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Applications of the Bartsch-Poppe duality approach

1 Introduction

In the papers [1], [2], [3] by R. Bartsch and H. Poppe a general duality system was defined and studied:

$$(X, Y, X^d, X^{dd}, J :\to X^{dd}) .$$

Here X, Y are spaces, X^d is the first dual space of X with respect to Y, X^{dd} denotes the second dual space of X w. r. t. Y and J is the canonical map as is known, from classical examples.

The map J we define by the evaluation map ω: let X, Y be nonempty sets,

$$\omega : X \times Y^X \to Y, \quad \forall (x, h) \in X \times Y^X : \omega(x, h) := h(x) .$$

Hence we find:

$$\forall x \in X : Jx = \omega(x, \cdot), \quad \omega(x, \cdot) : Y^X \to Y : \forall h \in Y^X : \omega(x, \cdot)(h) = \omega(x, h) = h(x) .$$

In short we call it the B/P duality approach.

In the papers [1], [2], [3] this general duality approach was applied to well known examples of representation theorems.

Let for instance X be a unital commutative Banachalgebra, or let X be a Boolean ring.

We used suitable spaces Y and defined then the dual spaces X^d and X^{dd} and proved the Gelfand and the Stone representation theorem respectively using the general B/P duality approach.

We also obtained new results. For example, [2], theorem 5.4 shows the embedding of a vector lattice X into X^{dd}, in [3], theorem 4.5 one finds the representation of an unital, noncommutative C^*-algebra.

What is the aim of this paper?

1. We want to improve the definitions of the first dual space X^d and the second dual space X^{dd} of a given space, X as were defined in [1]. For this purpose we will repeat in short the very basic definitions and some results of the B/P duality approach.
2. We apply the B/P duality approach to get new, well arranged proofs of
   (a) the representation of a nonunital commutative C^*-algebra (Gelfand-Naimark theorem)
   (b) the embedding theorem of Kadison.

2 The duality approach

2.1 Abstract definition

Let X, Y be sets or spaces. Y^X means of course the set of all functions from X to Y.

Now we will define an abstract scheme of duality.

Definition 2.1 1. *Let be $A \subseteq Y^X$, $A \neq \emptyset$.*

We call A to be the first dual space of X with respect to Y.

2. *We use here the definition of the map J. Let $B \subseteq Y^A$, $B \neq \emptyset$; let further be: $J : X \to Y^A$, hence as we know:*

$$\forall x \in X : Jx = \omega(x, \cdot), \omega(x, \cdot) : A \to Y : \forall h \in A : \omega(x, \cdot)(h) = \omega(x, h) = h(x)\,.$$

If $J(X) \subseteq B$, i. e. $\forall x \in X : \omega(x, \cdot) \in B$ then we call B to be the second (abstract) dual space of X w. r. t. Y.

Remarks 2.2 (a) If we in definition 2.1 only consider sets X, Y we cannot formulate nice properties or prove useful theorems concerning the abstract dual spaces A, B. But of course this is possible for spaces X, Y, where we can use the special properties of these spaces to give A, B concrete forms.

(b) We will consider spaces with algebraic, order, and topological structures, where topologies can be derived from metrics, norms or inner products. We also use measurable spaces.

We put emphasis on spaces with algebraic and topological structures.

2.2 Concrete definition of the first and of the second dual space

2.2.1 The first dual space X^d of a space X with respect to a space Y At first glance we can say:

X^d consists of homomorphism from X to Y. But this can only work if we state an assumption.

Basic Assumption 2.3 X and Y belong to the same class of spaces.

We consider three simple examples:

(a) X and Y are vector spaces over $\mathbb{R}$ If necessary we add: $\dim X = \dim Y$.

Then X and Y are in the same class of spaces.

(b) Let be X and Y C^*-algebras over $\mathbb{C}$; X and Y are commutative. If X has no unit element and Y has an unit then X and Y do not belong to the same class of spaces.

(c) Let X, Y be lattices. Then of course X and Y fullfill (2.3).

