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BEZUGSMÖGLICHKEITEN:

Universität Rostock
Universitätsbibliothek, Schriftentausch
18051 Rostock
Tel.: +49-381-498 22 81
Fax: +49-381-498 22 68
e-mail: maria.schumacher@ub.uni-rostock.de

Universität Rostock
Fachbereich Mathematik
18051 Rostock
Tel.: +49-381-498 1521
Fax: +49-381-498 1520
e-mail: romako@mathematik.uni-rostock.de

DRUCK: Universitätsdruckerei Rostock

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Isma Bouchemakh

On the König and the dual König property of order-interval hypergraphs of series-parallel posets

ABSTRACT. Let P be a finite poset. We consider the hypergraph $\mathcal{H}(P)$ whose vertices are the elements of P and whose edges are the maximal intervals of P. We prove the König and the dual König property of $\mathcal{H}(P)$ for the class of series-parallel posets.

1 Introduction

Let P be a finite poset. A subset I of P of the form $I = \{v \in P : p \leq v \leq q\}$ (denoted $[p, q]$) is called an *interval*. If p resp. q is a minimal resp. maximal element of P, then $[p, q]$ is called *maximal interval*. Let $\mathcal{I}(P)$ be the family of maximal intervals of P. The hypergraph $\mathcal{H}(P) = (P, \mathcal{I}(P))$, briefly denoted by $\mathcal{H} = (P, \mathcal{I})$, whose vertices are the elements of P and whose edges are the maximal intervals of P is said to be the *order-interval hypergraph of* P. The *representative graph* or *line-graph* $L(\mathcal{H})$ of $\mathcal{H}$ is a graph whose vertices are points $e_1, \dots, e_m$ representing the edges $I_1, \dots, I_m$ of $\mathcal{H}(P)$, the vertices e_i, e_j being adjacent iff $I_i \cap I_j \neq \emptyset$. The dual $\mathcal{H}^*$ of the order-interval hypergraph $\mathcal{H}$ is a hypergraph whose vertices $e_1, \dots, e_m$ correspond to intervals of P and with edges $X_i = \{e_j : \quad x_i \in I_j\}$. The line-graph $L(\mathcal{H}^*)$ of the dual of the order-interval hypergraph of P is called *the representative graph* of P. More precisely, the vertices of $L(\mathcal{H}^*)$ are the points of P and two vertices are joined iff they belong to the same interval of P.

Let α, ν, ρ and τ be the independence, matching, edge-covering, and vertex-covering number of a hypergraph $\mathcal{H}$, respectively. $\mathcal{H}$ has the König property if $\nu(\mathcal{H}) = \tau(\mathcal{H})$ and it has the dual König property if $\nu(\mathcal{H}^*) = \tau(\mathcal{H}^*)$ (or $\alpha(\mathcal{H}) = \rho(\mathcal{H})$ since $\alpha(\mathcal{H}) = \nu(\mathcal{H}^*)$ and $\rho(\mathcal{H}) = \tau(\mathcal{H}^*)$). Several interesting results exist about the matching, covering, independence and chromatic numbers of $\mathcal{H}$ such as the algorithmic complexity and min-max relations (see [2, 3, 4, 5]).

Let P and Q be two posets. The *disjoint sum* $P + Q$ of P and Q is the poset on the union $P \cup Q$, such that $x \leq y$ in $P + Q$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$. The *linear sum* $P \oplus Q$ of P and Q is the poset on the union $P \cup Q$, such that $x \leq y$ in $P \oplus Q$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$, or $x \in P$ and $y \in Q$. In [3], it is proved that for the class of interval orders, i.e. the posets which do not contain a subposet isomorphic to the disjoint sum of two chains of length 1, $\mathcal{H}$ has the König and the dual König properties. We are interested in this paper in another class of posets called *series-parallel*, namely they can be constructed from singletons using the operations of disjoint sum and linear sum. Series-parallel posets may be characterized by the fact that they do not contain a subset isomorphic to N, [8, 9] (N represents a subset of four elements $\{x, y, z, t\}$ such that $x < y > z < t$ and these are the only comparability relations). We prove that if P is series-parallel then $\mathcal{H}(P)$ has the König and dual König properties.

Theorem 1 *The representative graph $L(\mathcal{H}^*)$ of a series-parallel poset is perfect.*

Proof: We prove the perfectness of $L(\mathcal{H}^*)$ by using Seinsche's Theorem [7] which states that every graph with no induced P_4 (elementary chain of 4 vertices without chords) is perfect. Assume that there is an induced P_4, say $\mu = [x_1, x_2, x_3, x_4]$. In P, $x_1 \parallel x_3$, $x_1 \parallel x_4$ and $x_2 \parallel x_4$.
Let $I_1 = [p_1, q_1]$ be the interval containing both x_1 and x_2, $I_2 = [p_2, q_2]$ the interval containing both x_2 and x_3, and $I_3 = [p_3, q_3]$ the interval containing both x_3 and x_4. Let us distinguish three cases.

Case 1. $x_1 < x_2$.
Since the chain μ is chordless and the vertices x_2 and x_3 belong to the same interval in P, we have that $p_2 < x_2$, $x_3 < q_2$ with $x_2 \parallel x_3$ or $x_2 > x_3$.

Case 1.1. $x_2 \parallel x_3$.
Then the poset induced by $\{x_1, x_2, p_2, x_3\}$ and N are isomorphic. It suffices to verify that $p_2 \parallel x_1$ which is true because otherwise x_1x_3 is a chord of μ.

Case 1.2. $x_2 > x_3$.
Then the poset induced by $\{x_1, x_2, x_3, x_4\}$ and N are isomorphic if $x_3 < x_4$. In the other situation, i.e. when $x_3 \parallel x_4$, we have that $\{x_1, x_2, p_3, x_4\}$ and N are isomorphic. Indeed, $p_3 < x_2$ since $p_3 \leq x_3 < x_2$ and $x_1 \parallel p_3$ since $p_3 \leq x_1 < x_2$ implies the existence of the chord x_1x_3.

Case 2. $x_2 < x_1$.
By duality, this case is similar to the first case.

Case 3. $x_1 \parallel x_2$. We have three possibilities:

Case 3.1. $x_3 > x_2$.
Then the poset induced by $\{x_1, q_1, x_2, x_3\}$ is isomorphic to N since x_1x_3 is not a chord of μ, hence $q_1 \parallel x_3$.

Case 3.2. $x_3 < x_2$.
Then the poset induced by $\{x_3, x_2, p_1, x_1\}$ is isomorphic to N since $p_1 \leq x_3$ implies $p_1 \leq x_3 < x_2 \leq q_1$, i.e. x_1x_3 is a chord of μ.

Case 3.3. $x_3 \parallel x_2$.
Because it is not possible to have simultaneously $x_1 > p_2$ and $x_1 < q_2$, we may infer that the poset induced by $\{p_2, x_2, p_1, x_1\}$ is isomorphic to N if $x_1 \not> p_2$ and the poset induced by $\{x_1, q_1, x_2, q_2\}$ is isomorphic to N if $x_1 \not< q_2$. □

Let $\mathcal{H} = (E_1, \ldots, E_m)$ be a hypergraph. We say that $\mathcal{H}$ has the *Helly property* or is a *Helly hypergraph* if every intersecting family of $\mathcal{H}$ is a star, i.e. for $J \subset \{1, \ldots, m\}$, $E_i \cap E_j \neq \emptyset$, for $i, j \in J$ implies $\bigcap_{j \in J} E_j \neq \emptyset$. A good characterization of a Helly hypergraph, due to Berge and Duchet [1], is given by the following property:

For any three vertices a_1, a_2, a_3 the family of edges containing at least two of the vertices a_i has a non-empty intersection. This characterization will be used in the following theorem:

Theorem 2 *Let $\mathcal{H} = (P, \mathcal{I})$ be the order-interval hypergraph of a series-parallel order. Then $\mathcal{H}$ is a Helly hypergraph.*

Proof: Let $\mathcal{I} = \{I_1, \ldots, I_m\}$ be the family of maximal intervals of P. We suppose that there exist three vertices a_1, a_2, a_3 of P such that $\bigcap_{j \in J} I_j = \emptyset$, where $J = \{j : \quad |I_j \cap \{a_1, a_2, a_3\}| \geq 2\}$.
Then $|J| \geq 3$ and there exist three edges, say w.l.o.g., $I_1 = [p_1, q_1]$, $I_2 = [p_2, q_2]$ and $I_3 = [p_3, q_3]$, such that

$$\begin{aligned} a_2, a_3 \in I_1 \quad &\text{and} \quad a_1 \notin I_1 \\ a_1, a_3 \in I_2 \quad &\text{and} \quad a_2 \notin I_2 \\ a_1, a_2 \in I_3 \quad &\text{and} \quad a_3 \notin I_3 \end{aligned}$$

Since not simultaneously $p_3 \leq a_3$ and $a_3 \leq q_3$ let w.l.o.g. $p_3 \not\leq a_3$.
Case 1. $a_1 \parallel a_3$.
Then the poset induced by $\{p_3, a_1, p_2, a_3\}$ and N are isomorphic. It suffices to verify that $p_2 \neq p_3$ which is true because $a_3 \not\geq p_3$ but $a_3 > p_2$.
Case 2. $a_1 < a_3$.
Then we have $p_3 \leq a_1 < a_3$, a contradiction.
Case 3. $a_3 < a_1$.
We distinguish three subcases:
Case 3.1. $a_1 < a_2$.
Then $a_1 \in I_1$, a contradiction.
Case 3.2. $a_2 < a_1$.
Then the poset induced by $\{p_3, a_2, p_1, a_3\}$ is isomorphic to N. Indeed, we have $a_2 \parallel a_3$ since $a_2 < a_3$ gives $a_3 \in I_3$ and $a_3 < a_2$ gives $a_2 \in I_2$; $p_1 \neq p_3$ since $a_3 \not\geq p_3$ and $a_3 > p_1$.

Case 3.3. $a_1 \parallel a_2$.
Then the poset induced by $\{a_1, q_3, a_2, q_1\}$ is isomorphic to N. Indeed, we have $a_1 \not\leq q_1$ because $p_1 \leq a_3 < a_1 \leq q_1$ gives $a_1 \in I_1$; $q_1 \neq q_3$ since $a_1 < q_3$ and $a_1 \not\leq q_1$. □

Theorem 3 *The dual $\mathcal{H}^*$ of the order-interval hypergraph $\mathcal{H}$ of a series-parallel order has the Helly property.*

Proof: A necessary and sufficient condition for a hypergraph $\mathcal{H}^*$ to have the Helly property is that for any three edges I_1, I_2, I_3 of $\mathcal{H}$, there exists an edge (of $\mathcal{H}$) containing the set $(I_1 \cap I_2) \cup (I_1 \cap I_3) \cup (I_2 \cap I_3)$ [1]. For our hypergraph, we claim that $I_1 \cap I_2 \subseteq I_3$ or $I_1 \cap I_3 \subseteq I_2$ or $I_2 \cap I_3 \subseteq I_1$. Suppose not. Then we have $I_1 \cap I_2 \not\subseteq I_3$, $I_1 \cap I_3 \not\subseteq I_2$ and $I_2 \cap I_3 \not\subseteq I_1$. Let a_1, a_2 and a_3 be three vertices of P such that

$$\begin{aligned} a_1 \in I_2 \cap I_3 \quad &\text{and} \quad a_1 \notin I_1 \\ a_2 \in I_1 \cap I_3 \quad &\text{and} \quad a_2 \notin I_2 \\ a_3 \in I_1 \cap I_2 \quad &\text{and} \quad a_3 \notin I_3 \end{aligned}$$

i.e.

$$\begin{aligned} a_2, a_3 \in I_1 \quad &\text{and} \quad a_1 \notin I_1 \\ a_1, a_3 \in I_2 \quad &\text{and} \quad a_2 \notin I_2 \\ a_1, a_2 \in I_3 \quad &\text{and} \quad a_3 \notin I_3. \end{aligned}$$

But, we have already seen in the proof of Theorem 2, this leads to a contradiction. □

$\mathcal{H}$ is said to be normal if every partial hypergraph $\mathcal{H}'$ has the coloured edge property, i.e. it is possible to colour the edges of $\mathcal{H}'$ with $\Delta(\mathcal{H}')$ colours, where $\Delta(\mathcal{H}')$ represents the maximum degree of $\mathcal{H}'$. The normality of $\mathcal{H}$ induces the same property for all its partial hypergraphs. Several sufficient conditions exist for a hypergraph to have the König property [1]. One of them is its normality. A hypergraph $\mathcal{H}$ is normal iff it satisfies the Helly property and $L(\mathcal{H})$ is a perfect graph. This enables us to present the following corollary:

Corollary *If P is a series-parallel poset then every subhypergraph of $\mathcal{H}(P)$ has the dual König property.*

Proof: The line graph $L(\mathcal{H}^*)$ is a perfect graph by Theorem 1 and $\mathcal{H}^*$ has the Helly property by Theorem 3. Hence, $\mathcal{H}^*$ is normal and consequently every partial hypergraph is also normal. Since the dual of a partial hypergraph of $\mathcal{H}^*$ is a subhypergraph of $\mathcal{H}$, we may infer that every subhypergraph of $\mathcal{H}$ has the dual König property. □

Theorem 4 *If P is a series-parallel poset then $\mathcal{H}(P)$ has the König property.*

Proof: We are interested only in the case where P is connected, because if P has not this property, it can be decomposed into k connected subposets and the relations $\nu(\mathcal{H}(P)) = \nu(\mathcal{H}(P_1)) + ... + \nu(\mathcal{H}(P_k))$, $\tau(\mathcal{H}(P)) = \tau(\mathcal{H}(P_1)) + ... + \tau(\mathcal{H}(P_k))$ may be used.
Let P be a connected poset not reduced to a singleton. P can be decomposed into subposets using only the linear sum, say $P = \bigoplus_{i=1}^{k} P_i$, with $k \geq 2$.
The case $k \geq 3$ implies directly $\nu(\mathcal{H}) = \tau(\mathcal{H}) = 1$ because each singleton with one element of P_2 represents a vertex-covering of $\mathcal{H}$.
Let $k = 2$. Denote by $P_{11}, ..., P_{1k_1}$ and $P_{21}, ..., P_{2k_2}$ the connected components of P_1 and P_2, respectively. We suppose w.l.o.g. that $k_1 \leq k_2$.
Let $j_0 \in \{1, ..., k_1\}$ and a_{ij_0} (resp. b_{ij_0}) any minimal (resp. maximal) element of P_{ij_0}. Since P_{ij_0} is connected and series-parallel, each maximal element (of P_{ij_0}) is above every minimal element (of P_{ij_0}). It follows, $p < b_{1j_0}$ for all $p \in \min(P_{1j_0})$. Also, we have by the $\oplus$-operation $b_{1j_0} < p'$ for all $p' \in \min(P_2)$. Hence, each edge of $\mathcal{H}(P_{1j_0} \oplus P_2)$ contains the point b_{1j_0} and more generally, for $P = (P_{11} + ...+ P_{1k_1}) \oplus P_2$, the set $T = \{b_{1j}, j = 1, ..., k_1\}$ represents a point cover of $\mathcal{H}(P)$. On the other hand, it is not difficult to see that the family of edges $\{[a_{1j}, b_{2j}], j = 1, ..., k_1\}$ forms a matching of $\mathcal{H}$. It follows $\nu(\mathcal{H}) = \tau(\mathcal{H}) = \min\{k_1, k_2\}$. □

Remark An analogous reasoning produces another proof for the dual König property. Indeed, let $P = Q_1 + ... + Q_k$ be a series-parallel poset where each component Q_i is constructed from a $\oplus$-operation, say $Q_i = Q_{i1} \oplus Q_{i2}$. Denote by $\min(Q_i) = \{a_{i1}, \ldots, a_{ik}\}$ (resp. $\max(Q_i) = \{b_{i1}, \ldots, b_{il}\}$) the set of minimal (resp. maximal) elements of Q_i and consider the family of edges $\mathcal{I}_i^{(k)}$ of $\mathcal{H}(Q_i)$ such that $\mathcal{I}_i^{(k)} = \{[a_{ij}, b_{ij}],\ \ j = 1, \ldots, k\}$. Let $\mathcal{R}_i$ be $\mathcal{I}_i^{(k)}$ if $k = l$, $\mathcal{I}_i^{(k)} \cup \{[a_{ik}, b_{ij}],\ \ j = k+1, \ldots, l\}$ if $k < l$ and $\mathcal{I}_i^{(l)} \cup \{[a_{ij}, b_{il}],\ \ j = l+1, \ldots, k\}$ if $l < k$. It is not difficult to see that $\mathcal{R}_i$ is an edge-covering of $\mathcal{H}(Q_i)$ of size $\max\{|\min(Q_i)|, |\max(Q_i)|\}$. It follows, $\alpha(\mathcal{H}(Q_i)) = \rho(\mathcal{H}(Q_i)) = \max\{|\min(Q_i)|, |\max(Q_i)|\}$. Since $\alpha(\mathcal{H}(Q_i + Q_j)) = \alpha(\mathcal{H}(Q_i)) + \alpha(\mathcal{H}(Q_j))$ and $\rho(\mathcal{H}(Q_i + Q_j)) = \rho(\mathcal{H}(Q_i)) + \rho(\mathcal{H}(Q_j))$, we deduce

$\alpha(\mathcal{H}(P)) = \sum_{i=1}^{k} \alpha(\mathcal{H}(Q_i)) = \sum_{i=1}^{k} \max\{|\min(Q_i)|, |\max(Q_i)|\}$ and
$\rho(\mathcal{H}(P)) = \sum_{i=1}^{k} \rho(\mathcal{H}(Q_i)) = \sum_{i=1}^{k} \max\{|\min(Q_i)|, |\max(Q_i)|\}$.

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received: November 9, 2000

Author:

Isma Bouchemakh
U.S.T.H.B., Institut de Mathématiques,
B.P.32, El-Alia, Bab-Ezzouar, Alger
Algérie

Rostock. Math. Kolloq. **56**, 9–24 (2002)

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35 P.D.E.,35J10, 35J45

Laure Cardoulis

Existence of solutions for non necessarily cooperative systems involving Schrödinger operators

ABSTRACT. We study the existence of a solution for a non necessarily cooperative system of n equations involving Schrödinger operators defined on $I\!R^N$ and we study also a limit case (the Fredholm Alternative). We derive results for semilinear systems.

KEY WORDS AND PHRASES. Schrödinger operators, M-matrices, non necessarily cooperative systems.

1 Introduction

We consider the following elliptic system defined on $I\!R^N$:

$$(1)\begin{cases} \text{for } 1 \le i \le n, \\ (1i) \quad L_{q_i} u_i := (-\Delta + q_i) u_i = \sum_{j=1}^n a_{ij} u_j + f_i \text{ in } I\!R^N \end{cases}$$

where :

. n and N are two integers not equal to 0

. Δ is the Laplacian operator

(H1) for $1 \le i, j \le n,\ a_{ij} \in L^\infty(I\!R^N)$

(H2) for $1 \le i \le n$, q_i is a continuous potential defined on $I\!R^N$ such that:
$q_i(x) \ge 1, \quad \forall x \in I\!R^N$ and $q_i(x) \to +\infty$ when $|x| \to +\infty$

(H3) for $1 \le i \le n,\ f_i \in L^2(I\!R^N)$

We do not make here any assumptions on the sign of a_{ij}.
Recall that System (1) is called cooperative if $a_{ij} \ge 0$ *pp* for $i \ne j$.

Our paper is organized as follow:
- in Section 2, we recall some results about M-matrices and about the Maximum Principle for cooperative systems involving Schrödinger operators $-\Delta + q_i$ in $I\!R^N$.
- in Section 3, we show the existence of a solution for a non necessarily cooperative system of n equations. After that we study a limit case (the Fredholm Alternative) and finally we study the existence of a solution for a (non necessarily cooperative) semilinear system.

2 Recalls

2.1 M-matrix

We recall some results about the M-matrix([8]th2.3 p.134).
We say that a matrix is positive if all its coefficients are nonnegative and we say that a symmetric matrix is positive definite if all its principal minors are stictly positive.

Definition 2.1 ([8]) *A matrix $M = sI - B$ is called a non singular M-matrix if B is a positive matrix (i.e. with nonnegative coefficients) and $s > \rho(B) > 0$ the spectral radius of B.*

Proposition 2.2 ([8]) *If M is a matrix with nonpositive off-diagonal coefficients, the conditions $(P0), (P1), (P2), (P3), (P4)$ are equivalent.*

$(P0)$ *M is a non singular M-matrix,*

$(P1)$ *all the principal minors of M are strictly positive,*

$(P2)$ *M is semi-positive i.e.: $\exists X >> 0$ such that $MX >> 0$.*
$X >> 0$ signify $\forall i, X_i > 0$ if $X = (X_1, ..., X_n)$.

$(P3)$ *M has a positive inverse.*

$(P4)$ *there exists a diagonal matrix $D,\ D > 0$, such that $MD + D^tM$ is positive definite.*

Remark: If M is a non singular M-matrix, then ${}^tM$ is also a non singular M-matrix.
So $(P4) \Leftrightarrow (P5)$ with $(P5)$: there exists a diagonal matrix $D,\ D > 0$ such that ${}^tMD + DM$ is positive definite.

2.2 Schrödinger operators

Let $\mathcal{D}(I\!R^N) = \mathcal{C}_0^\infty(I\!R^N) = \mathcal{C}_c^\infty(I\!R^N)$ the set of functions $\mathcal{C}^\infty$ on $I\!R^N$ with compact support.
Let q a continuous potential defined on $I\!R^N$ such that :

$q(x) \geq 1, \quad \forall x \in I\!R^N$ and $q(x) \to +\infty$ when $|x| \to +\infty$. The variational space is $V_q(I\!R^N)$: the completion of $\mathcal{D}(I\!R^N)$ for the norm $\|.\|_q$ where $\|u\|_q = [\int_{I\!R^N} |\nabla u|^2 + q|u|^2]^{\frac{1}{2}}$.

$$V_q(I\!R^N) = \{u \in H^1(I\!R^N), \sqrt{q}u \in L^2(I\!R^N)\}$$

$(V_q(I\!R^N), \|.\|_q)$ is an Hilbert space.([1] prop.I.1.1)
Moreover:

Proposition 2.3 ([1] propI.1.1; [21]prop1, p.356)
The embedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact and with dense range.

To the form :

$$a(u,v) = \int_{I\!R^N} \nabla u.\nabla v + \int_{I\!R^N} quv, \qquad \forall (u,v) \in (V_q(I\!R^N))^2,$$

we associate the operator $L_q := -\Delta + q$ defined on $L^2(I\!R^N)$ by variational methods.
Here $D(L_q)$ denotes the domain of the operator L_q.
$D(L_q) = \{u \in V_q(I\!R^N), (-\Delta + q)u \in L^2(I\!R^N)\}$ ([5]th1.1, p4)
We have : $\forall u \in D(L_q), \forall v \in V_q(I\!R^N), a(u,v) = \int_{I\!R^N} L_q u.v.$
The embedding of $D(L_q)$ into $V_q(I\!R^N)$ is continuous and with dense range.
([1], p.24;[5], p.5,6)

Proposition 2.4 ([1], p.25 to 27;[5]th1.1, p.4,p.6,8,11; [6], p.3, th3.2 p.45; [14], p.488,489;[26], p.346 to 350; [28]thXIII.16, p.120,thXIII.47 p.207)
L_q, considered as an operator in $L^2(I\!R^N)$, is positive, selfadjoint, with compact inverse. Its spectrum is discrete and consists in an infinite sequence of positive eigenvalues tending to $+\infty$. The smallest one, denoted by $\lambda(q)$, is simple and associated with an eigenfunction ϕ_q which does not change sign in $I\!R^N$. We say that $\lambda(q)$ is a principal eigenvalue; it is positive and simple.
Furthermore, we have:

$$(R1) \begin{cases} L_q\phi_q = \lambda(q)\phi_q \ \ in \ I\!R^N \\ \phi_q(x) \to 0 \ \ when \ x \to +\infty; \phi_q > 0 \ \ in \ I\!R^N; \lambda(q) > 0 \end{cases}$$

$$(R2) \ \forall u \in V_q(I\!R^N), \lambda(q) \int_{I\!R^N} |u|^2 \leq \int_{I\!R^N} [|\nabla u|^2 + q|u|^2].$$

Moreover the equality holds iff u is colinear to ϕ_q.

If $a \in L^\infty(I\!R^N)$ let: $a^* = \sup_{x \in I\!R^N} a(x), \quad a_* = \inf_{x \in I\!R^N} a(x)$ and

$$\lambda(q-a) = \inf\{\frac{\int_{I\!R^N} [|\nabla \phi|^2 + (q-a)\phi^2]}{\int_{I\!R^N} \phi^2}; \phi \in \mathcal{D}(I\!R^N); \phi \not\equiv 0\}.$$

The operator $-\Delta + q - a$ in $I\!R^N$ has a unique selfadjoint realization ([6] p.3) in $L^2(I\!R^N)$ which is denoted L_{q-a}. (Indeed, q is a continuous potential, $a \in L^\infty(I\!R^N)$, so the condition in [6] $(q-a)_- \in L^p_{loc}(I\!R^N)$ for a $p > \frac{N}{2}$is satisfied.)
We note also that: $\lambda(q-a) \le \lambda(q) - a_*$ and:
$\forall m \in I\!R^{*+},\ \lambda(q-a+m) = \lambda(q-a) + m.$

The following theorem is classical:

Theorem 2.1 ([1],[11],[28]p.204) *We consider the equation:*
(E) $(-\Delta + q)u = au + f$ *in* $I\!R^N$ *where* $a \in I\!R,\ f \in L^2(I\!R^N),\ f \ge 0$
and q *is a continuous potential on* $I\!R^N$ *such that:*
$q \ge 1$ *and* $q(x) \to +\infty$ *when* $|x| \to +\infty$.
If $a < \lambda(q)$ *then* $\exists! u \in V_q(I\!R^N)$ *solution of* (E). *Moreover, we have:* $u \ge 0$.

2.3 Cooperative systems

We consider in this section System (1) and we assume here that it is cooperative, i.e.:
(H1*) $a_{ij} \in L^\infty(I\!R^N);\ a_{ij} \ge 0$ *pp* for $i \ne j$.
We recall here a sufficient condition for Maximum Principle and existence of solutions for such cooperative systems.
We say that System (1) satisfies the Maximum Principle if: $\forall f_i \ge 0$, $1 \le i \le n$, any solution $u = (u_1, ..., u_n)$ of (1) is nonnegative.
Let $E = (e_{ij})$ be the matrix $n \times n$ defined by:
$\forall 1 \le i \le n,\ e_{ii} = \lambda(q_i - a_{ii})$ and $\forall 1 \le i, j \le n,\ i \ne j \Rightarrow e_{ij} = -a^*_{ij}$.

Theorem 2.2 ([11]) *We assume that* $(H1*), (H2), (H3)$ *are satisfied.*
If E *is a non singular M-matrix, then System* (1) *satisfies the Maximum Principle.*

Theorem 2.3 ([11]) *We assume that* $(H1*), (H2), (H3)$ *are satisfied.*
If E *is a non singular M-matrix and if* $f_i \ge 0$ *for each* $1 \le i \le n$*, then System* (1) *has a unique solution which is nonnegative.*

3 Study of a non necessarily cooperative system

3.1 Study of a non necessarily cooperative system of n equations with bounded coefficients

We adapt here an approximation method used in [9] for problems defined on bounded domains.

We consider the following elliptic system defined on $I\!R^N$:

$$(1)\begin{cases}\text{for } 1 \le i \le n,\\ (1i) \quad L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^{n} a_{ij}u_j + f_i \text{ in } I\!R^N\,.\end{cases}$$

Let $G = (g_{ij})$ the matrix $n \times n$ defined by: $\forall 1 \le i \le n,\ g_{ii} = \lambda(q_i - a_{ii})$ and

$$\forall 1 \le i, j \le n,\ i \ne j \Rightarrow g_{ij} = -|a_{ij}|^* \text{ where } |a_{ij}|^* = \sup_{x \in I\!R^N} |a_{ij}(x)|.$$

We make the following hypothesis: (H) G is a non singular M-matrix.

Theorem 3.1 *We assume that* $(H1), (H2), (H3)$ *and* (H) *are satisfied. Then System* (1) *has a weak solution* $(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$.

First, we prove the following lemma:

Lemma 3.1 *We assume that* $(H), (H1), (H2), (H3)$ *are satisfied.*
Let $(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$ *solution of:*

$$(2)\begin{cases}\textit{for } 1 \le i \le n,\\ (2i) \quad L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^{n} a_{ij}u_j \textit{ in } I\!R^N\,.\end{cases}$$

Then: $(u_1, ..., u_n) = (0, ..., 0)$.

Proof of the lemma 3.1: Let $m \in I\!R^{*+}$ be such that:
$\forall 1 \le i \le n, m - a_{ii} > 0$. Let $q_i' = q_i + m - a_{ii} \ge 1$.
For any $1 \le i \le n$, we have:

$$\int_{I\!R^N} [|\nabla u_i|^2 + q_i'|u_i|^2] = \int_{I\!R^N} m|u_i|^2 + \sum_{j;j \ne i} \int_{I\!R^N} a_{ij}u_ju_i$$

$$\le \int_{I\!R^N} m|u_i|^2 + \sum_{j;j \ne i} \int_{I\!R^N} |a_{ij}u_ju_i|.$$

and by the characterization $(R2)$ of the first eigenvalue $\lambda(q_i')$ we get:
$(\lambda(q_i') - m) \int_{I\!R^N} |u_i|^2 \le \sum_{j;j \ne i} |a_{ij}|^* (\int_{I\!R^N} |u_j|^2)^{\frac{1}{2}} (\int_{I\!R^N} |u_i|^2)^{\frac{1}{2}}$.
So: $(\lambda(q_i') - m)(\int_{I\!R^N} |u_i|^2)^{\frac{1}{2}} \le \sum_{j;j \ne i} |a_{ij}|^* (\int_{I\!R^N} |u_j|^2)^{\frac{1}{2}}$.
Let

$$X = \begin{pmatrix} (\int_{I\!R^N} u_1^2)^{\frac{1}{2}} \\ . \\ . \\ . \\ (\int_{I\!R^N} u_n^2)^{\frac{1}{2}} \end{pmatrix}.$$

We have $X \geq 0$ and $GX \leq 0$. Since G is a non singular M-matrix, by Proposition 2.2 we deduce that $X \leq 0$. So $X = 0$ i.e. $\forall 1 \leq i \leq n, \;\; u_i = 0$.

Proof of Theorem 3.1: Let $m \in I\!R^{*+}$ such that : $\forall 1 \leq i \leq n, m - a_{ii} > 0$. Let $q_i' = q_i - a_{ii} + m \geq 1$. ($m$ exists because $\forall 1 \leq i \leq n, \; a_{ii} \in L^\infty(I\!R^N)$.)

First we note that:
$(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$ is a weak solution of (1) if and only if $(u_1, ..., u_n)$ is a weak solution of $(1')$ where:

$$(1') \left\{ \begin{array}{l} \text{for } 1 \leq i \leq n, \\ (1'i) \quad (-\Delta + q_i')u_i = mu_i + \sum_{j;j\neq i} a_{ij}u_j + f_i \text{ in } I\!R^N . \end{array} \right.$$

Let $\epsilon \in]0,1[, \; B_\epsilon = B(0, \frac{1}{\epsilon}) = \{x \in I\!R^N, \; |x| < \frac{1}{\epsilon}\}$ and 1_{B_ϵ} be the indicator function of B_ϵ.
Let $T : L^2(I\!R^N) \times ... \times L^2(I\!R^N) \to L^2(I\!R^N) \times ... \times L^2(I\!R^N)$ be defined by: $T(\xi_1, ..., \xi_n) = (\omega_1, ..., \omega_n)$ where for any $1 \leq i \leq n$,

$$(-\Delta + q_i')\omega_i = m\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij}\frac{\xi_j}{1+\epsilon|\xi_j|}1_{B_\epsilon} + f_i \text{ in } I\!R^N.$$

i) First we prove that T is well defined:

Let: $\forall (\xi_1, ..., \xi_n) \in L^2(I\!R^N) \times ... \times L^2(I\!R^N), \;\; \forall 1 \leq i \leq n,$

$$\psi_i(\xi_1, ..., \xi_n) = m\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij}\frac{\xi_j}{1+\epsilon|\xi_j|}1_{B_\epsilon}.$$

We have:

$$|\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}| = \frac{1}{\epsilon}|\frac{\epsilon\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}| \leq \frac{1}{\epsilon}1_{B_\epsilon}.$$

Since $1_{B_\epsilon} \in L^2(I\!R^N)$ and $a_{ij} \in L^\infty(I\!R^N)$, we deduce that for any $1 \leq i \leq n \;\; \psi_i(\xi_1, ..., \xi_n) \in L^2(I\!R^N)$. By $(H3)$, $f_i \in L^2(I\!R^N)$ and therefore $\psi_i(\xi_1, ..., \xi_n) + f_i \in L^2(I\!R^N)$.
By Theorem 2.1, we deduce the existence (and uniqueness) of $(\omega_1, ..., \omega_n) \in V_{q_1}(I\!R^N) \times ...V_{q_n}(I\!R^N)$. So T is well defined.

ii) We note that: $\forall (\xi_1, ..., \xi_n)$, $|\psi_i(\xi_1, ..., \xi_n)| \leq n \max_{j;j\neq i}(m, |a_{ij}|^*)\frac{1}{\epsilon}1_{B_\epsilon}$.

