfied for a certain index $\ell = \ell_m$, and by $\delta_m = 0$ elsewhere.*

Lemma 5.2 *If for given $t \in (0,1)$ it holds $\delta_m = 1$ $(m \in \mathbb{N})$ then $\delta_{m-1} = 1$, too, with the corresponding index $\ell_{m-1} = [\frac{\ell_m}{2}]$.*

Proof: We have $\delta_m = 1$ if and only if $t \in \overset{\circ}{F}_{\ell m}$ with $\ell = \ell_m$. But, according to (5.2), $t \in \overset{\circ}{F}_{\ell m}$ implies that $t \in \overset{\circ}{F}_{k,m-1}$ with $k = [\frac{\ell}{2}]$. This yields the assertion ■

Proposition 5.3 *The derivatives of the solution ϕ of (1.1)-(1.2) have the following property:*

1. *For $t \in M$, i.e. $t \in \overline{G}_{kn}$ with fixed k, n, it holds $\phi^{(m)}(t) \neq 0$ when $0 \leq m \leq n$ and $\phi^{(m)}(t) = 0$ when $m \geq n+1$.*

2. *For $t \in CM$ it holds $\phi^{(m)}(t) \neq 0$ for all $m \in \mathbb{N}_0$.*

Proof: **1.** In the case $a = 2$, where the interval $\overline{G}_{kn}$ degenerates to the point $t = \frac{2k+1}{2^{n+1}}$, it is known that $\phi^{(m)}(t) = 0$ for $m > n$, cf. [2, (4.8)] or [9, p.575]. For $a > 2$ and $t \in \overline{G}_{kn}$ the function ϕ is a polynomial of degree n according to (1.23) and hence $\phi^{(m)}(t) = 0$ for $m > n$, too. But (5.2) shows that it is also $t \in \overset{\circ}{F}_{kn}$ and hence $\phi^{(n)}(t) \neq 0$ in view of (5.4). Lemma 5.2 implies $\phi^{(m)}(t) \neq 0$ for $m \leq n$.

2. In view of (5.3) the supposition $t \in CM$ implies that for each m we have $t \in \overset{\circ}{F}_{\ell m}$ with $\ell = \ell_m$ defined above, and hence (5.4) implies $\phi^{(m)}(t) \neq 0$ ■

In order to determine the sequences δ_m and ℓ_m from Definition 5.1 explicitly for a given $t = \frac{\gamma_k}{a^n} \in (0, 1)$, which is necessary for a later application, we introduce the dyadic representation

$$k = d_p d_{p-1} \ldots d_1 d_0 \tag{5.6}$$

with $d_j \in \{0, 1\}$, i.e. $k = d_p 2^p + \ldots + d_1 2 + d_0$, where $p < n$ since $k < 2^n$. For convenience we extend the coefficients by $d_j = 0$ for $p + 1 \leq j \leq n$. In the next lemma we shall show that in Definition 5.1 it holds $\ell_m = [\frac{k}{2^{n-m}}]$, i.e.

$$\ell_m = d_{n-m} + d_{n-m+1} 2 + \ldots + d_n 2^m \tag{5.7}$$

when $m \in \{0, \ldots, n-1\}$.

Lemma 5.4 *Assume that $t = \frac{\gamma_k}{a^n} \in (0, 1)$ with k from (5.6) and $n \in \mathbb{N}$. If k has the form $k = 2^\sigma(2\kappa + 1)$ with $\sigma, \kappa \in \mathbb{N}_0$ then it holds $\delta_m = 1$ for $m \in \{0, \ldots, n - \sigma - 1\}$ with the corresponding index (5.7), and $\delta_m = 0$ for $m \geq n - \sigma$.*

Proof: With (1.15) and the above notations we have

$$a^m t - \gamma_{\ell_m} = \gamma_1 \left(\frac{d_0}{a^{n-m}} + \ldots + \frac{d_{n-m-1}}{a} \right). \tag{5.8}$$

The assumption $k = 2^\sigma(2\kappa + 1)$ means $d_\sigma = 1$ and in the case $\sigma > 0$ additionally $d_j = 0$ for $0 \leq j < \sigma$, so that (5.8) reduces to

$$a^m t - \gamma_{\ell_m} = \gamma_1 \left(\frac{1}{a^{n-\sigma-m}} + \frac{d_{\sigma+1}}{a^{n-\sigma-m-1}} + \ldots + \frac{d_{n-m-1}}{a} \right). \tag{5.9}$$

