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Universität Rostock
Fachbereich Mathematik
18051 Rostock
Tel.: (0381) 498 1521
Fax.: (0381) 498 1520

DRUCK: Universitätsdruckerei

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Ákos Császár

Cauchy Structures and Covering Structures[1]

To Professor H. Poppe on his 65th birthday

0 Introduction

Let X be a set. A *Cauchy structure* $\mathfrak{S}$ on X is a non-empty collection of filters in X such that

$$\dot{x} \in \mathfrak{S} \text{ for } x \in X, \tag{0.1}$$

$$\mathfrak{s} \in \mathfrak{S} \text{ and } \mathfrak{s} \subset \mathfrak{s}' \in \text{Fil } X \text{ imply } \mathfrak{s}' \in \mathfrak{S}, \tag{0.2}$$

$$\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S} \text{ and } \mathfrak{s} \Delta \mathfrak{s}' \in \mathfrak{S}. \tag{0.3}$$

Here $\dot{x} = \{S \subset X : x \in S\}$, Fil X is the collection of all filters (proper or not) in X, and $\mathfrak{s} \Delta \mathfrak{s}'$ iff $S \in \mathfrak{s}$, $S' \in \mathfrak{s}'$ imply $S \subset S' \neq \emptyset$ (see e.g. [3]). If $\mathfrak{S}$ satisfies (0.1) and (0.2), it is said to be a *screen* on X (see e.g. [2]).

In [17], [18], [19], [20], K. Morita has considered a non-empty collection $\mathfrak{B}$ of covers of X such that $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathfrak{B}$ implies the existence of $\mathfrak{c} \in \mathfrak{B}$ such that $\mathfrak{c} < \mathfrak{c}_1, \mathfrak{c} < \mathfrak{c}_2$ (i.e. $\mathfrak{c}$ refines both $\mathfrak{c}_1$ and $\mathfrak{c}_2$). According to W. Rinow ([23], [24]), a $\mathfrak{B}$ of this kind is said to be a *covering strukture* (Überdeckungsstruktur) on X (generalized uniform structure in [17], [18], [19], [20], or by H. Poppe ([21], [22])). In the terminology of [16], this is clearly the same as a base for a *merotopy* on X, composed of all covers refined by some $\mathfrak{c} \in \mathfrak{B}$.

In [23] or [24], a filter $\mathfrak{c}$ on X is said to be *fundamental* with respect to a covering structure $\mathfrak{B}$ iff, for every $\mathfrak{c} \in \mathfrak{B}$, there is $C \in \mathfrak{c} \cap \mathfrak{s}$, i.e. iff $\mathfrak{s}$ is a *Cauchy filter* with respect to the merotopy *generated* by $\mathfrak{B}$ (i.e. having $\mathfrak{B}$ for base). The collection of these filters is clearly a screen on X; in [11], conditions on $\mathfrak{B}$ are given for this screen being a Cauchy structure. The purpose of the present paper is to introduce and investigate more conditions of this kind.

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1 Preliminaries

We first translate the problem onto the terminologie of [11].

Let $\sum(X)$ be the collection of all non-empty subset of the power set $\exp X$ of X. A *loseness* on X is a set $\mathfrak{T}$ such that

$$\emptyset \neq \mathfrak{T} \subset \sum(X), \tag{1.1}$$

$$\emptyset \in \mathfrak{t} \in \sum(X) \text{ implies } \mathfrak{t} \in \mathfrak{T}, \tag{1.2}$$

$$\cap\, \mathfrak{t} = \emptyset \text{ for } \mathfrak{t} \in \mathfrak{T}, \tag{1.3}$$

$$\mathfrak{t} \in \mathfrak{T},\ \mathfrak{t} \ll \mathfrak{t}' \in \sum(X) \text{ imply } \mathfrak{t}' \in \mathfrak{T}, \tag{1.4}$$

$$\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \text{ implies } \mathfrak{t}(\cup)\mathfrak{t}' \in \mathfrak{T}; \tag{1.5}$$

here $\mathfrak{t} \ll \mathfrak{t}'$ means that, for $T \in \mathfrak{t}$, there is $T' \in \mathfrak{t}'$ such that $T \supset T'$, and

$$\mathfrak{t}(\cup)\mathfrak{t}' = \{T \cup T' \ : \ T \in \mathfrak{t}, T' \in \mathfrak{t}'\}.$$

The elememts of $\mathfrak{T}$ are called *loose* systems, those of $\sum(X) - \mathfrak{T}$ *tight* ones. It is easy to see that $\mathfrak{T}$ is a looseness on$X$ iff the collection of all covers $\{X - T \ : \ T \in \mathfrak{t}\}$, where $\mathfrak{t} \in \mathfrak{T}$, is a merotopy on X, called the merotopy $\mathfrak{K}(\mathfrak{T})$ *associated* with $\mathfrak{T}$ (see [4], [5], [6]).

Given a looseness $\mathfrak{T}$ on X, a filter $\mathfrak{s}$ in X is said to be $\mathfrak{T}$ *-compressed* iff $\mathfrak{t} \in \sum(X)$, $\mathfrak{t}\Delta\mathfrak{s}$ implies $\mathfrak{t} \notin \mathfrak{T}$ (see [5]). It is easily seen that a filter is $\mathfrak{T}$ -compressed iff it is fundamental with respect to the morotopy $\mathfrak{K}(\mathfrak{T})$ (or to a base generating it). The collection of all $\mathfrak{T}$ -compressed filters is denoted by $\mathfrak{S}(\mathfrak{T})$; this is clearly a screen. Our question is: for which loosenesses $\mathfrak{T}$ is the screen $\mathfrak{S}(\mathfrak{T})$ a Cauchy structure?

In [11], several properties of the looseness $\mathfrak{T}$ have been introduced for this purpose. In order to formulate them, let us denote by $<(\mathfrak{T})$ the following relation on $\exp X$:

$$A < (\mathfrak{T})B \text{ iff } \{A, X - B\} \in \mathfrak{T} \tag{1.6}$$

(see [11], 4); this is a (symmetric) *topogenous order* (cf. [1]) on X ([11], 4.2). In other words, we have for $<=<(\mathfrak{T})$ and $A, B, C, D \subset X$

$$\emptyset < \emptyset,\ X < X, \tag{1.7}$$

$$A < B \text{ implies } A \subset B, \tag{1.8}$$

$$A < B \text{ implies } X - B < X - A, \tag{1.9}$$

$$A \subset B < C \subset D \text{ implies } A < D, \tag{1.10}$$

$$A < B \text{ and } C < D \text{ imply } A \cup C < B \cup D \text{ and } A \cap C < B \cap D. \tag{1.11}$$

A map $\sigma : \mathfrak{t} \to \exp X$ is, for $\mathfrak{t} \in \sum(X)$, said to be a $\mathfrak{T}$ *-swelling* iff $T < (\mathfrak{T})\sigma(T)$ for each $T \in \mathfrak{t}$; σ is $free$ iff $\cap\sigma(\mathfrak{t}) = \emptyset$ for $\sigma(\mathfrak{t}) = \{\sigma(T) : T \in \mathfrak{t}\}$ nad $\mathfrak{T}$ *-loose* iff $\sigma(\mathfrak{t}) \in \mathfrak{T}$ (see [11], 3).

The looseness $\mathfrak{T}$ is said to be *Efremovich or strongly Efremovich* iff, for every $\mathfrak{t} \in \mathfrak{T}$, there exists a free or $\mathfrak{T}$ -loose $\mathfrak{T}$ -swelling $\sigma : \mathfrak{t} \to \exp X$, respectively.

By [11], 4.1, if $\mathfrak{T}$ is strongly Efremovich, then $\mathfrak{S}(\mathfrak{T})$ is a Cauchy structure. On the other hand, $\mathfrak{S}(\mathfrak{T})$ can be a Cauchy structure without $\mathfrak{T}$ being Efremovich ([11], 4.11).

Let $\mathfrak{T}$ still be a looseness on X. Define $c = c(\mathfrak{T})$ to be the following mapping from exp$X$ into itself:

$$c(A) = \{x \in X : \{\{x\},\ A\} \notin \mathfrak{T}\}. \tag{1.12}$$

Then c is a *closure* on X (see e.g. [4], 3), i.e.

$$c(\emptyset) = \emptyset, \tag{1.13}$$

$$A \subset B \subset X \text{ implies } c(A) \subset c(B), \tag{1.14}$$

$$A \subset c(A) \text{ for } A \subset X, \tag{1.15}$$

$$c(A \cup B) = c(A) \cup c(B) \text{ for } A,\ B \subset X. \tag{1.16}$$

Aclosure c on X is a *topology* iff $c(c(A)) = c(A)$ for $A \subset X$.

A looseness $\mathfrak{T}$ is *Riesz* iff $c(\mathfrak{t}) = \{c(T) : T \in \mathfrak{t}\}$ is free (i.e. $\cap\, c(\mathfrak{t}) = \emptyset) whenevert \in \mathfrak{T}$ and $c = c(\mathfrak{T})$ (see e.g. [4], 4). If $\mathfrak{T}$ is Riesz then the closure $c = c(\mathfrak{T})$ is S_1 (i.e. $x \notin c(A)$ implies $c(\{x\} \cap c(A) = \emptyset$ for $x \in X,\ A \subset X)$ (see e.g. [4], 4.3).

If c is an arbitrary closure on X, define $\mathfrak{T}(c)$ by

$$\mathfrak{t} \in \mathfrak{T}(c) \text{ iff } c(\mathfrak{t}) \text{ is free } (\mathfrak{t} \in \sum(X)). \tag{1.17}$$

Then clearly $\mathfrak{T}(c)$ is a looseness and $c = c(\mathfrak{T}(c))$ iff c is S_1; in this case, $\mathfrak{T}(c)$ is the largest Riesz looseness $\mathfrak{T}$ satisfying $c(\mathfrak{T}) = c$ ([4], 4.9).

If $\mathfrak{T}$ is an Efremovich looseness then $c(\mathfrak{T})$ is a completely regular topology ([4], 3; [11], 4.12 and 3.2).

2 Morita Loosenesses

Let $\mathfrak{T}$ be a looseness on X, $\mathfrak{t}$, $\mathfrak{t}' \in \sum(X)$. We denote $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$ iff, for any $T' \in \mathfrak{t}'$, there exists $T \in \mathfrak{t}$ such that $T' < (\mathfrak{T})T$.

Lemma 2.1 *If $\mathfrak{T}$ is a looseness and $A < (\mathfrak{T})B$ then $c(A) \subset B$ for $c = c(\mathfrak{T})$.*

Proof: $\{A,\ X - B\} \in \mathfrak{T}$ implies $\{A,\ X - B\} \ll \{A,\ \{x\}\}$ for $x \in X - B$, so $\{A,\ \{x\}\} \in \mathfrak{T}$ and $x \notin c(A)$ for these x.

Lemma 2.2 *If $\mathfrak{T}$ is alooseness on X, $\mathfrak{t}$, $\mathfrak{t}' \in \sum(X)$ and $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$ then $\mathfrak{t}' \ll c(\mathfrak{t})$ for $c = c(\mathfrak{T})$.*

Proof: If $T < (\mathfrak{T})T'$ then $c(T) \subset T'$ by 2.1.

Let us say that $\mathfrak{T}$ is *Morita or strongly Morita* iff $\mathfrak{t} \in \mathfrak{T}$ implies the existence of a $\mathfrak{t}' \in \sum(X)$ such that $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$ and $\cap\mathfrak{t}' = \emptyset$ or $\mathfrak{t}' \in \mathfrak{T}$, respectively.

The terminology is motivated by the fact that, in [17], [18], [19], [20], a regular convering structure $\mathfrak{B}$ is defined with the help of two properties, one of them precisely saying that the merotopy generated by $\mathfrak{B}$ is associated with a strongly Morita looseness.

Lemma 2.3 *If $\mathfrak{T}$ is a (strongly) Efremovich looseness then it is (strongly) Morita.*

Proof: If $\sigma\ :\ \mathfrak{t} \to \exp X$ is a $\mathfrak{T}$ -swelling then $\sigma(\mathfrak{t}) \ll (\mathfrak{T})\mathfrak{t}$.

Lemma 2.4 *A (strongly) Morita looseness is Riesz (Lodato).*

Proof: Let $\mathfrak{T}$ be (strongly) Morita and $\mathfrak{t} \in \mathfrak{T}$. Then there is a free $\mathfrak{t}'(\mathfrak{t}' \in \mathfrak{T})$ with $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$. By 2.2, we have $\mathfrak{t}' \ll c(\mathfrak{t})$ and $\cap c(\mathfrak{t}) = \emptyset$ ($c(\mathfrak{t}) \in \mathfrak{T}$) for $c = c(\mathfrak{T})$.

Proposition 2.5 **(cf. [17])** *If $\mathfrak{T}$ is a strongly Morita looseness then $c(\mathfrak{T})$ is a regular topology.*

Proof: By 2.4, $c = c(\mathfrak{T})$ is a topology. Assume $x \in X$ and let V be a neighbourhood of x, i.e. $x \notin c(X-V)$, $\{\{x\},\ X-V\} \in \mathfrak{T}$. There is $\mathfrak{t}' \in \mathfrak{T}$ such that $\mathfrak{t}' \ll (\mathfrak{T})\{\{x\},\ X-V\}$, i.e. $T' \in \mathfrak{t}'$ implies either $\{x\} < T'$ or $X - V < T'$ for $<=< (\mathfrak{T})$. By 2.4 again, $c(\mathfrak{t}') \in \mathfrak{T}$, so $\cap c(\mathfrak{t}') = \emptyset$, and there exists $T' \in \mathfrak{t}'$ such that $x \notin c(T')$, consequently $x \notin T'$, $X - V < T'$, $X - T' < V$. Hence $X - T'$ is a neighbourhood of x and $c(X - T') \subset V$ by 2.1.

On the over hand, we can say:

Proposition 2.6 **(cf. [17])** *For an S_1 closure c, $\mathfrak{T}(c)$ is a strongly Morita looseness iff c is a regular topology.*

Proof: Only if: 2.5. If: Let $\mathfrak{t} \in \mathfrak{T} = \mathfrak{T}(c)$; then $c(\mathfrak{t})$ is free by definition. For $x \in X$, choose $T \in \mathfrak{t}$ such that $x \notin c(T)$ and let V_x be an open neighbourhood of x satisfying $c(V_x) \cap c(T) = \emptyset$. Put $T_x = X - V_x$; then $\mathfrak{t}' = \{T_x : x \in X\}$ is a free system of closed sets, hence $\mathfrak{t}' \in \mathfrak{T}$ and $T < T_x$ for $<=< (\mathfrak{T})$, i.e. $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$.

Consequently the converse of 2.3 does not hold:

Proposition 2.7 *A looseness can be strongly Morita without being Efremovich.*

Proof: Let c be a regular but not completely regular topology on X. Then $\mathfrak{T}(c)$ is strongly Morita by 2.6, but it fails to be Efremovich because if it were, then $c = c(\mathfrak{T}(c))$ would be completely regular.

The following theorem generalizes a part of [11], 4.1:

Theorem 2.8 *If $\mathfrak{T}$ is a strongly Morita looseness then $\mathfrak{S}(\mathfrak{T}))$ is a cauchy structure.*

Proof: Suppose $\mathfrak{s}$ and $\mathfrak{s}'$ are $\mathfrak{T}$ -compressed filters and $\mathfrak{s}\Delta\mathfrak{s}'$. Then $\mathfrak{s}'' = \{S \cap S' : S \in \mathfrak{s},\ S' \in \mathfrak{s}'\}$ is a proper filter finer that $\mathfrak{s}$ and $\mathfrak{s}'$, hence $\mathfrak{T}$ -compressed as well. Define $\mathfrak{s}^* = \mathfrak{s} \cap \mathfrak{s}'$ and suppose $\mathfrak{t} \in \mathfrak{T},\ \mathfrak{t}\Delta\mathfrak{s}^*$. There is $\mathfrak{t}' \in \mathfrak{T}$ such that $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$; given any $T' \in \mathfrak{t}'$, there is $T \in \mathfrak{t}$ satisfying $T < (\mathfrak{T})T'$. By hypothesis, we have $T \in \sec \mathfrak{s}^*$, so that either $T \in \sec \mathfrak{s}$ or $T \in \sec \mathfrak{s}'$ and, $\mathfrak{s}$ and $\mathfrak{s}'$ being $\mathfrak{T}$ -compressed, either $T' \in \mathfrak{s}$ or $T' \in \mathfrak{s}'$ (e.g. $T \in \sec \mathfrak{s},\ X - T' \in \sec \mathfrak{s}$ would imply $\{T,\ X - T'\}\Delta\mathfrak{s}$ and $\{T,\ X - T'\} \notin \mathfrak{T}$, contrary to $T < (\mathfrak{T})T'$). In both cases $T' \in \mathfrak{s}'' \supset \mathfrak{s} \cup \mathfrak{s}'$. Now $T' \in \mathfrak{t}'$ being arbitrary, we have $\mathfrak{t}'\Delta\mathfrak{s}''$, hence $\mathfrak{t}' \notin \mathfrak{T}$: a contradiction. Therefore $\mathfrak{t}\Delta\mathfrak{s}^*$ implies $\mathfrak{t} \notin \mathfrak{T}$ and $\mathfrak{s}^*$ is $\mathfrak{T}$ -compressed.

Remark 2.9 a) The above proof is very similar to that of [11], 4.1.

b) In fact, a somewhat more general theorem, generalizing [11], 4.1, can be proved in the same way. Let $\mathfrak{T}_1$ denote the largest looseness $\mathfrak{T}'$ such that $\mathfrak{S}(\mathfrak{T}') = \mathfrak{S}(\mathfrak{T})$; it is composed of those $\mathfrak{t}_1 \in \sum(X)$ for which $\mathfrak{t}_1\Delta\mathfrak{s}$ does not hold for any $\mathfrak{s} \in \mathfrak{S}(\mathfrak{T})$ (see [10], 10). Define $\mathfrak{T}$ to be *super-Morita* iff $\mathfrak{t} \in \mathfrak{T}$ implies the existence of $\mathfrak{t}' \in \mathfrak{T}_1$ such that $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$. Clearly

$$\text{strongly Morita} \Rightarrow \text{super-Morita} \Rightarrow \text{Morita}.$$

Now, in 2.8, the condition if $\mathfrak{T}$ being strongly Morita can be replaced by that of being super-Morita; in the proof, $\mathfrak{t}'$ must be chosen from $\mathfrak{T}_1$ instead of $\mathfrak{T}$, and then $\mathfrak{t}'\Delta\mathfrak{s}''$ implies $\mathfrak{t}' \notin \mathfrak{T}_1$. The hypothesis of [11], 4.1 clearly implies that $\mathfrak{T}$ is super-Morita.

c) The example [11], 4.11 shows that $\mathfrak{S}(\mathfrak{T})$ can be a Cauchy structure without the looseness $\mathfrak{T}$ being Morita.

Corollary 2.10 *An Efremovich looseness need not be strongly Morita.*

Proof: In [11], 4.15, an Efremovich looseness $\mathfrak{T}$ is constructed such that $\mathfrak{S}(\mathfrak{T})$ fails to be a Cauchy structure.

Proposition 2.11 *For an S_1 closure c, $\mathfrak{T}(c)$ is a Morita looseness.*

Proof: Let $\mathfrak{t} \in \mathfrak{T} = \mathfrak{T}(c)$. For $x \in X$, put $T_x = X - \{x\}$; there is $T \in \mathfrak{t}$ such that $x \notin c(T)$, hence $c(\{x\}) \cap c(T) = \emptyset$. thus $\mathfrak{t}' = \{T_x : x \in X\}$ is free and $T < (\mathfrak{T})T_x$ for the above T. Hence $\mathfrak{t}' \ll (\mathfrak{T})\mathfrak{t}$.

Corollary 2.12 *A Morita looseness need not be Efremovich nor strongly Morita.*

Proof: Consider a non-regular S_1 topology (e.g. the cofinite topology on an infinite set) and $\mathfrak{T} = \mathfrak{T}(c)$. By 1.22, $\mathfrak{T}$ is Morita; by 2.5, is it not strongly Morita, and it is not Efremovich since c is not completely regular.

Corollary 2.13 *In the diagram*

strongly Efremovich ⇒ Efremovich ⇒ Morita,

strongly Efremovich ⇒ strongly Morita ⇒ Morita,

none of the implications is reversible.

Proof: 2.7, 2.10, 2.12.

Remark 2.14 The example [11], 4.15 shows that Efremovich ⇒ super-Morita (cf. 2.10 and 2.9b)).

Problem 2.15 Is there a looseness that is super-Morita without being strongly Morita?

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Author:

Á. Császár
Department of Analysis
Eötvös Loránd University
Budapest, Múzeum Krt. 6-8
1088, Hungary

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J. Cao and A. Mohamad; I.L. Reilly

Mapping Theorems of Some Topological Spaces*

* *Dedicated to Prof. Harry Poppe on his 65th anniversary*

Abstract: The aim of this paper is to study images and preimages of some topological spaces under open finite -to-one (compact) mappings and perfect mappings. Some questions of Gittings are answered negatively.

KEY WORDS and PHRASES. Open finite-to-one mapping, perfect mapping, S_1-space, quasi-metrizable, $w\Delta_2$.

1 Introduction

In the field of general topology, mappings and spaces are main themes. One of the main problems is: What topological properties are preserved under certain classes of mappings? Since the survey article [1] of Arhangel'skiĭ in the 1960s, a lot of nice mapping theorems have been established. Classification of spaces by mappings and classification of mappings by spaces have been investigated by many topologists. Among all mappings, two interesting classes are open finite-to-one mappings and perfect mappings. The main purpose of the present paper is to study the behaviour of $w\Delta_2$-spaces, $\sigma^{\#}$-spaces, spaces with the S_1-property, G_δ-diagonal and first countability under open finite-to-one, compact open , or perfect mappings. Several new mapping theorems are established.

Let X be a topological space, and let $\mathcal{G}$ be a collection of subsets of X. The topology on X is denoted by Top(X) and $St(x, \mathcal{G})$ stands for $\cup\{G : x \in G \in \mathcal{G}\}$. We say that X has a G_δ-*diagonal* if its diagonal $\Delta(X)$ is a G_δ-set in the product space X^2, i.e. $\Delta(X) = \cap_{n\in\omega} G_n$, where G_n are open sets of X^2. It is well-known that a space X has a G_δ-diagonal if and only if there is a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open covers such that $\cap_{n\in\omega} St(x, \mathcal{G}_n) = \{x\}$ for each point $x \in X$. A space X has a quasi-G_δ-diagonal if there is a sequence $\{\mathcal{G}_n : n \in \omega\}$

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of open collections (not necessarily covers) such that $\cap_{n\in\omega} St(x,\mathcal{G}_n) = \{x\}$ for each point $x \in X$, where $c(x) = \{n \in \omega : St(x,\mathcal{G}_n) \neq \emptyset\}$. Note that quasi-$G_\delta$-diagonals are called weak G_δ-diagonals in [14].

A collection $\mathcal{G}$ of subsets of a space X is *semi-open* [11] if for all $x \in X, x \in \text{Int } St(x,\mathcal{G})$. In addition, X is called a $w\Delta_2 -$*space* [7] if there is a sequence $\{\mathcal{G}_n : n \in \omega\}$ of semi-open covers such that for each $x \in X$ if $x_n \in St(x,\mathcal{G}_n)$ for each $n \in \omega$, then the sequence $\{x_n : n \in \omega\}$ has a cluster point (not necessarily x).

Recently, Hiremath [11] introduced some properties similar to the G_δ-diagonal property, namely S_1- and S_2-properties. Recall that a space X has the $S_1 -$ *property* (resp. $S_2 -$ *property*) [11] if there is a mapping $g : \omega \times X \to Top(X)$ such that $\cap_{n\in\omega} g(n,x) = \{x\}$ (resp. $\cap_{n\in\omega}\overline{g(n,x)} = \{x\}$) for each point $x \in X$, and for all $x_1, x_2 \in X$ and $n \in \omega, x_2 \in g(n,x_1)$ iff $x_1 \in g(n,x_2)$.

A cover $\mathcal{C}$ of a space X is called *point-separating* if for every pair of distinct points $x, y \in X$ there exists a $C \in \mathcal{C}$ such that $x \in C$ and $y \notin C$. A space with a σ-closure-preserving point-separating closed cover is called a $\sigma^\# -$ *space* [13].

By a compact (resp. quasi-compact) mapping $f : X \to Y$ from a space X into a space Y, we mean $f^{-1}(y)$ is compact (resp. countably compact) for each $y \in Y$. Moreover, we say f is a finite-to-one mapping if $|f^{-1}(y)| < \omega$ for each $y \in Y$. A mapping is called *perfect* (resp. *quasi-perfect*) if it is continuous, closed and compact (resp. quasi-compact). For other undefined notations, refer to [4] and [5].

2 Open finite-to-one images

Recall that a mapping $f : X \to Y$ is *pseudo-open* if for $y \in Y, y \in \text{Int}(f(G))$ whenever G is a neighbourhood of $f^{-1}(y)$. Clearly, both open mappings and closed mappings are pseudo-open.

Theorem 2.1 *Let $f : X \to Y$ be a continuous, finite-to-one and pseudo-open mapping from a space X onto a space Y. If X is $w\Delta_2$, then so is Y.*

Proof: Let $\{\mathcal{G}_n : n \in \omega\}$ be a sequence of semi-open covers of X satisfying all conditions in the definition of a $w\Delta_2$-space. For each $b \in \omega$, define $f(\mathcal{G}_n) = \{f(G) : G \in \mathcal{G}_n\}$. For each $y \in Y$ and $n \in \omega$, since $St(y, f(\mathcal{G})\} = f(St(f^{-1}(y),\mathcal{G}))$ and f is a pseudo-open mapping, $St(y, f(\mathcal{G}_n))$ is a neighbourhood of y. Thus, $\{f(\mathcal{G}_n) : n \in \omega\}$ is a sequence of semi-open covers of Y. In addition, if $y_n \in St(y, f(\mathcal{G}_n))$, then there is a $G_n \in \mathcal{G}_n$ such that $y, y_n \in f(G_n)$ for each $n \in \omega$. Choose $x_n, z_n \in G_n$ such that $f(x_n) = y$ and $f(z_n) = y_n$. Since f is finite-to-one, there is an $x \in f^{-1}(y)$ and a subsequence $\{x_{n_i} : i \in \omega\}$ of $\{x_n : n \in \omega\}$ such that $x_{n_i} = x$ for all $i \in \omega$. Furthermore, we have $z_{n_i} \in St(x, \mathcal{G}_{n_i})$ for each $i \in \omega$. Since X is a $w\Delta_2$-space,

$\{z_{n_i} : i \in \omega\}$ has a cluster point, and so does the sequence $\{z_n : n \in \omega\}$. It follows that $\{y_n : n \in \omega\}$ has a cluster point. Hence Y is a $w\Delta_2$-space. □

Corollary 2.2 *Let $f : X \to Y$ be a continuous and finit-to-open mapping from a $w\Delta_2$-space X onto a space Y. If f is either open or closed, then Y is a $w\Delta_2$-space.*

It is well-known that the G_δ-diagonal is an invariant under open finite-to-one (not necessarily continuous) mappings. Xia [15] proved that the quasi-G_δ-diagonal property is preserved by open finite-to-one mappings. Next we show that the S_1-property is also preserved by open finite-to-one mappings.

Theorem 2.3 *Let $f : X \to Y$ be a finite-to-one and open mapping from a space X onto a space Y. If X has the S_1-property, then so does Y.*

Proof: Suppose that X has the S_1-property. Then there is a mapping $g : \omega \times X \to Top(X)$ such that

(1) $\{x\} = \cap_{n\in\omega} g(n, x)$ for each $x \in X$;

(2) for all $x_1, x_2 \in X$ and $n \in \omega, x_2 \in g(n, x_1)$ iff $x_1 \in g(n, x_2)$.

Without loss of generality, we may assume that $g(n + 1, x) \subseteq g(n, x)$ for all $n \in \omega$ and all $x \in X$. For each $y \in Y$ and each $n \in \omega$, define $h(n, y) = \cup\{f(g(n, x)) : x \in f^{-1}(y)\}$.

We first show that $\cap_{n\in\omega} h(n, y) = \{y\}$ for each $y \in Y$. If $z \in Y$ and $z \neq y$, then $f^{-1}(z) \cap f^{-1}(y) = \emptyset$. Since $f^{-1}(z)$ and $f^{-1}(y)$ are finite subsets, there is an $n \in \omega$ such that $f^{-1}(z) \cap (\cup\{g(n, x) : x \in f^{-1}(y)\}) = \emptyset$. It follows that $z \notin h(n, y)$.

To complete the proof, we show that for all $y_1, y_2 \in Y$ and each $n \in \omega$, $y_1 \in h(n, y_2)$ iff $y_2 \in h(n, y_1)$. By the definition of h, $y_1 \in h(n, y_2)$ iff there exist $x_2 \in f^{-1}(y_2)$ and $x_1 \in g(n, x_2)$ such that $y_1 = f(x_1)$ which is equivalent to: there exist $x_2 \in f^{-1}(y_2)$ and $x_1 \in f^{-1}(y_1)$ such that $x_2 \in g(n, x_1)$. The last statement is equivalent to $y_2 \in h(n, y_1)$. □

By an argument similar to that of Theorem 2.3, we have the following result.

Theorem 2.4 *Let $f : X \to Y$ be a finite-to-one, open and closed mapping from a space X onto a space Y. If X has the S_2-property, then so does Y.*

In [7], Gittings asked whether the G_δ-diagonal property and $\sigma^{\#}$-spaces are inverse invariants under continuous, open finite-to-one mappings. The following simple example provides negative answers to these two questions. It also shows that the quasi-G_δ-diagonal property and the S_1-property are not inverse invariants under continuous, open finite-to-one mappings.

Example 2.5 A continuous, open (and closed) two-to-one mapping from a non-T_1 space X onto a metrizable space. Let Z be the set of all integers. Let $X = Z$ be the space with the topology generated by $\mathcal{B} = \{\emptyset\} \cup \{\{2n-1, 2n\} : n \in Z\}$. Let Y be the space of all even integers with the discrete topology. Define a mapping $f : X \to Y$ by $f(2n-1) = f(2n) = 2n$ for each $n \in Z$. It is easy to check that f is two-to-one, open and continuous, and Y is metrizable. Since X is not T_1, it is not a $\sigma^{\#}$-space, and does not have the quasi-G_δ-diagonal nor the S_1-property. Since f is closed, this example also shows that the S_2-property is not an inverse invariant under continuous, closed and open finite-to-one mappings. □

Remark 2.6 In [15], Xia proved that all quasi-perfect mappings preserve $\sigma^{\#}$-spaces. Note that the mapping defined in Example 2.5 is perfect. Therefore, $\sigma^{\#}$-spaces are not inverse invariants under perfect mappings.

3 Perfect preimages of some spaces

It is easy to give an example to show that first countability is not an inverse invariant of perfect mappings. Our next theorem, which generalizes some classical results, claims this problem has an affirmative answer under the additional hypothesis of having a domain with a quasi-G_δ-diagonal.

Lemma 3.1 [3, 14] *A countably compact space X with quasi-G_δ-diagonal is metrizable.*

Theorem 3.2 *Let $f : X \to Y$ be a quasi-perfect mapping from a regular space X onto a first countable space Y. If X has a quasi-G_δ-diagonal, then X is first countable.*

Proof: Let $g : \omega \times Y \to Top(Y)$ be a mapping such that for each $y \in Y$, $\{g(n, y) : n \in \omega\}$ is a local base for y. Since X has a quasi-G_δ-diagonal, then for each $y \in Y$ the subspace $f^{-1}(y)$ has a quasi-G_δ-diagonal. By Lemma 3.1, each $f^{-1}(y)$ is a metrizable subspace of X. Thus, there is a mapping $k_y : \omega \times f^{-1}(y) \to Top(f^{-1}(y))$ such that for each $x \in f^{-1}(y)$, $\{k_y(n, x) : n \in \omega\}$ is a local base for x in $f^{-1}(y)$. By the regularity of X, we can construct a mapping $h : \omega \times X \to Top(X)$ such that

(1) $h(n, x) \subseteq f^{-1}(g(n, f(x)))$ for each $x \in X$;

(2) $x \in h(n, x) \cap f^{-1}(f(x)) \subseteq k_{f(x)}(n, x)$ for each $x \in X$;

(3) $\overline{h(n+1, x)} \subseteq h(n, x)$ for each $x \in X$ and each $n \in \omega$.

We show that for each point $x \in X$, $\{h(n, x) : n \in \omega\}$ is a local base for x. Suppose the contrary. There are a point $x \in X$, a neighbourhood U of x and a sequence $\{x_n : n \in \omega\}$ with $x_n \in h(n, x) \backslash U$ for all $n \in \omega$. Since $\cap_{n\in\omega} h(n, x) \subseteq k_{f(x)}(n, x)$, then $\cap_{n\geq 1} \overline{\{x_m : m \geq n\}} \subseteq$

$\cap_{n\geq 1}\overline{h(x,n)\backslash U} \subseteq (\cap_{n\in\omega}k_{f(x)}(n,x))\backslash U = \emptyset$. This implies that $\{x_n : n \in \omega\}$ is a discrete subspace of X. Since f is closed, $\{f(x_n) : n \in \omega\}$ has no cluster points in Y which contradicts the fact that $f(x)$ is a cluster point of $\{f(x_n) : n \in \omega\}$ in Y. $\square$

Corollary 3.3 [12] *Let $f : X \to Y$ be a perfect mapping from a regular space X onto a first countable space Y. If X has a G_δ-diangonal, then X is first countable.*

For a topological space X, let $nw(X)$ stand for the network weight of X (see [4]).

Corollary 3.4 [4] *Let $f : X \to Y$ be a perfect mapping from a regular space X onto a first countable space Y. If $nw(X) \leq \omega$, then X is first countable.*

Proof: If $nw(X) \leq \omega$, then X is a σ-space (see [9] for definition). Since every σ-space has a G_δ-diagonal, it follows immediately from Theorem 3.2 that X is first countable. $\square$

Recall that a *quasi-metric* (resp. *non-Archimedean quasi-metric*) [5] on a set X is a mapping $d : X \times X \to \mathbf{R}^+$ such that (1) $d(x,y) = 0$ iff $x = y$ for all $x, y \in X$; (2) $d(x,z) \leq d(x,y) + d(y,z)$ (resp. $d(x,z) \leq \max\{d(x,y), d(y,z)\}$) for all $x, y, z \in X$. A space X is called *quasi-metrizable* (resp. *non-Archimedeanly quasi-metrizable*) if it admits a compatible quasi-metric (resp. non-Archimedean quasi-metric). In [2], Borges showed that a perfect preimage of a metrizable space is metrizable if and only if it has a G_δ-diagonal. Next we provide an example which says that there are no such analogues for quasi-metrizable spaces, non-Archimedeanly quasi-metrizable spaces and γ-spaces.