Let $h = \frac{n}{\epsilon}\max_{i,j;i\neq j}(m, |a_{ij}|^*)1_{B_\epsilon} \in L^2(I\!R^N)$.
$h + f_i \in L^2(I\!R^N)$, so, by the scalar case, we deduce that:
$\exists! \xi_i^0 \in V_{q_i}(I\!R^N)$ such that: $(-\Delta + q_i')\xi_i^0 = h + f_i$ in $I\!R^N$.
$(\xi_1^0, ..., \xi_n^0)$ is an uppersolution of $(1')$: $\forall 1 \leq i \leq n$,

$$(-\Delta + q_i')\xi_i^0 \geq \psi_i(\xi_1, ..., \xi_n) + f_i$$

By the same way, we construct a lowersolution of $(1')$: $\forall 1 \leq i \leq n$,
$\exists! \xi_{i,0} \in V_{q_i}(I\!R^N)$ such that: $(-\Delta + q_i')\xi_{i,0} = -h + f_i$ in $I\!R^N$.
$(\xi_{1,0}, ..., \xi_{n,0})$ is a lowersolution of $(1')$: $\forall 1 \leq i \leq n$,

$$(-\Delta + q_i')\xi_{i,0} \leq \psi_i(\xi_1, ..., \xi_n) + f_i.$$

We note that: $\forall i,\ \xi_{i,0} \leq \xi_i^0$ (because $(-\Delta + q_i')(\xi_i^0 - \xi_{i,0}) = 2h \geq 0$.)
We consider now the restriction of T, denoted by T^*, at $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$. We prove now that T^* has a fixed point by the Schauder Fixed Point Theorem.

iii) First we prove that $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ is invariant by T^*.
Let $(\xi_1, ..., \xi_n) \in [\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$. We put: $T^*(\xi_1, ..., \xi_n) = (\omega_1, ..., \omega_n)$.
We have: $(-\Delta + q_i')(\xi_i^0 - \omega_i) = h - \psi_i(\xi_1, ..., \xi_n) \geq O$.
By the scalar case, we deduce that: $\xi_i^0 \geq \omega_i\ pp$.
By the same way we get: $(-\Delta + q_i')(\omega_i - \xi_{i,0}) = \psi_i(\xi_1, ..., \xi_n) + h \geq 0$ and $\omega_i \geq \xi_{i,0}\ pp$.
So $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ is invariant by T^*.

iv) We prove that T^* is a compact continuous operator.
T^* is continuous if and only if $\forall i,\ \psi_i^*$ is continuous where ψ_i^* is the restriction of ψ_i to $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$.
Let $(\xi_1, ..., \xi_n) \in [\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$.
Let $(\xi_1^p, ..., \xi_n^p)_p$ a sequence in $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ converging to $(\xi_1, ..., \xi_n)$ for $\|.\|_{(L^2(I\!R^N))^n}$.
We have: $\forall 1 \leq i \leq n$,

$$\|\frac{\xi_i^p}{1+\epsilon|\xi_i^p|}1_{B_\epsilon} - \frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}\|_{L^2(I\!R^N)} \leq \frac{1}{\epsilon}\|\frac{\epsilon\xi_i^p}{1+\epsilon|\xi_i^p|} - \frac{\epsilon\xi_i}{1+\epsilon|\xi_i|}\|_{L^2(I\!R^N)}.$$

However the function l defined on $I\!R$ by: $\forall x \in I\!R,\ l(x) = \frac{x}{1+|x|}$ is Lipschitz and satisfies: $\forall x, y \in I\!R,\ |l(x) - l(y)| \leq |x - y|$.
So:

$$\|\frac{\xi_i^p}{1+\epsilon|\xi_i^p|} - \frac{\xi_i}{1+\epsilon|\xi_i|}\|_{L^2(I\!R^N)} \leq \frac{1}{\epsilon}\|\epsilon\xi_i^p - \epsilon\xi_i\|_{L^2(I\!R^N)} = \|\xi_i^p - \xi_i\|_{L^2(I\!R^N)}.$$

Hence:

$$\frac{\xi_i^p}{1+\epsilon|\xi_i^p|}1_{B_\epsilon} - \frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon} \to 0 \text{ in } L^2(I\!R^N) \text{ when } p \to +\infty.$$

So ψ_i^* is continuous and therefore $T^*$is a continuous operator.
Moreover, by Proposition 2.4, $(-\Delta + q_i')^{-1}$ is a compact operator.
So T^* is compact.

v) $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ is a closed convex subset.
Hence, by the Schauder Fixed Point Theorem, we deduce the existence of $(\xi_1, ..., \xi_n) \in$

$[\xi_{1,0},\xi_1^0]\times ... \times[\xi_{n,0},\xi_n^0]$ such that:
$T^*(\xi_1,...,\xi_n)=(\xi_1,...,\xi_n)$.
$\forall i,\ \xi_i$ depends of ϵ, so we denote: $\xi_i=u_{i,\epsilon}$.
$u_{1,\epsilon},...,u_{n,\epsilon}$ satisfy:

$$(S)\left\{\begin{array}{l}\forall 1\leq i\leq n\\ (Si)\quad (-\Delta+q_i')u_{i,\epsilon}=m\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}+\sum_{j;j\neq i}a_{ij}\frac{u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}+f_i \text{ in } I\!R^N.\end{array}\right.$$

vi) Now we prove that $\forall i,\ (\epsilon u_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i'}(I\!R^N)$.
Let $\|u\|_{q_i'}=[\int_{I\!R^N}|\nabla u|^2+q_i'|u|^2]^{\frac{1}{2}}$. We multiply (Si) by $\epsilon^2u_{i,\epsilon}$ and we integrate over $I\!R^N$.
We get:

$$\|\epsilon u_{i,\epsilon}\|_{q_i'}^2\leq m\int_{I\!R^N}|\frac{\epsilon u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}\epsilon u_{i,\epsilon}|+\sum_{j;j\neq i}|a_{ij}|^*\int_{I\!R^N}|\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}\epsilon u_{i,\epsilon}|$$
$$+\int_{I\!R^N}|\epsilon f_i\epsilon u_{i,\epsilon}|\,.$$

But: $\forall j,\ |\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}|<1$.
So there exists a strictly positive constant K such that: $\|\epsilon u_{i,\epsilon}\|_{q_i'}^2\leq K\|\epsilon u_{i,\epsilon}\|_{L^2(I\!R^N)}\leq K\|\epsilon u_{i,\epsilon}\|_{q_i'}$ and therefore: $\|\epsilon u_{i,\epsilon}\|_{q_i'}\leq K$.

vii) We prove now that $\epsilon u_{i,\epsilon}\to 0$ when $\epsilon\to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_{q_i'}(I\!R^N)$.
We know that the imbedding of $V_{q_i'}(I\!R^N)$ into $L^2(I\!R^N)$ is compact.
The sequence $(\epsilon u_{i,\epsilon})_\epsilon$ is bounded in $V_{q_i'}(I\!R^N)$ so (for a subsequence), we deduce that $\exists u_i^*$ such that:
$\epsilon u_{i,\epsilon}\to u_i^*$ when $\epsilon\to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_{q_i'}(I\!R^N)$.
Multiplying (Si) by ϵ, we get:

$$(-\Delta+q_i')\epsilon u_{i,\epsilon}=m\frac{\epsilon u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}+\sum_{j;j\neq i}a_{ij}\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}+\epsilon f_i \text{ in } I\!R^N.$$

But $\epsilon u_{i,\epsilon}\rightharpoonup u_i^*$ weakly in $V_{q_i}(I\!R^N)$.
So: $\forall\phi\in\mathcal{D}(I\!R^N)$,

$$\int_{I\!R^N}[\nabla(\epsilon u_{i,\epsilon}).\nabla\phi+q_i'\epsilon u_{i,\epsilon}\phi]\to\int_{I\!R^N}[\nabla u_i^*.\nabla\phi+q_i'u_i^*\phi] \text{ when } \epsilon\to 0.$$

Moreover: $\forall\phi\in\mathcal{D}(I\!R^N),\ \int_{I\!R^N}\epsilon f_i\phi\to 0$ when $\epsilon\to 0$.
Moreover we have: $\forall j$

$$\|\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}-\frac{u_j^*}{1+|u_j^*|}\|_{L^2(I\!R^N)}^2=\int_{B_\epsilon}[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}-\frac{u_j^*}{1+|u_j^*|}]^2$$

$$+\int_{I\!R^N-B_\epsilon}(\frac{u_j^*}{1+|u_j^*|})^2 .$$

Since: $|\frac{u_j^*}{1+|u_j^*|}| \le |u_j^*|$, $\frac{u_j^*}{1+|u_j^*|} \in L^2(I\!R^N)$, hence $\int_{I\!R^N-B_\epsilon}(\frac{u_j^*}{1+|u_j^*|})^2 \to 0$ when $\epsilon \to 0$.

Moreover: $\int_{B_\epsilon}[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - \frac{u_j^*}{1+|u_j^*|}]^2 \le \int_{I\!R^N}[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - \frac{u_j^*}{1+|u_j^*|}]^2$

$$\le \|\epsilon u_{j,\epsilon} - u_j^*\|^2_{L^2(I\!R^N)} .$$

But: $\epsilon u_{j,\epsilon} \to u_j^*$ when $\epsilon \to 0$ strongly in $L^2(I\!R^N)$.

So: $\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon} \to \frac{u_j^*}{1+|u_j^*|}$ when $\epsilon \to 0$ strongly in $L^2(I\!R^N)$.

Therefore we can pass through the limit and we get:

$$(S')\begin{cases} \forall 1 \le i \le n \\ (S'i) \quad (-\Delta + q_i')u_i^* = m\frac{u_i^*}{1+|u_i^*|} + \sum_{j;j\ne i} a_{ij}\frac{u_j^*}{1+|u_j^*|} \text{ in } I\!R^N . \end{cases}$$

We prove now that for any i, $u_i^* = 0$. We multiply $(S'i)$ by u_i^*, we integrate over $I\!R^N$ and we obtain:

$$\int_{I\!R^N}[|\nabla u_i^*|^2 + q_i'|u_i^*|^2] = \int_{I\!R^N} m\frac{|u_i^*|^2}{1+|u_i^*|} + \sum_{j;j\ne i}\int_{I\!R^N} a_{ij}\frac{u_j^* u_i^*}{1+|u_j^*|}$$

$$\le \int_{I\!R^N} m\frac{|u_i^*|^2}{1+|u_i^*|} + \sum_{j;j\ne i}\int_{I\!R^N} |a_{ij}|^*\frac{|u_j^*||u_i^*|}{1+|u_j^*|}.$$

But: $\forall j$, $\frac{1}{1+|u_j^*|} \le 1$.

So we get:

$$\lambda(q_i')\int_{I\!R^N}|u_i^*|^2 \le m\int_{I\!R^N}|u_i^*|^2 + \sum_{j;j\ne i}|a_{ij}|^*(\int_{I\!R^N}|u_j^*|^2)^{\frac{1}{2}}(\int_{I\!R^N}|u_i^*|^2)^{\frac{1}{2}}.$$

Replacing u_i by u_i^*, we proceed exactly as in lemma 3.1 and we get that $\forall 1 \le i \le n$, $u_i^* = 0$.

viii) We prove now by contradiction that $\forall 1 \le i \le n$, $(u_{i,\epsilon})_\epsilon$ is bounded in $V_{q_i}(I\!R^N)$.

We suppose that: $\exists i_0$, $\|u_{i_0,\epsilon}\|_{q_{i_0}} \to +\infty$ when $\epsilon \to 0$. Let: $\forall 1 \le i \le n$,

$$t_\epsilon = \max_i(\|u_{i,\epsilon}\|_{q_i}) \text{ and } v_{i,\epsilon} = \frac{1}{t_\epsilon}u_{i,\epsilon}.$$

We have $\|v_{i,\epsilon}\|_{q_i} \le 1$ so $(v_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i}(I\!R^N)$.

Since the imbedding of $V_{q_i}(I\!R^N)$ in $L^2(I\!R^N)$ is compact (Proposition 2.3), there exists v_i such that $v_{i,\epsilon} \to v_i$ when $\epsilon \to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_{q_i}(I\!R^N)$.

In a weak sense, we have: $\forall 1 \le i \le n$,

$$(-\Delta + q_i')v_{i,\epsilon} = m\frac{v_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon} + \sum_{j;j\ne i} a_{ij}\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon} + \frac{1}{t_\epsilon}f_i \text{ in } I\!R^N.$$

We have: $\forall \phi \in \mathcal{D}(I\!\!R^N)$,

$$\int_{I\!\!R^N} [\nabla v_{i,\epsilon}.\nabla \phi + q_i' v_{i,\epsilon}\phi] \to \int_{I\!\!R^N} [\nabla v_i.\nabla \phi + q_i' v_i \phi] \text{ when } \epsilon \to 0.$$

Moreover $t_\epsilon \to +\infty$ when $\epsilon \to 0$ so:
$\forall \phi \in \mathcal{D}(I\!\!R^N),\ \int_{I\!\!R^N} \frac{1}{t_\epsilon} f_i \phi \to 0$ when $\epsilon \to 0$.
We have also: $\forall 1 \le j \le n$,

$$\| \frac{v_{j,\epsilon}}{1+\epsilon |u_{j,\epsilon}|} 1_{B_\epsilon} - v_j \|^2_{L^2(I\!\!R^N)} = \int_{B_\epsilon} [\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j]^2 + \int_{I\!\!R^N - B_\epsilon} v_j^2 .$$

But: $v_j \in L^2(I\!\!R^N)$ so : $\int_{I\!\!R^N - B_\epsilon} v_j^2 \to 0$ when $\epsilon \to 0$.
Moreover:

$$\int_{B_\epsilon} [\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j]^2 \le \int_{I\!\!R^N} [\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j]^2$$

$$\le 2[\int_{I\!\!R^N} \frac{(v_{j,\epsilon}-v_j)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} + \int_{I\!\!R^N} \frac{(\epsilon v_j |u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2}] .$$

But: $1+\epsilon|u_{j,\epsilon}| \ge 1$. So: $\int_{I\!\!R^N} \frac{(v_{j,\epsilon}-v_j)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \le \int_{I\!\!R^N} (v_{j,\epsilon} - v_j)^2$.
Since $v_{j,\epsilon} \to v_j$ in $L^2(I\!\!R^N)$,we get: $\int_{I\!\!R^N} \frac{(v_{j,\epsilon}-v_j)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
Moreover:

$$\frac{(\epsilon v_j |u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \to 0 \ pp \text{ when } \epsilon \to 0$$

(at least for a subsequence because $\epsilon u_{j,\epsilon} \to 0$ when $\epsilon \to 0$.)
By using the Dominated Convergence Theorem, we deduce that:
$\int_{I\!\!R^N} \frac{(\epsilon v_j|u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
So we can pass through the limit and we get: $\forall 1 \le i \le n$,

$$(-\Delta + q_i')v_i = m v_i + \sum_{j;j\neq i} a_{ij} v_j \text{ in } I\!\!R^N.$$

By the precedent lemma, we deduce that: $\forall 1 \le i \le n,\ v_i = 0$.
However there exists a sequence (ϵ_n) such that $\exists i_1,\ \|v_{i_1,\epsilon_n}\|_{q_{i_1}} = 1$.
But $v_{i_1,\epsilon_n} \to v_{i_1}$ when $n \to +\infty$. So we get a contradiction.

ix) There exists u_i^0 such that :
$u_{i,\epsilon} \to u_i^0$ strongly in $L^2(I\!\!R^N)$ and weakly in $V_{q_i}(I\!\!R^N)$.
We have in a weak sense:

$$(-\Delta + q_i')u_{i,\epsilon} = m\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} 1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij} \frac{u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} 1_{B_\epsilon} + f_i \text{ in } I\!\!R^N.$$

But $u_{i,\epsilon} \rightharpoonup u_i^0$ when $\epsilon \to 0$ weakly in $V_{q_i}(I\!\!R^N)$. Hence $\forall \phi \in \mathcal{D}(I\!\!R^N)$,

$$\int_{I\!\!R^N} [\nabla u_{i,\epsilon}.\nabla\phi + q_i' u_{i,\epsilon}\phi] \to \int_{I\!\!R^N} [\nabla u_i^0.\nabla\phi + q_i' u_i^0\phi] \text{ when } \epsilon \to 0.$$

We have also:

$$\|\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon} - u_i^0\|^2_{L^2(I\!\!R^N)} = \int_{B_\epsilon}[\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0]^2 + \int_{I\!\!R^N - B_\epsilon} |u_i^0|^2 .$$

By $u_i^0 \in L^2(I\!\!R^N)$ we derive: $\int_{I\!\!R^N - B_\epsilon} |u_i^0|^2 \to 0$ when $\epsilon \to 0$.
Moreover:

$$\int_{B_\epsilon}[\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0]^2 \le \int_{I\!\!R^N}[\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0]^2$$
$$\le 2[\int_{I\!\!R^N} \frac{(u_{i,\epsilon} - u_i^0)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} + \int_{I\!\!R^N} \frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2}].$$

Since $1+\epsilon|u_{i,\epsilon}| \ge 1$ we get: $\int_{I\!\!R^N} \frac{(u_{i,\epsilon}-u_i^0)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \le \int_{I\!\!R^N} (u_{i,\epsilon} - u_i^0)^2$.
But: $u_{i,\epsilon} \to u_i^0$ in $L^2(I\!\!R^N)$. So: $\int_{I\!\!R^N} \frac{(u_{i,\epsilon}-u_i^0)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
Moreover:

$$\frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \to 0 \; pp \text{ when } \epsilon \to 0$$

(at least for a subsequence because $\epsilon u_{i,\epsilon} \to 0$ when $\epsilon \to 0$)
and $\frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \le |u_i^0|^2$ and $|u_i^0|^2 \in L^1(I\!\!R^N)$.
By using the Dominated Convergence Theorem, we deduce that:
$\int_{I\!\!R^N} \frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
So we can pass through the limit and we get: $\forall 1 \le i \le n$,

$$(-\Delta + q_i')u_i^0 = m u_i^0 + \sum_{j;j\ne i} a_{ij} u_j^0 + f_i \text{ in } I\!\!R^N.$$

So we get: $(-\Delta + q_i)u_i^0 = a_{ii}u_i^0 + \sum_{j;j\ne i} a_{ij}u_j^0 + f_i$ in $I\!\!R^N$.
$(u_1^0, ..., u_n^0)$ is a weak solution of (1).

3.2 Study of a limit case

We use again a method in [9]. We rewrite System (1), assuming $\forall 1 \le i \le n,\ q_i = q$:

$$(1)\begin{cases} \text{for } 1 \le i \le n, \\ (1i) \quad L_q u_i := (-\Delta + q)u_i = \sum_{j=1}^n a_{ij}u_j + f_i(x, u_1, ..., u_n) \text{ in } I\!\!R^N . \end{cases}$$

Each a_{ij} is a real constant.
We denote $A = (a_{ij})$ the $n \times n$ matrix, I the $n \times n$ identity matrix, ${}^tU = (u_1...u_n)$ and ${}^tF = (f_1...f_n)$.

Theorem 3.2 *We suppose that $(H1),(H2)$ and $(H3)$ are satisfied.*
We suppose that A has only real eigenvalues. We suppose also that $\lambda(q)$, the principal eigenvalue of $-\Delta+q$, is the largest eigenvalue of A and that it is simple.
Let $X\in I\!R^N$ such that: ${}^tX(\lambda(q)I-A)=0$.
Then System (1) has a solution if and only if $\int_{I\!R^N} {}^tXF\phi_q=0$, where ϕ_q is the eigenfunction associated to $\lambda(q)$.

Proof of Theorem 3.2: Let P be a $n\times n$ non singular matrix such that the last line of P is ${}^tX$ and such that $T=PAP^{-1}:=(t_{ij})$ where:
$t_{ij}=0$ if $i>j$; $t_{nn}=\lambda(q)$ and $\forall 1\le i\le n-1,\ t_{ii}<\lambda(q)$.
Let: $W=PU$.
System (1) is equivalent to System (2) : $\quad (-\Delta+q)W=TW+PF$.
Let: ${}^tW=(w_1...w_n)$ and $\pi_i=(\delta_{ij})$ where: $\delta_{ij}=0$ if $i\neq j$ and $\delta_{ii}=1$.
So System (2) is:

$$(2)\begin{cases}\text{for } 1\le i\le n,\\ (2i)\quad L_qw_i:=(-\Delta+q)w_i=t_{ii}w_i+\sum_{j;j>i}t_{ij}w_j+\pi_iPF \text{ in } I\!R^N.\end{cases}$$

We have: $(2n)\quad (-\Delta+q)w_n=\lambda(q)w_n+^t XF$ in $I\!R^N$. Equation $(2n)$ has a solution if and only if $\int_{I\!R^N} {}^tXF\phi_q=0$.
If $\int_{I\!R^N} {}^tXF\phi_q=0$ is satisfied, first we solve $(2n)$, then we solve $(2.n-1)$ until (2.1) because $\forall 1\le i\le n-1,\ t_{ii}<\lambda(q)$. Then we deduce U (because Matrix P is a non singular matrix).

3.3 Study of a non necessarily cooperative semilinear system of n equations

We rewrite System (1) :

$$(1)\begin{cases}\text{for } 1\le i\le n,\\ (1i)\quad L_{q_i}u_i:=(-\Delta+q_i)u_i=\sum_{j=1}^n a_{ij}u_j+f_i(x,u_1,...,u_n) \text{ in } I\!R^N.\end{cases}$$

We recall $G=(g_{ij})$ the $n\times n$ matrix defined by:
$\forall 1\le i\le n,\ g_{ii}=\lambda(q_i-a_{ii})$ and

$$\forall 1\le i,j\le n,\ i\neq j\Rightarrow g_{ij}=-|a_{ij}|^* \text{ where } |a_{ij}|^*=\sup_{x\in I\!R^N}|a_{ij}(x)|.$$

Let I be the identity matrix.

Theorem 3.3 *We assume that $(H1),(H2)$ and $(H3)$ are satisfied.*
We assume also that Hypothesis $(H4),(H5),(H6)$ are satisfied where

(H4) $\exists s>0$ *such that $F-sI$ is a non singular M-matrix.*

(H5) $\forall 1 \le i \le n,\ \exists \theta_i \in L^2(\mathbb{R}^N), \theta_i > 0,$ *such that :*
$\forall 1 \le i \le n,\ \forall u_1, ..., u_n,\ 0 \le f_i(x, u_1, ..., u_n) \le su_i + \theta_i;$

(H6) $\forall 1 \le i \le n,\ f_i$ *is Lipschitz for* $(u_1, ..., u_n)$*, uniformly in* x*.*

Then System (1) *has at least a solution.*

Proof of Thorem 3.3:

a) Construction of an upper and lower solution.

We consider the following system (S):

$$(S)\begin{cases} \forall 1 \le i \le n\,, \\ L_{q_i}u_i := (-\Delta + q_i)u_i = a_{ii}u_i + \sum_{j;j\neq i} |a_{ij}|u_j + su_i + \theta_i \text{ in } \mathbb{R}^N\,. \end{cases}$$

By Hypothesis $(H4), (H5)$ we can apply Theorem 2.3.

We deduce the existence of a positive solution $U^0 = (u_1^0, ..., u_n^0)$ in $V_{q_1}(\mathbb{R}^N) \times ... \times V_{q_n}(\mathbb{R}^N)$ for the system (S). U^0 is an upper solution of System (1).

Let: $U_0 = -U^0 = (-u_1^0, ..., -u_n^0)$.

We have: $\forall 1 \le i \le n,\ (-\Delta + q_i)(-u_i^0) = -(-\Delta + q_i)u_i^0$.

Hence: $(-\Delta + q_i)(-u_i^0) = -a_{ii}u_i^0 - \sum_{j;j\neq i} |a_{ij}|u_j^0 - su_i^0 - \theta_i$.

So: $\forall 1 \le i \le n$,

$$(-\Delta + q_i)(-u_i^0) \le a_{ii}(-u_i^0) + \sum_{j;j\neq i} a_{ij}(-u_j^0) + f_i(x, -u_1^0, ..., -u_n^0).$$

Therefore U_0 is a lower solution of System (1).

b) Definition of a compact operator.

Let $m \in \mathbb{R}^{*+}$ be such that : $\forall 1 \le i \le n,\ m - a_{ii} > 0$.

Let: $q_i' = q_i - a_{ii} + m$.

Let $T : (L^2(\mathbb{R}^N))^n \to (L^2(\mathbb{R}^N))^n$ defined by: $T(u_1, ..., u_n) = (w_1, ..., w_n)$ such that:

$$(S')\begin{cases} \forall 1 \le i \le n\,, \\ (S'i) \quad (-\Delta + q_i')w_i = mu_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, ..., u_n) \text{ in } \mathbb{R}^N\,. \end{cases}$$

We prove easily that T is a well defined operator by the scalar case, continuous by $(H6)$ and compact (because $(-\Delta + q_i')^{-1}$ is compact).

We prove now that $T([U_0, U^0]) \subset [U_0, U^0]$.

Let $U = (u_1, ..., u_n) \in [U_0, U^0]$.

We have: $\forall 1 \le i \le n,\ -u_i^0 \le u_i \le u_i^0$.

We have: $(-\Delta + q_i')(u_i^0 - w_i) = m(u_i^0 - u_i) + \sum_{j;j\neq i} |a_{ij}|u_j^0$

$-\sum_{j;j\neq i} a_{ij}u_j + su_i^0 + \theta_i - f_i(x,u_1,...,u_n)$.
We have: $m(u_i^0 - u_i) \geq 0$.
By $(H5)$, we have: $f_i(x,u_1,...,u_n) \leq su_i + \theta_i \leq su_i^0 + \theta_i$.
Moreover: $|a_{ij}u_j| \leq |a_{ij}|u_j^0$ so $a_{ij}u_j \leq |a_{ij}|u_j^0$.
So: $(-\Delta + q_i')(u_i^0 - w_i) \geq 0$ and by the scalar case: $u_i^0 - w_i \geq 0$.
In the same way, we have:
$(-\Delta + q_i')(w_i - (-u_i^0)) = m(u_i^0 + u_i) + \sum_{j;j\neq i} |a_{ij}|u_j^0$
$+\sum_{j;j\neq i} a_{ij}u_j + su_i^0 + \theta_i + f_i(x,u_1,...,u_n)$.
But $-u_i^0 \leq u_i$. So $m(u_i^0 + u_i) \geq 0$. Moreover: $-a_{ij}u_j \leq |a_{ij}|u_j^0$.
By using $(H5)$, we conclude that: $(-\Delta + q_i')(w_i + u_i^0) \geq 0$ and hence: $w_i \geq -u_i^0$. So: $T([U_0,U^0]) \subset [U_0,U^0]$.
$[U_0,U^0]$ is a convex, closed, and bounded subset of $(L^2(\mathbb{R}^N))^n$, so by the Schauder Fixed Point Theorem, we deduce that T has a fixed point.
Therefore System (1) has at least a solution.

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Author:

Laure Cardoulis
CEREMATH
Université des Sciences Sociales
21 Allé de Brienne
31042 Toulouse Cedex
France

e-mail: cardouli@math.univ-tlse1.fr

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Lothar Berg

On the Solution of Jordan's System of Difference Equations

ABSTRACT. By means of generalized Bernoulli numbers an explicit solution is given for the homogeneous system of difference equations in Jordan's normal form.

KEY WORDS. Difference equations, Jordan's normal form, Bernoulli numbers.

1 Introduction

In U. Krause and T. Nesemann [1], Lemma 3.9, there was determined the general solution of the system

$$x_i(t+1) = \lambda x_i(t) + x_{i+1}(t) \tag{1.1}$$

for $1 \leq i \leq n$, $t \in \mathbb{N}$ and $x_{n+1}(t) \equiv 0$. For $\lambda \neq 0$ it reads

$$x_i(t) = \lambda^{t+i-1} \Delta^{i-1} p(t)\,, \tag{1.2}$$

where p is an arbitrary polynomial of degree $\leq n-1$ and $\Delta p(t) = p(t+1) - p(t)$. The system (1.1) corresponds to a homogeneous matrix difference equation in Jordan's normal form with upper unities. After simple modifications, the result can easily be transferred to the system

$$y_i(t+1) = \lambda y_i(t) + y_{i-1}(t) \tag{1.3}$$

for $0 \leq i \leq n$ with $y_{-1}(t) \equiv 0$, where the solution turns over into

$$y_i(t) = \lambda^{t-i} \Delta^{n-i} p(t) \tag{1.4}$$

with an arbitrary polynomial p of degree $\leq n$. The system (1.3) corresponds to Jordan's normal form with lower unities.

The solution (1.4) has the disadvantage that its shape depends on n. In the sequel we derive a representation for the solution of (1.3) which is independent of n and in which, surprisingly, the Bernoulli numbers appear. The result can be transferred to inhomogeneous equations.

Let us mention that (1.2) and (1.4) are also the general solutions of (1.1) and (1.3), respectively, in the case $t \in \mathbb{R}$, if p denotes a polynomial in t with arbitrary 1-periodic coefficients. An analogous remark comes true for the latter solutions.

2 Solution of (1.3)

In order to solve (1.3) with $y_{-1}(t) \equiv 0$ and $\lambda \neq 0$ for $i = 0, 1, 2, \ldots$ we make the ansatz

$$y_i(t) = \lambda^{t-i} \sum_{j=0}^{i} \frac{1}{j!} a_{i,i-j} t^j \,. \tag{2.1}$$

The homogeneous equation, corresponding to (1.3) with fixed i, has the general solution $y_i = c_i \lambda^i$ with arbitrary c_i so that in (1.3) a_{ii} must remain arbitrarily. Comparing

$$y_i(t+1) - \lambda y_i(t) = \lambda^{t+1-i} \sum_{j=1}^{i} \frac{1}{j!} a_{i,i-j} \sum_{k=0}^{j-1} \binom{j}{k} t^k$$

with (2.1) for $i-1$ instead of i, equation (1.3) implies

$$\sum_{j=k+1}^{i} \frac{1}{(j-k)!} a_{i,i-j} = a_{i-1,i-1-k}$$

for $k = 0, 1, \ldots, i-1$ and therefore

$$\sum_{j=0}^{k} \frac{1}{(i+1-j)!} a_{ij} = a_{i-1,k} \tag{2.2}$$

for the same k. Defining auxiliary coefficients a_{ij} also for $j > i$, we introduce the generating functions

$$f_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j \tag{2.3}$$

as formal power series. Choosing the new a_{ij} for $i = 0$ arbitrarily and for $i \in \mathbb{N}$ in a suitable way, the equations (2.2) are equivalent to the recursions

$$\frac{e^z - 1}{z} f_i(z) = f_{i-1}(z)$$

with the solution

$$f_i(z) = \left(\frac{z}{e^z - 1}\right)^i f_0(z)\,. \tag{2.4}$$

Now, we define coefficients b_{ij} by

$$\left(\frac{z}{e^z - 1}\right)^i = \sum_{j=0}^{\infty} b_{ij} z^j \tag{2.5}$$

so that (2.3)-(2.4) imply the relations

$$a_{ij} = \sum_{k=0}^{j} b_{ik}\, a_{0,j-k}\,, \tag{2.6}$$

which, in view of (2.2), are valid for $0 \le j \le i-1$. But we can use (2.6) also for $j = i$, if we take a_{0i} as arbitrary constants instead of a_{ii}. The foregoing considerations can be summarized as follows:

Proposition *For $\lambda \neq 0$ the equations (1.3) with $i \in \mathbb{N}_0$ and $y_{-1}(t) \equiv 0$ have the general solution (2.1) with (2.6), the coefficients b_{ik} defined by (2.5), and arbitrary constants a_{0j}.*

Usually, the coefficients in (2.5) are written in the form

$$b_{ij} = \frac{1}{j!} B_j^{(i)}\,,$$

where $B_j^{(i)}$ are the generalized Bernoulli numbers, cf. [2], p. 145 or [3], p. 4. The generalized Bernoulli numbers are polynomials in i of degree j. For $0 \le j \le 12$ they are listed in [2], p. 459, from which we obtain in particular

$$b_{i0} = 1\,, \quad b_{i1} = -\frac{i}{2}\,, \quad b_{i2} = \frac{i}{24}(3i-1)\,, \quad b_{i3} = -\frac{i^2}{48}(i-1)\,.$$

These results can be checked by means of the recursions

$$b_{i+1,j} = \left(1 - \frac{j}{i}\right) b_{ij} - b_{i,j-1} \qquad (i,j \in \mathbb{N})\,,$$

cf. [2], p. 145, and the initial values $b_{i0} = 1$, $b_{1j} = \frac{1}{j!} B_j$ for $i,j \in \mathbb{N}_0$, where B_j are the ordinary Bernoulli numbers.

3 The inhomogeneous case

The foregoing considerations also allow to solve the inhomogeneous equation

$$y(t+1) = \lambda y(t) + t^{i-1} \lambda^t\,, \tag{3.1}$$

$i \in \mathbb{N}$, if we interprete (3.1) as equation (1.3) with $y(t) = y_i(t)$ and $t^{i-1}\lambda^t = y_{i-1}(t)$ with (2.1), i.e. in particular $a_{i-1,0} = (i-1)!\lambda^{i-1}$ and $a_{i-1,j} = 0$ elsewhere. Now, instead of (2.4) we only need the relation

$$f_i(z) = \frac{z}{e^z - 1} f_{i-1}(z)\,,$$

and since $f_{i-1}(z) = (i-1)!\lambda^{i-1}$, we obtain

$$a_{ij} = (i-1)!\lambda^{i-1}\frac{1}{j!}B_j \tag{3.2}$$

for $j = 0, 1, \ldots, i-1$, whereas a_{ii} is an arbitrary constant. Hence, we have found the solution $y(t) = y_i(t)$ of (3.1) with (2.1) and (3.2).

The solution of

$$y(t+1) = \lambda y(t) + p(t)\lambda^t\,, \tag{3.3}$$

where p is a polynomial of degree $i-1$, can be reduced to the solution of (3.1) by means of linear combinations. Successive application of the solution of (3.3) for $i = 1, 2, 3, \ldots$ again leads to the solution of the system (1.3) constructed in the foregoing preposition.

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received: March 27, 2001 **revised:** April 20, 2001

Author:

Prof. i.R. Dr. Lothar Berg
Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany

e-mail: lothar.berg@mathematik.uni-rostock.de

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Laure Cardoulis

Existence of solutions for some semilinear elliptic systems

ABSTRACT. We obtain results on the existence of solutions for the semilinear elliptic equation $(-\Delta + q)u = \lambda\rho u + f(x,u)$ in $I\!R^N$ under the hypothesis that $N \geq 3$, $q(x) \to +\infty$ as $|x| \to +\infty$ and $\rho \in L^{\frac{N}{2}}(I\!R^N)$. Similar results for semilinear systems of n equations are also established.

1 Introduction

In the present paper we will study the maximum principle and existence of solutions for the following elliptic equation:

$$(E_q) \quad L_q u := (-\Delta + q)u = \lambda\rho u + f(x,u) \text{ in } I\!R^N,$$

where $0 \leq \rho \in L^{\frac{N}{2}}(I\!R^N)$, $\lambda \in I\!R$, and $q \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R^N)$ is such that $q \geq constant\ c > 0$ and $q(x) \to +\infty$ as $|x| \to +\infty$. Such topic has been studied already in [8] for the case that $\rho \in L^\infty(I\!R^N)$ using the first eigenvalue $\lambda(q)$ of the operator $-\Delta + q$ in $L^2(I\!R^N)$. Here we obtain results on existence of solutions to (E_q) in terms of the first eigenvalue $\lambda(\rho)$ of the following problem studied in [6]:

$$(E_\rho) \quad \begin{cases} -\Delta u = \lambda\rho u \text{ in } I\!R^N \\ u(x) \to 0 \text{ as } |x| \to +\infty \end{cases}$$

We study also the following elliptic systems on $I\!R^N$:

$$L_{q_i} u_i := (-\Delta + q_i)u_i = \lambda_i\rho_i + \sum_{j=1; j\neq i}^{n} a_{ij}u_j + f_i \text{ in } I\!R^N \quad (1 \leq i \leq n).$$

Under mild assumptions we have obtained the existence of solutions to the above systems.

2 Recalls

2.1 About one equation

Let $\mathcal{D}(I\!R^N) = \mathcal{C}_0^\infty(I\!R^N) = \mathcal{C}_c^\infty(I\!R^N)$ be the set of $\mathcal{C}^\infty$ functions on $I\!R^N$ with compact support. Let $3 \leq N \in I\!N; 0 \leq \rho \in L^{\frac{N}{2}}(I\!R^N), \rho \neq 0$. Let $D^{1,2}$ be the completion of $\mathcal{D}(I\!R^N)$ under the norm $[\int_{I\!R^N} |\nabla u|^2]^{\frac{1}{2}}$.

We recall from [6] that the following equation

$$(E_\rho) \quad \begin{cases} -\Delta u = \lambda \rho u \text{ in } I\!R^N \\ u(x) \to 0 \text{ as } |x| \to +\infty \end{cases}$$

admits a simple and positive eigenvalue $\lambda(\rho)$, called the principal eigenvalue, associated with a positive eigenfunction ψ_ρ, such that

$$\lambda(\rho) \cdot \int_{I\!R^N} \rho u^2 \leq \int_{I\!R^N} |\nabla u|^2 \quad (\forall u \in D^{1,2}).$$

2.2 Schrödinger operators

Let $q \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R^N)$ be such that: $q \geq$ *constant* $c > 0$ and $q(x) \to +\infty$ as $|x| \to +\infty$, cf. [2, 3, p.3, 44, 68] and [15, Theorem XIII.47, p.207]. The variational space is the Hilbert space $V_q(I\!R^N)$ which is the completion of $\mathcal{D}(I\!R^N)$ under the norm $\|u\|_q = [\int_{I\!R^N} |\nabla u|^2 + q|u|^2]^{\frac{1}{2}}$.

Proposition 2.1 (see [1, Prop.I.1.1])
The embedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact and has dense range.

Proposition 2.2 (see e.g. [1, p.25 to 27])
$-\Delta + q$, considered as an operator in $L^2(I\!R^N)$, is positive, selfadjoint, with compact inverse. Its spectrum is discrete and consists of an infinite sequence of positive eigenvalues tending to $+\infty$. The smallest eigenvalue, called the principal eigenvalue and denoted by $\lambda(q)$, is simple and has a nonnegative eigenfunction ϕ_q. Moreover, there holds: $\forall u \in V_q(I\!R^N), \lambda(q) \int_{I\!R^N} |u|^2 \leq \int_{I\!R^N} [|\nabla u|^2 + q|u|^2]$.

2.3 M-matrix

We say that a matrix is *positive* if all of its entries are positive.