Choosing $m = n - \sigma$ we obtain $a^{n-\sigma} t - \gamma_{\ell_m} = 0$. In view of (1.20) this implies that $a^{n-\sigma} t - \gamma_\nu \geq 1$ for $\nu < \ell_m$ and that $a^{n-\sigma} t - \gamma_\nu \leq 0$ for $\nu \geq \ell_m$, i.e. $\delta_{n-\sigma} = 0$. Lemma 5.2 yields $\delta_m = 0$ for all $m \geq n - \sigma$. For $m \in \{0, \ldots, n - \sigma - 1\}$ equation (5.9) implies that

$$0 < \frac{a-1}{a^{n-\sigma-m}} \leq a^m t - \gamma_{\ell_m} \leq 1 - \frac{1}{a^{n-\sigma-m}} < 1.$$

Hence, for these m it holds $\delta_m = 1$ and the corresponding index reads (5.7) ■

6 More general recursions

The announced recursion formula for $\phi(\frac{\gamma_k}{a^n})$ in the case $a \geq 2$ is a consequence of the following

Theorem 6.1 *Assume that $n \in \mathbb{N}$ and that $k = 2^\sigma(2\kappa+1)$, $0 < k < 2^n$, has the dyadic representation (5.6). Then for $2-a \leq \tau \leq a-1$ it holds*

$$\phi\left(\frac{\gamma_k+\tau}{a^n}\right) = \varepsilon_k \phi\left(\frac{\tau}{a^n}\right) + \sum_{m=0}^{n-\sigma-1} \varepsilon_{\ell_m} \frac{\tau^m}{m!} b^m a^{\frac{m(m+1-2n)}{2}} \phi\left(\frac{\gamma_{r_m}}{a^{n-m}}\right) \tag{6.1}$$

where ℓ_m is given by (5.7) and

$$r_m = d_0 + d_1 2 + \ldots + d_{n-m-1} 2^{n-m-1}, \tag{6.2}$$

i.e. $k = 2^{n-m}\ell_m + r_m$.

Proof: Owing to Proposition 4.1, the function

$$f(\tau) = \phi\left(\frac{\gamma_k+\tau}{a^n}\right) - \varepsilon_k \phi\left(\frac{\tau}{a^n}\right)$$

is a polynomial of degree at most n when $2-a \leq \tau \leq a-1$. According to Taylor's formula and $\phi^{(m)}(0) = 0$ for all m, we get

$$f(\tau) = \sum_{m=0}^{n} \frac{1}{m!} \phi^{(m)}(t) \left(\frac{\tau}{a^n}\right)^m$$

where $t = \frac{\gamma_k}{a^n}$. Using Definition 5.1 and (5.4) we get equation

$$f(\tau) = \sum_{m=0}^{n} \delta_m \varepsilon_{\ell_m} \frac{\tau^m}{m!} b^m a^{\frac{m}{2}(m+1-2n)} \phi(a^m t - \gamma_{\ell_m}).$$

With (5.7) and (6.2), equation (5.8) can be written as

$$a^m t - \gamma_{\ell_m} = \frac{\gamma_{r_m}}{a^{n-m}}, \tag{6.3}$$

and the assertion follows from Lemma 5.4 ∎

Applying formula (6.1) with even k and $\tau = a-1$, then in view of $\gamma_k + a - 1 = \gamma_{k+1}$, cf. (1.18), and $b = \frac{a}{a-1}$ we obtain

Corollary 6.2 *Assume that $n \in \mathbb{N}$ and that $k = 2^\sigma(2\kappa+1)$ is even, $0 < k < 2^n$, with the dyadic representation (5.6). Then it holds*

$$\phi\left(\frac{\gamma_{k+1}}{a^n}\right) = \varepsilon_k \phi\left(\frac{\gamma_1}{a^n}\right) + \sum_{m=0}^{n-\sigma-1} \varepsilon_{\ell_m} \frac{1}{m!} a^{\frac{m}{2}(m+3-2n)} \phi\left(\frac{\gamma_{r_m}}{a^{n-m}}\right) \tag{6.4}$$

with the notations (5.7) and (6.2).

Note that the values $\phi(\frac{\gamma_1}{a^n})$ can be determined recursively by both formulas of Corollary 3.2 with $\tau = \gamma_1$. Using $\phi(\frac{\gamma_1}{a^n})$ as initial values, all further $\phi(\frac{\gamma_k}{a^n})$ with $1 < k < 2^n$ can be computed recursively by means of (6.4) in view of $\frac{\gamma_{2\ell}}{a^n} = \frac{\gamma_\ell}{a^{n-1}}$.

The formulas (6.4) are recursions for the right end points of the intervals $\overline{G}$ from (1.21). Left end points can be reduced to right ones by means of the symmetry of ϕ with respect to $\frac{1}{2}$.