Example 3.5 A non-γ-space with a G_δ-diagonal which is a perfect preimage of a non-Archimedeanly quasi-metrizable space. Let $X = \mathbf{R}^2$. Each point $(x,y) \in X$ with $y \neq 0$ has the usual neighbourhoods. Each point $(x,0) \in X$ has basic neighbourhoods of the form

$$U_\epsilon = \{(x,0)\} \cup \{(x_1,y_1) \in \mathbf{R}^2 : (x_1 - x)^2 + (y_1 \pm \epsilon)^2 < \epsilon^2\}.$$

Let Y be the Niemytzki upper half tangent plane. Since Y has a σ-interior-preserving base, it is non-Archimedeanly quasi-metrizable. Define $f : X \to Y$ by $f((x,y)) = (x,|y|)$. It is easy to see that f is a finite-to-one, continuous and closed mapping. Therefore, X is a perfect preimage of a non-Archimedeanly quasi-metrizable space. It follows from [5, Example 7.10] that X is a semi-stratifiable and non-development space. Since any semi-stratifiable γ-space is developable, X has a G_δ-diagonal, and it is not a γ-space. $\square$

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Authors:

Jiling Cao, A. Mohamad, I. L. Reilly
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland 1
New Zealand

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G. Di. Maio, Ľ. Holá, E. Meccariello

Notes on hit-and-miss topologies

Dedicated to Prof. Harry Poppe on his 65th anniversary

ABSTRACT. We give necessary conditions for normality of Δ-topologies, where Δ is stable under closed subsets. We have a complete solution of metrizability of proximal topology in uniform setting.

KEY WORDS AND PHRASES: Hit-and-miss, proximal, hypertopology, bounded, totally bounded, Urysohn family, Keesling family, uniform space, compatible metric, metrizability.

1 Introduction

Hypertopologies, i.e. topologies on thr hyperset $CL(X)$ of all closed and nonempty subsets of a topological space X, are intensively studied in the last decade, since they are a fundamental tool in some aspects of optimization theory and convex analysis.

Following the papers [6, 7, 9, 10, 11, 13, 21, 24] we will continue the study of hit-and-miss hyperspace topologies, more precisely the so called Δ and proximal Δ -topologies, where Δ is a fixed subfamily of $CL(X)$. The Vietoris topology and the proximal topology are the prototypes for hit-and-miss and proximal hit-and-miss topologies.

The Vietoris topology ([19], [8]) is the largest hit-and-miss topology and was very thoroughly studied since 1951, when the seminal paper of Michael ([19]) appeared.

The proximal Δ -topologies were deeply investigated in the last years, when their applicability to convex analysis has been found.

The proximal hit-and-miss topology parallels thr Vietoris topology (i.e. is the finest proximal hit-and-miss hypertopology) and has been studied in ([11], [3]).

The Vietoris topology and the proximal topology are too strong for many applications because they fail to work properly especially in metric spaces when sets under analysis are unbounded.

Partially in response to these shortcomings have been constructed on metric spaces the bounded Vietoris ([18], [4]) and the bounded proximal topology ([22], [5]).

Furthermore, the totally bounded proximal hypertopology is now recognized as a fundamental tool in the construction of the lattice of all hypertopologies on the matric space $(X,\ d)$. In fact, its upper part is the infimum of all upper Wijsman topologies corresponding to all metrics uniformly equivalent to d ([10]).

In our paper we study a large class of Δ and proximal Δ -topologies: those determined by subfamilies Δ of $CL(X)$ stable under closed subsets, i.e. families which contain all closed subsets of every its member. Notice that all the above mentioned topologies fulfill this condiction. We give necessary conditions for normality of such Δ and proximal Δ -topologies. We have the complete solution of matrizability of proximal topology in a uniform setting.

2 Preliminaries

Let X be a Hausdorff topological space. If $E \subset X$, we set:

$$E^- = \{A \in CL(X) : A \cap E \neq \emptyset\},$$
$$E^+ = \{A \in CL(X) : A \cap E\} = \{A \in CL(X) : A \cap E^c = \emptyset\},$$

where E^c denotes the cmplement of E.

The lower Vietoris topology τ_V^- on $CL(X)$ is generated by the sets of the form G^-, where G is open in X.

If Δ is a nonempty subfamily of $CL(X)$, then the upper Δ -topology τ_Δ^+ is generated by all collections of the form $(B^c)^+$, where $B \in \Delta$.

By the hit-and-miss topology determined by Δ , we will mean the topology $\tau_\Delta = \tau_V^- \vee \tau_\Delta^+$. The topologies τ_V^- and τ_Δ^+ are the lower and upper part of τ_Δ.

If $\Delta = CL(X)$ we obtain the familiar Vietoris topology τ_V, if $\Delta = K(X)$, the family of all compact subsets of X, we have the Fell topology τ_F.

Now, let $(X,\ \mathcal{U})$ be a uniform space. If $B \subset X$ put
$B^{++} = \{A \in CL(X) : \exists U \in \mathcal{U}$ whith $U[A] \subset B\}$.
We say that two subsets A and B are far if $A \in (B^c)^{++}$ (or what is the same $B \in (A^c)^{++}$) [20].

Let Δ be a prescribed nonempty subfamily of $CL(X)$, the upper proximal Δ-topology σ_Δ^+ has as a subbase all the sets of the form $(B^c)^{++}$, where $B \in \Delta$. The proximal Δ-topology $\sigma_\Delta = \tau_V^- \vee \sigma_\Delta^+$. If $\Delta = CL(X)$ we have the proximal topology $\sigma = \tau_V^- \vee \sigma_{CL(X)}^+$, the finest proximal hit-and-miss topology.

Since $(K^c)^+ = (K^c)^{++}$ for K compact, the Fell topology coincides with its proximal version.

We refer to [1] and [17] for all undefined terms.

3 Δ-topologies

In what follows let X be a Hausdorff topological space and let Δ be a nonempty subfamily of $CL(X)$. If $A \in CL(X)$ put $\Delta(A) = \{B \in CL(X) : B \in \Delta \text{ and } B \subset A\}$ By $\tau_{\Delta(A)}$ we mean the hit-and-miss topology on $CL(A) = \{D \in CL(X) : D \subset A\}$ associated to $\Delta(A)$, i.e. the topology which has as a subbase all sets of the form U^-, where U is an open set in A, plus all the sets $(B^c)^+$, where $B \in \Delta(A)$. We declare the family Δ to be *stable under closed subsets* iff it contains each closed subset of its members, i.e. if for each $B \in \Delta$ and $A \in CL(X)$ with $A \subset B$ we have also $A \in \Delta$.

Observe that the most familiar and well studied topologies are associated to subfamilies which are stable under closed subsets:

$CL(X)$, $K(X)$, in metric spaces the family of all closed totally bounded sets $TB(X)$, and the family of all closed bounded sets $CLB(X)$.

We start with a simple but useful Lemma.

Lemma 3.1 *Let X be a Hausdorff topological space, $\Delta \subset CL(X)$ be a nonempty family stable under closed subsets and $A \in CL(X)$. Then $\big(CL(A),\ \tau_{\Delta(A)}\big)$ coincides with $CL(A)$ equipped with the relative topology of $(CL(X),\ \tau_\Delta)$.*

Proof: Let U be A-open; there ia an open set V in X with $U = V \cap A$. We have $\{F \in CL(A) : F \cap U \neq \emptyset\} = \{F \in CL(A) : F \cap V \neq \emptyset\} = V^- \cap CL(A)$.

Now, suppose $K \in \Delta(A)$. Then $\{F \in CL(A) : F \cap K = \emptyset\} = (K^c)^+ \cap CL(A)$, and $(K^c)^+ \in \tau_\Delta$, since $K \in \Delta$.

On the other hand, suppose $K \in \Delta$ and consider the set $(K^c)^+ \cap CL(A)$. If $K \cap A = \emptyset$, then $(K^c)^+ \cap CL(A) = CL(A) \in \tau_{\Delta(A)}$. If $K \cap A \neq \emptyset$, then $K \cap A \in \Delta(A)$. Thus $(K^c)^+ \cap CL(A) = \{F \in CL(A) : F \cap (K \cap A) = \emptyset\} \in \tau_{\Delta(A)}$. ■

Lemma 3.2 *Let X be a Hausdorff topological space and $\Delta \subset CL(X)$ be a nonempty family stable under closed subsets. If $(CL(X),\ \tau_\Delta)$ is normal or second countable, then every $B \in \Delta$ must be compact.*

Proof: Let $B \in \Delta$. By Lemma 3.1 $\big(CL(B),\ \tau_{\Delta(B)}\big)$ coincides with $CL(B)$ equipped with the relative topology of $(CL(X),\ \tau_\Delta)$. Hence $(CL(B),\ \tau_{\Delta(B)})$ is normal since it is a closed subspace of the normal spyce $(CL(X),\ \tau_\Delta)$. Now, notice that $\tau_{\Delta(B)}$ coincides with the

Vietoris topology on $CL(B)$ because $B \in \Delta$ and Δ is stable under closed subsets. By the result of Veličko [23], B must be compact.

Similarly, if $(CL(X),\ \tau_\Delta$ is second countable, then $(CL(B),\ \tau_{\Delta(B)})$ is second countable and again $\tau_{\Delta(B)}$ coincides with the Vietoris topology on $CL(B)$. Hence B must be compact by the result of Michael [19]. ■

The following corollary shows that in the class of Δ-topologies, where Δ is stable under closed subsets, there is one, and only one, normal element: the Fell topology.

Corollary 3.3 *Let X be a Hausdorff topological space and $\Delta \subset CL(X)$ ba a nonempty family stable under closed subsets. If $(CL(X),\ \tau_\Delta)$ is normal, then $\tau_\Delta = \tau_F$.*

Proof: By the previous Lemma $\tau_\Delta \subset \tau_F$. Since τ_Δ is Hausdorff, it is finer than Fell topology ([1]). ■

Let $(X,\ d)$ be a metric space. If $\Delta = CLB(X)$ is the family of all closed and bounded subsets of X, the corresponding τ_Δ topology is called the bounded Vietoris Topology ([4], [18]) and is denoted by τ_{bV}. Furthermore, if $\Delta = TB(X)$, the family of all totally bounded sets of X , we have the topology τ_{TB} studied in [10].

Some efforts have been done in the literature to discribe metrizability of the bounded Vietoris topology τ_{bV} ([18]) and of the totally bounded topology τ_{TB} ([10]) in metric setting.

By applying the previous Lemma we give a deeper and more transparent description of metrizability of the above mentioned topologies: it is equivalent to normality.

Theorem 3.4 *Let $(X,\ d)$ be a metric space. The following are equivalent:*

(1) *$(CL(X),\ \tau_{bV})$ is second countable;*

(2) *$(CL(X),\ \tau_{bV})$ is metrizable;*

(3) *$(CL(X),\ \tau_{bV})$ is paracompact;*

(4) *$(CL(X),\ \tau_{bV})$ is normal;*

(5) *$(CL(X),\ \tau_{bV})$ is Polish;*

(6) *$(X,\ d)$ is boundedly compact.*

Proof: (1)⇒(2) τ_{bV} is a Tychonoff topology (Lemma 4.4.7 in [1]), thus second countability of τ_{bV} implies its metrizability.

(2)⇒(3) and (3)⇒(4) are clear, (4)⇒(6) by Lemma 3.2.

(6)⇒(5) If $(X,\ d)$ is boundedly compact, then the bounded Vietories topology τ_{bV} coincides with the Fell topology τ_F on $CL(X)$. $(CL(X),\ \tau_F)$ is Polish since X is locally compact and second countable (Theorem 5.1.5 in [1]).

(5)⇒(1) is clear. ■

The following theorem completes a result in [10].

Theorem 3.5 *Let $(X,\ d)$ ba a metric space. The following are equivalent:*

(1) $(CL(X),\ \tau_{TB})$ *is second countable;*

(2) $(CL(X),\ \tau_{TB})$ *is metrizable;*

(3) $(CL(X),\ \tau_{TB})$ *is paracompact;*

(4) $(CL(X),\ \tau_{TB})$ *is normal;*

(5) $(CL(X),\ \tau_{TB})$ *is Polish;*

(6) $(X,\ d)$ *is complete, second countable and locally compact.*

Proof: (1)⇒(2), (2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(6) by Lemma 3.2 every closed totally bounded subset must ba compact, hence $(X,\ d)$ is complete. If every $D \in TB(X)$ is compact, then $\tau_{TB} = \tau_F$ on $CL(X)$. Thus the normality of τ_{TB} guarantees the normality of τ_F which in turn implies both local compactness and Lindelöfness of X by a results of [14]. Thus X is second countable.

(6)⇒(5) If $(X,\ d)$ is complete, then $\tau_F = \tau_{TB}$ on $CL(X)$. But $(CL(X),\ \tau_F)$ is Polish since X is locally compact and second countable ([1]). ■

4 Proximal Δ-topologies

Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $A \in CL(X)$. We denote by $\sigma_{\Delta(A)}$ the proximal topology on $CL(A)$ which has as a subbase all sets of the form U^- and $(B^c)^{++}$, where $B \in \Delta(A)$ and U is open.

Lemma 4.1 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space, $\Delta \subset CL(X)$ be a nonempty family stable under closed subsets and $A \in \Delta$. Then $(CL(A),\ \sigma_{\Delta(A)})$ coincides with $CL(A)$ equipped with the relative topology of $(CL(X),\ \sigma_\Delta)$.*

Proof: Only the upper part of the involved topologies needs some comments. Let $B \in \Delta$ and $F \in (B^c)^{++} \cap CL(A)$. If there is $U \in \mathcal{U}$ with $U[B] \cap A = \emptyset$, then $F \in CL(A) \subset (B^c)^{++} \cap CL(A)$ and $CL(A) \in \sigma_{\Delta(A)}$.

On the contrary, suppose that we have $V[B] \cap A \neq \emptyset$ for every $V \in \mathcal{U}$. $F \in (B^c)^{++} \cap CL(A)$ implies that there exists $U \in \mathcal{U}$ with $U[B] \cap F = \emptyset$. Let $G \in \mathcal{U}$ be such that $G^4 \subset U$. Then $clG[B] \cap F = \emptyset$ and $F \in [(clG[B] \cap A)^c]^{++}$. Since Δ is stable under closed subsets and $A \in \Delta$ also $H = clG[B] \cap A \in \Delta(A)$. We claim that $\{C \in CL(A) : C \in (H^c)^{++}\} \subset (B^c)^{++} \cap CL(A)$.

By contradiction suppose that there is $L \in CL(A)$ such that $L \in (H^c)^{++}$ but $L \notin (B^c)^{++}$. Thus $G[B] \cap L \neq \emptyset$ and $G[B] \cap L \subset H = clG[B] \cap A$, a contradiction. On the other hand, let $D \in \Delta(A) \subset \Delta$. Observe that $(D^c)^{++} \cap CL(A) \subset \{C \in CL(A) : C \in (D^c)^{++}\}$, and the claim. ■

We now prove our main lemma of this section.

Lemma 4.2 *Let $(X, \mathcal{U})$ ba a Hausdorff uniform space and $\Delta \subset CL(X)$ be a notempty family stable under closed subsets. If $(CL(X), \sigma_\Delta)$ is normal or second countable, then every $B \in \Delta$ is totally bounded.*

Proof: Let $B \in \Delta$. By the previous Lemma $(CL(B), \sigma_{\Delta(B)})$ coincides with $CL(B)$ equipped with the relative topology of $(CL(X), \sigma_\Delta)$. Hence, it is normal as a closed subspace of a normal space. Suppose B is not totally bounded. There are a symmetric $U \in \mathcal{U}$ and an infinite subset $D = \{a_n : n \in Z^+\}$ of B, with $(a_n, a_m) \notin U$ for every n and m, $n \neq m$. Of course D is a closed subset of B and since Δ is stable under closed subsets also $D \in \Delta$. Thus $(CL(D), \sigma_{\Delta(D)})$ is a normal subspace. There are at most $2^{\mathcal{N}_0}$ real valued continuous functions on $CL(D)$ since it is separable. Now we use some ideas from the papers [15] and [16] by Keesling. For every $B \subset D$ set $\tilde{B} = \{a_{2n} : a_n \in B, n \in Z^+\} \cup \{a_{2n-1} : a_n \notin B, n \in Z^+\}$. The family $\mathcal{B} = \{\tilde{B} : \subset A\}$ is called the Keesling family relative to D. Note that $\mathcal{B}$ has cardinality $2^{\aleph_0}$ and it is $\sigma_{\Delta(D)}$- discrete and $\sigma_{\Delta(D)}$-closed. By Tietze's Theorem there are at least $2^{2^{\aleph_0}}$ real valued continuous functions on $CL(D)$, a contradiction.

To prove that also the second countability of $(CL(X), \sigma_\Delta)$ implies that every $B \in \Delta$ is totally bounded it suffices to observe that if total boundedness fails, then the constructed Keesling family has cardinality $2^{\aleph_0}$ and it is $\sigma_{\Delta(D)}$-discrete. ■

We apply the above Lemma to the families $CL(X)$ and $CLB(X)$ in metric spaces.

For $\Delta = CLB(X)$, the family of closed and bounded subsets of X, we obtain proximal bounded Vietoris topology $\sigma_{bV} = \sigma_{CLB(X)}$ ([5], [18]).

We improve results from [3] and [5] concerning the proximal topology σ and the bounded proximal topology $\sigma_{bV} = \sigma_{CLB(X)}$.

Theorem 4.3 *Let (X, d) be a metric space. The following are equivalent:*

(1) $(CL(X), \sigma)$ *is second countable;*

(2) $(CL(X), \sigma)$ *is metrizable;*

(3) $(CL(X), \sigma)$ *is paracompact;*

(4) $(CL(X), \sigma)$ *is normal;*

(5) (X, d) *is totally bounded.*

Proof: (1)⇔(2) and (2)⇔(5) are in Theorem 4.3 of [3].

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(5) By the previous Lemma normality of the hyperspace $(CL(X),\ \sigma)$ guarantees total boundedness of X. ■

Theorem 4.4 *Let $(X,\ d)$ be a metric space. The following are equivalent:*

(1) $(CL(X),\ \sigma_{bV})$ *is second countable;*

(2) $(CL(X),\ \sigma_{bV})$ *is metrizable;*

(3) $(CL(X),\ \sigma_{bV})$ *is paracompact;*

(4) $(CL(X),\ \sigma_{bV})$ *is normal;*

(5) *each bounded subset of X is totally bounded.*

Proof: (1)⇔(2) and (2)⇔(5) are proved in [5].

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(5) By Lemma 4.2 normality of $(CL(X),\ \sigma_{bV})$ implies that every closed bounded set is totally bounded. ■

To state an analogue of Corollary 3.3 for proximal Δ-topologies we need an extra condition on the family Δ . we recall that $\sum(\Delta)$ denotes the collection of all finite unions of elements from Δ .

If $\Delta = TB(X)$, we denote by σ_{TB} the corresponding proximal topology σ_Δ introduced and studied in [10].

Definition 4.5 *Let $(X,\ \mathcal{U})$ ba a Hausdorff uniform space. A nonempty subfamily of $CL(X)$ is declared a uniformly local family if there exists $U \in \mathcal{U}$ such that $clU[x] \in \sum(\Delta)$ for every $x \in X$.*

Corollary 4.6 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space, Δ be a nonempty uniformly local subfamily of $CL(X)$ stable under closed subsets. If $(CL(X),\ \sigma_\Delta)$ is normal, then $\sigma_\Delta = \sigma_{TB}$.*

Proof: Since σ_Δ is normal and Δ is stable under closed sets $\sigma_\Delta \subset \sigma_{TB}$ by the provious Lemma.

To prove the opposite inclusion $\sigma_{TB} \subset \sigma_\Delta$ it suffices to show $\sigma^+_{TB} \subset \sigma^+_\Delta$. Now, let $B \in TB(X)$ and $(B^c)^{++} \in \sigma^+_{TB}$. We prove that $(B^c)^{++} \in \sigma^+_\Delta$. Let $F \in (B^c)^{++}$, then there exists $V \in \mathcal{U}$ such that $V[B] \cap V[F] = \emptyset$. Let $W \in \mathcal{U}$ such that $W^4 \subset V$ and let $\tilde{U}$ ba the entourage which guarantees that Δ is a uniformly local family. Set $U = W \cap \tilde{U}$. Since B is totally bounded, there exist $b_1, \cdots, b_n \in B$ such that $B \subset \bigcup_{i=1}^{n} U[b_i]$. Put $T = \bigcup_{i=1}^{n} clU[b_i]$. Since Δ is stable under closed sets and $U \subset \tilde{U}$, it follows $clU[x] \in \sum(\Delta)$ for every $x \in X$. Hence $T \in \sum(\Delta)$. But $U \subset W,\ b_i \in B$ for $i = 1, \cdots, n$. So $\bigcup_{i=1}^{n} W[b_i] \subset W[B]$ and $T \in W^2[B]$. So, we have $F \in (T^c)^{++} \subset (B^c)^{++}$. ■

Remark 4.7 Example 4.16 in [10] shows that $(CL(X),\ \sigma_{TB})$ can be even metrizable, without the family $TB(X)$ being a uniformly local family.

We have seen that second countability, metrizability, paracompactness and normality are all equivalent for the proximal topology σ and bounded proximal Vietoris topology σ_{bV} constructed on the hyperset of a metric space. It is very easy to verify that also Lindelöfness is equivalent to the above mentioned properties.

We do not know whether paracompactness and normality are equivalent for the proximal totally bounded topology σ_{TB}. But, from the following we can argue that Lindelöfness and paracompactness are equivalent for σ_{TB}

Lemma 4.8 *Let $(X,\ d)$ ba a metric space. If $(CL(X),\ \sigma_{TB})$ is normal, then X is separable.*

Proof: Suppose X fails to be separable. There are a positive η and an uncountable set D such that $d(a,\ b) > 2\eta$ for every $a,\ b \in D,\ a \neq b$. Of course D is a closed set. The normality of $(CL(X),\ \sigma_{TB})$ implies that $CL(D)$ is normal as a closed subspace of $(CL(X),\ \sigma_{TB})$.

We show that the relative topology on $CL(D)$ coincides with $(CL(X),\ \sigma_{TB(D)})$. Let $F \in (B^c)^{++} \cap CL(D)$, where $B \in TB(X)$. There is a positive ε such that $clS_\varepsilon[B] \cap F = \emptyset$ (where $S_\varepsilon[B] = \{x \in X\ :\ d(x,\ b) < \varepsilon$ for some $b \in B\})$. Put $\alpha = \min\left\{\frac{\varepsilon}{4},\ \frac{\eta}{4}\right\}$.

We clain that $H = clS_{\frac{\alpha}{2}}[B] \cap D$ is finite. Suppose that H is infinite. For every $a \in H$ there is $a_B \in B$ with $d(a,\ a_B) < \alpha$. Let $a,\ b \in H$ two different elements. Let us compute $d(a_B,\ b_B)$. We have $d(a,\ b) \leq d(a,\ a_B) + d(a_B,\ b_B) + d(b_B, b) < 2\alpha + d(a_B,\ b_B)$ and $\eta < 2\eta - 2\frac{\eta}{4} \leq d(a,\ b) - 2\alpha \leq d(a_B,\ b_B)$.

So there are infinitely many elements in B such that the distance between any two of them is greater than η. This is a contradiction because B is totally bounded. Thus H is finite, i.e. $H \in TB(X)$ and it is very easy to verify that $F \in \{G \in CL(D)\ :\ G \in (H^c)^{++}\} \subset (B^c)^{++} \cap CL(D)$. It is also transparent that if $L \in TB(D)$, we have $(L^c)^{++} \cap CL(D) \subset \{G \in CL(D)\ :\ G \in (L^c)^{++}\}$.

Consequently, $(CL(D),\ \sigma_{TB(D)})$ is a normal space. Realize now that $\sigma_{TB(D)}$ is just the Fell topology on $CL(D)$. By a results of [14] D must be a Lindelöf space, a contradiction. ■

Theorem 4.9 *Let $(X,\ d)$ be a metric space. The following are equivalent:*

(1) $(CL(X),\ \sigma_{TB})$ *is paracompact;*

(2) $(CL(X),\ \sigma_{TB})$ *is Lindelöf.*

Proof: (2)⇒(1) Is clear.

(1)⇒(2) By Lemma 4.8 we derive that X is separable and so the hyperspace is separable too. But every separable paracompact space is Lindelöf by Corollary 5.1.26 [12]. ■

5 Second countability and metrizability of proximal Δ-topologies

In this part we complete results from [7] and [9] concerning second countability and metrizability of proximal Δ-topologies. Corresponding results for Δ-topologies can be found in [13]. Using some ideas and tools from [9] is possible to improve Theorem 3.10 of [7], where the attention is restricted to uniformizable hyperspace topologies.

Theorem 5.1 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and Δ be a subfamily of $CL(X)$. The following are equivalent:*

(1) $(CL(X),\ \sigma_\Delta)$ *is second countable;*

(2) *X is second countable anfd there is a countable family $\Delta' \subset \Delta$ such that for every $B \in \Delta$ and $U \in \mathcal{U}$ there is $S \in \sum(\Delta')$ with $B \subset S \subset U[B]$.*

We extend proposition 5.18 of [9] to uniform space.

Theorem 5.2 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ containing the singletons. The following are equivalent:*

(1) $(CL(X),\ \sigma_\Delta)$ *is metrizable;*

(2) $(CL(X),\ \sigma_\Delta)$ *is second countable and regular.*

The condition equivalent to regularity is described in [7], where it is shown that the regularity of σ_Δ is equivalent to the condition that Δ is a uniformly Urysohn family.

Definition 5.3 ([7]) *Let $(X,\ \mathcal{U})$ ba a Hausdorff uniform space and $\Delta \subset CL(X)$ be a family containing the singletons. Δ is called a uniformly Urysohn family provided whenever $B \in \Delta$ and $A \in CL(X)$ are far there exists $S \in \sum(\Delta)$ and $U \in \mathcal{U}$ with $U[B] \subset S \subset A^c$.*

Thus we can rewrite Theorem 5.2 as follows.

Theorem 5.4 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ be a family containing the singletons. The following are eqivalent:*

(1) $(CL(X),\ \sigma(\Delta)$ *is metrizable;*

(2) *there is a countable family $\Delta' \subset \Delta$ such that whenever $D \in \Delta$ and $A \in CL(X)$ are far, there is $S \in \sum(\Delta')$ and $V \in \mathcal{U}$ with $V[B] \subset S \subset A^c$.*

To investigate metrizability, we need some background material.

Let $\mathcal{F}$ be a family of real functionals defined on a set E. The weak topology $O(\mathcal{F})$ induced by $\mathcal{F}$on E is the weakest topology on E making each function in $\mathcal{F}$continuous. The uniformity $\mathcal{W}(\mathcal{F})$ on E determined by $\mathcal{F}$has as a subbase for its entourages all sets of the form:

$$\{(e_1,\ e_2)\ :\ |f(e_1) - f(e_2)| < \varepsilon\},$$

where $\varepsilon > 0$ and $f \in \mathcal{F}$. This uniformity not only is compatible with the weak topology $O(\mathcal{F})$ determined by $\mathcal{F}$, but also makes each element of $\mathcal{F}$uniformly continuous.

Now, given a uniform space $(X,\ \mathcal{U})$, $UC(X,\ I)$ denotes the family of uniformly continuous functions defined on $(X,\ \mathcal{U})$ with values in [0,1]. If $f \in UC(X,\ I)$ and $A \in CL(X)$, then $m_f(A) = \inf\{f(x)\ :\ x \in A\}$ is called the infimal value of f on A. For any $\alpha \in [0,1]$ we write $slv(f,\ \alpha) = \{x \in X\ :\ f(x) \leq \alpha\}$ for the sublevel set of f at height α. We may associate to every nonempty subfamily $\Delta \subset CL(X)$ a subfamily of $UC(X,\ I)$, namely $\mathcal{R}_\Delta = \{f \in UC(X,\ I)\ :$ whenever $\inf f < \alpha < \beta < \sup f,\ \exists S \in \sum(\Delta)$ with $slv(f,\ \alpha) \subset S \subset slv(f,\ \beta)\}$. The main result of [7] shows that if Δ is uniformly Urysohn family then σ_Δ is the weak topology determined by $\mathcal{R}^*_\Delta = \{m_f\ :\ f \in \mathcal{R}_\Delta\}$. Thus, $\mathcal{W}(\mathcal{R}^*_\Delta)$ is a compatible uniformity for σ_Δ.

We analyze the uniformity $\mathcal{W}(\mathcal{R}^*_\Delta)$ to create a better picture of $(CL(X),\ \sigma_\Delta)$. First, we need an introductory Lemma.

Theorem 5.5 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ be a family containing the singletons. If Δ is a uniformly Urysohn family, then i: $(X,\ \mathcal{U}) \rightarrow (CL(X),\ \mathcal{W}(\mathcal{R}^*_\Delta))$ defined by $i(x) = \{x\}$ is uniformly continuous.*

Proof: Let V be a subbasic element of $\mathcal{W}(\mathcal{R}^*_\Delta)$. So there are a function $f \in \mathcal{R}_\Delta$ and a positive ε such that

$$V = \{(A,\ B) \in CL(X) \times CL(X)\ :\ |m_f(A) - m_f(B)| < \varepsilon\}.$$

Since $f \in UC(X,\ I)$, there is $U \in \mathcal{U}$ with $|f(x) - f(y)| < \varepsilon$ for every $(x,\ y) \in U$. Thus for every $(x,\ y) \in U$, we have $(i(x),\ i(y)) \in V(|m_f(i(x)) - m_f(i(y))| = |f(x) - f(y)| < \varepsilon)$. Hence, the map i is uniformly continuous ([17]). ■

Remark 5.6 Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ be a uniformly Urysohn family containing the singletons. We can identify X with the set $\{\{x\}\ :\ x \in X\}$. Under this identification we can consider X as a subset of $(CL(X),\ \sigma_\Delta)$, since σ_Δ is an admissible topology. If we denote by $\mathcal{W}_X(\mathcal{R}^*_\Delta)$ the relative uniformity for X, then by Theorem 5.5 we have $\mathcal{W}_X(\mathcal{R}^*_\Delta) \subset \mathcal{U}$ and of course $\mathcal{W}_X(\mathcal{R}^*_\Delta)$ generates the orginal topology on X (induced by $\mathcal{U}$) since σ_Δ is an admissible topology.

If moreover σ_Δ is metrizable, Δ must satisfy the condition (2) from Theorem 5.4. Let Δ' be a countable subfamily of Δ described in (2) of Theorem 5.4. In the coincidence whith the proof of Theorem 3.10 in [7] put:

$$P = \left\{(S_1,\ S_2) \in \sum(\Delta') \times \sum(\Delta')\ :\ U[S_1] \subset S_2 \text{ for some } U \in \mathcal{U}\right\}.$$

Let $p = (S_1,\ S_2) \in P$. By Lemma 3.1 in [7] there is $f_p \in \mathcal{R}_\Delta$ such that $f_p(S_1) = 0$ and $f_p(clS_2^c) = 1$. Set $\mathcal{H}_\Delta = \{f_p \in \mathcal{R}_\Delta\ :\ p \in P\}$. Then $\mathcal{H}_\Delta$ is a countable subfamily of $\mathcal{R}_\Delta$ and it is shown in [7] that σ_Δ is the weak topology determined by $\mathcal{H}^*_\Delta = \{m_f\ :\ f \in \mathcal{H}_\Delta\}$. Thus $\mathcal{W}(\mathcal{H}^*_\Delta)$ is a compatible metrizable uniformity ([17]). Hence $\mathcal{W}_X(\mathcal{H}^*_\Delta)$ is a metrizable uniformity for X weaker than $\mathcal{U}$.

The following theorem shown that if $(X,\ \mathcal{U})$ is a uniform space and $\sigma(\sigma = \sigma(CL(X)))$ is metrizable, then $\mathcal{W}_X(\mathcal{H}^*_\Delta)$ is a metrizable uniformity compatible with $\mathcal{U}$. Thus if σ is a metrizable topology there is a totally bounded metric ϱ on X compatible with the uniformity $\mathcal{U}$ such that the proximal topology generated by $\varrho(\sigma_\varrho)$ and the proximal topology generated by $\mathcal{U}(\sigma_\mathcal{U})$ coincide. So, we have a complete and attractive solution to the metrization problem for the proximal topology σ.

Theorem 5.7 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space. The following are equivalent:*

(1) $(CL(X), \sigma)$ *is metrizable;*

(2) *there is a totally bounded metric ϱ on X compatible with $\mathcal{U}$.*

Proof: Only (1)⇒(2) needs proof since (2)⇒(1) is known ([3]).

(1)⇒(2) Put $\mathcal{H} = \mathcal{H}_{CL(X)}$ and $\mathcal{H}^* = \mathcal{H}^*_{CL(X)}$. From Remark 5.6 we know that $\mathcal{W}_X(\mathcal{H}^*)$ is a metrizable uniformity with $\mathcal{W}_X(\mathcal{H}^*) \subset \mathcal{U}$. We show that also the opposite inclusion $\mathcal{U} \subset \mathcal{W}_X(\mathcal{H}^*)$ holds.

Let $\{B_n\ :\ n \in Z^+\}$ be a countable base of $\mathcal{W}_X(\mathcal{H}^*)$. Without any loss of generality we may suppose that $B_{n+1} \subset B_n$ for every $n \in Z^+$. Suppose by contradiction that the inclusion $\mathcal{U} \subset \mathcal{W}_X(\mathcal{H}^*)$ fails. So there is a symmetric element $U \in \mathcal{U}$ with $B_n \not\subset U$ for every $n \in Z^+$. Let $(x_n,\ y_n) \in B_n \backslash U$ for every $n \in Z^+$ nad $V \in \mathcal{U}$ be a symmetric element with $V^4 \subset U$. By Efremovic Lemma there is an infinite set $J \subset Z^+$ such that $(x_p,\ y_q) \notin V$ for every $p,\ q \in J$ ([20]).