Definition 2.1 *A matrix $M = sI - B$ is called a non singular M-matrix if B is a positive matrix and $s > \rho(B) > 0$, where $\rho(B)$ denotes the spectral radius of B.*

Proposition 2.3 (see [4, Theorem 2.3, p.134])
If M is a matrix with nonpositive off-diagonal entries, then the following conditions $(P0)$, $(P1)$, $(P2)$, $(P3)$, $(P4)$ are equivalents:

$(P0)$ *M is a non singular M-matrix.*
$(P1)$ *All the principal minors of M are strictly positives.*
$(P2)$ *M is semi-positive i.e.: $\exists X >> 0$ such that $MX >> 0$.*
Here $X >> 0$ means that the entries of X are strictly positive.
$(P3)$ *M has a positive inverse.*
$(P4)$ *There exists a diagonale matrix D, $D > 0$, such that $MD + D^tM$ is positive definite.*

3 The scalar case

3.1 Case of a linear equation

We consider the following elliptic equation:

$$(E_q) \quad L_q u := (-\Delta + q)u = \lambda \rho u + f \text{ in } I\!R^N$$

where $N \geq 3$, $f \in L^2(I\!R^N)$, $0 \leq \rho \in L^{\frac{N}{2}}(I\!R^N), \rho \neq 0$, $\lambda \in I\!R$ and $q \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R^N)$ is such that $q \geq$ *constant* $c > 0$ and $q(x) \to +\infty$ as $|x| \to +\infty$. We say that u is a *weak solution* of (E_q) if $u \in V_q(I\!R^N)$ and

$$\int_{I\!R^N} (\nabla u.\nabla \phi + qu\phi) = \lambda \int_{I\!R^N} \rho u\phi + \int_{I\!R^N} f\phi \quad (\forall \phi \in D(I\!R^N)).$$

Theorem 3.1 *Assume $\lambda < \lambda(\rho)$. Then (E_q) admits a unique weak solution $u \in V_q(I\!R^N)$. Moreover, if $f \geq 0$, then $u \geq 0$.*

The proof of Theorem 3.1 is elmentary and thus the details are omitted.

By using a theorem obtained in [8] we can derive the following result for the asymptotic behaviour of the solutions.

Theorem 3.2 *Let u be a weak solution of (E_q). Then there holds the convergence that $\lim_{|x|\to\infty} u(x) = 0$ under each of the following conditions* **a)-b)***:*

a) *$N = 3$.*

b) *$N \geq 3$ and either $|f| \in H^m(I\!R^N)$ with $m > \frac{N}{2} - 2$ or $f \in L^{\frac{d}{2}}(I\!R^N)$ with $d > N$.*

3.2 Case of a semilinear equation

Of concern is the following semilinear elliptic equation

$$(E_q) \quad L_q u(x) := (-\Delta + q(x))u(x) = \lambda\rho(x)u(x) + f(x,u(x)) \text{ for } x \in I\!R^N.$$

In [8], in case the coefficient ρ is bounded, we have established already the existence of solutions for (E_q) by the sub and supper solutions method combining with the Schauder fixed point Theorem. Here we are interested in the more general case $\rho \in L^{\frac{N}{2}}(I\!R^N)$ to which the above method fails and thus we have to use the approximation method due to L. Boccardo, J. Fleckinger and F. de Thlin [5].

Theorem 3.3 *Let the following conditions $(h1)-(h4)$ be satisfied:*
$(h1)$ $N \geq 3$; $0 \leq \rho \in L^{\frac{N}{2}}(I\!R^N) \cap L^\infty_{loc}(I\!R^N), \rho \neq 0$; $\lambda \in I\!R$; *and* $q \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R^N)$ *is such that* $q \geq$ *constant* $c > 0$ *and* $q(x) \to +\infty$ *as* $|x| \to \infty$.
$(h2)$ $\exists \theta \in L^2(I\!R^N), \forall u \in L^2(I\!R^N), |f(x,u)| \leq \theta$.
$(h3)$ f *is Lipschitz respect to* u*, uniformly in* x.
$(h4)$ $\lambda < \lambda(\rho)$.
Then (E_q) *has a weak solution.*

Proof: Let $\epsilon \in]0,1[$ and $B_\epsilon = B(0,\frac{1}{\epsilon}) = \{x \in I\!R^N,\ |x| < \frac{1}{\epsilon}\}$. Let 1_{B_ϵ} be the indicator function of B_ϵ. Let $m \in I\!R^{*+}$ be such that $\lambda + m > 0$.
Let $T_\epsilon : L^2(I\!R^N) \to L^2(I\!R^N)$ be defined by $T_\epsilon(u) = v$, where u, v are determined as follows:

$$(E1) \quad (-\Delta + q + m\rho)v = (\lambda + m)\frac{\rho u}{1 + \epsilon\rho|u|}1_{B_\epsilon} + f(x,u) \text{ in } I\!R^N.$$

i) First we prove that T_ϵ is well defined.
By $(h2)$ and the scalar case, since $-m < 0 < \lambda(\rho)$, we deduce the existence (and uniqueness) of the weak solution $v \in V_q(I\!R^N)$ of $(E1)$.

ii) Construction of a sub and supper solution for $(E1)$.
We have: $(\lambda + m)\frac{1}{\epsilon}1_{B_\epsilon} + \theta \in L^2(I\!R^N)$ so by the scalar case, we deduce that: $\exists! \xi^0_\epsilon \in V_q(I\!R^N)$, $\xi^0_\epsilon \geq 0$ such that: $(-\Delta + q + m\rho)\xi^0_\epsilon = (\lambda + m)\frac{1}{\epsilon}1_{B_\epsilon} + \theta$ in $I\!R^N$. ξ^0_ϵ is a suppersolution of $(E1)$. By the same way, $\xi_{\epsilon,0} = -\xi^0_\epsilon$ is a subsolution of $(E1)$. Note that $T_\epsilon(\sigma_\epsilon) \subset \sigma_\epsilon$ where $\sigma_\epsilon = [\xi_{\epsilon,0}, \xi^0_\epsilon]$.

iii) We prove that T_ϵ is continuous.
Let $(u_n)_{n \in I\!N}$ be a sequence in $\sigma_\epsilon = [\xi_{\epsilon,0}, \xi^0_\epsilon]$ converging to u in $\|.\|_{L^2(I\!R^N)}$. Set $T_\epsilon(u_n) = v_n$ $(n \in I\!N)$ and $T_\epsilon(u) = v$. We have in a weak sense:
$(-\Delta + q + m\rho)(v_n - v) = (\lambda + m)[\frac{\rho u_n}{1+\epsilon\rho|u_n|} - \frac{\rho u}{1+\epsilon\rho|u|}]1_{B_\epsilon} + f(x,u_n) - f(x,u)$.
Multiplying by $v_n - v$, integrating over $I\!R^N$ and using the Cauchy-Schwartz inequality,

we get:
$\|v_n - v\|^2_{q+m\rho} \le (\lambda+m)[\int_{B_\epsilon}(\frac{\rho u_n}{1+\epsilon\rho|u_n|} - \frac{\rho u}{1+\epsilon\rho|u|})^2]^{\frac{1}{2}}\|v_n - v\|_{L^2(I\!R^N)}$
$+\|f(x,u_n) - f(x,u)\|_{L^2(I\!R^N)}\|v_n - v\|_{L^2(I\!R^N)}$.
But the function l on $I\!R$ defined by $l(x) := \frac{x}{1+|x|}$ $(\forall x \in I\!R)$ is Lipschitz and satisfies: $\forall x, y \in I\!R,\ |l(x) - l(y)| \le |x-y|$. So, by $(h3)$, we deduce that
$\|v_n - v\|_{q+m\rho} \le [(\lambda+m)[\int_{B_\epsilon}\rho^2(u_n-u)^2]^{\frac{1}{2}} + k\|u_n - u\|_{L^2(I\!R^N)}]\|v_n - v\|_{L^2(I\!R^N)}$ for some positive constant k. Since $\rho \in L^\infty_{loc}(I\!R^N)$ we find that $\|v_n - v\|_{q+m\rho} \le K(\epsilon)\|u_n - u\|_{L^2(I\!R^N)}$ for some positive constant $K(\epsilon)$. Hence $(v_n)_n$ is a sequence converging to v in $\|.\|_{L^2(I\!R^N)}$.

iv) Let $(v_n)_n$ be a sequence such that: $T_\epsilon(u_n) = v_n$.
We want to prove that there exists a subsequence of (v_n) converging in $L^2(I\!R^N)$. We have: $\forall n,\ (-\Delta + q + m\rho)v_n = (\lambda+m)\frac{\rho u_n}{1+\epsilon\rho|u_n|}1_{B_\epsilon} + f(x,u_n)$ in $I\!R^N$.
By the Cauchy-Schwartz inequality and by $(h2)$, we obtain:
$\|v_n\|^2_{q+m\rho} \le (\lambda+m)\frac{1}{\epsilon}[\int_{B_\epsilon}1]^{\frac{1}{2}}\|v_n\|_{L^2(B_\epsilon)} + \|\theta\|_{L^2(I\!R^N)}\|v_n\|_{L^2(I\!R^N)}$.
So the sequence (v_n) is bounded in $V_q(I\!R^N)$. Since the embedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact, we can find a subsequence of (v_n) which is convergent in $L^2(I\!R^N)$. Therefore $T_\epsilon(\sigma_\epsilon)$ is compact.

v) By the Schauder Fixed Point Theorem, we deduce the existence of $u_\epsilon \in \sigma_\epsilon$ such that: $T_\epsilon(u_\epsilon) = u_\epsilon$. Moreover, there holds

$$(-\Delta + q + m\rho)u_\epsilon = (\lambda+m)\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} + f(x,u_\epsilon) \text{ in } I\!R^N.$$

vi) Now we prove that $(\epsilon u_\epsilon)_\epsilon$ is a bounded sequence in $V_q(I\!R^N)$.
Multiplying by $\epsilon^2 u_\epsilon$, integrating over $I\!R^N$, using the Cauchy-Schwartz inequality and since $\frac{1_{B_\epsilon}}{1+\epsilon\rho|u_\epsilon|} \le 1$ we deduce that

$$\|\epsilon u_\epsilon\|^2_{q+m\rho} \le (\lambda+m)\int_{I\!R^N}\rho\epsilon^2u_\epsilon^2 + \|\theta\|_{L^2(I\!R^N)}\|\epsilon u_\epsilon\|_{L^2(I\!R^N)}.$$

Therefore, $\|\epsilon u_\epsilon\|^2_{q+m\rho} \le \frac{\lambda+m}{\lambda(\rho)+m}\|\epsilon u_\epsilon\|^2_{q+m\rho} + \|\theta\|_{L^2(I\!R^N)}\|\epsilon u_\epsilon\|_{L^2(I\!R^N)}$.
It follows that $0 \le \frac{\lambda(\rho)-\lambda}{\lambda(\rho)+m}\|\epsilon u_\epsilon\|^2_{q+m\rho} \le \|\theta\|_{L^2(I\!R^N)}\|\epsilon u_\epsilon\|_{L^2(I\!R^N)}$. Hence: $\exists K > 0$, K independent of ϵ such that $\|\epsilon u_\epsilon\|_{q+m\rho} \le K$.

vii) We prove now that $\epsilon u_\epsilon \to 0$ as $\epsilon \to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$. We know that the imbedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact. The sequence $(\epsilon u_\epsilon)_\epsilon$ is bounded in $V_q(I\!R^N)$ so (for a subsequence), we deduce that $\exists u^*$ such that: $\epsilon u_\epsilon \to u^*$ as $\epsilon \to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$. Multiplying by ϵ, we get:
$(-\Delta + q + m\rho)\epsilon u_\epsilon = (\lambda+m)\frac{\rho\epsilon u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} + \epsilon f(x,u_\epsilon)$ in $I\!R^N$.

We have: $\forall \phi \in \mathcal{D}(I\!R^N),\ |\int_{I\!R^N} \epsilon f(x,u_\epsilon)\phi| \leq \epsilon \int_{I\!R^N} \theta|\phi|$.
We deduce that: $\int_{I\!R^N} \epsilon f(x,u_\epsilon)\phi \to 0$ as $\epsilon \to 0$.
We have also: $[\frac{\epsilon\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} - \frac{\rho u^*}{1+\rho|u^*|}]\phi \to 0$ as $\epsilon \to 0$.
Since $\frac{\epsilon\rho|u_\epsilon|}{1+\epsilon\rho|u_\epsilon|} \leq 1$ and $\frac{\rho|u^*|}{1+\rho|u^*|} \leq 1$ we get

$$|[\frac{\epsilon\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} - \frac{\rho u^*}{1+\rho|u^*|}]\phi| \leq 2|\phi| \in L^1(I\!R^N).$$

Using the Dominated Convergence Theorem, we deduce that

$$\int_{I\!R^N} \frac{\epsilon\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}\phi \to \int_{I\!R^N} \frac{\rho u^*}{1+\rho|u^*|}\phi \text{ as } \epsilon \to 0.$$

So u^* is a weak solution of the equation $(-\Delta + q + m\rho)u^* = (\lambda+m)\frac{\rho u^*}{1+\rho|u^*|}$ in $I\!R^N$.
We prove $u^* = 0$. Multiplying by u^* and integrating over $I\!R^N$ yields

$$\int_{I\!R^N} [|\nabla u^*|^2 + (q+m\rho)|u^*|^2] = (\lambda+m)\int_{I\!R^N} \frac{\rho|u^*|^2}{1+\rho|u^*|} \leq (\lambda+m)\int_{I\!R^N} \rho u^{*2}.$$

Since $\lambda(\rho)\int_{I\!R^N} \rho u^{*2} \leq \int_{I\!R^N} |\nabla u^*|^2$ we have $(\lambda(\rho)+m)\int_{I\!R^N} \rho u^{*2} \leq (\lambda+m)\int_{I\!R^N} \rho u^{*2}$.
But $\lambda(\rho) - \lambda > 0$, we find that $\int_{I\!R^N} \rho u^{*2} = 0$ and thus $u^* = 0$ *a.e.*

viii) We prove now by contradiction that $(u_\epsilon)_\epsilon$ is bounded in $V_q(I\!R^N)$.
We suppose that (for a subsequence): $\|u_\epsilon\|_{q+m\rho} \to +\infty$ as $\epsilon \to 0$. Let $z_\epsilon = \frac{1}{\|u_\epsilon\|_{q+m\rho}}u_\epsilon$. Then $\|z_\epsilon\|_{q+m\rho} = 1$ and thus $(z_\epsilon)_\epsilon$ is a bounded sequence in $V_q(I\!R^N)$. There exists z such that $z_\epsilon \to z$ as $\epsilon \to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$. Furthermore: $\exists h \in L^2(I\!R^N), \forall\epsilon,\ \ |z_\epsilon| \leq h$ *a.e* (for a subsequence) (see [7, p.58]) In a weak sense, we have:
$(-\Delta + q + m\rho)z_\epsilon = \frac{(\lambda+m)}{\|u_\epsilon\|_{q+m\rho}}\frac{\rho u\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} + \frac{1}{\|u_\epsilon\|_{q+m\rho}}f(x,u_\epsilon)$ in $I\!R^N$.
We have: $\forall \phi \in \mathcal{D}(I\!R^N),\ \int_{I\!R^N} \frac{1}{\|u_\epsilon\|_{q+m\rho}}f(x,u_\epsilon)\phi \to 0$ as $\epsilon \to 0$.
We have also: $[\frac{\rho z_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} - \rho z]\phi \to 0$ as $\epsilon \to 0$ *a.e.*
$|[\frac{\rho z_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} - \rho z]\phi| \leq |\rho(z_\epsilon 1_{B_\epsilon} - z)||\phi| + |\frac{\epsilon\rho|u_\epsilon|}{1+\epsilon\rho|u_\epsilon|}\rho z\phi| \leq |\rho(|z|+|h|)\phi| + |\rho z\phi| \in L^1(I\!R^N)$ since $\rho \in L^\infty_{loc}(I\!R^N)$. Using the Dominated Convergence Theorem, we deduce that $\int_{I\!R^N} \frac{\rho z_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}\phi \to \int_{I\!R^N} \rho z\phi$ as $\epsilon \to 0$. So z is a weak solution of the equation $(-\Delta + q)z = \lambda\rho z$ in $I\!R^N$. Since $\lambda < \lambda(\rho)$ we find $z = 0$. Using $\|z_\epsilon\|_{L^2(I\!R^N)} = 1 \to \|z\|_{L^2(I\!R^N)} = 0$ as $\epsilon \to 0$, we get a contradiction.

ix) There exists u^0 such that $u_\epsilon \to u^0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$.
We have in a weak sense:
$(-\Delta + q + m\rho)u_\epsilon = (\lambda+m)\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} + f(x,u_\epsilon)$ in $I\!R^N$.
But $u_\epsilon \rightharpoonup u^0$ as $\epsilon \to 0$ weakly in $V_q(I\!R^N)$. Furthermore: $\exists h' \in L^2(I\!R^N),\ \forall\epsilon,\ |u_\epsilon| \leq h'$ *a.e* (for a subsequence.) By $(h3)$ we have also:

$$\|f(x,u_\epsilon) - f(x,u^0)\|^2_{L^2(I\!R^N)} \leq const \cdot \|u_\epsilon - u^0\|^2_{L^2(I\!R^N)}.$$

Therefore: $\forall \phi \in \mathcal{D}(I\!R^N)$, $\int_{I\!R^N} f(x, u_\epsilon)\phi \to \int_{I\!R^N} f(x, u^0)\phi$ as $\epsilon \to 0$.
Moreover: $[\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|} 1_{B_\epsilon} - \rho u^0]\phi \to 0$ as $\epsilon \to 0$ *a.e.* and
$|[\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|} 1_{B_\epsilon} - \rho u^0]\phi| \leq \rho(|h'| + 2|u^0|)|\phi| \in L^1(I\!R^N)$.
Using the Dominated Convergence Theorem, we deduce that $\int_{I\!R^N} \frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|} 1_{B_\epsilon}\phi \to \int_{I\!R^N} \rho u^0 \phi$ as $\epsilon \to 0$. So u^0 is a weak solution of the equation $(-\Delta + q)u^0 = \lambda\rho u^0 + f(x, u^0)$ in $I\!R^N$.

4 Study of a system

The system which we will study has the form

$$L_{q_i} u_i := (-\Delta + q_i)u_i = \lambda_i \rho_i + \sum_{j=1; j\neq i}^{n} a_{ij} u_j + f_i \text{ in } I\!R^N \quad (1 \leq i \leq n). \tag{1}$$

We assume $N \geq 3$ and impose the following conditions (H0)-(H3) for each index i :

(H0) $\lambda_i \in I\!R$.

(H1) $0 \leq \rho_i \in L^{N/2}(I\!R^N)$, $\rho_i \not\equiv 0$.

(H2) $q_i \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R)$ is such that $q_i \geq c_i$ for some positive constant c_i and $q_i(x) \to +\infty$ as $|x| \to \infty$.

(H3) $f_i \in L^2(I\!R^N)$.

We say that $(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$ is a weak solution of System (1) if: $\forall 1 \leq i \leq n, \forall \phi \in \mathcal{D}(I\!R^N)$,
$\int_{I\!R^N} [\nabla u_i.\nabla\phi + q_i u_i \phi] = \lambda_i \int_{I\!R^N} \rho_i u_i \phi + \int_{I\!R^N} \sum_{j=1; j\neq i}^{n} a_{ij} u_j \phi + \int_{I\!R^N} f_i \phi$.

We call System (1) *cooperative* if $a_{ij} \geq 0$ for all $i \neq j$. We say that System (1) satisfies the maximum principle if: $\forall f_i \geq 0$, $1 \leq i \leq n$, each solution $u = (u_1, ..., u_n)$ of (1) is nonnegative.

4.1 Study of a linear system

Theorem 4.1 *Assume conditions $(H0) - (H3)$ and the following ones*
$(H4)$ $\forall i, j$ $i \neq j \Rightarrow 0 \leq a_{ij} \leq k_{ij}\sqrt{\rho_i}\sqrt{\rho_j}$ with $k_{ij} \in I\!R^+$.
Let $D = (d_{ij})$ be the $n \times n$ matrix given by $d_{ii} = \lambda(\rho_i) - \lambda_i$ and $d_{ij} = -k_{ij}$ otherwise. If D is a non singular M-matrix, then System (1) satisfies the maximum principle.

Proof: Assume that: $\forall 1 \leq i \leq n, f_i \geq 0$. Let $(u_1, ..., u_n)$ be a weak solution of (1) and $u_i^- = \max(0, -u_i)$. Using $(H4)$ and the Cauchy-Schwartz inequality, we obtain

$$(\lambda(\rho_i) - \lambda_i)(\int_{I\!R^N} \rho_i |u_i^-|^2)^{\frac{1}{2}} - \sum_{j;j\neq i} k_{ij}(\int_{I\!R^N} \rho_j |u_j^-|^2)^{\frac{1}{2}} \leq 0.$$

Let ${}^tX = (x_1, ..., x_n)$ be such that: $\forall 1 \leq i \leq n,\ x_i = (\int_{I\!R^N} \rho_i |u_i^-|^2)^{\frac{1}{2}}$. We have $DX \leq 0$. Since D is a non singular M-matrix, by $(P3)$, D has a positive inverse. Hence, $X \leq 0$. It follows that $u_i^- \equiv 0$ for all i.

Theorem 4.2 *Assume conditions $(H0) - (H3)$ and the following ones*
$(H4*)$ $\forall i, j\ i \neq j \Rightarrow |a_{ij}| \leq k_{ij}\sqrt{\rho_i}\sqrt{\rho_j}$ *with* $k_{ij} \in I\!R^+$.
If D (given in Theorem 4.1) is a non singular M-matrix, then System (1) has a unique weak solution $(u_1, ..., u_n)$. Furthermore if System (1) is cooperative and if $f_i \geq 0$ for all i, then, by the maximum principle, we have that $u_i \geq 0$ for all i.

Proof: By $(P4)$, there exists a diagonale matrix E such that ${}^tDE + ED$ is positive definite, with: $E = (e_{ij})$ where $\forall 1 \leq i \leq n,\ e_{ii} = e_i > 0$. Let $\forall i,\ m_i \in I\!R^{*+}$ such that: $\forall 1 \leq i \leq n, \lambda_i + m_i > 0$. Let $l : (V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N))^2 \to I\!R$ be defined by
$l((u_1, ..., u_n), (v_1, ..., v_n)) = \sum_{i=1}^n e_i \int_{I\!R^N} [\nabla u_i.\nabla v_i + (q_i + m_i\rho_i)u_i v_i]$
$- \sum_{i=1}^n e_i(\lambda_i + m_i) \int_{I\!R^N} \rho_i u_i v_i - \sum_{i,j;i\neq j} e_i \int_{I\!R^N} a_{ij} u_j v_i$.
l is a bilinear continuous form. By $(H4*)$ and the Cauchy-Schwartz inequality, we prove that l is coercive. So, using the Lax-Milgram Theorem, we obtain a unique solution for System (1).

4.2 Study of a semilinear system

Theorem 4.3 *Keep conditions $(H0) - (H3)$ and $(H4*)$ and assume that the condition $(H1)$ is strengthened as follows: $\rho_i \in L^\infty_{loc}(I\!R^N) \cap L^1(I\!R^N)$ for each i. Assume further the following conditions $(H6) - (H7)$:*
$(H6)$ $\forall i,\ \exists \theta_i \in L^2(I\!R^N),\ \forall u_1, ..., u_n \in L^2(I\!R^N),\ |f_i(x, u_1, ..., u_n)| \leq \theta_i$.
$(H7)$ $\forall i,\ f_i$ *is Lipschitz respect to* u_i*, uniformly in* x.
If D (given in Theorem 4.1) is a non singular M-matrix, then System (1) has a weak solution $(u_1, ..., u_n)$.

Since the proof of Theorem 4.3 is very similar to that of Theorem 3.3, we omit the details.

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Author:

Laure Cardoulis
Univ. Toulouse 1
pl.A.France
31042 Toulouse Cedex

e-mail: cardouli@math.univ-tlse1.fr

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Nasreddine Megrez

A Nonlinear Elliptic Eigenvalue Problem in Unbounded Domain

KEY WORDS. Nonlinear eigenvalue problem, Schrödinger equation, P-Laplacian, Unbounded domain.

1 Introduction and main result

We find a localization of λ such that the problem

$$(P_\lambda): \quad \begin{cases} -\mathbf{\Delta}_p u + V(x)|u|^{p-2}u = \lambda f(x,u) \\ u_{|\partial\Omega} = 0 \\ \lim\limits_{|x|\longrightarrow\infty} u(x) = 0 \end{cases}$$

has a solution, where $\Omega \subset \mathbb{R}^N$ is an unbounded domain, $N > p \geq 2$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $V \in L^p_{loc}(\Omega)$ is a continuous potential on Ω satisfying

$$\liminf_{|x|\longrightarrow\infty} V(x) \geq \min_{x\in\Omega} V(x) > 0.$$

Several authors for example P. Felmer, M. Del Pino [1] and P. H. Rabinowitz [4] have studied the problem (P_λ) when $p = 2$ and $\lambda = 1$ under the following assumption:

$$H): \qquad \exists\theta > 2 \;/\; 0 \leq \theta F(s) < f(s)s, \text{ where } F(s) = \int_0^s f(t)dt.$$

They proved that the Palais-Smale sequence given by the mountain-pass lemma for the functional $J(u) = \frac{1}{2}\|u\|^2_H - \int_\Omega F(u)dx$, has a convergent subsequence, where the limit is a weak solution of the problem.

In [5], I. Schindler has studied the problem (P_λ) when $p = 2$ and $\lim\limits_{|x|\to\infty} V(x) = +\infty$, without assumption $H)$. He gave an interval I such that (P_λ) has a solution for almost every λ satisfying $\frac{1}{\lambda} \in I$.

In this paper, we will follow the ideas in [5], and prove two existence theorems under weaker assumptions on the potential V. The first result uses conditions on V similar to those in [1], the second uses conditions similar to those in [3].

We denote by $\mathcal{B}$ the Banach space defined by the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\mathcal{B}} := \left(\int_\Omega |\nabla u(x)|^p + V(x)|u(x)|^p dx\right)^{\frac{1}{p}}.$$

We use the following notations:

- $F(x,s) := \displaystyle\int_0^s f(x,t)dt.$
- $S_t := \{u \in \mathcal{B} : \ \|u\|_{\mathcal{B}}^p = t\}.$
- $g(u) := \displaystyle\int_\Omega F(x,u)dx$, and $\gamma(t) := \displaystyle\sup_{u\in S_t} g(u).$
- $I := \left(p \displaystyle\inf_{t\neq s} \frac{\gamma(t)-\gamma(s)}{t-s}, p\sup_{t\neq s} \frac{\gamma(t)-\gamma(s)}{t-s}\right).$

We will use the following assumptions:

H1) There exist a bounded subset K of Ω, and a real α such that

$$\forall x \in \Omega \setminus K, \ \forall s \in \mathbb{R} : \ f(x,s)s \leq \alpha V(x)|s|^p.$$

H2) $\displaystyle\limsup_{s\longrightarrow 0} \frac{f(x,s)}{|s|^{p-1}} = 0.$

H3) $\displaystyle\lim_{|s|\longrightarrow\infty} \frac{f(x,s)}{|s|^{p^*-1}} = 0$ uniformly in x.

H4) V belongs to the reverse Hölder class A_p.

We recall that a nonnegative locally L^q-integrable function V on $\mathbb{R}^N$ is said to belong to the reverse Hölder class A_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B |V(x)|^q dx\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|B|}\int_B |V(x)|dx\right) \tag{1}$$

holds for every ball B in $\mathbb{R}^N$; and we say that $V \in A_\infty$ if for any $\alpha, \ 0 < \alpha < 1$, there exists a $\beta, \ 0 < \beta < 1$ such that for all balls $B \subset \mathbb{R}^N$ and all subsets $E \subset B$,

$$|E| \geq \alpha|B| \Longrightarrow \int_E V(x)dx \geq \beta \int_B V(x)dx. \tag{2}$$

Note that if $V \in A_q$ then $V \in A_\infty$; and if $V \in A_\infty$, then there exists a $q, \ 1 < q < \infty$ such that $V \in A_q$.

H5) $\lim_{|x|\to\infty} m(x,V) = +\infty$, where $m(x,V)$ is an auxiliary function defined by:

$$\frac{1}{m(x,V)} := \sup\left\{ r > 0,\ \frac{1}{r^{N-p}} \int_{B(x,r)} V(y)dy \le 1 \right\}.$$

$B(x,r)$ is the ball of $\mathbb{R}^N$ of center x and radius r.

The function $m(x,V)$ was introduced by Z. Shen in [8] for $p=2$, to study the Neumann problem for the operator $-\mathbf{\Delta} + V(x)$ on the domain above a Lipschitzian graph. Recently, K. Kurata [3] has used this function to prove the existence of least energy solution for a Schrödinger equation with magnetic potential.

Note that $0 < m(x,V) < +\infty\ \forall x \in \mathbb{R}^N$, and if $r = \dfrac{1}{m(x,V)}$, then $\dfrac{1}{r^{N-p}} \displaystyle\int_{B(x,r)} V(y)dy = 1$.

The problem (P_λ) is equivalent to finding the values of $\rho = \dfrac{1}{\lambda}$ for which there exists u such that

$$g'(u) = \rho \|u\|_B^{p-2} u\,. \tag{3}$$

Let $G_\rho(u) := \dfrac{\rho}{p}\|u\|_{\mathcal{B}}^p - g(u)$.
Our main results are:

Theorem 1.1 *If* **H1** *and* **H3** *hold, then for almost every λ satisfying $\dfrac{1}{\lambda} \in I \cap (\alpha, +\infty)$ the problem (P_λ) has a solution in $\mathcal{B}\setminus\{0\}$.*
If each critical sequence of G_ρ is bounded, then (P_λ) has a solution for every λ satisfying $\dfrac{1}{\lambda} \in I \cap (\alpha, +\infty)$.

Theorem 1.2 *If* **H2**–**H5** *hold, then for almost every λ satisfying $\frac{1}{\lambda} \in I$ the problem (P_λ) has a solution in $\mathcal{B}\setminus\{0\}$.*
If each critical sequence of G_ρ is bounded, then (P_λ) has a solution for every λ satisfying $\frac{1}{\lambda} \in I$.

2 Mountain pass and impasse lemma

We use a modification of the usual Palais-Smale condition:

Definition 2.1 *We say that G_ρ satisfies the property (P) if any sequence $(u_n) \subset \mathcal{B}$ satisfying*

$$\|u_n\|_{\mathcal{B}}^p \xrightarrow[n\to\infty]{} t \neq 0 \tag{4}$$

$$G'_\rho(u_n) \xrightarrow[n\to\infty]{} 0 \tag{5}$$

has a converging subsequence.

Remark 2.1 If a sequence $(u_n)_n$ satisfying (5) and

$$G_\rho(u_n) \xrightarrow[n\to\infty]{} b \neq 0 \tag{6}$$

is bounded, then modulo a subsequence we have (4).

We recall mountain pass results due to K. Tintarev [11].

Definition 2.2 *We say that G_ρ has the mountain-pass geometry if:*

- $G_\rho(0) = 0.$
- $\exists t_0 > 0,\ \delta > 0$ *such that for* $\|u\|^p_{\mathcal{B}} = t_0,\ G_\rho(u) > \delta.$
- $\exists e \in \mathcal{B} \setminus \{0\},\ \|e\|^p_{\mathcal{B}} > t_0$ *such that* $G_\rho(e) \leq 0.$

We have the following mountain impasse lemma:

Lemma 2.1 *Assume that G_{ρ_0} has the mountain-pass geometry and satisfies the property (P). Then either there exists $u_0 \in \mathcal{B} \setminus \{0\}$ such that $G'_\rho(u_0) = 0$, or there exists a sequence $(u_n, h_n) \in \mathcal{B} \setminus \{0\} \times \mathbb{R}$ such that:*

$$0 < h_n \longrightarrow 0$$

$$\|u_n\|^p_{\mathcal{B}} = t_n \nearrow \infty$$

$$G_{\rho_0}(u_n) = c(t_n) \searrow c$$

$$G'_{\rho_0+h_n}(u_n) = 0.$$

Using results from [11] we have the following lemma [1]:

Lemma 2.2 *Let $\rho_0 \in \mathbb{R}$ such that G_{ρ_0} satisfies the property (P), and has the mountain-pass geometry. Then for a dense set of ρ in a neighborhood of ρ_0, there exists $u_\rho \in \mathcal{B} \setminus \{0\}$ such that*

$$G'_\rho(u_\rho) = 0$$

[1] Using a technique used by M. Struwe in [10] page 60, one should be able to improve the statement in lemma 2.2 to almost every ρ in a neighborhood of ρ_0. I. Schindler attempted to prove this in [5], but the proof is not complete.

3 Proof of the main results

The critical sequence converging to the solution of (P_λ) is given by the following lemma which is proved in [5] when $p = 2$. The proof for general p is similar and will be omitted.

Lemma 3.1 *Under assumption* **H3**, *if $\rho \in I$, then G_ρ has critical sequence $(u_n)_n$ which tends either to a local minimizer of G_ρ, or satisfies equations (6) and (5), and is of mountain-pass type. If the sequence tends to a local minimizer of G_ρ, then $\exists t_0 \neq 0 : \ \|u_n\|_{\mathcal{B}} \longrightarrow t_0$.*

To prove Theorem 1.1 and Theorem 1.2 it suffices to prove that G_ρ satisfies property (P):

Lemma 3.2 *Assume* **H1** *and* **H3**. *Then G_ρ satisfies property (P) for $\rho > \alpha$.*

Proof: Let $(u_n)_n$ be a sequence satisfying (4) and (5). By (4), the sequence $(u_n)_n$ is bounded, which means that there is a renumbered subsequence converging to a weak limit u. In fact, this convergence is strong. To prove this, it suffices to check that for each $\varepsilon > 0$ there exist $R > 0$ such that:

$$\limsup_{n\to\infty} \int_{\Omega\setminus B_R} \{|\nabla u_n|^p + V(x)|u_n|^p\}dx < \varepsilon.$$

Let R be chosen such that $K \subset B_{R/2}$, and let

$$\eta_R = \begin{cases} 0 \text{ on } B_{R/2} \\ 1 \text{ on } \Omega \setminus B_R \end{cases} \text{ with } 0 \le \eta_R \le 1 \ \text{ and } |\nabla \eta_R| \le \frac{C}{R}.$$

Since $(u_n)_n$ is a bounded sequence of Palais-Smale type, then

$$< G_\rho^{'}(u_n), \eta_R u_n >= o(1)$$

i.e:

$$\int_\Omega \{|\nabla u_n|^p + V(x)|u_n|^p\}\,\eta_R dx + \int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\eta_R dx = \int_\Omega \lambda f(x,u_n)u_n\eta_R dx + o(1)$$

Using **H1** we obtain

$$\begin{aligned}\int_\Omega \{|\nabla u_n|^p + V(x)|u_n|^p\}\eta_R dx + \int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\eta_R dx &\le \lambda\alpha \int_{\Omega\setminus B_{R/2}} V(x)|u_n|^p\eta_R dx \\ &\quad + o(1) \\ &\le \lambda\alpha \int_\Omega \{|\nabla u_n|^p + V(x) \\ &\quad \cdot|u_n|^p\}\eta_R dx + o(1).\end{aligned}$$

Since $\rho = \dfrac{1}{\lambda}$, we have

$$\begin{aligned}(\rho-\alpha)\int_\Omega \{|\nabla u_n|^p + V(x)|u_n|^p\}\eta_R dx &\leq -\rho\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\eta_R dx + o(1)\\ &\leq \left|\rho\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\eta_R dx\right| + o(1)\\ &\leq \rho\frac{C}{R}\int_\Omega |u_n||\nabla u_n|^{p-1}dx + o(1)\\ &\leq \rho\frac{C}{R}\left(\|u_n\|_{L^p(\Omega)}\|\nabla u_n\|^{\frac{p}{q}}_{L^p(\Omega)}\right) + o(1),\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Thus:

$$\int_{\Omega\backslash B_R}\{|\nabla u_n|^p + V(x)|u_n|^p\}dx \leq \frac{\rho C}{R(\rho-\alpha)}\left(\|u_n\|_{L^p(\Omega)}\|\nabla u_n\|^{\frac{p}{q}}_{L^p(\Omega)}\right) + o(1)$$

which completes the proof.