According to (6.3), Proposition 6.1 with $a = 2$ and $\tau = 1$ yields the

Corollary 6.3 *For $a = 2$, $n \in \mathbb{N}$ and $k = 1, 2, \ldots, 2^n - 1$, we have the equation*

$$\phi\left(\frac{k+1}{2^n}\right) = \varepsilon_k \phi\left(\frac{1}{2^n}\right) + \sum_{m=0}^{n-\sigma-1} \frac{1}{m!} \varepsilon_{\ell_m} 2^{\frac{m}{2}(m+3-2n)} \phi\left(\frac{k}{2^{n-m}} - \ell_m\right), \tag{6.5}$$

where $\ell_m = [\frac{k}{2^{n-m}}]$ and $k = 2^\sigma(2\kappa + 1)$.

Note that after a simple calculation, (6.5) for $k = 1$ and $n + 1$ instead of n yields

$$\phi\left(\frac{1}{2^n}\right) = \frac{1}{1 - 2^{1-n}} \sum_{m=2}^{n} \frac{1}{m!} 2^{\frac{m}{2}(m+1-2n)} \phi\left(\frac{1}{2^{n+1-m}}\right), \tag{6.6}$$

i.e. (3.1) with $a = 2$, cf. [10, (1.14)]. Therefore, (6.5) with the initial value $\phi(\frac{1}{2}) = 2$ is a recursion for all $\phi(\frac{k}{2^n})$ without additional knowledge where it suffices to use it only for even k with $k < 2^{n-2}$, considering the symmetry of ϕ and the relation

$$\phi(t) + \phi\left(\frac{1}{2} - t\right) = 2 \qquad \left(0 \le t \le \frac{1}{2}\right),$$

cf. (1.26) for $n = 0$ and $a = 2$. The first $\phi(\frac{k}{2^n})$ are calculated in [3, p.216].

7 Reduced polynomial representations

The polynomial representation (1.23) for ϕ is rather redundant, since the terms can be reduced by means of (1.12). One reduced formula was already set up in [2, (6.3)], to which we shall come back later on after some preliminaries. Though the following results are valid also for $a = 2$, they are only interesting in the case $a > 2$.

Let k, ℓ, m be even and u, v odd numbers from $\mathbb{N}_0$, such that, for some numbers p, q from $\mathbb{N}$, we have

$$k = 2^p + \ell \quad (0 \le \ell \le 2^p - 2), \qquad k + u = 2^{p+1} - 1 \tag{7.1}$$

and for the same or another odd u

$$u = 2^q + v \quad (1 \le v \le 2^q - 1), \qquad u + m = 2^{q+1} - 1. \tag{7.2}$$

In (7.1) it is always $k \geq 2$, and in (7.2) it is $u \geq 3$. For fixed $n \in \mathbb{N}$ we introduce the notations

$$\varphi_k = \phi\left(\frac{\gamma_k + \tau}{a^{n+1}}\right), \qquad \varphi_u = (-1)^n \phi\left(\frac{\gamma_u + 1 - \tau}{a^{n+1}}\right). \tag{7.3}$$

Proposition 7.1 *In the case (7.1) it holds*

$$\varphi_k = -\varphi_\ell + f_{n-p}\left(\frac{\gamma_k + \tau}{a^p}\right), \tag{7.4}$$

$$\varphi_k = (-1)^p \varphi_u + f_{n-p-1}\left(\frac{\gamma_k + \tau}{a^{p+1}}\right), \tag{7.5}$$

when $n \geq p$, and in the case (7.2)

$$\varphi_u = (-1)^q \varphi_m + (-1)^{q+1} f_{n-q-1}\left(\frac{\gamma_m + \tau}{a^{q+1}}\right), \tag{7.6}$$

$$\varphi_u = -\varphi_v + (-1)^q f_{n-q}\left(\frac{\gamma_m + \tau}{a^q} - \gamma_1\right), \tag{7.7}$$

when $n \geq q$, all formulas are valid for

$$0 \leq \tau \leq a. \tag{7.8}$$

Proof: Relation (7.4) is only another notation for (1.25). Replacing in (1.26) n by $n-p-1$ as well as τ by $\frac{\gamma_k+\tau}{a^{p+1}}$ and considering (1.19) we obtain (7.5). The condition concerning τ is equivalent to

$$a^{p+1}(2-a) - \gamma_k \leq \tau \leq a^{p+1}(a-1) - \gamma_k,$$

and these inequalities are satisfied in view of (7.8), $2 \leq a$, $k \leq 2^{p+1} - 2$, (1.20) and (1.17). Solving (7.5) with respect to φ_u and putting $k = m$ as well as $p = q$ we obtain (7.6). Given (7.2), we can write $u - 1 = k$, $v - 1 = \ell$, and we obtain the first relation of (7.1) with q instead of p. Replacing τ in (7.4) by $a - \tau$, whereby the condition (7.8) remains invariant, and considering

$$\gamma_k + a - \tau = \gamma_u + 1 - \tau$$

(cf.(1.18)) as well as (1.13) with $n-q$ instead of n and (1.19) concerning the second relation of (7.2), we obtain (7.7) ■

