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Estimates of the Solutions of Two-Scale Difference Equations

ABSTRACT. The solutions of special classes of two-scale difference equations are estimated in both directions.

KEY WORDS. Two-scale difference equations, de Rham's function, one-sided approximations

1 Introduction

A two-scale difference equation is a functional equation of the form

$$\varphi\left(\frac{t}{2}\right)=\sum_{\nu=0}^{n}c_\nu\varphi(t-\nu)\quad(t\in\mathbb{R})\,,\tag{1.1}$$

where c_ν are given real constants with $c_0c_n\neq 0$ and $n\in\mathbb{N}$, cf. [3]. As usual, we only seek non-trivial solutions of (1.1) which vanish for $t\leq 0$. Hence, by restriction to $t\leq 1$, equation (1.1) reduces to

$$\varphi\left(\frac{t}{2}\right)=a\varphi(t)\tag{1.2}$$

with $a=c_0$. By the further restriction to $t\geq 0$ and $a>0$, this equation has the general solution

$$\varphi(t)=t^\alpha P\left(\frac{\ln t}{\ln 2}\right)\tag{1.3}$$

where $\alpha=-\frac{\ln a}{\ln 2}$ and where $P(\cdot)$ is an arbitrary 1-periodic function. Hence, for a continuous (real) solution φ of (1.1) with $\varphi(1)\neq 0$ there always exist estimates of the form

$$\lambda_1\varphi(1)t^\alpha\leq\varphi(t)\leq\lambda_2\varphi(1)t^\alpha\quad(t\in[0,1])\tag{1.4}$$

with $\lambda_1 \leq 1 \leq \lambda_2$ in the case $\varphi(1) > 0$. The optimal constants are

$$\lambda_1 = \min \frac{P(\cdot)}{P(0)}, \quad \lambda_2 = \max \frac{P(\cdot)}{P(0)},$$

however since, usually speaking, the function P is unknown, we are going to determine estimates for these optimal constants. In order to prove the estimates (1.4) for concrete λ_1, λ_2 it suffices to prove them for $t \in \left(\frac{1}{2}, 1\right]$ only because, if (1.4) is satisfied for a fixed t then, in view of the periodicity of $P(\cdot)$, it is automatically satisfied for $2^{-k}t$ with arbitrary $k \in \mathbb{N}$, cf. (1.3).

In the following we shall determine λ_1, λ_2 in (1.3) for some classes of equations (1.1) up to $n = 3$. The special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-a)\varphi(t-1) \tag{1.5}$$

of (1.1) with $n = 1$ and $a \in (0, 1)$ was already treated in [1] in connection with investigations concerning the Hölder continuity of φ, but here we shall give a new prove which also works in more complicated cases. In the case (1.5) one of the constants λ_1, λ_2 in (1.4) is always equal to 1 but, in general, they are both different from 1.

2 Estimates for de Rham's function

We begin with the equation (1.5). After the normalization $\varphi(1) = 1$ it has the continuous solution $\varphi(t) = 0$ for $t \leq 0$, $\varphi(t) = 1$ for $t \geq 1$ and $\varphi(t) = \varphi_a(t)$ for $t \in [0, 1]$, where φ_a is de Rham's function, which is the simultaneous solution of (1.2) and

$$\varphi\left(\frac{t+1}{2}\right) = a + (1-a)\varphi(t), \tag{2.1}$$

both equations for $t \in [0, 1]$. In [1] it was proved:

Proposition 2.1 *In the case $a \in \left(0, \frac{1}{2}\right]$ de Rham's function satisfies the estimates*

$$C(a)t^\alpha \leq \varphi_a(t) \leq t^\alpha \ (t \in [0, 1]) \tag{2.2}$$

and in the case $a \in \left[\frac{1}{2}, 1\right)$ it satisfies

$$t^\alpha \leq \varphi_a(t) = C(a)t^\alpha \ (t \in [0, 1]) \tag{2.3}$$

where $\alpha = -\frac{\ln a}{\ln 2}$,

$$C(a) = \left(1 - q^{\frac{1}{1-\alpha}}\right)^{\alpha - 1} \tag{2.4}$$

and $q = \frac{1}{a} - 1$.

Proof: The case $a = \frac{1}{2}$ is elementary and can be excluded now, since $\alpha = C\left(\frac{1}{2}\right) = 1$ as well as $\varphi_{\frac{1}{2}}(t) = t$. We first deal with the case $a \in \left(0, \frac{1}{2}\right)$ where $\alpha > 1$. We assume that the inequalities (2.2) are satisfied for a fixed t and show that they are also satisfied for $\frac{t+1}{2}$ instead of t. From (2.1) and (2.2) we obtain

$$a + (1-a)C(a)t^\alpha \leq \varphi_a\left(\frac{t+1}{2}\right) \leq a + (1-a)t^\alpha. \tag{2.5}$$

In view of $\frac{1}{2^\alpha} = a$ we have to show that

$$aC(a)(t+1)^\alpha \leq a + (1-a)C(a)t^\alpha, \quad a + (1-a)t^\alpha \leq a(t+1)^\alpha$$

for $t \in [0,1]$, i.e. that

$$C(a) \leq \frac{1}{(t+1)^\alpha - qt^\alpha} \leqq 1\,, \tag{2.6}$$

since $(t+1)^\alpha \geq \frac{1}{\alpha}t^\alpha > qt^\alpha$. For this reason we introduce the auxiliary function $f(s) = \left(s^{\frac{1}{\alpha}} + 1\right)^\alpha - qs$ with $s = t^\alpha \in [0,1]$. We have $f''(s) = \frac{1-\alpha}{\alpha}\left(s^{\frac{1}{\alpha}} + 1\right)^{\alpha-2} s^{\frac{1}{\alpha}-2} \leq 0$ and $f(0) = f(1) = 1$ so that f is concave and the second inequality of (2.6) is proved. The maximum of f is attained at $s_0 = \left(q^{\frac{1}{\alpha-1}} - 1\right)^{-\alpha}$, i.e. the minimum of the quotient in (2.6) at $t_0 = \left(q^{\frac{1}{\alpha-1}} - 1\right)^{-1}$. A short calculation shows that this minimum is equal to (2.4) so that the first inequality of (2.6) is proved, too.

In the case $a \in \left(\frac{1}{2}, 1\right)$ where $\alpha \in (0,1)$, we assume that (2.3) is satisfied for a fixed t, conclude from (2.1) and (2.3) that

$$a + (1-a)t^\alpha \leq \varphi_a\left(\frac{t+1}{2}\right) \leq a + (1-a)C(a)t^\alpha\,, \tag{2.7}$$

and have to show that

$$a(t+1)^\alpha \leq a + (1-a)t^\alpha, \quad a + (1-a)C(a)t^\alpha \leq aC(a)(t+1)^\alpha$$

for $t \in [0,1]$, i.e.

$$1 \leq \frac{1}{(t+1)^\alpha - qt^\alpha} \leq C(a)\,. \tag{2.8}$$

In this case the auxiliary function f is convex in view of $\alpha \in (0,1)$, and the quotient in (2.8) attains its minimum at the same t_0 as before, i.e. also (2.8) is proved.

After these preliminaries we can prove the inequalities (2.2) and (2.3) by means of an inductive process. In the case $a \in \left(0, \frac{1}{2}\right)$ we see from (2.6) that $C(a) \leq 1$, and in the case $a \in \left(\frac{1}{2}, 1\right)$ we see from (2.8) that $1 \leq C(a)$. Hence according to $\varphi(1) = 1$, the inequalities

(2.2) and (2.3) are satisfied for $t=1$, and in view of $\alpha>0$ they are also satisfied for $t=0$. Now, we assume that the inequalities in question are satisfied for $t=\frac{\nu}{2^k}$, $\nu=0,1,\ldots,2^k$ and a fixed $k\in\mathbb{N}_0$. According to a remark in the introduction, they are also satisfied for $t=\frac{\nu}{2^{k+1}}$, $\nu=0,1,\ldots,2^k$, but in view of (2.5)-(2.8), they are moreover satisfied for $t=\frac{\nu}{2^{k+1}}$, $\nu=2^k+1,2^k+2,\ldots,2^{k+1}$. Hence by induction, the inequalities (2.2) and (2.3) are satisfied for all (rational) dyadic points, and by continuity for all points of $[0,1]$, so that the proposition is proved. ■

Improvement As already mentioned (but not proved) in [1], the constant $C(a)$ in Proposition 2.1 is not optimal for $a\neq\frac{1}{2}$. It can be improved by splitting the interval $[0,1]$ into several subintervals. In the simplest case of two subintervals we have to use the equation (2.1) only for $t\in\left[0,\frac{1}{2}\right]$ and additionally the consequence of (2.1)

$$\varphi\left(\frac{t+3}{4}\right)=2a-a^2+(1-a)^2\varphi(t)$$

for $t\in[0,1]$. In order to prove $Ct^\alpha\leq\varphi(t)$ for $a\in\left(0,\frac{1}{2}\right)$ inductively, we now have to fulfil the former inequality $aC(t+1)^\alpha\leq a+(1-a)Ct^\alpha$ only for $t\in\left[0,\frac{1}{2}\right]$, and additionally the inequality

$$a^2C(t+3)^\alpha\leq 2a-a^2+(1-a)^2Ct^\alpha$$

for $t\in[0,1]$. After determining maximal C_ν $(\nu=1,2)$ with $C_\nu\leq g_\nu(t)$ where

$$g_1(t)=\frac{1}{(t+1)^\alpha-qt^\alpha}\quad\left(t\in\left[0,\frac{1}{2}\right]\right),$$
$$g_2(t)=\frac{2q+1}{(t+3)^\alpha-q^2t^\alpha}\quad(t\in[0,1]),$$

we have to choose $C=\min(C_1,C_2)$. For $t\in[0,1]$ the minimum of g_1 is attained as before at $t_0=\left(q^{\frac{1}{\alpha-1}}-1\right)^{-1}$. For $a\in\left(\frac{1}{4},\frac{1}{2}\right)$ we have $t_0\in\left(\frac{1}{3},\frac{1}{2}\right)$, and we get no improvement. However, for $a\in\left(0,\frac{1}{4}\right)$ we have $t_0>\frac{1}{2}$, so that for $t\in\left[0,\frac{1}{2}\right]$ the function $g_1(t)$ attains its minimum at $t_1=\frac{1}{2}$, and it follows

$$C_1=\frac{1}{a(3^\alpha-q)}.$$

The function g_2 attains its minimum at $t_2=3\left(q^{\frac{2}{\alpha-1}}-1\right)^{-1}$ and this is equal to

$$C_2=\frac{2q+1}{3^\alpha q^2}\left(q^{\frac{2}{\alpha-1}}-1\right)^{\alpha-1}.$$

It turns out that $C_1>C_2$ in the case $a<0.10322$, and $C_1<C_2$ in the case $a>0.10322$, i.e. $C=C_2$ in the first and $C=C_1$ in the second case. However, the improvement in comparison with $C(a)$ is not essential and likewise not optimal.

Example 2.1 The special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + \varphi(t-1) + (1-a)\varphi(t-2) \tag{2.9}$$

of (1.1) with $n = 2$ and $a \in (0,1)$ can be reduced to the foregoing one, since (2.9) has a continuous solution with

$$\varphi(t) = \varphi_a(t), \quad \varphi(t+1) = \varphi_{1-a}(1-t)$$

for $t \in [0,1]$ and $\varphi(t) = 0$ for $t \notin (0,2)$, cf. [1] and in particular the relation $\varphi_{1-a}(1-t) = 1 - \varphi_a(t)$. Hence, besides of the estimates (2.2) resp. (2.3) we get the

Corollary 2.1 *In the case $a \in \left(0, \frac{1}{2}\right]$ the foregoing solution of (2.9) satisfies the estimates*

$$(1-t)^\beta \le \varphi(t+1) \le C(1-a)(1-t)^\beta \qquad (t \in [0,1])\,, \tag{2.10}$$

and in the case $a \in \left[\frac{1}{2}, 1\right)$ it satisfies

$$C(1-a)(1-t)^\beta \le \varphi(t+1) \le (1-t)^\beta \qquad (t \in [0,1])\,, \tag{2.11}$$

where $\beta = -\frac{\ln(1-a)}{\ln 2}$ and $C(\cdot)$ as in (2.4).

Example 2.2 The special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + a\varphi(t-1) + (1-a)\varphi(t-2) + (1-a)\varphi(t-3) \tag{2.12}$$

of (1.1) with $n = 3$, $a \in (0,1)$, can also be reduced to Proposition 2.1, since it has a continuous solution with

$$\varphi(t) = a\varphi_a(t),\ \varphi(t+1) = a + (1-2a)\varphi_a(t),\ \varphi(t+2) = (1-a)\varphi_{1-a}(1-t)$$

for $t \in [0,1]$, and $\varphi(t) = 0$ for $t \notin (0,3)$.

Example 2.3 Moreover, the special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-a)\varphi(t-1) + a\varphi(t-2) + (1-a)\varphi(t-3) \tag{2.13}$$

of (1.1) with $n = 3$, $a \in (0,1)$, can be reduced to Proposition 2.1, since it has a continuous solution Φ with the representation

$$\Phi(t) = \varphi(t) - \varphi(t-2)\,,$$

where φ is the solution of (1.5) studied before. This means in particular that

$$\Phi(t) = \varphi_a(t), \quad \Phi(t+1) = 1, \quad \Phi(t+2) = \varphi_{1-a}(1-t)$$

for $t \in [0,1]$ and $\Phi(t) = 0$ for $t \notin (0,3)$. In the last two cases the corresponding estimates are obvious in view of Proposition 2.1.

Example 2.4 Finally, we can estimate the solution

$$\varphi(t) = \int_0^t \varphi_a(t-\tau)\varphi_b(\tau)d\tau$$

of the equation

$$\varphi\left(\frac{t}{2}\right) = \frac{ab}{2}\varphi(t) + \left(\frac{a+b}{2} - ab\right)\varphi(t-1) + \frac{(1-a)(1-b)}{2}\varphi(t-2)\,, \tag{2.14}$$

$a, b \in (0,1)$, by means of Proposition 2.1. Namely, after writing the estimates (2.2) resp. (2.3) in the form

$$\lambda_1 t^\alpha \le \varphi_a(t) \le \lambda_2 t^\alpha\,, \quad \lambda_3 t^\beta \le \varphi_b(t) \le \lambda_4 t^\beta$$

with α as before and $\beta = -\frac{\ln b}{\ln 2}$, we immediately obtain

$$\lambda_1\lambda_3 \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} t^{\alpha+\beta+1} \le \varphi(t) \le \lambda_2\lambda_4 \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} t^{\alpha+\beta+1}$$

for $t \in [0,1]$. The method can easily be transferred to more complicated cases, but we do not deal with that.

3 The four-coefficient equation

Usually, two-scale difference equations (1.1) are considered under the so-called sum rules

$$\Sigma c_{2\nu} = 1, \quad \Sigma c_{2\nu+1} = 1\,, \tag{3.1}$$

cf. [3]. These are satisfied in the cases (2.9) and (2.12), but not in the cases (1.5), (2.13) and (2.14). The general form of (1.1) with $n = 3$ and (3.1) is a generalization of (2.12) and reads

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-b)\varphi(t-1) + (1-a)\varphi(t-2) + b\varphi(t-3)\,, \tag{3.2}$$

this equation was well investigated in [2]. We only look for compactly supported solutions, i.e. for solutions with $\varphi(t) = 0$ for $t \notin (0,3)$, so that it suffices to restrict ourselves to $t \in [0,3]$ in what follows. We assume that

$$0 < a\,, \quad b < 1\,. \tag{3.3}$$

Then there exists a continuous solution with

$$\varphi(1) = a\,, \quad \varphi(2) = b\,, \tag{3.4}$$

$$\varphi(t) + \varphi(t+1) + \varphi(t+2) = a + b \qquad (t \in [0,1])\,. \tag{3.5}$$

The equations (3.2) and (3.5) imply

$$\varphi\left(\frac{t+1}{2}\right) = a(a+b) + (1-a-b)\varphi(t) - a\varphi(t+2) \qquad (t \in [0,1])\,. \tag{3.6}$$

We begin with two special cases. In the case $b = \frac{1}{2} - a$ and $a \in \left(0, \frac{1}{2}\right)$ the solution of (3.2) is even differentiable, cf. [2], and satisfies the additional relation $\varphi(t) - \varphi(t+2) = a + \frac{1}{2}(t-1)$ for $t \in [0,1]$, so that (3.6) with $a + b = \frac{1}{2}$ simplifies to

$$\varphi\left(\frac{t+1}{2}\right) = a\left(a + \frac{t}{2}\right) + \left(\frac{1}{2} - a\right)\varphi(t) \qquad (t \in [0,1])\,. \tag{3.7}$$

Proposition 3.1 *The solution of (3.2) with $b = \frac{1}{2} - a$, $a \in \left(0, \frac{1}{2}\right)$ and $\varphi(1) = a$ satisfies the estimates (1.4) with $\alpha = -\frac{\ln a}{\ln 2}$ and*

$$\lambda_1 = \min f(t)\,, \quad \lambda_2 = \max f(t) \qquad (t \in [0,1]) \tag{3.8}$$

where

$$f(t) = \frac{2a+t}{2a(t+1)^\alpha - (1-2a)t^\alpha}\,. \tag{3.9}$$

Proof: Following the philosophy of the inductive proof of Proposition 2.1, and using (3.7) instead of (2.1), we only have to show that

$$a\lambda_1(t+1)^\alpha \le a + \frac{t}{2} + \left(\frac{1}{2} - a\right)\lambda_1 t^\alpha\,,$$
$$a + \frac{t}{2} + \left(\frac{1}{2} - a\right)\lambda_2 t^\alpha \le a\lambda_2(t+1)^\alpha$$

for $t \in [0,1]$. According to $2a(t+1)^\alpha - (1-2a)t^\alpha > (1+2a)t^\alpha$, the denominator of (3.9) is positive, so that the inequalities in question are equivalent to $\lambda_1 \le f(t) \le \lambda_2$. In view of (3.8) these inequalities are evident, and the proposition is proved. ■

a	t_1	t_2	λ_1	λ_2
.01	.938013	.153740	.975664	3.359434
.1	.791639	.163395	.972564	1.105538
.2	.672925	.114190	.991617	1.007689
.3	.057474	.568021	.998655	1.006186
.4	.010468	.473003	.999829	1.009885
.45	.000558	.428706	.999997	1.006748

Table 1: Some extremal points t_ν and the bounds $\lambda_\nu = f(t_\nu)$, $\nu = 1, 2$

Numerical results are contained in Table 1. The case $a = \frac{1}{4}$ with $\alpha = 2$ is elementary with $f(t) = \lambda_1 = \lambda_2 = 1$ and therefore $\varphi(t) = \frac{1}{4}t^2$ for $t \in [0,1]$. Elementary is also the limit case $a = \frac{1}{2}$ with $\alpha = 1$ and $\varphi(t) = \frac{1}{2}t$, where (3.2) reduces to (2.9) with $a = \frac{1}{2}$.

Second, we consider the symmetric case $b = a$ in (3.2), so that $\varphi(t) = \varphi(3-t)$, and (3.6) can be written in the form

$$\varphi\left(\frac{t+1}{2}\right) = 2a^2 + (1-2a)\varphi(t) - a\varphi(1-t) \qquad (t \in [0,1])\,. \tag{3.10}$$

Proposition 3.2 *The solution of (3.2) with $b = a$, $a \in \left(0, \frac{1}{2}\right)$ and $\varphi(1) = a$ satisfies the estimates (1.4) with $\alpha = -\frac{\ln a}{\ln 2}$, $\lambda_1 = 1 - \varepsilon$, $\lambda_2 = 1 + \varepsilon$ and*

$$\varepsilon = \max |f(t)| \qquad (t \in [0,1]) \tag{3.11}$$

where

$$f(t) = \frac{2a + (1-2a)t^\alpha - a(t+1)^\alpha - a(1-t)^\alpha}{a(t+1)^\alpha - a(1-t)^\alpha - (1-2a)t^\alpha}\,. \tag{3.12}$$

Proof: Once more, we follow the philosophy of the foregoing proofs, however, since (3.10) contains a negative coefficient, we have to show that

$$a\lambda_1(t+1)^\alpha \le 2a + (1-2a)\lambda_1 t^\alpha - a\lambda_2(1-t)^\alpha\,,$$
$$2a + (1-2a)\lambda_2 t^\alpha - a\lambda_1(1-t)^\alpha \le a\lambda_2(t+1)^\alpha\,.$$

For $t = 0$ these inequalities imply that $\lambda_1 + \lambda_2 \le 2 \le \lambda_1 + \lambda_2$ must be satisfied, i.e. $\lambda_1 = 1 - \varepsilon$ and $\lambda_2 = 1 + \varepsilon$ with a nonnegative ε. Hence, the foregoing inequalities turn over into

$$-\varepsilon N(t) \le 2a + (1-2a)t^\alpha - a(t+1)^\alpha - a(1-t)^\alpha \le \varepsilon N(t) \tag{3.13}$$

with $N(t) = a(t+1)^\alpha - a(1-t)^\alpha - (1-2a)t^\alpha$. Substituting $x = \frac{1}{t} \ge 1$, we obtain

$$t^{-\alpha}N(t) = a(x+1)^\alpha - a(x-1)^\alpha - (1-2a)\,.$$

According to $\alpha > 1$ and $2^\alpha = \frac{1}{a}$, the derivative of this function with respect to x reads $a\alpha\left((x+1)^{\alpha-1} - (x-1)^{\alpha-1}\right) > 0$, and the initial value for $x = 1$ is equal to $N(1) = 2a$. Hence $N(t) \ge 0$ for $t \in [0,1]$, and the inequalities (3.13) are equivalent to $-\varepsilon \le f(t) \le \varepsilon$. In view of (3.11) these inequalities are evident, and the proposition is proved. ■

a	t_1	$f(t_1)$
.1	.506816	-.030511
.2	.505478	-.013624

a	t_2	$f(t_2)$
.3	.490557	.012852
.4	.457641	.024888

Table 2: Some extremal points t_ν and the extrema $f(t_\nu)$, $\nu = 1, 2$

Numerical results are contained in Table 2, where $f \le 0$ for $0 < a \le \frac{1}{4}$ and $f \ge 0$ for $a \in \left[\frac{1}{4}, \frac{1}{2}\right)$, so that $\varepsilon = -f_{\min}$ in the first and $\varepsilon = f_{\max}$ in the second case. Again the cases $a = \frac{1}{4}$ and $a = \frac{1}{2}$, the last as a limit case, are elementary.

For $a \in \left(\frac{1}{2}, 1\right)$ the denominator $N(t)$ of (3.12) changes its sign so that the foregoing method fails. However, the method works again if (3.10) is replaced by

$$\varphi\left(\frac{t+1}{2}\right) = 2a^2(1-a) + (1-a)\varphi(t) + a(2a-1)\varphi(|2t-1|)\,,$$

but we are not concerned with this case.

Finally, we come back to the general four-coefficient equation (3.2), where besides of (1.4) with $\varphi(1) = a$ we also ask for estimates of the form

$$b\lambda_3(1-t)^\beta \le \varphi(t+2) \le b\lambda_4(1-t)^\beta \qquad (t \in [0,1]) \tag{3.14}$$

with $\beta = -\frac{\ln b}{\ln 2}$ and $\lambda_3 \le 1 \le \lambda_4$, cf. (3.4). The following fact can easily be seen: If $\varphi(t)$ is a solution of (3.2) with (3.4), then $\varphi(3-t)$ satisfies the same equations, only with interchanged coefficients. In particular, the inequalities (1.4) and (3.14) interchange, if we interchange a and b, and (3.6) interchanges with

$$\varphi\left(2+\frac{t}{2}\right) = b(a+b) + (1-a-b)\varphi(2+t) - b\varphi(t) \qquad (t \in [0,1])\,. \tag{3.15}$$

In the following we use the notations

$$\left.\begin{aligned} f(t,a,b) &= a+b+(1-a-b)t^\alpha - a(t+1)^\alpha - b(1-t)^\beta\,,\\ g(t,a,b) &= a(t+1)^\alpha - a(1-t)^\beta - (1-a-b)t^\alpha\,. \end{aligned}\right\} \tag{3.16}$$

Proposition 3.3 *Under the conditions* (3.3) *and* $a+b \le 1$ *the solution of* (3.2) *with* (3.4) *satisfies the estimates*

$$\varphi(t) \le a\lambda_2 t^\alpha\,, \quad b\lambda_3(1-t)^\beta \le \varphi(t+2) \qquad (t \in [0,1]) \tag{3.17}$$

with $\lambda_2 = 1 + b\varepsilon$, $\lambda_3 = 1 - a\varepsilon$ *and* $\varepsilon \ge 0$, *so long as both inequalities*

$$f(t,a,b) \le bg(t,a,b)\varepsilon\,, \quad -f(t,b,a) \le ag(t,b,a)\varepsilon \qquad (t \in [0,1]) \tag{3.18}$$

are satisfied. It satisfies the estimates

$$a\lambda_1 t^\alpha \le \varphi(t)\,, \quad \varphi(t+2) \le b\lambda_4(1-t)^\beta \qquad (t \in [0,1]) \tag{3.19}$$

with $\lambda_1 = 1 - b\varepsilon$, $\lambda_4 = 1 + a\varepsilon$ *and* $\varepsilon \ge 0$, *so long as both inequalities*

$$f(t,b,a) \le ag(t,b,a)\varepsilon\,, \quad -f(t,a,b) \le bg(t,a,b)\varepsilon \qquad (t \in [0,1]) \tag{3.20}$$

are satisfied.