Lemma 3.3 *If* **H2**–**H5** *hold, then G_ρ satisfies the property (P) for all $\rho \in \mathbb{R}$.*

For the proof, we use the concentration-compactness method. We recall the concentration-compactness lemma used in [5]:

Lemma 3.4 *Let $(\phi_n)_n$ be a sequence of non negatives functions in $L^1(\Omega)$ such that $\|\phi_n\|_{L^1(\Omega)} = t$. Then either:*

1. *$\forall R<\infty,\ \lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^N}\int_{B_R+y}\phi_n(x)dx = 0,$*
   or

2. *there exist a renumbered subsequence $(\phi_n)_n$, a $\beta\in(0,t]$, and sequences of nonnegative functions $(r_n)_n$, $(\phi_n^1)_n$, $(\phi_n^2)_n$ in $L^1(\mathbb{R}^N)$ such that $(\phi_n^1)_n$ is tight up to translations, and*

$$\phi_n = \phi_n^1 + \phi_n^2 + r_n,$$

$$\|\phi_n - (\phi_n^1+\phi_n^2)\|_{L^1(\Omega)} = \|r_n\|_{L^1(\Omega)} \xrightarrow[n\to\infty]{} 0,$$

$$\|\phi_n^1\|_{L^1(\Omega)} \xrightarrow[n\to\infty]{} \beta,\ \|\phi_n^2\|_{L^1(\Omega)} \xrightarrow[n\to\infty]{} t-\beta,$$

$$dist\left(Supp(\phi_n^1),\ Supp(\phi_n^2)\right) \xrightarrow[n\to\infty]{} \infty,$$

$$\phi_n^i r_n = 0\ a.e,\ i=1,2.$$

Furthermore, if there exists $R<\infty$ such that $\lim\limits_{n\to\infty}\int_{B_R}\phi_n dx = \delta > 0$, then we may assume the ϕ_n^1 to be tight.

Using the preceding lemma one can prove the following smooth version of concentration compactness:

Lemma 3.5 *Let (u_n) be a sequence in S_t, and let $\phi_n := |\nabla u_n|^p + V(x)|u_n|^p$. Assume $(\phi_n)_n$ is as in lemma 3.4 case 2. Then there exist two sequences $(u_n^1)_n$, $(u_n^2)_n$ in $\mathcal{B}$ such that*

$$\|u_n - (u_n^1 + u_n^2)\|_{\mathcal{B}} \xrightarrow[n\to\infty]{} 0,$$

$$dist\left(Supp(u_n^1), Supp(u_n^2)\right) \xrightarrow[n\to\infty]{} \infty,$$

$$\|u_n^1\|_{\mathcal{B}}^p = \beta,\ \|u_n^2\|_{\mathcal{B}}^p = t - \beta.$$

Moreover, $\psi_n^1 := |\nabla u_n^1|^p + V(x)|u_n^1|^p$ is tight up to translations.

We also recall

Lemma 3.6 *Let $(u_n)_n$ be a sequence in S_t, and assume that* **H2** *and* **H3** *hold. If $\int_\Omega |u_n(x)|^p dx \xrightarrow[n\to\infty]{} 0$, then*

$$(g'(u_n), u_n) \xrightarrow[n\to\infty]{} 0.$$

We need also the following lemma originally proved for polynomial V by C. Fefferman and D. H. Phong in [2], and extended to $V \in A_q$ with $1 < q < \infty$ and $p = 2$ by Z. Shen in [7, 8].

Lemma 3.7 *Let $u \in C_c^1(\mathbb{R}^N)$. Then:*

$$\int_{\mathbb{R}^N} |u(x)|^p m(x,V)^p dx \leq C \int_{\mathbb{R}^N} |\nabla u(x)|^p + V(x)|u(x)|^p dx$$

To prove this result, we need the following lemma which is the first statement of Lemma 1.8 in [8]. Its proof is based on the fact that $V \in A_p$.

Lemma 3.8 *For $x, y \in \mathbb{R}^N$, $m(x,V) \sim m(y,V)$ if $|x-y| \leq \dfrac{1}{m(x,V)}$.*

The proof of Lemma 3.7 is similar to that for $p = 2$ in [7, 8]. It is included for completeness without the confusing typographical error in [7, 8].

Proof of Lemma 3.7: Since $V \in A_p$, then $V \in A_\infty$. Hence, according to (2), there exists $\varepsilon \in]0,1[$ such that for each ball $B \subset \mathbb{R}^N$, we have

$$\left|\left\{y \in B :\ V(y) \geq \frac{\varepsilon}{|B|}\int_B V(x)dx\right\}\right| \geq \frac{1}{2}|B|.$$

Let $x_0 \in \mathbb{R}^N$, $r_0 = \dfrac{1}{m(x_0, V)}$, $B = B(x_0, r_0)$, and $E := \left\{ y \in B : \; V(y) \geq \dfrac{\varepsilon}{|B|} \displaystyle\int_B V(x)dx \right\}$, where ε is chosen such that $\dfrac{\varepsilon}{w_N} \leq 1$. ($w_N = |B(0,1)|$).

For $y \in E$ we have $V(y) \geq \dfrac{\varepsilon}{w_N r_0^N} \displaystyle\int_B V(x)dx = \dfrac{c_0}{r_0^p}$, where $c_0 = \dfrac{\varepsilon}{w_N}$.

Thus, $\displaystyle\int_B \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} dy \geq \int_E \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} dy \geq C r_0^{N-p}$.

On the other hand we have

$$\int_B |\nabla u(x)|^p dx \geq \frac{C}{r_0^{p+N}} \int_B \int_B |u(x) - u(y)|^p dx dy,$$

and

$$\int_B V(x)|u(x)|^p dx \geq \frac{C}{r_0^N} \int_B \int_B V(x)|u(x)|^p dx dy.$$

Summing those two inequalities, we obtain:

$$\begin{aligned} \int_B |\nabla u(x)|^p dx + \int_B V(x)|u(x)|^p dx &\geq \frac{C}{r_0^N} \int_B \int_B \left\{ \frac{c_0}{r_0^p}|u(x) - u(y)|^p + V(y)|u(y)|^p \right\} dx dy \\ &\geq \frac{C}{r_0^N} \int_B \int_B \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} (|u(x) - u(y)|^p + |u(y)|^p) \, dx dy \\ &\geq \frac{C}{r_0^N} \int_B \int_B \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} |u(x)|^p dx dy \\ &\geq C \int_B \frac{1}{r_0^p} |u(x)|^p dx \end{aligned}$$

Using Lemma 3.8 we obtain:

$$\int_B |u(x)|^p m(x,V)^p dx \leq C \int_B \{|\nabla u(x)|^p + V(x)|u(x)|^p\} \, dx \, .$$

Multiplying this last inequality by $m(x_0, V)^N$ and using Lemma 3.8 we obtain:

$$\begin{aligned} &\int_{\mathbb{R}^N} |u(x)|^p m(x,V)^{p+N} \chi_{B(x_0,r_0)}(x) dx \\ &\qquad \leq C \int_{\mathbb{R}^N} \{|\nabla u(x)|^p + V(x)|u(x)|^p\} \, m(x,V)^N \chi_{B(x_0,r_0)}(x) dx \, . \end{aligned}$$

Remarking that $\chi_{B(x_0,r_0)}(x) = \chi_{B(x,r_0)}(x_0)$, and integrating in x_0, we obtain:

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x)|^p m(x,V)^{p+N} \left(\int_{B(x,r_0)} dx_0 \right) dx \;&\leq\; C \left\{ \int_{\mathbb{R}^N} |\nabla u(x)|^p \, m(x,V)^N \left(\int_{B(x,r_0)} dx_0 \right) dx \right. \\ &+\; \left. \int_{\mathbb{R}^N} V(x)|u(x)|^p m(x,V)^N \left(\int_{B(x,r_0)} dx_0 \right) dx \right\} . \end{aligned}$$

With Lemma 3.8, we deduce that

$$\int_{|x_0-x|<r_0} dx_0 \sim \left(\frac{1}{m(x,V)}\right)^N$$

which completes the proof.

Proof of Lemma 3.3: Let $(u_n)_n$ be a sequence satisfying (4) and (5). We may assume that $\|u_n\|_{\mathcal{B}}^p = t$ for every n. Let $\phi_n := |\nabla u_n|^p + V(x)|u_n|^p$ which we claim to be in the case 2 of Lemma 3.4. Suppose this were not the case, then there is a relabeled subsequence $(u_n)_n$ such that

$$\int_{B_R} |\nabla u_n(x)|^p + V(x)|u_n(x)|^p dx \xrightarrow[n\to\infty]{} 0 \ \forall R < \infty \tag{7}$$

According to lemma 3.7, there exists $M > 0$ such that $\int_{\mathbb{R}^N} m(x,V)^p |u_n(x)|^p dx \le M$.

Since $\lim_{|x|\to\infty} m(x,V) = +\infty$, then $\forall \varepsilon > 0, \exists R > 0$ such that $m(x,V)^p \ge \dfrac{M}{\varepsilon}$ on $\{x : \ |x| \ge R\}$. Hence

$$\int_{\{|x|\ge R\}} |u_n(x)|^p dx \le \varepsilon \ \ \forall k. \tag{8}$$

From (7) and (8), we conclude that

$$\int_\Omega |u_n(x)|^p dx \xrightarrow[n\to\infty]{} 0. \tag{9}$$

According to lemma 3.6 we have:

$$(g'(u_n), u_n) \xrightarrow[n\to\infty]{} 0.$$

Furthermore, (5) and the fact that $(u_n)_n$ is bounded implies that

$$\rho\|u_n\|_{\mathcal{B}}^p - (g'(u_n), u_n) \xrightarrow[n\to\infty]{} 0.$$

Thus $\|u_n\|_{\mathcal{B}}^p = t = 0$, a contradiction. This completes the proof, see [5], [6].

Proof of Theorem 1.1 : By Lemma 3.1, Remark 2.1, and Lemma 2.2, we deduce the first statement. The second one is a consequence of Lemma 3.1 and the fact that G_ρ satisfies property (P).

The proof of Theorem 1.2 is similar.

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Author:

Nasreddine Megrez
CEREMATH-MIP/UT1 University of Toulouse1
21, allee de Brienne, Bat C, 31000 Toulouse
France

e-mail: megrez@math1.univ-tlse1.fr

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Ian Schindler, Kyril Tintarev[†]

A nonlinear Schrödinger equation with external magnetic field

ABSTRACT. The paper presents an existence result for a nonlinear Schrödinger equation with magnetic potential on unbounded domains.

1 Introduction

In this paper we study a Dirichlet boundary value problem for a nonlinear Schrödinger equation

$$\left(\frac{1}{i}\nabla + A(x)\right)^2 u + V(x)u = |u|^{q-1}u \tag{1}$$

with a subcritical q on $\Omega \subset \mathbb{R}^3$ an open, possibly unbounded, set. Equation (1) is a modification of the linear Schrödinger equation for standing waves $\psi(x,t) = e^{-iEt}u(x)$ with a magnetic potential and an energy term dependent on the magnitude of the wave function.

The problem (1) has been studied by [2], where existence was proved for the constant magnetic field $B = curl A$ on $\mathbb{R}^N$, $N = 2, 3$. In the case of variable magnetic field [2] gives only an implicit condition in terms of asymptotic values of the functional. The problem on $\mathbb{R}^N$ was also studied by [3]. This paper provides existence under explicit conditions on the electric and magnetic fields only in part overlapping with those [3], and for problems on more general domains than $\mathbb{R}^N$.

The proof of existence is variational: convergence of a critical sequence is obtained by application of an abstract concentration compactness framework ([7, 8]) which provides a functional-analytic generalization of original concentration compactness ([5, 6]).

2 Concentration compactness in Hilbert space and in Sobolev space

In what follows we quote results from [8].

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Definition 2.1 *Let H be a separable Hilbert space. A bounded set D of bounded linear operators on H is called a set of dislocations if the following properties are satisfied:*

(I) $g \in D \Rightarrow g^{-1} \in D,\ I \in D$;

(II) *if $g_k,\ h_k \in D$ and $g_k h_k^{-1}$ does not converge weakly to 0, then there exists a renamed subsequence of $g_k h_k^{-1}$ such that $u_k \rightharpoonup 0 \Rightarrow g_k h_k^{-1} u_k \rightharpoonup 0$;*

(III) $g_k \in D,\ u_k \rightharpoonup 0 \Rightarrow g_k^* g_k u_k \rightharpoonup 0$.

Definition 2.2 *Let $u, u_k \in H$. We will say that u_k converges to u* weakly with concentration *which we will denote as*

$$u_k \stackrel{cw}{\to} u,$$

if for all $\varphi \in H$,

$$\lim_{k\to\infty} \sup_{g\in D} (g(u_k - u), \varphi) = 0\,. \tag{1}$$

Theorem 2.3 *Let $u_k \in H$ be a bounded sequence. Then there exists $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$ such that for a renumbered subsequence*

$$w^{(n)} = \text{w-lim}\, {g_k^{(n)}}^{-1} u_k\,, \tag{2}$$

$${g_k^{(n)}}^{-1} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m\,, \tag{3}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2 \leq \limsup \|u_k\|^2 \tag{4}$$

and

$$u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)} \stackrel{cw}{\to} 0\,. \tag{5}$$

We now specify the implications of the theorem above for the case when $H = H^1(\mathbb{R}^N)$ and D is a group of shifts:

$$D = \{g_\alpha : u \mapsto u(\cdot + \alpha)\}_{\alpha \in Z^{\mathbb{N}}}\,. \tag{6}$$

Let $N > 1$ and let $2^* = \frac{2N}{N-2}$ (for $N = 2$ we set $2^* = \infty$). It is proved in [8] that D is a set of dislocations, so that Theorem 2.3 applies. Moreover ([8, 4, 6]), for bounded sequences concentrated weak convergence is same as $L^p - convergence$

Lemma 2.4 *Let H,D be as above, let $2 < p < 2^*$ and let u_k be a bounded sequence in H. Then $u_k \in H^1(\mathbb{R}^N) \stackrel{cw}{\to} 0$ if and only if $\|u_k\|_{L^p(\mathbb{R}^N)} \to 0$.*

An easy corollary from the above lemma is the following statement.

Lemma 2.5 *Let u_k, $w^{(n)}$, and $g_k^{(n)}$ be as in Theorem 2.3. If $F:\mathbb{R}\to\mathbb{R}$ is a continuous function and for every $\epsilon>0$ there is a $C_\epsilon<\infty$ and a p_ϵ such that $2<p_\epsilon<2^*$ and*

$$|F(s)|\le\epsilon(|s|^2+|s|^{p^*})+C_\epsilon|s|^{p_\epsilon} \tag{7}$$

then, on a renamed subsequence

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}F(|u_k|)=\sum_n\int_{\mathbb{R}^N}F(|w^{(n)}|). \tag{8}$$

Definition 2.6 *An open set $\Omega\subset\mathbb{R}^N$ will be called asymptotically contractive if for every sequence $u_k\in H_0^1(\Omega)$ and every sequence $\alpha_k\in\mathbb{R}^N$ such that $u_k(\cdot+\alpha_k)$ converges weakly in $H^1(\mathbb{R}^N)$ to some w, there exists a $\gamma\in\mathbb{R}^N$ such that $w(\cdot-\gamma)\in H_0^1(\Omega)$.*

A sufficient geometric condition for asymptotic contractiveness can be formulated in the terms of lower limit for sequences of sets,

$$\liminf\Omega_k:=\bigcup_{n\in\mathbb{N}}\bigcap_{k\ge n}\Omega_k\,, \tag{9}$$

namely,

Lemma 2.7 *Let $\Omega\subset\mathbb{R}^N$ be an open set such that $\partial\Omega=\partial(\mathbb{R}^N\setminus\bar{\Omega})$. It is asymptotically contractive if for every sequence $\alpha_k\in\mathbb{Z}^N$ there exist a set $Y\subset\mathbb{R}^N$ of zero measure and a convergent sequence $\gamma_k\in\mathbb{R}^N$ such that, on a renumbered subsequence,*

$$\liminf(\Omega+\alpha_k-\gamma_k)\subset\Omega\cup Y\,. \tag{10}$$

3 A nonlinear magnetic Schrödinger equation

Let $\Omega\subset\mathbb{R}^3$ be an open set. Let $V\in L^1_{loc}(\Omega)$ and assume that

$$\eta:=\inf_{x\in\Omega}(V(x)+\lambda_0)>0\,, \tag{1}$$

where $\lambda_0=\inf_{u\mathbb{C}_0^\infty(\Omega):\int|u|^2=1}\int|\nabla u|^2$. Note that λ_0 might be positive even for unbounded Ω. Let $A\in L^2_{loc}(\Omega;\mathbb{R}^3)$ satisfy

$$\exists\phi\in H^1_{loc}(\Omega):|A(x)-\nabla\phi|^2<\liminf_{|y|\to\infty}V(y)-V(x),x\in\Omega\,. \tag{2}$$

Theorem 3.1 *Let $q\in(1,5)$ and assume that A,V satisfy (2, 1). If $\Omega\subset\mathbb{R}^3$ is an asymptotically contractive set, then there is a solution $u\in H_0^1(\Omega;\mathbb{C})\setminus\{0\}$ satisfying the equation*

$$\left(\frac{1}{i}\nabla+A(x)\right)^2u+V(x)u=|u|^{q-1}u\,. \tag{3}$$

Proof: It suffices to prove existence of the minimizer for the following variational problem. Let

$$c = \inf_{\int_\Omega |u|^p = 1} I(u), \; p = q+1\,, \tag{4}$$

where

$$I(u) = \int_\Omega Pu \cdot \overline{Pu} + V(x)|u|^2, P = \frac{1}{i}\nabla + A(x)\,. \tag{5}$$

Without loss of generality we can assume that ϕ=0 in (2), since the value of c does not change when one replaces A by $A + \nabla\phi$. For smooth ϕ this is verified by replacing u by $e^{i\phi}u$, for general ϕ one can replace a minimizing sequence u_k by $e^{i\phi_k}u_k$ with appropriate smooth approximations ϕ_k of ϕ.

Let u_k be the maximizing sequence and let $u^{(0)} :=$ w-lim u_k. The weak limit here is understood with respect to the metric of the quadratic form $I(u)$. We will make use of the following inequality (cf. [2]):

$$\int |Pu|^2 \geq \int |\nabla|u||^2 \tag{6}$$

that together with (1) implies that $c > 0$ and that $v_k := |u_k - u^{(0)}| \in H^1_0(\Omega)$. We apply Theorem 2.3 in $H^1(\mathbb{R}^3)$ to the extensions of functions v_k (by zero on the complement of Ω). Since Ω is asymptotically contractive, the dislocated weak limits of v_k, $w^{(n)}$ (redefined by appropriate constant shifts) lie in $H^1_0(\Omega)$. Moreover, since $v_k \rightharpoonup 0$, one has $|\alpha_k^{(n)}| \to \infty$ for all n.

From Lemma 2.5 we have

$$\lim \int v_k^p = \sum_{n=1}^\infty \int w^{(n)p} \tag{7}$$

and, since all $|\alpha_k^{(n)}| \to \infty$,

$$1 = \lim \int |u_k|^p = \int |u^{(0)}|^p + \sum_{n=1}^\infty \int w^{(n)p}\,. \tag{8}$$

To continue with the proof we need the following inequality:

$$\int |Pu_k|^2 + V(x)|u_k|^2 \geq \int |Pu^{(0)}|^2 + V(x)|u^{(0)}|^2 + \sum_{n=1}^\infty \int (|\nabla w^{(n)}|^2 + \liminf_{|x|\to\infty} V(x) {w^{(n)}}^2)\,. \tag{9}$$

We start with:

$$\int |Pu_k|^2 = \int |Pu^{(0)}|^2 + \int |P(u_k - u^{(0)})|^2 - \int P(u_k - u^{(0)}) \cdot \overline{Pu^{(0)}} - \int Pu^{(0)} \cdot \overline{P(u_k - u^{(0)})}\,. \tag{10}$$

Passing to the limit we have

$$\liminf \int |Pu_k|^2 \geq \int |Pu^{(0)}|^2 + \liminf \int |P(u_k - u^{(0)})|^2\,. \tag{11}$$

Applying (6), we get

$$\liminf \int |Pu_k|^2 \geq \int |Pu^{(0)}|^2 + \liminf \int |\nabla v_k|^2 , \tag{12}$$

which in turn can be estimated from below by (4) if we apply Theorem 2.3 to the sequence v_k as a bounded sequence in $D^{1,2}$, so that

$$c = \liminf \int |Pu_k|^2 \geq \int |Pu^{(0)}|^2 + \sum_n \int |\nabla w^{(n)}|^2 . \tag{13}$$

A similar argument allows also to show that

$$\int V(x)|u_k|^2 \geq \int V(x)|u^{(0)}|^2 + \sum_{n=1}^{\infty} \liminf_{|x|\to\infty} V(x) \int {w^{(n)}}^2 . \tag{14}$$

This and (13) yield (9). Using the notations

$$t_n = \int {w^{(n)}}^p, t_0 = \int |u^{(0)}|^p , \tag{15}$$

$$c_n = t_n^{-2/p} \int |\nabla w^{(n)}|^2 + \liminf_{|x|\to\infty} V(x) \int {w^{(n)}}^2, c_0 = t_0^{-2/p} \int |Pu^{(0)}|^2 + V(x)|u^{(0)}|^2 , \tag{16}$$

one has from (8), (9)

$$\sum_{n=0}^{\infty} t_n = 1, \sum_{n=0}^{\infty} c_n t_n^{2/p} \leq c . \tag{17}$$

However, due to (2), $c_n > c$, while $c_0 \geq c$, so it is possible to satisfy (17) only with $t_0 = 1$, $t_n = 0$, $N = 1, 2 \ldots$ This also implies that the minimizing sequence converges to $u^{(0)}$ in norm.

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Authors:

Ian Schindler
Ceremath,University of Toulouse I
and
UMR MIP CNRS 5640.
Université Paul Sabatier
e-mail: ischindl@univ-tlse1.fr

Kyril Tintarev
Uppsala University
e-mail: kyril@math.uu.se

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Lothar Berg

On Polynomials Related with Generalized Bernoulli Numbers

ABSTRACT. The generalized Bernoulli numbers are connected with a sequence of polynomials, which are defined recursively and which can be represented by means of a generalized Rodrigues formula. A relation between the coefficients of a polynomial and corresponding moments is derived. A generating function with an unusual shape is constructed.

KEY WORDS. Bernoulli numbers, recursions, Rodrigues formula, moments, generating function

The generalized Bernoulli numbers $B_m^{(z)}$ are defined by

$$\left(\frac{w}{e^w - 1}\right)^z = \sum_{m=0}^{\infty} \frac{1}{m!} B_m^{(z)} w^m \ (z \in \mathbb{C}), \tag{1}$$

cf. [1], [2], they contain the ordinary Bernoulli numbers B_m as special case with $z = 1$. The generalized Bernoulli numbers satisfy the difference equation

$$B_m^{(z+1)} = \left(1 - \frac{m}{z}\right) B_m^{(z)} - m B_{m-1}^{(z)} \tag{2}$$

from which they can be calculated recursively for natural m, z subject to the initial values $B_0^{(z)} = 1$ and $B_m^{(1)} = B_m$. They are polynomials in z of degree m, the first of them are listed in [1, p. 459] for $0 \leq m \leq 12$.

The substitution

$$d_m(z) = (-1)^m \binom{z-1}{m} B_m^{(z)} \tag{3}$$

transfers (2) into the simpler recursion

$$d_m(z+1) - d_m(z) = z d_{m-1}(z)\,. \tag{4}$$

Under the initial values $d_0(z) = 1$, $d_m(1) = 0$ for $m \in \mathbb{N}$ the last has polynomial solutions of the form

$$d_m(z) = \sum_{\mu=1}^{m} c_{m\mu} \binom{z}{m+\mu} \tag{5}$$

from which, conversely, $B_m^{(z)}$ can be calculated by means of (3). Using the identity

$$z\binom{z}{n-1} = n\binom{z}{n} + (n-1)\binom{z}{n-1}$$

substitution of (5) into (4) yields the recursion

$$c_{m,\mu+1} = (m+\mu)(c_{m-1,\mu} + c_{m-1,\mu+1}) \quad (m \geq 2) \tag{6}$$

for $\mu = 0, 1, \ldots, m-1$ with $c_{11} = 1$ and $c_{m0} = c_{m,m+1} = 0$ $(m \in \mathbb{N})$. From (6) and the initial values we immediately obtain the values of $c_{m\mu}$ $(1 \leq \mu \leq m)$ in form of a Pascal triangle

m						$c_{m\mu}$					
1						1					
2					2		3				
3				6		20		15			
4			24		130		210		105		
5		120		924		2380		2520		945	
6	720		7308		26432		44100		34650		10395

and, moreover,

$$c_{mm} = (2m-1)!!,\ c_{m,m-1} = \frac{2}{3}(m-1)(2m-1)!!,\ c_{m1} = m!\,. \tag{7}$$

In order to learn more about the coefficients $c_{m\mu}$ we introduce the polynomials

$$f_n(z) = \sum_{\nu=0}^{n} c_{n+1,\nu+1} z^\nu \quad (n \in \mathbb{N}_0)\,. \tag{8}$$

The recursions (6) and the corresponding initial values imply

$$f_n(z) = ((n+2)z + n + 1)f_{n-1}(z) + z(z+1)f'_{n-1}(z) \quad (n \in \mathbb{N}) \tag{9}$$

and $f_0(z) = 1$. These equations yield immediately

$$f_n(0) = (n+1)!, \quad f_n(-1) = (-1)^n\,. \tag{10}$$

Equation (9) can also be written in the compact form

$$z^n f_n(z) = (z^{n+1}(z+1)f_{n-1}(z))' \quad (n \in \mathbb{N}) \tag{11}$$

from which, introducing the operator $D = \frac{d}{dz}[z^2(z+1)(.)]$, the generalized Rodrigues formula

$$f_n(z) = \frac{1}{z^n} D^n 1 \quad (n \in N_0) \tag{12}$$

can be derived. Using Rolle's theorem, formula (11) shows that f_n has exactly n simple zeros in (0,1), and that the zeros of f_{n-1} separate those of f_n.

For the next considerations the moments

$$J_{nk} = \int_{-1}^{0} z^k f_n(z) dz \quad (n, k \in \mathbb{N}_0) \tag{13}$$

are a crucial remedy. Using (11), partial integration yields

$$J_{nk} = (n-k)(J_{n-1,k+1} + J_{n-1,k}) \quad (n \in \mathbb{N}, k \in \mathbb{N}_0) \tag{14}$$

with the special cases

$$J_{nn} = 0,\ J_{n+1,n} = J_{n,n+1},\ J_{n+1,n-1} = 2J_{n,n-1} \quad (n \in \mathbb{N}) \tag{15}$$

whereas $J_{00} = 1$. Substituting (8) into (13) we see that

$$J_{nk} = (-1)^{n+k} \frac{k!}{(n+k+1)!} p_n(k) \tag{16}$$

with certain polynomials p_n of degree n, and by means of the DERIVE system it follows

$$p_0(k) = 1,\ p_1(k) = k-1,\ p_2(k) = (k-2)(k-3),\ p_3(k) = (k-2)(k-3)(k-11),$$
$$p_4(k) = (k-2)(k-4)(k-5)(k-31),\ p_5(k) = (k-4)(k-5)(k^3 - 90k^2 + 1019k - 1770),$$
$$p_6(k) = (k-4)(k-6)(k-7)(k^3 - 202k^2 + 4267k - 8490).$$

These special cases suggest:

Proposition 1 *For $n \geq 3$ the moments (13) vanish in the following cases:*

n odd and $k \in \{n-1, n\}$,
n even and $k \in \{n-2, n, n+1\}$.

Proof: In view of (15) we only have to show that $J_{n,n+1} = 0$ for positive even n. For this reason we introduce the abbreviations $\varphi(z) = z^2(z+1)$ and $\Phi_n(z) = z^n f_n(z)$ $(n \in \mathbb{N}_0)$, so that (11) turns over into

$$\Phi_n = (\varphi \Phi_{n-1})', \tag{17}$$

and (13) for $k = n + 1$ into

$$J_{n,n+1} = \int_{-1}^{0} z\Phi_n dz \,.$$

Using (17) and $\varphi = 0$ both for $z = 0$ and $z = -1$, partial integration yields

$$J_{n,n+1} = -\int_{-1}^{0} \varphi\Phi_{n-1} dz \tag{18}$$

and $k - 1$ further partial integrations yield

$$J_{n,n+1} = (-1)^k \int_{-1}^{0} \varphi\Phi_{k-1}\Phi_{n-k} dz \,.$$

For $k = n$ we obtain, in view of (18), $J_{n,n+1} = (-1)^{n+1} J_{n,n+1}$. This proves the assertion ■

From (15) and (16) we obtain the relations

$$p_{n+1}(n) = (n+1)p_n(n+1),\ p_{n+1}(n-1) = -(4n+2)p_n(n-1) \quad (n \in \mathbb{N})\,.$$

According to Proposition 1 the first of these are only interesting for n odd, and the last for n even.

Proposition 2 *The coefficients in (8) can be expressed by means of the polynomials in (16):*

$$c_{n+1,\nu+1} = (-1)^n \frac{1}{n!}\binom{n}{\nu} p_n(-\nu - 1)\,. \tag{19}$$

Proof: Substituting (8) into (13) we find

$$J_{nk} = \sum_{\nu=0}^{n} c_{n+1,\nu+1} \frac{(-1)^{k+\nu}}{k+\nu+1}\,,$$

and decomposition of (16) into partial fractions immediately yields (19) ■

According to (19) the relations (7) transfer into

$$p_n(-1) = (-1)^n n!(n+1)!,\ p_n(-n) = \frac{2}{3}(-1)^n n!(2n+1)!!\,,$$
$$p_n(-n-1) = (-1)^n n!(2n+1)!!\,.$$

Remark Here, we are interested in formula (19) only concerning the coefficients $c_{m\mu}$ from (5). But Proposition 2 is even valid for an arbitrary polynomial (8) with the moments (13) and polynomials $p_n(k)$ defined by (16).

Finally, we introduce the function

$$\psi = \ln\left(1+\frac{1}{z}\right) - \frac{1}{z}\,, \tag{20}$$

which is analytic for $z \notin \{-1, 0\}$. Both for $z < -1$ and for $z > 0$ the function ψ is real, whereas $\psi(z) = \ln\left|1+\frac{1}{z}\right| - \frac{1}{z} - i\pi$ for $-1 < z < 0$, cf. Figure 1. Differentiation of (20) yields $\psi' = \frac{1}{\varphi}$ where $\varphi = z^2(z+1)$ as before. Hence, for $z \notin \{-1, 0\}$ there exists the inverse function F with

$$F(\psi) = z\,. \tag{21}$$

We want to determine the generating function

$$G(z, w) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n(z) w^n\,, \tag{22}$$

which will have an unusual shape.

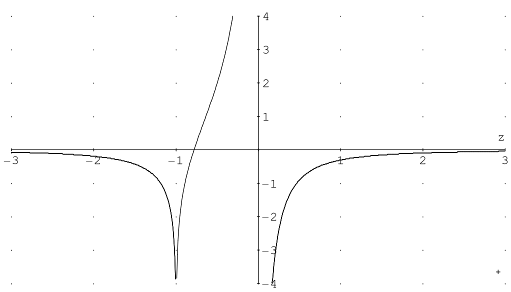

Figure 1: $\Re\left(\ln\left(1+\frac{1}{z}\right) - \frac{1}{z}\right)$

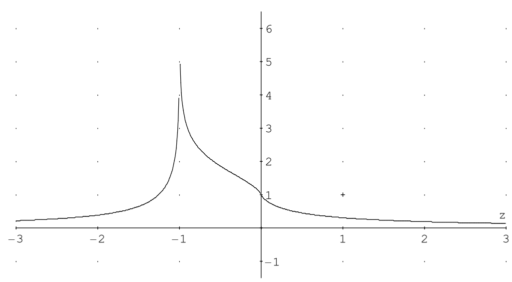

Figure 2: $\mathrm{abs}\left[z\ln\left(1+\frac{1}{z}\right) - 1\right]$

Proposition 3 *With the function F defined by (21) and (20) the generating function (22) has the representation*

$$G(z, w) = \frac{1}{z^2(z+1)} F'\left(\frac{w-1}{z} + \ln\left(1+\frac{1}{z}\right)\right) \quad (z \notin \{-1, 0\})\,, \tag{23}$$

and (22) converges for $|w| < |z\psi(z)|$, cf. Figure 2.

Proof: We give two proofs of this proposition, the first will be a constructive one.

The recursion (9) can be transferred into the partial differential equation

$$z(z+1)G_z + (w(z+1)-1)G_w + (3z+2)G = 0\,. \tag{24}$$

The corresponding characteristics satisfy the system

$$\frac{dz}{z(z+1)} = \frac{dw}{w(z+1)-1} = -\frac{dG}{(3z+2)G}$$

with the integrals

$$c_1 = \frac{w-1}{z} + \ln\left(1 + \frac{1}{z}\right), \quad c_2 = z^2(z+1)G\,.$$

Hence (24) has the general solution

$$G = \frac{1}{z^2(z+1)}\Phi\left(\frac{w}{z} + \psi(z)\right)$$

with (20) and an arbitrary differentiable function Φ. The initial condition $G(z,0) = 1$ yields

$$\Phi(\psi(z)) = z^2(z+1)\,.$$

Differentiation of (21) yields $\Phi = F'$ which proves (23).

Second we want to check (23) directly. The function F is singular only in the point 0. Hence, the Taylor expansion (22) converges for $|w| < |z\psi(z)|$. Comparing it with (22) we find that

$$f_n = \frac{1}{\varphi z^n} F^{(n+1)}(\psi)$$

for $n \in \mathbb{N}_0$ and therefore $F^{(n)}(\psi) = \varphi z^{n-1} f_{n-1}$ for $n \in \mathbb{N}$. Differentiation with respect to z yields

$$\left(\varphi z^{n+1} f_{n-1}\right)' = \frac{d}{dz}F^{(n)}(\psi) = \frac{1}{\varphi}F^{(n+1)}(\psi) = z^n f_n\,,$$

i.e. equation (11) which, together with $f_0 = \frac{1}{\varphi}F'(\psi) = 1$, defines f_n uniquely ■

In the excluded cases $z \in \{-1, 0\}$, where (23) is not applicable, the formulas (10) imply the special values of (22)

$$G(0,w) = \frac{1}{(1-w)^2}, \quad G(-1,w) = e^{-w}\,.$$

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Author:

Prof. i.R. Dr. Lothar Berg
Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany

e-mail: lothar.berg@mathematik.uni-rostock.de

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Lothar Berg, Manfred Krüppel

On Discrete Solutions of Two-Scale Difference Equations

ABSTRACT. Two-scale difference equations are considered at dyadic points. The corresponding solutions are called discrete solutions. They are calculated explicitly and accompanied by generating functions. As intermediate step there are also derived formulas which are valid for distributional solutions.

KEY WORDS. Two-scale difference equations, discrete solutions, distributional solutions, two-slanted matrices, generating functions

1 Introduction

Two scale difference equations

$$\varphi\left(\frac{t}{2}\right) = \sum_{n=0}^{N} c_n \varphi(t-n) \tag{1.1}$$

($t \in \mathbb{R}$) with $N \in \mathbb{N}$, $c_n \in \mathbb{C}$, $c_0 c_N \neq 0$, appear in wavelet theory and subdivision schemes, where nontrivial L^1-solutions are sought which are necessarily compactly supported, cf. [7], [9], [12]. A necessary condition for the existence of such solutions is

$$\sum_{n=0}^{N} c_n = 2^M \tag{1.2}$$

with $M \in \mathbb{N}$ (cf. [9]). Here, we require the two *boundary conditions* (cf. [3]):

(i) $\varphi(t) = 0$ for $t < 0$,

(ii) $\varphi(t)$ is equal to a polynomial $\pi(t)$ for $t > N$.

Then in (1.2) the foregoing condition $M \in \mathbb{N}$ weakens to $M \in \mathbb{Z}$ and, for $M \leq 0$, the polynomials π in (ii) are of degree $m = -M$, whereas they vanish identically for $M > 0$.