In case (a) the first dual space is well known. Let

$$Y = \mathbb{R}, X^d = \{h : X \to \mathbb{R} | h \text{ is linear } \}.$$

If X is a topological vector space, then we get:

$$X^d = \{h : X \to \mathbb{R} | h \text{ is linear and } h \text{ is continuous}\}.$$

Here we consider a continuous map as a topological homomorphism. In case (b) we cannot set $Y = \mathbb{C}$ since the C^*-algebra $\mathbb{C}$ has an unit and, hence $Y = \mathbb{C}$ contradicts assumption (2.3). We will later come back to this example.

In case (c) we at once can write:

$$X^d = \{h : X \to Y | h \text{ is a lattice-homomorphism}\}.$$

Let X, Y be spaces with algebraic or order operations. By the basic assumption (2.3) we find for each operation in X a corresponding operation in Y.

By $A(X, Y)$ we denote the set of all such pairs of operation in X and Y respectively.

We assume $\emptyset \neq A(X, Y)$ and $A(X, Y)$ is a finite set.

Definition 2.4 (a) *$H(X, Y) = \{h : X \to Y | h$ is a homomorphism for each pair of operations from $A(X, Y)\}$*

(b) *If both spaces X, Y have also a topology then we consider $H(X, Y) \cap C(X, Y)$, where $C(X, Y)$ is the space of all continuous functions from X to Y. $X^d = H(X, Y)$ or $X^d = H(X, Y) \cap C(X, Y)$ and we also find:*

$$X^d \subseteq H(X, Y) \text{ or } X^d \subseteq H(X, Y) \cap C(X, Y).$$

If this is possible and useful we provide X^d with a topology η, $\tau_p \leq \eta$ where τ_p denotes the pointwise topology.

We call X^d to be the first dual space of X with respect to Y. To define the pointwise topology τ_p for X^d we must have a topology for Y. As we soon will see, in some cases we indeed will use τ_p. Hence we come to:

Basic Assumption 2.5 Y always has a topology. If for Y no topology is given we will define: Y is provided with

$$\begin{cases}\text{the discrete topology, if } X \text{ has no topology}\\ \text{the trivial topology, if } X \text{ has a topology}\end{cases}$$

If we want that all $h \in H(X,Y)$ are continuous too and X has no topology we provide X also with the discrete topology. The elements of X^d are functions or maps. Using the operations in Y and in X we want to define corresponding operations in X^d too. In most cases we define these operations pointwise. For instance let be in X and in Y an addition is defined:

$$X = (X,+),\ Y = (Y,+)\,.$$

If now $h_1, h_2 \in X^d$:

$$h_1 + h_2 : \forall x \in X : (h_1+h_2)(x) := h_1(x) + h_2(x) \in Y\,.$$

If for example we have $X = Y$ than we can h_1, h_2 also compose: $h_1 \circ h_2$.

Definition 2.6 *If X, Y are spaces and we have defined X^d then for X^d there exists two possibilities:*

1. *X and X^d belong to the same class of spaces*
2. *X and X^d do not belong to the same class of spaces.*

Now let us consider some examples to clear up the situation.

Examples 2.7

1. Let X be a normed vector space over $\mathbb{R}$ and let be $Y = \mathbb{R}$.

   $$X^d = \{h : X \to \mathbb{R} \,|\, h \text{ is linear and } h \text{is continuous}\}\,.$$

   With pointwise defined vector operations and the sup-norm (on bounded sets) X^d is a normed vector space over $\mathbb{R}$ too. Hence X and X^d belong to the same class of spaces.

2. Let $X = (X, \|\cdot\|)$ a $\mathbb{R}$-normed space again, $Y = \mathbb{R}$ and $X^d = \{h : X \to \mathbb{R} \,|\, h$ is linear and continuous and $\|h\| = 1\}$. But here X^d is no vector space:

   we assume that X^d is a vector space, hence

   $$h \in X^d \Rightarrow 2h \in X^d\,,$$

   but $\|2h\| = 2\|h\| = 2 \neq 1$, a contradiction.

   Thus X and X^d do not belong to the same class of spaces.