Set $C = \{x_n : n \in J\}$ and $D = \{y_n : n \in J\}$. We claim that D and C are closed. It suffices to show that they are sequentially closed ($\mathcal{W}_X(\mathcal{H}^*)$ and $\mathcal{U}$ induce the same topology on X and $\mathcal{W}_X(\mathcal{R}^*)$ is metrizable). Suppose that C is not sequentially closed and let $\bar{x} \in cl(C) \backslash C$. Let $\{x_{n_k} : n_k \in J'\}$ be a subsequence of C converging to $\bar{x}$, where J' is an infinite subset of J. Since $(x_{n_k},\ y_{n_k}) \in B_{n_k}$ for any $n_k \in J'$, it follows that also the subsequence $\{y_{n_k} : n_k \in J'\}$ converges to $\bar{x}$, a contradiction since C and D are far sets (infact $V[C] \cap D = \emptyset$). Similarly, also D ia a closed set.

By (2) of Theorem 5.4 and Remark 5.6 there is $p = (S_1,\ S_2) \in P$ such that $C \subset S_1$ and $D \subset cl(S_2^c)$. Thus $f_p(C) = 0$ and $f_p(D) = 1$. Since the countable base $\{B_n : n \in Z^+\}$ is a nested family, there exists $n_0 \in Z^+$ such that

$$B_n \subset \left\{(A,\ B) \in CL(X) \times CL(X)\ :\ |m_{f_p}(A) - m_{f_p}(B)| < \frac{1}{4}\right\} \cap X \times X$$

for every $n \geq n_0$. This is a contradiction, since for $k \in J$ we have $(x_k,\ y_k) \in B_k$ and $|m_{f_p}(\{x_k\}) - m_{f_p}(\{y_k\})| = 1$, frequently.

Finally, we observe that if we apply Lemma 4.2 to $CL(X)$ we derive that $\mathcal{U}$must be totally bounded. Thus also $\mathcal{W}_X(\mathcal{H}^*)$ is totally bounded. ■

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Authors:

G. Di Maio
Instituto di Matematica
Seconda Università degli Studi di Napoli
Piazza Duomo
81100 Caserta
Italia

Ľ. Holá
Institute of Mathematics
Academy of Sciences
Štefanikova 49
81473 Bratislava
Slovakia

E. Meccariello
Università degli Studi di Salerno
gemmata di Benevento
Facoltà di Ingegneria Corso Garibaldi
Palazzo Bosco
82100 Benevento
Italia

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M. Arnold

A note on the uniform perturbation index[1]

ABSTRACT. For a given differential-algebraic equation (DAE) the perturbation index gives a measure for the sensitivity of a solution w. r. t. small perturbations. If we consider, however, *classes* of DAEs (e. g. all DAEs that arise as semi-discretizations of a given partial DAE by the method of lines) then the error bound in the definition of the perturbation index may become arbitrarily large even if the perturbation index does not exceed 1. We illustrate this fact by 2 examples and define as alternative the *uniform* perturbation index that gives simultaneously error bounds for *all* DAEs of a given class. We prove that in one example each individual DAE has perturbation index 1 but the uniform perturbation index is 2. Another example illustrates that the class of all finite difference semi-discretizations may even have *no* uniform perturbation index if the given partial DAE has perturbation index 2.

KEY WORDS: ifferential-algebraic equations, perturbation index, Baumgarte stabilization, partial DAEs, method of lines

1 Introduction

One main difficulty in the numerical integration of initial value problems for higher index differential-algebraic equations (DAEs)

$$F(x'(t), x(t), t) = 0 \ , \quad x(0) = x_0 \ , \quad t \in [0, T] \tag{1}$$

is the fact, that the solution does not depend continuously on small perturbations in the equations. The discrete analogue is the amplification of small errors during the numerical integration. Such errors arise e. g. as round-off errors or because of stopping the iterative solution of nonlinear equations. A quantitative measure of this effect is given by the perturbation index

[1]This paper is an extended version of a talk presented at the conference "DAEs, Related Fields and Applications" Oberwolfach (Germany), November 1995.

Definition 1 ([8, p. 478f]) *The DAE (1) has* perturbation index m *along a solution $x(t)$ on $[0,T]$, if m is the smallest integer such that, for all functions $\hat{x}(t)$ having a defect $F(\hat{x}'(t),\hat{x}(t),t)=\delta(t)$ there exists on $[0,T]$ an estimate*

$$\|\hat{x}(t)-x(t)\| \le C_0(\|\hat{x}(0)-x(0)\| + \max_{\tau\in[0,t]}\|\delta(\tau)\| + \ldots + \max_{\tau\in[0,t]}\|\delta^{(m-1)}(\tau)\|) \tag{2}$$

whenever the expression on the right hand side is sufficiently small.

If appropriate discretization methods are used then the error that is caused by the amplification of perturbations in the *numerical* solution is bounded by $C_0^* \cdot \frac{1}{h^{m-1}}\Delta$. Here Δ denotes an upper bound for the errors that arise in one single step of integration, h is the stepsize of integration. This term can be interpreted as discretization of the error bound (2) for the analytical solution (see e. g. [7], [1]).

As long as the constants C_0 and C_0^* are of moderate size the sensitivity of the solution w. r. t. perturbations can be completely characterized by the integer m in (2), i. e. by the perturbation index. These constants C_0 and C_0^* depend in general on bounds for partial derivatives of F in a neighbourhood of the analytical solution $x(t)$, for many applications they are $\mathcal{O}(1)$, [8, p. 480f]. The situation changes if we consider a class of DAEs with a parameter that may be arbitrarily small. Such a class appears e. g. if a partial DAE ([4], [5]) is discretized in space by the method of lines. The resulting semi-discretized DAEs depend on the space discretization. If the space discretization is refined then the constant C_0 in (2) may become arbitrarily large. Thus for practical computations the error bound $C_0^* \cdot \frac{1}{h^{m-1}}\Delta$ does not give any useful information about the amplification of errors during integration.

In Section 2 we study this effect in detail for a Baumgarte-like stabilization of differential-algebraic systems of index 2. For large values of the Baumgarte coefficient α the error bound of Definition 1 is useless since $\lim_{\alpha\to\infty} C_0 = \infty$. That is why we introduce in Section 3 the *uniform perturbation index* for a class of DAEs. In Section 4 this concept is applied to a system of 2 linear partial differential equations [5, Example 1]. This partial DAE has index 2 but the semi-discretization by finite differences on an equidistant grid is a DAE of perturbation index 1. The main result of Section 4 is that for this example — depending on the coefficients of the partial DAE — either

- the uniform perturbation index of the class of all these semi-discretizations is 2 and coincides thus with the index of the underlying partial DAE or

- the class of all these semi-discretizations has no uniform perturbation index at all.

2 A perturbation analysis for stabilized differential-algebraic systems of index 2

In this we consider the differential-algebraic system

$$\left.\begin{array}{rcl} y'(t) & = & f(y(t),z(t)) \\ 0 & = & g(y(t)) \end{array}\right\} \quad , \quad t \in [0,T] \quad , \quad y(0) = y_0 \quad , \quad z(0) = z_0 \tag{3}$$

that is supposed to have a solution $y : [0,T] \to \mathbb{R}^{n_y}$, $z : [0,T] \to \mathbb{R}^{n_z}$. We assume that in a neighbourhood of this solution functions f and g are sufficiently differentiable and satisfy the index-2 condition "$[g_y f_z](\eta,\zeta)$ non-singular".

The differential-algebraic system (3) has (perturbation and differential) index 2 [8, p. 480f]: We have

$$\begin{aligned} \|\hat{y}(t) - y(t)\| + &\|\hat{z}(t) - z(t)\| \leq \\ &\leq C_0(\|\hat{y}(0) - y(0)\| + \max_{\tau\in[0,t]} \|\delta(\tau)\| + \max_{\tau\in[0,t]} \|\theta(\tau)\| + \max_{\tau\in[0,t]} \|\theta'(\tau)\|) \end{aligned} \tag{4}$$

for all $t \in [0,T]$ if $\hat{y}'(t) = f(\hat{y},\hat{z}) + \delta(t)$ and $g(\hat{y}(t)) = \theta(t)$. Here the constant C_0 depends on the length T of the time interval and on upper bounds for $\|[(g_y f_z)^{-1}](\eta,\zeta)\|$ and for partial derivatives of f and g.

Similar to the stabilization of model equations for constrained mechanical systems that was introduced by Baumgarte ([3]) the index of (3) can be reduced to 1 if the algebraic constraints $g(y) = 0$ are substituted by

$$0 = \frac{1}{\alpha}\frac{d}{dt} g(y(t)) + g(y(t)) \tag{5}$$

with a constant $\alpha > 0$. With this substitution the analytical solution of (3) remains unchanged since $g(y(t)) = 0$ implies $\frac{d}{dt} g(y(t)) = 0$ and thus also (5). On the other hand consistent initial values for (3) satisfy $g(y(0)) = 0$ such that (5) results in $g(y(t)) = 0$, $(t \in [0,T])$.

As for (3) we study the sensitivity of the solution of the stabilized system w. r. t. small perturbations comparing $(y(t),z(t))$ with functions $(\hat{y}_\alpha(t),\hat{z}_\alpha(t))$ that satisfy for $t \in [0,T]$

$$\left.\begin{array}{rcl} \hat{y}_\alpha'(t) & = & f(\hat{y}_\alpha(t),\hat{z}_\alpha(t)) + \delta(t) \\ \theta(t) & = & \dfrac{1}{\alpha}\dfrac{d}{dt} g(\hat{y}_\alpha(t)) + g(\hat{y}_\alpha(t)) \end{array}\right\} \tag{6}$$

The key to these error bounds are the following estimates:

Lemma 1 *a) Let functions* $\tilde{\delta} \in \mathcal{C}[0,T]$, $\tilde{\theta} \in \mathcal{C}^1[0,T]$ *and a constant* $\alpha > 0$ *be given. The solutions of the linear differential equation*

$$w'(t) + \alpha w(t) = \tilde{\delta}(t) + \alpha\tilde{\theta}(t) \tag{7}$$

satisfy

$$\begin{aligned} |w(t) - \tilde{\theta}(t)| &\leq |w(0) - \tilde{\theta}(0)|e^{-\alpha t} + \frac{1}{\alpha} \cdot \max_{\tau\in[0,t]} (|\tilde{\delta}(\tau)| + |\tilde{\theta}'(\tau)|) \ , \\ |w'(t)| &\leq |w'(0)|e^{-\alpha t} + \max_{\tau\in[0,t]} (|\tilde{\delta}(\tau)| + |\tilde{\theta}'(\tau)|) \ . \end{aligned}$$

b) If $\tilde{\delta}(t) \equiv 0$, $\tilde{\theta}(t) = \Theta\cos\frac{t}{\varepsilon}$ *and* $w(0) = \varepsilon^2\alpha^2\Theta/(1+\varepsilon^2\alpha^2)$ *with (small) positive parameters* Θ*,* ε *then the solution* $w(t)$ *of (7) is*

$$w_\alpha(t) = (1 - \frac{1}{1+\varepsilon^2\alpha^2})\,\Theta\cos\frac{t}{\varepsilon} + \frac{\varepsilon\alpha}{1+\varepsilon^2\alpha^2}\,\Theta\sin\frac{t}{\varepsilon} \ . \tag{8}$$

Proof: The solution of (7) is given by

$$w(t) = w(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-\tau)}(\tilde{\delta}(\tau) + \alpha\tilde{\theta}(\tau))\,d\tau \ . \tag{9}$$

Integration by parts results in

$$\alpha\int_0^t e^{-\alpha(t-\tau)}\tilde{\theta}(\tau)\,d\tau = \left[e^{-\alpha(t-\tau)}\tilde{\theta}(\tau)\right]_0^t - \int_0^t e^{-\alpha(t-\tau)}\tilde{\theta}'(\tau)\,d\tau$$

and finally we have

$\int_0^t e^{-\alpha(t-\tau)}\,d\tau = \frac{1}{\alpha}(1-e^{-\alpha t}) < \frac{1}{\alpha}$,

i. e.

$$|\int_0^t e^{-\alpha(t-\tau)}(\tilde{\delta}(\tau) - \tilde{\theta}'(\tau))\,d\tau\,| < \frac{1}{\alpha}\cdot\max_{\tau\in[0,t]}(|\tilde{\delta}(\tau)| + |\tilde{\theta}'(\tau)|) \ .$$

The estimate for $|w'(t)|$ is obtained from $w'(t) = \tilde{\delta}(t) - \alpha(w(t) - \tilde{\theta}(t))$. To prove part b) of the lemma the given functions $\tilde{\delta}$, $\tilde{\theta}$ are inserted into (9). ■

We now return to Eqs. (6). Because of

$$\frac{d}{dt}g(y(t)) = g_y(y(t))y'(t) = [g_y f](y(t), z(t))$$

Eqs. (5) can be solved w. r. t. the algebraic components z and the stabilized system is of (differential and perturbation) index 1 ([8, p. 480]), we get

$$\begin{aligned}&\|\hat{y}_\alpha(t) - y(t)\| + \|\hat{z}_\alpha(t) - z(t)\| \leq \\ &\qquad \leq C_{0,\alpha}(\|\hat{y}_\alpha(0) - y(0)\| + \|\hat{z}_\alpha(0) - z(0)\| + \max_{\tau\in[0,t]} \|\delta(\tau)\| + \max_{\tau\in[0,t]} \|\theta(\tau)\|) \ .\end{aligned} \tag{10}$$

In general, however, this estimate can be satisfied for large values of α only, if $C_{0,\alpha} \to \infty$, ($\alpha \to \infty$). This is not surprising since (5) approximates for large values of α the algebraic constraint $g(y) = 0$ of the index-2 system (3) and there is (per definitionem) no estimate like (10) for systems of perturbation index 2.

Example 1 The system $y_1' = y_2' = z$, $0 = y_1 + y_2$ is of index 2. The solution of the initial value problem $y_1(0) = y_{1,0}$ is constant:

$$0 = y_1 + y_2 \ \Rightarrow \ 0 = y_1' + y_2' = 2z \ \Rightarrow \ z(t) \equiv 0 \ , \ y_1(t) \equiv y_{1,0} \ , \ y_2(t) \equiv -y_{1,0} \ .$$

Consider now functions $\hat{y}_\alpha$, $\hat{z}_\alpha$ that are defined by

$$\hat{y}_{\alpha,1}(t) = y_{1,0} + \frac{1}{2}(w_\alpha(t) - w_\alpha(0)) \ , \ \ \hat{y}_{\alpha,2}(t) = w_\alpha(t) - \hat{y}_{\alpha,1}(t) \ , \ \ \hat{z}_\alpha(t) = \frac{1}{2}w_\alpha'(t)$$

with $w_\alpha(t)$ from (8). These functions satisfy $g(\hat{y}_\alpha(t)) = \hat{y}_{\alpha,1}(t) + \hat{y}_{\alpha,2}(t) = w_\alpha(t)$ and we get in (6) $\delta(t) \equiv 0$, $\theta(t) = \Theta \cos \frac{t}{\varepsilon}$ (see Lemma 1), i. e. $\|\delta(t)\| = 0$, $\|\theta(t)\| = \mathcal{O}(\Theta)$, $\|\theta'(t)\| = \mathcal{O}(\frac{1}{\varepsilon}\Theta)$. Straightforward computations give

$$|\hat{z}_\alpha(t) - z(t)| = \frac{1}{2}|w_\alpha'(t)| = \frac{1}{2}| - \frac{\varepsilon\alpha^2}{1+\varepsilon^2\alpha^2}\sin\frac{t}{\varepsilon} + \frac{\alpha}{1+\varepsilon^2\alpha^2}\cos\frac{t}{\varepsilon}| \cdot \Theta$$

and for $\alpha \geq 1$ the constant $C_{0,\alpha}$ in (10) has to satisfy $C_{0,\alpha} \geq \frac{1}{12}\sqrt{\alpha}$ since the special choice $\varepsilon = \frac{1}{\sqrt{\alpha}}$ results in $\hat{y}_{\alpha,1}(0) = y_1(0)$, $|\hat{y}_{\alpha,2}(0) - y_2(0)| = |w_\alpha(0)| = \frac{\alpha}{1+\alpha}\Theta \leq \Theta$, $|\hat{z}_\alpha(0) - z(0)| = \frac{1}{2}\frac{\alpha}{1+\alpha}\Theta \leq \Theta$, $|\theta(t)| \leq \Theta$ and $|\hat{z}_\alpha(\frac{\pi}{2}\varepsilon) - z(\frac{\pi}{2}\varepsilon)| = \frac{1}{2}\frac{\alpha^{3/2}}{1+\alpha}\Theta \geq \frac{1}{4}\sqrt{\alpha}\Theta$.

I. e. standard perturbation index theory gives with (10) an error estimate for the stabilized system that grows rapidly for $\alpha \to \infty$. If $\max_{\tau\in[0,t]} \|\theta'(\tau)\|$ is of moderate size and $\alpha \gg 1$ then (10) overestimates the influence of small perturbations on $(y(t), z(t))$ substantially, (see Example 2).

It is known from the literature (e. g. [2]) that neither differential nor standard perturbation index is an appropriate measure for the difficulties that one has to expect in the numerical solution of Baumgarte-like stabilized differential-algebraic systems with large Baumgarte coefficients. Baumgarte stabilization reduces the index (in our example from 2 to 1) but because of boundary layers (see the terms $\ldots e^{-\alpha t}$ in Lemma 1) the numerical solution of the index-reduced system might be even more complicated than that of the original higher index system if the Baumgarte coefficients are large. Furthermore for large Baumgarte coefficients the index-reduced system is less robust against perturbations than the (low) perturbation index suggests.

3 The uniform perturbation index

The results of Section 2 motivate the extension of the perturbation analysis to classes of DAEs

$$F_\alpha(x'(t), x(t), t) = 0\ , \quad x(0) = x_0\ , \quad t \in [0, T] \tag{11}$$

where $\alpha \in M_\alpha$ denotes some free parameter. In this note we restrict ourselves to scalar parameters α, the dimension of x (and thus also the norm $\|\cdot\|$ in (12)) may vary with α (see Section 4).

Definition 2 *The class of DAEs (11) has* uniform perturbation index m *along solutions $x_\alpha(t)$ on $[0, T]$, if $m \geq 1$ is the smallest integer such that, for all $\alpha \in M_\alpha$ and for all functions $\hat{x}_\alpha(t)$ having a defect $F_\alpha(\hat{x}'_\alpha(t), \hat{x}_\alpha(t), t) = \delta(t)$ there exists on $[0, T]$ an estimate*

$$\|\hat{x}_\alpha(t) - x_\alpha(t)\| \leq C_0(\|\hat{x}_\alpha(0) - x_\alpha(0)\| + \max_{\tau\in[0,t]} \|\delta(\tau)\| + \ldots + \max_{\tau\in[0,t]} \|\delta^{(m-1)}(\tau)\|) \tag{12}$$

whenever the expression on the right hand side is sufficiently small. Here C_0 denotes a constant that is independent of α and $\delta(\tau)$.

Remarks 1 a) The condition $m \geq 1$ in Definition 2 can be relaxed to $m \geq 0$ if for $m = 0$ the term $\delta^{(m-1)}(\tau)$ is interpreted as $\int_0^\tau \delta(w)\,dw$ (cf. [8, p. 479]). I. e. the class of DAEs (11) has the uniform perturbation index $m = 0$ if instead of (12) the (stronger) estimate

$$\|\hat{x}_\alpha(t) - x_\alpha(t)\| \leq C_0(\|\hat{x}_\alpha(0) - x_\alpha(0)\| + \max_{\tau\in[0,t]} \|\int_0^\tau \delta(w)\,dw\|\,)$$

is satisfied.

b) The idea of error bounds that are independent of a (small) parameter is extensively used in the analysis of singular perturbation problems. As an example we refer to the work of Hairer et al. [6] and Lubich [9] who investigate in close connection to DAE-theory singularly perturbed ordinary differential equations (ODEs) that all have perturbation index 0 ([8, p. 479]). In terms of Definition 2 the class of singularly perturbed ODEs in [6] has uniform perturbation index 1. The class of singular singularly perturbed ODEs that is considered in [9] has even uniform perturbation index 3, i. e. the uniform perturbation index exceeds the classical one by 3.

c) Mattheij [10] and Wijckmans [12] analyse linear DAEs that are "close to a higher-index DAE" ([12, pp. 53ff, 73ff]) and study the sensitivity of the solution w. r. t. small perturbations. The present paper is closely related to their approach and uses with Lemma 1 the same basic tool in the proof of uniform error estimates. The extension of the well established concept of perturbation index from individual DAEs to classes of DAEs gives a unified framework for various case studies from the literature.

Example 2 Consider the class of all Baumgarte-like stabilized differential-algebraic systems of index 2 with parameter $\alpha \geq \alpha_0 > 0$ that was introduced in Section 2. If the parameter α is fixed then these stabilized systems have the classical perturbation index 1. If we consider, however, the class of all these systems then estimate (12) can not be satisfied with $m = 1$ and a constant C_0 that is independent of α (see Example 1).
Applying componentwise Lemma 1 to $\frac{d}{dt} g(\hat{y}_\alpha(t)) + \alpha g(\hat{y}_\alpha(t)) = \alpha\theta(t)$

$$\frac{d}{dt} g(\hat{y}_\alpha(t)) + \alpha g(\hat{y}_\alpha(t)) = \alpha\theta(t)$$

we get $g(\hat{y}_\alpha(t)) = \hat{\theta}(t)$ with

$$\begin{aligned} \|\hat{\theta}(t) - \theta(t)\| &\leq \|g(\hat{y}_\alpha(0)) - \theta(0)\| \cdot e^{-\alpha t} + \frac{1}{\alpha} \cdot \max_{\tau \in [0,t]} \|\theta'(\tau)\| \ , \\ \|\hat{\theta}'(t)\| &\leq \|\frac{d}{dt} g(\hat{y}_\alpha(t))\Big|_{t=0}\| \cdot e^{-\alpha t} + \max_{\tau \in [0,t]} \|\theta'(\tau)\| \ , \end{aligned}$$

Because of $\frac{d}{dt} g(\hat{y}_\alpha(t)) = g_y(\hat{y}_\alpha(t))\hat{y}'_\alpha(t) = [g_y f](\hat{y}_\alpha(t), \hat{z}_\alpha(t)) + g_y(\hat{y}_\alpha(t))\delta(t)$ and $g(y(0)) = [g_y f](y(0), z(0)) = 0$ we have

$$\begin{aligned} \hat{y}'_\alpha(t) &= f(\hat{y}_\alpha, \hat{z}_\alpha) + \delta(t) \\ \hat{\theta}(t) &= g(\hat{y}_\alpha(t)) \end{aligned}$$

with

$$\begin{aligned} \|\hat{\theta}(t)\| &\leq \|\theta(t)\| + \frac{1}{\alpha_0} \max_{\tau \in [0,t]} \|\theta'(\tau)\| + \mathcal{O}(1)(\|\hat{y}_\alpha(0) - y(0)\| + \|\theta(0)\|) \\ \|\hat{\theta}'(t)\| &\leq \max_{\tau \in [0,t]} \|\theta'(\tau)\| + \mathcal{O}(1)(\|\hat{y}_\alpha(0) - y(0)\| + \|\hat{z}_\alpha(0) - z(0)\| + \|\theta'(0)\| + \|\delta(0)\|) \ , \end{aligned}$$

(the constants in the $\mathcal{O}(.)$–terms are independent of α). Following the lines of standard perturbation index theory (see (4)) estimate (12) with $m = 2$ is proved. I. e., the class of all Baumgarte-like stabilized differential-algebraic systems of index 2 with parameter $\alpha \geq \alpha_0 > 0$ has uniform perturbation index 2.

Remarks 2 a) The uniform perturbation index remains unchanged if the class of DAEs that is considered in Example 2 is extended by the index-2 system (3), i. e. by the limit case $\alpha \to \infty$.

b) The discrete analogue of the uniform error bound in Example 2 is an error bound $C^*_{0,\infty} \cdot \frac{1}{h}\Delta$ with a constant $C^*_{0,\infty}$ that is independent of α. I. e., if appropriate discretization methods are used then the amplification of small errors Δ during integration is bounded by $\min(C^*_{0,\alpha}\Delta, C^*_{0,\infty}\cdot\frac{1}{h}\Delta)$ with $\lim_{\alpha\to\infty} C^*_{0,\alpha} = \infty$ and $C^*_{0,\infty} = \mathcal{O}(1)$. For large values of α and stepsizes h of moderate size the uniform error bound $C^*_{0,\infty}\cdot\frac{1}{h}\Delta$ is substantially smaller than the error bound $C^*_{0,\alpha}\Delta$ from standard perturbation index theory.

c) If the class of DAEs in Example 2 is restricted to systems with $\alpha < \overline{\alpha}$ and a fixed $\overline{\alpha}$ then the uniform perturbation index of the class is 1 since (12) with $m = 1$ can be proved with $C_0 = C_{0,\overline{\alpha}}$.

d) The analysis for the index-2 case is straightforwardly extended to prove that the classical Baumgarte stabilization for constrained mechanical systems ([2]) results in a class of index-1 DAEs that has uniform perturbation index 3.

4 Semidiscretizations of partial DAEs – a case study

Uniform error bounds found our special interest since recently partial DAEs and its semi-discretizations have been considered (e. g. [4]). With one example we illustrate in this that uniform error estimates in the sense of Definition 2 usually describe correctly the sensitivity of semi-discretized DAEs w. r. t. small perturbations.

Example 3 [5, Example 1]] Consider the system of 2 linear partial differential equations

$$\begin{aligned} u_t - \frac{1}{4}v_{xx} + \varrho v &= f^u(x,t) \\ -\frac{1}{4}u_{xx} + \frac{1}{4}v_{xx} + v &= f^v(x,t) \end{aligned} \tag{13}$$

for $0 \le x \le L$, $0 \le t \le T$ with initial conditions

$$u(x,0) = g^u(x)\ ,\ \ v(x,0) = g^v(x)\ ,\ \ (0 \le x \le L)$$

and homogenous Dirichlet boundary conditions, $f := (f^u, f^v)^T$, $g := (g^u, g^v)^T$. $\varrho \in \mathbb{R}$ denotes some (fixed) parameter, we will consider the cases $\varrho = 0$ and $\varrho = -2$ in detail. We suppose that functions f and g are sufficiently differentiable and that g satisfies the boundary conditions. Furthermore we suppose that u, v, f and g have series expansions

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x) u_n(t)\ ,\ \ v(x,t) = \sum_{n=1}^{\infty} \phi_n(x) v_n(t)\ ,\ \ \ldots$$

with $\phi_n(x) := \sin(\frac{n\pi x}{L})$. Under suitable smoothness assumptions the coefficients $u_n(t)$, $v_n(t)$, $(n \ge 1)$ are the solutions of the initial value problems

$$u_n(0) = g^u_n\ ,\ \ v_n(0) = g^v_n$$

for the linear constant-coefficient DAE

$$\begin{aligned} u_n'(t) + (\varrho + \frac{1}{4}\lambda_n^2) v_n(t) &= f_n^u(t) \\ \frac{1}{4}\lambda_n^2 u_n(t) + (1 - \frac{1}{4}\lambda_n^2) v_n(t) &= f_n^v(t) \end{aligned} \tag{14}$$

with $\lambda_n := \frac{n\pi}{L}$.
Consider now the discretization by finite differences on an equidistant grid $0 = x_0 < x_1 < \,< \ldots < x_N < x_{N+1} = L$, $x_i := ih$, $h = L/(N+1)$. Let $U(t) = (u_1^N(t), \ldots, u_N^N(t))^T$, $V(t) = (v_1^N(t), \ldots, v_N^N(t))^T$

$$U(t) = (u_1^N(t), \ldots, u_N^N(t))^T \ , \ \ V(t) = (v_1^N(t), \ldots, v_N^N(t))^T$$

with $u_i^N(t) \approx u(x_i, t)$, $v_i^N(t) \approx v(x_i, t)$, $(i = 1, \ldots, N)$ and

$$F^u(t) = (f^u(x_1, t), \ldots, f^u(x_N, t))^T \ , \ \ F^v(t) = (f^v(x_1, t), \ldots, f^v(x_N, t))^T \ ,$$

$$G^u = (g^u(x_1), \ldots, g^u(x_N))^T \ , \ \ G^v = (g^v(x_1), \ldots, g^v(x_N))^T \ .$$

The finite difference approximation satisfies $U(0) = G^u$, $V(0) = G^v$,

$$\begin{aligned} U'(t) - \frac{1}{4}A_h \cdot V(t) + \varrho \cdot V(t) &= F^u(t) \\ -\frac{1}{4}A_h \cdot U(t) + \frac{1}{4}A_h \cdot V(t) + V(t) &= F^v(t) \end{aligned} \tag{15}$$

with the symmetric tridiagonal matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & & & 0 \\ 1 & -2 & 1 & & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N \times N}$$

that has eigenvalues $\mu_i = -\frac{4}{h^2}\sin^2(\frac{i\pi}{2(N+1)})$ and eigenvectors

$$\Phi_i = (\,\sin(\frac{i\pi}{N+1})\,, \sin(\frac{2i\pi}{N+1})\,, \ldots, \sin(\frac{Ni\pi}{N+1})\,)^T \ , \ \ (i = 1, \ldots, N) \ .$$

We rewrite vectors $U(t)$, $V(t)$, $F^u(t)$, $F^v(t)$, G^u, G^v as linear-combinations of eigenvectors of A_h:

$$U(t) = \sum_{i=1}^{N} \frac{1}{\|\Phi_i\|_2}\Phi_i \cdot U_i(t) \quad , \quad V(t) = \sum_{i=1}^{N} \frac{1}{\|\Phi_i\|_2}\Phi_i \cdot V_i(t) \quad , \quad \ldots$$

Multiplying the equations (15) subsequently by $\Phi_1^T,\ \Phi_2^T,\ \dots,\ \Phi_N^T$ we get the equivalent system of equations

$$\begin{array}{rcl} U_i'(t) + (\varrho + \frac{1}{4}\Lambda_i^2)V_i(t) & = & F_i^u(t) \\ \frac{1}{4}\Lambda_i^2 U_i(t) + (1 - \frac{1}{4}\Lambda_i^2)V_i(t) & = & F_i^v(t) \end{array} \tag{16}$$

with $\Lambda_i := \sqrt{-\mu_i} = \frac{2}{h}\sin(\frac{i\pi}{2(N+1)})$, $(i = 1, \dots, N)$ since $A_h\Phi_i = \mu_i\Phi_i = -\Lambda_i^2\Phi_i$ and

$$\Phi_j^T\Phi_i = \begin{cases} \|\Phi_i\|_2^2 & \text{if } i = j\ , \\ 0 & \text{if } i \neq j\ . \end{cases}$$

This change of coordinates has no influence on the sensitivity of the solution w. r. t. small perturbations. For the perturbation analysis we prefer Eqs. (16) since they are quite similar to the corresponding equations (14) for the coefficients $u_n(t)$ of the solution of the partial DAE (13). If $i \le N$ is fixed then we get

$$\lim_{N\to\infty} \Lambda_i = \lambda_i = \frac{i\pi}{L}$$

and in the limit case $N \to \infty$ Eqs. (16) are transferred to (14).
For $\varrho = 0$ this analysis is carried out in [5]. They observe that (14) has (differential and perturbation) index 1 if $\lambda_n^2 \neq 4$ and index 2 if $\lambda_n^2 = 4$. Up to now there is no widely accepted index concept for partial DAEs (see [4] for a comprehensive study of this subject). But for the special example (13) it seems to be natural to call (13) a partial DAE of index 1 if the DAEs (14) have index 1 for *all* $n \in \mathbb{N}$ and a partial DAE of index 2 if there is one $n \in \mathbb{N}$ such that (14) has index 2 ([4, Example 2]). Since λ_n depends on L the index of the partial DAE varies with the length L of the domain.

If $\varrho \in \{0, -2\}$ and L is fixed then the finite difference approximation (16) has always index 1 if the discretization is sufficiently fine (i. e. h is sufficiently small): this follows in the case $\lambda_i^2 \neq 4$ from $\lim_{N\to\infty} \Lambda_i = \lambda_i$ and in the case $\lambda_i^2 = 4$ from $\Lambda_i \neq \lambda_i$. In this sense “the method of lines approximation … acts like a regularization” ([5]) if the partial DAE (13) has index 2. However, in view of the results of Section 2 we do not expect that the class of *all* semidiscretizations (15) has uniform perturbation index 1 if the partial DAE (13) has index 2, i. e. if there is an $n_0 \in \mathbb{N}$ with $\lambda_{n_0}^2 = 4$.

Therefore the most interesting case is given by DAEs (14) with $\lambda_n^2 \approx 4$ and $\lambda_n^2 \neq 4$. These problems can be interpreted as perturbations of an index-2 DAE (Eqs. (14) with $\lambda_n^2 = 4$), they were studied in great detail by Söderlind ([11]). He proved that the stability of the lower index system (i. e. (14) with $0 < |\lambda_n^2 - 4| \ll 1$) depends strongly on the sign of the perturbation. To analyse this phenomenon we solve the second equation in (14) w. r. t. $v_n(t)$, insert this expression into the first one and get (if $\varrho \in \{0, -2\}$)

$$\begin{aligned} u_n'(t) + \alpha u_n(t) &= f_n^u(t) + \alpha \cdot \frac{1}{\frac{1}{4}\lambda_n^2} f_n^v(t) \\ v_n(t) &= \frac{1}{\varrho + \frac{1}{4}\lambda_n^2}\left(f_n^u(t) - u_n'(t)\right) \end{aligned} \tag{17}$$

with

$$\alpha := -\frac{1}{4}\lambda_n^2 \frac{\varrho + \frac{1}{4}\lambda_n^2}{1 - \frac{1}{4}\lambda_n^2} \quad .$$

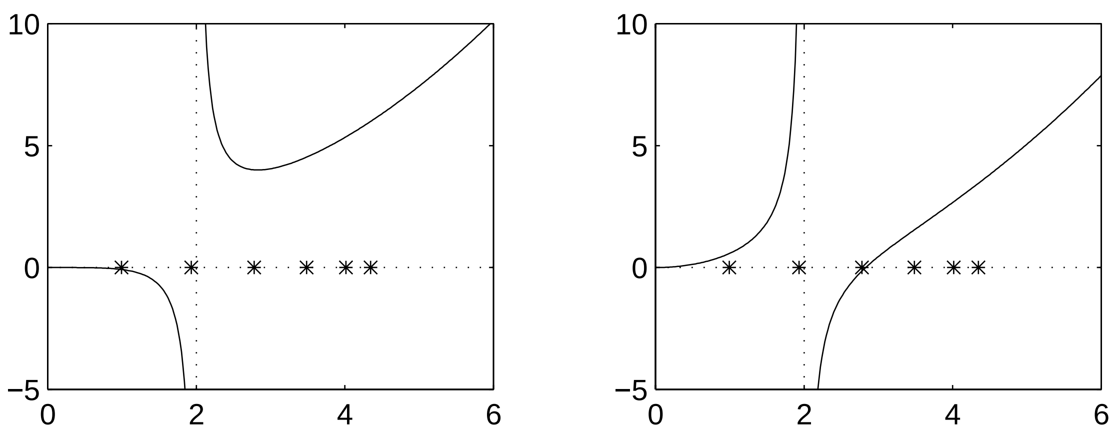

Figure 1: Coefficient α in (17) vs. λ_n for two values of ϱ. The asterisks at the abscissa mark the eigenvalues $\Lambda_1, \Lambda_2, \ldots, \Lambda_6$ of the semi-discretized problem (15) with $N = 6$ and $L = \pi$.