Vice versa, under the condition (1.2) with $M \in \mathbb{Z}$ there always exists a *distributional* solution of (1.1) with (i) and (ii), i.e. a solution which is a derivative of finite order of a continuous function. This solution is a *simple* one, i.e. it is nontrivial and uniquely determined up to a constant factor. For $M \le 0$ the polynomial π in condition (ii) can be calculated explicitly, and the natural extension of it to the whole set $\mathbb{R}$ is itself a simple solution of (1.1), cf. [6].

The case $N = 1$ in equation (1.1) is closely connected with de Rham's singular function which already is well investigated, cf. [1] and the literature quoted there. Hence we require in this paper

(iii) $N \ge 2$ and $M \in \mathbb{Z}$ in (1.2).

The natural domain for the solutions of (1.1) is not necessarily the whole set $\mathbb{R}$ but a subset containing with a number t also $t-1$, $t+1$, $\frac{t}{2}$ and $2t$. For this reason we introduce *discrete solutions* of (1.1), which satisfy this equation at dyadic points $t = \frac{j}{2^\ell}$ $(j \in \mathbb{Z}, \ell \in \mathbb{N}_0)$ and which satisfy the boundary conditions (at dyadic points) in the corresponding closed intervals, i.e.

(i') $\varphi(t) = 0$ for $t \le 0$,

(ii') $\varphi(t)$ is equal to a polynomial for $t \ge N$.

Discrete solutions determine uniquely corresponding continuous solutions and vice versa, but we do not require the existence of a nontrivial continuous solution. However, in the case of a discontinuous or even distributional solution there is in general no connection between such a solution and a discrete one. For discrete solutions it is convenient to restrict the arguments automatically to dyadic points. According to [6] it is also possible to calculate discrete solutions at the points $t = \frac{j}{2^\ell(2^k-1)}$ with a fixed $k \in \mathbb{N}$, but we are not concerned with this modification.

Our aim is to establish formulas for the explicit calculation of discrete solutions without claim that they are more effective than the usual subdivision algorithms. It suffices to determine the solutions in the interval $0 \le t \le 1$, since for $t > 1$ they can be extended by means of (1.1). The main result in this direction is the later formula (2.25). Preliminarily, we derive consequences of (1.1) which are even valid for distributional solutions, and if we speak about such solutions we tacitly assume that the sufficient conditions (i), (ii) and (iii) for their existence are satisfied. The main result concerning distributional solutions is contained in Theorem 2.4. Finally, discrete solutions are connected with corresponding generating functions.

Remark 1.1

1. Equalities in open intervals (with integer endpoints) for distributional solutions are valid for discrete solutions also in the endpoints. This follows from the differences in the boundary conditions.

2. Under the condition (i) equation (1.1) implies

$$\varphi(0) = c_0 \varphi(0).$$

This means for discrete solutions that the sharpening (i') of (i) is quite natural in the case $c_0 \neq 1$. In the case $c_0 = 1$ it would be possible to generalize the notion of a discrete solution allowing $\varphi(0)$ to be an arbitrary constant. An analogous generalization concerning $\varphi(N)$ would be possible in the case $c_N = 1$. We come back to these generalizations in Example 3.3.

In this paper we refer to polynomials $z_k = z_k(p, q)$, which are recursively determined by the system of two discrete two-scale difference equations

$$z_{2k} = p\, z_k, \quad z_{2k+1} = q\, z_k + z_{k+1} \quad (k \in \mathbb{N}) \tag{1.3}$$

and the initial condition

$$z_1 = 1. \tag{1.4}$$

These polynomials are intensively investigated in [2], and concerning their properties we refer to this paper. In particular, the polynomials z_k have the explicit representation

$$z_k = \sum_{j=0}^{k-1} \alpha^{\nu(j)}\, \beta^{\nu(k-1-j)} \tag{1.5}$$

where α and β are determined by

$$p = \alpha + \beta, \quad q = \alpha\, \beta \tag{1.6}$$

and where $\nu(j)$ is the binary sum-of-digits function, i.e. the number of $''1s''$ in the dyadic representation of j.

2 The main results

We begin with some notations. Equation (1.1) is connected with the polynomial

$$Q(z) = \sum_{n=0}^{N} c_n z^n \tag{2.1}$$

which we term *characteristic polynomial* of equation (1.1) in order to distinguish it from the so-called symbol in such cases where it is used with a factor $\frac{1}{2}$ or $\frac{1}{2}\sqrt{2}$. By means of (2.1) we can write condition (1.2) as $Q(1) = 2^M$. Moreover, we introduce the infinite vector

$$\Psi(t) = (\varphi(t), \varphi(t+1), \varphi(t+2), \ldots)^{\mathrm{T}} \tag{2.2}$$

and the infinite two-slanted matrix

$$\boldsymbol{A} = (c_{2j-k}) \qquad (j, k \geq 1) \tag{2.3}$$

with $c_j = 0$ for $j \notin \{0, \ldots, N\}$, so that the solution of (1.1) is equivalent to the solution of

$$\Psi\left(\frac{t+1}{2}\right) = \boldsymbol{A}\Psi(t) \tag{2.4}$$

for $t < 1$, both equations subject to (i), cf. [10], [5] and the literature quoted there. In the case (i') equation (1.1) is equivalent to (2.4) for $t \leq 1$. We also introduce the vector

$$\psi(t) = (\varphi(t), \varphi(t+1), \ldots, \varphi(t+N-2))^{\mathrm{T}} \tag{2.5}$$

and the matrix

$$A = (c_{2j-k}) \qquad (1 \leq j, k \leq N-1). \tag{2.6}$$

In the following existence theorem a *simple eigenvalue* means an eigenvalue with geometric multiplicity one:

Theorem 2.1 *Let (iii) be satisfied. For $M \geq 1$ in condition (1.2) let 1 be a simple eigenvalue of A whereas for $M \leq 0$ let 1 be no eigenvalue of A. Then (1.1) has exactly one simple discrete solution.*

Proof: Introducing suitable block matrices $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{O}$ where $\boldsymbol{O}$ is a zero matrix, we can split Ψ and $\boldsymbol{A}$ into

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \Psi(t+N-1) \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{O} & \boldsymbol{C} \end{pmatrix}.$$

Choosing $t = 1$ in (2.4), this equation turns over into the two equations

$$\psi(1) = A\psi(1) + \boldsymbol{B}\Psi(N), \qquad \Psi(N) = \boldsymbol{C}\Psi(N). \tag{2.7}$$

In the case $M \geq 1$ we have to look for a discrete solution with $\varphi(t) = 0$ for $t \geq N$. Hence $\Psi(N) = 0$, and $\psi(1)$ can be determined as a simple right eigenvector to the eigenvalue 1 of A out of the first equation in (2.7). In the case $M \leq 0$ there exists a simple polynomial solution of (1.1), cf. [6], Theorem 6.1, which determines $\Psi(N)$ as a simple vector satisfying the second equation of (2.7), and $\psi(1)$ can likewise be determined out of the first of these equations. Hence, in any case, we have a simple vector $\Psi(1)$, satisfying (2.4) for $t = 1$, and the values of Ψ at the dyadic points and therefore also the values of the discrete solution φ, follow recursively from (2.4) and (i') for $-1 \le t \le 1$. Finally, the simplicity of $\Psi(1)$ transfers to φ which finishes the proof ■

Remark 2.2 If 1 is an eigenvalue of A with the geometric multiplicity $r > 1$, then there exist r linearly independent eigenvectors and, in the case $M \ge 1$, therefore r linearly independent discrete solutions, but at most one of these can be extended to a continuous solution. Concerning the choice of the correct eigenvector $\psi(1)$ cf. [6]. If we speak about a discrete solution of (1.1), we tacitly assume that besides of the conditions (i') and (ii') either the conditions of Theorem 2.1 are satisfied or in the case of a multiple eigenvalue that a fixed eigenvector $\psi(1)$ is chosen for the construction of the discrete solution.

Concerning the multiplication of the matrix from the right $\boldsymbol{A}$ has always the eigenvalue c_N, since $c_N \neq 0$ is the entry of $\boldsymbol{A}$ for $j = k = N$, and $c_{2j-k} = 0$ for $j \ge N, k \le N$ provided that $(j,k) \neq (N,N)$.

Proposition 2.3 *To the eigenvalue c_N of $\boldsymbol{A}$ there belongs a left eigenvector*

$$\boldsymbol{x} = (x_1, x_2, x_3, \ldots) \tag{2.8}$$

with $x_1 = \ldots = x_{N-1} = 0$, $x_N = 1$. The generating function

$$G(z) = \sum_{k=0}^{\infty} x_{N+k} z^k \tag{2.9}$$

with $G(0) = 0$ has the product representation

$$G(z) = \prod_{j=0}^{\infty} \frac{z^{2^j N}}{c_N} Q\left(z^{-2^j}\right), \tag{2.10}$$

and both representations converge for $|z| < 1$.

Proof: The components of the left eigenvector (2.8) must satisfy the equations

$$\sum_{j=1}^{\infty} c_{2j-k} x_j = c_N \, x_k \tag{2.11}$$

for $k \in \mathbb{N}$. In the case $k \le N$ these equations are satisfied for $x_1 = \ldots = x_{N-1} = 0$, $x_N = 1$, and in the case $k > N$ they are recursions, which determine x_k uniquely. By multiplication with z^{k-N} and summation over k we obtain formally

$$c_N G(z) = z^N Q\left(\frac{1}{z}\right) G(z^2). \tag{2.12}$$

A continuous solution of this equation with $G(0) = 1$ must have the form (2.10) for $|z| < 1$. Vice versa, (2.10) converges uniformly for $|z| \le \varrho < 1$ and satisfies (2.12), hence also (2.9) is convergent for $|z| < 1$ ■

The next three values of x_j after $x_N = 1$ are

$$x_{N+1} = \frac{c_{N-1}}{c_N}, \quad x_{N+2} = \frac{c_{N-1} + c_{N-2}}{c_N}, \quad x_{N+3} = \left(\frac{c_{N-1}}{c_N}\right)^2 + \frac{c_{N-3}}{c_N}. \tag{2.13}$$

Besides of $\boldsymbol{x}$ from (2.8) we introduce the vector $\boldsymbol{y} = (y_1, y_2, y_3, \ldots)$, the components of which are determined by the initial values

$$y_1 = \ldots = y_{N-1} = 0, \quad y_N = 1 \tag{2.14}$$

and the recursions

$$\sum_{j=1}^{\infty} c_{N+k-2j}\, y_j = c_0\, y_k \qquad (k \in \mathbb{N}). \tag{2.15}$$

These recursions mean that

$$\left.\begin{aligned} c_{N-1}\, y_\ell + c_{N-3}\, y_{\ell+1} + \ldots &= c_0\, y_{2\ell-1}, \\ c_N\, y_\ell + c_{N-2}\, y_{\ell+1} + \ldots &= c_0\, y_{2\ell}, \end{aligned}\right\} \qquad (\ell \in \mathbb{N}). \tag{2.16}$$

The vector $\boldsymbol{y}$ arises from $\boldsymbol{x}$ if c_j is replaced by c_{N-j} for $j = 0, \ldots, N$, i.e. if (2.1) is replaced by the reversed polynomial $z^N Q(\frac{1}{z})$, cf. [3]. Hence for $|z| < 1$ we see from (2.10) that its generating function

$$F(z) = \sum_{k=0}^{\infty} y_{N+k} z^k \tag{2.17}$$

simplifies to

$$F(z) = \prod_{j=0}^{\infty} \frac{1}{c_0} Q\left(z^{2^j}\right), \tag{2.18}$$

and (2.12) simplifies to

$$F(z) = \frac{1}{c_0} Q(z) F\left(z^2\right). \tag{2.19}$$

Extending the components of the vector $\boldsymbol{y}$ by $y_j = 0$ for $j \le 0$ we can state:

Theorem 2.4 *For $\ell \in \mathbb{N}_0$ the matrix $\boldsymbol{A}^\ell$ has the entries*

$$c_0^\ell\; y_{2^\ell+N-1}, \quad c_0^\ell\; y_{2^\ell+N-2}, \ldots \tag{2.20}$$

in the first row, and the first $2^\ell + N - 1$ entries of the N^{th} row read

$$c_N^\ell\, x_1,\; c_N^\ell\, x_2,\; \ldots, c_N^\ell\, x_{2^\ell+N-1}. \tag{2.21}$$

Moreover, for non-negative integers $k < 2^\ell$ the distributional solution of (1.1) satisfies the equations

$$\varphi\left(\frac{k+t}{2^\ell}\right) = c_0^\ell \sum_{j=0}^{k} y_{N+k-j}\varphi(j+t) \qquad (t<1). \tag{2.22}$$

Proof: The statement about (2.21) is valid for $\ell = 0$ (and in view of (2.13) also for $\ell = 1$). If it is valid for a fixed $\ell \in \mathbb{N}$, then $\boldsymbol{A}^\ell \boldsymbol{A} = \boldsymbol{A}^{\ell+1}$ and (2.11) with $k \le 2^{\ell+1} + N - 1$ as well as $2j - k \le N$, i.e. $j \le 2^\ell + N - 1$, show its validity also for $\ell + 1$ instead of ℓ. Analogously, the statement about (2.20) is valid for $\ell = 0$ and proved in general, if

$$\sum_{j=1}^{\infty} c_{2j-k}\; y_{2^\ell+N-j} = c_0\; y_{2^{\ell+1}+N-k} \qquad (k \in \mathbb{N}).$$

But this equation is equivalent to (2.15).

The equation (2.4) implies that

$$\Psi\left(\frac{2^\ell+t-1}{2^\ell}\right) = \boldsymbol{A}^\ell \Psi(t) \qquad (t<1). \tag{2.23}$$

Replacing t by $t + k + 1 - 2^\ell$ with $k \le 2^\ell - 1$ and considering the properties of (2.20), we obtain for the first component of Ψ

$$\varphi\left(\frac{k+t}{2^\ell}\right) = c_0^\ell \sum_{j=1}^{\infty} y_{2^\ell+N-j}\varphi(k+t+j-2^\ell) \tag{2.24}$$

and therefore (2.22) in view of $y_n = 0$ for $n < N$ ■

Using equation (2.22) for $t = 0$ and according to Remark 1.1/1 also for $t = 1$, we obtain

Corollary 2.5 *The discrete solution of (1.1) (cf. Remark 2.2.) has the representation*

$$\varphi\left(\frac{k}{2^\ell}\right) = c_0^\ell \sum_{j=1}^{k} y_{N+k-j}\varphi(j) \tag{2.25}$$

$(0 \le k \le 2^\ell, \ell \in \mathbb{N}_0)$*, where the sequence y_j is defined by (2.14) and (2.16).*

Remark 2.6

1. Usually, one works with the matrices $T_0 = (c_{2j-k-1})$, $T_1 = (c_{2j-k})$, $j,\ k = 1,\dots,N$, instead of A, cf. [10]. The products of these matrices turn out to be submatrices of $\boldsymbol{A}^\ell$. Namely, if $K \in \mathbb{N}_0$ with $K < 2^\ell$ has the dyadic representation $K = d_\ell d_{\ell-1}\dots d_1$, $d_i \in \{0,1\}$, $i = 1,\dots,\ell$ ($d_\ell = 0$ is allowed), then the product $T_{d_\ell}T_{d_{\ell-1}}\dots T_{d_1}$ is equal to the submatrix of $\boldsymbol{A}^\ell$ with $1 \le j \le N$, $2^\ell - K \le k \le 2^\ell + N - K - 1$.

2. In the case $0 < t < 1$ the terms $\varphi(j+t)$ with $j \geq N$ in (2.22) can be replaced by the polynomials $\pi(j+t)$ known from condition (ii), so that we obtain for $k \geq N$

$$\varphi\left(\frac{k+t}{2^\ell}\right) = c_0^\ell \sum_{j=0}^{N-1} y_{N+k-j}\varphi(j+t) + c_0^\ell \sum_{j=N}^{k} y_{N+k-j}\pi(j+t) \qquad (0 < t < 1). \tag{2.26}$$

For $t = 0$ we get a modification of (2.25).

3. By means of a suitable interpretation, the foregoing results are also valid in the excluded case $N = 1$. In particular, (2.18) and (2.19) yield for $N = 1$ that $y_{k+1} = (\frac{c_1}{c_0})^{\nu(k)}$ with ν as at the end of Section 1, and (2.26) turns over into [1], (3.16).

Again for $N \geq 2$, the equations (2.22) have the disadvantage that they contain values of φ with arguments greater than 1. However, we can eliminate these values out of the first N equations or out of N arbitrary equations of (2.22) (in the version (2.26) for $k \geq N$). The result simplifies if we use additional equations as

$$\sum_{j=0}^{N-1} \xi_j \varphi(t+j) = S(t) \qquad (0 < t < 1) \tag{2.27}$$

with known coefficients ξ_j and known polynomials $S(t)$ (Concerning the existence and the number of linearly independent equations (2.27) cf. [4]). We only want to point out this in the most important case that we use only one additional relation, namely

$$\varphi(t) + \varphi(t+1) + \ldots + \varphi(t+N-1) = K \qquad (0 < t < 1), \tag{2.28}$$

which is valid in the case $M = 1$ in (1.2) almost everywhere for locally Lebesgue-integrable solutions of (1.1) where $\int_0^N \varphi(t)dt = K \neq 0$, cf. [5], [11]. In the case $M = 1$ equation (2.28), with a certain constant $K \neq 0$, is valid even for distributional solutions if and only if Q in (2.1) has a representation of the form $Q(z) = \frac{R(z^2)}{R(z)}P(z)$ where R and P are polynomials with $R(0)P(0) \neq 0$, $R(1) \neq 0$ and $P(-1) = 0$, cf. [3], Theorem 7.2. In this case it is $Q(1) = 2$ and therefore $M = 1$ in (1.2), too.

Proposition 2.7 *Let φ be a distributional solution of (1.1) such that (2.28) is satisfied. Then for fixed $k \in \mathbb{N}_0$ there exist constants Y_j $(j = 0, 1, \ldots, N-2)$, depending on k, such that for arbitrary $\ell \in \mathbb{N}$ with $2^\ell > k$ it holds*

$$\frac{1}{c_0^\ell}\varphi\left(\frac{k+t}{2^\ell}\right) = KY_{N-1} + \sum_{j=0}^{N-2} Y_j \frac{1}{c_0^{m_j}}\varphi\left(\frac{j+t}{2^{m_j}}\right) \qquad (0 < t < 1) \tag{2.29}$$

where $m_j = [\frac{\ln j}{\ln 2}] + 1$.

Proof: In view of $M = 1$ we have $\varphi(t) = 0$ for $t > N$. On the other side, $y_n = 0$ for $n < N$ and $y_N = 1$, so that for $k = 1, 2, \ldots, N-2$ and an arbitrary fixed k the equations (2.22), divided by c_0^ℓ, together with (2.28) yield the system

$$\begin{array}{rclcccccc}
c_0^{-\ell}\varphi\left(\frac{1+t}{2^\ell}\right) & = & y_{N+1}\varphi(t) & + & \varphi(t+1), & & & & \\
c_0^{-\ell}\varphi\left(\frac{2+t}{2^\ell}\right) & = & y_{N+2}\varphi(t) & + & y_{N+1}\varphi(t+1) & + & \varphi(t+2), & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
c_0^{-\ell}\varphi\left(\frac{N-2+t}{2^\ell}\right) & = & y_{2N-2}\varphi(t) & + & y_{2N-3}\varphi(t+1) & +\ldots+ & \varphi(t+N-2), & & \\
K & = & \varphi(t) & + & \varphi(t+1) & + & \ldots & + & \varphi(t+N-1), \\
c_0^{-\ell}\varphi\left(\frac{k+t}{2^\ell}\right) & = & y_{N+k}\varphi(t) & + & y_{N+k-1}\varphi(t+1) & + & \ldots & & +\, y_{k+1}\varphi(t+N-1)
\end{array}$$

with a Hessenberg matrix as coefficient matrix on the right-hand side. Choosing

$$(Y_1, Y_2, \ldots, Y_{N-1}, -1) \tag{2.30}$$

orthogonal to the last $N-1$ columns of this matrix, which is always possible, we obtain by multiplication from the left

$$c_0^{-\ell} \sum_{j=1}^{N-2} Y_j\, \varphi\left(\frac{j+t}{2^\ell}\right) + KY_{N-1} - c_0^{-\ell}\, \varphi\left(\frac{k+t}{2^\ell}\right) = -Y_0\, \varphi(t)$$

where

$$Y_0 = y_{N+k} - \sum_{j=1}^{N-2} Y_j y_{N+j} - Y_{N-1}. \tag{2.31}$$

Since $c_0^{-\ell}\varphi(\frac{j+t}{2^\ell})$ is independent of ℓ for $j \le 2^\ell - 1$, we finally obtain (2.29) for $1 \le k < 2^\ell$, but this equation is also valid for $k = 0$ with $Y_j = 0$ for $j > 0$ and $Y_0 = 1$ ■

Though k was fixed in Proposition 2.7, the result (2.29) is valid for arbitrary $k \in \{0, 1, \ldots, 2^\ell - 1\}$, where the constants Y_j are depending on k, but not on ℓ.

Proposition 2.8 *In the case $\ell > m_{N-2}$ the system (2.29) of 2^ℓ equations with $0 \le k \le 2^\ell - 1$ determines its distributional solution φ uniquely.*

Proof: The system (2.29) is an inhomogeneous one, since $K \neq 0$ and $Y_{N-1} = y_{k+1}$ is equal to 1 for $k = N-1$, and it is $N - 1 < 2^\ell$ in view of $N - 2 < 2^{m_{N-2}} < 2^\ell$. After n integrations with suitable great n (and arbitrary constants of integration) it defines a contractive operator in $C[0,1]$. Hence, the corresponding operator equation has a unique continuous solution, and the n^{th} (distributional) derivative of this is φ ■

As in the discrete case the solution φ can be extended from the interval (0,1) to greater t by means of (1.1) (only the integer values of t require an additional consideration). Note that the equations in (2.29) are nearly trivial for $0 \le k \le N-2$, since then we have $Y_k = 1$ and $Y_j = 0$ for $j \ne k$.

3 Four-coefficient equations

The foregoing results shall be illustrated by means of the equation

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-d)\varphi(t-1) + (1-a)\varphi(t-2) + d\varphi(t-3) \qquad (t \in \mathbb{R}) \tag{3.1}$$

($ad \ne 0$) with the characteristic polynomial

$$Q(z) = a + (1-d)z + (1-a)z^2 + dz^3 = (1+z)(a + (1-a-d)z + dz^2), \tag{3.2}$$

cf. (2.1), so that $N = 3$ in (1.1) and $M = 1$ in (1.2). Colella and Heil have determined the conditions for real a, d under which (3.1) has a nontrivial integrable compactly supported and, in particular, such a continuous solution, cf. Example 2 in [8]. Here we look for discrete and for distributional solutions.

The matrix

$$A = \begin{pmatrix} 1-d & a \\ d & 1-a \end{pmatrix},$$

cf. (2.6), has the eigenvalues 1 and $1-a-d$, and a right eigenvector $\psi(1)$ to the eigenvalue 1 reads

$$\begin{pmatrix} \varphi(1) \\ \varphi(2) \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix}. \tag{3.3}$$

It is simple and remains simple also in the case $d = -a\,(\ne 0)$. Hence the conditions of Theorem 2.1 are satisfied for arbitrary complex a, d and there exists a simple discrete solution of (3.1) without additional restrictions to the coefficients. According to (2.25) and (3.3) it has the representation

$$\varphi\left(\frac{k}{2^\ell}\right) = a^\ell(a\, y_{k+2} + d\, y_{k+1}), \tag{3.4}$$

$k \in \{0, 1, \ldots, 2^\ell\}$ ($\ell \in \mathbb{N}_0$), and the recursions (2.16) specialize to

$$y_{2k-1} = \frac{1-a}{a} y_k + y_{k+1}, \qquad y_{2k} = \frac{d}{a}\, y_k + \frac{1-d}{a}\, y_{k+1} \qquad (k \in \mathbb{N}). \tag{3.5}$$

The initial values (2.14) imply in particular $y_4 = \frac{1-d}{a}$.

Moreover, equation (3.1) possesses a distributional solution. In view of $Q(1) = 2$ and $Q(-1) = 0$ it satisfies (2.28) with a certain K, so that we can apply Proposition 2.7. Since the components of the vector (2.30) with $N = 3$ determine to

$$Y_1 = y_{k+2} - y_{k+1}, \qquad Y_2 = y_{k+1},$$

and since (2.31) turns over into

$$Y_0 = y_{k+3} - \frac{1-d}{a} y_{k+2} + \frac{1-a-d}{a} y_{k+1}, \tag{3.6}$$

we obtain the

Corollary 3.1 *For $k \in \{0, 1, \ldots, 2^\ell - 1\}$ ($\ell \in \mathbb{N}$), the distributional solution φ of (3.1) with (2.28) satisfies the equations*

$$\varphi\left(\frac{k+t}{2^\ell}\right) = a^\ell \left[K\, y_{k+1} + Y_0 \varphi(t) + \frac{1}{a}(y_{k+2} - y_{k+1})\varphi\left(\frac{t+1}{2}\right)\right] \quad (0 < t < 1) \tag{3.7}$$

with y_ℓ from (3.5) and Y_0 from (3.6).

It is possible to simplify the recursions (3.5) by means of the substitution

$$z_k = y_{k+2} - y_{k+1}, \tag{3.8}$$

so that $z_0 = 0$, $z_1 = 1$. A short calculation yields

$$z_{2k} = \frac{1-a-d}{a} z_k, \qquad z_{2k+1} = \frac{d}{a} z_k + z_{k+1}. \tag{3.9}$$

This means that we have the special case $z_k = z_k(p, q)$ of (1.3) and (1.4) with $p = \frac{1-a-d}{a}$, $q = \frac{d}{a}$. Since (3.8) implies

$$y_{k+1} = \sum_{j=1}^{k-1} z_j, \tag{3.10}$$

and (3.6) that $Y_0 = z_{k+1} - \frac{1-a-d}{a} z_k$, we obtain from (3.7)

$$\varphi\left(\frac{k+t}{2^\ell}\right) = a^\ell \left[K \sum_{j=1}^{k-1} z_j + \left(z_{k+1} - \frac{1-a-d}{a} z_k\right)\varphi(t) + \frac{1}{a} z_k \varphi\left(\frac{1+t}{2}\right)\right] \tag{3.11}$$

for $0 < t < 1$. According to Remark 1.1/1 the equations (3.11) can be used for a discrete solution also in the cases $t = 0$ and $t = 1$, where in view of $\varphi(\frac{1}{2}) = a^2$ and (3.9), if we choose $K = a + d$ (matching to (3.3)), we obtain the representation

$$\varphi\left(\frac{k}{2^\ell}\right) = a^{\ell+1} \sum_{j=1}^{k} z_{2j-1}, \tag{3.12}$$

$k \in \{0, 1, \ldots, 2^\ell\}$ ($\ell \in \mathbb{N}$), which is equivalent to (3.4) owing to (3.10). For $k = 2^\ell$ a simple consequence of (3.12) is

$$(p + q + 1)^\ell = \sum_{j=1}^{2^\ell} z_{2j-1}(p, q)$$

since $\varphi(1) = a$ and $\frac{1}{a} = p + q + 1$. Further properties of z_k are contained in [2].

Remark 3.2

1. In the case $d = -a$ we have $K = 0$, but (2.28) with this K is only satisfied at dyadic points, i.e. on a set with Lebesgue measure zero (whereas for a nontrivial distributional solution we recall that $K \neq 0$ there). The representation (3.12) simplifies to $\varphi(\frac{k}{2^\ell}) = a^{\ell+1} z_k$.

2. In the case $d = a$ equation (3.1) is symmetric (or self-reversed) so that $\varphi(t) = \varphi(3 - t)$ for all $t \in \mathbb{R}$ (cf. [3], Corollary 8.6). Hence, (3.1) and (2.28) with $N = 3$ and $K = 2a$ yield the relation

$$\varphi\left(\frac{1+t}{2}\right) = 2a^2 + (1 - 2a)\varphi(t) - a\varphi(1 - t) \quad (0 < t < 1)$$

which can be substituted into (3.11) with $d = a$.

Example 3.3 The case

$$\varphi\left(\frac{t}{2}\right) = \varphi(t) + \varphi(t - 3) \qquad (t \in \mathbb{R}) \tag{3.13}$$

is a specialization of (3.1) with $a = d = 1$. The recursions (3.5) read

$$y_{2\ell-1} = y_{\ell+1}, \qquad y_{2\ell} = y_\ell,$$

and subject to the initial values (2.14) they have the solution

$$y_{3k} = 1, \qquad y_{3k-1} = y_{3k-2} = 0 \qquad (k \in \mathbb{N}).$$

According to (3.3) we have $\varphi(1) = \varphi(2) = 1$, and (2.25) turns over into

$$\varphi\left(\frac{k}{2^\ell}\right) = y_{k+2} + y_{k+1} = \begin{cases} 0 & \text{for} \quad k \equiv 0 \mod 3 \\ 1 & \text{for} \quad k \not\equiv 0 \mod 3. \end{cases}$$

Equation (3.13) does not have a nontrivial continuous solution, but it has the discontinuous solution

$$\varphi(t) = \begin{cases} c & \text{for} \quad t = 0 \\ 1 & \text{for} \quad 0 < t < 3 \\ 1 - c & \text{for} \quad t = 3 \\ 0 & \text{elsewhere,} \end{cases}$$

with an arbitrary c. On the other side, it has a generalized discrete solution with conditions (i)-(ii), $\varphi(0) = c$, $\varphi(1) = \varphi(2) = 1$ and arbitrary $\varphi(3)$, which is interesting only in the case $\varphi(3) = 1 - c$.

Examples 3.4 Finally, we consider two examples with $N = 2$, which formally arise from the case $N = 3$ as limit case $c_3 \to 0$.

1. The three-coefficient equation

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + \varphi(t-1) + (1-a)\varphi(t-2) \qquad (t \in \mathbb{R})$$

is the limit case of (3.1) as $d \to 0$. For $0 < a < 1$ it has a nontrivial compactly supported continuous solution, cf. [13]. According to (1.5) with (1.6) the system (3.9) has the explicit solution

$$z_k = \left(\frac{1}{a} - 1\right)^{\nu(k-1)}.$$

The corresponding representation (3.12) is already known from [1] and [13], disregarding one factor a which stems from a different normalization.

2. The three-coefficient equation

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-2a)\varphi(t-1) + a\varphi(t-2) \qquad (t \in \mathbb{R}) \tag{3.14}$$

is neither a limit case of (3.1) nor it can be treated by means of Proposition 2.7 according to $M = 0$ in (1.2), but we can apply Theorem 2.4. Equation (3.14) possesses the constant solution $\varphi(t) = 1$, which we only use for $t \geq 2$ according to condition (ii'). For $t = 2$ we find from (3.14) and condition (i') that $\varphi(1) = \frac{1}{2}$ and therefore $\Psi(1) = (\frac{1}{2}, 1, 1, 1, \ldots)^{\mathrm{T}}$. From (2.25) we obtain the discrete solution

$$\varphi\left(\frac{k}{2^\ell}\right) = a^\ell \left(\frac{1}{2} y_{k+1} + y_k + \ldots + y_2\right), \tag{3.15}$$

and the recursions (2.16) read

$$y_{2\ell-1} = \left(\frac{1}{a} - 2\right) y_\ell, \qquad y_{2\ell} = y_\ell + y_{\ell+1}$$

with the initial values $y_1 = 0$, $y_2 = 1$. In this case we also have a connection to the sequence defined by (1.3) and (1.4), namely $y_{\ell+1} = z_\ell(p, q)$ with $p = \frac{1}{a} - 2$ and $q = 1$. Let us mention the curiosity that $z_k(p, 1)$ has the property

$$z_{3^n} = (p+1)^n \qquad \text{for } n = 0,\ 1,\ 2,\ 3 \text{ and } 6,$$

whereas for other integers n such a relation is unknown.

For the solutions of (3.14) we do not have a relation of the form (2.28). However, in view of the symmetry of (3.14) as well as (i), (ii) we have $\varphi(1+t)+\varphi(1-t)=1$ for $t\in\mathbb{R}$ (cf. [3], Corollary 8.6), so that we immediately obtain

$$\varphi\left(\frac{1+t}{2}\right) = a + (1-2a)\varphi(t) - a\varphi(1-t)$$

and after replacing t by $1-t$

$$\varphi\left(1-\frac{t}{2}\right) = a - a\varphi(t) + (1-2a)\varphi(1-t).$$

By means of these two equations and (3.14) for $0<t<1$ it can iteratively be shown for fixed $k\in\{0,1,\ldots,2^\ell-1\}$ ($\ell\in\mathbb{N}$) that $\varphi(\frac{k+t}{2^\ell})$ is a linear combination of 1, $\varphi(t)$ and $\varphi(1-t)$. But it is not necessary to carry out this procedure, since it is easy to see that $\varphi_3(t)=a\,[\varphi_2(t)-\varphi_2(t-1)]$ for all $t\in\mathbb{R}$, where φ_2 shall be the solution of (3.14) and φ_3 the solution of (3.1) with $d=a$. This means that $\varphi_2(t)=\frac{1}{a}\varphi_3(t)$ for $0<t<1$, so that the formulas (3.7) in the case $d=a$ immediately yield the formulas in question. In particular, the formulas (3.12) and (3.15) are modifications of each other.

4 Generating functions

Let φ be a discrete solution of (1.1). Condition (i') implies $\varphi\left(\frac{t}{2}\right)=c_0\varphi(t)$ for $t\le 1$. We define a function φ^* by

$$\varphi^*(t)=\varphi(t) \quad \text{for} \quad t\le 1, \tag{4.1}$$

and for $t>1$ we extend φ^* successively in the intervals $(2^{\ell-1},2^\ell]$ ($\ell\in\mathbb{N}$) by $\varphi^*(t)=\frac{1}{c_0}\varphi^*(\frac{t}{2})$, so that the equation

$$\varphi^*\left(\frac{t}{2}\right)=c_0\varphi^*(t) \tag{4.2}$$

is satisfied for all real (dyadic) t. Obviously, for $t\le 2^\ell$ ($\ell\in\mathbb{N}_0$) we have

$$c_0^{-\ell}\varphi\left(\frac{t}{2^\ell}\right)=\varphi^*(t), \tag{4.3}$$

and equation (2.25) can be written as

$$\varphi^*(k)=\sum_{j=1}^{k} y_{N+k-j}\varphi(j) \qquad (k\in\mathbb{N}_0). \tag{4.4}$$

In the following we consider the generating function

$$\Phi(z)=\sum_{k=1}^{\infty}\varphi(k)\,z^{k-1} \tag{4.5}$$

of the values $\varphi(k)$ (cf. [3], Lemma 2.9), as well as the generating function

$$\Phi^*(z) = \sum_{k=1}^{\infty} \varphi^*(k)\, z^{k-1} \tag{4.6}$$

of the values $\varphi^*(k)$ defined by (4.1) and (4.2). If $M \in \mathbb{N}$ in (1.2), then $\varphi(t) = 0$ for $t \geq N$, and (4.5) is a polynomial of degree $N-2$. Otherwise, (4.5) converges for $|z| < 1$ in view of (ii'). The convergence of (4.6) for the same z will follow from the next proposition.

Proposition 4.1 *The generating function (4.6) is representable in the form*

$$\Phi^*(z) = \Phi(z) F(z) \tag{4.7}$$

where F is given by (2.17), and the generating function (4.5) satisfies the equation

$$\Phi(z^2) = \frac{\Phi(z)\, Q(z) - \Phi(-z)\, Q(-z)}{2z} \tag{4.8}$$

with (2.1).

Proof: By multiplication of equation (4.4) with z^{k-1} and summation over k we find (4.7) with the functions (2.17) and (4.5). In view of $\varphi(t) = 0$ for $t \leq 0$, equation (1.1) with $t = 2k$ implies that

$$\varphi(k) = \sum_{j=1}^{2k} c_{2k-j}\, \varphi(j) \qquad (k \in \mathbb{N}).$$

By multiplication with z^{2k-2}, summation over k, and by means of

$$\Phi(z)Q(z) = \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} c_{\ell-j} \varphi(j) z^{\ell-1} \ ,$$

we obtain (4.8) after short calculations ■

Remark 4.2

1. The generating function Φ^* satisfies the equation

$$\frac{1}{c_0}\, \Phi^*(z^2) = \frac{\Phi^*(z) - \Phi^*(-z)}{2z} \tag{4.9}$$

which is equivalent to the specialization

$$\varphi^*(k) = c_0\, \varphi^*(2k)$$

of (4.2) for $t = 2k$, but which also follows directly from (2.19), (4.7) and (4.8).