Proof: The inequalities (3.17) are satisfied for $t = 0$ and $t = 1$. In order to prove them for all $t \in [0,1]$, we proceed analogously as before using (3.6) and (3.15), i.e. we have to show that both inequalities

$$a + b + (1 - a - b)\lambda_2 t^\alpha - b\lambda_3(1-t)^\beta \leq a\lambda_2(t+1)^\alpha$$

and

$$b\lambda_3(2-t)^\beta \leq a + b + (1-a-b)\lambda_3(1-t)^\beta - a\lambda_2 t^\alpha$$

are satisfied. Replacing t by $1 - t$ in the last inequality, it turns over into

$$b\lambda_3(t+1)^\beta \leq a + b + (1-a-b)\lambda_3 t^\beta - a\lambda_2(1-t)^\alpha \,.$$

For $t = 0$ we obtain from the first and the last inequality

$$a\lambda_2 + b\lambda_3 \leq a + b \leq a\lambda_2 + b\lambda_3$$

and therefore $a\lambda_2 + b\lambda_3 = a + b$. Choosing $\lambda_2 = 1 + b\varepsilon$ and $\lambda_3 = 1 - a\varepsilon$ with a non-negative ε and considering the notations (3.16), we see that the inequalities in question are equivalent with (3.18). The inequalities (3.20) can be derived analogously or by interchanging a and b. Hence, the proposition is proved. ■

There are two interesting special cases which shall be formulated for the conditions (3.18) only, but which can also be transferred to the conditions (3.20).

Corollary 3.1 *The conditions (3.18) of Proposition 3.3 simplify to*

$$f(t,a,b) \leq 0 \leq f(t,b,a) \qquad (t \in [0,1]) \tag{3.21}$$

in the case $\varepsilon = 0$, and to $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ with

$$\left.\begin{aligned} \varepsilon_1 &= \max \frac{f(t,a,b)}{bg(t,a,b)} \qquad (t \in [0,1])\,, \\ \varepsilon_2 &= \max -\frac{f(t,b,a)}{ag(t,b,a)} \qquad (t \in [0,1]) \end{aligned}\right\} \tag{3.22}$$

in the case that the denominators in (3.22) are non-negative.

Example 3.1 As an example to the case $\varepsilon = 0$ we shall show that the conditions (3.21) with $a + b = 1$, i.e. the inequalities

$$1 - \left(\frac{t+1}{2}\right)^\alpha - \left(\frac{1-t}{2}\right)^\beta \leq 0 \leq 1 - \left(\frac{t+1}{2}\right)^\beta - \left(\frac{1-t}{2}\right)^\alpha \qquad (t \in [0,1])\,, \tag{3.23}$$

are satisfied in the case $a \in \left(0, \frac{1}{2}\right)$ and therefore $\alpha > 1$, $b \in \left(\frac{1}{2}, 1\right)$, $\beta \in (0, 1)$. (Again, the case $a = \frac{1}{2}$ is elementary). Substituting $t + 1 = 2x$ so that $1 - t = 2(1 - x)$, we have to show that the auxiliary function $h(x) = (1 - (1 - x)^\beta)^{\frac{1}{\alpha}} - x$ satisfies $h(x) \geq 0$ for $x \in \left[0, \frac{1}{2}\right]$, and $h(x) \leq 0$ for $x \in \left[\frac{1}{2}, 1\right]$. From

$$h''(x) = \frac{\beta}{\alpha}(1 - (1 - x)^\beta)^{\frac{1}{\alpha} - 2}(1 - x)^{\beta - 2}\left(\left(\frac{\beta}{\alpha} - 1\right)(1 - x)^\beta + 1 - \beta\right)$$

we see that $h''(x)$ has exactly one zero in $(0, 1)$, and that $h''(x) < 0$ before as well as $h''(x) > 0$ afterwards. This means that $h(x)$ has exactly one turning point in $[0, 1]$, and $h(x)$ is concave before as well as convex afterwards. Together with $h(0) = h\left(\frac{1}{2}\right) = h(1) = 0$ this proves the assertion, and Corollary 3.1 implies the estimates (3.17) with $\lambda_2 = \lambda_3 = 1$ in the case $a \in \left(0, \frac{1}{2}\right)$.

Of course, these estimates are already known from Example 2.3 (where the normalization $\varphi(1) = 1$ was used instead of $\varphi(1) = a$ here), but the conditions (3.23) are only sufficient, and therefore no consequences of the results in Example 2.3.

Moreover, the conditions (3.21) and therefore also the estimates (3.17) with $\lambda_2 = \lambda_3 = 1$ are satisfied in a subdomain of $\left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right]$, which above and at the right is bounded by $a + b = 1$, at the left by the curve $a\alpha = b\beta$, and below by the envelope of the curves $f(t, a, b) = 0$ with the parameter $t \in [0, 0.61]$.

t		.6	.55	.5	.45	.4	.3	.2	.1	.05	
a	.25	.2557	.2839	.3103	.3350	.3583	.4006	.4382	.4714	.4864	.5
b	.5	.5015	.5078	.5121	.5148	.5163	.5165	.5138	.5083	.5045	.5

Table 3: Some points of the envelope $f(t, a, b) = f_t(t, a, b) = 0$

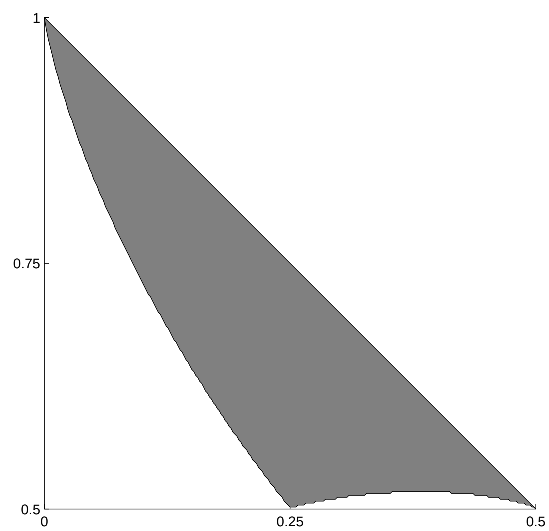

Figure 1: The level surface $\varepsilon = 0$

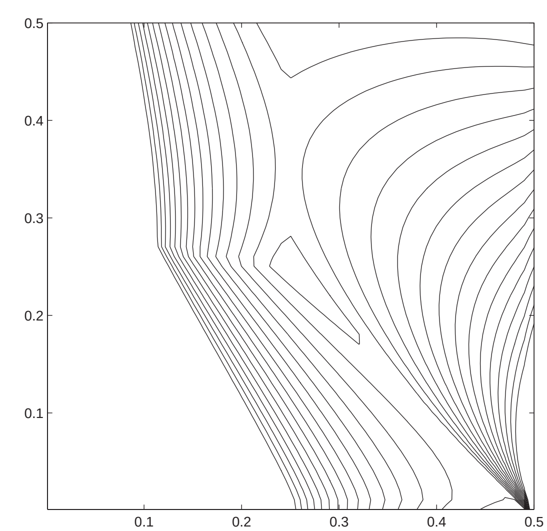

Figure 2: Some level lines for $\varepsilon > 0$

Numerically, this envelope is given by Table 3, where the corner points belong to all parameters t, and the complete subdomain is the black domain in Figure 1. The curves in the subdomain $\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$ of the (a, b)-plane, which are shown in Figure 2, are the level lines of ε from Corollary 3.1 for $\frac{1}{15}\left(\frac{1}{15}\right)1$ around the three points with $\varepsilon = 0$ in $\left(\frac{1}{4}, \frac{1}{4}\right)$ and $\left(\frac{1}{4}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, the last two are already known from Figure 1. Both Figures were worked out by K. Frischmuth [4].

In the case $\varepsilon > 0$ the estimates of the general Proposition 3.3 are worse than those following from the Propositions 3.1 and 3.2 in the special cases considered there, because there we have used more informations about the solutions.

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Some results on a nonunique fixed point

ABSTRACT. In the present paper, we obtain some nonunique fixed point theorems of single valued and multivalued maps in metric and generalized metric spaces, one of which generalizes the corresponding results of [1] and [2].

KEY WORDS AND PHRASES. Nonunique fixed point, T-orbitally continuous, T-orbitally complete, orbitally upper - semicontinuous.

1 Introduction

In [1], Pachpatte obtained some results on a nonunique fixed point in complete metric spaces and introduced an inequality as follows:

$$\begin{aligned} &\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), [d(y,Ty)]^2\} \\ &\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \le rd(x,Tx)d(y,Ty) \end{aligned} \tag{I}$$

for any x, y in X, where r is in $(0,1)$.

In [2], Liu generalized the above result for single valued maps and introduced the following:

$$\begin{aligned} &\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), d(x,y)d(y,TY), d(x,Tx)d(Tx,Ty), [d(y,Ty)]^2\} \\ &\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \le r \cdot \max\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \end{aligned}$$

for any x, y in X, where r is in $(0,1)$.

In the present paper, we obtain some results which generalize Theorem 1 of [1] and Theorem 1 of [2]. Furthermore, we give an example to show that our result indeed generalizes Theorem 1 of [1]. By the way, we show the example in [4] is false.

2 On a nonunique fixed point for single valued maps

Let $(X; d)$ be a metric space and T be a self map of X. T is called to be orbitally continuous if $\lim_i T^{n_i}x = u$ implies that $\lim_i TT^{n_i}x = Tu$ for each x in X. A metric space X is T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i}x\}_{i\geq 1}$ converges in X for x in X. Throughout this paper R^+ denotes the set of nonnegative real numbers.

Theorem 2.1 *Let (X, d) be a T-orbitally complete metric space, T be an orbitally continuous self map of X. If T satisfies the following condition*

$$\begin{aligned}
&\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty),\\
&\qquad d(x,Tx)d(y,Ty), d(y,Ty)d(Tx,Ty), [d(y,Ty)]^2\}\\
&\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \qquad (2.1)\\
&\leq r\cdot\max\{d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty), d(x,Tx)d(y,Ty),\\
&\quad d(x,Ty)d(y,Tx), d(y,Tx)d(Tx,Ty), d(y,Tx)d(y,Ty)\}
\end{aligned}$$

for any x, y in X, where r is in $(0,1)$, then T has a fixed point and for each x in X the sequence $\{T^n x\}_{n\geq 1}$ converges to a fixed point of T.

Proof: Let x be in X. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n \geq 0$, where $x_0 = x$. If $x_n = x_{n+1}$ for some $n \geq 0$, then the assertion follows immediately. Therefore we assume that $x_n \neq x_{n+1}$ for each $n \geq 0$. Put $d_n = d(x_n, x_{n+1})$ for $n \geq 0$. By (2.1) we obtain

$$\begin{aligned}
&\min\{d_{n+1}^2, d_n d_{n+1}, d_n d_{n+1}, d_n d_{n+1}, d_n d_{n+1}, d_{n+1}^2, d_{n+1}^2\}\\
&\quad - \min\{d_n d_{n+1}, d(x_n, x_{n+2})d(x_{n+1}, x_{n+1})\}\\
&\leq r\cdot\max\{d_n d_{n+1}, d_n d_{n+1}, d_n d_{n+1}, d_n d_{n+1}, d(x_n, x_{n+2})d(x_{n+1}, x_{n+1}),\\
&\qquad d(x_{n+1}, x_{n+1})d_{n+1}, d(x_{n+1}, x_{n+1})d_{n+1}\}
\end{aligned}$$

i.e.,

$$d_{n+1}^2 = \min\{d_{n+1}^2, d_2 d_{n+1}\} \leq r d_2 d_{n+1}$$

which implies that $d_{n+1} \leq r d_n$. It is easy to see that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. Since X is orbitally complete there is some u in X such that $u = \lim_n T^n x$. By the T-orbitally continuity of T, $Tu = \lim_n TT^n x = u$. This completes the proof.

Remark 2.1 Theorem 2.1 extends Theorem 1 of [1] and Theorem 1 of [2]. The following example shows that Theorem 2.1 is a proper generalization of Theorem 1 of [1].

Example 2.1 Let $X = \{0, 1, 2, 3, 4\}$, $d(x, y) = d(y, x)$ for all x in X and $d(x, y) = 0$ if and only if $x = y$, $d(0, 1) = 1$, $d(0, 2) = 2.5$, $d(0, 3) = 1$, $d(0, 4) = 1$, $d(1, 2) = 1.5$, $d(1, 3) = 2$,

$d(1,4) = 1$, $d(2,3) = 2$, $d(2,4) = 1.5$, $d(3,4) = 1$. Obviously, (X, d) is a complete metric space. Now let $T : X \to X$, $T0 = 1$, $T1 = 0$, $T2 = 3$, $T3 = 2$, $T4 = 4$. It is easy to verify that the conditions of Theorem 2.1 are satisfied for $r = 0.3$. But Theorem 1 of [1] is not applicable, because T doesn't satisfy (I) for $x = 0$, $y = 2$ and all r in $(0, 1)$.

Remark 2.2 In 1990, Ciric [4] gave an example to show that the corresponding results of Dhage [5], Mishra [6] and Pathak [7] are false. Unfortunately the example is false. In fact, through strictly examining the proofs of Dhage, Pathak and Mishra's results we assert that the results of [5], [6] and [7] are true.

Mishra [6], Daghe [5], Pathak [7] assume that T satisfies respectively the following conditions (A), (B) and (C):

(A) $\min\{d(Tx,Ty), d(x,Tx), d(y,Ty), d(Tx,T^2x), d(y,T^2x)\}$
$- \min\{d(x,ty), d(y,Tx), d(x,T^2x), d(Ty,T^2x)\} \le q\, d(x,y)$
for all x, y in X, where $0 \le q < 1$;

(B) $\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} + a \cdot \min\{d(x,Ty), d(y,Tx)\} \le qd(x,y) + pd(x,Tx)$
for all x, y in X, where $0 < p + q < 1$, a is a real number;

(C) $\min\{d(Tx,Ty), d(y,Ty)\} + a \cdot \min\{d(x,Ty), d(y,Tx)\} \le qd(x,y) + pd(x,Tx) + rd(x,Ty)$
for all x, y in X, where a, p, q and r are real numbers such that $0 \le r < 1$, $0 < p + q + 2r < 1$.

The example of Ciric [4] is as follows:

Let $M = \{0, 1, 3\}$ with the usual metric $d(x,y) = |x - y|$. Define the mapping T by $T0 = 1$, $T1 = 3$, $T3 = 0$.

Ciric [4] claimed that T satisfies each of conditions (A), (B) and C. We find that T doesn't satisfy any one of (A), (B) and (C), because if T satisfies (A), taking $x = 0$, $y = 1$, we have from (A)

$$\min\{d(1,3), d(0,1), d(1,3), d(1,3)\} - \min\{d(0,3), d(1,1), d(0,3), d(3,3)\} \le qd(0,1)$$

i.e., $1 \le q$. This contradicts the condition $0 \le q < 1$; if T satisfies (B), similarly we have $1 \le q + p < 1$, which is a contradiction, too; if T satisfies (C), we have $2 \le p + q + 3r$. Since $0 < p + q + 2r < 1$, it follows that $2 \le p + q + 3r < r + 1 < 2$, which is impossible.

Theorem 2.2 *Let (X,d) be a T-orbitally complete metric space, T be an orbitally continuous self map of X. If T satisfies the following condition*

$$\begin{aligned}
&a_1[d(Tx,Ty)]^2 + a_2 d(x,y)d(Tx,Ty) + a_3 d(x,y)d(y,Ty) + a_4 d(x,Tx)d(Tx.Ty)\\
&\quad + a_5 d(x,Tx)d(y,Ty) + a_6 d(y,Ty)d(Tx,Ty) + a_7[d(y,Ty)]^2\\
&\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \qquad (2.2)\\
&\le r\cdot\max\{d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty), d(x,Tx)d(y,Ty),\\
&\qquad\qquad d(x,Ty)d(y,Tx), d(y,Tx)d(Tx,Ty), d(y,Tx)d(y,Ty)\}
\end{aligned}$$

for all x, y in X, where $\sum_{i=1}^{7} a_i < 1$ and a_i is in R^+ for $i = 1, 2, \dots, 7$, then T has a fixed point and the sequence $\{T^n x\}_{n\ge 0}$ converges to a fixed point of T for x in X.

Proof: Note that (2.2) implies (2.1). Theorem 2.2 follows immediately from Theorem 2.1.

3 On a nonunique fixed point for multivalued maps

We recall that (X,d) is a generalized metric space if X is a set and $d : X\times X \to R^+\bigcup\{\infty\}$ satisfies all the properties of being a metric for X besides that d may have "infinite values". An orbit of F at the point x in X is a sequence $\{x_n : x_n \in Fx_{n-1}\}$, where $x_0 = x$. A multivalued map F on X is orbitally upper – semicontinuous if $x_n \to u \in X$ implies $u \in Fu$, whenever $\{x_n\}$ is an orbit of F at each x in X. A space X is F – orbitally complete if every orbit of F at all x in X which is a Cauchy sequence, converges in X. Let A and B be nonempty subsets of X. Denote

$$\begin{aligned}
D(A,B) &= \inf\{d(a,b) : a\in A, b\in B\},\\
CL(X) &= \{A : A\subset X, A \text{ is closed}\},\\
N(\varepsilon,A) &= \{x\in X : d(x,a)<\varepsilon \text{ for some } a \text{ in } A\},\ \varepsilon>0,\\
H(A,B) &= \begin{cases}\inf\{\varepsilon>0 : A\subseteq N(\varepsilon,B) \text{ and } B\subseteq N(\varepsilon,A)\}, & \text{if the infimum exists},\\ \infty, & \text{otherwise.}\end{cases}
\end{aligned}$$

Ciric [3] introduced the following inequality:

$$\min\{H(Fx,Fy), D(x,Fx), D(y,Fy)\} - \min\{D(x,Fy), D(y,Fx)\} \le q d(x,y)$$

for all x, y in M and some $q < 1$. Motivated by it, we obtain the following results.

Theorem 3.1 *Let (X,d) be a generalized metric space, $F : X \to CL(X)$ be orbitally upper–semicontinuous. If X is F-orbitally complete and F satisfies the following condition*

$$\begin{aligned}
&\min\{[H(Fx,Fy)]^2, d(x,y)H(Fx,Fy), d(x,y)D(y,Fy), D(x,Fx)H(Fx,Fy),\\
&\qquad D(x,Fx)D(y,Fy), D(y,Fy)H(Fx,Fy), [D(y,Fy)]^2\}\\
&\quad - \min\{D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx)\} \qquad (3.1)\\
&\le r\cdot\max\{d(x,y)d(Fx,Fy), d(x,y)D(y,Fy), D(x,Fx)D(Fx,Fy),\\
&\quad D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx), D(y,Fx)H(Fx,Fy), D(y,Fx)D(y,Fy)\}
\end{aligned}$$

for all x, y in X, where r is in $(0,1)$, then F has a fixed point.

Proof: Let $a > 0$ be a real number less than $1/2$. We define a single valued map $T : X \to X$ by letting $Tx = y \in Fx$ that satisfies

$$d(x,y) \le r^{-a}D(x,Fx)\,. \qquad (3.2)$$

Set $d_n = d(x_{n-1},x_n), D_n = D(x_n,Fx_n)$ and $H_n = H(Fx_{n-1},Fx_n)$ for $n \ge 0$. Now let's consider the following orbit of F at x in $X : x_0 = x, x_n = Tx_{n-1}$ for $n \ge 0$. We may assume that $x_{n-1} \ne x_n$ for any $n \ge 0$, otherwise the result is obtained at once. It follows from $x_n \in Fx_{n-1}$ that $D_n \le H_n, D(x_n,Fx_{n-1}) = 0$ and $D_{n-1} \le d_n$. By (3.1), we have

$$\begin{aligned}
&\min\{H_n^2, d_nH_n, d_nD_n, D_{n-1}H_n, D_{n-1}D_n, D_nH_n, D_n^2,\}\\
&\quad - \min\{D_{n-1}D_n, D(x_{n-1},Fx_n)D(x_n,Fx_{n-1})\}\\
&\le r\cdot\max\{d_nD(Fx_{n-1},Fx_n), d_nD_n, D_{n-1}D(Fx_{n-1},Fx_n), D_{n-1}D_n,\\
&\qquad D(x_{n-1},Fx_n)D(x_n,Fx_{n-1}), D(x_n,Fx_{n-1})H_n, D(x_n,Fx_{n-1})D_n\}
\end{aligned}$$

which implies that

$$\min\{D_n^2, D_{n-1}D_n\} = \min\{D_n^2, d_nD_n, D_{n-1}D_n\} \le r\cdot\max\{d_nD_n, D_{n-1}D_n\} = rd_nD_n\,,$$
$$\min\{r^{-2a}D_n^2, r^{-2a}D_{n-1}D_n\} \le r^{1-2a}d_nD_n \le r^{1-2a}d_nd_{n+1}\,,$$

on using (3.2)

$$\min\{d_{n+1}^2, d_nd_{n+1}\} \le r^{1-2a}d_nd_{n+1}\,.$$

Note that $0 < r^{1-2a} < 1$. If $d_n < d_{n+1}$, then

$$d_nd_{n+1} = \min\{d_{n+1}^2, d_nd_{n+1}\} \le r^{1-2a}d_nd_{n+1} < d_nd_{n+1}$$

a contradiction. Therefore $d_{n+1} < d_n$ and

$$d_{n+1}^2 = \min\{d_{n+1}^2, d_nd_{n+1}\} \le r^{1-2a}d_n^2$$

i.e., $d_{n+1} \leq bd_n$, where $b = r^{\frac{1}{2}-a}$. This implies $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. Since X is F-orbitally complete, there exists some point u in X such that $\lim_n x_n = u$. Thus the orbitally upper–semicontinuity of F implies $u \in Fu$. This completes the proof.

Theorem 3.2 *Let (X, d) be a generalized space, $F : X \to CL(X)$ be orbitally upper–semicontinuous. If X is F-orbitally complete and F satisfies the following condition*

$$\begin{aligned}
&a_1[H(Fx,Fy)]^2 + a_2 d(x,y)H(Fx,Fy) + a_3 d(x,y)D(y,Fy) + a_4 D(x,Fx)H(Fx,Fy) \\
&\quad + a_5 D(x,Fx)D(y,Fy) + a_6 D(y,Fy)H(Fx,Fy) + a_7[D(y,Fy)]^2 \\
&\quad - \min\{D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx)\} \\
&\leq r \cdot \max\{d(x,y)D(Fx,Fy), d(x,y)D(y,Fy), D(x,Fx)D(Fx,Fy), \\
&\qquad D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx), D(y,Fx)H(Fx,Fy), \\
&\qquad D(y,Fx)D(y,Fy)\}
\end{aligned} \tag{3.3}$$

for all x, y in X, where $\sum_{i=1}^{7} a_i < 1$ and a_i is in R^+ for $i = 1, 2, \ldots, 7$, then F has a fixed point.

Proof: Since (3.3) implies (3.1), Theorem 3.2 follows immediately from Theorem 3.1.

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Stationary points for set-valued mappings on two metric spaces

ABSTRACT. Stationary point theorems of set-valued mappings in complete and compact metric spaces are given. The corresponding results of Fisher [1] are generalized on set-valued mappings.