3. Let X be a vector lattice (a Riesz space), $\mathbb{R}$ with natural order also is a vector lattice. It is known that $X^d = \{h : X \to \mathbb{R} |\ h \text{ is linear and order bounded}\}$ is a vector lattice too, showing that X and X^d belong to the same class of spaces.

4. In the paper [2], definition 5.1 we find for a vector lattice

$$X : X^d = \{h : X \to \mathbb{R} \,|\, h \text{ is a linear lattice homomorphism}\} .$$

   the following example 5.3 shows that (in general) X^d is no vector lattice.

   Hence X and X^d do not belong to the same class of spaces.

Remarks 2.8 (a) If X and X^d belong to the same class of spaces we can define the second dual space by: $X^{dd} := (X^d)^d$. But otherwise we must find a suitable definition of X^{dd}.

(b) As special cases of definition (2.4) we get:

X and Y respectively have only:

(b.a) topologies

(b.b) algebraic operations

(b.c) lattice operations

Case (b.a) was treated in our paper [2], concerning (b.b) in [1], 5. Some examples and applications, [1], page 290 we considered two communicative rings X, Y with units.

(c) Let $(X, \underline{A}, \mu)$ be a measure space, where X is a set, $\underline{A}$ is a σ-algebra of subsets of X and $\mu : \underline{A} \to [0, +\infty]$ is a measure.

Let $p \in \mathbb{R}$, $1 \le p < \infty$, let $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ be measurable. Then the L^p-norm of f is given by

$$\|f\|_p = \left(\int_X |f|^p \right)^{\frac{1}{p}} .$$

$f : X \to \mathbb{R}$ is called p-integrable or a L^p-function if f is measureable and $\|f\|_p < \infty$.

$$L^p(\mu) = \{f : X \to \mathbb{R} \,|\, f \text{ is } \underline{A}\text{-measurable and } \|f\|_p < \infty\} .$$

$L^p(\mu) = (L^p(\mu), \|\cdot\|_p)$ is a normed space, even a Banach space. Hence

$$(L^p(\mu))^d = \{h : L^p(\mu) \to \mathbb{R} \,|\, h \text{ is linear and continuous}\}.$$

Now let be $p \in \mathbb{R}$ and $1 < p < \infty$ and let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$.

There exists a isomorphic and isometric map from $L^q(\mu)$ onto $(L^p(\mu))^d$.

This result has the advantage that we can much better work with $L^q(\mu)$ than with $(L^p(\mu))^d$. This situation we find by many dual spaces X^d, especially if the space X is a normed space. This procedure, where the dual space will be replaced by a *better space* we also will apply to the two following examples in this paper, where we will use the B/P duality approach. But here the starting spaces X are not only normed spaces.

Precise definitions and proofs of the above statements about $L^p(\mu)$-spaces one finds in modern books on measure and integration theory, for instance in [7].

Now we come back to 2.6. Following [1], definition 4.1, page 282 we define:

Definition 2.9 *We say that X^d has the defect D, D, if X and X^d do not belong to the same class of spaces; not the defect D, non D, otherwise.*

Now we can define the second dual space.

2.2.2 The second dual space X^{dd} with respect to a space Y.

Definition 2.10 *Let X, Y be spaces in the sense of 2.2, (b). X, Y fulfill basic assumption 2.3. According to basis assumption 2.5 X^d has a topology η with $\tau_p \leq \eta$, since τ_p is defined.*

Part 1

$$X^{dd} = \begin{cases} \big((X^d, \eta)^d, \mu\big) & \text{if non } D \\ \big(\mathcal{C}((X^d, \tau_p), (Y, \sigma)), \mu\big) & \text{if } D \end{cases}$$

where $\mathcal{C}(\cdot, \cdot)$ means the space of continuous maps.

Here we also assume:

$$\tau_p \leq \mu\,.$$

X^{dd} is called the second dual space of X w. r. t. Y, σ, η, μ.

By [1], lemma 4.1, page 283 and corollary 4.1, page 284 we know that $J(X) \subseteq X^{dd}$ holds.