2. Note the connection of the product representation (2.18) for the generating function (2.17) of the coefficients in (2.24) with respect to the product representation

$$\mathcal{L}\{\varphi\} = z^{M-1} \prod_{j=1}^{\infty} \frac{1}{2^M} Q\left(e^{-\frac{z}{2^j}}\right)$$

(cf. [9], p. 175) for the Laplace transform of the solution φ of (1.1) satisfying (2.22).

Example 4.3 In the simple case $Q(z) = 2\left(\frac{1+z}{2}\right)^{m+1}$ with $m = N-1 \in \mathbb{N}$ we have $y_j = x_j$ $(j \in \mathbb{N})$ in view of the symmetry of the coefficients. Formula (2.18) yields

$$F(z) = \prod_{j=0}^{\infty} \left(1 + z^{2^j}\right)^{m+1} = \frac{1}{(1-z)^{m+1}}. \tag{4.10}$$

Using the normalization $\varphi(1) = 1$, the solution of (1.1) is $\varphi(t) = m!\, N_m(t)$, where $N_m(t)$ is the B-spline of degree m. Hence, (4.5) turns over into the Euler-Frobenius polynomials

$$\Phi(z) = m! \sum_{k=1}^{m} N_m(k) z^{k-1} = E_m(z), \tag{4.11}$$

cf. [7], [14]. On the other side, we have $\varphi(t) = t^m$ for $0 \le t \le 1$ and therefore $\varphi^*(t) = t^m$ for all $t \ge 0$ so that (4.6) turns over into

$$\Phi^*(z) = \sum_{k=1}^{\infty} k^m z^{k-1}. \tag{4.12}$$

According to (4.10) and (4.11) the relation (4.7) expresses the well known fact that for $|z| < 1$ the series on the right-hand side of (4.12) has the sum

$$\frac{E_m(z)}{(1-z)^{m+1}}.$$

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Authors:

Prof. i.R. Dr. Lothar Berg, Prof. Dr. Manfred Krüppel
Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany

lothar.berg@mathematik.uni-rostock.de
manfred.krueppel@mathematik.uni-rostock.de

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Egbert Dettweiler

Characteristic Processes associated with a Discontinuous Martingale

1 Introduction

In [5] the following embedding theorem was proved (cf. [2] for a detailed and complete proof). Let $(M_k)_{k\geq 0}$ be an L^4-martingale on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then there exists an extension $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$ of Ω, a Brownian motion $(B_t)_{t\geq 0}$ relative to a filtration $(\mathcal{G}_t)_{t\geq 0}$ on $\bar{\Omega}$, and an increasing sequence $(T_n)_{n\geq 1}$ of $(\mathcal{G}_t)$-stopping times, such that the following properties hold:

(i) $M_n \;=\; B_{T_n}$ for all $n \geq 1$,

(ii) $\mathbf{E}\big\{T_n \,|M_0, M_1, \cdots\big\} \;=$

$$\tfrac{1}{3}\textstyle\sum_{k=1}^{n}(M_k - M_{k-1})^2 \;+\; \tfrac{2}{3}\sum_{k=1}^{n}\mathbf{E}\big\{(M_k - M_{k-1})^2 \,|M_0, \cdots, M_{k-1}\big\}\,,$$

(iii) $\mathrm{Var}\big\{T_n \,|M_0, M_1, \cdots\big\} \;=$

$$\begin{aligned} &\tfrac{2}{45}\textstyle\sum_{k=1}^{n}(M_k - M_{k-1})^4 \;+\; \tfrac{8}{45}\sum_{k=1}^{n}\mathbf{E}\big\{(M_k - M_{k-1})^4 \,|M_0, \cdots, M_{k-1}\big\}\\ &+ (\tfrac{4}{9} - c)\textstyle\sum_{k=1}^{n}\big(\mathbf{E}\big\{(M_k - M_{k-1})^2 \,|M_0, \cdots, M_{k-1}\big\}\big)^2\\ &+ c\textstyle\sum_{k=1}^{n}(M_k - M_{k-1})^2\mathbf{E}\big\{(M_k - M_{k-1})^2 \,|M_0, \cdots, M_{k-1}\big\}\\ &- c\textstyle\sum_{k=1}^{n}(M_k - M_{k-1})\mathbf{E}\big\{(M_k - M_{k-1})^3 \,|M_0, \cdots, M_{k-1}\big\}\,. \end{aligned}$$

The constant $c > 0$ in (iii) depends on the embedding and can be explicitely computed (cf. [2] and [5]).

In [5] a corresponding result is stated (without proof) for continuous time martingales. The roughly outlined idea in [5] is the approximation of a continuous time martingale $(M_t)_{t\geq 0}$ by the discrete time martingales $(M_{\frac{k}{2^m}})_{k\geq 0}$. But this raises the following problems, which are the subject of the present paper.

(A) In which sense do the expressions

$$\sum_{k=1}^{[2^m t]} \mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^2 \,|M_0, \cdots, M_{\frac{k-1}{2^m}}\big\}$$

up to

$$\sum_{k=1}^{[2^m t]} (M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})\mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^3 \,|M_0, \cdots, M_{\frac{k-1}{2^m}}\big\}$$

converge for $m \to \infty$?

It seems that convergence only holds under additional assumptions on $(M_t)_{t\geq 0}$. The reason is that the above conditioning is a conditioning relative to a filtration depending on m. If $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a given filtration on Ω such that $(M_t)_{t\geq 0}$ is an $\mathbf{F}$-martingale, then the following, slightly changed problem only depends on the fixed filtration $\mathbf{F}$:

(B) In which sense do the expressions

$$\sum_{k=1}^{[2^m t]} \mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^2 \mid \mathcal{F}_{\frac{k-1}{2^m}}\big\}$$

up to

$$\sum_{k=1}^{[2^m t]} (M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})\mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^3 \mid \mathcal{F}_{\frac{k-1}{2^m}}\big\}$$

converge for $m \to \infty$?

It turns out that (B) has a quite general solution. Convergence always takes place for the topology $\sigma(L^1, L^\infty)$. An inspection of the proof of the embedding theorem for continuous time martingales in [2] shows that this $\sigma(L^1, L^\infty)$-convergence is sufficient.

2 Statement of the Problem

Let Π denote the set of all partitions $\pi = (0 = t_0 < t_1 < \cdots)$ of $\mathbf{R}_+$ for which $\lim_{k\to\infty} t_k = \infty$ and $|\pi| := \sup_{k\geq 1}(t_k - t_{k-1}) < \infty$. We will tacitely identify such a partition π with its associated point set $\{t_k | k \geq 0\}$. A sequence $(\pi_n)_{n\geq 1}$ in Π will be called a *null-sequence of partitions*, if $\pi_m \subset \pi_n$ for $m \leq n$ and $\lim_{n\to\infty} |\pi_n| = 0$.

Now suppose that $M = (M_t)_{t\geq 0}$ is a right continuous martingale relative to a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$. More shortly, we will also say that $(M, \mathbf{F})$ is a martingale. $(M, \mathbf{F})$ is called a *discrete time martingale*, if there exists a partition $\pi = (0 = t_0 < t_1 < \cdots) \in \Pi$ such that $M_t = M_{t_k}$ and $\mathcal{F}_t = \mathcal{F}_{t_k}$ for $t_k \leq t < t_{k+1}$ and $k \geq 0$. More shortly, we will also say that $((M_{t_k}), (\mathcal{F}_{t_k}))$ is a discrete time martingale. If $(M, \mathbf{F})$ is a general right continuous martingale and $\pi = (0 = t_0 < t_1 < \cdots) \in \Pi$, and if we set $\pi(M) = (M_{t_k})_{k\geq 0}$ and $\pi(\mathbf{F}) = (\mathcal{F}_{t_k})_{k\geq 0}$, then clearly $(\pi(M), \pi(\mathbf{F}))$ is a discrete time martingale. Moreover, for every filtration $\mathbf{H} = (\mathcal{H}_{t_k})_{k\geq 0}$ such that

$$\sigma(M_{t_0}, \cdots, M_{t_k}) \subset \mathcal{H}_{t_k} \subset \mathcal{F}_{t_k}$$

for all $k \geq 0$, also $(\pi(M), \mathbf{H})$ is a discrete time martingale.

If $(\pi_n)_{n\geq 1}$ is a null-sequence of partitions, where $\pi_n = (0 = t_0^n < t_1^n < \cdots)$, then we will call a sequence $(\mathbf{H}^n)_{n\geq 1}$ of discrete filtrations $\mathbf{H}^n = (\mathcal{H}^n_{t_k^n})_{k\geq 0}$ an *approximating sequence of filtrations* for the given martingale $(M, \mathbf{F})$, if (i) M_t is $\mathcal{H}^n_t$-measurable for every $t \in \pi_n$ and $n \geq 1$, (ii) $\mathcal{H}^m_t \subset \mathcal{H}^n_t$ for $t \in \pi_m$ and $m \leq n$, and (iii) $\mathcal{F}_{t-} \subset \bigvee_{n\geq 1} \mathcal{H}^n_t \subset \mathcal{F}_t$ for all $t \geq 0$.

Motivated by the embedding result stated in the introduction, we give the following definition for discrete time martingales.

Definition 2.1 *Let $(M, \mathbf{F}) = ((M_{t_k}), (\mathcal{F}_{t_k}))$ be a given discrete time martingale.*
(1) *If $(M, \mathbf{F})$ is square integrable, then the processes $(M, \mathbf{F})^{(2)}$ and $(M, \mathbf{F})^{(\tilde{2})}$, defined by*

$$(M, \mathbf{F})^{(2)}_t := \sum_{k\geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})^2\,, \text{ and}$$
$$(M, \mathbf{F})^{(\tilde{2})}_t := \sum_{k\geq 1} 1_{[0,t]}(t_k)\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}\big\}\,,$$

will be called the first order characteristics *of $(M, \mathbf{F})$.*
(2) *If $(M, \mathbf{F})$ is in addition even an L^4-martingale, then the processes*

$$(M, \mathbf{F})^{(4)},\ (M, \mathbf{F})^{(\tilde{4})},\ (M, \mathbf{F})^{(2,\tilde{2})},\ (M, \mathbf{F})^{(\tilde{2},\tilde{2})},\ (M, \mathbf{F})^{(1,\tilde{3})}\,,$$

defined by

$$(M, \mathbf{F})^{(4)}_t := \sum_{k\geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})^4\,,$$
$$(M, \mathbf{F})^{(\tilde{4})}_t := \sum_{k\geq 1} 1_{[0,t]}(t_k)\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^4 \mid \mathcal{F}_{t_{k-1}}\big\}\,,$$
$$(M, \mathbf{F})^{(2,\tilde{2})}_t := \sum_{k\geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})^2\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}\big\}\,,$$
$$(M, \mathbf{F})^{(\tilde{2},\tilde{2})}_t := \sum_{k\geq 1} 1_{[0,t]}(t_k)\big(\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}\big\}\big)^2\,, \text{ and}$$
$$(M, \mathbf{F})^{(1,\tilde{3})}_t := \sum_{k\geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^3 \mid \mathcal{F}_{t_{k-1}}\big\}\,,$$

are called the second order characteristics *of* $(M, \mathbf{F})$.

Now we can formulate more precisely the problem which is the subject of the present paper: Let (π_n) be a fixed null-sequence in Π and $(M, \mathbf{F})$ be a given right continuous L^4-martingale. Suppose that $(\mathbf{H}^n)_{n\geq 1}$ is an approximating sequence of filtrations for $(M, \mathbf{F})$. Does there exists a "reasonable" topology, for which the processes $(\pi_n(M), \mathbf{H}^n)^{(2)}$, $(\pi_n(M), \mathbf{H}^n)^{(\tilde{2})}$, $(\pi_n(M), \mathbf{H}^n)^{(4)}$, $(\pi_n(M), \mathbf{H}^n)^{(\tilde{4})}$, $(\pi_n(M), \mathbf{H}^n)^{(2,\tilde{2})}$, $(\pi_n(M), \mathbf{H}^n)^{(\tilde{2},\tilde{2})}$, and $(\pi_n(M), \mathbf{H}^n)^{(1,\tilde{3})}$ converge for $n \to \infty$?

In case that $\mathbf{H}^n = \pi_n(\mathbf{F})$ for all $n \geq 1$, we will simply write $(\pi_n(M))^{(2)}$, $(\pi_n(M))^{\tilde{2}}$ etc. instead of $(\pi_n(M), \pi_n(\mathbf{F}))^{(2)}$, $(\pi_n(M), \pi_n(\mathbf{F}))^{\tilde{2}}$ etc.. Then the main result of this paper is the following theorem.

Theorem 2.2 *The limits*

$$
\begin{aligned}
(M)_t^{(2)} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(2)}, & (M)_t^{(\tilde{2})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{\tilde{2}}, \\
(M)_t^{(4)} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(4)}, & (M)_t^{(\tilde{4})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(\tilde{4})}, \\
(M)_t^{(2,\tilde{2})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(2,\tilde{2})}, & (M)_t^{(\tilde{2},\tilde{2})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(\tilde{2},\tilde{2})}, \text{ and} \\
(M)_t^{(1,\tilde{3})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(1,\tilde{3})}
\end{aligned}
$$

exist for all $t \geq 0$ *for the topology* $\sigma(L^1, L^\infty)$.

Definition 2.3 *The limit processes* $(M)^{(2)}$ *and* $(M)^{(\tilde{2})}$ *are again called the* first order characteristics *of* $(M, \mathbf{F})$*, and the limit processes* $(M)^{(4)}$, $(M)^{(\tilde{4})}$, $(M)^{(2,\tilde{2})}$, $(M)^{(\tilde{2},\tilde{2})}$, $(M)^{(1,\tilde{3})}$ *are called the* second order characteristics *of* $(M, \mathbf{F})$.

The first order characteristics $(M)^{(2)}$ and $(M)^{(\tilde{2})}$ are of course just the optional and the predictable quadratic variation $[M]$ and $<M>$, and we will use later also these more usual notations.

In the next section we present first some known results on the first order characteristics and give in that case also solutions for the above stated general approximation problem. In the last section we will finally prove theorem 2.2.

3 Compensators of increasing Processes

The existence of $[M] = (M)^{(2)}$ for the L^1-norm is well known and will not be discussed here. Since $<M>= (M)^{(\tilde{2})}$ is the so-called compensator of the increasing process $[M]$, we first recall some known results on increasing processes and their compensators.

Suppose that $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a fixed standard filtration on Ω, i.e. $\mathbf{F}$ is assumed to be right continuous and every $\mathcal{F}_t$ is assumed to contain all $\mathbf{P}$-null sets of the $\mathbf{P}$-completion of $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$. Let $X = (X_t)_{t\geq 0}$ be a given increasing, right continuous, $\mathbf{F}$-adapted process, and denote by $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ the *compensator* of X (also called the dual predictable projection of X). $\tilde{X}$ is the (up to $\mathbf{P}$-equality) unique increasing, right continuous, $\mathbf{F}$-predictable process with $\tilde{X}_0 = 0$ such that $X - \tilde{X}$ is a local martingale (see e.g. [1], theorem 15.2 for the existence and uniqueness of $\tilde{X}$).

The process X has the following general structure (cf. [1], ch.VI): there exist
(i) an increasing, continuous, $\mathbf{F}$-adapted process X^c, and
(ii) sequences $(S_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$ of predictable resp. totally inaccessible stopping times with pairwise disjoint graphs,
such that

$$X \;=\; X^c + \sum_{n\geq 1} \triangle X_{S_n}\, 1_{[S_n,\infty[} + \sum_{n\geq 1} \triangle X_{T_n}\, 1_{[T_n,\infty[}\,.$$

Let G_n $(n \geq 1)$ denote the (necessarily continuous) compensator of the jump process $\triangle X_{T_n}\, 1_{[T_n,\infty[}$. Then the compensator $\tilde{X}$ of X has the structure

$$\tilde{X} \;=\; \tilde{X}^c + \sum_{n\geq 1} \mathbf{E}\big\{\triangle X_{S_n}\,|\,\mathcal{F}_{S_n-}\big\}\, 1_{[S_n,\infty[}\,,$$

with $\tilde{X}^c = X^c + \sum_{n\geq 1} G_n$. As a consequence, one has the following equivalent assertions:
(a) X is regular (or quasi-left-continuous),
(b) X has no jumps at predictable stopping times, and
(c) $\tilde{X}$ is continuous.

The following result on the topological relations between increasing processes and their compensators can be found in [1] (ch.VII, Th.18 and 20).

Theorem 3.1 *Suppose that $(X^n)_{n\geq 0}$ is a sequence of right continuous, increasing, $\mathbf{F}$-adapted processes, which belongs uniformly to class (D). If for every stopping time T*

$$\lim_{n\to\infty} X^n_T \;=\; X^0_T \quad for \;\; \sigma(L^1, L^\infty)\,,$$

then also

$$\lim_{n\to\infty} \widetilde{X^n_T} \;=\; \widetilde{X^0_T} \quad for \;\; \sigma(L^1, L^\infty)\,,$$

where $(\widetilde{X^n})_{n\geq 0}$ denotes the associated sequence of compensators. If $(X^n)_{n\geq 0}$ is a sequence such that X^0 is regular (so that $\widetilde{X^0}$ is continuous), and if $(X^n)_{n\geq 1}$ is increasing with limit X^0, then even

$$\lim_{n\to\infty} \widetilde{X^n_T} \;=\; \widetilde{X^0_T}$$

for the L^1-norm.

As a consequence of the above theorem one gets that the compensator of an increasing process can be obtained as the limit of compensators of discrete increasing processes (cf. [1], Th.21). We give a different proof than in [1], which seems to be more elementary. Moreover, our proof can serve at the same time as an existence proof for the compensator.

Theorem 3.2 *Let $X = (X_t)_{t\geq 0}$ be a given increasing, right continuous, integrable and $\mathbf{F}$-adapted process. Let further $(\pi_n)_{n\geq 1}$ be a null-sequence of partitions of $\mathbf{R}_+$, and define for every $n \geq 1$ and $t \geq 0$*

$$C^n(X)_t \ := \ \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}\,|\,\mathcal{F}_{t_{k-1}^n}\}\,.$$

Then there exists an increasing, right continuous, integrable and predictable process $C(X)$ such that

$$\lim_{n\to\infty} C^n(X)_t \ = \ C(X)_t \quad for \quad \sigma(L^1, L^\infty)$$

for all $t \geq 0$, and such that $C(X)$ is the compensator $\tilde{X}$ of X.

Proof: Let $Z \in L^\infty(\Omega)$ be given and suppose that $(Y_t)_{t\geq 0}$ is a cadlag-modification of $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\geq 0}$. Then we get for every fixed $t \geq 0$

$$\begin{aligned}
&\mathbf{E}\{Z\,C^n(X)_t\} \\
&\quad = \ \mathbf{E}\big\{Z\sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}\,|\,\mathcal{F}_{t_{k-1}^n}\}\big\} \\
&\quad = \ \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\big\{\,Z\,\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}\,|\,\mathcal{F}_{t_{k-1}^n}\}\big\} \\
&\quad = \ \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\big\{\mathbf{E}\{Z|\mathcal{F}_{t_{k-1}^n}\}(X_{t_k^n} - X_{t_{k-1}^n})\big\} \\
&\quad = \ \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\big\{Y_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n})\big\} \\
&\quad = \ \mathbf{E}\big\{\sum_{k\geq 1} 1_{[o,t]}(t_k^n)\,Y_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n})\big\}\,.
\end{aligned}$$

If we define

$$F_n \ := \ \sum_{k\geq 1} 1_{[o,t]}(t_k^n)Y_{t_{k-1}^n}\,1_{]t_{k-1}^n,t_k^n]}\,,$$

the above equation just reads

$$\mathbf{E}\{Z\,C^n(X)_t\} \ = \ \mathbf{E}\big\{\int_0^t F_n(s)\,dX_s\big\}\,.$$

By the definition of F_n and Y we have $\lim_{n\to\infty} F_n(s) = Y_{s-}$ for every $s > 0$, and Lebesgue's theorem yields

$$\lim_{n\to\infty} \mathbf{E}\{Z\,C^n(X)_t\} \ = \ \mathbf{E}\big\{\int_0^t Y_{s-}\,dX_s\big\}\,.$$

Especially, this implies that $(C^n(X)_t)_{n\geq 1}$ is a $\sigma(L^1, L^\infty)$-Cauchy sequence. Because of

$$\mathbf{E}\, C^n(X)_t \ \leq \ \mathbf{E}\, X_t \ =: \ r_t$$

we have that $(C^n(X)_t)_{n\geq 1}$ is contained in the L^1-ball $\{X \in L^1 |\, \mathbf{E}|X| \leq r_t\}$, which is weakly sequentially complete (cf. e.g. [6], p.121). This now implies that there exists a $\bar{C}(X)_t \in L^1$ such that

$$\lim_{n\to\infty} \mathbf{E}\{Z\, C^n(X)_t\} \ = \ \mathbf{E}\{Z\, \bar{C}(X)_t\}$$

for all $Z \in L^\infty$.

Now consider the so obtained process $(\bar{C}(X)_t)_{t\geq 0}$. It is easy to see that $(\bar{C}(X)_t)_{t\geq 0}$ is $\mathbf{F}$-adapted and that

$$\bar{C}(X)_s \ \leq \ \bar{C}(X)_t$$

$\mathbf{P}$-a.s. for all $s \leq t$. Clearly, $(\bar{C}(X)_t)_{t\geq 0}$ is in general not necessarily right continuous. So let us show that $(\bar{C}(X)_t)_{t\geq 0}$ has a right continuous, increasing modification.

We choose a null set N such that for all $\omega \notin N$ and all $s, t \in \mathbf{Q}_+$ with $s \leq t$ we have

$$\bar{C}(X)_s(\omega) \ \leq \ \bar{C}(X)_t(\omega)\,.$$

Then we define

$$C(X)_t \ := \ 1_{N^c} \inf_{u>t, u\in\mathbf{Q}} \bar{C}(X)_u\,.$$

It follows from the properties of $\mathbf{F}$ that the process $(C(X)_t)_{t\geq 0}$ is an $\mathbf{F}$-adapted, increasing and right continuous process. Furthermore,

$$\begin{aligned}\mathbf{E}\{Z\, C(X)_t\} \ &= \inf\nolimits_{u>t, u\in\mathbf{Q}} \mathbf{E}\{Z\, \bar{C}(X)_u\} \ &&= \inf_{u>t, u\in\mathbf{Q}} \mathbf{E}\{\int_0^u Y_{s-}\, dX_s\} \\ &= \ \mathbf{E}\{\textstyle\int_0^t Y_{s-}\, dX_s\} &&= \ \mathbf{E}\{Z\, \bar{C}(X)_t\}\end{aligned}$$

for all $t \geq 0$ and all $Z \in L^\infty$ with $Z \geq 0$. It follows that

$$\mathbf{P}\{\, C(X)_t \ = \ \bar{C}(X)_t\,\} \ = \ 1$$

for all $t \geq 0$, i.e. $(C(X)_t)_{t\geq 0}$ is a modification of $(\bar{C}(X)_t)_{t\geq 0}$.

Altogether we have proved up to now that there exists an increasing, right continuous, integrable and $\mathbf{F}$-adapted process $(C(X)_t)_{t\geq 0}$ such that

$$\lim_{n\to\infty} \mathbf{E}\{Z\, C^n(X)_t\} \ = \ \mathbf{E}\{Z\, C(X)_t\}$$

for all $t \geq 0$ and all $Z \in L^\infty$.

It is an easy exercise to show that $X - C(X)$ is an $\mathbf{F}$-martingale. This follows from the $\sigma(L^1, L^\infty)$-convergence and the observation that for every $n \geq 1$ the process

$(X_{t_k^n} - C^n(X)_{t_k^n})_{k\geq 0}$ is a martingale relative to the filtration $(\mathcal{F}_{t_k^n})_{k\geq 0}$. Hence for the proof of the theorem it remains to show that $C(X)$ is predictable. We do this by proving the equivalent assertion that $C(X)$ is a natural process. This means that we have to show that

$$\mathbf{E}\{Y_t\, C(X)_t\} \;=\; \mathbf{E}\{\int_0^t Y_{s-}\, dC(X)_s\}$$

holds for every $t \geq 0$ and every non-negative, bounded cadlag $\mathbf{F}$-martingale $(Y_t)_{t\geq 0}$.

The idea is to apply to $C(X)$ the same procedure which we applied to X. For $n \geq m > 1$ we have the following simple relation:

$$\begin{aligned}
&C^m(C^n(X))_t \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\big\{C^n(X)_{t_j^m} - C^n(X)_{t_{j-1}^m} \,|\, \mathcal{F}_{t_{j-1}^m}\big\} \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\Big\{\sum_{k\in I_j^m} \mathbf{E}\big\{X_{t_k^n} - X_{t_{k-1}^n} \,|\, \mathcal{F}_{t_{k-1}^n}\big\} \,|\, \mathcal{F}_{t_{j-1}^m}\Big\} \\
&\qquad\qquad\qquad\qquad (I_j^m \;:=\; \{k | t_{j-1}^m < t_k^n \leq t_j^m\}) \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m) \sum_{k\in I_j^m} \mathbf{E}\big\{X_{t_k^n} - X_{t_{k-1}^n} \,|\, \mathcal{F}_{t_{j-1}^m}\big\} \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\big\{X_{t_j^m} - X_{t_{j-1}^m} \,|\, \mathcal{F}_{t_{j-1}^m}\big\} \\
&\quad= C^m(X)_t\,.
\end{aligned}$$

Now let $(Y_t)_{t\geq 0}$ be a given non-negative, bounded $\mathbf{F}$-cadlag-martingale. Then we have for every $t \geq 0$ on one side

(i) $\qquad \lim_{m\to\infty} \mathbf{E}\{Y_t\, C^m(X)_t\} \;=\; \mathbf{E}\{Y_t\, C(X)_t\}\,,$

and on the other side we get

$$\begin{aligned}
\mathbf{E}\{Y_t\, C^m(X)_t\} &\;=\; \mathbf{E}\{Y_t\, C^m(C^n(X))_t\} \quad (n \geq m) \\
&= \mathbf{E}\Big\{Y_t \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\big\{C^n(X)_{t_j^m} - C^n(X)_{t_{j-1}^m} \,|\, \mathcal{F}_{t_{j-1}^m}\big\}\Big\} \\
&= \mathbf{E}\Big\{\sum_{j\geq 1} 1_{[0,t]}(t_j^m) Y_{t_{j-1}^m}\big(C^n(X)_{t_j^m} - C^n(X)_{t_{j-1}^m}\big)\Big\}\,,
\end{aligned}$$

which gives for $n \to \infty$

$$\mathbf{E}\{Y_t\, C^m(X)_t\} \;=\; \mathbf{E}\Big\{\sum_{j\geq 1} 1_{[0,t]}(t_j^m) Y_{t_{j-1}^m}\big(C(X)_{t_j^m} - C(X)_{t_{j-1}^m}\big)\Big\}\,,$$

and hence

(ii) $\qquad \lim_{m\to\infty} \mathbf{E}\{Y_t\, C^m(X)_t\} \;=\; \mathbf{E}\{\int_0^t Y_{s-}\, dC(X)_s\}\,,$

and we have proved that $C(X)$ is natural. □

Remark 3.3 If X is continuous, then one can prove that $\lim_{n\to\infty} C^n(X)_t = C(X)_t = X_t$ even for the L^1-norm. But in general this is not true. There is a counterexample of Dellacherie and Doléans (cf. [1] for a reference).

Let $(\mathbf{H}^n)_{n\geq 1}$ be an increasing sequence of filtrations $\mathbf{H}^n = (\mathcal{H}^n_t)_{t\geq 0}$. Define $\mathbf{H}^\infty$ by $\mathcal{H}^\infty_t := \bigvee_{n\geq 1} \mathcal{H}^n_t$ for $t \geq 0$ and let $\mathbf{F}$ be the standard filtration generated by $\mathbf{H}^\infty$. In analogy to theorem 3.2 one could ask the following question (which is related to the general approximation problem stated in section 2):

Let $X = (X_t)_{t\geq 0}$ be an increasing, right continuous, $\mathbf{F}$-adapted process. Define for every $n \geq 1$

$$C^n(X)_t := \sum_{k\geq 1} 1_{[0,t]}(t^n_k)\mathbf{E}\big\{X_{t^n_k} - X_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\,.$$

Is it again true that

$$\lim_{n\to\infty} C^n(X)_t = \tilde{X}_t$$

for every $t \geq 0$ for the topology $\sigma(L^1, L^\infty)$?

Looking at the proof of theorem (3.2), there is the following problem. If $Z \in L^\infty$ is given, and if the function F_n is now defined as

$$F_n = \sum_{k\geq 1} 1_{[0,t]}(t^n_k)\mathbf{E}\big\{Z \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\, 1_{]t^n_{k-1},t^n_k]}\,,$$

then clearly

$$F_n(s) \longrightarrow Y_{s-} := \mathbf{E}\{Z|\mathcal{F}_{s-}\}$$

$\mathbf{P}$-a.s. (!) for every $s \geq 0$. But it is not at all clear (and probably not true) that outside *one* fixed null-set one has the convergence $F_n(s) \to Y_{s-}$ for all $s \geq 0$. The reason is a kind of regularity problem for two-parameter martingales: Consider

$$\big(\mathbf{E}\{Z|\mathcal{H}^n_t\}\big)_{1\leq n\leq\infty, t\geq 0}\,.$$

This is a two-parameter martingale. Does there exist a modification $(Y_{n,t})_{1\leq n\leq\infty,t\geq 0}$, such that

$$\lim_{n\to\infty, s\nearrow t} Y_{n,s} = Y_{\infty,t}$$

for all $t \geq 0$?

The following result indicates that even in case that X is a continuous process the situation differs from the situation, where just one filtration $\mathbf{F}$ is involved.

Theorem 3.4 *Let $(\pi_n)_{n\geq 1}$ be a null-sequence of partitions and $\mathbf{H}^n$ $(n \geq 1)$, $\mathbf{F}$ filtrations as above. If X is an increasing, continuous, $\mathbf{F}$-adapted process, and if*

$$C^n(X)_t := \sum_{k\geq 1} 1_{[0,t]}(t^n_k)\mathbf{E}\big\{X_{t^n_k} - X_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}$$

for $n \geq 1$ and $t \geq 0$, then for every $t \geq 0$

$$\lim_{n\to\infty} C^n(X)_t \;=\; X_t$$

for the topolgy $\sigma(L^1, L^\infty)$.

Proof: (1) For every integrable process $Y = (Y_t)_{t\geq 0}$ we set

$$C^n(X)_t \;:=\; \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\,.$$

Now assume that Y is even square integrable. Then we will first prove an elementary inequality for $\mathbf{E}\{C^n(Y)_t^2\}$. For a shorter notation we set

$$m(t) := m(t,n) := \max\{k \geq 1 |\, t_k^n \leq t\}\,.$$

Then we get

$$\begin{aligned}
\mathbf{E}\{C^n(Y)_t^2\} \;&=\; \mathbf{E}\Big(\sum_{k=1}^{m(t)} \mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\Big)^2 \\
&= \;\mathbf{E}\sum_{k=1}^{m(t)} \big(\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\big)^2 \\
&\quad + 2\sum_{k<j} \mathbf{E}\big(\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t_j^n} - Y_{t_{j-1}^n} \mid \mathcal{H}^n_{t^n_{j-1}}\big\}\big) \\
&= \;\mathbf{E}\sum_{k=1}^{m(t)} \big(\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\big)^2 \\
&\quad + 2\sum_{k<j} \mathbf{E}\big(\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t_j^n} - Y_{t_{j-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\big) \\
&\leq \; 2\sum_{k\leq j} \mathbf{E}\big(\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t_j^n} - Y_{t_{j-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\big) \\
&= \; 2\sum_{k=1}^{m(t)} \mathbf{E}\big(\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t^n_{m(t)}} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\big) \\
&= \; 2\sum_{k=1}^{m(t)} \mathbf{E}\big((Y_{t^n_{m(t)}} - Y_{t_{k-1}^n})\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\big)\;\; .
\end{aligned}$$

and we have obtained the inequality

$$\mathbf{E}\{C^n(Y)_t^2\} \leq 2\mathbf{E}\Big(\max_{0\leq k<m(t)} |Y_{t^n_{m(t)}} - Y_{t^n_k}| \sum_{k=1}^{m(t)} \big|\mathbf{E}\big\{Y_{t_k^n} - Y_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\big\}\big|\Big)\,. \qquad (3.4.1)$$

If Y is increasing we have especially

$$\begin{aligned}\mathbf{E}\{C^n(Y)_t^2\} &\le 2\,\mathbf{E}\big((Y_t - Y_0)\,C^n(Y)_t\big)\\ &\le 2\,\big(\mathbf{E}(Y_t - Y_0)^2\big)^{\frac{1}{2}}\big(\mathbf{E}(C^n(Y)_t^2)\big)^{\frac{1}{2}},\end{aligned}$$

which implies

$$\mathbf{E}\{C^n(Y)_t^2\} \le 4\,\mathbf{E}(Y_t - Y_0)^2\,. \tag{3.4.2}$$

Now suppose that $Z = X - Y$, where X and Y are increasing and square integrable processes. Then we get from (3.4.1)

$$\begin{aligned}&\mathbf{E}\{C^n(Y)_t^2\}\\ &\quad\le 2\,\mathbf{E}\Big(\max_{0\le k<m(t)} |Z_{t^n_{m(t)}} - Z_{t^n_k}|\,(C^n(X)_t + C^n(Y)_t)\Big)\\ &\quad\le 2\,\Big(\mathbf{E}\big\{\max_{0\le k<m(t)} (Z_{t^n_{m(t)}} - Z_{t^n_k})^2\big\}\Big)^{\frac{1}{2}}\,\Big(\mathbf{E}\big\{C^n(X)_t + C^n(Y)_t\big\}^2\Big)^{\frac{1}{2}},\end{aligned}$$

and (3.4.2) gives

$$\begin{aligned}&\mathbf{E}\{C^n(Y)_t^2\}\\ &\quad\le 4\,\Big(\mathbf{E}\big\{\max_{0\le k<m(t)} (Z_{t^n_{m(t)}} - Z_{t^n_k})^2\big\}\Big)^{\frac{1}{2}}\Big(\big(\mathbf{E}(X_t - X_0)^2\big)^{\frac{1}{2}} + \big(\mathbf{E}(Y_t - Y_0)^2\big)^{\frac{1}{2}}\Big).\end{aligned} \tag{3.4.3}$$

This inequality will be used in the next step of the proof.