KEY WORDS AND PHRASES. Set-valued mappings, stationary points, complete metric spaces, compact metric spaces.

1 Preliminaries

In [1] and [2], Fisher and Popa have proved fixed point theorems for single valued mappings on two metric spaces. The purpose of this paper is to generalize these results for set-valued mappings. In this paper we show stationary point results of set-valued mappings in complete and compact metric spaces.

Let (X, d) and (Y, ρ) be complete metric spaces and $B(X)$ and $B(Y)$ be two families of all nonempty bounded subsets of X and Y, respectively. The function $\delta(A, B)$ with A and B in $B(X)$ is defined as follows:

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}\,.$$

Define $\delta(A) = \delta(A, A)$. Similarly, the function $\delta'(C, D)$ with C and D in $B(Y)$ is defined as follows:

$$\delta'(C, D) = \sup\{\rho(c, d) : c \in C, d \in D\}\,.$$

$\{A_n : n = 1, 2, \dots\}$ that is a sequence of sets in $B(X)$ converges to the set A in $B(X)$ if

(i) each point a in A is the limit of some convergent sequence $\{a_n \in A_n : n = 1, 2, \dots\}$;

(ii) for arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subset A_\varepsilon$ for $n > N$, where A_ε is the union of all open spheres with centers in A and radius ε.

Let T be a set-valued mapping of X into $B(X)$, z is a stationary point of T if $Tz = \{z\}$. T is continuous at x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx in $B(X)$. If T is continuous at each point x in X, then T is a continuous mapping of X into $B(X)$.

The following Lemmas 1.1 and 1.2 were proved in [3] and [4], respectively.

Lemma 1.1 *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets A and B respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.2 *Let $\{A_n\}$ be a sequence of nonempty subsets of X and x be a point of X such that $\lim_{n\to\infty} \delta(A_n, x) = 0$. Then the sequence $\{A_n\}$ converges to the set $\{x\}$.*

2 Stationary point results

Now we prove the following theorem for set-valued mappings.

Theorem 2.1 *Let (X, d) and (Y, ρ) be complete metric spaces. If T is a continuous mapping of X into $B(Y)$ and S is a continuous mapping of Y into $B(X)$ satisfying the inequalities*

$$\delta(STx, STy) \le c \cdot \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}\,, \tag{1}$$

$$\delta'(TSx', TSy') \le c \cdot \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\} \tag{2}$$

for all x, y in X and x', y' in Y, where $0 \le c < 1$, then ST has a stationary point z in X and TS has a stationary point w in Y. Further $Tz = \{w\}$ and $Sw = \{z\}$.

Proof: From (1) and (2), it is easy to see that

$$\delta(STA, STB) \le c \cdot \max\{\delta(A, B), \delta(A, STA), \delta(B, STB), \delta'(TA, TB)\}\,, \tag{1'}$$

$$\delta'(TSA', TSB') \le c \cdot \max\{\delta'(A', B'), \delta'(A', TSA'), \delta'(B', TSB'), \delta(SA', SB')\} \tag{2'}$$

for all A, B in $B(X)$ and A', B' in $B(Y)$.

Let x be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in $B(X)$ and $B(Y)$ respectively by choose a point x_n in $(ST)^n x = X_n$ and choose a point y_n in $T(ST)^{n-1}x = Y_n$ for $n = 1, 2, \ldots$. From (1′) and (2′) we have

$$\begin{aligned}\delta(X_n, X_{n+1}) &= \delta(STX_{n-1}, STX_n)\\ &\le c \cdot \max\{\delta(X_{n-1}, X_n), \delta(X_{n-1}, X_n), \delta(X_n, X_{n+1}), \delta'(Y_n, Y_{n+1})\}\\ &\le c \cdot \max\{\delta(X_{n-1}, X_n), \delta'(Y_n, Y_{n+1})\}\,.\end{aligned}$$

Similarly $\delta'(Y_n, Y_{n+1}) \le c \cdot \max\{\delta'(Y_{n-1}, Y_n), \delta(X_{n-1}, X_n)\}$. Put $M = \max\{\delta(x, X_1), \delta'(Y_1, Y_2)\}$. From the above inequalities, we obtain immediatly

$$\delta(X_n, X_{n+1}) \le c^n \cdot M\,, \tag{3}$$
$$\delta'(Y_n, Y_{n+1}) \le c^n \cdot M \tag{4}$$

for $n = 1, 2, \ldots$. It follows from (2) that

$$\begin{aligned}\delta(X_n, X_{n+r}) &\le \delta(X_n, X_{n+1}) + \cdots + \delta(X_{n+r-1}, X_{n+r}) \\ &\le (c^n + \cdots + c^{n+r-1})M \\ &\le \frac{c^n}{1-c} M\,.\end{aligned}$$

Since $c < 1$, then $\delta(X_n, X_{n+r}) \to 0$ as $n \to \infty$. So $d(x_n, x_{n+r}) \le \delta(X_n, X_{n+r}) \to 0$ as $n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X implies that there exists z in X such that $x_n \to z$ as $n \to \infty$. Further

$$\delta(z, X_n) \le \delta(z, x_n) + \delta(x_n, X_n) \le \delta(z, x_n) + \delta(X_n, X_n) \le \delta(z, x_n) + 2\delta(X_n, X_{n+1})$$

which implies that $\delta(z, X_n) \to 0$ as $n \to \infty$. Similarly, there exists w in Y such that $y_n \to w$ and $\delta'(w, Y_n) \to 0$ as $n \to \infty$. Then

$$\delta'(w, Tx_n) \le \delta'(w, TX_n) = \delta'(w, Y_{n+1})\,.$$

By the continuity of T and Lemma 1.1, we have $\delta'(w, Tz) \to 0$ as $n \to \infty$. From Lemma 1.2 it follows that $Tz = \{w\}$. Further

$$\delta(STz, x_n) \le \delta(STz, X_n) \le c \cdot \max\{\delta(z, X_{n-1}), \delta(z, STz), \delta(X_{n-1}, X_n), \delta'(Tz, TX_{n-1})\}\,.$$

Letting n tend to infinity, we have $\delta(STz, z) \le c \cdot \max\{\delta(STz, z), 0\}$, which implies that $STz = \{z\} = Sw$. Similarly, we can show w is a stationary point of TS. This completes the proof of the theorem.

Theorem 2.2 *Let (X, d) be a complete metric space, S and T be continuous mappings of X into $B(X)$ and map bounded set into bounded set. If S and T satisfy the inequalities*

$$\delta(STx, STy) \le c \cdot \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta(x, STy), \delta(y, STx), \delta(Tx, Ty)\}, \tag{5}$$
$$\delta(TSx, TSy) \le c \cdot \max\{\delta(x, y), \delta(x, TSx), \delta(y, TSy), \delta(x, TSy), \delta(y, TSx), \delta(Sx, Sy)\} \tag{6}$$

for all x, y in X, where $0 \le c < 1$, then ST has a stationary point z and TS has a stationary point w. Further $Tz = \{w\}$ and $Sw = \{z\}$. If $z = w$, then z is the unique common stationary point of S and T.

Proof: Let x be an arbitrary point in X, we define a sequence of sets $\{X_n\}$ by $T(ST)^{n-1}x = X_{2n-1}$, $(ST)^n x = X_{2n}$ for $n = 1, 2, \dots$, and $X_0 = \{x\}$.

Now suppose that $\{\delta(X_n)\}$ is unbounded. Then the real-valued sequence $\{a_n\}$ is unbounded, where $a_{2n-1} = \delta(X_{2n-1}, X_3)$, $a_{2n} = \delta(X_{2n}, X_2)$ for $n = 1, 2, \dots$, and so there exists an integer k such that

$$a_k > \frac{c}{1-c}\max\{\delta(x, X_2), \delta(X_1, X_3)\}\,, \tag{7}$$

$$a_k > \max\{a_1, \dots, a_{k-1}\}\,. \tag{8}$$

Suppose that k is even, put $k = 2n$. From (7) and (8) we have

$$c\cdot\delta(X_{2r}, x) \le c\cdot[\delta(X_{2r}, X_2) + \delta(X_2, x)] < \delta(X_{2n}, X_2)\,,$$
$$c\cdot\delta(X_{2r-1}, X_1) \le c\cdot[\delta(X_{2r-1}, X_3) + \delta(X_3, X_1)] < \delta(X_{2n}, X_2)\,.$$

That is

$$\delta(X_{2n}, X_2) > c\cdot\max\{\delta(X_{2r}, x), \delta(X_{2r-1}, X_1) : 1 \le r \le n\}\,. \tag{9}$$

We prove that the following (10) is true for $m \ge 1$:

$$\delta(X_{2n}, X_2) \le c^m\cdot\max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\}\,. \tag{10}$$

From (5) we have

$$\begin{aligned}\delta(X_{2n}, X_2) &= \delta(STX_{2n-2}, STx)\\ &\le c\cdot\max\{\delta(X_{2n-2}, x), \delta(X_{2n-2}, X_{2n}), \delta(x, X_2), \delta(x, X_{2n}),\\ &\qquad \delta(X_{2n-2}, X_2), \delta(X_{2n-1}, X_1)\}\,.\end{aligned}$$

It follows from (8) and (9) that $\delta(X_{2n}, X_2) \le c\cdot\delta(X_{2n-2}, X_{2n})$.

Now suppose that (10) is true for some m. From (5), (6), (8) and (9) we have

$$\begin{aligned}\delta(X_{2n}, X_2) &\le c^m\cdot\max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\}\\ &\le c^{m+1}\cdot\max\{\delta(X_{2r-2}, X_{2s-2}), \delta(X_{2r-2}, X_{2r}), \delta(X_{2s-2}, X_{2s}), \delta(X_{2r-2}, X_{2s})\,,\\ &\qquad \delta(X_{2s-2}, X_{2r}), \delta(X_{2r-1}, X_{2s-1}), \delta(X_{2r'-3}, X_{2s'-3})\,,\\ &\qquad \delta(X_{2r'-3}, X_{2r'-1}), \delta(X_{2s'-3}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\}\\ &\le c^{m+1}\cdot\max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\}\,.\end{aligned}$$

So (10) is true for all $m \ge 1$. Letting m tend to infinity, from (8) and (9) we have $0 < \delta(X_{2n}, X_2) \le 0$, which is impossible. Similarly, when k is odd, $2n-1$, say, we also have $0 < \delta(X_{2n-1}, X_3) \le 0$, which is also impossible. Hence $\{\delta(X_n)\}$ is bounded.

Let $M = \sup\{\delta(X_r, X_s) : r, s = 0, 1, 2, \dots\} < \infty$. For arbitrary $\varepsilon > 0$, choose a positive integer N such that $c^N \cdot M < \varepsilon$. Thus for m, n greater than $2N$ with m and n both even or both odd, from (5) and (6) we have

$$\begin{aligned}\delta(X_m, X_n) &\leq c \cdot \max\{\delta(X_{m-2}, X_{n-2}), \delta(X_{m-2}, X_m), \delta(X_{n-2}, X_n), \delta(X_{m-2}, X_n),\\ &\qquad \delta(X_{n-2}, X_m), \delta(X_{m-1}, X_{n-1})\}\\ &\leq c \cdot \max\{\delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : m-2 \leq r, r' \leq m, n-2 \leq s, s' \leq n\}\\ &\leq c^N \cdot \max\{\delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : m-2N \leq r, r' \leq m,\\ &\qquad\qquad n-2N \leq s, s' \leq n\}\\ &\leq c^N \cdot M < \varepsilon\,.\end{aligned}$$

So $\delta(X_{2n})$ and $\delta(X_{2n+1}) \to 0$ as $n \to \infty$. Take a point x_n in X_n for $n \geq 1$. Since $d(x_{2n}, x_{2n+2p}) \leq \delta(X_{2n}, X_{2n+2p}) \to 0$ as $n \to \infty$, hence $\{x_{2n}\}$ is a Cauchy sequence. Completeness of X implies that $\{x_{2n}\}$ has a limit z in X. Further

$$\delta(z, X_{2n}) \leq \delta(z, x_{2n}) + \delta(x_{2n}, X_{2n}) \leq \delta(z, x_{2n}) + \delta(X_{2n})\,.$$

That is $\delta(z, X_{2n}) \to 0$ as $n \to \infty$. Similarly $\{x_{2n+1}\}$ converges to some point w in X and $\delta(w, X_{2n+1}) \to 0$ as $n \to \infty$. Since $\delta(w, TX_{2n}) = \delta(w, X_{2n+1})$, by the continuity of T and Lemma 1.1, we have $\delta(w, Tz) \to 0$ as $n \to \infty$. From Lemma 1.2 it follows that $Tz = \{w\}$. Further

$$\begin{aligned}\delta(STz, x_{2n}) &\leq \delta(STz, X_{2n})\\ &\leq c \cdot \max\{\delta(z, X_{2n-2}), \delta(z, STz), \delta(X_{2n-2}, X_{2n}), \delta(z, X_{2n}),\\ &\qquad \delta(X_{2n-2}, STz), \delta(Tz, X_{2n-1})\}\end{aligned}$$

which implies that $\delta(STz, z) \leq c \cdot \max\{\delta(z, STz), 0\}$ as $n \to \infty$. Since $c < 1$, $\delta(STz, z) = 0$. Therefore $STz = \{z\} = Sw$ and $TSw = Tz = \{w\}$.

Now suppose that $z = w$ and that z' is a second common stationary point of S and T. Using (5)

$$\begin{aligned}\delta(z, z') = \delta(STz, STz') &\leq c \cdot \max\{\delta(z, z'), \delta(z, STz), \delta(z', STz'), \delta(z', STz),\\ &\qquad \delta(z, STz'), \delta(Tz, Tz')\}\\ &\leq c \cdot \delta(z, z')\,.\end{aligned}$$

So $z = z'$ and this completes the proof of the theorem.

Remark 2.1 If we use single valued mappings in place of set-valued mappings in Theorems 2.1 and 2.2, Theorems 2 and 3 of Fisher [1] can be attained.

Remark 2.2 The following example 2.1 demonstrates that the continuity of S and T in Theorems 2.1 and 2.2 is necessary.

Example 2.1 Let $X = \{0\} \cup \left\{\frac{1}{n} : n \geq 1\right\} = Y$ with the usual metric. Define mappings S, T by $T0 = \{1\}$, $T\frac{1}{n} = \left\{\frac{1}{2n}\right\}$ for $n \geq 1$ and $S = T$. It is easy to prove that all the conditions of Theorems 2.1 and 2.2 are satisfied except that the mappings S and T are continuous. But ST and TS have no stationary points.

Now we give the following theorem for the compact metric spaces.

Theorem 2.3 *Let (X, d) and (Y, ρ) be compact metric spaces. If T is a continuous mapping of X into $B(Y)$ and S is a continuous mapping of Y into $B(X)$ satisfying the following inequaltities*

$$\delta(STx, STy) < \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}\,, \tag{11}$$

$$\delta'(TSx', TSy') < \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\} \tag{12}$$

for all distinct x, y in X and distinct x', y' in Y, then ST has a stationary point z and TS has a stationary point w. Further $Tz = \{w\}$ and $Sw = \{z\}$.

Proof: Let us suppose that the right-hand sides of inequalities (11) and (12) are positive for all distinct x, y in X and distinct x', y' in Y. Define the real valued function $f(x, y)$ in $X \times X$ as follows:

$$f(x, y) = \delta(STx, STy)/\max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}\,.$$

Since S and T are continuous, f is continuous and achieves the maximum value s on the compact metric space $X \times X$. Inequality (11) implies $s < 1$, that is

$$\delta(STx, STy) \leq s \cdot \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\} \tag{13}$$

for all distinct x, y in X. It is obvious that inequality (13) is also true for $x = y$. Similarly, there exists $t < 1$ such that

$$\delta'(TSx', TSy') \leq t \cdot \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\} \tag{14}$$

for all x', y' in Y. So Theorem 2.3 follows immediately from Theorem 2.1.

Now suppose there exist z, z' in X such that

$$\max\{\delta(z, z'), \delta(z, STz), \delta(z', STz'), \delta'(Tz, Tz')\} = 0$$

which implies $\{z\} = \{z'\} = STz$ and $Tz = Tz'$, a singleton, $\{w\}$, say. Therefore we have $STz = Sw = \{z\}$, $TSw = Tz = \{w\}$. If there exist w, w' in Y such that

$$\max\{\delta'(w, w'), \delta'(w, TSw), \delta'(w', TSw'), \delta(Sw, Sw')\} = 0\,,$$

similarly, we also have $STz = Sw = \{z\}$, $TSw = Tz = \{w\}$. This completes the proof of the theorem.

Remark 2.3 Theorem 4 of Fisher [1] is a particular case of our Theorem 2.3 if set-valued mappings are replaced by single valued mappings in Theorem 2.3.

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On Vector Quasi-variational Inequality Problems in H-spaces

ABSTRACT. The purpose of this paper is to study the existence problem of solutions of the vector quasi-variational inequality for vector-valued functions in H-spaces.

KEY WORDS. H-space [1], H-convex set [1], transfer opened (transfer closed) [1], H^*-concave vector function [2], (weakly) vector minimal point [3].

1 Introduction and Preliminaries

In 1991, Chang and Shu [4] firstly introduced a quasi-variational inequality in locally convex Hausdorff linear topological spaces. Recently, Chang et al [1] obtained some existence theorems of solutions for the quasi-variational inequality in H-spaces, and Lee et al [3] studied the vector quasi-variational inequality for vector-valued functions in a real locally convex Hausdorff linear topological space.

The purpose of this paper is to study the existence problem of solutions of the vector quasi-variational inequality for vector-valued functions in H-spaces. The results presented in this paper generalize some important results in [1].

For the convenience we first give some definitions and preliminary results.

Definition 1.1 *Let E be a real topological vector space, K a nonempty closed convex subset of E satisfying*

(i) $\lambda x \in K$ *for all* $x \in K$ *and all* $\lambda \geq 0$*,*

(ii) $x \in K$*,* $-x \in K$ *implies* $x = \theta$*, where* θ *is the zero-element of* E*.*

Then K is said to be a cone. We denote the interior of K by K°. A cone K is said to be a body cone, if $K^\circ \neq \emptyset$.

Lemma 1.1 *Let E be a real topological vector space, $K \subset E$ a body cone. Then*

(i) $K + K = \{x + y : x \in K, y \in K\} = K$,

(ii) $\lambda + K \subset K^\circ$ *for any* $\lambda \in K^\circ$,

(iii) $K + K^\circ = K^\circ + K^\circ = K^\circ$.

Proof: The conclusions are obvious.

Definition 1.2 *Let $(X, \{\Gamma_A\})$ be an H-space, E a real topological vector space with a body cone $K \subset E$. A function $f : X \to E$ is said to be H^*-convex, if $-f : X \to E$ is H^*-concave.*

Lemma 1.2 *Let $f : X \to E$ be an H^*-convex function. Then the set $M_\lambda = \{x \in X : f(x) \notin \lambda + K\}$ is H-convex for any $\lambda \in E$.*

Proof: Since $-f$ is H^*-concave, it follows from Proposition 10.1.1 in Chang [2] that

$$\{x \in X : (-\lambda) - (-f(x)) \notin K\} = \{x \in X : f(x) \notin \lambda + K\} = M_\lambda$$

is H-convex. This completes the proof.

Lemma 1.3 [1, Lemma 2.1] *Let X be a compact topological space, $(Y, \{\Gamma_B\})$ an H-space. Let $G : X \to 2^Y$ be a multifunction with nonempty H-convex values and G^{-1} is transfer open-valued. Then there exists a continuous selection of G, i.e., there exists a continuous function $f : X \to Y$ such that $f(x) \in G(x)$ for all $x \in X$.*

Lemma 1.4 [1, Theorem 2.5] *Let $(X, \{\Gamma_A\})$ be a compact H-space, $T : X \to 2^X$ a multifunction such that*

(i) *$T(x)$ is a nonempty H-convex set for any $x \in X$,*

(ii) *$T^{-1} : X \to 2^X$ is transfer open-valued.*

Then there exists an $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$.

Lemma 1.5 [3, Lemma 1.1] *Let E be a real topological vector space with a body cone $K \subset E$, C a nonempty compact subset of E. Then $\mathrm{Min}_K C \neq \emptyset$ and $\mathrm{WMin}_K C \neq \emptyset$, where $\mathrm{Min}_K C$ is the set of all the vector minimal points of C and $\mathrm{WMin}_K C$ is the set of all the weakly vector minimal points of C.*

Lemma 1.6 [3, Lemma 1.2] *Let X and Y be two Hausdorff topological spaces, and $F : X \to 2^Y$ be a multifunction.*

(i) *If F is u.s.c. and compact-valued, then F is closed.*

(ii) *If Y is compact, and F is closed, then F is u.s.c..*

(iii) *If X is compact, and F is u.s.c. and compact-valued, then $F(X) = \bigcup_{x \in X} F(X)$ is compact.*

2 The Main Results

Theorem 2.1 *Let $(X, \{\Gamma_A\})$ be a compact H-space, $(Y, \{\Gamma_B\})$ be an H-space. Let E be a real topological vector space with a body cone $K \subset E$. Suppose that*

(i) *$S : X \to 2^X$ is a continuous multifunction with nonempty compact H-convex values and $S^{-1}(x)$ is open for any $x \in X$,*

(ii) *$T : X \to 2^Y$ is a multifunction with nonempty H-convex values and $T^{-1} : Y \to 2^X$ is transfer open-valued,*

(iii) *$\varphi : X \times Y \times X \to E$ is a continuous function satisfying*

  (a) *$\varphi(x, y, x) \notin K^\circ$ for all $x \in X$ and all $y \in T(x)$,*

  (b) *the function $z \mapsto \varphi(x, y, z)$ is H^*-convex.*

Then there exists $\overline{x} \in S(\overline{x})$, $\overline{y} \in T(\overline{x})$ such that

$$\varphi(\overline{x}, \overline{y}, x) \notin -K^\circ \qquad \text{for all } x \in S(\overline{x}).$$

Proof: By Lemma 1.3 and condition (ii), there exists a continuous selection $f : X \to Y$ of the multifunction T. Next for any $\lambda \in K^\circ$, we define a multifunction $F_\lambda : X \to 2^X$ by

$$F_\lambda(x) = \{z \in S(x) : \varphi(x, f(x), s) - \varphi(x, f(x), z) \notin -\lambda - K,\ \forall s \in S(x)\} \quad \text{for all } x \in X. \tag{2.1}$$

Let

$$P_\lambda(x) = \{z \in X : \varphi(x, f(x), s) - \varphi(x, f(x), z) \notin -\lambda - K,\ \forall s \in S(x)\} \quad \text{for all } x \in X. \tag{2.2}$$

Then

$$F_\lambda(x) = S(x) \bigcap P_\lambda(x) \quad \text{for all } x \in X \tag{2.3}$$

and

$$F_\lambda^{-1}(z) = S^{-1}(z) \bigcap P_\lambda^{-1}(z) \quad \text{for all } z \in X. \tag{2.4}$$

It is obvious that

$$\begin{aligned} P_\lambda(x) &= \bigcap_{s \in S(x)} \{z \in X : \varphi(x, f(x), z) \notin \varphi(x, f(x), s) + \lambda + K\} \\ &= \{z \in X : \varphi(x, f(x), S(x)) - \varphi(x, f(x), z) \bigcap -\lambda - K = \emptyset\} \\ &\supset \{z \in X : \varphi(x, f(x), S(x)) - \varphi(x, f(x), z) \bigcap -K^\circ = \emptyset\}. \end{aligned} \tag{2.5}$$

(I) $F_\lambda(x)$ is nonempty for all $x \in X$.

In fact, since S is compact valued and φ is continuous, $\varphi(x, f(x), S(x))$ is a compact subset of E. By Lemma 1.5, $\mathrm{WMin}_K \varphi(x, f(x), S(x)) \neq \emptyset$, i.e., there exists a $z \in S(x)$, such that $\varphi(x, f(x), s) - \varphi(x, f(x), z) \notin -K^\circ$ for each $s \in S(x)$ and hence $z \in F_\lambda(x)$ by (2.3) and (2.5).

(II) $F_\lambda(x)$ is H-convex.

By Lemma 1.2 and condition (iii)(b), the set $\{z \in X : \varphi(x, f(x), z) \notin \varphi(x, f(x), s) + \lambda + K\}$ is H-convex, and hence $P_\lambda(x)$is H-convex by (2.5). On the other hand, by condition (i), $S(x)$ is H-convex. Therefore $F_\lambda(x)$ is H-convex by (2.3).