Basic Assumption 2.11 X and X^{dd} are in the same class of spaces.

Part 2

$$X^{dd} = \begin{cases} X^{dd} \text{ as defined in part 1, if (2.11) holds} \\ J(X) \text{ otherwise} \end{cases}$$

Remark 2.12 The operations in X^{dd} we define pointwise using the operations in X^d and in Y. See also [1], page 283.

3 The Gelfand-Naimark theorem for nonunital commutative C^*-algebras

At first we will repeat some well-known definitions and results:

Let X be a commutative nonunital C^*-algebra.

Then $X_1 = X \times \mathbb{C}$ is a commutative C^*-algebra with unit, if we provide X_1 with the defined algebraic operations and the C^*-norm for X_1.

The unit for X_1 is then $(0,1) \in X \times \mathbb{C}$.

The map: $x \to (x,0)$ from X to X_1 is a *-isomorphic, isometric homomorphism onto $X \times \{0\}$ with $\|(x,0)\| = \|x\|$.

Thus we can identify X with $X \times \{0\} \subseteq X_1$ and by this way X can be considered as a subspace of X_1.

$x \to (x,0)$ is also an uniform bijective map implying that $X \times \{0\}$ is complete since X is complete hence $X \times \{0\}$ is a closed subspace of X_1; this set is even a maximal ideal in X_1.

We state:

Proposition 3.1 *$X \times \{0\} = \{(x,0) | x \in X\}$ is a nonunital C^*-subalgebra of $X_1 = X \times \mathbb{C}$.*

3.1 The first dual spaces of X, X_1 and the second dual space of X

According to definition 2.4 we can define:

$$\begin{aligned} X^d &= \{h : X \to \mathbb{C} \,|\, h \text{ is a } {}^*\text{-homomorphism and } h \text{ is continuous}\} \\ &= \{h : X \to \mathbb{C} \,|\, h \text{ is a } {}^*\text{-homomorphism}\}, \\ X_1^d &= \{g : X_1 \to \mathbb{C} \,|\, g \text{ is } {}^*\text{-homomorphismus}\}. \end{aligned}$$

If 0 is the zero-homomorphism, by definition 3.2 of [1], page 281, $0 \in X^d$, but by lemma 4.2 of [1], page 288, $0 \notin X_1^d$, hence $X_1^d \backslash \{0\}$ is the new dual space.

For X_1 we know the second dual space

$$X_1^{dd} = \left(C((X_1^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|}\right) ,$$

where τ_p is the pontwise topology and $\tau_{\|\cdot\|} = \tau_u$ is the uniform topology generated by the sup-norm, [1], [3], and the Gelfand-Naimark theorem for unital algebras. We can use the sup-norm here because $((X_1^d \backslash \{0\}), \tau_p)$ is compact and Hausdorff and thus X^{dd} consists of bounded functions:

$$X_1^{dd} = \left(C_b((X_1^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_u\right) .$$

Remark 3.2 Concerning the Gelfand-Naimark theorem for unital C^*-algebras look at [8] and relevant books and papers.

3.2 Preliminaries

At first we show a result, which is important for our purposes: the dual spaces (X^d, τ_p) and $(X_1^d \backslash \{0\}, \tau_p)$ are homeomorphic. Moreover we consider a simple criterion that a continuous function already belongs to the space of continuous functions vanishing at infinity.

And we need the Stone-Weierstrass theorem for the case that the basic space is not compact but only locally compact.

It is nearby that there exists a connection between the first dual space X^d of X and the first dual space X_1^d of the unital extension, $X_1 = X \times \mathbb{C}$, of X.

Indeed:

$$\forall (h, (x, \alpha)) \in X^d \times X_1 : \tilde{h} : \tilde{h}(x, \alpha) = h(x) + \alpha$$

By the following proposition we show that holds:

$$\forall h \in X^d : \tilde{h} \in X_1^d \,.$$

This proposition is well known (see for instance [4]). We will not prove the proposition.