(2) Now let X be the increasing, continuous, $\mathbf{F}$-adapted process, for which we want to prove the assertion of the theorem. In this step of the proof we also assume in addition that X is square integrable. For every $m \ge 1$ we define the process $X^m = (X^m_t)_{t\ge 0}$ by

$$X^m := \sum_{j\ge 1} X_{t^m_{j-1}}\, 1_{[t^m_{j-1}, t^m_j[}\,.$$

To prove the asserted $\sigma(L^1, L^\infty)$-convergence, let Z be an arbitrary given element of L^∞. Then

$$\mathbf{E}\{Z\,C^n(X)_t\} = \mathbf{E}\{Z\,C^n(X^m)_t\} + R^{m,n}_1(t)\,, \tag{3.4.4}$$

where

$$R^{m,n}_1(t) = \mathbf{E}\{Z\,(C^n(X)_t - C^n(X^m)_t)\}\,.$$

First we compute $\mathbf{E}\{Z\,C^n(X^m)_t\}$:

$$\begin{aligned}&\mathbf{E}\{Z\,C^n(X^m)_t\}\\ &\quad= \mathbf{E}\Big\{Z\sum_{k=1}^{m(t)} \mathbf{E}\{X_{t^n_k} - X_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\}\Big\}\\ &\quad= \mathbf{E}\Big\{\sum_{k=1}^{m(t)} \mathbf{E}\{Z \,|\, \mathcal{H}^n_{t^n_{k-1}}\}(X_{t^n_k} - X_{t^n_{k-1}})\Big\}\,,\end{aligned}$$

and with the definition

$$F_n := \sum_{k=1}^{m(t)} \mathbf{E}\{Z \mid \mathcal{H}^n_{t^n_{k-1}}\} \, 1_{[t^m_{k-1}, t^m_k[}$$

we have an **F**-predictable function such that

$$\mathbf{E}\{Z\, C^n(X^m)_t\} \;=\; \mathbf{E}\Big\{ \int_0^t F_n(s)\, dX^m_s \Big\} \,. \tag{3.4.5}$$

The increasing process X^m corresponds to the measure

$$\sum_{j\geq 1} (X_{t^m_j} - X_{t^m_{j-1}})\, \delta_{t^m_j} \,.$$

Hence, if $k(j)$ denotes that integer for which

$$t^n_{k(j)} \;=\; t^m_j \quad (n \geq m)\,,$$

then

$$\begin{aligned} &\int_0^t F_n(s)\, dX^m_s \\ &\quad = \sum_{j\geq 1} F_n(t^n_{k(j)})(X_{t^m_j} - X_{t^m_{j-1}}) \\ &\quad = \sum_{j\geq 1} 1_{[0,t]}(t^m_j)\mathbf{E}\{Z|\mathcal{H}^n_{t^n_{k(j)-1}}\}(X_{t^m_j} - X_{t^m_{j-1}})\,. \end{aligned}$$

Thus we have

$$\mathbf{E}\{Z\, C^n(X^m)_t\} = \mathbf{E}\Big(\sum_{j\geq 1} 1_{[0,t]}(t^m_j)\mathbf{E}\{Z|\mathcal{F}_{t^m_j -}\}(X_{t^m_j} - X_{t^m_{j-1}}) \Big) \;+\; R^{m,n}_2(t)\,, \tag{3.4.6}$$

with

$$R^{m,n}_2(t) := \mathbf{E}\Big(\sum_{j\geq 1} 1_{[0,t]}(t^m_j)\big[\mathbf{E}\{Z|\mathcal{H}^n_{t^n_{k(j)-1}}\} \;-\; \mathbf{E}\{Z|\mathcal{F}_{t^m_j -}\}\big]\,(X_{t^m_j} - X_{t^m_{j-1}}) \Big)\,.$$

The continuity of X implies that every X_t is $\mathcal{F}_{t-}$-measurable. This gives

$$\mathbf{E}\big(\mathbf{E}\{Z|\mathcal{F}_{t^m_j -}\}(X_{t^m_j} - X_{t^m_{j-1}})\big) \;=\; \mathbf{E}\big\{Z\,(X_{t^m_j} - X_{t^m_{j-1}})\big\}\,,$$

and we get from (3.4.6) that

$$\mathbf{E}\{Z\, C^n(X^m)_t\} \;=\; \mathbf{E}\{Z\,(X_t - X_0)\} \;+\; R^{m,n}_2(t)$$

and altogether we have proved that

$$\mathbf{E}\{Z\, C^n(X)_t\} = \mathbf{E}\{Z\,(X_t - X_0)\} \;+\; R^{m,n}_1(t) \;+\; R^{m,n}_2(t) \tag{3.4.7}$$

for $m \le n$. So we have to prove for the asserted $\sigma(L^1, L^\infty)$-convergence that for every $\varepsilon > 0$ there exists an $m \ge 1$ such that for every $n \ge m$ we have

$$|R_1^{m,n}(t)| < \varepsilon \quad \text{and} \quad |R_2^{m,n}(t)| < \varepsilon\,.$$

For $R_1^{m,n}(t)$ we use the inequality (3.4.3). With $D := \|Z\|_\infty$ we get

$$\begin{aligned} |R_1^{m,n}(t)| &\le D\,\mathbf{E}|C^n(X)_t - C^n(X^m)_t| \\ &= D\,\mathbf{E}\,C^n(X - X^m)_t \\ &\le D\left(\mathbf{E}\{C^n(X-X^m)_t\}^2\right)^{\frac{1}{2}} \end{aligned}$$

and hence

$$\begin{aligned} &R_1^{m,n}(t)^4 \\ &\le D^4\left(\mathbf{E}\{C^n(X-X^m)_t\}^2\right)^2 \\ &\le 2^4 D^4 \mathbf{E}\Big\{\max_{0\le k<m(t)}\left((X-X^m)_{t^n_{m(t)}} - (X-X^m)_{t^n_k}\right)^2\Big\} \\ &\qquad\qquad \cdot\left([\mathbf{E}(X_t - X_0)^2]^{\frac{1}{2}} + [\mathbf{E}(X_t^m - X_0^m)^2]^{\frac{1}{2}}\right)^2 \\ &\le 2^6 D^4 \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{0\le k<m(t)}\left((X-X^m)_{t^n_{m(t)}} - (X-X^m)_{t^n_k}\right)^2\Big\} \\ &\le 2^7 D^4 \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{0\le k\le m(t)}\left((X-X^m)_{t^n_k}\right)^2\Big\} \\ &\le 2^7 D^4 \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{j\ge 1} 1_{[0,t]}(t_j^m)(X_{t_j^m} - X_{t_{j-1}^m})^2\Big\}\,. \end{aligned}$$

Since X is assumed to be continuous, it follows easily that

$$\lim_{m\to\infty}\mathbf{E}\Big\{\max_{j\ge 1} 1_{[0,t]}(t_j^m)(X_{t_j^m} - X_{t_{j-1}^m})^2\Big\} \;=\; 0\,,$$

and thus we have proved that there exists an $m_1 = m_1(\varepsilon)$ such that for all $n \ge m \ge m_1$ we have $|R_1^{m,n}(t)| < \varepsilon$.

For $R_2^{m,n}(t)$ we get from the definition

$$R_2^{m,n}(t)^2 \le \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{j\ge 1}\Big(1_{[0,t]}(t_j^m)\big(\mathbf{E}\{Z\,|\mathcal{H}^n_{t^n_{k(j)-1}}\} \;-\; \mathbf{E}\{Z|\mathcal{F}_{t_j^m -}\}\big)\Big)^2\Big\}\,.$$

Since $t^n_{k(j)-1} \nearrow t_j^m$ for $n \to \infty$, we have

$$\bigvee_{n\ge 1}\mathcal{H}^n_{t^n_{k(j)-1}} \;=\; \mathcal{H}^\infty_{t_j^m -}\,,$$

and it is easy to show that $\mathbf{E}\{Z|\mathcal{H}^\infty_{t-}\} = \mathbf{E}\{Z|\mathcal{F}_{t-}\}$ for all $t > 0$. Hence

$$\lim_{n\to\infty}\mathbf{E}\{Z|\mathcal{H}^n_{t^n_{k(j)-1}}\} \;=\; \mathbf{E}\{Z|\mathcal{F}_{t_j^m -}\}$$

$\mathbf{P}$-a.s. for all $j \geq 1$. Since $Z \in L^\infty$, it follows from Lebesgue's theorem that $\lim_{n\to\infty} R_2^{m,n}(t) = 0$. Hence for a given $\varepsilon > 0$ and $m \geq 1$ there exists an $n(\varepsilon, m) \geq m$ such that

$$|R_2^{m,n}(t)| \,<\, \varepsilon$$

for all $n \geq n(\varepsilon, m)$.

So let $\varepsilon > 0$ be given. Then first we can choose an $m(\varepsilon)$ such that $|R_1^{m(\varepsilon),n}(t)| < \varepsilon$ for all $n \geq m(\varepsilon)$. Then we choose an $n(\varepsilon) := n(\varepsilon, m(\varepsilon)) \geq m(\varepsilon)$ such that $|R_2^{m(\varepsilon),n}(t)| \,<\, \varepsilon$ for all $n \geq n(\varepsilon)$. Then we obtain from (3.4.7)

$$|\,\mathbf{E}\{Z\,C^n(X)_t\} \,-\, \mathbf{E}\{Z\,(X_t - X_0)\}\,| \,<\, 2\varepsilon$$

for all $n \geq n(\varepsilon)$. Since $\varepsilon > 0$ and $Z \in L^\infty$ were arbitrarily chosen, we have proved the asserted $\sigma(L^1, L^\infty)$-convergence under the assumption that X is square integrable.

(3) Now just suppose that X is integrable. For $Z \in L^\infty$ and $\varepsilon > 0$ given we choose a constant $c > 0$ such that

$$\mathbf{E}\{X_t \,-\, (X \wedge c)_t\} \,<\, \frac{\varepsilon}{2\|Z\|_\infty}\,.$$

Then we have

$$\mathbf{E}\{Z\,C^n(X)_t\} - \mathbf{E}\{Z\,(X_t - X_0)\} = \mathbf{E}\{Z\,C^n(X \wedge c)_t\} - \mathbf{E}\{Z((X \wedge c)_t - (X \wedge c)_0)\} + R_t^{n,c}\,,$$

where

$$R_t^{n,c} \;=\; \mathbf{E}\{Z(C^n(X)_t \,-\, C^n(X \wedge c)_t)\} \,+\, \mathbf{E}\{Z\,((X_t - (X \wedge c)_t) - (X_0 - (X \wedge c)_0))\}\,.$$

Then

$$|R_t^{n,c}| \,\leq 2\|Z\|_\infty \mathbf{E}(X \,-\, (X \wedge c))_t \,<\, \varepsilon\,.$$

This proves finally the assertion of the theorem for the general case that X is only assumed to be integrable. □

One special application of the theorem is formulated in the following corollary.

Corollary 3.5 *Let $M = (M_t)_{t\geq 0}$ be a square integrable martingale and let $\mathbf{F}$ be the standard filtration generated by the canonical filtration of M. Let as before $(\pi_n)_{n\geq 1}$ denote a null-sequence of partitions and define for every $n \geq 1$ the filtration $\mathbf{H}^n = (\mathcal{H}_t^n)_{t\geq 0}$ by*

$$\mathcal{H}_t^n \;:=\; \sigma\big(\{M_{t_k^n}\,|\,t_k^n \leq t\}\big)\;.$$

If the predictable quadratic variation $<M>$ of M is continuous, then

$$\lim_{n\to\infty} \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2\,|\,\mathcal{H}_{t_{k-1}^n}^n\big\} \;= <M>_t$$

for every $t > 0$ in the topology $\sigma(L^1, L^\infty)$.

To get convergence in the L^1-norm, it seems that in general one has to assume a stronger condition on the limit process X than just continuity. In principle, it will be a condition implying that there is a kind of uniform nearness of the conditional expectations $\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}|\mathcal{H}^n_{t_{k-1}^n}\}$ to $\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}|\mathcal{F}_{t_{k-1}^n}\}$. The condition which we impose on X in the following theorem is a condition which is usually fulfilled in applications (survival analysis, queuing theory, risk theory etc.).

Theorem 3.6 *Let $(\mathbf{H}^n)_{n\geq 1}$ and $\mathbf{F}$ be given as before, and let X be an increasing, right continuous, $\mathbf{F}$-adapted and integrable process. Suppose that in addition X has $\mathbf{P}$-a.s. a density Z relative to a non-random measure μ on $\mathbf{R}_+$, i.e.*

$$X_t \;=\; \int_0^t Z_s\,\mu(ds)$$

for every $t > 0$ $\mathbf{P}$-a.s.. Then $\lim_{n\to\infty} C^n(X)_t \;=\; X_t$ in the L^1-norm for every $t > 0$.

Proof: (1) We will denote by $\mathcal{P}$ the predictable σ-algebra on $\mathbf{R}_+\times\Omega$ for the filtration $\mathbf{F}$. For every $n\geq 1$ we define then $\mathcal{P}_n$ as the sub-σ-algebra of $\mathcal{P}$ generated by the sets $\{0\}\times A$, where $A\in\mathcal{H}^n_0$ and by the sets $]t^n_{k-1},t^n_k]\times A$, where $A\in\mathcal{H}^n_{t^n_{k-1}}$ and $k\geq 1$. Furthermore, we define a sequence $(M_n)_{n\geq 1}$ of maps $M_n : \mathbf{R}_+\times\Omega\;\to\;\mathbf{R}_+$ by

$$M_n \;:=: \sum_{k\geq 1}\frac{\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}|\mathcal{H}^n_{t_{k-1}^n}\}}{\mu(]t^n_{k-1},t^n_k])}\,1_{]t^n_{k-1},t^n_k]}\,, \tag{3.6.1}$$

with the (admissible) convention that

$$\frac{\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}|\mathcal{H}^n_{t_{k-1}^n}\}}{\mu(]t^n_{k-1},t^n_k])} \;:=\; 0\;\;,$$

in case that $\mu(]t^n_{k-1},t^n_k]) \;=\; 0$. Now let $t > 0$ be given. We set $\mathcal{P}^t := \mathcal{P}\cap([0,t]\times\Omega)$ and $\mathcal{P}^t_n := \mathcal{P}_n\cap([0,t]\times\Omega)$ for $n\geq 1$. Then we define the probability space

$$\Omega^t \;:=\; \Big([0,t]\times\Omega,\mathcal{P}^t,\frac{\mu}{\mu([0,t])}\otimes\mathbf{P}\Big)\,.$$

In the following we will view $(M_n)_{n\geq 1}$ as a process on Ω^t. By definition, every M_n is $\mathcal{P}^t_n$-measurable, i.e. $(M_n)_{n\geq 1}$ is a $(\mathcal{P}^t_n)$-adapted process. It is also clear that $(M_n)_{n\geq 1}$ is integrable. We will prove now that $(M_n)_{n\geq 1}$ is even a $(\mathcal{P}^t_n)$-martingale.

We set $\mu_t \;:=\; \dfrac{\mu}{\mu([0,t])}$. For $m < n$ we take a set $A\in\mathcal{H}^m_{t^m_{j-1}}$. Furthermore, we set $I^m_j := \{k\geq 1|\,t^m_{j-1} < t^n_k \leq t^m_j\}$ and

$$Y^n_k \;:=\; \frac{\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}|\mathcal{H}^n_{t_{k-1}^n}\}}{\mu(]t^n_{k-1},t^n_k])}\,.$$

Then

$$\begin{aligned}
&\int_{]t^m_{j-1},t^m_j]\times A} M_n\, d(\mu_t\otimes \mathbf{P}) \\
&= \int 1_{]t^m_{j-1},t^m_j]\times A}\Big(\sum_{k\in I^m_j} Y^n_k\, 1_{]t^n_{k-1},t^n_k]}\Big)\, d(\mu_t\otimes\mathbf{P}) \\
&= \sum_{k\in I^m_j}\int_\Omega\int_{\mathbf{R}_+} 1_A\, Y^n_k\, 1_{]t^n_{k-1},t^n_k]}\, d\mu_t\, d\mathbf{P} \\
&= \sum_{k\in I^m_j}\mu_t(]t^n_{k-1},t^n_k])\int_A Y^n_k\, d\mathbf{P} \\
&= \sum_{k\in I^m_j}\frac{\mu_t(]t^n_{k-1},t^n_k])}{\mu(]t^n_{k-1},t^n_k])}\int_A \mathbf{E}\{X_{t^n_k}-X_{t^n_{k-1}}|\mathcal{H}^n_{t^n_{k-1}}\}\, d\mathbf{P} \\
&= \mu([0,t])^{-1}\int_A \mathbf{E}\{X_{t^m_j}-X_{t^m_{j-1}}|\mathcal{H}^m_{t^m_{j-1}}\}\, d\mathbf{P} \\
&= \frac{\mu(]t^m_{j-1},t^m_j])}{\mu([0,t])}\int_A\frac{\mathbf{E}\{X_{t^m_j}-X_{t^m_{j-1}}|\mathcal{H}^m_{t^m_{j-1}}\}}{\mu(]t^m_{j-1},t^m_j])}\, d\mathbf{P} \\
&= \int_\Omega\int_{\mathbf{R}_+} 1_A\, 1_{]t^m_{j-1},t^m_j]}\, d\mu_t\, d\mathbf{P} \\
&= \int_{]t^m_{j-1},t^m_j]\times A} M_m\, d(\mu_t\otimes\mathbf{P})\,.
\end{aligned}$$

Since the sets $]t^m_{j-1},t^m_j]\times A$ ($j\geq 1$, $A\in\mathcal{H}^m_{t^m_{j-1}}$) together with the sets $\{0\}\times A$ ($A\in\mathcal{H}^m_0$) form a $\cap$-stable system generating $\mathcal{P}_m$, it follows from the just proved equation that $(M_n)_{n\geq 1}$ is a $(\mathcal{P}^t_n)$-martingale.

Since in addition, $(M_n)_{n\geq 1}$ is non-negative, the martingale convergence theorem implies that $(M_n)_{n\geq 1}$ is a.s. convergent.

(2) Now we prove that $(M_n)_{n\geq 1}$ also converges in $L^1([0,t]\times\Omega)$. By (1) it is necessary and sufficient to show that $(M_n)_{n\geq 1}$ is uniformly integrable. We prove this under the momentary extra condition that the process $(Z_s)_{0\leq s\leq t}$ belongs to $L^2([0,t]\times\Omega)$. Then it is sufficient to prove that

$$\sup_{n\geq 1}\int M_n^2\, d(\mu_t\otimes\mathbf{P})\;<\;\infty\,.$$

For every $s\in]t^n_{k-1},t^n_k]$ we have

$$\begin{aligned}
M_n(s,\cdot) &= \frac{\mathbf{E}\{X_{t^n_k}-X_{t^n_{k-1}}|\mathcal{H}^n_{t^n_{k-1}}\}}{\mu(]t^n_{k-1},t^n_k])} \\
&= \frac{\int_{]t^n_{k-1},t^n_k]}\mathbf{E}\{Z_r\,|\mathcal{H}^n_{t^n_{k-1}}\}\,\mu(dr)}{\mu(]t^n_{k-1},t^n_k])}\,,
\end{aligned}$$

and hence

$$M_n(s,\cdot)^2 \;\le\; \frac{\int_{]t^n_{k-1},t^n_k]} \big(\mathbf{E}\{Z_r\,|\mathcal{H}^n_{t^n_{k-1}}\}\big)^2\,\mu(dr)}{\mu(]t^n_{k-1},t^n_k])} \;\le\; \frac{\int_{]t^n_{k-1},t^n_k]} \mathbf{E}\{Z_r^2\,|\mathcal{H}^n_{t^n_{k-1}}\}\,\mu(dr)}{\mu(]t^n_{k-1},t^n_k])}\,.$$

Therefore we obtain

$$\begin{aligned}
&\int M_n^2\,d(\mu_t\otimes\mathbf{P})\\
&\le \int_{[0,t]} \Big(\sum_{k\ge1}\mu(]t^n_{k-1},t^n_k])^{-1}1_{]t^n_{k-1},t^n_k]}(u)\int_{]t^n_{k-1},t^n_k]}\mathbf{E}\big(\mathbf{E}\{Z_r^2\,|\mathcal{H}^n_{t^n_{k-1}}\}\big)\,\mu(dr)\Big)\,\mu_t(du)\\
&= \int_{[0,t]} \Big(\sum_{k\ge1}\mu(]t^n_{k-1},t^n_k])^{-1}1_{]t^n_{k-1},t^n_k]}(u)\big(\int_{]t^n_{k-1},t^n_k]}\mathbf{E}\,Z_r^2\,\mu(dr)\big)\Big)\,\mu_t(du)\\
&= \int_{[0,t]} Z^2\,d(\mu_t\otimes\mathbf{P})\,,
\end{aligned}$$

and it follows that $(M_n)_{n\ge1}$ is uniformly integrable. Since $(M_n)_{n\ge1}$ is also $(\mu_t\otimes\mathbf{P})$-a.s. convergent, there exists an $L^1([0,t]\times\Omega)$-limit M of $(M_n)_{n\ge1}$. Because of

$$\mathbf{E}\,\Big|\int_{[0,t]} M_n\,d\mu_t \;-\; \int_{[0,t]} M\,d\mu_t\Big| \;\le\; \mathbf{E}\int_{[0,t]}|M_n-M|\,d\mu_t$$

and the observation that

$$\begin{aligned}
&\int_{[0,t]} M_n\,d\mu_t\\
&= \mu([0,t])^{-1}\sum_{k\ge1}1_{[0,t]}(t^n_k)\mathbf{E}\{X_{t^n_k}-X_{t^n_{k-1}}|\mathcal{H}^n_{t^n_{k-1}}\}\\
&= \mu([0,t])^{-1}\,C^n(X)_t\,,
\end{aligned}$$

we have especially proved that

$$\lim_{n\to\infty} C^n(X)_t \;=\; X_t$$

in the L^1-norm for every $t>0$.

(3) Finally, if $(Z_s)_{0\le s\le t}$ is not assumed to be square integrable, we set for every constant $c>0$

$$X_t^c \;:=\; \int_{[0,t]}(Z_s\wedge c)\,\mu(ds)\,.$$

For any given $\varepsilon>0$ it is then possible to find a $c=c(\varepsilon)$ such that

$$\mathbf{E}\,X_t \;-\; \mathbf{E}\,X_t^c \;<\; \varepsilon\,.$$

If we denote by M_n^c the map of step (2) associated with X^c, then $(M_n^c)_{n\geq 1}$ converges in $L^1([0,t]\times\Omega)$ by step (2). Furthermore, we have

$$\begin{aligned}
&\int_{[0,t]\times\Omega} (M_n \,-\, M_n^c)\, d(\mu_t \otimes \mathbf{P}) \\
&\leq \int_{[0,t]} \Bigg(\sum_{k\geq 1} \mu(]t_{k-1}^n, t_k^n])^{-1} 1_{]t_{k-1}^n, t_k^n]}(u) \\
&\qquad \cdot \int_\Omega \int_{]t_{k-1}^n, t_k^n]} \big| \mathbf{E}\{Z_r | \mathcal{H}^n_{t_{k-1}^n}\} \,-\, \mathbf{E}\{Z_r \wedge c | \mathcal{H}^n_{t_{k-1}^n}\}\big|\, \mu(dr)\, d\mathbf{P} \Bigg)\, \mu_t(du) \\
&\leq \int_\Omega \int_{[0,t]} (Z_r \,-\, Z_r \wedge c)\, \mu_t(dr)\, d\mathbf{P} \\
&= \mu([0,t])^{-1} \mathbf{E}(X_t \,-\, X_t^c)\,.
\end{aligned}$$

Now it follows easily that $(M_n)_{n\geq 1}$ is a Cauchy sequence in $L^1([0,t]\times\Omega)$ and hence convergent. As in the last part of step (2) one finally gets $\lim_{n\to\infty} C^n(X)_t \,=\, X_t$ in $L^1(\Omega)$. This finishes the proof of the theorem. □

4 Existence of the Second Order Characteristics

In this section we will prove that in the sense of theorem 2.2 all second order characteristics exist for every L^4-martingale. So let $M \,=\, (M_t)_{t\geq 0}$ be in the following a fixed cadlag $\mathbf{F}$-martingale such that $M_t \in L^4(\Omega)$ for all $t \geq 0$. We will often make use of the following structure of M (cf. [3], ch.4). First we have the decomposition

$$M \,=\, M^c \,+\, M^d$$

into two L^4-martingales. M^c is a continuous martingale, called the *continuous part* of M. The martingale M^d, called the *purely discontinuous part* of M, has the structure

$$M^d \,=\, M^p \,+\, M^t\,,$$

where the martingales M^p and M^t are called the *predictable* and the *totally inaccessible part* resp. of M. We have

$$M^p \,=\, \sum_{i\geq 1} X_i\, 1_{[S_i,\infty[}\,,$$

where the S_i are predictable stopping times and $X_i \,=\, \triangle M_{S_i}$. M^t has the structure

$$M^t \,=\, \sum_{j\geq 1} \big(Y_j\, 1_{[T_j,\infty[} \,-\, \tilde{A}^j \big)\,.$$

Here the T_j are totally inaccessible stopping times, $\tilde{A}^j$ denotes the (continuous) compensator of $Y_j\, 1_{[T_j,\infty[}$ and $Y_j \,=\, \triangle M_{T_j}$. Moreover, the family

$$\big\{\, T \,|\, T = S_i \text{ or } T = T_j,\ i,j \in \mathbf{N} \,\big\}$$

may and will be assumed to have pairwise disjoint graphs. Then the following orthogonality relation holds:

$$\mathbf{E}\, M_t^2 \;=\; \mathbf{E}\,(M_t^c)^2 \;+ \sum_{i\geq 1} \mathbf{E}\big(X_i^2\, 1_{[S_i\leq t]}\big) \;+ \sum_{j\geq 1} \mathbf{E}\big(Y_j\, 1_{[T_j\leq t]} \;-\; \tilde{A}_t^j\big)^2 .$$

Theorem 4.1 (Existence of $(M)^{(4)}$) *For every $t \geq 0$,*

$$(M)_t^{(4)} \;:=\; \lim_{n\to\infty} \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\,(M_{t_k^n} \;-\; M_{t_{k-1}^n})^4$$

exists in L^1 and (with the notations introduced above)

$$(M)_t^{(4)} \;=\; \sum_{i\geq 1} X_i^4\, 1_{[S_i\leq t]} \;+\; \sum_{j\geq 1} Y_j^4\, 1_{[T_j\leq t]} \,.$$

Proof: (1) First we assume that M^c and M^d are bounded by some constant $C > 0$. For every $n \geq 1$ we set as before

$$m(t) = m_n(t) := \max\{k \geq 1 | t_k^n \leq t\}\,.$$

Then we have

$$\sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n})^4 \;= \sum_{k=1}^{m(t)} (M^d_{t_k^n} - M^d_{t_{k-1}^n})^4 \;+\; \sum_{i=1}^{4} A_i^n(t)\,,$$

where

$$\begin{aligned}
A_1^n(t) \;&:=\; 4\sum_{k=1}^{m(t)} (M^c_{t_k^n} - M^c_{t_{k-1}^n})(M^d_{t_k^n} - M^d_{t_{k-1}^n})^3\,,\\
A_2^n(t) \;&:=\; 6\sum_{k=1}^{m(t)} (M^c_{t_k^n} - M^c_{t_{k-1}^n})^2(M^d_{t_k^n} - M^d_{t_{k-1}^n})^2\,,\\
A_3^n(t) \;&:=\; 4\sum_{k=1}^{m(t)} (M^c_{t_k^n} - M^c_{t_{k-1}^n})^3(M^d_{t_k^n} - M^d_{t_{k-1}^n})\,, \text{ and}\\
A_4^n(t) \;&:=\; \sum_{k=1}^{m(t)} (M^c_{t_k^n} - M^c_{t_{k-1}^n})^4\,.
\end{aligned}$$

We assert that $\lim_{n\to\infty} A_i^n(t) = 0$ in $L^1(\Omega)$ for $i = 1, 2, 3, 4$. Since the proofs are very similar, we only show that $\lim_{n\to\infty} A_1^n(t) = 0$ in $L^1(\Omega)$. For the proof of that assertion we will use the obvious inequality

$$|A_1^n(t)| \leq 8C \max_k |M^c_{t_k^n} - M^c_{t_{k-1}^n}| \sum_{k=1}^{m(t)} (M^d_{t_k^n} - M^d_{t_{k-1}^n})^2\,.$$

Now let $\varepsilon > 0$ be given. Since

$$\lim_{n\to\infty} \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^2 \;=\; [M^d]_t$$

in $L^1(\Omega)$, we can find an $n_1 \in \mathbf{N}$ such that

$$\mathbf{E}\Big| \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^2 \;-\; [M^d]_t\Big| \;<\; \frac{\varepsilon}{48\,C^2}$$

for all $n \geq n_1$. For every $m \geq 1$ and $\eta > 0$ we set

$$\Omega_{m,\eta} := \Big\{\, \omega \mid \max_k |M^c_{t^n_k}(\omega) - M^c_{t^n_{k-1}}(\omega)| \;<\; \eta \text{ for all } n \geq m \,\Big\}.$$

Then the sequence $(\Omega_{m,\eta})_{m\geq 1}$ is increasing with $\bigcup_{m\geq 1} \Omega_{m,\eta} \;=\; \Omega$. We choose

$$\eta \;=\; \frac{1}{24\,C\,\mathbf{E}\,[M^d]_t} \quad .$$

Then there exists an $m \in \mathbf{N}$ such that

$$\mathbf{E}\big\{\, 1_{\Omega^c_{m,\eta}} [M^d]_t \big\} \;<\; \frac{\varepsilon}{48\,C^2}\,.$$

With $n_0 := \max(n_1, m)$ we have for all $n \geq n_0$

$$\begin{aligned}
&\mathbf{E}\,|A^n_1(t)| \\
&\leq\; 16\,C^2\,\mathbf{E}\Big|\sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^2 - [M^d]_t\Big| + 8\,C\,\mathbf{E}\big(\max_k |M^c_{t^n_k} - M^c_{t^n_{k-1}}| \cdot [M^d]_t\big) \\
&<\; \frac{\varepsilon}{3} \;+\; 8\,C \int_{\Omega_{m,\eta}} \max_k |M^c_{t^n_k} - M^c_{t^n_{k-1}}| \cdot [M^d]_t \, d\mathbf{P} \;+\; 16\,C^2 \int_{\Omega^c_{m,\eta}} [M^d]_t \, d\mathbf{P} \\
&<\; \frac{\varepsilon}{3} \;+\; 8\,C\,\eta\,\mathbf{E}[M^d]_t \;+\; 16\,C^2\,\mathbf{E}\big\{1_{\Omega^c_{m,\eta}} [M^d]_t\big\} \;\;<\; \varepsilon
\end{aligned}$$

by the choice of n_0. Since $\varepsilon > 0$ was arbitrary, we have proved that $\lim_{n\to\infty} A^n_1(t) \;=\; 0$ in $L^1(\Omega)$ and $\lim_{n\to\infty} A^n_i(t) \;=\; 0$ $(i = 2, 3, 4)$ follows similarly.

It remains to prove the existence of

$$\lim_{n\to\infty} \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^4$$

in $L^1(\Omega)$. This is proved similarly to the proof of the existence of $[M^d]$ in [3] (th. 18.6). So let us assume first that the number of jumps of M on $[0, t]$ is bounded, which means that

$$M^d_s \;=\; \sum_{i=1}^{k_1} X_i\, 1_{[S_i \leq s]} \;+\; \sum_{j=1}^{k_2} (Y_j\, 1_{[T_i \leq s]} - \tilde{A}^j_s)$$

for $k_1, k_2 \in \mathbf{N}$ and $s \leq t$. Since $(M^d_s)_{s\leq t}$ is now a process of finite variation having only finite many jumps, one can prove that even pathwise

$$\lim_{n\to\infty}\sum_{k=1}^{m(t)}(M^d_{t^n_k} - M^d_{t^n_{k-1}})^4 \;=\; \sum_{i=1}^{k_1} X_i^4, 1_{[S_i\leq t]} \;+\; \sum_{j=1}^{k_2} Y_j^4\, 1_{[T_i\leq t]}\,,$$

and the boundedness assumption on M implies that this convergence also takes place in the L^1-norm.

If the number of jumps of M on $[0,t]$ is not bounded, we set

$$M^{d,m} \;:=\; \sum_{i=1}^{m} Z_i\,,$$

where every Z_i is either of the form $X_i 1_{[S_i,\infty[}$ or of the form $(Y_j 1_{[T_j,\infty[} - \tilde{A}^j)$. For a shorter notation we will write

$$(N)^{(4),n}_t \;:=\; \sum_{k=0}^{m(t)}(N_{t^n_k} - N_{t^n_{k-1}})^4$$

for every L^4-martingale $N = (N_t)_{t\geq 0}$. Now set

$$\bar{M}^{d,m} \;:=\; M^d - M^{d,m}\,.$$

Then

$$(M^d)^{(4),n}_t \;-\; (M^{d,m})^{(4),n}_t \;=\; \sum_{i=1}^{4} B_i^{n,m}\,,$$

where

$$\begin{aligned}
B_1^{n,m}(t) &:= 4\sum_{k=1}^{m(t)}(M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^3(\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})\,,\\
B_2^{n,m}(t) &:= 6\sum_{k=1}^{m(t)}(M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^2(\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})^2\,,\\
B_3^{n,m}(t) &:= 4\sum_{k=1}^{m(t)}(M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})(\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})^3\,, \text{ and}\\
B_4^{n,m}(t) &:= \sum_{k=1}^{m(t)}(\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})^4\,,
\end{aligned}$$

and we prove that $\lim_{m\to\infty} B_i^{n,m} = 0$ in L^1 for $i = 1,2,3,4$ uniformly in n. By the boundedness assumption we have

$$\begin{aligned}
|B_1^{n,m}| &\leq 4\,C\Big(\sum_{k=1}^{m(t)}(M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^2\Big)\max_k|\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}}|\\
&\leq 8\,C\sup_{s\leq t}|\bar{M}^{d,m}_s|\sum_{k=1}^{m(t)}(M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^2\,.
\end{aligned}$$

It follows from the Burkholder-Davis-Gundy-inequality (applied to $M^{d,m}$) that

$$\mathbf{E}|B_1^{n,m}| \;\le\; K\,\big(\mathbf{E}\sup_{s\le t}(M_s^{d,m})^4\big)^{\frac{1}{2}}\big(\mathbf{E}\sup_{s\le t}(\bar{M}_s^{d,m})^2\big)^{\frac{1}{2}}$$

for some constant $K>0$. Since

$$\lim_{m\to\infty}\mathbf{E}\sup_{s\le t}(\bar{M}_s^{d,m})^2 \;=\; 0\,,$$

we have proved that $\lim_{m\to\infty} B_1^{n,m} = 0$ in L^1 uniformly in n. Similar proofs show that also $\lim_{m\to\infty} B_i^{n,m} = 0$ ($i=2,3,4$) in L^1 uniformly in n. Hence under the boundedness assumption on M we have proved altogether that the sequence $((M)_t^{(4),n})_{n\ge1}$ converges in L^1 and has the asserted limit.

(2) If M^c and M^d are not necessarily bounded, we introduce for every $C>0$ the set

$$F_C \;:=\; \big\{\,\omega \mid \sup_{s\le t}|M_s^c(\omega)\le C \;\text{ and }\; \sup_{s\le t}|M_s^d(\omega)\le C\,\big\}\,.$$

Similar as in (1) one can prove that

$$\lim_{n\to\infty} 1_{F_C}\,(M)_t^{(4),n}$$

exists in L^1. Since $\lim_{C\nearrow\infty}\mathbf{P}\,F_C^c = 0$, it follows that the sequence $((M)_t^{(4),n})_{n\ge1}$ converges in probability. For the asserted L^1-convergence it is therefore sufficient to show that $((M)_t^{(4),n})_{n\ge1}$ is uniformly integrable. Now, for every $A\in\mathcal{F}$ we have

$$\begin{aligned}
&\mathbf{E}\big(1_A\sum_{k=1}^{m(t)}(M_{t_k^n}-M_{t_{k-1}^n})^4\big)\\
&\le\; 2\,\mathbf{E}\big(1_A\sup_{s\le t}M_s^2\sum_{k=1}^{m(t)}(M_{t_k^n}-M_{t_{k-1}^n})^2\big)\\
&\le\; 2\,K\,\big(\mathbf{E}\,1_A\sup_{s\le t}M_s^4\big)^{\frac{1}{2}}\big(\mathbf{E}\sup_{s\le t}M_s^4\big)^{\frac{1}{2}}\,,
\end{aligned}$$

where the last line follows again from the Burkholder-Davis-Gundy-inequality. The uniform integrability is now obvious and the theorem is proved. □

Theorem 4.2 (Existence of $(M)^{(\tilde{4})}$) *For every $t\ge0$,*

$$(M)_t^{(\tilde{4})} \;:=\; \lim_{n\to\infty}\sum_{k=1}^{m(t)}\mathbf{E}\big\{(M_{t_k^n}-M_{t_{k-1}^n})^4 \mid \mathcal{F}_{t_{k-1}^n}\big\}$$

exists for the topology $\sigma(L^1,L^\infty)$. Moreover,

$$(M)^{(\tilde{4})} \;=\; \sum_{i\ge1}\mathbf{E}\{X_i^4\mid\mathcal{F}_{S_i-}\}1_{[S_i,\infty[} \;+\; \sum_{j\ge1}\tilde{B}^j\,,$$

where the processes $\tilde{B}^j$ are the (continuous) compensators of the $Y_j^4\,1_{[T_j,\infty[}$.