(III) $F_\lambda^{-1}(z)$ is open for each $z \in X$.

By condition (i) and (2.4), it is sufficient to prove that $P_\lambda^{-1}(z)$ is open. Taking $x \in P_\lambda^{-1}(z)$, by (2.5), we have

$$\varphi(x, f(x), S(x)) - \varphi(x, f(x), z) \bigcap -\lambda - K = \emptyset\,.$$

Let $H_z(x) = \varphi(x, f(x), S(x)) - \varphi(x, f(x), z)$. It follows from φ is continuous and S is u.s.c. and compact valued that $H_z : X \to 2^E$ is u.s.c., and hence there exists a neighborhood $N(x)$ of x such that

$$H_z(y) \bigcap -\lambda - K = \emptyset \quad \text{for all } y \in N(x).$$

This shows that $N(x) \subset P_\lambda^{-1}(z)$ and $P_\lambda^{-1}(z)$ is open.

Combining (I), (II) and (III), by Lemma 1.4, there exists $x_\lambda \in X$ such that

$$x_\lambda \in F_\lambda(x_\lambda)\,. \tag{2.6}$$

For any $\lambda_1, \lambda_2 \in K^\circ$, we define $\lambda_1 \prec \lambda_2$ if and only if $\lambda_1 \in \lambda_2 + K$ and $x_{\lambda_1} \prec x_{\lambda_2}$ if and only if $\lambda_1 \prec \lambda_2$. Then $\{x_\lambda\}_{\lambda \in K^\circ} \subset X$ is a net. Since X is compact, without loss of generality, we can assume that $x_\lambda \to \overline{x} \in X$ and so $y_\lambda := f(x_\lambda) \to f(\overline{x}) := \overline{y} \in T(\overline{x})$.

On the other hand, from the definition of F_λ and (2.6) we have

$$x_\lambda \in S(x_\lambda) \quad \text{and} \quad x_\lambda \in P_\lambda(x_\lambda)\,.$$

Since S is compact-valued and continuous, by Lemma 1.6, the graph of S is closed and so $\overline{x} \subset S(\overline{x})$.

For any $\lambda \in K^\circ$, it follows from $x_\lambda \in P_\lambda(x_\lambda)$ that

$$\varphi(x_\lambda, y_\lambda, S(x_\lambda)) - \varphi(x_\lambda, y_\lambda, x_\lambda) \bigcap -\lambda - K = \emptyset$$

and hence

$$\varphi(x_\lambda, y_\lambda, S(x_\lambda)) - \varphi(x_\lambda, y_\lambda, x_\lambda) \bigcap -\lambda - K^\circ = \emptyset\,.$$

Taking $\lambda_0 \in K^\circ$. For any $\lambda \in K^\circ$ with $\lambda_0 \prec \lambda$, we have

$$\lambda_0 + K^\circ \subset \lambda + K + K^\circ = \lambda + K^\circ$$

and so

$$\varphi(x_\lambda, y_\lambda, S(x_\lambda)) - \varphi(x_\lambda, y_\lambda, x_\lambda) \bigcap -\lambda_0 - K^\circ = \emptyset \quad \text{for all } \lambda \succ \lambda_0\,. \tag{2.7}$$

Now we prove that

$$\varphi(\overline{x}, \overline{y}, S(\overline{x})) - \varphi(\overline{x}, \overline{y}, \overline{x}) \bigcap -\lambda_0 - K^\circ = \emptyset\,. \tag{2.8}$$

Suppose that this is not the case, then there exists $\overline{x} \in S(\overline{x})$ such that

$$\varphi(\overline{x}, \overline{y}, \overline{s}) - \varphi(\overline{x}, \overline{y}, \overline{x}) \in -\lambda_0 - K^\circ\,. \tag{2.9}$$

Since S is l.s.c., there exists a net $\{s_\lambda\}$ such that $\{s_\lambda\} \to \overline{s}$ and $s_\lambda \in S(x_\lambda)$ for each $\lambda \in K^\circ$. By the continuity of φ and (2.9), for λ large enough

$$\varphi(x_\lambda, y_\lambda, s_\lambda) - \varphi(x_\lambda, y_\lambda, x_\lambda) \in -\lambda_0 - K^\circ\,.$$

This contradicts (2.7), and (2.8) is proved. Hence

$$\varphi(\overline{x}, \overline{y}, S(\overline{x})) - \varphi(\overline{x}, \overline{y}, \overline{x}) \bigcap \left(\bigcup_{\lambda_0 \in K^\circ} (-\lambda_0 - K^\circ) \right) = \varphi(\overline{x}, \overline{y}, S(\overline{x})) - \varphi(\overline{x}, \overline{y}, \overline{x}) \bigcap -K^\circ = \emptyset\,.$$

Therefore,

$$\varphi(\overline{x}, \overline{y}, x) - \varphi(\overline{x}, \overline{y}, \overline{x}) \notin -K^\circ \quad \text{for all } x \in S(\overline{x})\,.$$

By condition (iii)(a), $\varphi(\overline{x}, \overline{y}, \overline{x}) \in -K^\circ$ and so

$$\varphi(\overline{x}, \overline{y}, x) \notin -K^\circ \quad \text{for all } x \in S(\overline{x})\,.$$

This completes the proof.

Theorem 2.2 *Let $(X, \{\Gamma_A\})$ be a compact H-space, $(Y, \{\Gamma_B\})$ be an H-space. Let E be a real topological vector space with a body cone $K \subset E$. Suppose that*

(i) $T : X \to 2^Y$ *is a multifunction with nonempty H-convex values and $T^{-1} : Y \to 2^X$ is transfer open-valued,*

(ii) $S : X \to 2^X$ *is a l.s.c. multifunction with nonempty H-convex values and $S^{-1} : X \to 2^X$ is closed-valued,*

(iii) $\varphi : X \times Y \times X \to E$ *is a continuous function satisfying*

  (a) $\varphi(x, y, x) \notin -K^\circ$ *for all $x \in X$ and all $y \in T(x)$,*

  (b) *for each $(x, y) \in X \times Y$, the set*

$$\{z \in X : \varphi(x, y, z) \in \mathrm{WMin}_K \varphi(x, y, S(x))\}$$

  is H-convex,

(iv) *for any continuous function $f : X \to Y$, there exists a finite subset $A \subset X$ such that for any $x \in X$*

$$\varphi(x, f(x), A \cap S(x)) \bigcap \mathrm{WMin}_K \varphi(x, f(x), S(x)) \neq \emptyset.$$

Then there exists $\overline{x} \in S(\overline{x}), \overline{y} \in T(\overline{x})$ such that

$$\varphi(\overline{x}, \overline{y}, x) \notin -K^\circ \quad \text{for all } x \in S(\overline{x})\,.$$

Proof: By Lemma 1.3 and condition (i), there exists a continuous selection $f : X \to Y$ of T. Now we define a multifunction $F : X \to 2^X$ by

$$F(x) = \{z \in S(x) : \varphi(x, f(x), z) \in \mathrm{WMin}_K \varphi(x, f(x), S(x))\}\,.$$

Since f is continuous and S is H-convex-valued, by condition (iii)(b) and (iv), F is nonempty H-convex valued.

Now we prove that for each $z \in X, F^{-1}(z)$ is closed. Let $\{x_\alpha\}_{\alpha \in I} \subset F^{-1}(z)$ be any net which converges to x. Then for each $\alpha \in I$, $z \in F(x_\alpha)$ and so

$$z \in S(x_\alpha) \quad \text{and} \quad \varphi(x_\alpha, f(x_\alpha), z) \in \mathrm{WMin}_K \varphi(x_\alpha, f(x_\alpha), S(x_\alpha))\,. \tag{2.10}$$

Since $S^{-1}(z)$ is closed, hence $x \in S^{-1}(z)$ and so $z \in S(x)$. Suppose that $x \notin F^{-1}(z)$, then

$$\varphi(x, f(x), z) \notin \mathrm{WMin}_K \varphi(x, f(x), S(x))$$

and hence there exists an $s \in S(x)$ such that

$$\varphi(x, f(x), s) - \varphi(x, f(x), z) \in -K^\circ .$$

Since S is l.s.c., there exists a net $\{s_\alpha\} \to s$ with $s_\alpha \in S(x_\alpha)$ for all $\alpha \in I$. Again, φ is continuous and $-K^\circ$ is open, for α large enough

$$\varphi(x_\alpha, f(x_\alpha), s_\alpha) - \varphi(x_\alpha, f(x_\alpha), z) \in -K^\circ .$$

This contradics (2.10). Hence $x \in F^{-1}(z)$ and so $F^{-1}(z)$ is closed for all $z \in X$.

On the other hand, by condition (iv), there exists a finite subset $A \subset X$, such that for any $x \in X$, $A \cap F(x) \neq \emptyset$ and hence

$$F^{-1}(A) = \bigcup_{z \in A} F^{-1}(z) = X .$$

From Corollary 3.6.4. in Chang [2], it follows that there exists $\overline{x} \in X$ such that $\overline{x} \in F(\overline{x})$, i.e.,

$$\overline{x} \in S(\overline{x}) \quad \text{ and } \quad \varphi(\overline{x}, f(\overline{x}), \overline{x}) \in \mathrm{WMin}_K \varphi(\overline{x}, f(\overline{x}), S(\overline{x})) .$$

Since f is a continuous selection of T, $f(\overline{x}) \in T(\overline{x})$. Letting $\overline{y} = f(\overline{x})$, we have $\overline{y} \in T(\overline{x})$ and

$$\varphi(\overline{x}, \overline{y}, \overline{x}) \in \mathrm{WMin}_K \varphi(\overline{x}, \overline{y}, S(\overline{x})) ,$$

i.e.,

$$\varphi(\overline{x}, \overline{y}, x) - \varphi(\overline{x}, \overline{y}, \overline{x}) \notin -K^\circ \quad \text{for all } x \in S(\overline{x}) .$$

By condition (iii)(a), we have

$$\varphi(\overline{x}, \overline{y}, x) \notin -K^\circ \quad \text{for all } x \in S(\overline{x}) .$$

This completes the proof.

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On common fixed point theorems for families of mappings

ABSTRACT. In this paper we prove some common fixed point theorems for families of mappings on metric spaces which are generalizations of the results due to Fisher [2, 3, 4], Kasahara [5], Kim and Leem [6], Kim, Kim, Leem and Ume [7], Ohta and Nikaido [8], Park and Rhoades [9, 10], Taskovic [12] and others.

KEY WORDS AND PHRASES. Common fixed point, fixed point, family of mappings, closed mapping, cluster point, metric space, compact metric space.

1 Introduction and preliminaries

Let ω, N denote the sets of nonnegative integers and positive integers, respectively. Let f, g, h and t be self mappings of a metric space (X, d) and $C_f = \{s : s : X \to X \text{ and } sf = fs\}$. For $x, y \in X$ and $A, B \subseteq X$, define $O_f(x) = \{f^n x : n \in \omega\}$, $O_f(x, y) = O_f(x) \cup O_f(y)$, $O_{f,g}(x) = \{f^i g^j x : i, j \in \omega\}$, $O_{f,g}(x, y) = O_{f,g}(x) \cup O_{f,g}(y)$, $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$, $\delta(A, A) = \delta(A)$ and $\delta(x, B) = \delta(\{x\}, B)$. It is easy to verify that $\{f^n : n \in \omega\} \subseteq C_f$. For each $t \in [0, +\infty)$, $[t]$ denotes the largest integer not exceeding t. Let

$$\Phi_1 = \{\varphi : \varphi : [0, +\infty) \to [0, +\infty) \text{ is nondecreasing, continuous from the right and satisfies } \varphi(t) < t \text{ for } t > 0\},$$

$$\Phi_2 = \{\varphi : \varphi : [0, +\infty) \to [0, +\infty) \text{ is nondecreasing, upper semicontinuous from the right and satisfies } \varphi(t) < t \text{ for } t > 0\},$$

$$\Phi = \{\varphi : \varphi : [0, +\infty) \to [0, +\infty) \text{ is nondecreasing, upper semicontinuous and satisfies } \varphi(t) < t \text{ for } t > 0\}.$$

Clearly, $\Phi \subseteq \Phi_2$. Chang [1] and Wong [13] proved that $\Phi_2 \subseteq \Phi$ and $\Phi_1 = \Phi_2$. Therefore $\Phi_2 = \Phi$.

The Banach contraction principle has long been one of the most important tools in the study of nonlinear problems. Motivated by this fact, during the past three decades, there has grown an extensive literature devoted to sharper forms of the principle. Recently, Fisher [2, 3, 4], Kasahara [5], Kim and Leem [6], Kim, Kim, Leem and Ume [7], Otha and Nikaido [8], Park and Rhoades [9, 10], Taskovic [12] and others have established the existence of fixed and common fixed points for the following contractive type mappings:

$$\begin{aligned}&\exists r \in [0,1) \text{ such that for all } x, y \in X\,,\\ &\qquad d(fx, fy) \le r\delta(O_f(x,y)) \qquad \text{(Taskovic [12])}\,;\end{aligned} \tag{1.1}$$

$$\begin{aligned}&\exists k \in N \text{ and } r \in [0,1) \text{ such that for all } x, y \in X\,,\\ &\qquad d(f^k x, f^k y) \le r\delta(O_f(x,y)) \qquad \text{(Ohta and Nikaido [8])}\,;\end{aligned} \tag{1.2}$$

$$\begin{aligned}&\exists k \in N \text{ and } r \in [0,1) \text{ such that for all } x, y \in X\,,\\ &\qquad d(f^k x, g^k y) \le r\delta(O_{f,g}(x,y)) \qquad \text{(Kim and Leem [6])}\,;\end{aligned} \tag{1.3}$$

$$\begin{aligned}&\exists k \in N \text{ and } r \in [0,1) \text{ such that for all } x, y \in X\,,\\ &\qquad d((fg)^k x, (fg)^k y) \le r\delta(O_{f,g}(x,y)) \qquad \text{(Kim and Leem [6])}\,;\end{aligned} \tag{1.4}$$

$$\begin{aligned}&\exists m, n \in N \text{ and } r \in [0,1) \text{ such that for all } x, y \in X\,,\\ &\qquad d((fg)^m x, (fg)^n y) \le r\delta(O_{f,g}(x,y)) \qquad \text{(Kim, Kim, Leem and Ume [7])}\,;\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\exists \varphi \in \Phi_1 \text{ such that for all } x, y \in X\,,\\ &\qquad d(fx, f^2 y) \le \varphi(\delta(O_f(x, fy))) \qquad \text{(Park and Rhoades [9])}\,;\end{aligned} \tag{1.6}$$

$$\begin{aligned}&\exists \varphi \in \Phi_1 \text{ such that for all } x, y \in X\,,\\ &\qquad d(gx, gy) \le \varphi(\delta(O_f(x,y))) \qquad \text{(Park and Rhoades [10])}\,;\end{aligned} \tag{1.7}$$

$$\begin{aligned}&\exists \varphi \in \Phi_2 \text{ such that for all } x, y \in X\,,\\ &\qquad d(fx, fy) \le \varphi(\delta(O_f(x,y))) \qquad \text{(Kasahara [5])}\,.\end{aligned} \tag{1.8}$$

Now we list contractive type conditions to be considered:

$$\begin{aligned}&\exists \varphi \in \Phi, p, q, m, n \in \omega \text{ with } p+q, m+n \in N \text{ such that for all } x, y \in X\,,\\ &\qquad d(f^p g^q x, f^m g^n y) \le \varphi(\delta(O_{f,g}(x,y)))\,;\end{aligned} \tag{1.9}$$

$$\begin{aligned}&\exists \varphi \in \Phi, m \in \{1,2\} \text{ such that for all } x, y \in X\,,\\ &\qquad d(fx, f^m y) \le \varphi(\delta(O_f(x,y)))\,;\end{aligned} \tag{1.10}$$

$$\begin{aligned}&\exists p, q, m, n \in \omega \text{ with } p+q, m+n \in N \text{ such that for all } x, y \in X\\ &\qquad \text{with } f^p g^q x \ne f^m g^n y\,,\ d(f^p g^q x, f^m g^n y) < \delta\Big(\bigcup_{h \in C_f \cap C_g} h O_{f,g}(x,y)\Big)\,;\end{aligned} \tag{1.11}$$

$$\begin{aligned}&\exists p, q, m, n \in \omega \text{ with } p+q, m+n \in N \text{ such that for all } x, y \in X\\ &\qquad \text{with } f^p g^q x \ne h^m t^n y\,,\ d(f^p g^q x, h^m t^n y) < \delta\Big(\bigcup_{u \in C_{fg}} u O_{fg}(x), \bigcup_{v \in C_{ht}} v O_{ht}(y)\Big)\,.\end{aligned} \tag{1.12}$$

It is easy to see that the following diagrams of implications hold:

$$(1.9) \Leftarrow (1.7) \Leftarrow (1.8) \Leftarrow (1.1) \Rightarrow (1.2) \Rightarrow (1.3) \Rightarrow (1.99)\,,$$
$$(1.4) \Rightarrow (1.5) \Rightarrow (1.9)\,,$$
$$(1.6) \Rightarrow (1.10) \Rightarrow (1.9) \Rightarrow (1.11)\,.$$

In this paper we establish several fixed and common fixed point theorems involving hypotheses weak enough to include a number of results due to Fisher [2, 3, 4], Kasahara [5], Kim and Leem [6], Kim, Kim, Leem and Ume [7], Ohta and Nikaido [8], Park and Rhoades [9, 10], Taskovic [12] and others as special cases.

The following lemmas were introduced by Kim, Kim, Leem and Ume [7] and Sing and Meade [11], respectively.

Lemma 1.1 [7]. *Let f and g be commuting mappings from a compact metric space (X, d) into itself. Assume that fg is closed. If $A = \bigcap_{n\in N}(fg)^n X$, then*

(i) $hA \subseteq A$ *for all* $h \in C_{fg}$,

(ii) $A = fA = gA \neq \phi$,

(iii) A *is a compact subset of* X.

Lemma 1.2 [11]. *Let $\varphi \in \Phi$. Then*

(i) $\lim_{n\to\infty} \varphi^n(t) = 0$ *for all* $t > 0$,

(ii) $t = 0$ *provided that* $t \leq \varphi(t)$ *for some* $t \geq 0$.

2 Main results

Our main results are as follows.

Theorem 2.1 *Let f and g be commuting mappings from a metric space (X, d) into itself such that fg is closed. Suppose that there exists $u \in X$ such that the sequence $\{(fg)^i u\}_{i\in N}$ has a cluster point $w \in X$ for which (1.9) holds for all $x, y \in O_{f,g}(u, w)$, where $\delta(O_{f,g}(u, w)) < +\infty$. Then w is a common fixed point of f and g and*

$$d((fg)^i f^a g^b u, w) \leq \varphi^{\left[\frac{i}{k}\right]}(\delta(O_{f,g}(u)))$$

for all $i \in N$ and $a, b \in \{0, 1\}$, where $k = \max\{p, q, m, n\}$.

Proof: For any $i, j, l, s, t \in \omega$, it follows from (1.9) that

$$\begin{aligned} d(f^{i+k+j}g^{i+k+l}u, f^{i+k+s}g^{i+k+t}u) &\le \varphi(\delta(O_{f,g}(f^{i+k-p+j}g^{i+k-q+l}u, f^{i+k-m+s}g^{i+k-n+t}u))) \\ &\le \varphi(\delta(O_{f,g}(f^{i+j}g^{i+l}u, f^{i+s}g^{i+t}u))) \\ &\le \varphi(\delta(O_{f,g}((fg)^i u)))\,. \end{aligned}$$

This implies that

$$\delta(O_{f,g}((fg)^{i+k}u)) \le \varphi(\delta(O_{f,g}((fg)^i u))) \tag{2.1}$$

for all $i \in \omega$. We claim that

$$d((fg)^i u, (fg)^{i+t}u) \le \varphi^{\left[\frac{i}{k}\right]}(\delta(O_{f,g}(u))) \tag{2.2}$$

for all $i, t \in N$. We can write $i = ck + l$ uniquely for some $c, l \in \omega$ with $l \le k-1$. Using (2.1),

$$\begin{aligned} d((fg)^i u, (fg)^{i+t}u) &\le \delta(O_{f,g}((fg)^{ck+l}u)) \\ &\le \varphi(\delta(O_{f,g}((fg)^{(c-1)k+l}u))) \\ &\le \cdots \\ &\le \varphi^c(\delta(O_{f,g}((fg)^l u))) \\ &\le \varphi^c(\delta(O_{f,g}(u)))\,. \end{aligned}$$

That is, (2.2) holds. Lemma 1.2 ensures that $\{(fg)^i u\}_{i\in N}$ is a Cauchy sequence and since it has a cluster point $w \in X$, so $w = \lim_{i\to\infty}(fg)^i u$. Note that fg is closed. Then

$$w = \lim_{i\to\infty}(fg)^i u = \lim_{i\to\infty} fg(fg)^i u = fgw\,. \tag{2.3}$$

For any $i, j, s, t \in \omega$, by (1.9) and(2.3) we obtain

$$\begin{aligned} d(f^i g^j w, f^s g^t w) &= d(f^{i+k}g^{j+k}w, f^{s+k}g^{j+k}w) \\ &\le \varphi(\delta(O_{f,g}(f^{i+k-p}g^{j+k-q}w, f^{s+k-m}g^{t+k-n}w))) \\ &\le \varphi(\delta(O_{f,g}(w))) \end{aligned}$$

which implies that

$$\delta(O_{f,g}(w)) \le \varphi(\delta(O_{f,g}(w)))\,.$$

It follows from Lemma 1.2 that $\delta(O_{f,g}(w)) = 0$. That is, $w = fw = gw$.

In view of (2.1) we have

$$\begin{aligned} d((fg)^i f^a g^b u, (fg)^{i+t}u) &\le \delta(O_{f,g}((fg)^i u)) \\ &\le \varphi(\delta(O_{f,g}(fg)^{i-k}u)) \\ &\le \varphi^{\left[\frac{i}{k}\right]}(\delta(O_{f,g}(u))) \end{aligned}$$

for all $i, t \in N$ and $a, b \in \{0, 1\}$. Letting t tend to infinity we get

$$d((fg)^i f^a g^b u, w) \leq \varphi^{\left[\frac{i}{k}\right]}(\delta(O_{f,g}(u)))$$

for all $i \in N$ and $a, b \in \{0, 1\}$. This completes the proof.

Remark 2.1 Theorem 2.1 includes Theorems 3 and 4 of Kim and Leem [5], Theorems 1 and 2 of Park and Rhoades [10] as apecial cases.

From Theorem 2.1 we immediately have

Theorem 2.2 *Let f and g be commuting mappings from a bounded complete metric space (X, d) into itself such that fg is closed. Suppose that (1.9) holds. Then f and g have a unique common fixed point $w \in X$ and*

$$d((fg)^i f^a g^b x, w) \leq \varphi^{\left[\frac{i}{k}\right]}(\delta(O_{f,g}(x)))$$

for all $x \in X$, $i \in N$ and $a, b \in \{0, 1\}$, where $k = \max\{p, q, m, n\}$.

Remark 2.2 Theorem 2.2 includes Theorem 2.1 of Kim, Kim, Leem and Ume [6] as a special case.

Corollary 2.1 *Let f be a closed mapping from a metric space (X, d) into itself. Suppose that there exists $u \in X$ such that the sequence $\{(fg)^i u\}_{i \in N}$ has a cluster point $w \in X$ for which the following*

$$d(f^p x, f^m y) \leq \varphi(\delta(O_f(x, y))) \tag{2.4}$$

holds for all $x, y \in O_f(u, w)$, where $\delta(O_f(u, w)) < +\infty$, $\varphi \in \Phi$ and $p, m \in N$. Then f has a fixed point w and satisfies

$$d(f^i u, w) \leq \varphi^{\left[\frac{i}{k}\right]}(\delta(O_f(u)))$$

for all $i \in N$, where $k = \max\{p, m\}$.

Remark 2.3 In case $\varphi(t) = rt$ and $p = m$, Corollary 2.1 reduces to a result which extends Theorem 3 of Ohta and Nikaido [7].

The following result reveals that the condition that T be closed is unnecessary if $p = 1$ and $m \in \{1, 2\}$.