Proposition 3.3 1. *$\tilde{h}(0, 1) = 1$*

2. *$\tilde{h}$ is uniquely determined by h*
3. *$\tilde{h}$ is a $*$-homomorphism and thus $\tilde{h}$ is continuous*
4. *$\tilde{h}$ is an extension of h*
5. *If $0 \in X^d$ is the zero-element then $\tilde{0}$ is not, the zero-element of*

$$X_1^d : \forall (x, \alpha) \in X_1 : \tilde{0}(x, \alpha) = \hat{0}(x) + \alpha = 0 + \alpha = \alpha \,.$$

6. *If $g \in X_1^d \backslash \{0\}$ then $g|(X \times \{0\}) \in X^d$*

Now we can define the map

$$G : X^d \to X_1^d \backslash \{0\} : \forall h \in X^d : G(h) = \tilde{h} \,.$$

By proposition 3.3 we know that $\tilde{h} \in X_1^d \backslash \{0\}$

Theorem 3.4 (a) *The map G is bijective*

(b) *G is neither linear nor multiplicative*

(c) *$G : (X^d, \tau_p) \to (X_1^d \backslash \{0\}, \tau_p)$ is continuous*

(d) *$G : (X^d, \tau_p) \to (X_1^d \backslash \{0\}, \tau)$ is open*

Proof. (a) G is injective:

$$\forall (h_1, h_2) \in X^d \times X^d,\ h_1 \neq h_2$$

and we assume

$$\begin{aligned} G(h_1) = G(h_2);\ h_1 \neq h_2 \Rightarrow \exists x_0 \in X : h_1(x_0) \neq h_2(x_0),\ \tilde{h}_1 = \tilde{h}_2 \\ \tilde{h}_1(x_0, 0) = \tilde{h}_2(x_0, 0) \Rightarrow h_1(x_0) + 0 = h_2(x_0) + 0 \\ \Rightarrow h_1(x_0) = h_2(x_0)\,, \end{aligned}$$

a contradiction.

G is surjective too:

$$\forall\ f \in X_1^d \backslash \{0\},\ f \neq 0\,,$$

(a.a) $f = \tilde{0}$: we know:

$0 \in X^d$ and hence $G(0) = \tilde{0} = f$;

(a.b) $f \neq \tilde{0}$, by 3.3, 6.:

$$\begin{aligned} f|X \times \{0\} \in X^d, & G(f|X \times \{0\}) \\ & = (f|\widetilde{X \times \{0\}}) :\ \forall (x, \alpha) \in X_1 : (f|X \times \{0\})(x, \alpha) \\ & = (f|X \times \{0\})(x) + \alpha = f(x, 0) + \alpha = f(x, 0) + 1\alpha \\ & = f(x, 0) + \alpha f(0, 1) = f(x, 0) + f(0, \alpha) = f(x, \alpha)\,. \end{aligned}$$

Thus G is bijective

(b) Let be $f, g \in X^d$ and $f + g \in X^d$, $f \neq 0$, $g \neq 0$;

$$G(f + g) = \widetilde{f + g}\,;$$

let be $(x, \alpha) \in X_1$, $\alpha \neq 0$,

$$\begin{aligned} (\widetilde{f + g})(x, \alpha) = (f + g)(x) + \alpha = f(x) + g(x) + \alpha \\ \neq \tilde{f}(x, \alpha) + \tilde{g}(x, \alpha) = (f(x) + \alpha) + (g(x) + \alpha)\,. \end{aligned}$$

Analogously one shows that G is not multiplicative too.

(c) Let (h_i) be a net from X^d, $h \in X^d$ and $h_i \overset{\tau_p}{\to} h$,

$$\forall (x, \alpha) \in X_1;\ h_i(x) \to h(x) \Rightarrow h_i(x) + \alpha \to h(x) + \alpha \text{ in } \mathbb{C}\,; \Rightarrow \tilde{h}_i(x, \alpha) \to \tilde{h}(x, \alpha)\,,$$

hence $G(h_i) \overset{\tau_p}{\to} G(h)$.