The way in that errors are propagated in (17) is determined by α. If $\alpha > 0$ then these equations have the same basic structure as the Baumgarte-like stabilized systems of Section 2 ($u_n(t) \to g(y(t))$, $v_n(t) \to z(t)$), the perturbation analysis of Sections 2 and 3 can be carried over straightforwardly. If, however, $\alpha < 0$ then errors may grow like $e^{-\alpha t}$. This is still acceptable if $0 \geq \alpha \geq -\alpha_0$ with a positive constant α_0 of moderate size (i. e. $e^{-\alpha t} \leq e^{\alpha_0 t}$), but if $\lambda_n^2 \to 4$ then α may become arbitrarily small. Fig. 1 shows α vs. λ_n in the two cases $\varrho = 0$ (left) and $\varrho = -2$ (right). Depending on ϱ we get the following results:

Lemma 2 *Let a positive constant* $\Delta_\lambda \in (0, 1]$ *be given.*

a) If $\varrho \in \{0, -2\}$ *then the class of all DAEs* (14) *with* $|\lambda_n^2 - 4| \geq \Delta_\lambda > 0$ *has uniform perturbation index 1.*

b) If $\varrho = 0$ *then the class of all DAEs* (14) *with* $\lambda_n^2 \in (0, 4 - \Delta_\lambda] \cup [4, \infty)$ *has uniform perturbation index 2.*

c) If $\varrho = -2$ *then the class of all DAEs* (14) *with* $\lambda_n^2 \in (0, 4] \cup [4 + \Delta_\lambda, \infty)$ *has uniform* *perturbation index 2.*

d) Neither for $\varrho = 0$ *nor for* $\varrho = -2$ *the class of all DAEs* (14) *(with arbitrary* λ_n^2*) has a uniform perturbation index.*

Proof: If $|\lambda_n^2 - 4| \geq \Delta_\lambda$ then $\alpha \geq -\alpha_0$ with a constant α_0 that is independent of λ_n but depends on Δ_λ. Following standard perturbation index theory estimate (2) with $m = 1$ is proved ($C_0 = \mathcal{O}(e^{\alpha_0 t})$ and $\lim_{\Delta_\lambda \to 0} C_0 = \infty$). In the stripe $\{ \lambda_n : |\lambda_n^2 - 4| \leq \Delta_\lambda \}$ the way in that errors are propagated depends on the sign of α (and thus on ϱ, see Fig. 1): if $\alpha > 0$ the results of Sections 2 and 3 can be applied to prove the uniform error estimate (12) with $m = 2$, in the case $\alpha < 0$ there is no estimate (12) at all since $C_0 = \mathcal{O}(e^{-\alpha t})$ and $\alpha \to -\infty$. ■

Söderlind [11] points out that DAEs of the form (14) with $\lambda_n^2 = 4$ are not isolated higher-index problems that are difficult to solve numerically but these problems separate a class of index-1 DAEs that are in the limit case $\lambda_n^2 \to 4$ very similar to index-2 DAEs from another class of index-1 DAEs that exhibit an essential instability. This statement is carried over straightforwardly to partial DAEs (13) if the length L of the domain is such that the index of the partial DAE is 2.

Remarks 3 a) For the partial DAE (13) we consider the set of *all* DAEs (14) and for the semi-discretized DAEs the set of *all* DAEs (16), the dimension of $U(t)$, $V(t)$ varies with N. Uniform error estimates make sense only, if the norms in (12) are compatible for varying N. Throughout this we use the $\mathcal{L}^2$–norm on $[0, L]$ in the partial DAE case and the discrete analogue $\|U\|_{2,N} := \left(\frac{1}{N} \sum_{i=1}^N (U_i^N)^2 \right)^{1/2}$ for the semi-discretized DAEs, i. e.

$$\|U(t)\| = \left(\frac{1}{N} \sum_{i=1}^N U_i^2(t) \right)^{1/2} \quad , \quad \|V(t)\| = \left(\frac{1}{N} \sum_{i=1}^N V_i^2(t) \right)^{1/2} \quad .$$

b) Because of $\lim_{N\to\infty} \Lambda_i = \lambda_i$ Lemma 2a proves that for a given partial DAE (13) of index 1 with $\varrho \in \{0, -2\}$ the class of all (sufficiently fine) finite difference approximations (15) has uniform perturbation index 1.

c) If the partial DAE (13) has index 2 (i. e. $\frac{2L}{\pi} \in \mathbb{N}$) then the class of all (sufficiently fine) finite difference approximations (15) has either uniform perturbation index 2 (if $\varrho = -2$, see Lemma 2c) or no uniform perturbation index at all (if $\varrho = 0$, see Lemma 2d). This follows from $\lim_{N\to\infty} \Lambda_i = \lambda_i$ and $\Lambda_i < \lambda_i$, $(i = 1, \ldots, N)$, see also the asterisks for $N = 6$ in Fig. 1. If $\varrho = 0$ and $\lambda_n^2 = 4$ then errors in (16) may be amplified by $\exp(3(\frac{N+1}{L})^2 t)$ since $\Lambda_n^2 = (\frac{2}{h} \sin h)^2 = 4(1 - \frac{1}{3}h^2) + \mathcal{O}(h^4)$.

5 Summary

Classes of DAEs may consist of DAEs that all have perturbation index 1 but a (in some sense well-defined) limit is of higher index. We illustrated with 2 examples that the numerical solution of such *index-1* DAEs may cause problems that are typical of *higher index* DAEs. Furthermore, the error bounds from standard perturbation index theory do not give useful information about the sensitivity of the solution w. r. t. perturbations. The uniform perturbation index describes for DAEs close to the higher index DAE the influence of perturbations on the solution correctly.

In the case of partial DAEs with an index that depends on the domain the existence of a uniform perturbation index can not be guaranteed. In one example the analysis of an index-2 partial DAE results in an (ordinary) index-2 DAE that separates a class of DAEs with uniform perturbation index 2 from a class of DAEs that has no uniform perturbation index at all. The same phenomenon is found analysing the sensitivity of the solutions of the semi-discretized systems w. r. t. small perturbations.

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Author:

M. Arnold
Institute of Robotics and System Dynamics
DLR Research Center Oberpfaffenhofen
Postfach 1116
D – 82230 Weßling
Germany

email: martin.arnold@dlr.de

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G. I. Balikov

Notes on the continuous linear extension operator and the basis in some DFS–spaces of ultradifferentiable functions on interval[1]

ABSTRACT. We consider relation between a basis and continuous linear extension operator of the functions from outside a interval.

This work considers DFS–spaces[2] of ultradifferentiable functions on interval. Unlike the results of R.Meise's and A.Taylor's works [8, 9], we give a method for concretely construction of continuous linear extension operator. We also considered the relations between a basis and a continuous linear extension operator of the functions outside an interval.

1 Introduction

Let $M = \{M_{pk}\}_{p,k=0}^{\infty}$ be a matrix of positives numbers satisfying the following conditions:

$$1 \leq M_{pk} \leq M_{pk+1} \quad \text{for any} \quad p, k = 0, 1, \dots \tag{1}$$

$$M_{pk}^2 \leq M_{pk-1} M_{pk+1} \qquad \text{for} \qquad p = 0, 1, \dots ; k = 1, 2, \dots \tag{2}$$

For any p there exists q such that

$$\sup_k (k^k M_{pk}/M_{qk}) < +\infty. \tag{3}$$

For any p there exists q such that

$$\sup_k (M_{p2k}/M_{qk}) < +\infty. \tag{4}$$

An example of a matrix, satisfying the conditions (1–4) is $M = \{k^{pk}\}_{p,k=0}^{\infty}$. We shall use the following spaces, connected with the matrix M :
Let K be compact set regular according [6] (further we consider only such compact sets),

[1]Partially supported by BNSF,contract MM409/94

[2]An inductive limit of F spaces with compact embedding operator.

contained in R, and p is a non-negative number.
Let

$$E\{p,K,M\} = \{f \in C^\infty(K), \|f\|_p = \sup_{K,s} |f^{(s)}(t)|/M_{ps} < +\infty\}. \tag{5}$$

It is easy to see, that the space $E\{p,K,M\}$ is a B–space and it is known that for $p<q$, the space $E\{p,K,M\}$ is embedded in the space $E\{q,K,M\}$ with a nuclear embedding operator (see [1]).
Let

$$E\{K,M\} = \lim_p \, ind \, E\{p,K,M\}, \tag{6}$$

it is known that $E\{K,M\}$ is a nuclear DFS–space. Let K and L are compact sets and let $K \subset \ Int\, L \subset R$.
Let

$$E\{L,K,M\} = \{f \in E\{L,M\}, f_{|K} \equiv 0\} \tag{7}$$

considered with relative topology induced by the space $E\{L,M\}$. From the conditions, imposed on the matrix M, it follows that the space $E\{L,K,M\}$ is not trivial. From the theorems of the Whitney type for ultradifferentiable functions (see [5]), it follows that the sequence (8) is exact.

$$0 \longrightarrow E\{L,K,M\} \stackrel{I}{\longrightarrow} E\{L,M\} \stackrel{J}{\longrightarrow} E\{K,M\} \longrightarrow 0 \quad , \tag{8}$$

where I is the operator of embedding and J is the operator of restriction.

2 The main result.

Theorem 1 *Let in the exact sequence (8) the space $E\{K,M\}$ has an absolute basis $\{f_s\}_{s=0}^\infty$. The continuous linear right inverse of the operator J exists then $\Longleftrightarrow$ to exist functions $\{\hat{f}_s\}_{s=0}^\infty$ satisfying the following conditions:*

$$\hat{f}_s \in E\{L,M\} \text{ and } \hat{f}_{s|K} \equiv f_s, \quad s = 0,1,2,\dots ;$$

and for any q exist p and a constant C that depends on p and q, but does not depend on s, for which

$$\|\hat{f}_s\|_{E\{p,L,M\}} \le C \, \|f_s\|_{E\{q,K,M\}}, \quad s = 0,1,2,\dots \tag{9}$$

Proof: The spaces $E\{L,M\}$ and $E\{K,M\}$ as DFS–spaces are regular inductive limits (*see* [4]). Let J_r^{-1} be a continuous linear right inverse for J of the exact sequence (8) and let $\hat{f}_s = J_r^{-1} f_s$. From the regularity of the spaces $E\{L,M\}$ and $E\{K,M\}$, it follows that for any q there exists p, such that J_r^{-1} is a continuous operator.

$$J_r^{-1} : E\{q,K,M\} \longrightarrow E\{p,L,M\} \quad .$$

It easily follows that for any $q > q_0$, $f_s \in E\{q,K,M\}$.
On the other hand, let $\{f_s\}_{s=0}^{\infty}$ be an absolute basis in the space $E\{K,M\}$ and the functions $\{\hat{f}_s\}_{s=0}^{\infty}$ are with the indicated characteristics. Let $f \in E\{K,M\}$ then exists $p_0 \geq 0$, so that $f \in E\{p_0,K,M\}$ i.e. $\|f\|_{p_0} < +\infty$. Let's $f = \sum_s \xi_s f_s$. From this that the functions $\{f_s\}_{s=0}^{\infty}$ are an absolute basis in the space $E\{K,M\}$ and it is a regular inductive limit, it follows that

$$\exists q \ , \ \sum_s |\xi_s| \|f_s\|_q \leq C_1 \|f\|_{p_0}. \tag{10}$$

We define $\hat{f} = J_r^{-1} f = \sum_s \xi_s \hat{f}_s$ and $p(q)$ by the condition (9), then

$$\|\hat{f}\|_{p(q)} \leq \sum_s |\xi_s| \|\hat{f}\|_{p(q)} \leq C_2 \sum_s |\xi_s| \|f_s\|_q \leq C_3 \|f\|_{p_0}. \tag{11}$$

From (10) and (11) it follows that the operator defined in this way is a continuous linear right inverse for the operator J from (8).

3 Note I.

Upper bound of the derivatives of one useful function.
Let $0<a<1$ and $b>0$ and

$$B(a,b,t) = \begin{cases} 0 & \text{when} \quad -1 \leq t \leq -a \ , \\ \exp(-\frac{ba^4}{t^2(a+t)^2}) & \text{when} \quad -a < t \leq 0 \ , \\ 0 & \text{when} \quad 0 \leq t \leq 1 \ . \end{cases} \tag{12}$$

Let define the useful function

$$A(a,b,t) = \begin{cases} \frac{\int_{-1}^{t} B(a,b,t)dt}{\int_{-1}^{1} B(a,b,t)dt} & \text{when} \quad -1 \leq t \leq 0 \ , \\ A(a,b,-t) & \text{when} \quad 0 < t \leq 1 \ , \\ 0 & \text{when} \quad t \notin [-1,1] \ . \end{cases} \tag{13}$$

We shall bound the derivatives of the function $A(a,b,t)$. A Function like A, for the first time was used by Dzanasija (*see* [3]).

First we shall lower bound $\int_{-1}^{1} B(a,b,t)dt$ in the follow way. From this that the function $B(a,b,t)$ is increasing when $-1 \le t < -\frac{a}{2}$ and decreasing symmetrically when $-\frac{a}{2} < t < 0$ it follows that

$$\int_{-1}^{1} B(a,b,t)dt > \frac{a}{2} B(a,b,-\frac{a}{4})$$

and

$$\int_{-1}^{1} B(a,b,t)dt > \frac{a}{2} \exp(-36b) \quad . \tag{14}$$

For the bound of $|B^{(n)}(a,b,t)|$ we shall use, by analogy with [3], that $B(a,b,t)$ is analytical in a interval $(-a,0)$ and we shall use the Cauchy's formula for contour with a center in the point $t \in [-\frac{a}{2},0)$ and radius ht, where $0<h<1$

$$B^{(n)}(a,b,t) = \frac{n!}{2\pi} \int_0^{2\pi} \frac{B(a,b,t-hte^{i\varphi})}{(-hte^{i\varphi})^n} d\varphi \quad , \tag{15}$$

$$|B^{(n)}(a,b,t)| \le \frac{n!}{2\pi h^n |t|^n} \int_0^{2\pi} |B(a,b,t-hte^{i\varphi})| d\varphi \quad , \tag{16}$$

$$|B(a,b,t-the^{i\varphi})| = \left| exp\left(-\frac{ba^4}{t^2(1-he^{i\varphi})(a+t-th^{i\varphi})^2} \right) \right| \tag{17}$$

$$\le exp\left(-\frac{a^4 b \cos(2\gamma+2\beta)}{t^2(1+h^2-2h\cos\varphi)((a+t+th\cos\varphi)^2+t^2h^2\sin^2\varphi)} \right) \quad ,$$

where

$$\sin\beta = \frac{h\sin\varphi}{\sqrt{(1-2h\cos\varphi+h^2)}} \quad ,$$

$$\sin\gamma = \frac{th\sin\varphi}{\sqrt{((a+t+th\cos\varphi)^2+t^2h^2\sin^2\varphi)}} \quad . \tag{18}$$

When h is small and fixed, we have:

$$\frac{\cos(2\gamma+2\beta)}{(1+h^2-2h\cos\varphi)((a+t+th\cos\varphi)^2+t^2h^2\sin^2\varphi)} > \frac{1}{18a^2} \quad . \tag{19}$$

From the last it follows that

$$|B(a,b,t(1-he^{i\varphi}))| \le exp\left(-\frac{a^2 b}{18t^2} \right) \quad . \tag{20}$$

From (16) and (20), we have:

$$|B^{(n)}(a,b,t)| \le \frac{n!}{h^n |t|^n} exp\left(-\frac{a^2 b}{18t^2} \right) \quad . \tag{21}$$

It is easy to see, that

$$\max_t \left(\frac{1}{|t|^n} exp\left(-\frac{a^2 b}{18t^2} \right) \right) \leq \frac{3^n n^{\frac{n}{2}} e^{-\frac{n}{2}}}{a^n b^{\frac{n}{2}}} \quad . \tag{22}$$

From (21) and (22) we get

$$|B^{(n)}(a,b,t)| \leq \frac{3^n n! n^{\frac{n}{2}} e^{-\frac{n}{2}}}{h^n a^n b^{\frac{n}{2}}} \quad . \tag{23}$$

From (13), (14) and (23) we get the following bound for the derivatives of the function $A(a,b,t)$:

$$|A^{(n)}(a,b,t)| \leq 2\exp(-36b) \left(\frac{3}{h}\right)^n \frac{n^{(n-1)} n^{\frac{(n-1)}{2}}}{a^n b^{\frac{(n-1)}{2}}} \tag{24}$$

when $t \in [-1,1]$.

4 Note II.

In the exact sequence (25)

$$0 \longrightarrow E\{[-2,2],[-1,1],M\} \xrightarrow{I} E\{[-2,2],M\} \xrightarrow{J} E\{[-1,1],M\} \longrightarrow 0 \tag{25}$$

the operator J has a continuous linear right inverse and the sequence is split. In fact, in [2] it is proved that Jacobian's polynamials $J_k^{\alpha\beta}(t)$, when $\alpha, \beta \geq -\frac{1}{2}$ are fixed, form an absolute basis in the space $E\{[-1,1],M\}$. We shall define the functions

$$\hat{J}_k^{\alpha\beta}(t) = J_k^{\alpha\beta}(t) A_k(t) \quad , \quad t \in [-2,2] \quad , \tag{26}$$

where $A_0(t) = 1$ when $k = 1, 2, \ldots$

$$A_k(t) = \begin{cases} A\left(k^{-2}, 1, t+1\right) & \text{when} \quad t \in [-2,-1] \quad , \\ 1 & \text{when} \quad t \in [-1,1] \quad , \\ A\left(k^{-2}, 1, t-1\right) & \text{when} \quad t \in [1,2] \quad , \end{cases} \tag{27}$$

where $A(.,.,.)$ is the functions from (13). We shall show that the functions $\hat{J}_k^{\alpha\beta}$ satisfy the conditions of the Theorem 1. For this, we shall bound the derivatives of the functions $\hat{J}_k^{\alpha\beta}(t)$. As $A_k(t) \equiv 0$ when $t \notin \left[-1-\frac{1}{k^2}, 1+\frac{1}{k^2}\right]$, then

$$\begin{aligned} \left|\hat{J}_k^{\alpha\beta}(t)^{(n)}\right| &\leq \sum_{s=0}^{n} \binom{n}{s} \left|J_n^{\alpha\beta(n-s)}(t)\right| \left|A_k^{(s)}(t)\right| \leq \\ &\leq \sum_{s=0}^{n} \binom{n}{s} \max_{t\in\left[-1-\frac{1}{k^2}, 1+\frac{1}{k^2}\right]} \left|J_k^{\alpha\beta(n-s)}(t)\right| \left|A_k^{(s)}(t)\right| . \end{aligned} \tag{28}$$

To bound $\max\left|J_k^{\alpha\beta(n-s)}(t)\right|$ when $t\in\left[-1-\frac{1}{k^2},1+\frac{1}{k^2}\right]$ we shall use the Chebyshev's inequality (see [11] p.79), if $p(t)$ is a polynomial with $\deg p\le k$ and $|x_0|>1$, then

$$|p(x_0)|\le(|x_0|+\sqrt{x_0^2-1})^k\max_{t\in[-1,1]}|p(t)| \quad . \tag{29}$$

As above from Markov's inequality (see [11] p.179), i.e. if $p(t)$ is a polynomial with $\deg p\le k$ then

$$\max_{t\in[a,b]}|p'(t)|\le 2k^2\max_{t\in[a,b]}|p(t)|\,/(b-a) \tag{30}$$

and from the bound

$$\max_{t\in[-1,1]}\left|J_k^{\alpha\beta}(t)\right|\le C_1k^{\sigma+\frac{1}{2}} \tag{31}$$

where $\sigma=\max(\alpha,\beta)$ and $C_1>0$ is a constant which does not depend on k (see [12]). From (29), (30) and (31) we have

$$\begin{aligned}\max_{t\in\left[-1-\frac{1}{k^2},1+\frac{1}{k^2}\right]}\left|J_k^{\alpha\beta(s)}(t)\right|&\le\left(1+\frac{3}{k}\right)^{k-s}\max_{t\in[-1,1]}\left|J_k^{\alpha\beta(s)}(t)\right|\\&\le e^3\max_{t\in[-1,1]}\left|J_k^{\alpha\beta(s)}(t)\right|\le e^3\left\|J_k^{\alpha\beta}\right\|_{E\{p,[-1,1],M\}}M_{ps} \quad .\end{aligned} \tag{32}$$

Let be as above $a_{p0}=1$,

$$a_{pk}=\sup_s\log(k^sM_{p0}\,/M_{ps})\,,k=0,1,2,\dots \tag{33}$$

Then (see [7]) from (2) it follows that

$$\sup_k(k^q/e^{a_{pk}})\sim\frac{C_2}{M_{p0}}M_{pq}. \tag{34}$$

From (34), (30) and (31) it follows that for any natural number p_1 there exist natural number p_2 and a positive constant $C_1>0$ such that :

$$\left\|J_k^{\alpha\beta}\right\|_{E\{p_1,[-1,1],M\}}\le C_1e^{a_{p_2k}},k=0,1,2,\dots \tag{35}$$

and for any natural number q_1 there exist a natural number q_2 and a positive constant C_2 such that

$$\left\|J_k^{\alpha\beta}\right\|_{E\{q_2,[-1,1],M\}}\ge C_2e^{a_{q_2k}}$$

From (28) and (32) and when $t\in[-2,2]$, we have :

$$\begin{aligned}\left|\hat{J}_k^{\alpha\beta\,(l)}(t)\right|&\le e^3\sum_{s=0}^{l}\binom{l}{s}\max_{t\in[-1,1]}\left|J_k^{\alpha\beta\,(l-s)}(t)\right|\left|A_k^{(s)}(t)\right|\le\\&C_3T^lk^{2l+\sigma+\frac{1}{2}}l^{2l} \quad ,\end{aligned} \tag{36}$$

where C_3 and T are positive constants which do not depend on k and l. From last and (35), for any natural number p we have :

$$\left|\hat{J}_k^{\alpha\beta\,(l)}(t)\right| \le C_3 T^l l^{2l} \sup_k \left(\frac{k^{pl}}{l^{a_{pk}}}\right) l^{a_{pk}} \le C_4 T^l l^{2l} M_{ps_l} \left\| J_k^{\alpha\beta} \right\|_{E\{p,[-1,1],M\}} . \tag{37}$$

And from (3), (4) and (37) it follows that for any natural number p there exist a natural number q and a positive constant C_5 such that :

$$\left|\hat{J}_k^{\alpha\beta\,(l)}\right| \le C_5 M_{ql} \left\| J_k^{\alpha\beta} \right\|_{E\{p,[-1,1],M\}} \quad , i.e.$$

$$\left\| \hat{J}_k \right\|_{E\{q,[-2,2],M\}} \le C_5 \left\| J_k^{\alpha\beta} \right\|_{E\{p,[-1,1],M\}}. \tag{38}$$

From (38) it follows that the functions $\left\{\hat{J}_k^{\alpha\beta}\right\}_{k=0}^{\infty}$ satisfy all the conditions of the Theorem 1.

5 Note III.

In the exact sequence (39), the operator J has a linear and uninterupted right inverse and the sequence is split.

$$0 \longrightarrow E\{[-1,1],\{0\},M\} \xrightarrow{I} E\{[-1,1],M\} \xrightarrow{J} E\{\{0\},M\} \longrightarrow 0 \quad . \tag{39}$$

In fact, the space $E\{\{0\},M\}$ is a space of sequences

$$E\{\{0\},M\} = \left\{ \xi = \{\xi_k\}, \exists p : \|\xi\|_p = \sup_k |\xi_k| / M_{pk} < +\infty \right\} \quad . \tag{40}$$

In the space $E\{\{0\},M\}$ the sequences $e_k = \{\delta_{i,k}\}_{i,k=0}^{\infty}$, where $\delta_{ik} = \begin{cases} 0 & i \ne k \\ 1 & i = k \end{cases}$ form a basis. Let

$$m_k = \max_{0\le i,j\le k} M_{ij}.$$

We shall define the functions $\hat{e}_k$

$$\hat{e}_k = A\left(m_k^{-1}, m_k^2, t\right) t^k / k! \quad , k = 0,1,2,\dots,a \quad . \tag{41}$$

$A(.,.,.)$ is the function from (13). We will bound the derivatives of the functions $\hat{e}_k(t)$. As *supp* $A\left(m_k^{-1}, m_k^2, t\right) \in \left[-m_k^{-1}, m_k^{-1}\right]$, so that

$$\left|\hat{e}_k{}^{(l)}(t)\right| \le \sum_{s=0}^{l} \binom{l}{s} \left| A\left(m_k^{-1}, m_k^2, t\right)^{(l-s)} \right| \frac{1}{m_k^{(k-s)}(k-s)!} \le$$

$$\le C_5 l^{-36 m_k^2} \sum_{s=0}^{l} \binom{l}{s} \left(\frac{3}{h}\right)^{(l-s)} \frac{(l-s)^{(l-s-1)} (l-s)^{\frac{l-s-1}{2}} m_k^{(l-s)}}{m_k^{2^{\frac{l-s-1}{2}}} m_k^{(k-s)} (k-s)!} \le$$

$$\le C_6 l^{-36 m_k^2} \frac{m_k^{(l+1)} k^l l^{2l} T_1^l}{m_k^k k^k} \quad , \tag{42}$$

where C_5, C_6 and T_1 are positive constants that do not depend on k and l.
If we take under consideration that $\|e_k\|_p = M_{pk}^{-1}$ then by (42) and a fixed natural number p we have :
I.When $k = l$

$$\left|\hat{e}_l{}^{(l)}(t)\right| \leq C_7 l^{-36m_l^2} m_l l^{2l} T_1^l M_{pe} / M_{pl} \leq$$
$$\leq \frac{C_8 m_l M_{pl} l^{2l} T_1^l}{m_l^2 M_{pl}} \leq C_9 M_{q_1 l} \frac{1}{M_{pl}} \quad , \tag{43}$$

where q_1 is a natural number such that

$$\sup_l \left(\frac{M_{pl} l^{2l} T_1^l}{M_{q_1 l}} \right) < \infty \quad .$$

II.When $k > l$

$$\left|\hat{e}_k{}^{(l)}(t)\right| \leq C_{10} l^{-36m_k^2} \frac{m_k^{(l)}}{m_k^k} \leq$$
$$C_{11} \frac{k^l M_{pk}}{m_k^{k-l+3} M_{pk}} \frac{l^l T_1^l}{M_{q_1 l}} M_{q_1 l} \leq$$
$$C_{12} \sup_k \frac{k^l}{m_k^{k-l+2}} \frac{C_{13}}{M_p k} M_{q_1 l} \leq C_{14} \frac{M_{q_1 l}}{M_{pk}} \quad . \tag{44}$$

III.When $k < l$

$$\left|\hat{e}_k{}^{(l)}(t)\right| \leq C_{15} l^{-36m_k^2} m_k^{l-k+1} k^{l-k} l^{2l} T_1^l \leq$$
$$C_{16} l^{-36m_k^2} m_k^l l^{3l} T_1^l \leq C_{17} \frac{l^{-36m_k^2} m_k^{l+1}}{M_{pk}} \frac{l^{3l} T_1^l}{M_{q_2 l}} M_{q_2 l} \ , \tag{45}$$

where q_2 is such that

$$\sup_l \left(l^{4l} T_1^l / M_{q_2 l} \right) < \infty \quad .$$

As $l^{-36m_k^2} m_k^{l+1} \leq C_{18} l^{\frac{l}{2}} T_2^l$ by (43),(44) and (45), it follows that for any natural number p there exists a natural number q such that :

$$\|\hat{e}_k\|_{E\{q,[-1,1]\}} \leq C_{20} \|e_k\|_p \quad ,$$

i.e. the conditions of the theorem are fulfilled and it follows that the continuous linear and continuous extension operator exists.

6 Note IV.

If we use that $[E\{\{0\},M\}]^2 = E\{\{0\},M\}$ and the results of Note III, then it follows that:

$$E\{R,M\} = \lim_{N} \mathrm{ind} \mathrm{E}\{[-\mathrm{N},\mathrm{N}],\mathrm{M}\} \cong \prod_{\mathrm{s}=1}^{\infty} \mathrm{E}\{[-1,1],\mathrm{M}\} \quad ,$$

and as $E\{[-1,1],M\}$ has a basis, it follows that $E\{R,M\}$ has also a basis.

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Author:

Gurko I. Balikov
Institute of Mathematics,
Bulgarian Academy of Sciences
Acad. Bonchev str., bl. 8,
1113 Sofia
Bulgaria

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Friedrich Liese; Andreas Wienke

A note on the Hajek-Le Cam bound

Abstract: Using Rüschendorf's approach to the uniform weak compactness lemma for tests under the LAN-condition a simplified approach to the Hajek-Le Cam bound is given if the decision space is compact.

1 Notations and results

The concept of convergence of statistical experiments is a basic notion in the asymptotic decision theory. This convergence is defined by a suitable metric in the space of all experiments. But it turns out that this type of convergence can be equivalently characterized by the weak convergence of the finite dimensional distributions of the corresponding likelihood processes. If the limit experiment is a Gaussian shift experiment then the convergence of experiments may be reduced to the weak convergence of the one dimensional distributions of the likelihood processes. The convergence of experiments is then also called LAN-property (local asymptotic normality). To formulate the LAN-property we recall to the notion of the likehood ratio. Let P, Q be probability measures, say on $(\Omega, \mathcal{F})$. We set $\mu = \frac{1}{2}(P+Q)$ and $f = \frac{dP}{d\mu}, g = \frac{dQ}{d\mu}$. The likelihood ratio $\frac{dP}{dQ}$ is defined by

$$\frac{dP}{dQ} = \frac{f}{g} I_{\{g>0\}} + \infty I_{\{f>0, g=0\}},$$

where I_A is the indicator function of the event A and we used the convention $0 \cdot \infty = 0$. Note that $\frac{dP}{dQ}$ takes on values in $[0, \infty]$ and $\ln \frac{dP}{dQ}$ has values in $[-\infty, \infty]$. Let Σ be a fixed positive definite symmetric $k \times k$ matrix and denote by $N_h, h \in \mathbb{R}^k$ the normal distribution on $(\mathbb{R}^k, \mathcal{B}^k)$ with expectation h and covariance matrix Σ. Then the log likelihood ratio is given by

$$\ln \frac{dN_h}{dN_0}(x) = h^T \Sigma^{-1} x - \frac{1}{2} h^T \Sigma^{-1} h.$$

To simplify the notation we introduce a suitable scalar product $\langle \cdot, \cdot \rangle$ by $\langle x, y \rangle = x^T \Sigma^{-1} y$ and set $||x|| = \langle x, x \rangle^{\frac{1}{2}}$. Then

$$\ln \frac{dN_h}{dN_0}(x) = \langle x, h \rangle - \frac{1}{2} ||h||^2.$$

Assume $H_1 \subseteq H_2 \subseteq \ldots \subseteq \mathbb{R}^k$ and set $H = \bigcup_{n=1}^{\infty} H_n$. The sequence of experiments

$$\mathcal{E}_n = (\Omega_n, \mathcal{F}_n, P_{n,h}, h \in H_n)$$

is called asymptotically normal or $\{P_{n,h}, h \in H_n\}$ has the LAN-property if for every fixed $h \in H$ the logarithm of the likelihood ratio admits the following representation

$$\ln \frac{dP_{n,h}}{dP_{n,0}} = \langle Z_n, h \rangle - \frac{1}{2}||h||^2 + R_n,$$

where the remainder term R_n tends $P_{n,0}$-stochastically to zero and the distribution $\mathcal{L}(Z_n|P_{n,0})$ of Z_n under $P_{n,0}$ tends weakly to N_0. We suppose that the decision space $\mathbb{D}$ is a Polish space and $\mathcal{D}$ is the σ-algebra of Borel sets.

By a stochastic kernel $K : (\Omega, \mathcal{F}) \Longrightarrow (\mathbb{D}, \mathcal{D})$ we shall mean a mapping $K : \Omega \times \mathcal{D} \longrightarrow [0,1]$ such that $\omega \longmapsto K(\omega, B)$ is $\mathcal{F} - \mathcal{D}$-measurable for every $B \in \mathcal{D}$ and $K(\omega, \cdot)$ is a probability measure on $(\mathbb{D}, \mathcal{D})$ for every $\omega \in \Omega$. Denote by $\mathbb{P}$ the set of all probability measures equipped with the topology of weak convergence. A metrization of this topology is given by Prokhorov metric ρ (see Prokhorov [5], p. 167). $(\mathbb{P}, \rho)$ is again a Polish space. Let $\mathcal{P}$ be the σ-algebra of Borel sets of $\mathbb{P}$. Then obviously for every bounded and continuous φ the mapping $\mu \longmapsto \int \varphi \, d\mu$, $\mu \in \mathbb{P}$, is continuous and consequently $\mathcal{P} - \mathcal{B}^1$-measurable. By a pointwise approximation of the indicator function I_B of a closed set B one can see that $\mu \longmapsto \mu(B)$ is $\mathcal{P} - \mathcal{B}^1$-measurable. Let $\mathfrak{M}$ be the set of all $A \in \mathcal{D}$ such that $\mu \longmapsto \mu(A)$ is $\mathcal{P} - \mathcal{B}^1$-measurable. It is easy to see that $\mathfrak{M}$ is a Dynkinsystem which contains the $\cap$-closed system of closed sets. Hence $\mathcal{D} = \mathfrak{M}$ which means that for every Borel set $A \in \mathcal{D}$ the mapping $\mu \longmapsto \mu(A)$ is $\mathcal{P} - \mathcal{B}^1$-measurable. Using this fact we can say that a mapping $K : \Omega \times \mathcal{D} \rightarrow [0,1]$ is a stochastic kernel iff the mapping $\omega \longmapsto K(\omega, \cdot)$, also denoted by K, from Ω into $\mathbb{P}$ is $\mathcal{F} - \mathcal{P}$-measurable.