Proof: We take an arbitrary, fixed $Z \in L^\infty$ and denote by $(Y_t)_{t\geq 0}$ a cadlag-modification of $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\geq 0}$. Then we have

$$\begin{aligned}
&\mathbf{E}\, Z\, (M)_t^{(\tilde{4}),n} \\
&:= \mathbf{E}\, Z \sum_{k=1}^{m(t)} \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^4 \mid \mathcal{F}_{t_{k-1}^n}\big\} \\
&= \mathbf{E} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M_{t_k^n} - M_{t_{k-1}^n})^4 \,.
\end{aligned}$$

As in the proof of theorem 4.1 one can now show that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M_{t_k^n} - M_{t_{k-1}^n})^4 \\
&= \lim_{n\to\infty} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M^d_{t_k^n} - M^d_{t_{k-1}^n})^4 \\
&= \int_{[0,t]} Y_{s-}\, d(M)_s^{(4)}
\end{aligned}$$

in L^1. The process

$$\widetilde{(M)^{(4)}} = \sum_{i\geq 1} \mathbf{E}\{X_i^4 \mid \mathcal{F}_{S_i -}\} 1_{[S_i,\infty[} + \sum_{j\geq 1} \tilde{B}^j$$

is the compensator of $(M)^{(4)}$, and hence

$$\mathbf{E} \int_{[0,t]} Y_{s-}\, d(M)_s^{(4)} = \mathbf{E} \int_{[0,t]} Y_{s-}\, d\widetilde{(M)^{(4)}}_s \,.$$

From

$$\begin{aligned}
&\big|\mathbf{E}\, Z\, (M)_t^{(\tilde{4}),n} - \mathbf{E}\, Z\, \widetilde{(M)^{(4)}}_t \big| \\
&= \big|\mathbf{E}\, Z\, (M)_t^{(\tilde{4}),n} - \mathbf{E} \int_{[0,t]} Y_{s-}\, d\widetilde{(M)^{(4)}}_s \big| \\
&\leq \mathbf{E} \Big| \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M_{t_k^n} - M_{t_{k-1}^n})^4 - \int_{[0,t]} Y_{s-}\, d(M)_s^{(4)} \Big|
\end{aligned}$$

we get that

$$\lim_{n\to\infty} (M)_t^{(\tilde{4}),n} = \widetilde{(M)^{(4)}}_t$$

for $\sigma(L^1, L^\infty)$ and the theorem is proved. □

Theorem 4.3 (Existence of $(M)^{(2,\tilde{2})}$) *For every $t \geq 0$,*

$$(M)_t^{(2,\tilde{2})} := \lim_{n\to\infty} \sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n})^2 \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2 \mid \mathcal{F}_{t_{k-1}^n}\big\}$$

exists for the topology $\sigma(L^1, L^\infty)$, *and*

$$(M)^{(2,\tilde{2})} = \sum_{i\geq 1} X_i^2 \mathbf{E}\{X_i^2 \mid \mathcal{F}_{S_i-}\} 1_{[S_i,\infty[} .$$

Proof: We set $\bar{M}^p := M - M^p$. Then we have

$$\begin{aligned}
&(M)_t^{(2,\tilde{2}),n} \\
&:= \sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n})^2 \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2 \mid \mathcal{F}_{t_{k-1}^n}\big\} \\
&= \sum_{k=1}^{m(t)} \big((\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n}) + (M^p_{t_k^n} - M^p_{t_{k-1}^n})\big)^2 \\
&\qquad\qquad \cdot \mathbf{E}\big\{\big((\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n}) + (M^p_{t_k^n} - M^p_{t_{k-1}^n})\big)^2 | \mathcal{F}_{t_{k-1}^n}\big\} \\
&= \sum_{k=1}^{m(t)} (M^p_{t_k^n} - M^p_{t_{k-1}^n})^2 \mathbf{E}\big\{ <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n} | \mathcal{F}_{t_{k-1}^n}\big\} \\
&\qquad\qquad + C_1^n + C_2^n + 2C_3^n ,
\end{aligned}$$

where

$$\begin{aligned}
C_1^n &:= \sum_{k=1}^{m(t)} (\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n})^2 \mathbf{E}\big\{ <M>_{t_k^n} - <M>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\} , \\
C_2^n &:= \sum_{k=1}^{m(t)} (M^p_{t_k^n} - M^p_{t_{k-1}^n})^2 \mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\} , \text{ and} \\
C_3^n &:= \sum_{k=1}^{m(t)} (\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n})(M^p_{t_k^n} - M^p_{t_{k-1}^n}) \mathbf{E}\big\{ <M>_{t_k^n} - <M>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\} .
\end{aligned}$$

First we prove that $\lim_{n\to\infty} C_i^n = 0$ in L^1 $(i = 1, 2, 3)$. We have

$$\begin{aligned}
&\mathbf{E}\,|C_1^n| \\
&\quad= \mathbf{E}\Big(\sum_{k=1}^{m(t)} \mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\big(<M>_{t_k^n} - <M>_{t_{k-1}^n} \big)\Big) \\
&\quad\leq \mathbf{E}\Big(\max_k \mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\} <M>_t \Big) \\
&\quad\leq \Big[\mathbf{E}\Big(\max_k \big(\mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\big)^2\Big)\Big]^{\frac{1}{2}} \big[\mathbf{E} <M>_t^2 \big]^{\frac{1}{2}} .
\end{aligned}$$

Now

$$\begin{aligned}
&\mathbf{E} \max_k \left(\mathbf{E}\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} | \mathcal{F}_{t_{k-1}^n} \} \right)^2 \\
&\leq \mathbf{E} \sum_{k=1}^{m(t)} \left(\mathbf{E}\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} | \mathcal{F}_{t_{k-1}^n} \} \right)^2 \\
&\leq \mathbf{E} \sum_{k=1}^{m(t)} \left(<\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} \right)^2,
\end{aligned}$$

and the last expression tends to zero for $n \to \infty$, since $<\bar{M}^p>$ is continuous. So we have proved that $\lim_{n\to\infty} C_1^n = 0$ in L^1.

Again using the continuity of $<\bar{M}^p>$, one can prove similarly that also $\lim_{n\to\infty} C_2^n = 0$ in L^1.

For the proof that $\lim_{n\to\infty} C_3^n = 0$ we first show that one can reduce the problem to the case where the number of jumps of M on $[0, t]$ is bounded. Then

$$C_3^n = D_1^n + D_2^n$$

with

$$D_1^n := \sum_{k=1}^{m(t)} (M^c_{t_k^n} - M^c_{t_{k-1}^n})(M^p_{t_k^n} - M^p_{t_{k-1}^n}) \mathbf{E}\{ <M>_{t_k^n} - <M>_{t_{k-1}^n} | \mathcal{F}_{t_{k-1}^n} \}$$

and

$$D_2^n := \sum_{k=1}^{m(t)} (M^t_{t_k^n} - M^t_{t_{k-1}^n})(M^p_{t_k^n} - M^p_{t_{k-1}^n}) \mathbf{E}\{ <M>_{t_k^n} - <M>_{t_{k-1}^n} | \mathcal{F}_{t_{k-1}^n} \}.$$

Then $\lim_{n\to\infty} \mathbf{E} D_1^n = 0$ because of the continuity of M^c, and $\lim_{n\to\infty} \mathbf{E} D_2^n = 0$, since the finite many jumps of M^t and M^p have pairwise disjoint graphs.

As far we have proved that for the existence of $(M)^{(2,\tilde{2})}$ we only have to prove that

$$(M^p)_t^{(2,\tilde{2})} = \lim_{n\to\infty} (M^p)_t^{(2,\tilde{2}),n}$$

exists for $\sigma(L^1, L^\infty)$, where

$$(M^p)_t^{(2,\tilde{2}),n} := \sum_{k=1}^{m(t)} (M^p_{t_k^n} - M^p_{t_{k-1}^n})^2 \, \mathbf{E}\{ <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n} | \mathcal{F}_{t_{k-1}^n} \}.$$

Again, one first proves that this problem can be reduced to the case that M^p has only finite many jumps. We omit this proof and assume now that

$$M^p = \sum_{i=1}^{m} X_i \, 1_{[S_i, \infty[}.$$

To prove the assertion for an M^p of the above form, we need the following lemma.

Lemma 4.4 *Let $X = (X_t)_{t\geq 0}$ be an increasing, right continuous, integrable, $\mathbf{F}$-predictable process. For every stopping time T we set*

$$\tilde{X}_T^n \;:=\; \sum_{k\geq 1}\Big(\sum_{j=1}^{k}\mathbf{E}\{X_{t_j^n} - Xt_{j-1}^n | \mathcal{F}_{t_{j-1}^n}\}\Big)\, 1_{]t_{k-1}^n, t_k^n]}(T)\,,\ \text{and}$$
$$\bar{X}_T^n \;:=\; \sum_{k\geq 1}\Big(\sum_{j=1}^{k-1}\mathbf{E}\{X_{t_j^n} - Xt_{j-1}^n | \mathcal{F}_{t_{j-1}^n}\}\Big)\, 1_{]t_{k-1}^n, t_k^n]}(T)$$

for every $n \geq 1$. If T is predictable, then

$$\lim_{n\to\infty} (\tilde{X}_T^n - \bar{X}_T^n) = \triangle X_T$$

for the topoloy $\sigma(L^1, L^\infty)$.

Proof of the lemma: It follows from theorem 3.2 that $\lim_{n\to\infty} \tilde{X}_T^n = X_T$ for $\sigma(L^1, L^\infty)$. Hence it remains to prove that $\lim_{n\to\infty} \bar{X}_T^n = X_{T-}$ for $\sigma(L^1, L^\infty)$.

Let $(T_i)_{i\geq 1}$ be an announcing sequence for T. From theorem 3.2 we have $\lim_{n\to\infty} \tilde{X}_{T_i}^n = X_{T_i}$ for $\sigma(L^1, L^\infty)$ uniformly in $i \in \mathbf{N}$. Moreover, the $\sigma(L^1, L^\infty)$-convergence of $(\tilde{X}_{T_i}^n)_{n\geq 1}$ implies (cf. [4],p.20) that for every fixed non-negative $Z \in L^\infty$ the sequence $(Z\,\tilde{X}_{T_i}^n)_{n\geq 1}$ is uniformly integrable for every $i \geq 1$. Hence for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, i)$ such that

$$\mathbf{E}\, 1_{[T - T_i \leq \delta]}\, Z\, \tilde{X}_{T_i}^n \;<\; \varepsilon\,.$$

Now, for every $\omega \in [T - T_i > \delta]$ there exists an $n(\delta)$ such that

$$\tilde{X}_{T_i(\omega)}^n(\omega) \;\leq\; \bar{X}_{T(\omega)}^n(\omega)$$

for $n \geq n(\delta)$. Therefore,

$$\mathbf{E}\, Z\, \tilde{X}_{T_i}^n \;\leq\; \mathbf{E}\, 1_{[T - T_i > \delta]}\, Z\, \tilde{X}_{T_i}^n + \varepsilon \;\leq\; \mathbf{E}\, Z\, \bar{X}_T^n + \varepsilon$$

for every $n \geq n(\delta)$. It follows that

$$\mathbf{E}\, Z\, X_{T_i} \;=\; \lim_{n\to\infty} \mathbf{E}\, Z\, \tilde{X}_{T_i}^n \;\leq \limsup_{n\to\infty} \mathbf{E}\, Z\, \bar{X}_T^n + \varepsilon\,.$$

On the other side, we have for every fixed $n \geq 1$

$$\mathbf{E}\, Z\, \tilde{X}_T^n \;=\; \lim_{i\to\infty} \mathbf{E}\, Z\, \tilde{X}_{T_i}^n\,,$$

and hence

$$\mathbf{E}\, Z\, \bar{X}_T^n \;\leq\; \mathbf{E}\, Z\, \tilde{X}_T^n \;\leq\; \mathbf{E}\, Z\, \tilde{X}_{T_i}^n + \varepsilon$$

for all $i \geq i(n, \varepsilon)$. Since

$$\lim_{n\to\infty} \mathbf{E}\, Z\, \tilde{X}^n_{T_i} \;=\; \mathbf{E}\, Z\, X_{T_i}$$

uniformly in $i \geq 1$, there exists an $n(\varepsilon)$ such that for $n \geq n(\varepsilon)$ and for an $i(n)$,

$$\mathbf{E}\, Z\, \tilde{X}^n_{T_{i(n)}} \;\leq\; \mathbf{E}\, Z\, X_{T_{i(n)}} \;+\; \varepsilon\,.$$

Hence we have

$$\limsup_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, X_{T_{i(n)}} \;+\; 2\,\varepsilon\,.$$

So far we have proved

$$\begin{aligned}
\mathbf{E}\, Z\, X_{T-} \;&=\; \lim_{i\to\infty} \mathbf{E}\, Z\, X^n_{T_i} \\
&\leq\; \liminf_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;+\; \varepsilon \\
&\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;+\; \varepsilon \\
&\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, X_{T_{i(n)}} \;+\; 3\,\varepsilon\,,
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{E}\, Z\, X_{T-} \;-\; \varepsilon \;&\leq\; \liminf_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \\
&\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \\
&\leq\; \mathbf{E}\, Z\, X_{T-} \;+\; 2\,\varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen,

$$\lim_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;=\; \mathbf{E}\, Z\, X_{T-}$$

and the lemma is proved. □

Now we complete the proof of theorem 4.3 using the above lemma. We set

$$M^j \;=\; X_j\, 1_{[S_j,\infty[}\,.$$

Then $M^p = \sum_{j=1}^{m(t)} M^j$. For a fixed $j = 1, \cdots, m$ we set $T := S_j$ and $X := <M^p>$. Then

$$\begin{aligned}
&\sum_{k=1}^{m(t)} (M^j_{t^n_k} - M^j_{t^n_{k-1}})^2\, \mathbf{E}\big\{ <M^p>_{t^n_k} \;-\; <M^p>_{t^n_{k-1}} \,|\mathcal{F}_{t^n_{k-1}} \big\} \\
&\quad=\; \sum_{k=1}^{m(t)} X_j^2\, 1_{]t^n_{k-1},t^n_k]}(T)\mathbf{E}\{X_{t^n_k} - X t^n_{k-1} | \mathcal{F}_{t^n_{k-1}}\} \\
&\quad=\; X_j^2\, (\tilde{X}^n_T \;-\; \bar{X}^n_T)\, 1_{[T\leq t]}\,.
\end{aligned}$$

Suppose first that the jump height X_j is bounded. Then we get from the lemma that

$$\lim_{n\to\infty} X_j^2\,(\tilde{X}_T^n \,-\, \bar{X}_T^n)\,1_{[T\le t]} \;=\; X_j^2 \triangle X_T\, 1_{[T\le t]}$$
$$=\; X_j^2 \triangle <M^p>_{S_j}\; 1_{[S_j\le t]} \;=\; X_j^2\,\mathbf{E}\{X_j^2|\mathcal{F}_{S_j-}\}\,1_{[S_j\le t]}$$

for $\sigma(L^1, L^\infty)$. By a uniform integrability argument, this limit relation also holds, if X_j is not necessarily bounded, and we have thus proved that for $\sigma(L^1, L^\infty)$

$$\lim_{n\to\infty} (M^p)_t^{(2,\tilde{2}),n} \;=\; \sum_{i=1}^{m} X_i^2\,\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\,1_{[S_i\le t]}\,.$$

This finishes the proof of the theorem. □

Theorem 4.5 (Existence of $(M)^{(\tilde{2},\tilde{2})}$) *For every $t \ge 0$,*

$$(M)_t^{(\tilde{2},\tilde{2})} \;:=\; \lim_{n\to\infty} \sum_{k=1}^{m(t)} \Big(E\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2 \,|\, \mathcal{F}_{t_{k-1}^n}\big\}\Big)^2$$

exists for the topology $\sigma(L^1, L^\infty)$ and

$$(M)^{(\tilde{2},\tilde{2})} \;=\; \sum_{i\ge 1} \big(\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\big)^2\, 1_{[S_i,\infty[}\,.$$

Proof: With similar arguments as in the proof of theorem 4.3 one first shows that the $\sigma(L^1, L^\infty)$-convergence of

$$(M)_t^{(\tilde{2},\tilde{2}),n} \;:=\; \sum_{k=1}^{m(t)} \Big(E\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2 \,|\, \mathcal{F}_{t_{k-1}^n}\big\}\Big)^2$$

can be reduced to the problem of the $\sigma(L^1, L^\infty)$-convergence of

$$(M^p)_t^{(\tilde{2},\tilde{2}),n} \;:=\; \sum_{k=1}^{m(t)} \Big(E\big\{(M^p_{t_k^n} - M^p_{t_{k-1}^n})^2 \,|\, \mathcal{F}_{t_{k-1}^n}\big\}\Big)^2,$$

and that one can even assume that M^p has only a finite number of jumps, i.e. we assume that

$$M^p \;=\; \sum_{i=1}^{m} X_i\, 1_{[S_i,\infty[}\,.$$

Now let Z be a fixed non-negative element of $L^\infty(\Omega)$ and denote by $(Y_t)_{t\ge 0}$ a cadlag-modification of the martingale $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\ge 0}$. Then we have

$$\mathbf{E}\,Z\,(M^p)_t^{(\tilde{2},\tilde{2}),n}$$
$$=\; \mathbf{E}\,Z \sum_{k=1}^{m(t)} \Big(E\big\{\, <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n} \;|\, \mathcal{F}_{t_{k-1}^n}\big\}\Big)^2$$
$$=\; \mathbf{E} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n}\big(\, <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n}\,\big)\mathbf{E}\big\{\, <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n} \;|\, \mathcal{F}_{t_{k-1}^n}\big\}\,.$$

Since

$$<M^p> = \sum_{i=1}^{m} \mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\, 1_{[S_i,\infty[}\,,$$

we have to prove for the convergence of the sequence $(\mathbf{E}\, Z\,(M^p)_t^{(\tilde{2},\tilde{2}),n})_{n\geq 1}$ that for every fixed $i = 1,\cdots,m$ the sequence

$$\left(\mathbf{E}\sum_{k=1}^{m(t)} Y_{t_{k-1}^n}\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\, 1_{]t_{k-1}^n,t_k^n]}(S_i)\mathbf{E}\Big\{\, <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n}\ |\,\mathcal{F}_{t_{k-1}^n}\Big\}\right)_{n\geq 1}$$

converges. As in the proof of the last theorem we show the convergence of that sequence by proving that the sequence

$$\left(\sum_{k=1}^{m(t)} Y_{t_{k-1}^n}\, 1_{]t_{k-1}^n,t_k^n]}(S_i)\mathbf{E}\Big\{\, <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n}\ |\,\mathcal{F}_{t_{k-1}^n}\Big\}\right)_{n\geq 1}$$

converges for $\sigma(L^1, L^\infty)$.

We set $T := S_i$ and $X :=<M^p>$ and define for every $t \geq 0$,

$$\tilde{Y}_t^n \ := \sum_{k\geq 1}\Big(\sum_{j=1}^{k} Y_{t_{j-1}^n}\mathbf{E}\{X_{t_j^n} - X_{t_{j-1}^n}|\,\mathcal{F}_{t_{j-1}^n}\}\Big)\, 1_{]t_{k-1}^n,t_k^n]}(t)\,.$$

Then we assert that the sequence $(\tilde{Y}_t^n)_{n\geq 1}$ converges for $\sigma(L^1, L^\infty)$. We take an arbitrary $V \in L^\infty$ and a cadlag-modification $(U_t)_{t\geq 0}$ of $(\mathbf{E}\{V|\mathcal{F}_t\})_{t\geq 0}$. Then

$$\begin{aligned}
&\lim_{n\to\infty}\mathbf{E}\, V\,\tilde{Y}_t^n\\
&\quad= \lim_{n\to\infty}\ \mathbf{E}\sum_{k=1}^{m(t)} U_{t_{k-1}^n}Y_{t_{k-1}^n}\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}|\,\mathcal{F}_{t_{k-1}^n}\}\\
&\quad= \lim_{n\to\infty}\ \mathbf{E}\sum_{k=1}^{m(t)} U_{t_{k-1}^n}Y_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n})\\
&\quad= \mathbf{E}\int_0^t U_{s-}\, Y_{s-}\, dX_s\\
&\quad= \mathbf{E}\int_0^t U_{s-}\, dW_s\\
&\qquad\quad \text{(where } W \text{ is the predictable process given by } W_t = \int_0^t Y_{s-}dX_s)\\
&\quad= \mathbf{E}\, U_t\int_0^t Y_{s-}\, dX_s\\
&\quad= \mathbf{E}\, V\int_0^t Y_{s-}\, dX_s\,,
\end{aligned}$$

and we have proved that

$$\lim_{n\to\infty} \tilde{Y}^n_t = \int_0^t Y_{s-}\, dX_s$$

for the $\sigma(L^1, L^\infty)$-topology. more general, if T is a stopping time such that $X_T \in L^1$, then

$$\lim_{n\to\infty} \tilde{Y}^n_S = \int_0^S Y_{s-}\, dX_s$$

for $\sigma(L^1, L^\infty)$ uniformly on the set of all stopping times S with $S \le T$.

Now we define for every $n \ge 1$ and $t \ge 0$,

$$\bar{Y}^n_t := \sum_{k\ge 1} \Big(\sum_{j=1}^{k-1} Y_{t^n_{j-1}} \mathbf{E}\{X_{t^n_j} - X_{t^n_{j-1}} \,|\, \mathcal{F}_{t^n_{j-1}}\} \Big)\, 1_{]t^n_{k-1}, t^n_k]}(t)\,.$$

Then one can prove exactly as in the proof of lemma 4.4 that

$$\lim_{n\to\infty} (\tilde{Y}^n_T - \bar{Y}^n_T) = \int_0^T Y_{s-}\, dX_s - \int_0^{T-} Y_{s-}\, dX_s = Y_{T-} \triangle X_T$$

for $\sigma(L^1, L^\infty)$, using the predictability of T. So we have proved that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=1}^{m(t)} Y_{t^n_{k-1}}\, 1_{]t^n_{k-1}, t^n_k]}(S_i) \mathbf{E}\big\{ <M^p>_{t^n_k} - <M^p>_{t^n_{k-1}} \,|\, \mathcal{F}_{t^n_{k-1}} \big\} \\
&= Y_{S_i-} \triangle <M^p>_{S_i}\, 1_{[S_i \le t]} \\
&= Y_{S_i-}\, \mathbf{E}\{X_i^2 | \mathcal{F}_{S_i-}\}\, 1_{[S_i \le t]}
\end{aligned}$$

for $\sigma(L^1, L^\infty)$. Assuming first that the jump X_i is bounded, the just proved relation implies

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\Big(\sum_{k=1}^{m(t)} Y_{t^n_{k-1}} \mathbf{E}\{X_i^2 | \mathcal{F}_{S_i-}\}\, 1_{]t^n_{k-1}, t^n_k]}(S_i) \cdot \mathbf{E}\big\{ <M^p>_{t^n_k} - <M^p>_{t^n_{k-1}} \,|\, \mathcal{F}_{t^n_{k-1}} \big\} \Big) \\
&= \mathbf{E}\Big(Y_{S_i-} \big(\mathbf{E}\{X_i^2 | \mathcal{F}_{S_i-}\}\big)^2\, 1_{[S_i \le t]} \Big) \\
&= \mathbf{E}\Big(Z \big(\mathbf{E}\{X_i^2 | \mathcal{F}_{S_i-}\}\big)^2\, 1_{[S_i \le t]} \Big)\,.
\end{aligned}$$

Again, one can prove by a uniform integrability argument that the last limit relation also holds for general X_i, and the theorem is proved. □

Theorem 4.6 (Existence of $(M)^{(1,\tilde{3})}$) *For every $t \ge 0$,*

$$(M)^{(1,\tilde{3})}_t := \lim_{n\to\infty} \sum_{k=1}^{m(t)} (M_{t^n_k} - M_{t^n_{k-1}})\, \mathbf{E}\big\{ (M_{t^n_k} - M_{t^n_{k-1}})^3 \,|\, \mathcal{F}_{t^n_{k-1}} \big\}$$

exists for the $\sigma(L^1, L^\infty)$-topology, and

$$(M)^{(1,\tilde{3})} = \sum_{i\ge 1} X_i \mathbf{E}\{X_i^3 \,|\, \mathcal{F}_{S_i-}\}\, 1_{[S_i, \infty[}\,.$$

Proof: Partially,we just indicate the ideas of the proof, since the arguments are often similar to those in the proofs of the foregoing results.
(1) First one shows that

$$(M)^{(3)}_t \;:=\; \lim_{n\to\infty} \sum_{k=1}^{m(t)} (M_{t^n_k} - M_{t^n_{k-1}})^3$$

exists in L^1 and that

$$(M)^{(3)} \;=\; \sum_{i\geq 1} X_i^3 \, 1_{[S_i,\infty[} \;+\; \sum_{j\geq 1} Y_j^3 \, 1_{[T_j,\infty[} \,.$$

The proof is similar to the proof of theorem 4.1. Now let $\tilde{C}^j$ denote the continuous compensator of $Y_j^3 \, 1_{[T_j,\infty[}$. Then the compensator of $(M)^{(3)}$ is given by

$$(M)^{(\tilde{3})} := \sum_{i\geq 1} \mathbf{E}\{X_i^3 | \mathcal{F}_{S_i-}\} \, 1_{[S_i,\infty[} \;+\; \sum_{j\geq 1} \tilde{C}^j \,.$$

(2) For every $n \geq 1$ and $t \geq 0$ we set

$$(M)^{(\tilde{3}),n}_t \;:= \sum_{k=1}^{m(t)} \mathbf{E}\big\{(M_{t^n_k} - M_{t^n_{k-1}})^3 \mid \mathcal{F}_{t^n_{k-1}}\big\} \,.$$

Let $Z \in L^\infty$ be given and denote again by $(Y_t)_{t\geq 0}$ a cadlag-modification of $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\geq 0}$. Then

$$\mathbf{E}\, Z \, (M)^{(\tilde{3}),n}_t \;=\; \mathbf{E} \sum_{k=1}^{m(t)} Y_{t^n_{k-1}} (M_{t^n_k} - M_{t^n_{k-1}})^3 \,.$$

As in the proof of theorem 4.2 it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\, Z \, (M)^{(\tilde{3}),n}_t \\
&\quad= \lim_{n\to\infty} \mathbf{E}\, Z \, (M^d)^{(\tilde{3}),n}_t \\
&\quad= \mathbf{E} \int_0^t Y_{s-} \, d(M^d)^{(3)}_s \\
&\quad= \mathbf{E} \int_0^t Y_{s-} \, d(M^d)^{(\tilde{3})}_s \\
&\quad= \mathbf{E}\, Z \, (M^d)^{(\tilde{3})}_t \,,
\end{aligned}$$

and hence we have proved that

$$\lim_{n\to\infty} (M)^{(\tilde{3}),n}_t \;=\; (M)^{(\tilde{3})}_t \;=\lim_{n\to\infty} (M^d)^{(\tilde{3}),n}_t \;=\; (M^d)^{(\tilde{3})}_t$$

for $\sigma(L^1, L^\infty)$.
Consider now the sequence $((M)_t^{(1,\tilde{3}),n})_{n\geq 1}$ defined by

$$(M)_t^{(1,\tilde{3}),n} := \sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n}) \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^3 \mid \mathcal{F}_{t_{k-1}^n}\big\}\,.$$

It is not difficult to see that the $\sigma(L^1, L^\infty)$-convergence of that sequence is equivalent to the $\sigma(L^1, L^\infty)$-convergence of the corresponding sequence $((M^d)_t^{(1,\tilde{3}),n})_{n\geq 1}$, where M is replaced by M^d. For analyzing the limit behaviour of the last sequence, let us first assume that M^d is of the simple form

$$M^d \;=\; X\,1_{[S,\infty[} \;+\; Y\,1_{[T,\infty[}$$

with $X, Y \in L^4$ and S, T stopping times. We get

$$\begin{aligned} &(M^d)_t^{(1,\tilde{3}),n} \\ &= \sum_{k=1}^{m(t)} \Big[\Big(X\,1_{]t_{k-1}^n, t_k^n]}(S) \;+\; Y\,1_{]t_{k-1}^n, t_k^n]}(T)\Big) \\ &\qquad\qquad \cdot \mathbf{E}\big\{X^3\,1_{]t_{k-1}^n, t_k^n]}(S) \;+\; Y^3\,1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n}\big\}\Big] \\ &\quad + A^n\,, \end{aligned}$$

where

$$\begin{aligned} A^n \;:=\; \sum_{k=1}^{m(t)} \Big[&\Big(X\,1_{]t_{k-1}^n, t_k^n]}(S) \;+\; Y\,1_{]t_{k-1}^n, t_k^n]}(T)\Big) \\ &\cdot \mathbf{E}\big\{3(XY^2 \;+\; X^2Y)1_{]t_{k-1}^n, t_k^n]}(S)\,1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n}\big\}\Big]\,. \end{aligned}$$

For A^n we get (with $Z := 3(XY^2 + X^2Y)$)

$$\begin{aligned} &\mathbf{E}\,|A^n| \\ &\leq\; \mathbf{E} \sum_{k=1}^{m(t)} \Big[\Big(|X|\,1_{]t_{k-1}^n, t_k^n]}(S) \;+\; |Y|\,1_{]t_{k-1}^n, t_k^n]}(T)\Big) \\ &\qquad\qquad \cdot \mathbf{E}\big\{\,|Z|\,1_{]t_{k-1}^n, t_k^n]}(S)\,1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n}\big\}\Big] \\ &=\; \mathbf{E} \sum_{k=1}^{m(t)} \Big[\mathbf{E}\Big\{|X|\,1_{]t_{k-1}^n, t_k^n]}(S) \;+\; |Y|\,1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n}\Big\} \\ &\qquad\qquad \cdot \Big(|Z|\,1_{]t_{k-1}^n, t_k^n]}(S)\,1_{]t_{k-1}^n, t_k^n]}(T)\Big)\Big]\,. \end{aligned}$$

If the stopping times S and T have disjoint graphs, it is not difficult to conclude from the above inequality that

$$\lim_{n\to\infty} \mathbf{E}\,|A^n| \;=\; 0\,.$$

Suppose now that S is predictable and T is totally inaccessible and denote by $\tilde{D}^i$ $(i = 1, 2)$ the continuous compensators of $|Y|\, 1_{T,\infty[}$ and $|Y^3|\, 1_{T,\infty[}$ resp.. Then we have

$$\begin{aligned} &(M^d)_t^{(1,\tilde{3}),n} \;-\; A^n \\ &= \sum_{k=1}^{m(t)} X\, 1_{]t_{k-1}^n, t_k^n]}(S) \mathbf{E}\big\{X^3\, 1_{]t_{k-1}^n, t_k^n]}(S) \mid \mathcal{F}_{t_{k-1}^n}\big\} \\ &\quad + B^n \;+\; C^n \end{aligned}$$

with

$$B^n \;:=\; \sum_{k=1}^{m(t)} \Big(Y\, 1_{]t_{k-1}^n, t_k^n]}(T) \mathbf{E}\big\{X^3\, 1_{]t_{k-1}^n, t_k^n]}(S) \;+\; Y^3\, 1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n}\big\} \Big)$$

and

$$C^n \;:=\; \sum_{k=1}^{m(t)} X\, 1_{]t_{k-1}^n, t_k^n]}(S) \mathbf{E}\big\{Y^3\, 1_{]t_{k-1}^n, t_k^n]}(S) \mid \mathcal{F}_{t_{k-1}^n}\big\}\,.$$

We get

$$\begin{aligned} &\mathbf{E}\,|B^n| \\ &\le\; \mathbf{E} \sum_{k=1}^{m(t)} \Big(|Y|\, 1_{]t_{k-1}^n, t_k^n]}(T) \\ &\qquad\qquad \cdot\, \mathbf{E}\big\{|X^3|\, 1_{]t_{k-1}^n, t_k^n]}(S) \;+\; |Y^3|\, 1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n}\big\} \Big) \\ &=\; \mathbf{E} \sum_{k=1}^{m(t)} \Big(|Y|\, 1_{]t_{k-1}^n, t_k^n]}(T) \\ &\qquad\qquad \cdot\, \mathbf{E}\Big\{ \mathbf{E}\big\{|X^3| \big| \mathcal{F}_{S-}\big\}\, 1_{]t_{k-1}^n, t_k^n]}(S) \;+\; (\tilde{D}^2_{t_k^n} \;-\; \tilde{D}^2_{t_{k-1}^n}) \,\Big|\, \mathcal{F}_{t_{k-1}^n} \Big\} \Big) \\ &=\; \mathbf{E} \sum_{k=1}^{m(t)} \Big(\mathbf{E}\big\{|Y|\, 1_{]t_{k-1}^n, t_k^n]}(T) \,\big|\, \mathcal{F}_{t_{k-1}^n}\big\} \\ &\qquad\qquad \cdot\, \Big[\mathbf{E}\big\{|X^3| \big| \mathcal{F}_{S-}\big\}\, 1_{]t_{k-1}^n, t_k^n]}(S) \;+\; (\tilde{D}^2_{t_k^n} \;-\; \tilde{D}^2_{t_{k-1}^n}) \Big] \Big) \\ &=\; \mathbf{E} \sum_{k=1}^{m(t)} \Big(\mathbf{E}\big\{(\tilde{D}^1_{t_k^n} \;-\; \tilde{D}^1_{t_{k-1}^n}) \big| \mathcal{F}_{t_{k-1}^n}\big\} \\ &\qquad\qquad \cdot\, \Big[\mathbf{E}\big\{|X^3| \big| \mathcal{F}_{S-}\big\}\, 1_{]t_{k-1}^n, t_k^n]}(S) \;+\; (\tilde{D}^2_{t_k^n} \;-\; \tilde{D}^2_{t_{k-1}^n}) \Big] \Big)\,. \end{aligned}$$

Using the continuity of $\tilde{D}^1$, it is now easy to show that

$$\lim_{n\to\infty} \mathbf{E}\,|B^n| \;=\; 0\,.$$

Similarly, one can show by using the continuity of $\tilde{D}^2$ that

$$\lim_{n\to\infty} \mathbf{E}\,|C^n| \;=\; 0\,.$$

We assumed that M^d had just two jumps mainly to avoid overburdening of notation. If M^d has more than two jumps, similar arguments as above give that the convergence of the sequence $((M^d)_t^{(1,\bar{3}),n})_{n\geq 1}$ is equivalent to the convergence of the sequence

$$\Big(\sum_{k=1}^{m(t)} \big(\sum_{i\geq 1} X_i \, 1_{]t^n_{k-1},t^n_k]}(S_i)\big)\mathbf{E}\big\{\sum_{i\geq 1} X_i^3 \, 1_{]t^n_{k-1},t^n_k]}(S_i) \mid \mathcal{F}_{t^n_{k-1}}\big\}\Big)_{n\geq 1} .$$

Now we can proceed as in the proof of theorem 4.3 (cf. especially lemma 4.4) to obtain that the last sequence is convergent for $\sigma(L^1, L^\infty)$ and has the limit

$$\sum_{i\geq 1} X_i \, \mathbf{E}\{X_i^3 | \mathcal{F}_{S_i-}\} \, 1_{[S_i\leq t]} .$$

This finishes the proof of the theorem. □

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Author:

Egbert Dettweiler,
Mathematisches Institut der Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
Germany

E.Dettweiler@web.de

Rostock. Math. Kolloq. **56**, 117–117 (2002)

Pavel G. Todorov

Errata

in the paper "Taylor Series of the Pick Function and the Loewner Variation within the Class *S* and Applications" by Pavel G. Todorov, Rostock. Math. Kolloq. 54, 91-99 (2000):

1) Page 91, Formula (4):
   It is written: $G(e^{\lambda}g(z))$
   It must be: $G(e^{-\lambda}g(z))$

2) Page 94, Formula (21), in the last line:
   It is written: $= e^{-p\nu}$
   It must be: $= e^{-p\lambda}$

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Author:

Pavel Georgiev Todorov,
20 Maritsa Avenue,
4002 Plovdiv,
Bulgaria

e-mail:pgtodorov@abv.bg

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