(d) Let be $H \subseteq X^d$ be τ_p-open, we will show that $G(H)$ is τ_p-open in

$$X_1^d \backslash \{0\} : \forall f \in G(H) \exists h \in H : f = \tilde{h} = G(h)\,;$$

now let be (f_i) a net from $G(H)$, $f_i \overset{\tau_p}{\to} f$;

$$f_i = \tilde{h}_i, h_i \in H.\ \forall \times \in X : f_i(x,0) \to f_i(x,0)\,,$$

hence $\tilde{h}_i(x,0) \to \tilde{h}(x,0) \Rightarrow h_i(x) \to h(x)$, hence $h_i \overset{\tau_p}{\to} h$; but then there exists i_o:

$$\forall_i \geq i_o : h_i \in H \Rightarrow \forall i \geq i_o : f_i = G(h_i) \in G(H)\,.$$

Thus $G(H)$ is τ_p-open in $X_1^d \backslash \{0\}$.

Corollary 3.5 *The map G is a topological map from (X^d, τ_p) onto $((X_1^d \backslash \{0\}), \tau_p)$.*

Remark 3.6 The two dual spaces (X^d, τ_p) and $(X_1^d \backslash \{0\}, \tau_p)$ respectively are topologically equivalent, but (in general) not algebraically. We see here once more that in our duality approach the essential space is the second dual space X^{dd} of X and not the first dual space X^d of X. Of course X^d is necessary to construct X^{dd}, but in some sense X^d is not so important.

When does a continuous function already vanish at infinity?

It is not hard to find an answer to this question.

Let X be a locally compact, non-compact Hausdorff space, and let $\alpha X = X \cup \{\infty\}$, $\infty \notin X$, be the one-point – compactificativen of X. If $f \in C(X, \mathbb{K})$, we define:

$$f_\infty : \alpha X \to \mathbb{K} :$$
$$f_\infty(x) = \begin{cases} f(x), & x \in X \\ 0, & x = \infty\,. \end{cases}$$

By the definition of a continuous function vanishing at infinity and by the definitions of the topology for αX we see at once:

Proposition 3.7 (a) *$f \in C_0(X, \mathbb{K}) \Leftrightarrow f_\infty$ is continuous in $x = \infty \Leftrightarrow$*

(b) *For each net (x_i) from αX, $\forall i : x_i \neq \infty$, $x_i \to \infty$ in $\alpha X \Rightarrow f(x_i) \to 0$ in $\mathbb{K}$.*

A Stone-Weierstrass theorem

Theorem 3.8 *Let X be a locally compact noncompact Hausdorff space. Suppose A is a closed, selfadjoint subalgebra of $C_0(X, \mathbb{C})$. If A separates the points of X and for every $x \in X$ there exists $f \in A$ with $f(x) \neq 0$ then $A = C_0(X, \mathbb{C})$.*

The homorphic image units

Theorem 3.9 *Let X, Y be rings and $h : X \to Y$ a ring-homomorphism*

(a) *If h is surjective and e is a (multiplicative) unit in X then $h(e)$ is an unit in Y.*

(b) *Let h be bijective and let e_y be a unit in Y.*

Then $h^{-1}(e_y)$ is a unit in X.

We do not prove this proposition.

3.3 The second dual space of X

X is our starting space: X is a nonunital C^*-algebra. As in the case of an unital Banachalgebra or an unital C^*-algebra here also X^d has by Definition 2.9 the defect D and hence by definition 2.10 we get:

$$X^{dd} = \big(C((X^d, \tau_p), \mathbb{C}), \mu\big) \ ,$$

where the topology μ still must be determined. And we have the canonical map

$$J : \ X \to X^{dd} .$$

The constant function

$$1 : \ \forall h \in X^d : 1(h) = 1$$

is a multiplicative unit in X^{dd} . But this means that X and X^{dd} do not belong to the same class of spaces. Hence according to definition 2.10 we must look at $J(X) \subseteq X^{dd}$ and show that X and $J(X)$ belong to the same class of spaces.