Let $\mathcal{E} = (\Omega, \mathcal{F}, P_h, h \in H)$ be any experiment. By an randomized decision we shall mean a stochastic kernel $K : (\Omega, \mathcal{F}) \Longrightarrow (\mathbb{D}, \mathcal{D})$. By the distribution of K under P_h we shall mean $P_h \circ K^{-1}$ which is defined on $(\mathbb{P}, \mathcal{P})$. In this sense every randomized decision with decision space $(\mathbb{D}, \mathcal{D})$ can be identified with a nonrandomized decision with decision space $(\mathbb{P}, \mathcal{P})$. Let now $M : (\Omega, \mathcal{F}) \Longrightarrow (\mathbb{P}, \mathcal{P})$ be a randomized decision with decision space $(\mathbb{P}, \mathcal{P})$. We set for $\omega \in \Omega, A \in \mathcal{D}$

$$K(\omega, A) := \int \mu(A) M(\omega, d\mu). \tag{1}$$

It is easy to see that $K : (\Omega, \mathcal{F}) \Longrightarrow (\mathbb{D}, \mathcal{D})$ is a stochastic kernel. Consequently every stochastic kernel M defines a randomized decision K.

If the sequence $\{P_{n,h}, h \in H_n\}$ has the LAN-property then this family is an approximativ exponential family so that the central sequence is asymptotically sufficient. The main idea of Rüschendorf's approach (see Rüschendorf [6]) to the hypothesis testing problem under the LAN-condition is a suitable version of this asymptotic sufficiency. We generalize this idea to more general decision spaces. To be more precise we denote by $\mathcal{E} = (\mathbb{R}^k, \mathcal{B}^k, N_h, h \in R^k)$ the standard Gauss shift experiment. Let $K : (\mathbb{R}^k, \mathcal{B}^k) \Longrightarrow (\mathbb{D}, \mathcal{D})$ be a stochastic kernel. Denote by Z the projection of $\mathbb{R}^k \times \mathbb{P}$ onto $\mathbb{R}^k$.

Proposition: *Suppose the family $\{P_{n,h},\ h \in H_n\}$ has the LAN-property, the decision space $\mathbb{D}$ is a Polish space and $K_n : (\Omega^n, \mathcal{F}^n) \Longrightarrow (\mathbb{D}, \mathcal{D})$ is a sequence of randomized decisions for the experiments $\mathcal{E}_n$. If the distributions $\mathcal{L}((Z_n, K_n)|P_{n,0})$ converge weakly to $\mathcal{L}((Z, K)|P_0)$ as $n \to \infty$ then $\mathcal{L}((Z_n, K_n)|P_{n,h})$ converge weakly to, say Q_h, for every $h \in \mathbb{R}^k$. The statistic $Z : \mathbb{R}^k \times \mathbb{P} \longrightarrow \mathbb{R}^k$ is sufficient for the family $\{Q_h,\ h \in \mathbb{R}^k\}$.*

Assume now $L :\ H \times \mathbb{D} \longrightarrow [0, \infty)$ is a bounded and continuous loss function. We introduce the risks by

$$R_n(h, K_n) = \int \Big(\int L(h, t) K_n(\omega, dt) \Big) P_{n,h}(d\omega) \tag{2}$$

and

$$R(h, K) = \int \Big(\int L(h, t) K(\omega, dt) \Big) N_h(d\omega).$$

The next lemma is an essential step in the proof of the Hajek-Le Cam bound. This lemma states that under the LAN-condition every sequence of randomized decisions is sequentially compact. It is a generalization of Theorem 2.7 in Rüschendorf [6].

Lemma *Suppose the decision space $\mathbb{D}$ is compact and $L : H \times \mathbb{D} \longrightarrow [0, \infty)$ is bounded and continuous. If $\{P_{n,h},\ h \in H_n\}$ has the LAN-property and $K_n : (\Omega^n, \mathcal{F}^n) \Longrightarrow (\mathbb{D}, \mathcal{D})$ is a sequence of randomized decisions for $\mathcal{E}_n$, then there exist a subsequence $\{n_l\}$ and a randomized decision $K : (\mathbb{R}^k, \mathcal{B}^k) \Longrightarrow (\mathbb{D}, \mathcal{D})$ for the experiment $\mathcal{E} = (\mathbb{R}^k, \mathcal{B}^k, N_h, h \in \mathbb{R}^k)$ such that*

$$\lim_{l \to \infty} R_{n_l}(h, K_{n_l}) = R(h, K)$$

for every $h \in \mathbb{R}^k$.

Now we are ready to formulate the main result of this paper.

Theorem *Assume the decision space $\mathbb{D}$ is a compact Polish space and the loss function L is bounded and continuous. If $\{P_{n,h},\ h \in H\}$ has the LAN-property and $K_n : (\Omega^n, \mathcal{F}^n) \Longrightarrow (\mathbb{D}, \mathcal{D})$ is any sequence of randomized decisions then for every subset $H_0 \subseteq \mathbb{R}^k$*

$$\liminf_{n \to \infty} \sup_{h \in H_0 \cap H_n} R_n(h, K_n) \geq \inf_K \sup_{h \in H_0} R(h, K).$$

Remark 1 The value $\inf_K \sup_{h \in H_0} R(h, K)$ is called the Hajek-Le Cam bound. It is the minimal value of the maximum risk which can be asymptotically attained by a sequence of decisions. Therefore a sequence $\{K_n\}$ is called asymptotically minimax if $\{K_n\}$ realizes equality in the above inequality.

Remark 2 A similar statement can be also derived if the sequence of experiments $\mathcal{E}_n$ converges to a limit experiment which is not a Gaussian shift. For this more general statement we refer to Le Cam [3]. But the intention of this note is to give a simplified approach which can be used in lectures. A simplified approach to Le Cam's result was given in Liese, Steinke [4] in the general situation. The methods there are completely different from that in this paper.

2 Proofs

Proof of the Proposition: At first we have to show that there exists a distribution Q_h on $(\mathbb{R}^k \times \mathbb{P}, \mathcal{B}^k \otimes P)$ such that for every bounded and continuous $\varphi : \mathbb{R}^k \times \mathbb{P} \longrightarrow \mathbb{R}$

$$\int \varphi(Z_n, K_n)\, dP_{n,h} \xrightarrow[n\to\infty]{} \int \varphi\, dQ_h.$$

Assume φ has compact support. By assumption

$$\int \varphi(Z_n, K_n)\, dP_{n,h} = \int \varphi(Z_n, K_n) \exp\{\langle Z_n, h\rangle - \frac{1}{2}||h||^2 + R_n\}\, dP_{n,0}.$$

As φ has compact support the function $\psi : \mathbb{R}^k \times \mathbb{P} \longrightarrow \mathbb{R}^1$ defined by

$$\psi(x, \mu) = \varphi(x, \mu) \exp\{\langle x, h\rangle - \frac{1}{2}||h||^2\}$$

is bounded and continuous. As $R_n \longrightarrow 0$, $P_{n,0}$-stochastically we get from the weak convergence of $\mathcal{L}((Z_n, K_n)|P_{n,0})$ to Q_0 and the LAN-property that

$$\begin{aligned} \lim_{n\to\infty} \int \varphi(Z_n, K_n)\, dP_{n,h} &= \lim_{n\to\infty} \int \varphi(Z_n, K_n) \exp\{\langle Z_n, h\rangle - \frac{1}{2}||h||^2\}\, dP_{n,0} \\ &= \int \varphi(x, \mu)\, Q_h(dx, d\mu), \end{aligned} \tag{3}$$

where the measure Q_h is defined by

$$Q_h(A) = \int I_A(x, \mu) \exp\{\langle x, h\rangle - \frac{1}{2}||h||^2\} Q_0(dx, d\mu).$$

As Q_h is the limit of $\mathcal{L}((Z_n, K_n)|P_{n,h})$ only in the vague topology we have to show that $Q_h(\mathbb{R}^k \times \mathbb{P}) = 1$. As the sequence $\{P_{n,h},\ h \in H_n\}$ has the LAN property the sequence $\{P_{n,h}\}$ is contiguous with respect to $\{P_{n,0}\}$. Consequently the tightness of $\mathcal{L}((Z_n, K_n)|P_{n,0})$ implies the tightness of $\mathcal{L}((Z_n, K_n)|P_{n,h})$. But tightness and vague convergence imply weak convergence of $\mathcal{L}((Z_n, K_n)|P_{n,h})$ to Q_h which is therefore a probability measure. If $Z : \mathbb{R}^k \times \mathbb{P} \longrightarrow \mathbb{R}^k$ is the projection onto $\mathbb{R}^k$ we see that the density

$$\frac{dQ_h}{dQ_0} = \exp\{\langle Z, h\rangle - \frac{1}{2}||h||^2\}$$

is only a function of Z. Hence Z is sufficient which completes the proof. ■

Proof of the Lemma: Let $Q_{n,h}$ be the distribution $\mathcal{L}((Z_n, K_n)|P_{n,h})$. Set $h = 0$ and note $\{Q_{n,0}\}$ is tight as $\mathcal{L}(Z_n|P_{n,0}) \Longrightarrow N_0$ and $\mathbb{P}$ is compact. Let $\{n_l\}$ be a subsequence such that $Q_{n_l,0}$ converges, say to Q_0. Then by the Proposition for every bounded and continuous $\psi : \mathbb{R}^k \times \mathbb{P} \longrightarrow \mathbb{R}^1$

$$\lim_{l\to\infty} \int \psi(z,\mu)\, Q_{n_l,h}(dz, d\mu) = \int \psi(z,\mu)\, Q_h(dz, d\mu)$$

where

$$\frac{dQ_h}{dQ_0} = \exp\{\langle Z, h\rangle - \frac{1}{2}||h||^2\}.$$

The weak convergence of $\mathcal{L}(Z_n|P_{n,0})$ to N_0 implies that $Q_0 \circ Z^{-1} = N_0$. Furthermore by the third lemma of Le Cam $\mathcal{L}(Z_n|P_{n,h}) \Longrightarrow N_h$ so that N_h is the marginal distribution on $\mathbb{R}^k$ of Q_h. As $\mathbb{P}$ is a Polish space there is a desintegration of Q_0 w.r.t. N_0, i.e. a stochastic kernel

$$M : (\mathbb{R}^k, \mathcal{B}^k) \Longrightarrow (\mathbb{P}, \mathcal{P})$$

with

$$\int_A M(x, B)\, N_0(dx) = Q_0(A \times B)$$

for every $A \in \mathcal{B}^k, B \in \mathcal{P}$. By the sufficiency of Z established in the Proposition the kernel M is also a conditional distribution for Q_h. Consequently,

$$\int_A M(x, B)\, N_h(dx) = Q_h(A \times B).$$

Hence for every bounded and measurable $\varphi : \mathbb{P} \longrightarrow \mathbb{R}^1$

$$\iint \varphi(\mu) M(z, d\mu) N_h(dz) = \iint \varphi(\mu)\, Q_h(dz, d\mu). \tag{4}$$

Similar as in (1) we introduce $K : (\mathbb{R}^k, \mathcal{B}^k) \Longrightarrow (\mathbb{D}, \mathcal{D})$ by

$$K(x, A) = \int \mu(A) M(x, d\mu). \tag{5}$$

Note that by the continuity and boundedness of L the function

$$\varphi_L(h,\mu) = \int L(h,t)\,\mu(dt)$$

is bounded and continuous on $\mathbb{R}^k \times \mathbb{P}$. Consequently by (2), (4) and (5)

$$\begin{aligned}
\lim_{l\to\infty} R_{n_l}(h,K_{n_l}) &= \lim_{l\to\infty} \iint L(h,t)K_{n_l}(x,dt)\,P_{n_l,h}(dx) \\
&= \lim_{l\to\infty} \iint L(h,t)\mu(dt)\mathcal{L}(K_{n_l}|P_{n_l,h})(d\mu) \\
&= \lim_{l\to\infty} \iiint L(h,t)\mu(dt)\mathcal{L}((Z_{n_l},K_{n_l})|P_{n_l,h})(dz,d\mu) \\
&= \lim_{l\to\infty} \iint \varphi_L(h,\mu)Q_{n_l,h}(dz,d\mu) \\
&= \iint \varphi_L(h,\mu)Q_h(dz,d\mu) \\
&= \iint \varphi_L(h,\mu)M(z,d\mu)N_h(dz) \\
&= \iiint L(h,t)\mu(dt)M(z,d\mu)N_h(dz) \\
&= \iint L(h,t)K(z,dt)N_h(dz) \\
&= R(h,K).
\end{aligned}$$

This completes the proof. ■

Proof of the Theorem: There exists subsequence $\{n'\} \subseteq \{n\}$ with

$$\liminf_{n\to\infty} \sup_{h\in H_0\cap H_n} R_n(h,K_n) = \lim_{n'\to\infty} \sup_{h\in H_0\cap H_n} R_{n'}(h,K_{n'}).$$

By the Lemma there exists a subsequence $\{n''\} \subseteq \{n'\}$ and a randomized decision K, such that for every $h \in \mathbb{R}^k$

$$\lim_{n''\to\infty} R_{n''}(h,K_{n''}) = R(h,K).$$

Fix $\varepsilon > 0$. Then there exists a h_ε with

$$R(h_\varepsilon,K) \geq \sup_{h\in H_0} R(h,K) - \varepsilon.$$

Consequently

$$\begin{aligned}
\liminf_{n\to\infty} \sup_{h\in H_0\cap H_n} R_n(h,K_n) &\geq \liminf_{n\to\infty} R_n(h_\varepsilon,K_n) \\
&= \lim_{n''\to\infty} R_{n''}(h_\varepsilon,K_{n''}) \\
&= R(h_\varepsilon,K) \\
&\geq \sup_{h\in H_0} R(h,K) - \varepsilon.
\end{aligned}$$

Take the infimum about K and then $\varepsilon \longrightarrow 0$ to complete the proof. ■

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received: January 20, 1997

Authors:

Friedrich Liese
Universität Rostock
Fachbereich Mathematik
Universitätsplatz 1
18051 Rostock
Bundesrepublik Deutschland
email: friedrich.liese@mathematik.uni-rostock.de

Andreas Wienke
Max Planck Institut
for Demographic Research
Doberanerstraße 114
18057 Rostock
Bundesrepublik Deutschland

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P. Dencker

A Limit Theorem for empirical processes from planned experiments

ABSTRACT. In this paper the data $Y_1, \dots, Y_m$ have the following structure. The measurements are taken at $t_i \in (0,1)^q$ and there is a family of distributions Q_η, $\eta \in (a,b)$ and $g_0 : [0,1]^q \to (a,b)$ such that $\mathcal{L}(Y_i) = Q_{g_0(t_i)}$. The main results of this paper is a limit theorem for the sequence of random fields

$$W_m(t) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} (Y_j - g_0(t_j))\, I_{(0,t]}(t_j) \ .$$

Limit theorems for functionals of W_m are used to construct an asymptotic α-test for $H_0 : g = g_0$ versus $H_A : g \neq g_0$. For a regression model the asymptotic power is investigated under local alternatives.

KEY WORDS. Random fields, Limit Theorems, Wiener field, Goodness of fit test.

1 Introduction

We consider a sequence of independent random variables $Y_1, Y_2, \dots, Y_m$ whose distributions depend on covariables $t_1, \dots, t_m$ from $(0,1)^q$ in the following way. Assume there is a one parameter family Q_η, $\eta \in (a,b) \subseteq \mathbb{R}_1$ and some function g such that

$$\mathcal{L}(Y_i) = Q_{g(t_i)} \ .$$

We suppose that Q_η has finite second moments and is parametrized by its expectation, i. e. it holds

$$\int x\, Q_\eta(dx) = \eta \ .$$

If Q is a distribution with mean zero and finite variance then $Q_\eta(B) = Q(B - \eta)$ leads to the regression model

$$Y_i = g(t_i) + \varepsilon_i$$

where ε_i are i. i. d. with $E\varepsilon_i = 0$ and $\mathrm{V}(\varepsilon_i) = \sigma^2 < \infty$.

Let Q_η be an exponential family on $(\mathbb{R}_1, \mathcal{B}_1)$ with mean value parametrization and dominating measure ν. Then

$$\frac{dQ_\eta}{d\nu}(x) = \exp\{c(\eta)x - K(\eta)\}$$

and $\int x\, Q_\eta(dx) = \eta$. If $g_0 : [0,1]^q \to (a,b)$ is a function which links the covariables t_i with the parameter values η_i then the model

$$Q_{g_0(t_1)}, \ldots, Q_{g_0(t_m)}$$

is nearly related to the generalized linear model in which it is assumed that $g_0(t) = h(\langle\beta_0, t\rangle)$ where $h : \mathbb{R}_1 \to \mathbb{R}_1$ and β_0 is some unknown parameter vector. $\langle\cdot,\cdot\rangle$ is the scalar product, I_A is the indicator function of the set A.

We are interested in testing the hypothesis $H_0 : g = g_0$ versus $H_A : g \neq g_0$. To prove limit theorems for a suitable sequence of test statistics we introduce random fields

$$W_m(t) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} (Y_j - g_0(t_j))\, I_{(0,t]}(t_j)\ .$$

Note that W_m is a random variable taking values in $(D_q, \mathcal{D}_q)$. $D_q := D[0,1]^q$ is the Skohorod space of all functions $w : [0,1]^q \to \mathbb{R}_1$ which are continuous from above in the sense of Neuhaus [6]. $\mathcal{D}_q$ ist the σ-algebra of Borel sets generated by the Skohorod topology, which is defined in Neuhaus [6]. The main result of this paper is a limit theorem for sequences of random fields W_m. The continuous mapping theorem then implies for continuous functionals a limit theorem which is used to construct asymptotic α-tests for the hypothesis $H_0 : g = g_0$ versus $H_A : g \neq g_0$.

2 Results

To each random field

$$W(t),\ t \in [0,1]^q$$

we assign a set function on the set of all q-dimensional intervals $(s,t] \subseteq [0,1]^q$ by

$$W((s,t]) = \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_q=0}^{1} (-1)^{q-\sum_{p=1}^{q}\varepsilon_p} W(s_1 + \varepsilon_1(t_1 - s_1), \ldots, s_q + \varepsilon_q(t_q - s_q))$$

where $s = (s_1, \ldots, s_q), t = (t_1, \ldots, t_q)$.

We say that W vanishes on the lower boundary of $W(t) = 0$ if a least one t_i is zero. In this case we have

$$W(t) = W((0,t]) \ .$$

Assume now ν is a finite measure on the Borel sets of $[0,1]^q$. It is called continuous if the corresponding distribution function

$$B(t) = \nu([0,t])$$

is a continuous function.

Denote by $N(\mu, \sigma^2)$ the normal distribution with expectation μ and variance $\sigma^2 \geq 0$ where $N(\mu, 0) = \delta_\mu$ is the δ-distribution concentrated at μ.

A random field $W_B(t)$, $t \in [0,1]^q$ with continuous paths is called a *Wiener field* with variance B if W vanishes on the lower boundary, B is continuous, $\mathcal{L}\left(W((s,t])\right) = N\left(0, \nu((s,t])\right)$ and for every disjoint $(s_1,t_1], \ldots, (s_m,t_m]$ the random variables $W\left((s_1,t_1]\right), \ldots, W\left((s_m,t_m]\right)$ are independent.

A simple calculation shows that the covariance function

$$K(s,t) = \operatorname{cov}\left(W(s), W(t)\right)$$

of a Wiener field is given by

$$K(s,t) = B(s \wedge t)$$

where $s \wedge t = (\min(s_1,t_1), \ldots, \min(s_q,t_q))$ for $s = (s_1, \ldots, s_q), t = (t_1, \ldots, t_q)$.

Let $Y_{n,j}$, $1 \leq j \leq m_n$, $n = 1, 2, \ldots$ a double array of random variables and $t_{n,j} \in (0,1)^q$, $1 \leq j \leq m$, $n = 1, 2, \ldots$ is an associate array of covariables. We assume that for every fixed n the r. v.

$$Y_{n,1}, \ldots, Y_{n,m_n} \text{ are independent and } EY_{n,i}^2 < \infty,\ i = 1, \ldots, m_n \ . \tag{1}$$

Set $V(Y_{n,j}) = E\left(Y_{n,j} - EY_{n,j}\right)^2$. Introduce the random fields W_n by

$$W_n(t) = \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} \left(Y_{n,j} - EY_{n,j}\right) I_{(0,t]}(t_{n,j}) \ .$$

We suppose that there is a continuous function $B:\ [0,1]^q \to [0,\infty)$ such that

$$\frac{1}{m_n} \sum_{j=1}^{m_n} \mathrm{V}(Y_{n,j}) I_{(0,t]}(t_{n,j}) \ \stackrel{n\to\infty}{\longrightarrow}\ B(t) \tag{2}$$

for every $t \in [0,1]^q$. Let W_B be a Wiener field with variance B.

Theorem 1 *Assume the conditions (1), (2) are fulfilled. If the Lindeberg condition*

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{j=1}^{m_n} E\left(Y_{n,j} - EY_{n,j}\right)^2 I_{\{|Y_{n,j}-EY_{n,j}|\geq\varepsilon\sqrt{m_n}\}} = 0 \tag{3}$$

for every $\varepsilon > 0$ is fulfilled then

$$\mathcal{L}(W_n) \stackrel{n\to\infty}{\Longrightarrow} \mathcal{L}(W_B)$$

in the sense of weak convergence of distributions on $(D_q, \mathcal{D}_q)$.

Remark 3 The existence of a Wienere field, i.e. the existence of a Gaussian field with the properties formulated above follows from the proof of Theorem 1 (see Lemma 1).

We now ask under which assumptions the condition (1), (2), (3) are satisfied if the r. v. $Y_{n,j}$ belong to a model from the introduction. To be more precise we assume that

$$\mathcal{L}(Y_{n,j}) = Q_{g_0(t_{n,j})} \tag{4}$$

where Q_η is a family of distributions with

$$\int x\, Q_\eta(dx) = \eta$$

and

$$\lim_{N\to\infty} \sup_{a_1\leq\eta\leq a_2} \int\limits_{\{|x|>N\}} x^2\, Q_\eta(dx) = 0 \tag{5}$$

for every finite a_1, a_2 with $[a_1, a_2] \subseteq (a, b)$. Condition (5) implies that $EY_{n,j}^2 < \infty$ and moreover that the $Y_{n,j}^2$ are uniformly integrable. We suppose that Q_η depends continuous on η in the sense of weak convergence, i. e.

$$Q_{\eta_n} \;\Rightarrow\; Q_\eta, \text{ as } \eta_n \to \eta\ . \tag{6}$$

To describe the asymptotic behaviour of the sequence of experimental designs $\{t_{n,1}, \ldots, t_{n,m_n}\}$ we introduce the corresponding empirical measure μ_n by

$$\mu_n = \frac{1}{m_n} \sum_{j=1}^{m_n} \delta_{t_{n,j}}$$

and suppose that

$$\mu_n \stackrel{n\to\infty}{\Longrightarrow} \mu\ . \tag{7}$$

Theorem 2 *Assume the conditions (4), (5), (6), (7) are fulfilled. If g_0 and μ are continuous then*

$$\mathcal{L}(W_n) \ \Rightarrow \ \mathcal{L}(W_B)$$

where

$$B(t) = \int\limits_{[0,t]} \left(\int (x - g_0(s))^2 \, Q_{g_0(s)}(dx) \right) \mu(ds) \ .$$

Now we give an example which shows that the conditions (5), (6), (7) are fulfilled in many situations. Let Q be a distribution on $(\mathbb{R}_1, \mathcal{B}_1)$ with

$$\int x^2 \, Q(dx) < \infty \tag{8}$$

and

$$\int x \, Q(dx) = 0 \ .$$

Set $Q_\eta(B) = Q(B - \eta)$. Then

$$\int \varphi(t) \, Q_\eta(dt) = \int \varphi(t + \eta) \, Q(dt)$$

for every bounded φ. Moreover, φ is continuous then

$$\lim_{n \to \infty} \int \varphi(t) Q_{\eta_n}(dt) = \int \varphi(t) \, Q_\eta(dt)$$

if $\eta_n \to \eta$ by the Theorem of Lebesgue. Hence (6) is fulfilled. To show (5) we note that

$$\int\limits_{\{|x|>N\}} x^2 \, Q_\eta(dx) \le \int\limits_{\{|x|>N-|\eta|\}} (x+\eta)^2 \, Q(dx) \le 2 \int\limits_{\{|x|>N-|\eta|\}} (x^2 + \eta^2) \, Q(dx) \ .$$

Hence with $c = \max(|a|, |b|)$

$$\sup_{a \le \eta \le b} \int\limits_{\{|x|>N\}} x^2 \, Q_\eta(dx) \le 2c^2 Q_\eta(|x| > N - c) + 2 \int\limits_{\{|x|>N-c\}} x^2 \, Q(dx) \ \xrightarrow{N \to \infty} \ 0$$

for every fixed a, b by condition (8) and the Theorem of Lebesgue.

3 Applications

Suppose the r. v. $Y_{n,1},\ldots,Y_{n,m_n}$ are independent and $\mathcal{L}(Y_{n,j}) = Q_{g_0(t_{n,j})}$. Then under the assumptions formulated in Theorem 2 we have

$$\mathcal{L}(W_n) \;\Rightarrow\; \mathcal{L}(W_B)$$

where

$$\begin{aligned} W_n(t) &= \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} (Y_{n,j} - g_0(t_{n,j}))\, I_{(0,t]}(t_{n,j}) \\ B(t) &= \lim_{n\to\infty} \frac{1}{m_n} \sum_{j=1}^{m_n} \mathrm{V}(Y_{n,j}) I_{(0,t]}(t_{n,j}) = \int_{[0,t]} \left(\int (x - g_0(s))^2\, Q_{g_0(s)}(dx) \right) \mu(ds)\ . \end{aligned}$$

We introduce the sequence of test statistics T_n by

$$T_n = \int W_n^2(t)\, dt\ .$$

The continuous mapping theorem yields

$$\mathcal{L}(T_n) \;\Rightarrow\; \mathcal{L}(T) \tag{9}$$

as $n \longrightarrow \infty$ where

$$T = \int W_B^2(t)\, dt\ .$$

Note that the covariance function of W_B is given by $K(s,t) = B(s \wedge t)$. As the function B is continuous the kernel K is a Hilbert-Schmidt operator, i. e.

$$\int\int K^2(s,t)\, dsdt < \infty\ .$$

If B is not identical zero which is assumed in the sequel we have

$$\int\int K^2(s,t)\, dsdt > 0\ .$$

Let $f_0, f_1, \ldots$ be denote the complete system of orthogonal eigenfunctions belonging to the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$ of K. Then

$$W_B(t) \stackrel{d}{=} \sum_{l=0}^{\infty} X_l \sqrt{\lambda_l} f_l(t) \tag{10}$$

where the X_i are independent and standard normal and $\stackrel{d}{=}$ is the symbol for equality in distribution. The representation (10) yields

$$\int W_B^2(t)\, dt \stackrel{d}{=} \sum_{l=0}^{\infty} \lambda_l X_l^2\ .$$

The characteristic function of $\int W_B^2(t)\,dt$ is given by

$$\varphi(s) = \prod_{l=0}^{\infty} (1 - 2i\lambda_l s)^{-\frac{1}{2}} .$$

Let F be the corresponding distribution function and denote by $z_{1-\alpha}$ the $(1-\alpha)$-quantil of F. Note that $z_{1-\alpha}$ is uniquely determined.

To test the hypothesis $H_0 : g = g_0$ versus $H_A : g \neq g_0$ we introduce a sequence of tests φ_n by

$$\varphi_n = I_{(z_{1-\alpha},\infty)}(T_n) .$$

The statement (9) yields that the sequence $\{\varphi_n\}$ is an asymptotic α-test.

Now we consider a situation in which the sequence of eigenvalues can be explicitely evaluated. Let

$$Y_i = g_0(t_i) + \varepsilon_i$$

be the nonparametric regression model with i. i. d. errors ε_i which fulfill $E\varepsilon_i = 0, \sigma^2 = \mathrm{V}(\varepsilon_i) < \infty$. Using the notations above we get

$$B(t) = \lim_{n\to\infty} \frac{1}{m_n} \sum_{i=1}^{m_n} \mathrm{V}(Y_{n,i}) I_{(0,t]}(t_{n,i}) = \int\limits_{[0,t]} \sigma^2\, \mu(ds) = \sigma^2 \mu([0,t]) \tag{11}$$

and $K(s,t) = \sigma^2 \mu([0, s \wedge t])$. We now assume in addition that μ is the Lebesgue measure. Then

$$K(s,t) = \sigma^2 \prod_{i=1}^{q} (s_i \wedge t_i) .$$

Set $\widetilde{K}(x,y) = x \wedge y,\ 0 \leq x, y \leq 1$. It is well known that the eigenvalues and eigenfunctions of $\widetilde{K}$ are given by

$$\lambda_l = \left(l + \frac{1}{2}\right)^{-2} \frac{1}{\pi^2}$$

and $\varphi_l(t) = \sqrt{2} \sin\left(l + \frac{1}{2}\right)\pi t$, respectively. Due to the product structure of K the system of eigenvalues and eigenfunctions are given by

$$\begin{aligned} \lambda_\alpha &= \sigma^2 \lambda_{l_1} \cdots \lambda_{l_q} \\ \varphi_\alpha(t) &= \varphi_{l_1}(t_1) \cdots \varphi_{l_q}(t_q) \end{aligned}$$

where $\alpha = (l_1, \ldots, l_q)$. The characteristic function φ of T has the respresentation

$$\varphi(s) = \prod_{\alpha} (1 - 2i\lambda_\alpha s)^{-\frac{1}{2}} .$$

Now we investigate the asymptotic power of the test under local alternatives for the regression model. More precisely we suppose that

$$Y_{n,j} = g_0(t_{n,j}) + \frac{1}{\sqrt{m_n}} h(t_{n,j}) + \varepsilon_{n,j},\ \ j = 1, \ldots, m_n$$

where $h:\ [0,1]^q \to \mathbb{R}_1$ is a continuous function. Note that

$$\begin{aligned}
W_n(t) &= \frac{1}{\sqrt{m_n}} \sum_j \left(Y_{n,j} - g_0(t_{n,j})\right) I_{(0,t]}(t_{n,j}) \\
&= \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} \left(Y_{n,j} - g_0(t_{n,j}) - \frac{1}{\sqrt{m_n}} h(t_{n,j})\right) I_{(0,t]}(t_{n,j}) + \int\limits_{[0,t]} h(s)\, \mu_n(ds) \\
&= \frac{1}{\sqrt{m_n}} \sum_j \varepsilon_{n,j} I_{(0,t]}(t_{n,j}) + \int\limits_{[0,t]} h(s)\, \mu_n(ds)\ .
\end{aligned}$$

As μ is supposed to be continuous we get that for the boundary $\partial[0,t]$ of $[0,t]$ holds $\mu(\partial[0,t]) = 0$. Hence for every fixed t the function $h(s)I_{[0,t]}(s)$ is continuous μ-a. s. . Hence by $\mu_n \Rightarrow \mu$

$$\int\limits_{[0,t]} h(s)\, \mu_n(ds) \ \longrightarrow \ \int\limits_{[0,t]} h(s)\, \mu(ds) =: H(t)\ .$$

Assume the $\varepsilon_{n,j}$ are i. i. d. with $E\varepsilon_{n,j} = 0, \sigma^2 = E\varepsilon_{n,j}^2 < \infty$. Then the application of Theorem 1 yields that $\mathcal{L}(W_n) \Rightarrow \mathcal{L}(W_B + H)$, where B is given by (11). To investigate the asymptotic power we note that

$$W_B + H \stackrel{d}{=} \sum_{l=0}^{\infty} \sqrt{\lambda_l}(X_l + \delta_l)\varphi_l$$

where $\delta_l = \frac{1}{\sqrt{\lambda_l}} \int H(t)\varphi_l(t)\mu(dt)$. Consequently

$$T \stackrel{d}{=} \sum_{l=0}^{\infty} \lambda_l (X_l + \delta_l)^2$$

and the asymptotic power of the sequence of tests φ_n is given by

$$\lim_{n\to\infty} E_h\varphi_n = P\left(T > z_{1-\alpha}\right)\ .$$

4 Proofs

Given $t_1, \dots, t_l \in [0,1]^q$ we denote by $\pi_{t_1,\dots,t_l} : D_q \to \mathbb{R}^l$ the projection defined by

$$\pi_{t_1,\dots,t_l}(x) = (x(t_1), \dots, x(t_l)) \ .$$

If P is a distribution on $(D_q, \mathcal{D}_q)$ and $t_1, \dots, t_l,\ l = 1, 2, \dots$ are arbitraryly choosen then the system of distributions $P \circ \pi_{t_1,\dots,t_l}$ is called the family of all finite dimensional distributions. For $x \in D_q$ the modul of continuity is defined by

$$w_x(\delta) = \sup_{|s-t|<\delta} |x(s) - x(t)| \, , \ 0 < \delta \le 1 \ .$$

The following criteria for weak convergence can be found in Neuhaus [6]. Given $x \in D_q$ we set

$$\|x\| := \sup_{\tau \in [0,1]^q} |x(\tau)| \ .$$

Lemma 1 *Let $P_1, P_2, \dots$ be a sequence of distributions on $(D_q, \mathcal{D}_q)$ such that for every fixed $t_1, \dots, t_l$ the distributions $P_{n;t_1,\dots,t_l}$ converge weakly say to $P_{t_1,\dots,t_l}$. If*

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P_n \left(\{x : \ w_x(\delta) > \varepsilon\} \right) = 0$$

for every $\varepsilon > 0$ then there exists a uniquely determined distribution P with finite dimensional distributions $P_{t_1,\dots,t_l}$ such that

$$P_n \ \Rightarrow \ P \quad \textit{as } n \to \infty \ .$$

Moreover $P(C_q) = 1$.