In the following we restrict (7.8) to the inequality

$$1 \leq \tau \leq a - 1$$

which includes the condition $a \geq 2$. In view of (1.14), (7.6) and (7.3) it holds for these τ

$$\varphi_0 = f_n(\tau), \qquad \varphi_1 = f_n(\tau - \gamma_1). \tag{7.9}$$

By means of the formulas of Proposition 7.1 we can reduce the index k of φ_k successively down to 1 or 0, where we have the representations (7.9). In this way it is possible to arrive at formulas of the type

$$\varphi_k = \sum_{j=0}^{p+1} \sigma_j f_{n-j}(\cdot) \tag{7.10}$$

with $\sigma_j \in \{-1, 0, 1\}$ and suitable arguments by the polynomials f. It would be sufficient to carry out this reduction only by means of (7.4). Then (7.10) is the already mentioned formula [2, (6.3)] and the signs of the non-vanishing terms in (7.10) alternate. But there are further possibilities.

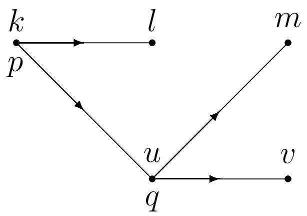

Figure 1: Graph of (7.1) and (7.2)

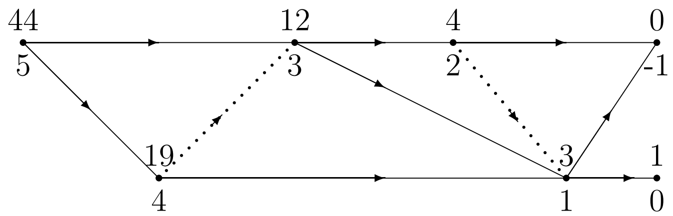

Figure 2: Graph in the case $k = 44$

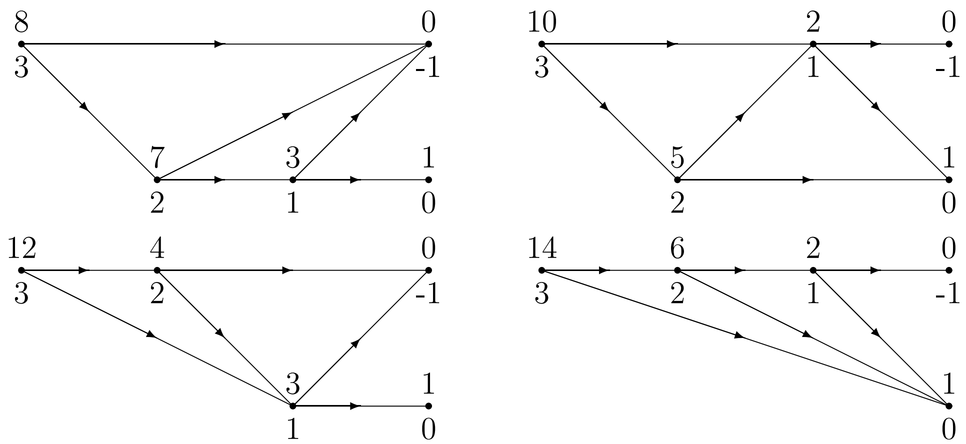

Figure 3: The cases $8 \leq k < 16$

In order to describe them in detail we identify u in (7.1) and (7.2), visualize these relations by the directed graph in Figure 1, and proceed with the nodes ℓ, m, v analogously down to the endpoints 1 and 0, respectively. There are two possibilities to label the nodes, namely, either by means of the numbers k and u (written over the node), or by means of the exponents p and q (written down the node) corresponding to them in (7.1-2). For the end points 1 and 0 we define $q = 0$ in the first and $p = -1$ in the second case. For example, Figure 2 shows the graph in the case $k = 44$ $(p = 5)$, and Figure 3 the graphs in the cases $8 \le k < 16$ $(p = 3)$. In the following we mainly characterize the nodes by means of the exponential labels p, q.

In general for $p \in \mathbb{N}$, $d_j \in \{0,1\}$, assume that (5.6) is the dyadic representation of a given even number k with $d_p = 1$ and $d_0 = 0$. For $0 \le j \le p$ we introduce the extended notations

$$k_j = d_j d_{j-1} \dots d_0, \qquad u_j = \overline{d}_j \overline{d}_{j-1} \dots \overline{d}_0 \tag{7.11}$$

with d_j from (5.6) and $\overline{d}_j = 1 - d_j$. The directed graph belonging to $k \ge 2$ has the following structure. It has $p+2$ nodes $p, \dots, 1, 0, -1$ with the root p and two end points 0, -1. For convenience the nodes j are placed on a first line with the end point -1 when $d_j = 1$, whereas they are placed on a second line with the endpoint 0 when $\overline{d}_j = 1$. The corresponding number (7.11) belonging to a fixed node j is k_j on the first line $(k_{-1} = 0)$ and u_j on the second one. Every node, which is no end point, is the start point of exactly two arcs, one to the next smallest j on the same line, and one to the next smallest j on the other line. In particular, for every $j \ge 1$ there is an arc from j to $j-1$.