Theorem 2.3 *Let f be a mapping from a metric space (X, d) into itself. Suppose that there exists $u \in X$ such that the sequence $\{f^i u\}_{i \in N}$ has a cluster point $w \in X$ for which (1.10) holds for all $x, y \in O_f(u, w)$, where $\delta(O_f(u, w)) < +\infty$. Then w is a fixed point of f and satisfies*

$$d(f^i u, w) \leq \varphi^{\left[\frac{i}{m}\right]}(\delta(O_f(u))) \tag{2.5}$$

for all $i \in N$.

Proof: It follows from the proof of Theorem 2.1 that

$$\delta(O_f(f^{i+m}u)) \le \varphi(\delta(O_f(f^i u))) \tag{2.6}$$

for all $i \in \omega$,

$$d(f^i u, f^{i+t}u) \le \varphi^{[\frac{i}{m}]}(\delta(O_f(u))) \tag{2.7}$$

for all $i, t \in N$ and $w = \lim_{i\to\infty} f^i u$. Letting t tend to infinity in (2.7), we easily conclude that (2.5) holds.

For every $\varepsilon > 0$ there exists an integer $k > 2m$ such that $i > k - m$ implies $d(f^i u, w) < \varepsilon$. For any $n, p \in N$ with $p > k$, by (1.9) we have

$$\begin{aligned} d(w, f^n w) &\le d(w, f^p u) + d(f^p u, f^n w) \\ &\le \varepsilon + \varphi(\delta(O_f(f^{p-1}u, f^{n-m}w))) \\ &\le \varepsilon + \varphi(\max\{2\varepsilon, \delta(O_f(w)) + \varepsilon\}) \end{aligned}$$

which implies that

$$\delta(w, O_f(w)) \le \varepsilon + \varphi(\max\{2\varepsilon, \delta(O_f(w)) + \varepsilon\}) .$$

Thus we have

$$\begin{aligned} \delta(w, O_f(w)) &\le \limsup_{\varepsilon\to 0}\{\varepsilon + \varphi(\max\{2\varepsilon, \delta(O_f(w)) + \varepsilon\})\} \\ &\le \limsup_{\varepsilon\to 0} \varphi(\max\{2\varepsilon, \delta(O_f(w)) + \varepsilon\}) \\ &\le \varphi(\delta(O_f(w))) . \end{aligned}$$

That is,

$$\delta(w, O_f(w)) \le \varphi(\delta(O_f(w))) . \tag{2.8}$$

For $m = 1$, by (2.6) and (2.8) we have

$$\delta(O_f(w)) = \max\{\delta(w, O_f(fw)), \delta(O_f(fw))\} \le \varphi(\delta(O_f(w))) .$$

Lemma 1.2 ensures that $\delta(O_f(w)) = 0$. Hence $w = fw$.

For $m = 2$, by (2.6), (1.10) and(2.8) we get

$$\begin{aligned} \delta(O_f(w)) &= \max\{\delta(w, O_f(fw)), \delta(fw, O_f(f^2w)), \delta(O_f(f^2w))\} \\ &\le \max\{\varphi(\delta(O_f(w))), \sup_{t\in\omega} \varphi(\delta(O_f(w, f^t w))), \varphi(\delta(O_f(w)))\} \\ &= \varphi(\delta(O_f(w))) . \end{aligned}$$

It follows from Lemma 1.2 that $\delta(O_f(w)) = 0$. That is, $w = fw$. This completes the proof.

Remark 2.4 Theorem 2.3 extends, improves and unifies Theorem 2 of Park and Rhoades [8] and Theorem 1 of Kasahara [4].

Question 2.1 Does Theorem 2.3 hold for $m \geq 3$?

Theorem 2.4 *Let f and g be commuting mappings from a compact metric space (X, d) into itself such that fg is closed. If (1.11) holds, then f and g have a unique common fixed point $w \in X$. Moreover, $w = hw$ for all $h \in C_f \cap C_g$.*

Proof: Let $A = \bigcap_{i \in N} (fg)^i X$. Lemma 1.1 implies that $A = fA = gA \neq \phi$. We assert that $A = \{w\}$ for some $w \in X$. Otherwise $\delta(A) > 0$. It follows from the compactness of A that $\delta(A) = d(u, v)$ for some $u, v \in A$. Obviously there exist $x, y \in A$ such that $u = f^p g^q x$, $v = f^m g^n y$. By (1.11) and Lemma 1.1 we conclude that

$$\begin{aligned} \delta(A) &= d(f^p g^q x, f^m g^n y) \\ &< \delta(\bigcup_{h \in C_f \cap C_g} hO_{f,g}(x, y)) \\ &\leq \delta(\bigcup_{h \in C_f \cap C_g} hA) \\ &\leq \delta(A) \end{aligned}$$

which is a contradiction. Thus A is a singleton and $A = \{w\}$ for some $w \in X$. Therefore $w = fw = gw$. That is, w is a common fixed point of f and g.

Now suppose that f and g have a second common fixed point u. Then $u = (fg)^i u$ for all $i \in N$. This implies that $u \in A = \{w\}$. That is, $u = w$. This proves the uniqueness of w, which implies that $w = hw$ for all $h \in C_f \cap C_g$. This completes the proof.

Remark 2.5 Theorem 4 of Fisher [1], Theorem 5 of Fisher [2] and Theorem 2 of Fisher [3] are special cases of Theorem 2.4.

Theorem 2.5 *Let f, g, h and t be self mappings of a compact metric space (X, d) such that (1.12) holds. If fg, ht are closed and $f \in C_g$, $h \in C_t$, then f, g, h and t have a unique common fixed point $w \in X$. Moreover, $w = uw = vw$ for $u \in C_{fg}$ and $v \in C_{ht}$.*

Proof: Put $A \bigcap_{i \in N} (fg)^i X$ and $B = \bigcap_{i \in N} (ht)^i X$. From Lemma 1.1 it follows that

$$A = fA = gA \neq \phi\,, \quad B = hB = tB \neq \phi\,,$$

and that A, B are compact. We claim that $\delta(A, B) = 0$. If not, then $\delta(A, B) > 0$. By the compactness of A, B there exist $a \in A$, $b \in B$ such that $\delta(A, B) = d(a, b)$. Since $f^p g^q A = A$ and $h^m t^n B = B$, we can find $x \in A$ and $y \in B$ with $f^p g^q x = a$ and $h^m t^n y = b$. It follows from (1.12) and Lemma 1.1 that

$$\begin{aligned}\delta(A,B) &= d(f^p g^q x, h^m t^n y)\\ &< \delta(\bigcup_{u\in C_{fg}} uO_{fg}(x), \bigcup_{v\in C_{ht}} vO_{ht}(y))\\ &\le \delta(\bigcup_{u\in C_{fg}} uA, \bigcup_{v\in C_{ht}} vB)\\ &= \delta(A,B)\end{aligned}$$

which is absurd and hence $\delta(A,B)=0$, which implies that $A=B=\{w\}$ for some $w\in X$. It is easy to see that

$$w = fw = gw = hw = tw = uw = vw$$

for all $u\in C_{fg}$ and $v\in C_{ht}$.

If z is another common fixed point of f,g,h and t, then $z=f^i g^i z = h^i t^i z$ for all $i\in N$. That is, $z\in A=B=\{w\}$, which proves the uniqueness of w. This completes the proof.

The following examples show that the closedness assumptions in Theorems 2.1, 2.2, 2.4 and 2.5 and Corollary 2.1 are necessary for $p+q\ge 2$ and $m+n\ge 2$.

Example 2.1 Let $X=[0,1]$ with the usual metric d. Define mappings $f,g: X\to X$ by $f0=1$, $fx=\frac{1}{3}x$ for $x\in(0,1]$ and $gx=x$ for $x\in X$. Take $q=n=0$ and $\varphi(t)=\frac{1}{3}t$ for $t\in[0,+\infty)$. It is easy to verify that

$$d(f^p x, f^m y) = \frac{1}{3} d(f^{p-1}x, f^{m-1}y) \le \varphi(\delta(O_f(x,y)))$$

for all $x,y\in X$, where $p,m\ge 2$. Thus (1.9) and (2.4) are satisfied. Note that $\frac{1}{n}\to 0$ and $f\frac{1}{n}=\frac{1}{3n}\to 0$ as $n\to\infty$ and that $0\ne 1=f0$. Hence f and fg are not closed. Thus the conditions of Theorems 2.1 and 2.2 and Corollary 2.1 are satisfied except the closedness assumption; f however has no fixed point.

Example 2.2 Let $(X,d), f,g,p,q,m,n$ be as in Example 2.1. Take $h=f$ and $t=g$. Then

$$d(f^p x, f^m y) = \frac{1}{3} d(f^{p-1}x, f^{m-1}y) < \delta(O_f(x,y))$$

for all $x,y\in X$ with $f^p x\ne f^m y$, where $p,m\ge 2$. Thus the conditions of Theorems 2.4 and 2.5 are satisfied except the closedness assumption; f however has no fixed point.

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Characterizations of common fixed points in 2-metric spaces

ABSTRACT. In this paper we obtain a few necessary and sufficient conditions for the existence of a common fixed point of a pair of mappings in 2-metric spaces. Our results generalize, improve and unify a number of fixed point theorems given by Cho [1], Constantin [2], Khan and Fisher [11], Kubiak [14], Rhoades [25], Singh, Tiwari and Gupta [31] and others.

KEY WORDS AND PHRASES. 2-metric spaces, common fixed points, compatible mappings.

1 Introduction

Gähler [4] introduced the concept of 2-metric space. A 2-metric space is a set X with a function $d : X \times X \times X \to [0, \infty)$ satisfying the following conditions:

(G1) for two distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,

(G2) $d(x, y, z) = 0$ if at least two of x, y, z are equal,

(G3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,

(G4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

It has been shown by Gähler [4] that a 2-metric d is a continuous function of any one of its three arguments but it need not be continuous in two arguments. If it is continuous in two arguments, then it is continuous in all three arguments. A 2-metric d which is continuous in all of its arguments will be called continuous.

Iséki [7], for the first time, developed a fixed point theorem in 2-metric spaces. Since then a quite number of authors ([1]-[3], [5]-[36]) have extended and generalized the result of Iséki and various other results involving contractive and expansive type mappings. Especially, Murthy, Chang, Cho and Sharma [17] introduced the concepts of compatible mappings and compatible mappings of type (A) in 2-metric spaces, derived some relations between these mappings and proved common fixed point theorems for compatible mappings of type (A) in 2-metric spaces.

On the other hand, Cho [1], Constantin [2], Khan and Fisher [11] and Kubiak [14] established some necessary and sufficient conditions which guarantee the existence of a common fixed point for a pair of continuous mappings in 2-metric spaces.

In this paper we establish criteria for the existence of a common fixed point of a pair of mappings in 2-metric spaces. Our results generalize, improve and unify the corresponding results of Cho [1], Constantin [2], Khan and Fisher [11], Kubiak [14], Rhoades [25], Singh, Tiwari and Gupta [31] and others.

2 Preliminaries

Throughout this paper, N and ω denote the sets of positive and nonnegative integers, respectively. Let $R^+ = [0, \infty)$ and

$$W = \{w : w : R^+ \to R^+ \text{ is continuous and satisfies } 0 < w(t) < t \text{ for } t > 0\}.$$

We consider the family Φ of all continuous functions $\varphi : (R^+)^5 \to R^+$ with the following properties:

(i) φ is non-decreasing in the 4$^{\text{th}}$ and 5$^{\text{th}}$ variables,

(ii) if $u, v \in R^+$ with $u \le \max\{\varphi(v, v, u, u+v, 0), \varphi(v, u, v, u+v, 0), \varphi(v, v, u, 0, u+v), \varphi(v, u, v, 0, u+v)\}$, then $u \le cv$ for some $c \in (0, 1)$,

(iii) if $u \in R^+$ with $u \le \max\{\varphi(u, 0, 0, u, u), \varphi(0, u, 0, u, u), \varphi(0, 0, u, u, u)\}$, then $u = 0$.

Let f be a mapping of a 2-metric space (X, d) into itself and $B \subset X$. Define $d(x, B, a) = \inf_{b \in B} d(x, b, a)$ for $x, a \in X$ and $F(f) = \{t : t = ft \in X\}$.

Definition 2.1 *A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X, d) is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} d(x_n, x, a) = 0$ for all $a \in X$. The point x is called the limit of the sequence $\{x_n\}_{n \in N}$ in X.*

Definition 2.2 *A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X, d) is said to be a Cauchy sequence if $\lim_{m,n \to \infty} d(x_m, x_n, a) = 0$ for all $a \in X$.*

Definition 2.3 *A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Note that, in a 2-metric space (X, d), a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X ([19]).

Definition 2.4 *Let f and g be mappings from a 2-metric space (X, d) into itself. f and g are said to be compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n, a) = 0$$

for all $a \in X$, whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$; f and g are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n, a) = \lim_{n\to\infty} d(gfx_n, ffx_n, a) = 0$$

for all $a \in X$, whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.5 *A mapping f from a 2-metric space (X, d) into itself is said to be continuous at $x \in X$ if for every sequence $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$, $\lim_{n\to\infty} d(fx_n, fx, a) = 0$. f is called continuous on X if it is so at all points of X.*

Lemma 2.1 ([17]) *Let f and g be compatible mappings from a 2-metric space (X, d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = ggt = gft = fft$.*

Lemma 2.2 ([17]) *Let f and g be compatible mappings from a 2-metric space (X, d) into itself. If f is continuous at some $t \in X$ and if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, then $\lim_{n\to\infty} gfx_n = ft$.*

3 Characterizations of common fixed points

Theorem 3.1 *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mappings of X into itself. Then the following conditions are equivalent:*

(1) *h and t have a common fixed point;*

(2) *there exist $r \in (0, 1)$, $f : X \to t(X)$ and $g : X \to h(X)$ such that*

  (a1) *the pairs f, h and g, t are compatible,*

  (a2) *one of f, g, h and t is continuous,*

  (a3) $d(fx, gy, a) \le r \max\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a),$
  $\frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\}$ *for all $x, y, a \in X$;*

(3) *there exist* $w \in W, f : X \to t(X)$ *and* $g : X \to h(X)$ *satisfying* (a1), (a2) *and* (a4):

(a4) $$d(fx, gy, a) \le \max\Big\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\Big\} - w\Big(\max\Big\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\Big\}\Big)$$

for all $x, y, a \in X$*;*

(4) *there exist* $\varphi \in \Phi, f : X \to t(X)$ *and* $g : X \to h(X)$ *satisfying* (a1), (a2) *and* (a5):

(a5) $d(fx, gy, a) \le \varphi(d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), d(hx, gy, a), d(ty, fx, a))$ *for all* $x, y, a \in X$.

Proof: (1) $\Rightarrow$ (2) and (4). Let z be a common fixed point of h and t. Define $f : X \to t(X)$ and $g : X \to h(X)$ by $fx = gx = z$ for all $x \in X$. Then (a1) and (a2) hold. For each $r \in (0, 1)$ and $\varphi \in \Phi$, (a3) and (a5) also hold.

(2) $\Rightarrow$ (3). Take $w(t) = (1 - r)t$. Then $w \in W$ and (a3) implies (a4).

(3) $\Rightarrow$ (1). Let x_0 be an arbitrary point in X. Since $f(X) \subset t(X)$ and $g(X) \subset h(X)$, there exist sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ in X satisfying $y_{2n} = tx_{2n+1} = fx_{2n}$, $y_{2n+1} = hx_{2n+2} = gx_{2n+1}$ for $n \in \omega$. Define $d_n(a) = d(y_n, y_{n+1}, a)$ for $a \in X$ and $n \in \omega$. We claim that for any $i, j, k \in \omega$

$$d(y_i, y_j, y_k) = 0\,. \tag{3.1}$$

Suppose that $d_{2n}(y_{2n+2}) > 0$. Using (a4), we have

$$\begin{aligned} d(fx_{2n+2}, gx_{2n+1}, y_{2n}) \le \max\Big\{&d(hx_{2n+2}, tx_{2n+1}, y_{2n}), d(hx_{2n+2}, fx_{2n+2}, y_{2n}), \\ &d(tx_{2n+1}, gx_{2n+1}, y_{2n}), \\ &\frac{1}{2}[d(hx_{2x+2}, gx_{2n+1}, y_{2n}) + d(tx_{2n+1}, fx_{2n+2}, y_{2n})]\Big\} \\ - w\Big(\max\Big\{&d(hx_{2n+2}, tx_{2n+1}, y_{2n}), d(hx_{2n+2}, fx_{2n+2}, y_{2n}), \\ &d(tx_{2n+1}, gx_{2n+1}, y_{2n}), \\ &\frac{1}{2}[d(hx_{2n+2}, gx_{2n+1}, y_{2n}) + d(tx_{2n+1}, fx_{2n+2}, y_{2n})]\Big\}\Big) \end{aligned}$$

which implies that

$$\begin{aligned} d_{2n}(y_{2n+2}) &\le \max\{0, d_{2n}(y_{2n+2}), 0, 0\} - w(\max\{0, d_{2n}(y_{2n+2}), 0, 0\}) \\ &= d_{2n}(y_{2n+2}) - w(d_{2n}(y_{2n+2})) < d_{2n}(y_{2n+2}) \end{aligned}$$

which is a contradiction. Hence $d_{2n}(y_{2n+2}) = 0$. Similarly, we have $d_{2n+1}(y_{2n+3}) = 0$. Consequently, $d_n(y_{n+2}) = 0$ for all $n \in \omega$. Note that

$$\begin{aligned} d(y_n, y_{n+2}, a) &\le d_n(y_{n+2}) + d_n(a) + d_{n+1}(a) \\ &= d_n(a) + d_{n+1}(a)\,. \end{aligned} \tag{3.2}$$

By (a4) and (3.2) we have

$$\begin{aligned} d_{2n+1}(a) &= d(fx_{2n+2}, gx_{2n+1}, a) \\ &\le \max\big\{d(hx_{2n+2}, tx_{2n+1}, a), d(fx_{2n+2}, hx_{2n+2}, a), d(gx_{2n+1}, tx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n+2}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n+2}, a)]\big\} \\ &\quad - w\big(\max\big\{d(hx_{2n+2}, tx_{2n+1}, a), d(fx_{2n+2}, hx_{2n+2}, a), d(gx_{2n+1}, tx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n+2}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n+2}, a)]\big\}\big) \\ &= \max\big\{d_{2n}(a), d_{2n+1}(a), d_{2n}(a), \frac{1}{2}[0 + d(y_{2n}, y_{2n+2}, a)]\big\} \\ &\quad - w\big(\max\big\{d_{2n}(a), d_{2n+1}(a), d_{2n}(a), \frac{1}{2}[0 + d(y_{2n}, y_{2n+2}, a)]\big\}\big) \\ &= \max\big\{d_{2n}(a), d_{2n+1}(a), \frac{1}{2}[d_{2n}(a) + d_{2n+1}(a)]\big\} \\ &\quad - w\big(\max\big\{d_{2n}(a), d_{2n+1}(a), \frac{1}{2}[d_{2n}(a) + d_{2n+1}(a)]\big\}\big) \\ &= \max\big\{d_{2n}(a), d_{2n+1}(a)\big\} - w\big(\max\{d_{2n}(a), d_{2n+1}(a)\}\big)\,. \end{aligned}$$

Suppose that $d_{2n+1}(a) > d_{2n}(a)$. Then $d_{2n+1}(a) \le d_{2n+1}(a) - w(d_{2n+1}(a)) < d_{2n+1}(a)$, which is a contradiction. Hence $d_{2n+1}(a) \le d_{2n}(a)$ and so $d_{2n+1}(a) \le d_{2n}(a) - w(d_{2n}(a)) < d_{2n}(a)$. Similarly, we have $d_{2n}(a) \le d_{2n-1}(a)$. That is, for all $n \in N$

$$d_{n+1}(a) \le d_n(a)\,. \tag{3.3}$$

Let n, m be in ω. If $n \ge m$, then $0 = d_m(y_m) \ge d_n(y_m)$; if $n < m$, then

$$\begin{aligned} d_n(y_m) &\le d_n(y_{m-1}) + d_{m-1}(y_n) + d_{m-1}(y_{n+1}) \\ &\le d_n(y_{m-1}) + d_n(y_n) + d_n(y_{n+1}) \\ &\le d_n(y_{m-1}) \le d_n(y_{m-2}) \le \cdots \le d_n(y_{n+1}) = 0\,. \end{aligned}$$

Thus, for any $n, m \in \omega$

$$d_n(y_m) = 0\,. \tag{3.4}$$

For all $i, j, k \in \omega$, we may, without loss of generality, assume that $i < j$. It follows from (3.4) that

$$\begin{aligned} d(y_i, y_j, y_k) &\le d_i(y_j) + d_i(y_k) + d(y_{i+1}, y_j, y_k) \\ &= d(y_{i+1}, y_j, y_k) \le d(y_{i+2}, y_j, y_k) \le \dots \\ &\le d(y_{j-1}, y_j, y_k) = d_{j-1}(y_k) = 0\,. \end{aligned}$$

Therefore (3.1) holds.

By virtue of (a4), (3.3) and (3.4) we have

$$\begin{aligned} d_{2n}(a) &= d(fx_{2n}, gx_{2n+1}, a) \\ &\le \max\big\{d(hx_{2n}, tx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), d(tx_{2n+1}, gx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n}, a)]\big\} \\ &\quad - w\big(\max\big\{d(hx_{2n}, tx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), d(tx_{2n+1}, gx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n}, a)]\big\}\big) \\ &= \max\big\{d_{2n-1}(a), 0, d_{2n}(a), \frac{1}{2}d(y_{2n-1}, y_{2n+1}, a)\big\} \\ &\quad - w\big(\max\big\{d_{2n-1}(a), 0, d_{2n}(a), \frac{1}{2}d(y_{2n-1}, y_{2n+1}, a)\big\}\big) \\ &= \max\big\{d_{2n-1}(a), d_{2n}(a), \frac{1}{2}[d_{2n-1}(a) + d_{2n}(a) + d_{2n+1}(y_{2n-1})]\big\} \\ &\quad - w\big(\max\big\{d_{2n-1}(a), d_{2n}(a), \frac{1}{2}[d_{2n-1}(a) + d_{2n}(a) + d_{2n+1}(y_{2n-1})]\big\}\big) \\ &= \max\{d_{2n-1}(a), d_{2n}(a)\} - w(\max\{d_{2n-1}(a), d_{2n}(a)\}) \\ &= d_{2n-1}(a) - w(d_{2n-1}(a))\,. \end{aligned}$$

Similarly, we have $d_{2n+1}(a) \le d_{2n}(a) - w(d_{2n}(a))$. It follows that

$$\sum_{i=0}^{n} w(d_i(a)) \le \sum_{i=0}^{n}[d_i(a) - d_{i+1}(a)] = d_0(a) - d_{n+1}(a) \le d_0(a)\,.$$

So the series of nonnegative terms $\sum\limits_{n=0}^{\infty} w(d_n(a))$ is convergent. This means that

$$\lim_{n\to\infty} w(d_n(a)) = 0\,. \tag{3.5}$$

(3.3) ensures that $\{d_n(a)\}_{n\in\omega}$ converges to some $r \ge 0$. In view of the continuity of w and (3.5) we have

$$w(r) = \lim_{n\to\infty} w(d_n(a)) = 0$$

which implies that $r = 0$. Hence

$$\lim_{n\to\infty} d_n(a) = 0\,. \tag{3.6}$$

In order to show that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence, by (3.6), it is sufficient to show that $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\omega}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and $a \in X$ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(y_{2m(k)}, y_{2n(k)}, a) \geq \varepsilon$.