Given $t = (t_1, \dots, t_q) \in [0,1]^q$ and $0 \le \tau_i \le 1$ we set

$$\begin{aligned} x^{(p)}_{\tau_i}(t) &= x(t_1, \dots, t_{p-1}, \tau_i, t_{p+1}, \dots, t_q) \\ \left\| x^{(p)}_{\tau_1} - x^{(p)}_{\tau_2} \right\| &= \sup_{\substack{0 \le t_j \le 1 \\ j \ne p}} \left| x^{(p)}_{\tau_1}(t) - x^{(p)}_{\tau_2}(t) \right| \end{aligned} \tag{12}$$

and for $0 \le \tau_1 < \tau_2 < \tau_3 \le 1$

$$\begin{aligned} m_p(\tau_1, \tau_2, \tau_3, x) &= \min \left(\left\| x^{(p)}_{\tau_1} - x^{(p)}_{\tau_2}, \right\| , \left\| x^{(p)}_{\tau_2} - x^{(p)}_{\tau_3} \right\| \right) \\ \mathcal{M}(x) &= \max_{1 \le p \le q} \sup_{0 \le \tau_1 \le \tau_2 \le \tau_3 \le 1} m_p(\tau_1, \tau_2, \tau_3, x) \ . \end{aligned}$$

Suppose that $x(t_1, \dots, t_q) = 0$ if at least one at the t_l is zero. Set $u_p = (1, \dots, 1, t_{p+1}, \dots, t_q)$, $\widetilde{u}_p = (1, \dots, 1, 0, t_{p+2}, \dots, t_q)$ and note that

$$\begin{aligned} |x(u_{p-1})| &\le \min \left(|x(u_{p-1}) - x(\widetilde{u}_{p-1})|, |x(u_p) - x(u_{p-1})| \right) + |x(u_p)| \\ &\le \mathcal{M}(x) + |x(u_p)| \ . \end{aligned}$$

Hence

$$\|x\| \le q\mathcal{M}(x) + |x(1,\dots,1)| \ . \tag{13}$$

To each $x \in D_q$ we associate a set function on the system of q-dimensional intervals $(s,t] = (s_1,t_1] \times \cdots \times (s_q,t_q]$ (where $s = (s_1,\dots,s_q), t = (t_1,\dots,t_q)$) by

$$x((s,t]) = \sum_{\varepsilon_1=0,1} \cdots \sum_{\varepsilon_q=0,1} (-1)^{q-\sum\limits_p \varepsilon_p} x\left(s_1 + \varepsilon_1(t_1-s_1),\dots,s_q+\varepsilon_q(t_q-s_q)\right) \ .$$

Assume now S is a stochastic process with paths in D_q. The estimation of $\mathcal{M}(S)$ in the next Lemma is crucial for all further considerations. The statement formulated below is a special case of a more general result established in Bickel, Wischura [1].

Lemma 2 *If $\gamma > 0$ and*

$$E\left|S((s,t])S((u,v])\right|^\gamma \le \mu_{\gamma,S}((s,t])\mu_{\gamma,S}((u,v]) \tag{14}$$

for every disjoint $(s,t],(u,v]$ with some finite measure $\mu_{\gamma,S}$ on the Borel sets of $[0,1]^q$ then there is a constant K depending only on γ such that

$$P\left(\mathcal{M}(S) \ge \lambda\right) \le K\lambda^{2\gamma}\left(\mu_{\gamma,S}([0,1]^q)\right)^2 \ . \tag{15}$$

Assume $\xi_{n,1},\dots,\xi_{n,m_n},\ n = 1,2,\dots,$ is a double array of random variables which are independent for every fixed n and fulfill $E\xi_{n,i} = 0, \sigma^2_{n,i} = E\xi^2_{n,i} < \infty$. If in addition $E\xi^4_{n,i} < \infty$ then we set $\varkappa_{n,i} = E\xi^4_{n,i}$. Let $t_{n,1},\dots,t_{n,m_n},\ n = 1,2,\dots,\ t_{n,j} \in [0,1]^q$ the corresponding double array of covariables. Denote by $t_{n,j,1},\dots,t_{n,j,q}$ the components of $t_{n,j}$. Introduce the sequence of stochastic processes W_n by

$$W_n(t) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \xi_{n,i} I_{(0,t]}(t_{n,i}) \ . \tag{16}$$

We suppose that $t_{n,j} \in (0,1)^q$ for every n,j,i. Hence $W_n(t) = 0$ if a least one component of t is zero and W_n is random element of D_q. The definition of W_n yields

$$W_n((s,t]) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \xi_{n,i} I_{(s,t]}(t_{n,i}) \ .$$

Note that by the independence of $\xi_{n,1},\dots,\xi_{n,m_n}$ the random variables $W_n((s,t])$ and $W_n((u,v])$ are independent again for disjoint $(s,t],(u,v]$. To describe the moments of $W_n((s,t])$ we introduce the following measures on the Borel sets of $(0,1]^q$

$$\nu_{W_n} = \frac{1}{m_n} \sum_{i=1}^{m_n} \sigma^2_{n,i} \delta_{t_{n,i}} \tag{17}$$

$$\varkappa_{W_n} = \frac{1}{m_n^2} \sum_{i=1}^{m_n} \varkappa_{n,i} \delta_{t_{n,i}} \ . \tag{18}$$

Note that for every independent random variables $X_1, \ldots X_m$ with $EX_i = 0$ it holds

$$E\left(\sum_{i=1}^{m} X_i\right)^2 = \sum_{i=1}^{m} EX_i^2 \tag{19}$$

and for $EX_i^4 < \infty$

$$E\left(\sum_{i=1}^{m} X_i\right)^4 = \sum_{i=1}^{m} EX_i^4 + 3\left(\sum_{i\neq j} EX_i^2 EX_j^2\right) . \tag{20}$$

The application of (19) and (20) to $W_n((s,t])$ yields

$$\begin{aligned} EW_n^2((s,t]) &= \nu_{W_n}((s,t]) \\ EW_n^4((s,t]) &\leq \varkappa_{W_n}((s,t]) + 3\left(\nu_{W_n}((s,t])\right)^2 . \end{aligned} \tag{21}$$

Lemma 3 *Let W_n be defined by (16) and assume $E\xi_{n,i}^2 < \infty, E\xi_{n,i} = 0$. Suppose that $0 < t_{n,j,i} < 1$ and*

$$c_1 = \sup_n \nu_{W_n}\left((0,1]^q\right) < \infty. \tag{22}$$

For every $\lambda > 0$ there is a constant $d_1(c_1, \lambda)$ such that

$$P\left(\|W_n\| \geq \lambda\right) \leq d_1(c_1,\lambda)\nu_{W_n}\left((0,1]^q\right)$$

If in addition $E\xi_{n,i}^4 < \infty$ and

$$c_2 = \sup_n \varkappa_{W_n}\left((0,1]^q\right) < \infty \tag{23}$$

then there is $d_2(c_1, c_2, \lambda)$ such that

$$P\left(\|W_n\| \geq \lambda\right) \leq d_2(c_1,c_2,\lambda)\left(\varkappa_{W_n}\left((0,1]^q\right) + 3\left(\nu_{W_n}\left((0,1]^q\right)\right)^2\right) .$$

Proof: We obtain from inequality (13) that

$$\|W_n\| \leq q\mathcal{M}(W_n) + |W_n(1,\ldots,1)| .$$

Hence

$$P\left(\|W_n\| \geq \lambda\right) \leq P\left(\mathcal{M}(W_n) \geq \frac{\lambda}{q+1}\right) + P\left(|W_n(1,\ldots,1)| \geq \frac{\lambda}{q+1}\right) . \tag{24}$$

Set $\gamma = 2$. Due to the independence of $W_n((s,t]), W_n((u,v])$ for disjoint $(s,t], (u,v]$ the condition (14) in Lemma 2 is fulfilled with

$$\mu_{2,W_n} = \nu_{W_n} .$$

Hence by (22)

$$P\left(\mathcal{M}(W_n) \geq \frac{\lambda}{q+1}\right) \leq K\left(\frac{\lambda}{q+1}\right)^4 \left(\nu_{W_n}\left((0,1]^q\right)\right)^2$$
$$\leq K\left(\frac{\lambda}{q+1}\right)^4 c_1 \nu_{W_n}\left((0,1]\right)^q \ .$$

Moreover, Chebyshev's inequality yields

$$P\left(|W_n(1,\dots,1)| \geq \frac{\lambda}{q+1}\right) \leq \frac{(q+1)^2}{\lambda^2} \nu_{W_n}\left((0,1]^q\right) \ .$$

The last two inequalities yield the first statement. To prove the second statement we set $\gamma = 4$ and note that by (21) the condition (14) is fulfilled with

$$\mu_{4,W_n}((s,t]) = \varkappa_{W_n}((s,t]) + 3\nu_{W_n}\left((0,1]^q\right)\nu_{W_n}((s,t]) \ .$$

The rest of the proof is similar as for the first statement. ■

To investigate the modul of continuity of W_n we study differences $W_n(\widetilde{u}_1) - W_n(\widetilde{u}_2)$ if the two vectors $\widetilde{u}_1, \widetilde{u}_2$ are different only in one component. More precisely we suppose that $0 < s_1 < s_2 < 1$ are fixed. Then

$$\sup_{\substack{s_1<t\leq s_2 \\ 0<u_j\leq 1, j\neq i}} |W_n(u_1,\dots,u_{i-1},t,u_{i+1},\dots,u_q) - W_n(u_1,\dots,u_{i-1},s_1,u_{i+1},\dots,u_q)| \quad (25)$$
$$= \sup_{\substack{s_1<t\leq s_2 \\ 0<u_j<1, j\neq i}} \left| \frac{1}{\sqrt{m_n}} \sum_{l=1}^{m_n} \xi_{n,l} I_{(s_1,t]}(t_{n,l,i}) \prod_{j\neq i} I_{(0,u_j]}(t_{n,l,j}) \right|$$
$$= \sup_{\substack{s_1<t\leq s_2 \\ 0<u_j<1, j\neq i}} \left| \frac{1}{\sqrt{m_n}} \sum_{l=1}^{m_n} \xi_{n,l} I_{(s_1,s_2]}(t_{n,l,i}) I_{(0,t]}(t_{n,l,i}) \prod_{j\neq i} I_{(0,u_j]}(t_{n,l,j}) \right|$$
$$= \sup_{\substack{0<u_j<1 \\ j=1,\dots,q}} \left| \frac{1}{\sqrt{m_n}} \sum_{l=1}^{m_n} \widetilde{\xi}_{n,l} \prod_{j=1}^{q} I_{(0,u_j]}(t_{n,l,j}) \right|$$
$$= \|\Delta_i(W_n, s_1, s_2)\|$$

where $\widetilde{\xi}_{n,l} = \xi_{n,l} I_{(s_1,s_2]}(t_{n,l,i})$ and the process $\Delta_i(W_n, s_1, s_2)$ is defined by

$$\Delta_i(W_n, s_1, s_2)(u) = \frac{1}{\sqrt{m_n}} \sum_{l=1}^{m_n} \widetilde{\xi}_{n,l} I_{(0,u]}(t_{n,l}) \ .$$

If ϱ is a measure on the Borel sets of $(0,1]^q$ we denote by $\varrho^{(i)}$ the i-th marginal measure defined by

$$\varrho^{(i)}((a,b]) = \varrho\left((0,1] \times \cdots \times (0,1] \times (a,b] \times (0,1] \times \cdots \times (0,1]\right)$$

where $0 \le a < b \le 1$.

Now we estimate the measures $\nu_{\Delta_i(W_n,s_1,s_2)}$ and $\varkappa_{\Delta_i(W_n,s_1,s_2)}$. The definitions (17), (18) and the definition of $\widetilde{\xi}_{n,l}$ provide

$$\begin{aligned} \nu_{\Delta_i(W_n,s_1,s_2)}\left((0,1]^q\right) &= \nu^{(i)}_{W_n}\left((s_1,s_2]\right) \\ \varkappa_{\Delta_i(W_n,s_1,s_2)}\left((0,1]^q\right) &= \varkappa^{(i)}_{W_n}\left((s_1,s_2]\right) \ . \end{aligned}$$

The application of Lemma 3 to the process $\Delta_i(W_n,s_1,s_2)$ yields

Lemma 4 *If $E\xi^4_{n,i} < \infty$, (22) and (23) are fulfilled then*

$$P\left(\|\Delta_i(W_n,s_1,s_2)\| \ge \lambda\right) \le d_2(c_1,c_2,\lambda)\left(\varkappa^{(i)}_{W_n}((s_1,s_2]) + 3\left(\nu^{(i)}_{W_n}((s_1,s_2])\right)^2\right) \ .$$

To get precise estimates of the modul of continuity we apply a truncation technique. To be more precise we set for $a > 0$

$$\overline{\xi}_{n,i} = \xi_{n,i} I_{\{|\xi_{n,i}| \le a\sqrt{m_n}\}} - E\xi_{n,i} I_{\{|\xi_{n,i}| \le a\sqrt{m_n}\}} \ .$$

Note that $E\overline{\xi}_{n,i} = 0$,

$$E\overline{\xi}^2_{n,i} \le E\xi^2_{n,i} = \sigma^2_{n,i}$$

and by $(a+b)^4 \le 8(a^4+b^4)$, $|EX|^4 \le EX^4$

$$\begin{aligned} E\overline{\xi}^4_{n,i} &\le 8\left(a^2 m_n E\xi^2_{n,i} + E\xi^4_{n,i} I_{\{|\xi_{n,i}| \le a\sqrt{m_n}\}}\right) \\ &\le 8\left(a^2 m_n \sigma^2_{n,i} + a^2 m_n \sigma^2_{n,i}\right) = 16 a^2 m_n \sigma^2_{n,i} \ . \end{aligned}$$

Set

$$W_{n,a}(t) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \overline{\xi}_{n,i} I_{(0,t]}(t_{n,i}).$$

Then

$$\nu_{W_{n,a}}((s,t]) \le \nu_{W_n}((s,t])$$

and

$$\varkappa_{W_{n,a}}((s,t]) \le \frac{1}{m_n^2} \sum_{i=1}^{m_n} 16 a^2 m_n \sigma^2_{n,i} \delta_{t_{n,i}}((s,t]) = 16 a^2 \nu_{W_n}((s,t])$$

and $c_2 = \sup\limits_n \varkappa_{W_{n,a}}\left((0,1]^q\right) \le 16 c_1$ if $a \in (0,1]$. The application of Lemma 4 to $W_{n,a}$ yields

Lemma 5 *If (22) is fulfilled then for* $a \in (0,1]$

$$P\left(\|\Delta_i(W_{n,a},s_1,s_2)\| \geq \lambda\right) \leq d_2(c_1,16c_1,\lambda)\left(16a^2\nu^{(i)}_{W_n}((s_1,s_2]) + 3\left(\nu^{(i)}_{W_n}((s_1,s_2])\right)^2\right) \ .$$

We now show that under the Lindeberg condition the modul of continuity of the processes W_n fulfills the assumptions in Lemma 1.

Lemma 6 *Assume that* $\nu_{W_n} \Rightarrow \nu$ *as* $n \to \infty$ *where* ν *is some continuous measure on* $(0,1]^q$*. If the Lindeberg condition*

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{i=1}^{m_n} E\xi^2_{n,i} I_{\{|\xi_{n,i}|>\varepsilon\sqrt{m_n}\}} = 0 \tag{26}$$

is fulfilled for every $\varepsilon > 0$ *then*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\left(w_{W_n}(\delta) > \varepsilon\right) = 0$$

for every $\varepsilon > 0$.

Proof: The proof is devided into several steps
1) We show that W_n can be approximated by the process $W_{n,a}$ in the uniform metrik. To this end we note that by $E\xi_{n,i} = 0$

$$\begin{aligned}
\overline{W}_n(t) &= W_n(t) - W_{n,a}(t) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \left(\xi_{n,i} I_{\{|\xi_{n,i}|>a\sqrt{m_n}\}} - E\xi_{n,i} I_{\{|\xi_{n,i}|>a\sqrt{m_n}\}}\right) \\
&= \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \overline{\overline{\xi}}_{n,i} \ .
\end{aligned}$$

The inequality

$$E(\overline{\overline{\xi}}_{n,i})^2 \leq E\xi^2_{n,i} I_{\{|\xi_{n,i}|>a\sqrt{m_n}\}}$$

implies

$$\nu_{\overline{W}_n}\left((0,1]^q\right) \leq \frac{1}{m_n} \sum_{i=1}^{m_n} E\xi^2_{n,i} I_{\{|\xi_{n,i}|>a\sqrt{m_n}|\}} \overset{n\to\infty}{\longrightarrow} 0$$

by the Lindeberg condition. Hence by the first statement in Lemma 3

$$\limsup_{n\to\infty} P\left(\|W_n - W_{n,a}\| > \lambda\right) = 0 \ . \tag{27}$$

2) To estimate the modul of continuity of $W_{n,a}$ we remark that

$$\begin{aligned}
|W_{n,a}(s) - W_{n,a}(t)| &= |W_{n,a}(s_1,\ldots,s_q) - W_{n,a}(t_1,\ldots,t_q)| \\
&\leq |W_{n,a}(s_1,\ldots,s_q) - W_{n,a}(t_1,s_2,\ldots,s_q)| \\
&\quad + \ldots + |W_{n,a}(t_1,\ldots,t_{q-1},s_q) - W_{n,a}(t_1,\ldots,t_q)| \ .
\end{aligned}$$

Hence by the definition of $\Delta_i(W_{n,a}, s_1, s_2)$: for fixed $s = (s_1, \ldots, s_q)$ and $\delta > 0$

$$\sup_{t:\,|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| \leq \sum_{i=1}^{q} \|\Delta_i\left(W_{n,a}, (s_i - \delta) \vee 0, (s_i + \delta) \wedge 1\right)\| \tag{28}$$

where $a \wedge b = \min(a, b), a \vee b = \max(a, b)$. From the representation of $\|\Delta_i(W_{n,a}, s_1, s_2)\|$ in (25) we see that

$$\|\Delta_i(W_{n,a}, u_1, u_2)\| \geq \|\Delta_i(W_{n,a}, v_1, v_2)\| \tag{29}$$

if $u_1 < v_1, u_2 > v_2$. Then for every $m = 1, 2, \ldots$

$$\sup_{0 \leq s_i \leq 1} \|\Delta_i\left(W_{n,a}, (s_i - \delta) \vee 0, (s_i + \delta) \wedge 1\right)\| \leq \sup_{l=0,1,\ldots,m} \left\|\Delta_i\left(W_{n,a}, \frac{l}{m}, \frac{l + k(m, \delta)}{m}\right)\right\|$$

and

$$\begin{aligned} &P\left(\sup_{0 \leq s_i \leq 1} \|\Delta_i\left(W_{n,a}, (s_i - \delta) \vee 0, (s_i + \delta) \wedge 1\right)\| > \alpha\right) \\ &\leq \sum_{l=0}^{m} P\left(\left\|\Delta_i\left(W_{n,a}, \frac{l}{m}, \frac{l + k(m, \delta)}{m}\right)\right\| > \alpha\right) , \end{aligned}$$

where $k(m, \delta)$ is the integer which fulfills the inequality

$$\frac{k(m, \delta)}{m} \geq 2\delta > \frac{k(m, \delta) - 1}{m} .$$

Applying this inequality to (28) we get

$$P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda\right) \leq \sum_{i=1}^{q} \sum_{l=0}^{m} P\left(\left\|\Delta_i\left(W_{n,a}, \frac{l}{m}, \frac{m + k(m, \delta)}{m}\right)\right\| > \frac{\lambda}{q}\right) .$$

The assumption $\nu_{W_n} \Rightarrow \nu$ implies

$$c_1 = \sup_n \nu_{W_n}\left((0, 1]^q\right) < \infty .$$

Hence by Lemma 5

$$\begin{aligned} &P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda\right) \\ &\leq \sum_{i=1}^{q} \sum_{l=0}^{m} d_2(c_1, 16c_1, \lambda) \left(16a^2 \nu_{W_n}^{(i)}\left(\left(\frac{l}{m}, \frac{l + k(m, \delta)}{m}\right]\right)\right. \\ &\quad \left. +3\left(\nu_{W_n}^{(i)}\left(\left(\frac{l}{m}, \frac{l + k(m, \delta)}{m}\right]\right)\right)^2\right) . \end{aligned}$$

The assumption $\nu_{W_n} \Rightarrow \nu$ in conjunction with the continuity of ν yields

$$\lim_{n\to\infty} \nu_{W_n}^{(i)}((a,b]) = \nu^{(i)}((a,b])$$

for every $0 \le a < b \le 1$. Hence

$$\begin{aligned}&\limsup_{a\downarrow 0}\limsup_{n\to\infty} P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda\right)\\ &\le\ 3d_2\left(c_1, 16c_1, \frac{\lambda}{q}\right)\sum_{i=1}^{q}\sum_{l=0}^{m}\left(\nu^{(i)}\left(\left(\frac{l}{m}, \frac{l+k(m,\delta)}{m}\right]\right)\right)^2 .\end{aligned}$$

Taking $\delta \downarrow 0$ we get

$$\begin{aligned}&\lim_{\delta\downarrow 0}\limsup_{a\downarrow 0}\limsup_{n\to\infty} P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda\right)\\ &\le\ 3d_2\left(c_1, 16c_1, \frac{\lambda}{q}\right)\sum_{i=1}^{q}\sum_{l=0}^{m}\left(\nu^{(i)}\left(\left(\frac{l}{m}, \frac{l+1}{m}\right]\right)\right)^2\\ &\le\ 3d_2\left(c_1, 16c_1, \frac{\lambda}{q}\right)\max_{i=1,\dots,q}\max_{1\le l\le m-1}\nu^{(i)}\left(\left(\frac{l}{m}, \frac{l+1}{m}\right]\right)\sum_{i=1}^{q}\nu^{(i)}((0,1]) .\end{aligned}$$

The continuity of the $\nu^{(i)}$ implies that

$$\sup_{1\le l\le m-1}\nu^{(i)}\left(\left(\frac{l}{m}, \frac{l+1}{m}\right]\right) \overset{m\to\infty}{\longrightarrow} 0 ,$$

which yields

$$\lim_{\delta\downarrow 0}\lim_{a\downarrow 0}\limsup_{n\to\infty} P\left(w_{W_{n,a}}(\delta) > \lambda\right) = 0 . \tag{30}$$

3) It holds

$$P\left(w_{W_n}(\delta) \ge \lambda\right) \le P\left(\|W_n - W_{n,a}\| \ge \frac{\lambda}{3}\right) + P\left(w_{W_{n,a}}(\delta) \ge \frac{\lambda}{3}\right) .$$

Taking in this inequality at first $n \to \infty$ then $a \downarrow 0$ and finally $\delta \downarrow 0$ we get from (27) and (30) that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P\left(w_{W_n}(\delta) \ge \lambda\right) = 0$$

which completes the proof.

Lemma 7 *If the Lindeberg condition (26) in Lemma 6 is fulfilled and $\nu_{W_n} \Rightarrow \nu$ then the finite dimensional distributions of W_n converge weakly to the corresponding finite dimensional distributions of a Wiener field W with variance B where $B(t) = \nu((0,t])$.*

Proof: Note that both W_n and W vanishes on the lower boundary. Hence $W_n(t) = W_n((0,t]), W(t) = W((0,t])$ and instead of proving the weak convergence of the finite dimensional distributions of W_n we may equivalently prove that for any disjoint $(s_1,t_1],\ldots,(s_m,t_m]$ the distribution of

$$W_n((s_1,t_1]),\ldots,W_n((s_m,t_m])$$

converges weakly to the distribution of

$$W((s_1,t_1]),\ldots,W((s_m,t_m]) \; .$$

Due to the independence of $W_n((\cdot,\cdot])$ and $W((\cdot,\cdot])$, respectively, on disjoint intervals it remains to prove that for any fixed s,t

$$\mathcal{L}\left(W_n((s,t])\right) \overset{n\to\infty}{\Longrightarrow} \mathcal{L}\left(W((s,t])\right) \; . \tag{31}$$

Consider first the case $\nu((s,t]) = 0$. Then

$$V\left(W_n((s,t])\right) = \nu_{W_n}((s,t]) \;\to\; \nu((s,t]) = 0.$$

$W_n((s,t])$ converges in probability to $0 = W((s,t])$ and hence (31).

Consider now $\nu((s,t]) > 0$. Set

$$b_n^2 = \sum_{i=1}^{m_n} \sigma_{n,i}^2 I_{(s,t]}(t_{n,i})$$

and $\eta_{n,i} = \xi_{n,i} I_{(s,t]}(t_{n,i})$. We obtain

$$\frac{b_n^2}{m_n} = \frac{1}{m_n}\mathrm{V}\left(\sum_{i=1}^{m_n}\eta_{n,i}\right) = \nu_{W_n}((s,t]) \overset{n\to\infty}{\longrightarrow} \nu((s,t]) \; . \tag{32}$$

There are $\alpha > 0$ and n_0 such that for $n \geq n_0$

$$b_n \geq \alpha\sqrt{m_n} > 0 \; .$$

We note that

$$W_n((s,t]) = \frac{b_n}{\sqrt{m_n}}\frac{1}{b_n}\sum_{i=1}^{m_n}\eta_{n,i} \; . \tag{33}$$

It holds

$$\begin{aligned}
&\frac{1}{b_n^2}\sum_{i=1}^{m_n} E\left|\eta_{n,i}\right|^2 I_{\{|\eta_{n,i}|>\varepsilon b_n\}} \\
&\quad\leq \frac{m_n}{b_n^2}\frac{1}{m_n}\sum_{i=1}^{m_n} E\xi_{n,i}^2 I_{\{|\xi_{n,i}|>\varepsilon\alpha\sqrt{m_n}\}} \overset{n\to\infty}{\longrightarrow} 0 \; .
\end{aligned}$$

The central limit theorem implies

$$\mathcal{L}\left(\frac{1}{b_n}\sum_{i=1}^{m_n}\eta_{n,i}\right) \;\Rightarrow\; N(0,1)\ . \tag{34}$$

(32), (33), (34) yield (31).

Proof of Theorem 2:
Set

$$a_1 = \inf_{s\in[0,1]^q} g_0(s),\ \ a_2 = \sup_{s\in[0,1]^q} g_0(s)\ .$$

By the continuity of g_0 we get $-\infty < a_1 \le a_2 < \infty$. Set

$$\psi(s) = \int (x-g_0(s))^2\, Q_{g_0(s)}(dx) = \int x^2\, Q_{g_0(s)}(dx) - g_0^2(s)\ .$$

Denote by $\chi_N(x)$ a picewise linear function with $0 \le \chi_N(x) \le 1$ such that $\chi_N(x) = 1, -N \le x \le N; \chi_N(x) = 0, x < -(N+1), x > N+1$. As

$$\sup_{s\in[0,1]^q}\int x^2\, Q_{g_0(s)}(dx) \le \sup_{a_1\le\eta\le a_2}\int x^2\, Q_\eta(dx)$$

we see from (5) that ψ is bounded. Furthermore for $s_n \to s$

$$\begin{aligned}|\psi(s)-\psi(s_n)| \;\;&\le\;\; 2\sup_{a_1\le\eta\le a_2}\int x^2\,(1-\chi_N(x))\,Q_\eta(dx)\\ &\quad+\left|\int x^2\chi_N(x)\,Q_{g_0(s)}(dx) - \int x^2\chi_N(x)\,Q_{g_0(s_n)}(dx)\right|\ .\end{aligned}$$

Given $\varepsilon > 0$ we choose N sucht that the first term does not exceed $\frac{\varepsilon}{2}$. By condition (6) there is n_0 such that

$$\left|\int x^2\chi_N(x)\,Q_{g_0(s)}(dx) - \int x^2\chi_N(x)\,Q_{g_0(s_n)}(dx)\right| < \frac{\varepsilon}{2}$$

for even $n \ge n_0$. Hence

$$|\psi(s)-\psi(s_n)| < \varepsilon$$

for every $n \ge n_0$ which proves the continuity of ψ. As μ is continuous the function $f_t(s) = I_{[0,t]}(s)\psi(s)$ is μ-a. e. continuous for every t and bounded. This implies the continuity of B. We have

$$\begin{aligned}\frac{1}{m_n}\sum_{j=1}^{m_n} V(Y_{n,j})I_{(0,t]}(t_{n,j}) \;\;&=\;\; \int\limits_{[0,t]}\left(\int (x-g_0(s))^2\,Q_{g_0(s)}(dx)\right)\mu_n(ds)\\ &=\;\; \int I_{(0,t]}(s)\psi(s)\,\mu_n(ds)\\ &=\;\; \int f_t(s)\,\mu_n(ds) \;\overset{n\to\infty}{\longrightarrow}\; \int f_t(s)\,\mu(ds)\end{aligned}$$

where the last statement follows from assumption (7).

To prove the Lindeberg condition we note that

$$\{|Y_{n,j} - EY_{n,j}| > \varepsilon\sqrt{m_n}\} \subseteq \{|Y_{n,j}| > \varepsilon\sqrt{m_n} - \max(|a_1|, |a_2|)\} \ .$$

Set $b_n(\varepsilon) = \varepsilon\sqrt{m_n} - \max(|a_1|, |a_2|)$. Then

$$\begin{aligned}
&\sup_{1\le j\le m_n} E|Y_{n,j} - EY_{n,j}|^2 I_{\{|Y_{n,j}-EY_{n,j}|>\varepsilon\sqrt{m_n}\}} \\
&\le \sup_{1\le j\le m_n} EY_{n,j}^2 I_{\{|Y_{n,j}|>b_n(\varepsilon)\}} + \sup_{1\le j\le m_n} |EY_{n,j}|^2 I_{\{|Y_{n,j}|>b_n(\varepsilon)\}} \\
&\le 2 \sup_{1\le j\le m_n} EY_{n,j}^2 I_{\{|Y_{n,j}|>b_n(\varepsilon)\}} \\
&\le 2 \sup_{a_1\le\eta\le a_2} \int\limits_{\{|x|>b_n(\varepsilon)\}} x^2\, Q_\eta(dx) \ \overset{n\to\infty}{\longrightarrow}\ 0
\end{aligned}$$

by assumption (5). Hence the Lindeberg condition is established.

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Author:

P. Dencker
Fachbereich Mathematik
Universität Rostock
D-18051 Rostock
Germany

email: peter.dencker@stud.uni-rostock.de

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D. Lau

Die maximalen Klassen von $\bigcap_{\varrho\in Q} Pol_3\varrho$ für $Q \subseteq \mathfrak{P}(\{0,1,2\})$, Teil II

In Fortsetzung von Teil I dieser Arbeit sollen nachfolgend die maximalen Klassen von Teilklassen der Form $\bigcap_{\varrho\in Q} Pol_3\varrho$ mit $Q \in \{\{\{a,b\},\{c\}\},\{\{a,b\},\{a\},\{b\}\},\{\{a,b\},\{a\},\{c\}\}, \{\{a,b\},\{a,c\}\},\{\{a,b\},\{a,c\},\{b\}\}\}$, $\{a,b,c\} := \{0,1,2\}$ ermittelt werden.

Lemma 1 *Seien*

$$\varrho_1 := \begin{pmatrix} 0\ 2\ 1\ 2 \\ 2\ 0\ 2\ 1 \end{pmatrix} \qquad \varrho_2 := \begin{pmatrix} 0\ 1\ 2\ 0\ 1 \\ 0\ 1\ 2\ 1\ 0 \end{pmatrix}, \quad \varrho_3 := \begin{pmatrix} 0\ 1\ 2\ 0\ 1 \\ 0\ 1\ 2\ 2\ 2 \end{pmatrix},$$
$$\varrho_4 := \begin{pmatrix} 0\ 1\ 2\ 0\ 2\ 1\ 2 \\ 0\ 1\ 2\ 2\ 0\ 2\ 1 \end{pmatrix}, \ \varrho_5 := \begin{pmatrix} 0\ 1\ 0\ 2\ 1\ 2 \\ 0\ 1\ 2\ 0\ 2\ 1 \end{pmatrix}, \ \varrho_6 := \begin{pmatrix} 0\ 0\ 1\ 1\ 2\ 0\ 1 \\ 0\ 1\ 0\ 1\ 2\ 2\ 2 \end{pmatrix}.$$

Dann gilt:

(a) $T_{0,1;2} \cap Pol_3\varrho_1 \subseteq T_{0,1;2} \cap Pol_3\varrho_2$,

(b) $T_{0,1;2} \cap Pol_3\varrho_3 \subseteq T_{0,1;2} \cap Pol_3\varrho_4$,

(c) $T_{0,1;2} \cap Pol_3\varrho_5 \subseteq T_{0,1;2} \cap Pol_3\varrho_4$,

(d) $T_{0,1;2} \cap Pol_3\varrho_6 \subseteq T_{0,1;2} \cap Pol_3\varrho_2$.

Beweis: Bezeichne $\circ$ das Relationenprodukt und für binäre Relationen ϱ sei $\tau\varrho := \{(y,x) \mid (x,y) \in \varrho\}$. Obige Behauptungen ergeben sich dann aus $\varrho_1 \circ \varrho_1 = \varrho_2$, $(\tau\varrho_3) \circ \varrho_3 = \varrho_4$, $((E_3 \times E_2) \cap \varrho_5) \circ (\varrho_5 \cap (E_2 \times E_3)) = \varrho_4$, $\varrho_6 \cap (\tau\varrho_6) = \varrho_2$ und (1). □

Satz 2 *Sei $\{a,b,c\} := E_3$. $T_{a,b;c} := Pol_3\{a,b\} \cap Pol_3\{c\}$ besitzt genau 11 maximale Klassen:*

(1) $T_{a,b;c} \cap Pol_3\{a\}$

(2) $T_{a,b;c} \cap Pol_3\{b\}$

(3) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} a & a & b \\ a & b & b \end{pmatrix}$

(4) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

(5) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} a & a & a & b & b & a & b & b \\ a & a & b & b & a & b & a & b \\ a & b & a & a & b & b & a & b \\ a & b & b & a & a & a & b & b \end{pmatrix}$

(6) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} a & b & c & a & b \\ a & b & c & b & a \end{pmatrix}$

(7) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} a & b & c & a & c & b & c \\ a & b & c & c & a & c & b \end{pmatrix}$

(8) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} a & a & b & b & a & c & b & c \\ a & b & a & b & c & a & c & b \end{pmatrix}$

(9) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} a & a & b & b & a & a & b & b \\ a & a & b & b & a & b & a & b \\ a & b & a & b & c & c & c & c \end{pmatrix}$

(10) $T_{a,b;c} \cap Pol_3\ E_2^3 \cup \begin{pmatrix} a & b \\ a & b \\ c & c \end{pmatrix}$

(11) $T_{a,b;c} \cap Pol_3 \begin{pmatrix} c & a & c & b & c \\ c & c & a & c & b \end{pmatrix}$.