Let ℓ be the length of a fixed path from p to one of the end points, obviously $1 \le \ell \le p$, where there always exist two paths of maximal length $\ell = p$. But we are interested in shortest paths.

Proposition 7.2 *(i) For $j = p, p-1, \dots, 1$ let $\widehat{jx_j}$, $\widehat{jy_j}$ be the arcs of the graph belonging to a given even integer k. We get a shortest path, if we choose successively the arcs $\widehat{jz_j}$ with $z_j = \min(x_j, y_j)$.*

(ii) Suppose that in the representation (5.6) of k there are $\ell - 1 \ge 0$ disjunct pairs (d_{j+1}, d_j) of the form $(1,0)$ or $(0,1)$ for $j = p-2, \dots, 1$. Then ℓ is the length of the shortest path.

Proof: (i) Let $\ell(j)$ be the length of a shortest path from p to j, so that $\ell(p) = 0$. Let J be the set of the nodes j belonging to the path with the arcs $\widehat{jz_j}$. This path is a shortest path if Bellman's equation

$$\ell(z_j) = \min_{\widehat{iz_j}} \ell(i) + 1 \tag{7.12}$$

is satisfied for all $j \in J$ with $j \ge 1$, cf. [5, p. 101]. For all these j it is $\max(x_j, y_j) = j - 1$ and therefore $z_j \le j - 2$. This means $z_j + 1 \le j - 1$, where z_j and $z_j + 1$ lie on different lines. Hence, for all $j \in J$ with $j < p$ the nodes j and $j+1$ lie on different lines. For the first arc $\widehat{pz_p}$ it is $\ell(z_p) = 1 \le \ell(z_p+1)$. If $\ell(j) \le \ell(j+1)$ for a fixed $j \in J$ with $j < p$, then $\ell(z_j) = \ell(j) + 1 = \ell(j-1) \le \ell(z_j+1)$, which implies that $\ell(j) \le \ell(j+1)$ for all $j \in J$ with $j < p$. Moreover, we see that the possible i in (7.12) are either $j, j-1, \ldots, z_j+1$ or $j+1, j, \ldots, z_j+1$, cf. Figure 4 or an analogous figure with interchanged lines, and Bellman's equation (7.12) is satisfied indeed.

(ii) To every arc $\widehat{jz_j}$ of the just constructed shortest path with $p-2 \ge z_j \ge 1$ we consider the nodes i with $j > i \ge z_j$. These nodes contain the pair (z_j+1, z_j) with nodes on different lines, but no other such pairs which are disjoint to (z_j+1, z_j), cf. Figure 4. These pairs correspond to the pairs (d_{j+1}, d_j) of the proposition. Since we have to consider also the arc with the end point -1 or 0 the number ℓ of all arcs exceeds the number of the just mentioned pairs by 1 ■

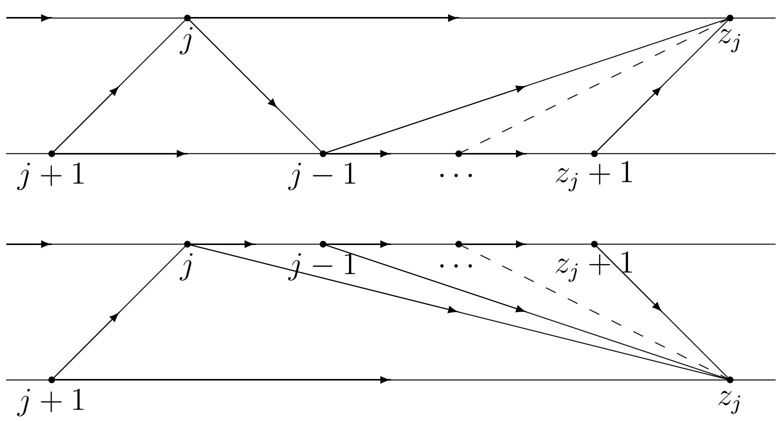

Figure 4: The neighbourhood of $\widehat{jz_j}$

As a simple consequence of Proposition 7.2/(ii) we get

Corollary 7.3 *The length ℓ_k of the shortest path belonging to the even k from (7.1) satisfies the estimate*

$$\ell_k \le \left[\frac{p+1}{2}\right]. \tag{7.13}$$

The smallest numbers k such that $\ell_k = n \in \mathbb{N}$ are $\pi_n = \frac{2}{3}(4^n - 1)$ since these are the numbers $k = 2^p + 2^{p-2} + \ldots + 2 = \frac{2}{3}(2^{p+1} - 1)$ with odd p and $n = \frac{p+1}{2}$.