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality, so that

$$d(y_{2m(k)-2}, y_{2n(k)}, a) \leq \varepsilon\,, \quad d(y_{2m(k)}, y_{2n(k)}, a) > \varepsilon\,. \tag{3.7}$$

For each even integer $2k$, by (3.1) and (3.7) we have

$$\begin{aligned}
\varepsilon &< d(y_{2m(k)}, y_{2n(k)}, a)\\
&\leq d(y_{2m(k)-2}, y_{2n(k)}, a) + d(y_{2m(k)}, y_{2m(k)-2}, a) + d(y_{2m(k)}, y_{2n(k)}, y_{2m(k)-2})\\
&\leq \varepsilon + d(y_{2m(k)-2}, y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2m(k)-1}, a) + d(y_{2m(k)-1}, y_{2m(k)}, a)\\
&= \varepsilon + d_{2m(k)-2}(a) + d_{2m(k)-1}(a)
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}, a) = \varepsilon\,. \tag{3.8}$$

It follows from (3.7) that

$$\begin{aligned}
0 &< d(y_{2n(k)}, y_{2m(k)}, a) - d(y_{2n(k)}, y_{2m(k)-2}, a)\\
&\leq d(y_{2m(k)-2}, y_{2m(k)}, a)\\
&\leq d_{2m(k)-2}(a) + d_{2m(k)-1}(a)\,.
\end{aligned}$$

In view of (3.6) and (3.8) we immediately obtain

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-2}, a) = \varepsilon\,. \tag{3.9}$$

Note that

$$\begin{aligned}
\big|d(y_{2n(k)}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)}, a)\big| &\leq d_{2m(k)-1}(a) + d_{2m(k)-1}(y_{2n(k)})\,,\\
\big|d(y_{2n(k)+1}, y_{2m(k)}, a) - d(y_{2n(k)}, y_{2m(k)}, a)\big| &\leq d_{2n(k)}(a) + d_{2n(k)}(y_{2m(k)})\,,\\
\big|d(y_{2n(k)+1}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)-1}, a)\big| &\leq d_{2n(k)}(a) + d_{2n(k)}(y_{2m(k)-1})\,.
\end{aligned}$$

It is easy to see that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}, a) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}, a) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}, a) = \varepsilon\,. \tag{3.10}$$

It follows from (a4) that

$$\begin{aligned} d(y_{2m(k)}, y_{2n(k)+1}, a) &= d(fx_{2m(k)}, gx_{2n(k)+1}, a) \\ &\le \max\big\{d(hx_{2m(k)}, tx_{2n(k)+1}, a), d(hx_{2m(k)}, fx_{2m(k)}, a), \\ &\qquad d(tx_{2n(k)+1}, gx_{2n(k)+1}, a), \\ &\qquad \tfrac{1}{2}[d(hx_{2m(k)}, gx_{2n(k)+1}, a) + d(tx_{2n(k)+1}, fx_{2m(k)}, a)]\big\} \\ &\quad - w\big(\max\big\{d(hx_{2m(k)}, tx_{2n(k)+1}, a), d(hx_{2m(k)}, fx_{2m(k)}, a), \\ &\qquad d(tx_{2n(k)+1}, gx_{2n(k)+1}, a), \\ &\qquad \tfrac{1}{2}[d(hx_{2m(k)}, gx_{2n(k)+1}, a) + d(tx_{2n(k)+1}, fx_{2m(k)}, a)]\big\}\big) \\ &= \max\big\{d(y_{2m(k)-1}, y_{2n(k)}, a), d_{2m(k)-1}(a), d_{2n(k)}(a), \\ &\qquad \tfrac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)+1}, a) + d(y_{2n(k)}, y_{2m(k)}, a)]\big\} \\ &\quad - w\big(\max\big\{d(y_{2m(k)-1}, y_{2n(k)}, a), d_{2m(k)-1}(a), d_{2n(k)}(a), \\ &\qquad \tfrac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)+1}, a) + d(y_{2n(k)}, y_{2m(k)}, a)]\big\}\big)\,. \end{aligned}$$

Letting $k \to \infty$, by (3.10), (3.8), and (3.6) we have

$$\varepsilon \le \max\{\varepsilon, 0, 0, \varepsilon\} - w(\max\{\varepsilon, 0, 0, \varepsilon\}) = \varepsilon - w(\varepsilon) < \varepsilon$$

which is a contradiction. Therefore $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence in X.

It follows from completeness of (X, d) that $\{y_n\}_{n\in\omega}$ converges to a point $u \in X$. Now, suppose that t is continuous. Since g and t are compatible and $\{gx_{2n+1}\}_{n\in\omega}$ and $\{tx_{2n+1}\}_{n\in\omega}$ converge to the point u, by Lemma 2.2 we get that gtx_{2n+1}, $tgx_{2n+1} \to tu$ as $n \to \infty$. In virtue of (a4) we have

$$\begin{aligned} d(fx_{2n}, gtx_{2n+1}, a) &\le \max\big\{d(hx_{2n}, ttx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), d(ttx_{2n+1}, gtx_{2n+1}, a), \\ &\qquad \tfrac{1}{2}[d(hx_{2n}, gtx_{2n+1}, a) + d(ttx_{2n+1}, fx_{2n}, a)]\big\} \\ &\quad - w\big(\max\big\{d(hx_{2n}, ttx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), \\ &\qquad d(ttx_{2n+1}, gtx_{2n+1}, a), \\ &\qquad \tfrac{1}{2}[d(hx_{2n}, gtx_{2n+1}, a) + d(ttx_{2n+1}, fx_{2n}, a)]\big\}\big)\,. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(u,tu,a) &\le \max\left\{d(u,tu,a), d(u,u,a), d(tu,tu,a), \frac{1}{2}[d(u,tu,a)+d(ttu,u,a)]\right\} \\ &\quad - w\Big(\max\left\{d(u,tu,a), d(u,u,a), d(tu,tu,a), \frac{1}{2}[d(u,tu,a)+d(ttu,u,a)]\right\}\Big) \\ &= d(u,tu,a) - w(d(u,tu,a)) \end{aligned}$$

which implies that $w(d(u,tu,a)) \le 0$. This means that $u = tu$.

It follows from (a4) that

$$\begin{aligned} d(fx_{2n}, gu, a) \le \max\Big\{& d(hx_{2n}, tu, a), d(hx_{2n}, fx_{2n}, a), d(tu, gu, a), \\ & \frac{1}{2}[d(hx_{2n}, gu, a) + d(tu, fx_{2n}, a)]\Big\} \\ - w\Big(\max\Big\{& d(hx_{2n}, tu, a), d(hx_{2n}, fx_{2n}, a), d(tu, gu, a), \\ & \frac{1}{2}[d(hx_{2n}, gu, a) + d(tu, fx_{2n}, a)]\Big\}\Big). \end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned} d(u,gu,a) &\le \max\left\{d(u,u,a), d(u,u,a), d(u,gu,a), \frac{1}{2}[d(u,gu,a)+d(u,u,a)]\right\} \\ &\quad - w\Big(\max\left\{d(u,u,a), d(u,u,a), d(u,gu,a), \frac{1}{2}[d(u,gu,a)+d(u,u,a)]\right\}\Big) \\ &= d(u,gu,a) - w(d(u,u,a)) \end{aligned}$$

which implies that $u = gu$. It follows from $g(X) \subset h(X)$ that there exists $v \in X$ with $u = gu = hv$. From (a4) we get

$$\begin{aligned} d(fv,u,a) &= d(fv,gu,a) \\ &\le \max\left\{d(hv,tu,a), d(hv,fv,a), d(tu,gu,a), \frac{1}{2}[d(hv,gu,a)+d(tu,fv,a)]\right\} \\ &\quad - w\Big(\max\Big\{d(hv,tu,a), d(hv,fv,a), d(tu,gu,a), \\ &\qquad \frac{1}{2}[d(hv,gu,a)+d(tu,fv,a)]\Big\}\Big) \\ &= d(u,fv,a) - w(d(u,fv,a)). \end{aligned}$$

Therefore, $u = fv$. Lemma 2.1 ensures that $fu = fhv = hfv = hu$. By (a4) we obtain again

$$\begin{aligned} d(fu,u,a) &= d(fv,gu,a) \\ &\le \max\left\{d(hu,tu,a), d(hu,fu,a), d(tu,gu,a), \frac{1}{2}[d(hu,gu,a)+d(tu,fu,a)]\right\} \\ &\quad - w\Big(\max\Big\{d(hu,tu,a), d(hu,fu,a), d(tu,gu,a), \\ &\qquad \frac{1}{2}[d(hu,gu,a)+d(tu,fu,a)]\Big\}\Big) \\ &= d(fu,u,a) - w(d(fu,u,a)). \end{aligned}$$

Hence $u = fu$. That is, u is a common fixed point of f, g, h and t. Similarly, we can complete the proof when f or g or h is continuous.

(4) $\Rightarrow$ (1). Let $\{x_n\}_{n\in\omega}$, $\{y_n\}_{n\in\omega}$, $d_n(a)$ be as in the proof of (3) $\Rightarrow$ (1). Analogous we conclude that for all $n, m \in \omega$

$$d_n(a) \le hd_{n-1}(a) \le \cdots \le h^n d_0(a) \tag{3.11}$$

and

$$d_n(y_m) = 0\,. \tag{3.12}$$

For any $n, m \in \omega$, by (3.11) and (3.12) we have

$$\begin{aligned} d(y_n, y_{n+m}, a) &\le d_n(a) + d(y_{n+1}, y_{n+m}, a) + d_n(y_{n+m}) \\ &= d_n(a) + d(y_{n+1}, y_{n+m}, a) \le \dots \\ &\le \sum_{k=n}^{n+m-1} d_k(a) \le \sum_{k=n}^{n+m-1} h^k d_0(a) \le \frac{h^n}{1-h} d_0(a) \end{aligned}$$

which implies that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence. The remainder of the proof follows from the proof process of (3) $\Rightarrow$ (1).

This completes the proof.

From Theorem 3.1 we immediately have

Theorem 3.2 *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mappings of X into itself. Then (1) is equivalent to each of the following conditions:*

(5) *there exist $r \in (0, 1)$, $f : X \to t(X) \cap h(X)$ such that*

- (a6) *the pairs f, h and f, t are compatible,*
- (a7) *one of f, t and h is continuous,*
- (a8) $d(fx, fy, a) \le r \max\big\{d(hx, ty, a), d(hx, fx, a), d(ty, fy, a), \tfrac{1}{2}[d(hx, fy, a) + d(ty, fx, a)]\big\}$ *for all $x, y, a \in X$;*

(6) *there exist $w \in W$, $f : X \to t(X) \cap h(X)$ satisfying* (a6), (a7) *and* (a9):

(a9) $$d(fx, fy, a) \le \max\big\{d(hx, ty, a), d(hx, fx, a), d(ty, fy, a), \frac{1}{2}[d(hx, fy, a) + d(ty, fx, a)]\big\} - w\Big(\max\big\{d(hx, ty, a), d(hx, fx, a), d(ty, fy, a), \frac{1}{2}[d(hx, fy, a) + d(ty, fx, a)]\big\}\Big)$$

for all $x, y, a \in X$;

(7) *there exist* $\varphi \in \Phi, f : X \to t(X) \cap h(X)$ *satisfying* (a6), (a7) *and* (a10):

(a10) $d(fx, fy, a) \le \varphi(d(hx, ty, a), d(hx, fx, a), d(ty, fy, a), d(hx, fy, a), d(ty, fx, a))$ *for all* $x, y, a \in X$.

Remark 3.1 Theorems 3.1 and 3.2 are still true even though the condition of the compatibility is replaced by the compatibility of type (A).

Remark 3.2 Theorems 3.1 and 3.2 generalize, improve and unify Theorem 4.8 of Cho [1], the Theorem of Constantin [2], Theorem 2 of Khan and Fisher [11], Theorem 1 of Kubiak [14], Theorem 4 of Rhoades [24], Theorem 1 of Singh, Tiwari and Gupta [30].

Theorem 3.3 *Let* f *be a mapping of a complete* 2*-metric space* (X, d) *into itself satisfying*

(a11) $$d(fx, fy, a) \le \max\big\{d(x, y, a), \frac{1}{2}[d(x, fx, a) + d(y, fy, a)], \frac{1}{2}[d(x, fy, a) + d(y, fx, a)], \frac{1}{2}[d(y, fy, a) + d(y, fx, a)], \frac{1}{2}[d(y, fy, a) + d(x, fy, a)]\big\}$$

for all $x, y, a \in X$.

Then the following conditions are equivalent:

(8) $F(f)$ *is nonempty and for each* $x \in X$, *the sequence of iterates* $\{f^n x\}_{n\in\omega}$ *converges to some point in* $F(f)$;

(9) *there exists a closed subset* G *of* X *such that*

(a12) $d(fx, p, a) \le d(x, p, a)$ *for* $x, a \in X$, $p \in G$,

(a13) $\liminf\limits_{n\to\infty} d(f^n x, G, a) = 0$ *for* $a \in X$.

Proof: (8) $\Rightarrow$ (9). Take $G = F(f)$ and $x \in X$. Then $\{f^n x\}_{n\in N}$ converges to some point $w \in G$. It follows that

$$\begin{aligned}\liminf_{n\to\infty} d(f^n x, G, a) &= \liminf_{n\to\infty}\big\{\inf_{p\in G} d(f^n x, p, a)\big\} \le \liminf_{n\to\infty} d(f^n x, w, a)\\ &= \lim_{n\to\infty} d(f^n x, w, a) = 0\end{aligned}$$

which implies that $\liminf_{n\to\infty} d(f^n, G, a) = 0$ for $a \in X$.

Let $x, a \in X$ and $p \in G$. In view of (a11) we have

$$\begin{aligned}d(fx, p, a) &= d(fx, fp, a)\\ &\le \max\big\{d(x,p,a), \frac{1}{2}d(x,fx,a), \frac{1}{2}[d(x,p,a)+d(p,fx,a)],\\ &\qquad \frac{1}{2}d(p,fx,a), \frac{1}{2}d(x,p,a)\big\}\\ &= \max\big\{d(x,p,a), \frac{1}{2}(x,fx,a), \frac{1}{2}[d(x,p,a)+d(p,fx,a)]\big\}\,.\end{aligned} \tag{3.13}$$

Using (3.13) we obtain

$$d(fx,p,x) \le \max\big\{0, 0, \frac{1}{2}d(p,fx,x)\big\} = \frac{1}{2}d(fx,p,x)$$

which implies that $d(fx,p,x) = 0$. Hence

$$\begin{aligned}d(x,fx,a) &\le d(x,fx,p) + d(x,p,a) + d(p,fx,a)\\ &= d(x,p,a) + d(p,fx,a)\,.\end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$d(fx,p,a) \le \max\big\{d(x,p,a), \frac{1}{2}[d(x,p,a)+d(p,fx,a)]\big\}$$

which implies that

$$d(fx,p,a) \le d(x,p,a)\,. \tag{3.15}$$

Assume that $\{y_n\}_{n\in N} \subset G$ and $y_n \to y \in X$ as $n \to \infty$. It follows from (3.15) that

$$d(fy,y,a) \le d(fy,y,y_n) + d(fy,y_n,a) + d(y_n,y,a) \le 2d(y_n,y,a)\,.$$

Letting $n \to \infty$, we easily conclude that $d(fy,y,a) = 0$ for all $a \in X$. Therefore $y = fy \in G$. That is, G is a closed.

(9) $\Rightarrow$ (8). Let x_0 and a be in X and $x_n = f^n x_0$ for all $n \in N$. It follows from (a12) that

$$d(x_n, G, a) = \inf_{p\in G}\big\{d(x_n,p,a)\big\} \le \inf_{p\in G}\big\{d(x_{n-1},p,a)\big\} = d(x_{n-1},G,a)\,.$$

That is, $\{d(x_n, G, a)\}_{n\in N}$ is non-increasing. Thus (a13) implies that $\lim_{n\to\infty} d(x_n, G, a) = \liminf_{n\to\infty} d(x_n, G, a) = 0$. Therefore, for any $\varepsilon > 0$, there exists $n_0(a) \in N$ such that for all $n \geq n_0(a)$

$$d(x_n, G, a) < \frac{1}{2}\varepsilon\,. \tag{3.16}$$

Now, suppose that $n > m \geq n_0(a)$ and $p \in G$. By (a12) we have for $k \geq n_0(a)$

$$d(x_n, x_m, p) = d(fx_{n-1}, p, x_m) \leq d(x_{n-1}, p, x_m) \leq \cdots \leq d(x_m, p, x_m) = 0 \tag{3.17}$$

and

$$d(x_k, p, a) = d(fx_{k-1}, p, a) \leq d(x_{k-1}, p, a) \leq \cdots \leq d(x_{n_0(a)}, p, a)\,. \tag{3.18}$$

In virtue of (3.17) and (3.18) we get

$$\begin{aligned} d(x_n, x_m, a) &\leq d(x_n, x_m, p) + d(x_n, p, a) + d(p, x_m, a) \\ &\leq 2d(x_{n_0(a)}, p, a) \end{aligned}$$

which implies that

$$d(x_n, x_m, a) \leq 2d(x_{n_0(a)}, G, a)\,.$$

By (3.16) and the above inequality we have for $n > m \geq n_0(a)$

$$d(x_n, x_m, a) < \varepsilon\,.$$

Hence $\{x_n\}_{n\in N}$ is a Cauchy sequence in X and has a limit $w \in X$ since (X, d) is complete. Letting $k \to \infty$ in (3.18), we have

$$d(w, p, a) \leq d(x_{n_0(a)}, p, a)$$

which implies that

$$d(w, G, a) = \inf_{p\in G} d(w, p, a) \leq d(x_{n_0(a)}, G, a)\,. \tag{3.19}$$

In view of (3.16) and (3.19) we obtain that $d(w, G, a) < \varepsilon$, which implies that $d(w, G, a) = 0$ for all $a \in X$. Since G is closed, so $w \in G$. It follows from (a12) that $d(fw, w, a) \leq d(w, w, a) = 0$ for all $a \in X$. This means that $w = fw \in F(f) \neq \phi$.

This completes the proof.

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Common Fixed Points in Compact Metric Spaces

ABSTRACT. Common fixed point theorems for contractive type mappings of compact metric spaces are given. Our works generalize known results of Edelstein, Fisher, Leader, Jungck and Liu.

KEY WORDS AND PHRASES. Contractive type mappings, common fixed points, compact metric spaces.

1 Introduction

Throughout this paper, we assume that (X, d) is a compact metric space and that f, g, a and b are self mappings of (X, d). N and ω denote the sets of positive integers and nonnegative integers, respectively. For $x, y \in X$, define $O(x, f) = \{f^n x \,|\, n \in \omega\}$ and $O(x, y, f, g) = O(x, f) \cup O(y, f) \cup O(x, g) \cup O(y, g)$. Put $\delta(A) = \sup\{d(a, b) \,|\, a, b \in A\}$ for $A \subseteq X$. Define $C_f = \{h \,|\, h : X \to X \text{ and } hf = fh\}$, $H_f = \{h \,|\, h : X \to X \text{ and } h \bigcap_{n\in N} f^n X \subseteq \bigcap_{n\in N} f^n X\}$. Clearly $H_f \supseteq C_f \supseteq \{f^n \,|\, n \in \omega\}$.

In 1962, Edelstein [1] proved the following

Theorem E *Let f satisfy $d(fx, fy) < d(x, y)$ for all distinct $x, y \in X$. Then f has a unique fixed point.*

Fisher [2, 3, 4, 5], Leader [6], Jungck [7] and Liu [8] et al. gave a number of generalizations of Theorem E, some of which deal with contractive type mappings as follows:

1) $$d(fx, fy) < \max\{d(x, y), \frac{1}{2}[d(x, fx) + d(y, fy)], \frac{1}{2}[d(x, fy) + d(y, fx)]\}$$
for all distinct $x, y \in X$ (Fisher [2]).

2) There exist $p, q \in N$ such that
$$d(f^p x, f^q y) < \max\{d(f^m x, f^n y), d(f^m x, f^i x), d(f^n y, f^j y) \,|\, 0 \le m, i \le p, 0 \le n, j \le q\} \tag{I}$$
for all $x, y \in X$ for which the right-hand side of (I) is positive (Fisher [3]).

3) There exist $p, q, r, s \in \omega$, $p+q \geq 1$, $r+s \geq 1$ such that

$$d(f^p g^q x, f^r g^s y) < \max\{d(f^i g^j x, f^t g^k y), d(f^i g^j x, f^m g^n x), d(f^t g^k y, f^u g^v y) \mid 0 \leq i, m \leq p, 0 \leq j, n \leq q, 0 \leq t, u \leq r, 0 \leq k, v \leq s\} \tag{II}$$

for all $x, y \in X$ for which the right-hand side of (II) is positive (Fisher [4]).

4)
$$d(fx, gy) < \max\{d(ax, by), d(ax, fx), d(by, gy), d(ax, gy), d(by, fx)\} \tag{III}$$

for all $x, y \in X$ for which the right-hand side of (III) is positive (Fisher [5]).

5) There exists $p \in N$ such that

$$d(f^p x, f^p y) < \delta(O(x, f) \cup O(y, f))$$

for all distinct $x, y \in X$ (Leader [6]).

6)
$$d(fx, fy) < \delta(\bigcup_{h \in C_{gf}} h\{x, y\})$$

for all $x, y \in X$ with $fx \neq gy$ (Jungck [7]).

7)
$$d(fx, gy) < \delta(\bigcup_{h \in H_{gf}} h\{x, y\})$$

for all $x, y \in X$ with $fx \neq gy$ (Liu [8]).

The main purpose of this paper is to investigate the existence of common fixed points of self mappings f, g of (X, d) satisfying the following contractive type conditions:

8) There exist $p, q, r, s \in \omega$, $p+q \geq 1$, $r+s \geq 1$, such that

$$d(f^p g^q x, f^r g^s y) < \delta(\bigcup_{i,j,m,n \in \omega, h \in H_{gf}} hO(x, y, g^i f^j, f^n g^m))$$

for all $x, y \in X$ with $f^p g^q x \neq f^r g^s y$.

9) There exist $p, q \in \omega$, $p+q \geq 1$, such that

$$d(f^p g^q x, f^p g^q y) < \delta(\bigcup_{i,j,m,n \in \omega, h \in H_{gf}} hO(x, y, g^i f^j, f^n g^m)) \tag{IV}$$

for all distinct $x, y \in X$.

In Section 2, we prove some contractive type conditions are equivalent and establish several common fixed point theorems which extend and unify some known results in [1]-[8]. In Section 3, we give five examples to show that our results are more general than the results in [1]-[8].

2 Common fixed points

We first define contractive type conditions as follows:

2)′ There exist $p, q \in N$ such that (I) holds for all $x, y \in X$ with $f^p x \neq f^q y$.

3)′ There exist $p, q, r, s \in \omega$, $p+q \geq 1$, $r+s \geq 1$, such that (II) holds for all $x, y \in X$ with $f^p g^q x \neq f^r g^s y$.

4)′ (III) holds for all $x, y \in X$ with $fx \neq gy$.

9)′ There exist $p, q \in \omega$, $p+q \geq 1$, such that (IV) holds for all $x, y \in X$ with $f^p g^q x \neq f^p g^q y$.

We now prove the following

Lemma 1 *Conditions* 2) *and* 2)′ *are equivalent.*

Proof: Let $R(x, y)$ and $L(x, y)$ denote the right-hand and left-hand sides of 2) respectively. Assume that 2) holds. For $x, y \in X$, if $L(x, y) > 0$, then $R(x, y) > 0$. By assumption it follows that (I) holds for $x, y \in X$ with $L(x, y) > 0$. This proves that 2) implies 2)′.

Conversely, for $x, y \in X$, if $R(x, y) > 0$, we consider two cases:

(i) $L(x, y) = 0$. Clearly $L(x, y) = 0 < R(x, y)$.

(ii) $L(x, y) > 0$. Since 2)′ holds, $L(x, y) < R(x, y)$. It follows that 2)′ implies 2).

Similarly we have

Lemma 2 *Conditions* 3) *and* 3)′ *are equivalent.*

Lemma 3 *Conditions* 4) *and* 4)′ *are equivalent.*

Lemma 4 *Conditions* 9) *and* 9)′ *are equivalent.*

The following lemma was given in [6, 7].