Beweis: O.B.d.A. seien $a = 0$, $b = 1$ und $c = 2$. Mit A bezeichnen wir in diesem Beweis eine Teilmenge von $T_{0,1;2}$, die keine Teilmenge der unter (1) bis (11) aufgezählten Teilklassen von $T_{0,1;2}$ ist. Dann gehören zu $[A]$ gewisse Funktionen $f_1, f_2, ..., f_{11}$ mit der im Teil I vereinbarten Eigenschaft (*).
Wegen Lemma 3.6 können wir als Superpositionen über den Funktionen f_6 und f_7 gewisse Funktionen f_{12}^4, f_{13}^5, f_{14}^6 und f_{15}^7 mit den Eigenschaften

$$f_{12} \begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix} \in \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}, \quad f_{14} \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 0 & 2 & 1 \end{pmatrix} \in \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix},$$
$$f_{13} \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix} \in \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad f_{15} \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 & 2 \end{pmatrix} \in \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

erhalten.
Mit Hilfe von Lemma 2.1 (Teil I) sieht man leicht ein, daß jede Funktion aus P_2 die Einschränkung einer Funktion aus $[A]$ sein muß. Damit sind auch alle einstelligen Funktionen aus $T_{0,1;2}$ Superpositionen über A, d.h., $\{u_2, s_1, s_3, v_2\} \subseteq [A]$.
Als nächstes soll gezeigt werden, daß alle Funktionen aus $T_{0,1;2}$, die nur Werte aus $\{0, 2\}$ oder $\{1, 2\}$ annehmen, zu $[A]$ gehören.
Wir beginnen mit dem Nachweis gewisser zweistelliger Funktionen

$$k,\ d \in [A] \cap P_{3;\{0,2\}} \qquad (1)$$

mit den Eigenschaften

$$\begin{array}{cc|cc} x & y & k(x,y) & d(x,y) \\ \hline 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{array}.$$

Es gilt $f'_{12}(x,y) := u_2(f_{12}(x,y,v_2(x),v_2(y))) \in [A]$, wobei $W(f'_{12}) = \{0,2\}$ und

$$f'_{12}\begin{pmatrix} 0\,0 \\ 0\,2 \\ 2\,0 \\ 2\,2 \end{pmatrix} \in \begin{pmatrix} 0\,0 \\ 0\,2 \\ 0\,2 \\ 2\,2 \end{pmatrix}.$$

Fall 1: $f'_{12}(0,2) = f'_{12}(2,0) = 0$ (d.h., $f'_{12} = k$).
In diesem Fall können wir die Funktion
$f'_{14}(x,y) := f_{14}(f'_{12}(x,y), v_2(f'_{12}(x,y)), x, y, v_2(x), v_2(y))$ in $[A]$ nachweisen, für die wir o.B.d.A. $f'_{14}\begin{pmatrix} 0\,2 \\ 2\,0 \end{pmatrix} \in \begin{pmatrix} 0\,2 \\ 1\,2 \end{pmatrix}$ annehmen können. Falls $f'_{14}\begin{pmatrix} 0\,2 \\ 2\,0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gilt, ist
$f'_8(x,y) := u_2(f_8(f'_{12}(x,y), f'_{14}(x,y), f'_{14}(y,x), v_2(f'_{12}(x,y)), x, y, v_2(x), v_2(y))) \in [A]$ die gesuchte Funktion d.
Im Fall $f'_{14}\begin{pmatrix} 0\,2 \\ 2\,0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ können wir als d die Funktion $u_2 \star f'_{14}$ wählen, womit (1) im Fall 1 gezeigt ist.

Fall 2: $f'_{12}(0,2) = f'_{12}(2,0) = 2$ (d.h., $f'_{12} = d$).
Die noch fehlende Funktion $k \in [A]$ erhält man in diesem Fall durch
$k(x,y) := u_2(f_{11}(f'_{12}(x,y), x, y, v_2(x), v_2(y)))$. Also gilt (1) auch im Fall 2.

Um Lemma 2.6 aus Teil I (für 2 anstelle von 1 und $P_{3,\{0,2\}}$ anstelle von $P_{3,2}$) anwenden zu können, benötigen wir noch eine auf $\{0,2\}$ nicht monotone Funktion aus $T_{0,1;2} \cap P_{3,\{0,2\}}$.
Für die Funktion $f'_{13}(x,y,z) := f_{13}(x, v_2(x), z, y, v_2(y))$ gilt $f'_{13}\begin{pmatrix} 0\,0\,2 \\ 0\,2\,2 \end{pmatrix} \in \begin{pmatrix} 0\,1\,2\,2 \\ 1\,0\,0\,1 \end{pmatrix}$, für die wir wegen $s_3 \in [A]$ o.B.d.A. $f'_{13}\begin{pmatrix} 0\,0\,2 \\ 0\,2\,2 \end{pmatrix} \in \begin{pmatrix} 0\,2 \\ 1\,0 \end{pmatrix}$ annehmen können. Falls $f'_{13}\begin{pmatrix} 0\,0\,2 \\ 0\,2\,2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ist, erhält man durch
$f'_{15}(x,y,z) := u_2(f_{15}(x, f'_{13}(x,y,z), s_3(f'_{13}(x,y,z)), v_2(x), z, y, v_2(y)) \in [A]$ eine Funktion mit der Eigenschaft $f'_{15}\begin{pmatrix} 0\,0\,2 \\ 0\,2\,2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Also ist auf $\{0,2\}$ entweder f'_{13} oder f'_{15} nicht monoton.

Mit Hilfe von Lemma 2.6 (Teil I) ist nach diesen Vorbereitungen leicht zu zeigen, daß zu jeder Funktion g^m aus $Pol_{\{0,2\}}\{0\} \cap Pol_{\{0,2\}}\{2\}$ in $[A]$ eine Funktion G^m mit

$$\forall \mathbf{x} \in \{0,2\}^m : \; g(\mathbf{x}) = G(\mathbf{x})$$

existiert.

Zwecks Nachweis von $T_{0,1;2} \cap P_{3,\{0,2\}} \subseteq [A]$ bilden wir als nächstes die Superposition $h(x,y) := u_2(f_6(u_2(x), v_2(x), y, x, s_3(x)))$ über A, für die wir o.B.d.A. $h\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ annehmen können.

Bildet man nun zu beliebigen Tupeln $\mathbf{x} := (x_1, x_2, ..., x_n) \in E_3^n \backslash E_2^n$ die Tupel

$$\begin{aligned} T_{\mathbf{x}} := (&h(x_1,x_2), h(x_1,x_3), ..., h(x_1,x_n),\\ &h(x_2,x_1), h(x_2,x_3), ..., h(x_2,x_n),\\ &...,\\ &h(x_n,x_1), h(x_n,x_2), ..., h(x_n,x_{n-1}),\\ &u_2(x_1), u_2(x_2), ..., u_2(x_n)), \end{aligned}$$

so sieht man leicht, daß für beliebige Tupel $\mathbf{x}, \mathbf{x}' \in E_3^n \backslash E_2^n$

$$\mathbf{x} \neq \mathbf{x}' \implies T_{\mathbf{x}} \neq T_{\mathbf{x}'}$$

gilt. Folglich und nach dem oben Gezeigten findet man zu einer beliebigen Funktion $g^n \in T_{0,1;2} \cap P_{3,\{0,2\}}$ in $[A]$ eine Funktion f_g mit

$$\begin{aligned} g(x_1, ..., x_n) = f_g(&h(x_1,x_2), h(x_1,x_3), ..., h(x_1,x_n),\\ &h(x_2,x_1), h(x_2,x_3), ..., h(x_2,x_n),\\ &...,\\ &h(x_n,x_1), h(x_n,x_2), ..., h(x_n,x_{n-1}),\\ &u_2(x_1), u_2(x_2), ..., u_2(x_n)). \end{aligned}$$

(Man beachte dabei, daß solche Funktionen g auf E_2^n nur den Wert 0 annehmen.)

Folglich haben wir $T_{0,1;2} \cap P_{3,\{0,2\}} \subseteq [A]$ und $\{\; v_2 \star f \mid f \in T_{0,1;2} \cap P_{3,\{1,2\}} \;\} = T_{0,1;2} \cap P_{\{1,2\}} \subseteq [A]$.

Als nächstes soll gezeigt werden, wie man alle Funktionen aus $T_{0,1;2}$, die nur auf dem Tupel $(2,2,...,2)$ den Wert 2 annehmen, als Superpositionen über Funktionen aus A darstellen kann. Wir beginnen mit der Konstruktion einiger Hilfsfunktionen:

Aus f_7 ist zunächst $f_7'(x,y) := f_7(h_1(x,y), h_2(x,y), h_3(x,y), x, y, v_2(x), v_2(y))$ mit $h_1 \in [A] \cap P_{\{0,2\}}$, $h_2, h_3 \in [A] \cap P_{\{1,2\}}$ und

$$h_1\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; h_2\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; h_3\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

bildbar, wobei wir (wegen $s_3 \in [A]$) $f_7' \begin{pmatrix} 0\,2 \\ 2\,0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ annehmen können. Als Superposition über f_7' und u_2 erhält man dann $r(x,y) := f_7'(u_2(x), u_2(y))$ mit der Eigenschaft

$$r \begin{pmatrix} 0\,0 \\ 0\,1 \\ 1\,0 \\ 1\,1 \\ 0\,2 \\ 2\,0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \\ 0 \\ 1 \end{pmatrix}$$

und $\alpha \in \{0,1\}$. Der Einfachheit halber, können wir $\alpha = 0$ annehmen. (Im Fall $\alpha = 1$ hat die Funktion $r'(x,y) := s_3(r(y,x))$ die gewünschten Eigenschaften.)

Bezeichne q^n eine beliebige Funktion aus $T_{0,1;2}$, die auf Tupeln aus E_2^n nur den Wert 0 annimmt und den Wert 2 nur auf $(2,2,...,2)$. Zu $[A]$ gehören die n-stelligen Funktionen

$$q_1(\mathbf{x}) := \begin{cases} 0 \text{ für } & q(\mathbf{x}) = 0, \\ 2 \text{ sonst} & \end{cases}$$

und

$$q_2(\mathbf{x}) := \begin{cases} 2 \text{ für } & (q(\mathbf{x}) = 0 \;\wedge\; \mathbf{x} \notin E_2^n) \;\vee\; \mathbf{x} = \mathbf{2}, \\ 0 \text{ sonst,} & \end{cases}$$

womit $q(\mathbf{x}) = r(q_1(\mathbf{x}), q_2(\mathbf{x})) \in [A]$ gilt. Wegen $s_3 \in [A]$ sind auch beliebige n-stellige Funktionen, die auf Tupeln aus E_2^n nur den Wert 1 annehmen und den Wert 2 nur auf $(2,2,...,2)$, Superpositionen über A.

Als nächstes bilden wir die Funktion

$f_9'(x,y) := f_9(u_2(x), x, s_3(x), v_2(x), u_2(y), y, s_3(y), v_2(y)) \in [A]$, für die wir (wegen $s_3 \in [A]$) $f_9' \begin{pmatrix} 0\,0 \\ 0\,1 \\ 1\,2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \beta \end{pmatrix}$ mit gewissem $\beta \in E_2$ annehmen können. Bezeichne p eine n-stellige Funktion, die auf Tupeln aus E_2^n beliebige Werte aus E_2 annehmen kann und den Wert 2 nur auf dem Tupel $(2,2,...,2)$. (Die Verteilung der Werte 0 und 1 auf den Tupeln $E_3^n \backslash (E_2^n \cup \{\mathbf{2}\})$ ist für die sich anschließende Konstruktion unwichtig und sie hängt ab von der weiter unten beschriebenen Superpositionsbildung in $[A]$.) Eine Funktion p mit diesen Eigenschaften läßt sich mit Hilfe einer in $[A]$ bereits nachgewiesene n-stellige Funktion p' mit $p'(\mathbf{x}) = p(\mathbf{x})$ für alle $\mathbf{x} \in E_2^n$, der n-stelligen Funktion $p'' \in [A]$, die definiert ist durch

$$p''(\mathbf{x}) := \begin{cases} 0 \text{ für } & p'(\mathbf{x}) \in E_2, \\ 2 \text{ für } & \mathbf{x} = \mathbf{2}, \\ 1 \text{ sonst} & \end{cases}$$

und der Funktion f_9' wie folgt als Superposition über A gewinnen: $p(\mathbf{x}) := f_9'(p''(\mathbf{x}), p'(\mathbf{x}))$. Seien $E_2^n = \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_{2^n}}\}$ und $E_3^n \backslash (E_2^n \cup \{\mathbf{2}\}) = \{\mathbf{a_{2^n+1}}, \mathbf{a_{2^n+2}}, ..., \mathbf{a_{3^n-1}}\}$. Superpositionen über A sind gewisse n-stellige Funktionen g_i $(i = 1, 2, 3, ..., 3^n - 1)$ mit den Eigenschaften:

$$g_i(\mathbf{x}) = \begin{cases} 1 \text{ für } \mathbf{x} = \mathbf{a_i}, \\ 0 \text{ für } \mathbf{x} \in E_2^n \backslash \{\mathbf{a_i}\} \end{cases}$$

für $i = 1, 2, ..., 2^n$ und

$$g_j(\mathbf{x}) = \begin{cases} 1 \text{ für } & \mathbf{x} = \mathbf{a_j}, \\ 2 \text{ für } & \mathbf{x} = \mathbf{2}, \\ 0 \text{ sonst} \end{cases}$$

für $j = 2^n + 1, 2^n + 2, .., 3^n - 1$. Durch $(g_1(\mathbf{x}), g_2(\mathbf{x}), ..., g_{3^n-1}(\mathbf{x}))$ mit $\mathbf{x} \in E_3^n$ lassen sich dann 3^n paarweise verschiedene Tupel beschreiben, die bis auf $(2, 2, 2, ..., 2)$ sämtlich zu $E_2^{3^n-1}$ gehören. Folglich findet man in $[A]$ zu jeder n-stelligen Funktion $f_* \in T_{0,1;2}$, die nur auf $(2, 2, ..., 2)$ den Wert 2 annimmt, eine gewisse $(3^n - 1)$-stellige Funktion h_{f_*} mit $f_*(\mathbf{x}) = h_{f_*}(g_1(\mathbf{x}), g_2(\mathbf{x}), ..., g_{3^n-1}(\mathbf{x}))$. Durch Einsetzen gewisser solcher gerade konstruierten Funktionen in f_{10} läßt sich in $[A]$ eine zweistellige Funktion f_{10}' mit $f_{10}'\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ nachweisen.

Bezeichne nun f^n eine beliebige Funktion aus $T_{0,1;2}$. Wie oben gezeigt wurde gehören zu $[A]$ folgende zwei n-stelligen Funktionen:

$$t_1(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{für} & f(\mathbf{x}) \in E_2, \\ 2 & \text{für} & \mathbf{x} = \mathbf{2}, \\ 0 & \text{sonst} \end{cases}$$

und

$$t_2(\mathbf{x}) = \begin{cases} 2 \text{ für } & f(\mathbf{x}) = 2, \\ 0 \text{ sonst.} \end{cases}$$

Damit gilt $f(\mathbf{x}) = f_{10}'(t_1(\mathbf{x}), t_2(\mathbf{x})) \in [A]$ und $[A] = T_{0,1;2}$ ist gezeigt. Hieraus und aus Tabelle 1 folgt dann die Behauptung von Satz 2. Die Funktionen $g_1, ..., g_{10}$ aus Tabelle 1 sind in Tabelle 2 definiert. Für die Funktion g_{11}^3 sei

$$g_{11}(\mathbf{x}) := \begin{cases} 2 \text{ für } & \mathbf{x} \in E_3^3 \backslash E_2^3, \\ 1 \text{ für } & \mathbf{x} \in \{(0,1,1), (1,0,1), (1,1,0), (1,1,1)\}, \\ 0 \text{ sonst.} \end{cases}$$

□

	u_2	v_2	s_3	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	h
(1)	+	−	−	+	−	+	−	+	−	+	+	+	+	+
(2)	−	+	−	−	+	−	−	+	+	+	−	+	+	+
(3)	+	+	−	+	+	−	−	+	−	+	+	+	+	+
(4)	−	−	+	−	−	−	+	+	−	−	−	+	−	+
(5)	+	+	+	+	+	−	+	+	−	−	+	+	−	−
(6)	+	+	+	−	−	−	−	−	+	+	+	−	+	+
(7)	+	+	+	+	+	−	−	−	−	+	+	−	+	+
(8)	+	+	+	−	+	−	+	+	+	−	−	−	−	−
(9)	+	+	+	+	+	−	−	−	−	+	+	+	−	+
(10)	+	+	+	+	+	−	−	−	−	−	−	−	+	−
(11)	+	+	+	−	−	−	−	−	+	+	+	+	−	+

Tabelle 1

x_1	x_2	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	1	1	0	0	1	0	0	0
1	0	0	1	0	0	1	1	1	0	1	0
1	1	0	1	0	0	1	1	1	0	1	1
0	2	0	1	0	1	0	1	2	2	1	0
1	2	2	2	2	2	2	0	2	2	2	0
2	0	0	1	1	1	1	2	2	2	2	0
2	1	2	1	2	1	1	2	2	2	2	0
2	2	2	2	2	2	2	2	2	2	2	2

Tabelle 2

Satz 3 *Sei* $\{a, b, c\} := E_3$. $T_{a,b;a;b} := Pol_3\{a, b\} \cap Pol_3\{a\} \cap Pol_3\{b\}$ *besitzt genau 11 maximale Klassen:*

(1) $T_{a,b;a;b} \cap Pol_3 \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

(2) $T_{a,b;a;b} \cap Pol_3 \begin{pmatrix} a & a & b \\ a & b & a \end{pmatrix}$

(3) $T_{a,b;a;b} \cap Pol_3 \begin{pmatrix} b & a & b \\ b & b & a \end{pmatrix}$

(4) $T_{a,b;a;b} \cap Pol_3 \begin{pmatrix} a & a & b \\ a & b & b \end{pmatrix}$

(5) $T_{a,b;a;b} \cap Pol_3\{c\}$

(6) $T_{a,b;a;b} \cap Pol_3\{a, c\}$

(7) $T_{a,b;a;b} \cap Pol_3\{b, c\}$

(8) $T_{a,b;a;b} \cap Pol_3 \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0 & 1 & 2 & b & a \end{pmatrix}$

(9) $T_{a,b;a;b} \cap Pol_3 \begin{pmatrix} a & a & b & b & a & c & b & c \\ a & b & a & b & c & a & c & b \end{pmatrix}$

(10) $Pol_3 \begin{pmatrix} a & a & b & b & a \\ a & b & a & b & c \end{pmatrix}$

(11) $Pol_3 \begin{pmatrix} a & a & b & b & b \\ a & b & a & b & c \end{pmatrix}$.

Beweis: O.B.d.A. seien $a = 0$, $b = 1$ und $c = 2$. Mit A bezeichnen wir in diesem Beweis eine Teilmenge von $T_{0,1;0;1}$, die keine Teilmenge der unter (1) bis (11) aufgezählten Teilklassen von $T_{0,1;0;1}$ ist. Dann gehören zu $[A]$ gewisse Funktionen $f_1, f_2, ..., f_{11}$ mit der im Teil I vereinbarten Eigenschaft (*).

Da $(E_2 \times E_3) \cap \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}$ gilt, haben wir $T_{0,1;0;1} \cap Pol_3 \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{pmatrix} \subseteq T_{0,1;0;1} \cap Pol \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}$. Wegen $f_{10} \in A$ ist folglich auch eine gewisse Funktion f_{12}^6 mit $f_{12} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{pmatrix} \in \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$ eine Superposition über A.

Mit Hilfe von Lemma 2.3, Teil I sieht man leicht ein, daß jede Funktion aus $T_0 \cap T_1$ $(\subset P_2)$

die Einschränkung einer passend gewählten Funktion aus $[A]$ sein muß.
Da $f_5 \in \{j_1, j_5\}$, $f_6(j_1(x), x) = j_5(x)$ und $f_7(j_5(x), x) = j_1(x)$, sind die Funktionen j_1 und j_5 Superpositionen über A.
Mit Hilfe dieser Funktionen und Lemma 2.6, Teil I sieht man dann, daß $A \cap P_{3,2} \subseteq [A]$ gilt.

Wir konstruieren als nächstes einige Hilfsfunktionen.
Zu $[A]$ gehören Funktionen g_1^2, g_2^2 mit $g_1 \begin{pmatrix} 1\,0 \\ 0\,2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ und $g_2 \begin{pmatrix} 1\,0 \\ 0\,2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Folglich hat die Funktion $f'_{10}(x,y) := f_{10}(j_1(y), g_1(x,y), x, g_2(x,y), y) \in [A]$ die Eigenschaft $f'_{10} \begin{pmatrix} 1\,0 \\ 0\,2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
Mit Hilfe der Funktion f_{11} und gewissen Funktionen aus $P_{3,2} \cap [A]$ zeigt man analog, daß zu $[A]$ eine Funktion f'_{11} mit $f'_{11} \begin{pmatrix} 0\,1 \\ 1\,2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ gehört.
Außerdem haben wir $f_8 \begin{pmatrix} 0\,1\,2\,0\,1 \\ 0\,1\,2\,1\,0 \end{pmatrix} \in \begin{pmatrix} 0\,1\,2\,2 \\ 2\,2\,0\,1 \end{pmatrix}$. O.B.d.A. können wir $f_8 \begin{pmatrix} 0\,1\,2\,0\,1 \\ 0\,1\,2\,1\,0 \end{pmatrix} \in \begin{pmatrix} 0\,1 \\ 2\,2 \end{pmatrix}$ annehmen, da wir in den anderen Fällen zur Funktion $f_8^*(x_1, ..., x_5) := f_8(x_1, x_2, x_3, x_5, x_4)$ übergehen könnten, die die gewünschte Eigenschaft besitzt.
Bildet man nun $f'_8(x,y) := f'_{11}(g_3(x,y), f_8(j_1(x), j_5(x), x, y, g_4(x,y)))$, wobei g_3 und g_4 gewisse Funktionen aus $[A]$ mit

$$g_3(x,y) := \begin{cases} 1 \text{ für} & (x,y) \in \{(1,1),(2,1)\}, \\ 0 \text{ sonst} \end{cases}$$

und

$$g_4(x,y) := \begin{cases} 1 \text{ für} & (x,y) \in \{(1,1),(2,0)\}, \\ 0 \text{ sonst} \end{cases}$$

bezeichnen, so hat diese Funktion die Eigenschaft

$$f'_8 \begin{pmatrix} 0\,0 \\ 0\,1 \\ 1\,0 \\ 1\,1 \\ 2\,0 \\ 2\,1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

Einsetzen gewisser Funktionen $g_3, g_4, g_5, g_6 \in [A] \cap P_{3,2}$ in f_{12} liefert eine zweistellige Funktion $f'_{12}(x,y) := f_{12}(g_3(x,y), g_4(x,y), g_5(x,y), g_6(x,y), x, y)$ mit $f'_{12} \begin{pmatrix} 0\,2 \\ 2\,0 \end{pmatrix} \in \begin{pmatrix} 1\,2\,2 \\ 2\,1\,2 \end{pmatrix}$. Da man anstelle von f'_{12} auch die Funktion $f''_{12}(x,y) := f'_{12}(y,x)$ verwenden kann, genügt es $f'_{12}\begin{pmatrix}0\,2\\2\,0\end{pmatrix} \in \begin{pmatrix}1\,2\\2\,2\end{pmatrix}$ weiter zu betrachten. Falls $f'_{12}\begin{pmatrix}0\,2\\2\,0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, erhält man durch $f'_9(x,y) := f_9(g_3(x,y), g_4(x,y), g_5(x,y), g_6(x,y), x, y, f'_{12}(x,y), f'_{12}(y,x))$ eine Funktion aus $[A]$, für die $f'_9\begin{pmatrix}0\,2\\2\,0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$ gilt. Also können wir o.B.d.A. $f'_{12}\begin{pmatrix}0\,2\\2\,0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$ annehmen.

Mit Hilfe der Funktionen f'_8, f'_{12} und Funktionen aus $[A] \cap P_{3,2}$ wollen wir jetzt gewisse n-stellige Funktionen $f_{\mathbf{a_1},\mathbf{a_2},...,\mathbf{a_r}} \in T_{0,1;0;1}$ ($\{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_r}\} \subseteq E_3^n \backslash E_2^n$) mit

$$f_{\mathbf{a_1},\mathbf{a_2},...,\mathbf{a_r}}(\mathbf{x}) = \begin{cases} 2 \text{ für } & \mathbf{x} \in \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_r}\},\\ 1 \text{ für } & \mathbf{x} = \mathbf{1},\\ 0 \text{ sonst} \end{cases}$$

konstruieren.

Falls $\mathbf{a} := (a_1, a_2, ..., a_n) \in E_3^n$ und $a_i = 2$ für ein gewisses $i \in \{1, 2, ..., n\}$, erhält man die n-stellige Funktion $f_{\mathbf{a}}$ als Superposition über A mit Hilfe der n-stelligen Funktion

$$t_{\mathbf{a}}(\mathbf{x}) := \begin{cases} 1 \text{ für } & \mathbf{x} \in \{\mathbf{a}, \mathbf{1}\},\\ 0 \text{ sonst} \end{cases}$$

aus $[A] \cap P_{3,2}$ mittels $f_{\mathbf{a}}(\mathbf{x}) = f'_8(x_i, t_{\mathbf{a}}(\mathbf{x}))$.

$f_{\mathbf{a_1},\mathbf{a_2},...,\mathbf{a_r}} \in [A]$ für $r > 1$ und $\{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_r}\} \subseteq E_3^n \backslash E_2^n$ folgt aus $f_{\mathbf{a_1},\mathbf{a_2},...,\mathbf{a_r}}(\mathbf{x}) = f'_{12}(f_{\mathbf{a_1},\mathbf{a_2},...,\mathbf{a_{r-1}}}(\mathbf{x}), f_{\mathbf{a_r}}(\mathbf{x}))$.

Eine beliebige n-stellige Funktion f aus $T_{0,1;0;1}$ ist dann wie folgt als Superposition über A darstellbar: $f(\mathbf{x}) := f'_{10}(g_f(\mathbf{x}), t_{\mathbf{a_1},...,\mathbf{a_r}}(\mathbf{x}))$, wobei $\{\mathbf{a_1}, ..., \mathbf{a_r}\}$ die Menge aller Tupel bezeichnet, auf denen f den Wert 2 annimmt, und g_f durch

$$g_f(\mathbf{x}) := \begin{cases} f(\mathbf{x}) \text{ für } & f(\mathbf{x}) \in \{0, 1\},\\ 1 & \text{sonst,} \end{cases}$$

definiert ist. Folglich gilt $[A] = T_{0,1;0;1}$.

Die Behauptung von Satz 3 ergibt sich dann aus Tabelle 3, wobei die Funktionen $g_1, ..., g_{10}$ aus Tabelle 3 in Tabelle 4 definiert sind. Für die Funktion h_0^3, h_1^3 und h_2^3 gelte

$$h_0(\mathbf{x}) := \begin{cases} x + y + z \ (mod\, 2) \text{ für } & \mathbf{x} \in E_2^3,\\ 0 & \text{sonst,} \end{cases}$$

$$h_1(\mathbf{x}) := \begin{cases} x \wedge (y \vee \overline{z}) \ (mod\, 2) \text{ für } & \mathbf{x} \in E_2^3,\\ 1 & \text{sonst,} \end{cases}$$

und

$$h_2(\mathbf{x}) := \begin{cases} x \vee (y \wedge \overline{z}) \ (mod\, 2) \text{ für } & \mathbf{x} \in E_2^3,\\ 2 & \text{sonst.} \end{cases}$$

	j_1	j_5	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	h_0	h_1	h_2
(1)	+	+	−	−	+	+	+	+	−	−	+	−	+	−	−
(2)	+	+	+	−	+	+	+	+	−	+	+	+	−	+	−
(3)	+	+	−	+	+	+	+	+	+	−	+	−	−	−	+
(4)	+	+	+	+	+	+	+	+	+	+	+	+	−	−	−
(5)	−	−	−	−	+	−	+	+	+	+	+	−	−	−	+
(6)	+	−	+	−	−	−	−	−	+	+	−	+	+	−	+
(7)	−	+	−	+	+	+	+	−	+	+	+	−	−	+	+
(8)	+	+	−	−	−	−	−	+	+	+	−	+	+	+	+
(9)	+	+	−	+	−	+	+	+	−	−	−	+	+	+	−
(10)	+	+	+	−	−	+	−	+	−	+	−	+	+	+	−
(11)	+	+	−	+	−	+	−	+	+	−	−	+	+	+	−

Tabelle 3

x_1	x_2	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	1	1	0	0	1	0	0	0
1	0	0	1	0	0	1	1	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1
0	2	0	1	0	1	0	1	2	2	1	0
1	2	2	2	2	2	2	0	2	2	2	0
2	0	0	1	1	1	1	2	2	2	2	0
2	1	2	1	2	1	1	2	2	2	2	0
2	2	0	1	2	1	2	2	2	2	2	0

Tabelle 4

□

Lemma 4 *Seien*

$$\varrho_1 := \begin{pmatrix} 0\,0\,1\,1\,0\,2 \\ 0\,1\,0\,1\,2\,0 \end{pmatrix}, \varrho_2 := \begin{pmatrix} 0\,0\,1\,1\,0 \\ 0\,1\,0\,1\,2 \end{pmatrix}, \varrho_3 := \begin{pmatrix} 0\,0\,1\,1\,2\,0\,1 \\ 0\,1\,0\,1\,2\,2\,2 \end{pmatrix},$$
$$\varrho_4 := \begin{pmatrix} 0\,1\,2\,1\,0 \\ 0\,1\,2\,0\,1 \end{pmatrix}, \quad \varrho_5 := \begin{pmatrix} 0\,2\,1\,2 \\ 2\,0\,2\,1 \end{pmatrix}.$$

Dann gilt

(a) $T_{0,1;0;2} \cap Pol_3\varrho_1 \subseteq T_{0,1;0;2} \cap Pol_3\varrho_2$*;*

(b) $T_{0,1;0;2} \cap Pol_3\varrho_3 \subseteq T_{0,1;0;2} \cap Pol_3\varrho_4$*;*

(c) $T_{0,1;0;2} \cap Pol_3\varrho_5 \subseteq T_{0,1;0;2} \cap Pol_3\varrho_4$*.*

Beweis: (a) folgt aus $\varrho_1 \cap (E_2 \times E_3) = \varrho_2$ und (b) aus $\varrho_3 \cap (\tau\varrho_3) = \varrho_4$.
(c) ist eine Folgerung aus Lemma 1, (a). □

Satz 5 *Sei* $\{a,b,c\} := E_3$. $T_{a,b;a;c} := Pol_3\{a,b\} \cap Pol_3\{a\} \cap Pol_3\{c\}$ *besitzt genau 11 maximale Klassen:*

(1) $T_{a,b;a;c} \cap Pol_3\{b\}$

(2) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & a & b \\ a & b & b \end{pmatrix}$

(3) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & b & a \\ a & a & b \end{pmatrix}$

(4) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & a & a & b & b & a & b & b \\ a & a & b & b & a & b & a & b \\ a & b & a & a & b & b & a & b \\ a & b & b & a & a & a & b & b \end{pmatrix}$

(5) $T_{a,b;a;c} \cap Pol_3\{a, c\}$

(6) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & b & c & a & b \\ a & b & c & b & a \end{pmatrix}$

(7) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & b & c & a & c & b & c \\ a & b & c & c & a & c & b \end{pmatrix}$

(8) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & a & b & b & a & c & b & c \\ a & b & a & b & c & a & c & b \end{pmatrix}$

(9) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} c & a & c & b & c \\ c & c & a & c & b \end{pmatrix}$

(10) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & a & a & b \\ a & b & c & c \end{pmatrix}$

(11) $T_{a,b;a;c} \cap Pol_3 \begin{pmatrix} a & a & b & b & a \\ a & b & a & b & c \end{pmatrix}$.

Beweis: O.B.d.A. seien $a = 0$, $b = 1$ und $c = 2$. Mit A bezeichnen wir in diesem Beweis eine Teilmenge von $T_{0,1;0;2}$, die keine Teilmenge der unter (1) bis (11) aufgezählten Teilklassen von $T_{0,1;0;2}$ ist. Dann gehören zu $[A]$ gewisse Funktionen $f_1, f_2, ..., f_{11}$ mit der im Teil I vereinbarten Eigenschaft (*).
Wegen Lemma 4 gibt es in $[A]$ als Superpositionen über f_6 und f_{11} gewisse Funktionen f_{12}, f_{13} und f_{14} mit

$$f_{12}\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{pmatrix} \in \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \; f_{13}\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 & 2 \end{pmatrix} \in \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \; f_{14}\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix} \in \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

Mit Hilfe von Lemma 2.2, Teil I sieht man leicht ein, daß jede Funktion aus $T_0 \subset P_2$ die Einschränkung einer Funktion aus $[A]$ sein muß. Damit sind auch alle einstelligen Funktionen aus $T_{0,1;0;2}$ Superpositionen über A: $u_2,\ s_1\ \in\ [A]$.
Es gilt $f_5(0, 2) = 1$ und damit $f_5 \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \in \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.
Fall 1: $f_5(2, 0) = 0$.
Als erstes soll gezeigt werden, daß alle Funktionen aus $T_{0,1;0;2}$, die nur Werte aus $\{0, 2\}$ annehmen, zu $[A]$ gehören.
Wir beginnen mit dem Nachweis gewisser zweistelliger Funktionen $k,\ d \in [A] \cap P_{3;\{0,2\}}$, mit den Eigenschaften

x y	$k(x,y)$	$d(x,y)$
0 0	0	0
0 2	0	2
2 0	0	2
2 2	2	2

.