For a given $k \ge 2$ formula (7.10) or a more complicated formula arises, if we construct the corresponding graph, choose a path from p to -1 or 0 and apply the formulas of Proposition 7.1 as well as (7.9). If we take the path along the first line, then we only have to apply formula (7.4). This possibility is preferable if many of the d_j in (5.6) vanish. In the case that many $\overline{d}_j$ vanish it is preferable to use the path from p down to and then along the second line, i.e. to apply first formula (7.5) and then always (7.7). Another possibility yields the zigzag path, where the formulas (7.5), (7.6) are applied alternately.

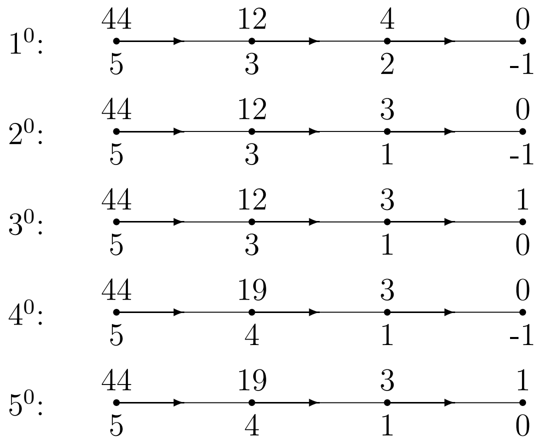

Figure 5: Shortest paths of Figure 2

We call (7.10) a *minimal formula* if we have used a shortest path for the construction. The graph in Figure 2 for $k = 44 = 2^5 + 2^3 + 2^2$ and $u = 19 = 2^4 + 2^1 + 2^0$ has five shortest paths which we obtain if we disregard the dotted arcs, and which are shown in Figure 5. To these shortest paths belong the formulas

$$1^0 : \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-5}\left(\tfrac{\gamma_{44}+\tau}{a^5}\right) - f_{n-3}\left(\tfrac{\gamma_{12}+\tau}{a^3}\right) + f_{n-2}\left(\tfrac{\gamma_4+\tau}{a^2}\right) - f_n(\tau),$$

$$2^0 : \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-5}\left(\tfrac{\gamma_{44}+\tau}{a^5}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4}\right) + f_{n-2}\left(\tfrac{\tau}{a^2}\right) - f_n(\tau),$$

$$3^0 : \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-5}\left(\tfrac{\gamma_{44}+\tau}{a^5}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4}\right) + f_{n-1}\left(\tfrac{\tau}{a} - \gamma_1\right) - f_n(\tau - \gamma_1),$$

$$4^0 : \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-6}\left(\tfrac{\gamma_{44}+\tau}{a^6}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4} - \gamma_1\right) + f_{n-2}\left(\tfrac{\tau}{a^2}\right) - f_n(\tau),$$

$$5^0 : \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-6}\left(\tfrac{\gamma_{44}+\tau}{a^6}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4} - \gamma_1\right) + f_{n-1}\left(\tfrac{\tau}{a} - \gamma_1\right) - f_n(\tau - \gamma_1)$$

with $n \geq 5$. The equivalence of these formulas can be checked by means of (1.12). The first formula is that one where only (7.4) is applied. The second one is the formula corresponding to the path of Proposition 7.2 and the last one is that where after the first step only (7.7) is applied. It is remarkable that all these minimal formulas are alternating. The zigzag case $44 - 19 - 12 - 3 - 0$ does not yield a minimal formula.

A minimal formula (7.10) is called *optimal formula*, if the indices j with $\sigma_j \neq 0$ are maximal, i.e. if the degrees of the polynomials are minimal. In the foregoing examples formula 4^0 is optimal. However, since the practical advantage of optimal formulas is small, we do not investigate existence and uniquiness of them.

8 Formulas for a greater domain of a

Finally, we give up the general assumption $a \geq 2$.

8.1. Recursions in the case $a \geq \frac{3}{2}$. First we remark that for $\tau = \frac{1}{2}$ equation (1.26), which is valid for $a \geq \frac{3}{2}$, implies

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = \frac{1}{2} f_{2n}\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}_0) \tag{8.1}$$

and $n = 0$ yields

$$\phi\left(\frac{1}{2a}\right) = \frac{b}{2} \tag{8.2}$$

in view of $f_0(t) = b$.