Lemma 5 *Let f be continuous and $B = \bigcap_{n \in N} f^n X$. Then B is compact, $B = fB \neq \emptyset$ and $\delta(f^n X) \to \delta(B)$ as $n \to \infty$. Further $hB \subseteq B$ for $h \in C_f$.*

Our main results are as follows:

Theorem 1 *Let gf be continuous and $f \bigcap_{n \in N} (gf)^n X = \bigcap_{n \in N} (gf)^n X$. If f and g satisfy* 8)*, then f and g have a unique common fixed point u and $(gf)^n x$ converges to u uniformly on X. Further, $u = hu$ for $h \in H_{gf}$.*

Proof: Let $B = \bigcap_{n\in N}(gf)^n X$. It follows from Lemma 5 that B is a nonempty compact subset of X and $gfB = B$. Consequently $gB = gfB = B$ since $fB = B$. Suppose that $\delta(B) > 0$. By the compactness of B, there exist $u, v \in X$ such that $\delta(B) = d(u, v)$. Since $fB = gB = B$, there exist $x, y \in B$ such that $u = f^p g^q x$, $v = f^r g^s y$. Obviously $\bigcup_{i,j,m,n\in\omega,h\in H_{gf}} hO(x, y, g^i f^j, f^n g^m) \subseteq B$. Using 8) we get

$$\begin{aligned}\delta(B) &= d(f^p g^q x, f^r g^s y)\\ &< \delta(\bigcup_{i,j,m,n\in\omega,h\in H_{gf}} hO(x, y, g^i f^j, f^n g^m))\\ &\le \delta(B)\end{aligned}$$

which is impossible and hence $\delta(B) = 0$; i.e., B is a singleton, say, $B = \{u\}$ for some $u \in X$. It follows from $fB = gB = B$ that $fu = gu = u$; i.e., f and g have a common fixed point u.

Suppose that f and g have a second common fixed point v. Then $fv = gv = v = (gf)^n v \in (gf)^n X$ for $n \in N$. This means that $v \in \bigcap_{n\in N}(gf)^n X = B = \{u\}$. Therefore f and g have a unique common fixed point u. Note that $hB \subseteq B$ for $h \in H_{gf}$. Consequently $u = hu$ for $h \in H_{gf}$.

By Lemma 5, we have for $x \in X$ and $n \in N$

$$d((gf)^n x, u) \le \delta((gf)^n X) \to \delta(B) = 0 \text{ as } n \to \infty\,,$$

i.e., $(gf)^n x$ converges to u uniformly on X. This completes the proof.

From Theorem 1 we get immediately

Corollary 1 [8, Theorem 1]. *Let gf be continuous and $f\bigcap_{n\in N}(gf)^n X = \bigcap_{n\in N}(gf)^n X$. If f and g satisfy 7), then the conclusion of Theorem 1 remains unchanged.*

Proof: In case $p = s = 1$, $q = r = 0$, Corollary 1 follows from Theorem 1.

As a consequence of Corollary 1 we have

Corollary 2 [7, Theorem 4.2]. *Let gf be continuous and f and g commute. If f and g satisfy 6), then f and g have a unique common fixed point u. Further, $u = hu$ for $h \in C_{gf}$.*

Remark 1 Example 1 reveals that Theorem 1 is indeed a proper extension of Theorem 1 of Liu [8] and Theorem 4.2 of Jungck [7].

Corollary 3 *Let gf be continuous and $f\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}(gf)^nX$. If f and g satisfy*

10)
$$d(f^pg^qx, f^rg^sy) < \delta(\bigcup_{i,j,m,n\in\omega} O(x, y, g^if^j, f^ng^m))$$

for all $x, y \in X$ with $f^pg^qx \neq f^rg^sy$, where $p, q, r, s \geq 0$ are fixed integers with $p+q \geq 1$, $r + s \geq 1$,

then f and g have a unique common fixed point.

Proof: Since 10) implies 8), Corollary 3 follows from Theorem 1.

Corollary 4 *Let gf be continuous and $f\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}(gf)^nX$. If f and g satisfy 3), then f and g have a unique common fixed point u. Further, if $q = 0$ or $s = 0$, then u is the unique fixed point of f and if $p = 0$ or $r = 0$, then u is the unique fixed point of g.*

Proof: Note that 3)$'$ implies 10). It follows from Lemma 2 and Corollary 3 that f and g have a unique common fixed point u. Suppose that $q = 0$ and that v is a second distinct fixed point of f. From 3) we have

$$0 < d(v, u) = d(f^pv, f^rg^su) < d(v, u)$$

giving a contradiction. This proves the uniqueness of u.

Similarly, we can prove that u is the unique fixed point of f if $s = 0$, and that u is the unique fixed point of g if $p = 0$ or $r = 0$.

Remark 2 Theorem 5 of Fisher [4] is a special case of Corollary 4. Example 2 proves that Corollary 4 is a substantial generalization of the result of Fisher.

Theorem 2 *Let fg be continuous and $g\bigcap_{n\in N}(fg)^nX = \bigcap_{n\in N}(fg)^nX$. If f and g satisfy*

11)
$$d(f^pg^qx, f^rg^sy) < \delta(\bigcup_{i,j,m,n\in\omega,h\in H_{fg}} hO(x, y, g^if^j, f^ng^m))$$

for all $x, y \in X$ with $f^pg^qx \neq f^rg^sy$, where $p, q, r, s \geq 0$ are fixed integers and $p+q \geq 1$, $r + s \geq 1$,

then f and g have a unique common fixed point u and $(fg)^nx$ converges to u uniformly on X. Further, $u = hu$ for $h \in H_{fg}$.

The proof of this theorem goes in a similar fashion as that of Theorem 1, so we omit the proof.

Corollary 5 *Let fg be continuous, $g\bigcap_{n\in N}(fg)^nX = \bigcap_{n\in N}(fg)^nX$ and $a, b \in H_{fg}$. If f, g, a and b satisfy 4), then f, g, a and b have a unique common fixed point u. Further, u is the unique common fixed point of f and a and of g and b.*

Proof: By Lemma 3 we conclude easily that 4) is a particular case of 11). It follows from Theorem 2 that f, g, a and b have a unique common fixed point u. Suppose that f and a have a second distinct common fixed point v. Using 4)

$$0 < d(v, u) = d(fv, gu) < d(v, u)$$

which is a contradiction. This proves the uniqueness of u. Similarly we can prove that u is the unique common fixed point of g and b.

Remark 3 By setting $g =$identity mapping in Theorem 2, we get an improved version of Theorem 4 of Fisher [3]. Example 3 demonstrates that Theorem 2 extends properly Theorem 4 of Fisher [3]. Example 4 reveals that Corollary 5 is indeed a generalization of Theorem 2 of Fisher [5].

From Theorem 1 and Lemma 4 we have

Theorem 3 *Let gf be continuous and $f\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}(gf)^nX$. If f and g satisfy 9), then the conclusion of Theorem 1 remains unchanged.*

Remark 4 Theorem E, Theorems 7, 8, 9 and 10 of Fisher [2] and Corollaries 2 and 3 of Leader [6] are particular cases of Theorem 3. Example 5 proves that Theorem 3 extends properly the results of Edelstein, Fisher and Leader.

3 Examples

Example 1 Let $X = \{1, 2, 3, 4\}$ with the usual metric, $f1 = f4 = 1$, $f2 = g2 = g4 = 2$, $f3 = g3 = 4$ and $g1 = 3$. Take $p = r = s = 2$, $q = 1$. It is easy to check that the conditions of Theorem 1 are satisfied. But Theorem 1 of Liu [8] and Theorem 4.2 of Jungck [7] are not applicable since 6) and 7) do not hold for $x = 1$ and $y = 3$.

Example 2 Let $X = [0, 1]$ with the usual metric, $fx = x/3$ for $x \in X$, $g1 = 0$ and $gx = x/2$ for $x \in [0, 1)$. Clearly $\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}[0, 1/6^n] = \{0\}$. Set $p = s = 1$, $q = r = 0$. It is simple to verify that the assumptions of Corollary 4 are satisfied. But Theorem 5 of Fisher [4] is not applicable since $fg1 = 0 \neq 1/6 = gf1$.

Example 3 Let X and f be as in Example 1. For any $p, q \in N$, f does not satisfy 2) since

$$d(f^p 1, f^q 2) = 1 = \max\{d(f^m 1, f^n 2), d(f^m 1, f^i 1), d(f^n 2, f^j 2) \,|\, 0 \leq m, i \leq p, 0 \leq n, j \leq q\}$$

and hence Theorem 4 of Fisher [3] is not applicable. Define a self mapping g of (X, d) by $g1 = 3$, $g2 = g4 = 2$ and $g3 = 4$. Take $p = r = s = 2$, $g = 1$. It is easily seen that the hypotheses of Theorem 2 are valid.

Example 4 Let $X = [0, 1]$ with the usual metric, $f1 = 0$, $fx = x^2/2$ for $x \in [0, 1)$, $gx = x/3$, $ax = bx = x$ for $x \in [0, 1]$. Then $\bigcap_{n \in N} (fg)^n X = \bigcap_{n \in N} [0, 1/18^{2^n - 1}] = \{0\}$. It is easy to check that the conditions of Corollary 5 are satisfied. Theorem 2 of Fisher [5] however is not applicable since $fg1 = 1/18 \neq 0 = gf1$.

Example 5 Let $X = \{1, 2, 3, 7\}$ with the usual metric, $f1 = 1$, $f2 = f7 = 2$ and $f3 = 7$. Then Theorem E, Theorems 7, 8, 9 and 10 of Fisher [2] and Corollaries 2 and 3 of Leader [6] are not applicable since 5) does not hold for every $p \in N$ and $x = 1$, $y = 2$. Define a self mapping g of X by $g1 = g2 = g3 = 2$ and $g7 = 1$. Choose $p = q = 1$. It is easy to show that the conditions of Theorem 3 are satisfied.

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Nontrivial Positive Solutions for Systems with the p-Laplacian in $I\!R^N$ Involving Critical Exponents

ABSTRACT. In this paper we give some results for the existence of positive solutions of p-Laplacian systems in $I\!R^N$ involving critical exponents. These solutions can be obtained by variational methods, more precisely by the mountain pass lemma and the concentration compactness lemma.

1 Introduction

In this paper we are interested in the existence of nontrivial positive solutions for Systems of q and p Laplacian involving critical exponents in $I\!R^N$. Our study follows several works for the case of a simple equation which take their origin in the paper of Brézis and Nirenberg [2] concerning a semilinear equation of the following form

$$(EC) \qquad \begin{cases} -\Delta u = u^{2^*-1} + \lambda u & \text{in } \Omega \subset I\!R^N, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain and 2^* is the Sobolev exponent. We recall that for $1 \leq p < +\infty$, $p^* = \frac{Np}{N-p}$.
They give existence of positive solutions when $N \geq 3$ and $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue for the Laplacian operator.
Recently, several authors were interested by this type of problems in the case of the p-Laplacian (i.e. $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$). See [10], [5], [6], [8] and the references therein.
For $1 < p < N$, Drabeck and Huang in [6] consider the equation

$$(EC1) \qquad -\Delta_p u = \lambda g(x)|u|^{p-2}u + f(x)|u|^{p^*-2}u, \quad x \in I\!R^N.$$

They introduce $\lambda_1 > 0$, principal eigenvalue for the problem

$$\begin{cases} -\Delta_p u = \lambda g(x)|u|^{p-2}u & x \in I\!R^N, \\ \displaystyle\int_{I\!R^N} g(x)|u|^p dx > 0, \end{cases}$$

and they prove that Equation $(EC1)$ admits at least one weak positive solution if $\lambda \in (0, \lambda_1)$. This solution lies in $\mathcal{D}^{1,p}(I\!R^N)$, i.e. the closure of $\mathcal{C}_0^\infty(I\!R^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(I\!R^N)} = \|\nabla u\|_{L^p(I\!R^N)}.$$

In this paper we extend the results above ([6], [2]) to the system

$$(SC) \qquad \begin{cases} -\Delta_p u = a(x)|u|^{p^*-2}u + \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1}, \\ -\Delta_q v = c(x)|v|^{q^*-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v \qquad \text{in } I\!R^N, \\ \lim\limits_{|x|\to+\infty} u(x) = \lim\limits_{|x|\to+\infty} v(x) = 0, \\ u > 0, \ v > 0, \end{cases}$$

with

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1, \quad \alpha+\beta+2 < N, \tag{1.1}$$

$$p > 1, \quad q > 1, \quad \max\{p, q\} < N, \quad \alpha \geq 0, \quad \beta \geq 0. \tag{1.2}$$

We prove the existence of a positive solution for any $\lambda \in (0, \lambda_1)$. Here λ_1 is the principal eigenvalue associated to the problem

$$(VP) \qquad \begin{cases} -\Delta_p u = \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1}, \\ -\Delta_q v = \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v \qquad \text{in } I\!R^N, \\ u > 0, \ v > 0, \\ \lim\limits_{|x|\to+\infty} u(x) = \lim\limits_{|x|\to+\infty} v(x) = 0. \end{cases}$$

J. Fleckinger, R.F. Manàsevich, N.M. Stavrakakis and F. de Thélin [7] prove that under hypotheses (1.1), (1.2) and

(H1) b is a $\mathcal{C}^{0,\gamma}(I\!R^N)$ function with $\gamma \in (0,1)$ which belongs to $L^{\frac{N}{\alpha+\beta+2}}(I\!R^N) \cap L^\infty(I\!R^N)$,

there exists a positive eigenvalue denoted by λ_1.
Results concerning the subcritical case have been obtained independently by [1] and [11]. But, from our knowledge, there are no results concerning the critical case for systems in $I\!R^N$. In the case of a bounded domain and $p = q$ see [4].

Now we come back to the critical system (SC). Here we shall prove the existence of positive solutions by a variational method, more precisely by the Montain Pass Theorem [3]. One of the difficulties in the critical case is that the Palais Smale conditions will be satisfied only for $c > 0$ satisfying

$$c < \min\left\{\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(\mathbb{R}^N)}^{\frac{p-N}{p}}, \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-N}{q}}\right\}.$$

We recall that $(u,v) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,q}(\mathbb{R}^N)$ is a weak solution if it satisfies (SC) in weak sense. And the Sobolev constant $\mathcal{S}_p$ is given by

$$\mathcal{S}_p = \inf\left\{\frac{\|\nabla u\|_{L^p(\mathbb{R}^N)}^p}{\|u\|_{L^{p^*}(\mathbb{R}^N)}^p} \quad u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}\right\}.$$

Now, we introduce the hypotheses below what we need in our proofs:

(H2) a and c are nonnegative functions in $L^\infty(\mathbb{R}^N)$.

(H3) There exist $R > 0$ and $b_0 > 0$ such that

$$b(x) \geq b_0 > 0 \qquad \text{in } B_R.$$

(H4) If $\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(\mathbb{R}^N)}^{\frac{p-N}{p}} \leq \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-N}{q}}$ we assume

- $\frac{(N-p)(\alpha+1)}{p} > \frac{pq}{p-1}$,
- $a(x) = a(0) + O(|x|^{k_1})$ near 0, where $k_1 > \frac{\delta p}{p-1}$ and $\delta = N - \frac{N-p}{p}(\alpha+1) + \frac{N-p}{p^2}(\alpha+1)$,
- $a(0) = \|a\|_{L^\infty(\mathbb{R}^N)}$ and $a(x) > 0$ in B_{2R}.

If $\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(\mathbb{R}^N)}^{\frac{p-N}{p}} \geq \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-N}{q}}$ we assume

- $\frac{(N-q)(\beta+1)}{q} > \frac{pq}{q-1}$,
- $c(x) = c(0) + O(|x|^{k_2})$ near 0, where $k_2 > \frac{\gamma q}{q-1}$ and $\gamma = N - \frac{N-q}{q}(\beta+1) + \frac{N-q}{q^2}(\beta+1)$,
- $c(0) = \|c\|_{L^\infty(\mathbb{R}^N)}$ and $c(x) > 0$ in B_{2R}.

Example: If $a(x) = a(0)$ on B_R, $a(0) > \left\{\frac{\alpha+1}{\beta+1}\frac{\mathcal{S}_p^{\frac{N}{p}}}{\mathcal{S}_q^{\frac{N}{q}}}\|c\|_\infty^{\frac{N-q}{q}}\right\}^{\frac{p}{N-p}}$ and $N - p > \frac{pp'q}{\alpha+1}$, then the hypothesis (**H**4) is satisfied.

Theorem 1.1 *If the hypotheses* (**H**1) - (**H**4), (1.1) *and* (1.2) *are satisfied, then System* (SC) *admits a weak solution* $(u,v) \neq (0,0)$.

Remark 1.2 In fact, we prove that $u \neq 0$ and $v \neq 0$ (see Corollary 3.1).

2 Preliminaries

Lemma 2.1 *Assume that hypotheses* (**H**1), (1.1) *and* (1.2) *are satisfied; then for all* $\lambda \in (0, \lambda_1)$, *there exists* $\alpha_0 > 0$ *(depending of* λ*) such that* $\forall (u,v) \in \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ *we have*

$$\begin{aligned} &\frac{\alpha+1}{p}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} - \lambda \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx \\ &\geq \alpha_0 \left(\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} \right). \end{aligned} \tag{2.3}$$

■

Now, we introduce the functional

$$\begin{aligned} J_\lambda(u,v) = &\frac{\alpha+1}{p}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} - \frac{\alpha+1}{p^*}\int_{I\!R^N} a(x)|u|^{p^*}dx \\ &- \frac{\beta+1}{q^*}\int_{I\!R^N} c(x)|v|^{q^*}dx - \lambda \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx. \end{aligned} \tag{2.4}$$

Classical arguments show that J_λ is well defined.
In the following lemma we recall Lions's concentration-compactness Lemma.

Lemma 2.2 [6], [9], [8].
Let $\{u_n\}_n$ *converge weakly to* u *in* $\mathcal{D}^{1,p}(I\!R^N)$ *such that* $\{|u_n|^{p^*}\}_n$ *and* $\{|\nabla u_n|^p\}_n$ *converge weakly to the nonnegative measure* ν, μ *on* $I\!R^N$ *respectively. Then*

1. $\nu = |u|^{p^*} + \sum_{j \in J} \nu_j \delta_{x_j}$ *with* $\nu_j > 0 \quad \forall j \in J$,
2. $\mu \geq |\nabla u|^p + \sum_{j \in J} \mu_j \delta_{x_j}$ *with* $\mu_j > 0 \quad \forall j \in J$,
3. $\mathcal{S}_p \nu_j^{\frac{p}{p^*}} \leq \mu_j, \quad \forall j \in J$,

where $x_j \in I\!R^N, \delta_x$ *is the Dirac measure,* J *is a countable set and* $\mathcal{S}_p$ *is the Sobolev constant.*

Proposition 2.3 *Let* $\{u_n\}_n$ *converge weakly to* u_0 *in* $\mathcal{D}^{1,p}(I\!R^N)$. *Then for all* $h \in L^\infty(I\!R^N)$, *there exists a subsequence of* $\{u_n\}_n$ *denoted again by* $\{u_n\}_n$ *such that for all* $\phi \in \mathcal{C}_0^\infty(I\!R^N)$

$$\int_{I\!R^N} h|u_n|^{p^*-2}u_n\phi dx \to \int_{I\!R^N} h|u_0|^{p^*-2}u_0\phi dx,$$

and

$$\int_{I\!R^N} h|u_n|^{p-2}u_n\phi dx \to \int_{I\!R^N} h|u_0|^{p-2}u_0\phi dx.$$

See [6] for the proof of Proposition 2.3.

Proposition 2.4 *Let $\{(u_n, v_n)\}_n$ be a bounded sequence converging weakly to (u, v) in $\mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ and satisfying*

$$J_\lambda(u_n, v_n) \to C, \tag{2.5}$$

and

$$J'_\lambda(u_n, v_n) \to 0. \tag{2.6}$$

Then there exists a subsequence still denoted by $\{(u_n, v_n)\}_n$ such that

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \qquad \text{weakly in } \left(L^p(I\!R^N)\right)', \tag{2.7}$$

$$|\nabla v_n|^{q-2}\nabla v_n \to |\nabla v|^{q-2}\nabla v \qquad \text{weakly in } \left(L^q(I\!R^N)\right)'. \tag{2.8}$$

Proof: It is clear from Sobolev embedding that $\{(u_n, v_n)\}_n$ converges strongly in $L^p_{loc}(I\!R^N) \times L^q_{loc}(I\!R^N)$ and so there exists a subsequence still denoted by $\{(u_n, v_n)\}_n$ such that

$$(u_n, v_n) \to (u, v) \qquad \text{a.e. in } I\!R^N.$$

Observe, that all hypotheses of Lemma 2.2 are satisfied, and thus, there exist two countable sets J and $\bar{J}$ such that $(u_n)_n$ and $(v_n)_n$ satisfy the properties 1, 2 and 3 (Lemma 2.2) in J and $\bar{J}$ respectively.
First, we show that J and $\bar{J}$ are finite.
From (2.4) we have

$$\begin{aligned} J'_\lambda(u,v)(w,z) &= (\alpha+1)\int_{I\!R^N} |\nabla u|^{p-2}\nabla u.\nabla w dx + (\beta+1)\int_{I\!R^N} |\nabla v|^{q-2}\nabla v.\nabla z dx \\ &- (\alpha+1)\int_{I\!R^N} a(x)|u|^{p^*-2}uw dx - (\beta+1)\int_{I\!R^N} c(x)|v|^{q^*-2}vz dx \\ &- \lambda\int_{I\!R^N} \left[(\alpha+1)b(x)|u|^{\alpha-1}uw|v|^{\beta+1} + (\beta+1)b(x)|u|^{\alpha+1}|v|^{\beta-1}vz\right] dx, \end{aligned} \tag{2.9}$$

for all $(u, v),\ (w, z) \in \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$.
Now, for $x_j \in \{x_i,\ i \in J\}$ we introduce a function $\phi_j \in \mathcal{C}_0^\infty(I\!R^N)$ satisfying $\phi_j \equiv 1$ in $B(x_j, \epsilon)$, $\phi_j \equiv 0$ in $I\!R^N/B(x_j, 2\epsilon)$ and $|\nabla\phi_j| \le \frac{c}{\epsilon}$.
Substituting $w = \phi_j u_n$, $z \equiv 0$ in (2.9) and using (2.6) we obtain

$$\begin{aligned} &\lim_{n\to+\infty}\int_{I\!R^N} |\nabla u_n|^{p-2}u_n(\nabla u_n.\nabla\phi_j)dx \\ &\quad = \lim_{n\to\infty}\left[\int_{I\!R^N} a(x)|u_n|^{p^*}\phi_j dx + \lambda\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\phi_j dx - \int_{I\!R^N} |\nabla u_n|^p\phi_j dx\right]. \end{aligned} \tag{2.10}$$

Classical arguments (see [1]) show that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\phi_j dx = \int_{\mathbb{R}^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}\phi_j dx. \tag{2.11}$$

And from Lemma 2.2, (2.11) and (2.10), we obtain

$$\begin{aligned}\lim_{n\to+\infty}&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}u_n(\nabla u_n.\nabla\phi_j)dx \\ &= \int_{\mathbb{R}^N} a(x)\phi_j d\nu + \lambda\int_{\mathbb{R}^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}\phi_j dx - \int_{\mathbb{R}^N}\phi_j d\mu.\end{aligned} \tag{2.12}$$

From [6] we have

$$\left|\lim_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}u_n(\nabla u_n.\nabla\phi_j)dx\right| \to 0, \tag{2.13}$$

as $\epsilon \to 0$.
Letting ϵ go to 0 in (2.12) and using (2.13), we obtain

$$a(x_j)\nu_j = \mu_j. \tag{2.14}$$

We deduce from this last equality

$$a(x_j) > 0 \quad \forall j \in J, \tag{2.15}$$

$$\nu_j \geq (\frac{\mathcal{S}_p}{a(x_j)})^{\frac{N}{p}}, \tag{2.16}$$

$$\text{J is finite.} \tag{2.17}$$

Indeed, if J is infinite we deduce from (2.16) and Lemma 2.2 that $\int_{\mathbb{R}^N}|u_n|^{p^*}dx \to +\infty$; a contradiction.
Now, we let $J = \{1, ..., m\}$ and $\Omega_{\epsilon_0} = \{x \in \mathbb{R}^N : \ dist(x_j, x) > \epsilon_0, \forall j = 1, ..., m\}$.
Arguing as in [8] we prove that (see the details in [1])

$$\int_{\Omega_{\epsilon_0}} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx \to 0 \quad \forall \epsilon_0 > 0. \tag{2.18}$$

Classical arguments show that

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad a.e.\ in\ \Omega_{\epsilon_0} \quad 1 \leq i \leq N.$$

Since ϵ_0 is arbitrary, we conclude by a diagonal process that there exists a subsequence denoted again by (u_n) such that

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad a.e.\ in\ \mathbb{R}^N \quad 1 \leq i \leq N.$$

Since $\{|\nabla u_n|^{p-2}\dfrac{\partial u_n}{\partial x_i}\}$ is bounded in $(L^p(I\!R^N))'$, we conclude that

$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \to |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \textit{weakly in} \;\; (L^p(I\!R^N))', \quad 1 \le i \le N.$$

Hence (2.7).
Similarly we obtain (2.8). ■

In the following lemma we prove Palais Smale conditions.