Da zu $[A]$ gewisse zweistellige Funktionen h_0, h_1 mit $h_0\begin{pmatrix}0\ 1\\1\ 0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ und $h_1\begin{pmatrix}0\ 1\\1\ 0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ gehören, gilt
$f'_{12}(x,y) := f_{12}(h_0(f_5(y,x), f_5(x,y)), f_5(y,x), f_5(x,y), h_1(f_5(y,x), f_5(x,y)), x, y) \in [A]$ mit $f'_{12}\begin{pmatrix}0\ 2\\2\ 0\end{pmatrix} \in \begin{pmatrix}1\ 2\ 2\\2\ 1\ 2\end{pmatrix}$. Falls $f'_{12}\begin{pmatrix}0\ 2\\2\ 0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$, erhält man durch $f''_{12}(x,y) := f_5(x, f'_{12}(x,y))$ eine Funktion mit der Eigenschaft $f''_{12}\begin{pmatrix}0\ 2\\2\ 0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$. Also können wir o.B.d.A. $f'_{12}\begin{pmatrix}0\ 2\\2\ 0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ annehmen.
Die Funktion
$$f'_8(x,y) := u_2(f_8(h_0(f_5(y,x), f_5(x,y)), f_5(y,x), f_5(x,y), h_1(f_5(x,y), f_5(x,y)), x, y, f'_{12}(x,y), f'_{12}(y,x)))$$
hat dann die gewünschten Eigenschaften von d und als Funktion k können wir $k(x,y) := u_2(f_9(f'_{12}(x,y), x, y, f'_{12}(x,y), f'_{12}(y,x)))$ wählen.
Um Lemma 2.3, Teil I anwenden zu können, benötigen wir noch eine auf $\{0,2\}$ nicht monotone Funktion aus $T_{0,1;0;2} \cap P_{3,\{0,2\}}$.
Zu $[A]$ gehört eine Funktion q mit der Eigenschaft $q\begin{pmatrix}0\ 1\\1\ 1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. Damit haben wir $q'(x,y,z) := q(f_5(x,y), f_5(x,z)) \in [A]$ mit $q'\begin{pmatrix}0\ 0\ 2\\0\ 2\ 2\end{pmatrix} \in \begin{pmatrix}1\\0\end{pmatrix}$. Folglich erhält man durch $f'_{13}(x,y,z) := u_2(f_{13}(x, f_5(x,y), q'(x,y,z), f_5(x,z), z, y, f'_{12}(y,z))) \in [A]$ eine Funktion mit $f'_{13}\begin{pmatrix}0\ 0\ 2\\0\ 2\ 2\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$.
Mit Hilfe von Lemma 2.3, Teil I ist nach diesen Vorbereitungen leicht zu zeigen, daß zu jeder Funktion g^m aus $Pol_{\{0,2\}}\{0\} \cap Pol_{\{0,2\}}\{2\}$ in $[A]$ eine Funktion G^m mit

$$\forall \mathbf{x} \in \{0,2\}^m : \ g(\mathbf{x}) = G(\mathbf{x})$$

existiert.
Als nächstes soll eine Fuktion h^2 mit $W(h) = \{0,2\}$ und

$$h\begin{pmatrix}0\ 2\\1\ 2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} \tag{2}$$

konstruiert werden. Wir betrachten dazu wieder die Funktion f_5, für die $f_5(0,2) = 1$ gilt. Falls $f_5(1,2) = 2$ ist, erhält man h durch $u_2 \star f_5$. Ist $f_5(1,2) = 0$, so kann man als h die Funktion $f'_6(x,y) := u_2(f_6(u_2(x), t_1(f_5(x,y), x), y, x, f_5(x,y)) \in [A]$ wählen, wobei für

$t_1 \in [A]$ $t_1(1,0) = t_1(0,1) = 1$ gilt und o.B.d.A. $u_2\left(f_6\begin{pmatrix}0\,1\,2\,0\,1\\0\,1\,2\,1\,0\end{pmatrix}\right) = \begin{pmatrix}0\\2\end{pmatrix}$ angenommen wurde.

Falls $f_5(1,2) = 1$ ist, kann man mit Hilfe einer Funktion $t_2 \in [A]$ mit $t_2(1,0) = 1$ und $t_2(1,1) = 0$ die Funktion $f_5'(x,y) := t_2(f_5(x,y),x)$ bilden, die $f_5'\begin{pmatrix}0\,2\\1\,2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ erfüllt, womit wir wie bei dem vorher betrachteten Fall weiter verfahren können.

Also existiert in $[A]$ eine Funktion h mit der oben angegebenen Eigenschaft (2). Wie im Beweis von Satz 2 läßt sich damit $T_{0,1;0;2} \cap P_{3,\{0,2\}} \subseteq [A]$ zeigen.

Ebenfalls analog zum Beweis von Satz 2 kann man sich überlegen, daß alle Funktionen aus $T_{0,1;0;2}$, die auf Tupeln aus E_2^n stets 0 sind und nur auf dem Tupel $(2,2,...,2)$ den Wert 2 annehmen, Superpositionen über Funktionen aus A sind, da die Funktion $r(x,y) := f_5(u_2(y),u_2(x))$ die Eigenschaft $r\begin{pmatrix}0\,0\\0\,2\\2\,0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ hat.

Wegen $q_1, q_2 \in [A] \cap P_{\{0,2\}}$ mit $q_1\begin{pmatrix}0\,0\\0\,1\\1\,2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $q_2\begin{pmatrix}0\,0\\0\,1\\1\,2\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix}$ haben wir außerdem

$f_{10}'(x,y) := f_{10}(q_1(x,y),x,q_2(x,y),y) \in [A]$ mit $f_{10}'\begin{pmatrix}0\,0\\0\,1\\1\,2\end{pmatrix} \in \begin{pmatrix}0\,0\\1\,1\\0\,1\end{pmatrix}$.

Mit passend gewählten Funktionen $q_3^2, q_4^2, q_5^2 \in [A]$, die die Eigenschaften

$$q_3\begin{pmatrix}0\,0\\1\,0\\0\,2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},\ q_4\begin{pmatrix}0\,0\\1\,0\\0\,2\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix},\ q_5\begin{pmatrix}0\,0\\1\,0\\0\,2\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

besitzen, läßt sich weiterhin $f_{11}'(x,y) := f_{11}(q_3(x,y),q_4(x,y),x,q_5(x,y),y)$ mit der Eigenschaft $f_{11}'\begin{pmatrix}0\,0\\1\,0\\0\,2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}$ nachweisen.

Damit lassen sich analoge Konstruktionen wie im Beweis von Satz 2 durchführen und folglich $[A] = T_{0,1;0;2}$ im Fall 1 zeigen.

Fall 2: $f_5(2,0) = 1$.

Für die Funktion $f_5''(x,y) := f_5(u_2(f_5(x,y)),y) \in [A]$ haben wir dann $f_5''\begin{pmatrix}0\,2\\2\,0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, womit Fall 2 weiter wie Fall 1 behandelt werden kann.

Fall 3: $f_5(2,0) = 2$.

Da $u_2\left(f_{14}\begin{pmatrix}0\,2\,1\,2\\2\,0\,2\,1\end{pmatrix}\right)\in\begin{pmatrix}0\,2\\0\,2\end{pmatrix}$, hat die Funktion $f'_{14}(x,y):=f_{14}(x,y,f_5(x,y),f_5(y,x))\in[A]$ die Eigenschaft $f'_{14}\begin{pmatrix}0\,2\\2\,0\end{pmatrix}\in\begin{pmatrix}0\,2\\0\,2\end{pmatrix}$. Falls $f'_{14}(0,2)=f'_{14}(2,0)=0$ ist, kann man die Funktion $f_5'''(x,y):=f_5(f'_{14}(x,y),y)$ bilden, für die $f_5'''\begin{pmatrix}0\,2\\2\,0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ gilt, womit weiter wie unter Fall 1 verfahren werden kann.
Der Fall $f'_{14}(0,2)=f'_{14}(2,0)=2$ läßt sich durch Bildung von $f'_9(x,y):=u_2(f_9(f'_{14}(x,y),x,y,f_5(x,y),f_5(y,x)))$ auf den Fall $f'_{14}(0,2)=f'_{14}(2,0)=0$ zurückführen, da $f'_9\begin{pmatrix}0\,2\\2\,0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ gilt.

Folglich ist in allen möglichen Fällen für f_5: $[A]=T_{0,1;0;2}$. Die Behauptung von Satz 5 ergibt sich dann aus Tabelle 5, wobei die Funktionen $g_1,...,g_{12}$ in Tabelle 6 definiert sind und für die Funktionen h_1^3, h_2^3, h_3^3 gelte

$$h_1(x,y,z):=\begin{cases}xy\vee xz\vee yz \text{ für } & x,y,z\in E_2,\\ 2 \text{ sonst;} & \end{cases}$$

$$h_2(x,y,z):=\begin{cases}x+y+z\ (mod\,2) \text{ für } & x,y,z\in E_2,\\ 2 \text{ sonst;} & \end{cases}$$

$$h_3(x,y,z):=\begin{cases}x+y+z\ (mod\,2) \text{ für } & x,y,z\in E_2,\\ 2 \text{ für } & x=y=z=2,\\ 1 \text{ sonst.} & \end{cases}$$

	u_2	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	h_1	h_2	h_3
(1)	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$+$
(2)	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
(3)	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
(4)	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$
(5)	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$-$
(6)	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$
(7)	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$+$
(8)	$+$	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$+$
(9)	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$-$
(10)	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$+$	$-$
(11)	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$+$	$-$	$-$	$-$	$+$

Tabelle 5

x_1	x_2	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	1	1	0	0	1	0	0	0	0	0
1	0	0	1	0	0	1	1	1	0	1	0	0	1
1	1	0	1	0	0	1	1	1	0	1	1	0	1
0	2	0	2	0	1	0	1	2	2	1	0	0	2
1	2	2	1	2	2	2	0	2	2	2	0	0	2
2	0	0	1	1	1	1	2	2	2	2	0	0	0
2	1	2	1	2	1	1	2	2	2	2	0	1	2
2	2	2	2	2	2	2	2	2	2	2	2	2	2

Tabelle 6

$\square$

Satz 6 *Sei* $\{a,b,c\} := E_3$. $T_{a,b;a,c} := Pol_3\{a,b\} \cap Pol_3\{a,c\}$ *besitzt genau 14 maximale Klassen:*

(1) $T_{a,b;a,c} \cap Pol_3\{b\}$

(2) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & a & b \\ a & b & b \end{pmatrix}$

(3) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & b & a \\ a & a & b \end{pmatrix}$

(4) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & a & a & b & b & a & b & b \\ a & a & b & b & a & b & a & b \\ a & b & a & a & b & b & a & b \\ a & b & b & a & a & a & b & b \end{pmatrix}$

(5) $T_{a,b;a,c} \cap Pol_3\{c\}$

(6) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & a & c \\ a & c & c \end{pmatrix}$

(7) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & c & a \\ a & a & c \end{pmatrix}$

(8) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & a & a & c & c & a & c & c \\ a & a & c & c & a & c & a & c \\ a & c & a & a & c & c & a & c \\ a & c & c & a & a & a & c & c \end{pmatrix}$

(9) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & b & a \\ a & b & c \end{pmatrix}$

(10) $T_{a,b;a,c} \cap Pol_3 \begin{pmatrix} a & c & a \\ a & c & b \end{pmatrix}$

(11) $Pol_3 \begin{pmatrix} a & a \\ a & b \\ a & c \end{pmatrix}$

(12) $Pol_3 \begin{pmatrix} a & a & a \\ a & b & b \\ a & a & c \end{pmatrix}$

(13) $Pol_3 \begin{pmatrix} a & a & a \\ a & a & b \\ a & c & c \end{pmatrix}$

(14) $Pol_3 \begin{pmatrix} a & a & a \\ a & b & a \\ a & a & c \end{pmatrix}$.

Beweis: O.B.d.A. seien $a = 0$, $b = 1$ und $c = 2$. Mit A bezeichnen wir in diesem Beweis eine Teilmenge von $T_{0,1;0,2}$, die keine Teilmenge der unter (1) bis (14) aufgezählten Teilklassen von $T_{0,1;0,2}$ ist. Dann gehören zu $[A]$ gewisse Funktionen $f_1, f_2, ..., f_{14}$ mit der im Teil I vereinbarten Eigenschaft (*).
Wegen Lemma 2.2, Teil I existiert zu jeder Funktion $g^m \in Pol_3\{0\}(\subset P_2)$ und jeder Funktion $h^m \in Pol_3\{0\}(\subset P_{\{0,2\}})$ Funktionen $G^m, H^m \in [A]$ mit

$$(\forall \mathbf{x} \in E_2^m : \; g(\mathbf{x}) = G(\mathbf{x})) \;\wedge\; (\forall \mathbf{x} \in \{0,2\}^m : \; h(\mathbf{x}) = H(\mathbf{x})).$$

Folglich gehören zu $[A]$ auch einstellige Funktionen t_1, t_2 mit $t_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = t_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, womit $t_1 \star t_2 = c_0 \in [A]$ gilt. Folglich ist auch $f'_{11}(x) := f_{11}(c_0, x) \in \{j_1, u_2\}$ eine Superposition über A. Da $f_{12}(c_0, j_1(x), x) = u_2(x)$ und $f_{13}(c_0, u_2(x), x) = j_1(x)$, haben wir $T^1_{0,1;0,2} \subseteq [A]$.
Als nächstes soll gezeigt werden, daß alle Funktionen aus $T_{0,1;0,2}$, die nur Werte aus $\{0,1\}$ oder $\{0,2\}$ annehmen, als Superpositionen über A darstellbar sind.
Die Funktion $f'_9(x,y) := j_1(f_9(c_0(x), x, y)) \in [A]$ hat die Eigenschaft: $f'_9 \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \in \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Wir bilden nun zu jedem $\mathbf{x} \in E_3^n$ ein Tupel

$$\begin{aligned}\mathfrak{T}_{\mathbf{x}} := (&q_{\mathbf{a_1}}(\mathbf{x}), q_{\mathbf{a_2}}(\mathbf{x}), ..., q_{\mathbf{a_{2^n-1}}}(\mathbf{x}),\\ &j_1(x_1), j_1(x_2), ..., j_1(x_n),\\ &..., f_9'(x_i, x_j), ...)\end{aligned}$$

der Länge $t := 2^n - 1 + n + n \cdot (n-1)$, wobei $\{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_{2^n-1}}\} = E_2^n \backslash \{\mathbf{0}\}$, $q_{\mathbf{a_i}} \in [A]$,

$$q_{\mathbf{a_i}}(\mathbf{x}) := \begin{cases} 1 & \text{für} \quad \mathbf{x} = \mathbf{a_i}, \\ 0 & \text{für} \quad \mathbf{x} \in \{0,2\}^n, \\ \in \{0,1\} & \text{sonst} \end{cases}$$

$(i = 1, 2, ..., 2^n - 1)$, $\{i, j\} \subseteq \{1, 2, 3, ..., n\}$, i, j paarweise verschieden und $\{i, j\}$ sämtliche Möglichkeiten durchläuft.
Man prüft nun leicht nach, daß für beliebige $\mathbf{x}, \mathbf{y} \in E_3^n$ die folgenden Implikationen gelten:

$$\begin{aligned}&(\mathbf{x}, \mathbf{y} \in E_2^n \ \wedge \ \mathbf{x} \neq \mathbf{y}) \implies \mathfrak{T}_{\mathbf{x}} \neq \mathfrak{T}_{\mathbf{y}},\\ &(\mathbf{x} \in E_2^n \ \wedge \ \mathbf{y} \in E_3^n \backslash (E_2^n \cup \{0,2\}^n)) \implies \mathfrak{T}_{\mathbf{x}} \neq \mathfrak{T}_{\mathbf{y}},\\ &(\mathbf{x}, \mathbf{y} \in E_3^n \backslash (E_2^n \cup \{0,2\}^n) \ \wedge \ \mathbf{x} \neq \mathbf{y}) \implies \mathfrak{T}_{\mathbf{x}} \neq \mathfrak{T}_{\mathbf{y}}.\end{aligned}$$

Folglich existiert zu jeder n-stelligen Funktion $f \in T_{0,1;0,2} \cap P_{3,2}$ in $[A] \cap P_{3,2}$ eine t-stellige Funktion q_f mit $q_f(\mathfrak{T}_{\mathbf{x}}) = f(\mathbf{x})$, woraus sich $T_{0,1;0,2} \cap P_{3,2} \subseteq [A]$ ergibt.
Mit Hilfe der Funktion f_{10} zeigt man analog, daß $T_{0,1;0,2} \cap P_{3,\{0,2\}} \subseteq [A]$ richtig ist.

Bildet man $f_{14}'(x, y) := f_{14}(c_0(x), x, y) \in [A]$, so hat diese Funktion die Eigenschaft: $f_{14}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Eine beliebige Funktion $g^m \in T_{0,1;0,2}$ läßt sich dann mit Hilfe der m-stelligen Funktionen

$$g_1(\mathbf{x}) := \begin{cases} g(\mathbf{x}) & \text{für} \quad g(\mathbf{x}) \in \{0,1\}, \\ 0 & \text{sonst}, \end{cases} \qquad g_2(\mathbf{x}) := \begin{cases} g(\mathbf{x}) & \text{für} \quad g(\mathbf{x}) \in \{0,2\}, \\ 0 & \text{sonst} \end{cases}$$

in der Form $g(\mathbf{x}) = f_{14}(g_1(\mathbf{x}), g_2(\mathbf{x}))$ darstellen, woraus $[A] = T_{0,1;0,2}$ folgt.
Die paarweise Unvergleichbarkeit der Klassen (9) - (14) ist der Tabelle 7 zu entnehmen, wobei

$$h_1(x, y, z) := \begin{cases} x + y + z (mod\, 4) & \text{für} \quad x, y, z \in \{0,2\}, \\ x & \text{sonst}, \end{cases}$$

$$h_2(x, y, z) := \begin{cases} x + y + z (mod\, 2) & \text{für} \quad x, y, z \in \{0,1\}, \\ x & \text{sonst} \end{cases}$$

und die Funktionen g_1, g_2, g_3 in Tabelle 8 definiert sind.
Die paarweise Unvergleichbarkeit dieser Mengen mit den restlichen Klassen und die der restlichen Klassen untereinander ist leicht nachzuprüfen.

	h_1	h_2	g_1	g_2	g_3	j_1	u_2
(9)	+	−	−	−	−	+	+
(10)	−	+	−	+	−	+	+
(11)	−	−	+	+	+	−	−
(12)	−	−	−	+	+	+	−
(13)	−	−	−	+	+	−	+
(14)	−	−	+	−	+	+	+

Tabelle 7

x_1	x_2	g_1	g_2	g_3
0	0	0	0	0
0	1	1	1	0
1	0	0	1	1
1	1	0	1	1
0	2	2	2	0
2	0	0	2	2
2	2	0	2	2
1	2	0	2	0
2	1	0	2	0

Tabelle 8

□

Satz 7 *Sei $\{a,b,c\} := E_3$. $T_{a,b;a,c;b} := Pol_3\{a,b\} \cap Pol_3\{a,c\} \cap Pol_3\{b\}$ besitzt genau 12 maximale Klassen:*

(1) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

(2) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & a & b \\ a & b & b \end{pmatrix}$

(3) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & b & a \\ a & a & b \end{pmatrix}$

(4) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} b & b & a \\ b & a & b \end{pmatrix}$

(5) $T_{a,b;a,c;b} \cap Pol_3\{c\}$

(6) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & a & c \\ a & c & c \end{pmatrix}$

(7) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & c & a \\ a & a & c \end{pmatrix}$

(8) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & a & a & c & c & a & c & c \\ a & a & c & c & a & c & a & c \\ a & c & a & a & c & c & a & c \\ a & c & c & a & a & a & c & c \end{pmatrix}$

(9) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & b & a \\ a & b & c \end{pmatrix}$

(10) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & c & a \\ a & c & b \end{pmatrix}$

(11) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & b & a \\ a & a & c \end{pmatrix}$

(12) $T_{a,b;a,c;b} \cap Pol_3 \begin{pmatrix} a & b & b \\ a & a & c \end{pmatrix}$.

Beweis: O.B.d.A. seien $a = 0$, $b = 1$ und $c = 2$. Mit A bezeichnen wir in diesem Beweis eine Teilmenge von $T_{0,1;0,2;1}$, die keine Teilmenge der unter (1) bis (12) aufgezählten Teilklassen von $T_{0,1;0,2;1}$ ist. Dann gehören zu $[A]$ gewisse Funktionen $f_1, f_2, ..., f_{12}$ mit der im Teil I

vereinbarten Eigenschaft (*). Insbesondere gilt $f_5 = j_1 \in [A]$.
Wegen Lemma 2.3 und 2.2 aus Teil I existieren zu jeder Funktion $g^m \in Pol_2\{0\} \cap Pol_2\{1\} (\subset P_2)$ und jeder Funktion $h^m \in Pol_3\{0\} (\subset P_{\{0,2\}})$ Funktionen $G^m, H^m \in [A]$ mit

$$(\forall \mathbf{x} \in E_2^m : \ g(\mathbf{x}) = G(\mathbf{x})) \ \wedge \ (\forall \mathbf{x} \in \{0,2\}^m : \ h(\mathbf{x}) = H(\mathbf{x})).$$

Als nächstes soll gezeigt werden, daß alle Funktionen aus $T_{0,1;0,2;1}$, die nur Werte aus $\{0,1\}$ annehmen, als Superpositionen über A darstellbar sind.
Die Funktion $f_9'(x,y) := j_1(f_9(j_1(y),x,y)) \in [A] \cap P_{3,2}$ hat die Eigenschaft: $f_9' \begin{pmatrix} 1\,0 \\ 1\,2 \end{pmatrix} \in \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$. Wir bilden nun zu jedem $\mathbf{x} \in E_3^n$ ein Tupel

$$\begin{aligned} \mathfrak{T}_{\mathbf{x}} := (&q_{\mathbf{a_1}}(\mathbf{x}), q_{\mathbf{a_2}}(\mathbf{x}), ..., q_{\mathbf{a_{2^n-2}}}(\mathbf{x}), \\ &j_1(x_1), j_1(x_2), ..., j_1(x_n), \\ &..., f_9'(x_i,x_j), ...) \end{aligned}$$

der Länge $t := 2^n - 2 + n + n \cdot (n-1)$, wobei $\{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_{2^n-2}}\} = E_2^n \backslash \{\mathbf{0}, \mathbf{1}\}$, $q_{\mathbf{a_i}} \in [A]$,

$$q_{\mathbf{a_i}}(\mathbf{x}) := \begin{cases} 1 & \text{für} \quad \mathbf{x} = \mathbf{a_i}, \\ 0 & \text{für} \quad \mathbf{x} \in \{0,2\}^n, \\ \in \{0,1\} & \text{sonst} \end{cases}$$

$(i = 1,2,...,2^n-2)$, $\{i,j\} \subseteq \{1,2,3,...,n\}$, $i \neq j$ und $\{i,j\}$ sämtliche Möglichkeiten durchläuft.
Man prüft nun leicht nach, daß für beliebige $\mathbf{x}, \mathbf{y} \in E_3^n$ die folgenden Implikationen gelten:

$$\begin{aligned} &(\mathbf{x}, \mathbf{y} \in E_2^n \ \wedge \ \mathbf{x} \neq \mathbf{y}) \implies \mathfrak{T}_{\mathbf{x}} \neq \mathfrak{T}_{\mathbf{y}}, \\ &(\mathbf{x} \in E_2^n \ \wedge \ \mathbf{y} \in E_3^n \backslash (E_2^n \cup \{0,2\}^n)\ ) \implies \mathfrak{T}_{\mathbf{x}} \neq \mathfrak{T}_{\mathbf{y}}, \\ &(\mathbf{x}, \mathbf{y} \in E_3^n \backslash (E_2^n \cup \{0,2\}^n) \ \wedge \ \mathbf{x} \neq \mathbf{y}) \implies \mathfrak{T}_{\mathbf{x}} \neq \mathfrak{T}_{\mathbf{y}}. \end{aligned}$$

Folglich existiert zu jeder n-stelligen Funktion $f \in T_{0,1;0,2;1} \cap P_{3,2}$ in $[A] \cap P_{3,2}$ eine t-stellige Funktion mit $q_f(\mathfrak{T}_{\mathbf{x}}) = f(\mathbf{x})$, woraus sich $T_{0,1;0,2;1} \cap P_{3,2} \subseteq [A]$ ergibt.
Als nächstes sollen alle Funktionen aus $T_{0,1;0,2;1}$ in $[A]$ nachgewiesen werden, die nur auf $\mathbf{1}$ den Wert 1 annehmen. Wir beginnen mit der Konstruktion einiger Hilfsfunktionen:
Wählt man $g_1^2 \in T_{0,1;0,2;1} \cap P_{3,2}$ mit $g_1 \begin{pmatrix} 2\,0 \\ 2\,1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so hat die Funktion $f_{10}'(x,y) := f_{10}(g_1(x,y),x,y)$ die Eigenschaft $f_{10}' \begin{pmatrix} 2\,0 \\ 2\,1 \end{pmatrix} \in \begin{pmatrix} 0\,2\,2 \\ 2\,0\,1 \end{pmatrix}$. Falls $f_{10}' \begin{pmatrix} 2\,0 \\ 2\,1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ist, erhält man durch $f_{12}'(x,y) := f_{12}(g_1(x,y),y,f_{10}'(x,y))$ eine Funktion aus $[A]$ mit $f_{12}' \begin{pmatrix} 2\,0 \\ 2\,1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Also genügt es, $f'_{10}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \in \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ weiter zu betrachten.

Wie bereits gezeigt wurde, gehört zu $[A]$ eine Funktion g_2^2 mit $g_2\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, mit der man $f''_{10}(x,y) := g_3(f'_{10}(x,y),x) \in [A]$ erhält. O.B.d.A. können wir damit $f'_{10}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ und $f''_{10}\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ annehmen.

Wählt man $g_3^2, g_4^2, g_5^2 \in [A] \cap P_{3,2}$ wie folgt:

$$g_3(x,y) := \begin{cases} 1 \text{ für} & (x,y) \in \{(1,1),(2,1)\}, \\ 0 \text{ sonst}, & \end{cases}$$

$$g_4(x,y) := \begin{cases} 1 \text{ für} & (x,y) = (1,1), \\ 0 \text{ sonst} & \end{cases}$$

und

$$g_5(x,y) := \begin{cases} f''_{10}(x,y) \text{ für} & (x,y) \in E_2^2, \\ 0 \qquad\qquad \text{sonst}, & \end{cases}$$

so haben die Funktionen $r(x,y) := f'_{12}(g_3(x,y), f'_{10}(x,y))$ und $s(x,y) := f_{12}(g_4(x,y), g_5(x,y), f''_{10}(x,y))$ die Eigenschaften $r\begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ und $s\begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Nach diesen Vorbereitungen können wir jetzt n-stellige Funktionen der Form

$$g_{\mathbf{a}}(\mathbf{x}) := \begin{cases} 2 \text{ für} & \mathbf{x} = \mathbf{a}, \\ 1 \text{ für} & \mathbf{x} = \mathbf{1}, \\ 0 \text{ sonst} & \end{cases}$$

für $\mathbf{a} \in E_3^n \backslash E_2^n$ konstruieren:

Sei zunächst $\mathbf{a} \in \{0,2\}^n \backslash \{\mathbf{0}\}$. Mit Hilfe der Funktion $h_{\mathbf{a}}^n \in [A]$ mit $h_{\mathbf{a}}^n(\mathbf{a}) = 2$, $h_{\mathbf{a}}^n(\mathbf{x}) = 0$ für alle $\mathbf{x} \in \{0,2\}^n \backslash \{\mathbf{0}\}$ und der Funktion

$$t_{\mathbf{a}}(\mathbf{x}) := \begin{cases} 1 \text{ für} & \mathbf{x} = \mathbf{1} \ \vee \ (h_{\mathbf{a}}(\mathbf{x}) = 2 \ \wedge \ \mathbf{x} \neq \mathbf{a}), \\ 0 \text{ sonst} & \end{cases}$$

erhalten wir: $g_{\mathbf{a}}(\mathbf{x}) = s(h_{\mathbf{a}}(\mathbf{x}), t_{\mathbf{a}}(\mathbf{x}))$. Falls $\mathbf{a}$ zu $E_3^n \backslash (E_2^n \cup \{0,2\}^n)$ gehört und $a_i = 2$ ist, erhält man $g_{\mathbf{a}}$ als Superposition über r und der Funktion

$$u_{\mathbf{a}}(\mathbf{x}) := \begin{cases} 1 \text{ für} & \mathbf{x} \in \{\mathbf{a}, \mathbf{1}\}, \\ 0 \text{ sonst} \end{cases}$$

wie folgt: $g_{\mathbf{a}}(\mathbf{x}) = r(x_i, u_{\mathbf{a}}(\mathbf{x}))$. Offenbar sind alle n-stelligen Funktionen aus $T_{0,1;0,2;1} \backslash P_{3,2}$, die nur auf $\mathbf{1}$ den Wert 1 annehmen, Superpositionen über Funktionen der Form $g_{\mathbf{a}}$ mit $\mathbf{a} \in E_3^n \backslash E_2^n$ und einer gewissen Funktion $d^2 \in [A]$ mit $d \begin{pmatrix} 0\,2 \\ 2\,0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.
Indem man die Funktion $f'_{11}(x,y) := f_{11}(g_4(x,y), x, y) \in [A]$ bildet, die die Eigenschaft $f'_{11} \begin{pmatrix} 1\,0 \\ 0\,2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ hat, läßt sich eine beliebige Funktion $f^n \in T_{0,1;0,2;1}$ wie folgt als Superposition über den oben konstruierten Funktionen darstellen: $f(\mathbf{x}) = f'_{11}(t(\mathbf{x}), g(\mathbf{x}))$, wobei

$$t(\mathbf{x}) := \begin{cases} f(\mathbf{x}) \text{ für} & f(\mathbf{x}) \in \{0,1\}, \\ 0 & \text{sonst} \end{cases}$$

und

$$g(\mathbf{x}) := \begin{cases} 1 \text{ für} & \mathbf{x} = \mathbf{1}, \\ 2 \text{ für} & f(\mathbf{x}) = 2, \\ 0 \text{ sonst.} \end{cases}$$

Die paarweise Unvergleichbarkeit der Klassen (9) - (12) ist der Tabelle 9 zu entnehmen, wobei

$$h_1(x,y,z) := \begin{cases} x + y + z (mod\, 4) \text{ für} & x,y,z \in \{0,2\}, \\ x & \text{sonst,} \end{cases}$$

$$h_2(x,y,z) := \begin{cases} x + y + z (mod\, 2) \text{ für} & x,y,z \in \{0,1\}, \\ x & \text{sonst} \end{cases}$$

und die Funktionen h_3, h_4, h_5 in Tabelle 10 definiert sind.
Die paarweise Unvergleichbarkeit dieser Mengen mit den restlichen Klassen und die der

restlichen Klassen untereinander ist leicht nachzuprüfen.

	h_1	h_2	h_3	h_4	h_5
(9)	$+$	$-$	$+$	$-$	$-$
(10)	$-$	$+$	$-$	$+$	$-$
(11)	$-$	$-$	$+$	$-$	$+$
(12)	$-$	$-$	$-$	$+$	$+$

Tabelle 9

x_1	x_2	h_3	h_4	h_5
0	0	0	0	0
0	1	0	1	0
1	0	0	1	1
1	1	1	1	1
0	2	2	2	0
2	0	2	2	2
2	2	0	2	2
1	2	0	2	0
2	1	0	2	0

Tabelle 10

$\square$

Literatur

[1] **Lau, D. :** *Die maximalen Klassen von* $Pol_3\{\varrho|\varrho \in Q\}$ *für* $Q \subseteq \mathfrak{P}(\{0,1,2\})$, *Teil I*. Rostock. Math. Kolloq. **51**, 111-126 (1997). Teil III, Eingereicht bei Rostock. Math. Kolloq.

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Autor:

Dietlinde Lau
Fachbereich Mathematik
Universität Rostock
D-18051 Rostock
Germany

e-mail: dietlinde.lau@mathematik.uni-rostock.de

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Insbesondere beachte man:

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3. **Textbreite/Texthöhe** des Manuskripts sollten sich an den Maßen **160mm/230mm** orientieren.

4. Der Manuskripttext ist **eineinhalbzeilig, linksbündig**, wenn möglich links-und rechtsbündig zu schreiben. Beim Auftreten von Formeln im laufenden Text ist der Zeilenabstand entsprechend zu vergrößern.

5. Der Platz für **Abbildungen** sollte beim Schreiben ausgespart werden; die Abbildungen selbst sind in der dem ausgesparten Platz entsprechenden Größe gesondert beizufügen.

6. **Literaturzitate** sind im Text durch laufende Nummern (vgl. [3], [4]; [7, 8, 10]) zu kennzeichnen und am Schluß der Arbeit unter der Zwischenüberschrift **Literatur** bzw. **References** zusammenzustellen. Hierbei ist die durch die nachfolgenden Beispiele veranschaulichte Form einzuhalten (die Zeitschriftenabkürzungen erfolgen nach Mathematical Reviews).

   [3] **Zariski, O.**, and **Samuel, P.:** *Commutative Algebra.* Princeton 1958

   [4] **Steinitz, E.:** *Algebraische Theorie der Körper.* J. Reine Angew. Math. **137**, 167-309 (1920)

   [8] **Gnedenko, B.W.:** *Über die Arbeiten von C.F. Gauß zur Wahrscheinlichkeitsrechnung.* In: Reichard, H. (Ed.): C.F. Gauß, Gedenkband anläßlich des 100. Todestages. S. 193-204, Leipzig 1967

   Die Angaben erfolgen in Originalsprache; bei kyrillischen Buchstaben sollte die (bibliothekarische) Transliteration bzw. eine Übersetzung lt. Mathematical Reviews verwendet werden.

7. Die aktuelle, vollständige Adresse des Autors sollte enthalten: Vornamen Name / Institution / Struktureinheit / Straße Hausnummer / Postleitzahl Ort / Land.

Weiterhin besteht die Möglichkeit, mit dem **Satzsystem LaTeX** oder unter anderen Textprogrammen erstellte Manuskripte auf unter **MS-DOS** formatierten Disketten (**3.5″, 0.72MB, 1.44MB**) einzureichen oder per e-mail an eine der folgenden Adressen zu schicken:

`susann.dittmer@mathematik.uni-rostock.de`

`heike.schubert@mathematik.uni-rostock.de`