Proposition 8.1 *For $a \geq \frac{3}{2}$ and $n \geq 1$ we have the recursions*

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = \frac{1}{2n} \sum_{\nu=1}^{n} \frac{1}{(2\nu)!} B_{2\nu} \frac{a^{\nu(2\nu+1-4n)}}{a^{2\nu}-1} \phi\left(\frac{1}{2a^{2n-2\nu+1}}\right) \tag{8.3}$$

and

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = \frac{a^{2n}}{a^{2n}-1} \sum_{\nu=1}^{n} \frac{1-2^{1-2\nu}}{(2\nu)!} B_{2\nu} a^{\nu(2\nu-1-4n)} \phi\left(\frac{1}{2a^{2n-2\nu+1}}\right) \tag{8.4}$$

both with the initial value (8.2).

Proof: Substituting (8.1) into (1.11) with $t = \frac{1}{2}$ and $2n$ instead of n, we get (8.3). From (2.1) with $t = s = \frac{1}{2}$ and $2n$ instead of n, we obtain analogously

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = a^{2n} \sum_{\nu=0}^{n} \frac{1}{(2\nu)!} B_{2\nu}\left(\frac{1}{2}\right) a^{\nu(2\nu-1-4n)} \phi\left(\frac{1}{2a^{2n-2\nu+1}}\right),$$

and by means of the well-known relation $B_\nu\left(\frac{1}{2}\right) = -\left(1 - 2^{1-\nu}\right) B_\nu$, cf. [6, p.22], it follows (8.4) ∎

8.2. The maximum value. Equation (1.27) yields for $a \geq \frac{4}{3}$ the relation

$$\phi\left(t - \frac{1}{b}\right) + \phi(t) + \phi\left(t + \frac{1}{b}\right) = b \qquad \left(\frac{2}{a} - 1 \leq t \leq 2 - \frac{2}{a}\right). \tag{8.5}$$

Putting $t = \frac{1}{2}$ in (8.5), we obtain for the maximum value of the solution ϕ of (1.1)-(1.2) that

$$\phi\left(\frac{1}{2}\right) = b - 2\phi\left(\frac{1}{a} - \frac{1}{2}\right) \qquad \left(a \geq \frac{4}{3}\right), \tag{8.6}$$

since $\frac{1}{a}+\frac{1}{b}=1$ and ϕ is symmetric. In order to give an application for (8.1), we define α_n ($n\in\mathbb{N}$) as the real solution of $a^{2n}(2-a)=1$ which is different from 1 where $\alpha_1=\frac{1}{2}(1+\sqrt{5})=1.618\ldots$, $\alpha_n<\alpha_{n+1}<2$ and

$$\alpha_n=2-\frac{1}{4^n}+\mathcal{O}\left(\frac{n}{16^n}\right)\qquad(n\to\infty).$$

Hence, by (8.1) with $a=\alpha_n>\frac{3}{2}$, formula (8.6) turns over into the explicit formula

$$\phi\left(\frac{1}{2}\right)=b-f_{2n}\left(\frac{1}{2}\right)\qquad(a=\alpha_n,\ n\in\mathbb{N}).$$

For arbitrary $a>1$ it follows from (1.27) with $t=\frac{1}{2}$ that the maximum value $\phi(\frac{1}{2})$ has the form

$$\phi\left(\frac{1}{2}\right)=c(a)b$$

where $c(a)=1$ for $a\geq 2$, and where $0<c(a)<1$ for $1<a<2$. Moreover, $c(a)\to 0$ as $a\to 1$ in view of

$$c(a)=\frac{1}{b}\phi\left(\frac{1}{2}\right)=1-\frac{2}{b}\sum_{\nu=1}^{\infty}\phi\left(\frac{1}{2}-\frac{\nu}{b}\right)\to 1-2\int_0^{1/2}\phi(t)dt=0,$$

where we have used (1.27), $\frac{1}{b}\to 0$, the symmetry of ϕ and (1.2).

On the other side, $\phi(\frac{1}{2})\to\infty$ as $a\to 1$, since otherwise we would get a contradiction to the solution $\phi(t)=\delta(t-\frac{1}{2})$ of (1.1)-(1.2) for $a=1$, cf. [1, p.164].

8.3. Special series. We denote by a_p ($p\in\mathbb{N}_0$) the positive solution of $a^p(2-a)=a-1$. Then $a_0=\frac{3}{2}$, $a_1=\alpha_1$ and $a_p<a_{p+1}<2$. Moreover, it is

$$a_p=2-\frac{1}{2^p}+\mathcal{O}\left(\frac{p}{4^p}\right)\qquad(p\to\infty)$$

and $a_{2n}>\alpha_n$ for $n\in\mathbb{N}$.