Lemma 2.5 *Suppose that the hypotheses* (**H**1), (**H**2), (1.1), (1.2) *hold and* $\lambda \in (0, \lambda_1)$. *We define*

$$\mathcal{S}_0 = \min\left\{\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}}, \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(I\!R^N)}^{\frac{q-N}{q}}\right\}. \tag{2.19}$$

Then the function J_λ *satisfies the Palais Smale conditions at* C *for all* $C < \mathcal{S}_0$.

Proof: Let $\{(u_n, v_n)\}_n \subset \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ be so that (2.5) and (2.6) are satisfied. First, we will prove that $\{(u_n,\ v_n)\}_n$ is bounded.
Suppose the contrary, so $\|(u_n,\ v_n)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)} \to +\infty$.
We deduce from (2.4) and (2.9)

$$\begin{aligned} &J_\lambda(u_n, v_n) - J'_\lambda(u_n, v_n)(\frac{u_n}{p}, \frac{v_n}{q}) \\ &= \frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u_n|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v_n|^{q^*}dx \\ &= C + o(1)\|(u_n,\ v_n)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)}. \end{aligned} \tag{2.20}$$

Thus

$$\frac{\alpha+1}{p}\|u_n\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|v_n\|^q_{\mathcal{D}^{1,q}(I\!R^N)} - \lambda\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx \le c\|(u_n, v_n)\|, \tag{2.21}$$

and with Lemma 2.1, it follows

$$\alpha_0(\|u_n\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \|v_n\|^q_{\mathcal{D}^{1,q}(I\!R^N)}) \le c\|(u_n, v_n)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)}, \tag{2.22}$$

which implies that $\{(u_n,\ v_n)\}_n$ is a bounded sequence.
Now, we prove that we can extract a converging subsequence.

It is clear that there exists a subsequence denoted again by $\{(u_n, v_n)\}_n$ which converges weakly to (u, v) in $\mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$, and satisfies

$$\begin{aligned} |u_n|^{p^*} &\to \nu, \qquad |\nabla u_n|^p \to \mu, \\ |v_n|^{q^*} &\to \bar{\nu}, \qquad |\nabla v_n|^q \to \bar{\mu}, \end{aligned}$$

weakly.
From Propositions 2.3, 2.4 we have

$$J'_\lambda(u, v) = 0. \tag{2.23}$$

It is easy to show that the hypotheses of Lemma 2.2 are satisfied and from the proof of Proposition 2.4 that

$$a(x_j)\nu_j = \mu_j \quad \forall j \in J, \qquad b(x_j)\bar{\nu}_j = \bar{\mu}_j \quad \forall j \in \bar{J}, \tag{2.24}$$

and

$$\nu_j \geq [\frac{\mathcal{S}_p}{a(x_j)}]^{\frac{N}{p}} \quad \forall j \in J, \qquad \bar{\nu}_j \geq [\frac{\mathcal{S}_q}{c(x_j)}]^{\frac{N}{q}} \quad \forall j \in \bar{J}. \tag{2.25}$$

J and $\bar{J}$ are finit sets.
We will prove that J and $\bar{J}$ are empty sets.
Suppose that $J \cup \bar{J} \neq \emptyset$.
Since $\{(u_n, v_n)\}_n$ converges weakly to (u, v) and

$$\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx \to \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx,$$

we deduce that for any $\sigma > 0$ and n sufficiently large

$$\begin{aligned} C + \sigma &> J_\lambda(u_n, v_n) - J'_\lambda(u_n, v_n)(\frac{u_n}{p^*}, \frac{v_n}{q^*}) \\ &= \frac{\alpha+1}{N}\|u_n\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{N}\|v_n\|^q_{\mathcal{D}^{1,q}(I\!R^N)} \\ &\quad - \lambda(\frac{\alpha+\beta+2}{N})\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx. \end{aligned} \tag{2.26}$$

Taking $n \to +\infty$, we obtain

$$\begin{aligned} C + \sigma \geq\ & \frac{\alpha+1}{N}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{N}\|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} \\ & - \lambda(\frac{\alpha+\beta+2}{N})\int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx \\ & + \frac{\alpha+1}{N}\sum_{j\in J}\mu_j + \frac{\beta+1}{N}\sum_{j\in\bar{J}}\bar{\mu}_j. \end{aligned} \tag{2.27}$$

Taking into account (2.23), (2.24) and (2.25) we obtain

$$\begin{aligned} C+\sigma &\geq \frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v|^{q^*}dx \\ &+ \frac{\alpha+1}{N}\sum_{j\in J} a(x_j)\nu_j + \frac{\beta+1}{N}\sum_{j\in \bar{J}} c(x_j)\bar{\nu}_j \\ &> \frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v|^{q^*}dx \\ &+ \min\left\{\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}}, \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(I\!R^N)}^{\frac{q-N}{q}}\right\}. \end{aligned}$$

Since $C<\mathcal{S}_0$ and σ is arbitrary this implies

$$\frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v|^{q^*}dx < 0, \tag{2.28}$$

which contradicts (**H**2).
Therefore J and $\bar{J}$ are empty, hence

$$\int_{I\!R^N} |u_n|^{p^*}dx \to \int_{I\!R^N} |u|^{p^*}dx \quad \text{and} \quad \int_{I\!R^N} |v_n|^{q^*}dx \to \int_{I\!R^N} |v|^{q^*}dx.$$

With the weak convergence of $\{(u_n,v_n)\}_n$ to (u,v) in $\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)$, we have

$$(u_n,v_n)\to(u,v) \qquad \text{strongly in } L^{p^*}(I\!R^N)\times L^{q^*}(I\!R^N). \tag{2.29}$$

■

Lemma 2.6 *Suppose that the hypotheses* (**H**1), (**H**2), (1.1) *and* (1.2) *are satisfied, then there exist two constants* $\delta>0, \rho>0$ *such that*

$$J_\lambda(u,v)\geq\rho, \qquad \textit{for all } \|(u,v)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)}=\delta \quad \textit{and} \quad \lambda\in(0,\lambda_1). \tag{2.30}$$

Proof: We deduce from (2.3) and the Sobolev inequality that

$$J_\lambda(u,v)\geq\alpha_0(\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)}+\|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)}) - c\|u\|^{p^*}_{\mathcal{D}^{1,p}(I\!R^N)} - c\|v\|^{q^*}_{\mathcal{D}^{1,q}(I\!R^N)}.$$

So, for δ sufficiently small we have

$$J_\lambda(u,v)\geq\rho.$$

■

Let $R>\epsilon>0$, $0<r\leq R$ and ϕ_r be a nonnegative function in $\mathcal{C}_0^\infty(I\!R^N)$ satisfying $0\leq\phi_r\leq1$, $\phi_r\equiv1$ in $B(0,r)$ and $\phi\equiv0$ in $I\!R^N/B(0,2r)$.
We define

$$u_\epsilon(x)=\frac{\phi_R(x)}{(\epsilon+|x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}, \qquad w_\epsilon(x)=\frac{u_\epsilon}{\|u_\epsilon\|_{L^{p^*}(I\!R^N)}}, \qquad z_\epsilon(x)=\phi_\epsilon(x).$$

Lemma 2.7 *We have*

1) $\|u_\epsilon\|^p_{L^{p^*}(I\!R^N)} = c\epsilon^{\frac{p-N}{p}} + O(1).$

For ϵ sufficiently small

2) $\dfrac{1}{\|u_\epsilon\|^{p^*}_{L^{p^*}(I\!R^N)}} \displaystyle\int_{I\!R^N} \frac{a(0)-a(x)}{(\epsilon+|x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}dx = O(\epsilon^{k_1(\frac{p-1}{p})}).$

3) $\displaystyle\int_{I\!R^N} w_\epsilon(x)^{\alpha+1}\phi_\epsilon(x)^{\beta+1}dx \geq c\epsilon^\delta,$
where $\delta = N - (\frac{N-p}{p})(\alpha+1) + (\frac{N-p}{p^2})(\alpha+1).$

Proof: For the proof of 1) and 2) see [2].
3) We have

$$\begin{aligned}
\int_{I\!R^N} u_\epsilon^{\alpha+1}\phi_\epsilon^{\beta+1}dx \geq \int_{B_\epsilon} u_\epsilon^{\alpha+1}dx &= \int_0^\epsilon \frac{s^{N-1}}{\left(\epsilon+s^{\frac{p}{p-1}}\right)^{\frac{(N-p)(\alpha+1)}{p}}}ds \\
&= \epsilon^{(\frac{p-1}{p})(N)-(\frac{N-p}{p})(\alpha+1)} \int_0^{\epsilon^{\frac{1}{p}}} \frac{r^{N-1}}{\left(1+r^{\frac{p}{p-1}}\right)^{\frac{(N-p)(\alpha+1)}{p}}}dr \qquad (2.31)\\
&\geq c\epsilon^{(\frac{p-1}{p})(N)-(\frac{N-p}{p})(\alpha+1)} \int_0^{\epsilon^{\frac{1}{p}}} r^{N-1}dr \\
&\geq c\epsilon^{N-(\frac{N-p}{p})(\alpha+1)}.
\end{aligned}$$

On the other hand, for ϵ sufficiently small

$$\frac{1}{\|u_\epsilon\|^{\alpha+1}_{L^{p^*}(I\!R^N)}} \geq c\epsilon^{(\frac{N-p}{p^2})(\alpha+1)}. \qquad (2.32)$$

Therefore

$$\int_{I\!R^N} w_\epsilon^{\alpha+1}\phi_\epsilon^{\beta+1}dx \geq c\epsilon^{N-(\frac{N-p}{p})(\alpha+1)+(\frac{N-p}{p^2})(\alpha+1)}. \qquad (2.33)$$

Lemma 2.8 *Suppose that the hypotheses* (**H**1) - (**H**4), (1.1) *and* (1.2) *are satisfied, then there exist $\epsilon > 0$ and $t_1 > 0$ such that for all $\lambda \in (0, \lambda_1)$* $J_\lambda(t_1^{\frac{1}{p}}w_\epsilon, t_1^{\frac{1}{q}}\phi_\epsilon) < 0$ *and*

$$0 < \sup_{t>0} J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon) < \frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_\infty^{\frac{p-N}{p}}. \qquad (2.34)$$

Proof: We have

$$J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon)$$
$$= t\left(\frac{\alpha+1}{p}\|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!R^N)} - \lambda\int_{I\!R^N} b(x)|w_\epsilon|^{\alpha+1}|\phi_\epsilon|^{\beta+1}dx\right) \tag{2.35}$$
$$- \frac{\alpha+1}{p^*}t^\tau\int_{I\!R^N} a(x)|w_\epsilon|^{q^*}dx - \frac{\beta+1}{q^*}t^{\bar\tau}\int_{I\!R^N} c(x)|\phi_\epsilon|^{p^*}dx.$$

Since $t \to -J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon)$ is a convex differentiable function it admits a minimum at $t_\epsilon \in [0, +\infty)$ satisfying

$$\frac{dJ_\lambda}{dt}(t_\epsilon^{\frac{1}{p}}w_\epsilon, t_\epsilon^{\frac{1}{q}}\phi_\epsilon) = 0.$$

Therefore

$$\frac{\alpha+1}{p}t_\epsilon^{\tau-1}\int_{I\!R^N} a(x)|w_\epsilon|^{p^*}dx + \frac{\beta+1}{q}t_\epsilon^{\bar\tau-1}\int_{I\!R^N} c(x)|\phi_\epsilon|^{q^*}dx$$
$$\le \frac{\alpha+1}{p}\|w_\epsilon\|^p_{L^{p^*}(I\!R^N)} + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{L^{q^*}(I\!R^N)} - \lambda\int_{I\!R^N} b(x)|w_\epsilon|^{\alpha+1}|\phi_\epsilon|^{\beta+1}dx. \tag{2.36}$$

First, we show that t_ϵ is bounded from above and below.

• Indeed, it follows from (2.36), (**H**1), (**H**2) and (**H**4)

$$\frac{\alpha+1}{p}t_\epsilon^{\tau-1}a(0)\int_{I\!R^N}|w_\epsilon|^{p^*}dx - \frac{\alpha+1}{p}t_\epsilon^{\tau-1}\int_{I\!R^N}(a(0)-a(x))|w_\epsilon|^{p^*}dx$$
$$\le \frac{\alpha+1}{p}\|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!R^N)}.$$

From 2) in Lemma 2.7, the fact that $\|w_\epsilon\|_{\mathcal{D}^{1,p}(I\!R^N)} = \mathcal{S}_p$, and $\int_{I\!R^N}|w_\epsilon|^{p^*} = 1$, we have

$$\frac{\alpha+1}{p}[a(0) - O(\epsilon^{k_1(\frac{p-1}{p})})]t_\epsilon^{\tau-1} \le \frac{\alpha+1}{p}\mathcal{S}_p + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!R^N)}.$$

By taking ϵ small enough such that $O(\epsilon^{k_1(\frac{p-1}{p})}) \le \dfrac{a(0)}{2}$ and remarking that $\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!R^N)} \le c\epsilon^{N-q}$ we obtain

$$\frac{\alpha+1}{2p}a(0)t_\epsilon^{\tau-1} \le \frac{\alpha+1}{p}\mathcal{S}_p + c. \tag{2.37}$$

This proves that t_ϵ is bounded from above.

• On the other hand, observe that

$$J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon) \ge \alpha_0 t\left(\|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!R^N)}\right)$$
$$- \frac{\alpha+1}{p^*}t^\tau\int_{I\!R^N} a(x)|w_\epsilon|^{p^*}dx - \frac{\beta+1}{q^*}t^{\bar\tau}\int_{I\!R^N} c(x)|\phi_\epsilon|^{q^*}dx \tag{2.38}$$
$$\ge \alpha_0 t\mathcal{S}_p^p - \frac{\alpha+1}{p^*}\|a\|_{L^\infty(I\!R^N)}t^\tau - \frac{\beta+1}{q^*}\|c\|_{L^\infty(I\!R^N)}t^{\bar\tau} = \Theta(t).$$

It is clear that

$$\frac{dJ_\lambda}{dt}(0) \geq \Theta'(0) = \alpha_0 \mathcal{S}_p^p = k > 0,$$

and by continuity there exists $\eta > 0$ such that $J'_\lambda(t^{\frac{1}{p}} w_\epsilon, t^{\frac{1}{q}} \phi_\epsilon) \geq \Theta'(t) > \frac{k}{2}$ for any $|t| < \eta$. Hence $t_\epsilon \geq \eta$.

On the other hand, we deduce from (2.38) that for $\epsilon > 0$ sufficiently small $J_\lambda(t^{\frac{1}{p}} w_\epsilon, t^{\frac{1}{q}} \phi_\epsilon) > 0$ which implies that $J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{q}} \phi_\epsilon) > 0$. Now, we write $J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{q}} \phi_\epsilon)$ as below

$$J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{q}} \phi_\epsilon) = \sup_{t \geq 0} J_\lambda(t^{\frac{1}{p}} w_\epsilon, t^{\frac{1}{q}} \phi_\epsilon) = E_\epsilon - F_\epsilon,$$

where

$$\begin{aligned} E_\epsilon = & \, t_\epsilon \frac{\alpha+1}{p} \|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!R^N)} - \frac{\alpha+1}{p} a(0) t_\epsilon^\tau \int_{I\!R^N} w_\epsilon^{p^*} dx \\ & + t_\epsilon \frac{\beta+1}{q} \|\phi_\epsilon\|^q_{\mathcal{D}^{1,p}(I\!R^N)} - \frac{\beta+1}{q^*} t_\epsilon^{\bar{\tau}} \int_{I\!R^N} c(x) \phi_\epsilon^{q^*} dx, \end{aligned}$$

$$F_\epsilon = \lambda t_\epsilon \int_{I\!R^N} b(x) |w_\epsilon|^{\alpha+1} |\phi_\epsilon|^{\beta+1} dx - \frac{\alpha+1}{p^*} t_\epsilon^\tau \int_{I\!R^N} (a(0) - a(x)) w_\epsilon^{p^*} dx.$$

We consider the function

$$m(t) = Mt - \frac{L}{\tau} t^\tau,$$

m attains its maximum at $t_1 = \left(\frac{M}{L}\right)^{\frac{N-p}{p}}$. Since t_ϵ is bounded from above, since $\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!R^N)}$ $= O(\epsilon^{N-q})$ and c is nonnegative, we deduce that

$$\begin{aligned} E_\epsilon & \leq \frac{\alpha+1}{p} \left(\frac{\|w_\epsilon\|^N_{\mathcal{D}^{1,p}(I\!R^N)}}{a(0)^{\frac{N-p}{p}}} - \frac{1}{\tau} \frac{\|w_\epsilon\|^N_{\mathcal{D}^{1,p}(I\!R^N)}}{a(0)^{\frac{N-p}{p}}} \right) + O(\epsilon^{N-q}) \\ & = \frac{\alpha+1}{N} \|a\|_\infty^{\frac{p-N}{p}} \mathcal{S}^{\frac{N}{p}} + O(\epsilon^{N-q}). \end{aligned} \tag{2.39}$$

(Remark that $\|w_\epsilon\|^N_{\mathcal{D}^{1,p}(I\!R^N)} = \mathcal{S}_p^{\frac{N}{p}}$.)

Since t_ϵ is bounded from below and above, it follows from Lemma 2.7 that

$$\begin{aligned} \lambda t_\epsilon \int_{I\!R^N} b(x) |w_\epsilon|^{\alpha+1} |\phi_\epsilon|^{\beta+1} & \geq \lambda t_\epsilon \int_{I\!R^N} b_0 |w_\epsilon|^{\alpha+1} |\phi_\epsilon|^{\beta+1} dx \\ & \geq c\epsilon^\delta. \end{aligned}$$

and

$$\frac{\alpha+1}{p} t_\epsilon^\tau \int_{I\!R^N} (a(0) - a(x)) w_\epsilon^{p^*} dx = O(\epsilon^{k_1(\frac{p-1}{p})}) \tag{2.40}$$

Observing that $\delta < k_1(\frac{p-1}{p})$, this implies

$$F_\epsilon \geq c\epsilon^\delta.$$

Therefore

$$J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{p}} \phi_\epsilon) \leq \frac{\alpha+1}{N} \mathcal{S}^{\frac{N}{p}} \|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}} - c\epsilon^\delta + O(\epsilon^{N-q}).$$

On the other hand, it follows from (**H**4) that $\delta < N - q$. Thus, we obtain

$$J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{p}} \phi_\epsilon) \leq \frac{\alpha+1}{N} \mathcal{S}^{\frac{N}{p}} \|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}} - c\epsilon^\delta,$$

hence (2.34).
By taking t sufficiently large in (2.35), we have $J_\lambda(t^{\frac{1}{p}} w_\epsilon,\ t^{\frac{1}{q}} \phi_\epsilon) < 0$, and thus there exists a t_1 such that $J_\lambda(t_1^{\frac{1}{p}} w_\epsilon,\ t_1^{\frac{1}{q}} \phi_\epsilon) < 0$, which concludes the proof of Lemma 2.8.

■

3 Proof of Theorem 1.1

We let $X = \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$, $u_0 = (t_1^{\frac{1}{p}} w_\epsilon, t_1^{\frac{1}{q}} z_\epsilon)$ and $C = \inf_{h\in\Gamma} \sup_{t\in[0,1]} J(h(t))$, where

$$\Gamma = \{h \in \mathcal{C}([0,1], X) :\ h(0) = 0,\ h(1) = u_0\}.$$

It is clear from Lemma 2.8 that $C < \dfrac{\alpha+1}{N} \mathcal{S}_p^{\frac{N}{p}} \|a\|_\infty^{\frac{p-N}{p}}$, furthermore (**H**4) and Lemma 2.5 imply that J_λ satisfies Palais Smale conditions at C. Thus, by Lemmas 2.8, 2.6 and the Mountain Pass Lemma we deduce that (SC) admits a solution $(u, v) \neq (0, 0)$. Moreover this solution is nonnegative because $J_\lambda(h(t)) = J_\lambda(|h(t)|)$.

■

Corollary 3.1 *The solution (u, v) obtained in Theorem 1.1 satisfies $u \neq 0$ and $v \neq 0$.*

Proof: It is clear from Theorem 1.1 that there exist other solutions than $(0, 0)$. Suppose (for example) that $u \neq 0$ and $v = 0$. In this case System (SC) is reduced to the simple equation

$$-\Delta_p u = a(x)|u|^{p^*-2} u. \tag{3.41}$$

Multiplying equation (3.41) by u, and integrating over $I\!R^N$, we have

$$\|u\|_{\mathcal{D}^{1,p}(I\!R^N)}^p = \int_{I\!R^N} a(x)|u|^{p^*} dx \leq \|a\|_{L^\infty(I\!R^N)} \int_{I\!R^N} |u|^{p^*} dx. \tag{3.42}$$

From the definition of $\mathcal{S}_p$

$$\mathcal{S}_p^{\frac{p^*}{p}} \int_{I\!R^N} |u|^{p^*} dx \le \|u\|^{p^*}_{\mathcal{D}^{1,p}(I\!R^N)}. \tag{3.43}$$

Combining (3.42) and (3.43), we have

$$\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} \le \|a\|_{L^\infty(I\!R^N)} \int_{I\!R^N} |u|^{p^*} dx \le \mathcal{S}_p^{-\frac{p^*}{p}} \|a\|_{L^\infty(I\!R^N)} \|u\|^{p^*}_{\mathcal{D}^{1,p}(I\!R^N)}. \tag{3.44}$$

Therefore

$$\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} \ge \|a\|^{\frac{p-N}{p}}_{L^\infty(I\!R^N)} \mathcal{S}_p^{\frac{N}{p}}. \tag{3.45}$$

Using (3.42), we have

$$J_\lambda(u,0) = \frac{\alpha+1}{p}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} - \frac{\alpha+1}{p^*}\int_{I\!R^N} a(x)|u|^{p^*} dx = \frac{\alpha+1}{N}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)}. \tag{3.46}$$

So, we deduce from (3.45) and (3.46) that

$$J_\lambda(u,0) \ge \frac{\alpha+1}{N}\|a\|^{\frac{p-N}{p}}_{L^\infty(I\!R^N)} \mathcal{S}_p^{\frac{N}{p}}. \tag{3.47}$$

But from Theorem 1.1 $C = J_\lambda(u,0) < \dfrac{\alpha+1}{N}\|a\|^{\frac{p-N}{p}}_{L^\infty(I\!R^N)} \mathcal{S}_p^{\frac{N}{p}}$, which contradicts (3.47).

■

Corollary 3.2 *Suppose that the functions a, b, c are in $L^\infty(I\!R^N)$, $q \ge p$ and (1.1), (1.2) are satisfied. Then for each solution $(u,v) \in \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ we have the following assertions:*

- $\lim\limits_{|x|\to+\infty} u(x) = \lim\limits_{|x|\to+\infty} v(x) = 0.$
- *Let $x \in I\!R^N$ and $R > 0$ be such that*

$$\begin{aligned} &\max\left(\|a\|_\infty, \|c\|_\infty, \|b\|_\infty\right) \max\left\{2^p \mathcal{S}_p \tau^{p-1}, 2^{2q-p}\mathcal{S}_q |B_1|^{\frac{q-p}{N}} R^{q-p}\tau^{q-1}\right\} \times \\ &\times \left(\|u\|^{p(\tau-1)}_{L^{p^*}(B_{2R}(x))} + \|v\|^{q(\tau-1)}_{L^{q\tau}(B_{2R}(x))}\right) < 1. \end{aligned} \tag{3.48}$$

Then each solution (u,v) of (SC) belongs to $\mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N) \cap \left(\mathcal{C}^{1,\alpha}(B_{\frac{R}{2}}(x)\right)^2$.

The proof of this corollary is a consequence of Theorem 4.2.1 in [1] and Tolksdorfs Theorem [12].

■

Corollary 3.3 *Under the hypotheses of Corollary 3.2 the solution (u, v) of (SC) is positive.*

Observe that (3.48) is satisfied for all $x \in I\!R^N$ and R sufficiently small. So by applying the strong Maximum Principle of Vazquez [13] in the ball $B_{\frac{R}{2}}(x)$, we deduce that $u(x)$ and $v(x)$ are positive; since x is arbitrary this implies that u and v are positive everywhere on $I\!R^N$.

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