Lemma 8.2 *For $a\geq a_p$ and $n\in\mathbb{N}_0$ we have*

$$\phi\left(\frac{1}{a^n(a^p+1)}\right)=(-1)^n\phi\left(\frac{a^p}{a^n(a^p+1)}\right)+f_{n-1}\left(\frac{1}{a^p+1}\right).\tag{8.7}$$

Proof: Applying (1.26) with $\tau=\frac{1}{a^p+1}$ and $n-1$ instead of n yields (8.7) in view of $1-\tau=\frac{a^p}{a^p+1}$, when

$$2-a\leq\frac{1}{a^p+1}\leq a-1.$$

The first inequality is equivalent to

$$a^p(2-a)\leq a-1\tag{8.8}$$

and therefore valid for $a \geq a_p$. The second inequality is equivalent to $a^p(1-a) \leq a-2$ which follows from (8.8) in view of $a^p > 1$ ■

In the case $a \geq 2$ equation (8.7) is valid for all $p \in \mathbb{N}_0$ since $a_p < 2$. Owing to $\phi(0) = 0$ and (1.13), $p \to \infty$ yields the known formula $\phi\left(\frac{1}{a^n}\right) = f_{n-1}(1)$, cf. (1.14).

Proposition 8.2 *Assume that $a \geq a_p$ with $p \in \mathbb{N}$, $q \in \mathbb{Z}$ and $q \leq p$. Then the solution ϕ of (1.1)-(1.2) has the expansion*

$$\phi\left(\frac{a^q}{a^p+1}\right) = -\sum_{\nu=1}^{\infty} \eta_\nu f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right) \tag{8.9}$$

where

$$\eta_\nu = (-1)^{\frac{\nu(\nu+1)p}{2}+\nu q}. \tag{8.10}$$

Proof: For $\nu \in \mathbb{N}$ we have $n = \nu p - q \in \mathbb{N}_0$ and equation (8.7) reads

$$\phi\left(\frac{a^q}{a^{\nu p}(a^p+1)}\right) = (-1)^{\nu p-q}\phi\left(\frac{a^q}{a^{(\nu-1)p}(a^p+1)}\right) + f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right).$$

Multiplication with η_ν from (8.10) yields the relation

$$\eta_\nu\phi_\nu = \eta_{\nu-1}\phi_{\nu-1} + \eta_\nu f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right) \tag{8.11}$$

where $\phi_\nu = \phi(\frac{a^q}{a^{\nu p}(a^p+1)})$. In view of $\eta_0 = 1$ and $\phi_\nu \to 0$ as $\nu \to \infty$ we obtain by summation over $\nu \geq 1$ that

$$0 = \phi_0 + \sum_{\nu=1}^{\infty} \eta_\nu f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right)$$

and this implies the assertion ■

Remark 8.4 **1.** The coefficients η_ν, given by (8.10), are 4-periodic with $\eta_1 = (-1)^{p+q}$, $\eta_2 = (-1)^p$, $\eta_3 = (-1)^q$ and $\eta_4 = 1$. By means of (1.5) and (1.7) it can be shown that, for $0 \leq t \leq 1$, the polynomial f_n satisfies the inequality $|f_n(t)| \leq \frac{1}{n+1}c_n$ with c_n from (1.10). This means that the series (8.9) are rapidly convergent.

2. In the case $a \geq a_1$ equation (8.9) for $p = 1$ and $q = 0$ yields

$$\phi\left(\frac{1}{a+1}\right) = \sum_{\nu=0}^{\infty}(-1)^{\frac{\nu(\nu+3)}{2}} f_\nu\left(\frac{1}{a+1}\right).$$

In view of (1.12) with $t = \frac{a^2}{a+1}$ and (1.13) with $t = \frac{a}{a+1}$ it is easy to see that the foregoing equation is equivalent to

$$\phi\left(\frac{1}{a+1}\right) = \sum_{\nu=0}^{\infty}(-1)^\nu f_{2\nu+1}\left(\frac{a^2}{a+1}\right),$$

i.e. [3, (5.9)] is not only true for $a \geq 2$ but even for $a \geq a_1$.

3. Since the number $x = \frac{1}{a+1}$ has the expansion

$$\frac{1}{a+1} = \gamma_1 \sum_{\nu=1}^{\infty} \frac{1}{a^{2\nu}}$$

it follows by [3, Proposition 4.4] that x belongs to CM. This means for $a \geq 2$ that $\frac{1}{a+1}$ never lies in one of the intervals $\overline{G}_{kn}$, so that $\phi(\frac{1}{a+1})$ cannot be calculated by means of the formulas in [1] or [2]. Analogously, this comes true for the more general left-hand side of (8.9).

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Es besteht die Möglichkeit, mit dem **Satzsystem LaTeX** oder unter anderen Textprogrammen erstellte Manuskripte auf unter **MS-DOS** formatierten Disketten (**3.5″, 1.44 MB**) einzureichen oder per e-mail an folgende Adresse zu schicken:

`romako@mathematik.uni-rostock.de`