X_1 is an unital C^*-algebra and hence $(X_1^d \backslash \{0\}, \tau_p)$ is compact and Hausdorff yielding by corollary 3.5 that (X^d, τ_p) is compact and Hausdorff too. This implies that hold

$$X^{dd} = \big(C_b((X^d, \tau_p), \mathbb{C}), \mu\big)$$

But now we can choose $\mu = \tau_{\|\cdot\|_{\text{sup}}}$: the uniform topology generated by the sup-norm.

Proposition 3.10

$$J(X) \subseteq \big(C_b((X^d, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{sup}}\big)$$
$$\text{and} \ \ J : X \to J(X)$$

is an isomorphy and an isometry.

Proof.

$$J_1 : \ X_1 \to X_1^{dd}, \quad J_1(x, \alpha) = \omega((x, \alpha), \cdot) \, .$$
$$J_1 : \ X_1 \to \big(C_b((X_1^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\text{sup}}}\big) = X_1^{dd}$$

is a bijective, isomorphic and isometric map. $X \times \{0\}$ is a C^*-subalgebra of X_1.

$$(J_1|(X \times \{0\}))(x, \alpha) = J_1(x, 0) = \omega((x, 0), \cdot) = J(x, 0) \in J(X)\,.$$

Hence J maps X isomorphically and isometrically to $J(X) = J(X \times \{0\})$.

We consider $0 \in X^d$; 0 is either a τ_p-isolated point or a τ_p-accumulation point (clusterpoint) of (X^d, τ_p).

If 0 is isolated then $(X^d \backslash \{0\}, \tau_p)$ is still a compact Hausdorff space implying that

$$X^{dd} = \left(C_b((X^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\text{sup}}}\right)\,.$$

Since $1 \in X^{dd}$, X and X^{dd} do not belong to the same class of spaces. Hence $0 \in X^d$ must be a τ_p accumulation point.

3.4 Proof of the Gelfand-Naimark theorem

Theorem 3.11

1. *X^d has enough elements*
2. *$(X^d \backslash \{0\}, \tau_p)$ is a Hausdorff, locally compact, noncompact topological space and $(X \backslash \{0\}) \cup \{0\}$ is the onepoint-compactification of $(X^d \backslash \{0\}, \tau_p)$*
3. *$J:\ X \to \left(C_b((X^d \backslash \{0\}, \tau_p), \mathbb{C}, \tau_{\|\cdot\|_{sup}}\right)$ and $J(X) = \left(C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{sup}}\right)$*
4. *J is an isomorphic and isometric map from X onto $C_0(X^d \backslash \{0\}, \tau_p), \mathbb{C})$*
5. *X and $J(X) = C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C})$ belong to the same class of spaces.*

Proof.

1. X_1 is a commutative, unital C^*-algebra, hence we know that $X_1^d \backslash \{0\}$ has enough elements. But by the theorem 3.4 we get for the cardinal numbers:
$$|X^d| = |X_1^d \backslash \{0\}|\,.$$
2. $0 \in X^d$ is a τ_p-accumulation point and hence it is well-known that 2. holds, since (X^d, τ_p) is compact and Hausdorff.
3. At first we show that
$$\begin{aligned} &J(X) \subseteq C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C}) : \\ &J(X) = \{\omega(x, \cdot) | x \in X\}\,, \\ &\omega(x, \cdot) : X^d \to \mathbb{C} :\ \forall h \in X^d :\ \omega(x, \cdot)(h) = \omega(x, h) = h(x)\,. \end{aligned}$$
We consider $\omega(x, \cdot)$ for some $x \in X$; the zerohomomorphism fromX^d is the point at infinity of $(X^d \backslash \{0\}, \tau_p)$. Let (h_i) be an arbitrary net from $X^d \backslash \{0\}$ and $h_i \xrightarrow{\tau_p} 0$, then
$$h_i(x) \mapsto 0(x) = 0 \in \mathbb{C} \Rightarrow \omega(x_i, \cdot)(h_i) \to 0$$

showing by proposition 3.7 that

$$\omega(x,\cdot) = Jx \in C_0((X^d\backslash\{0\},\tau_p),\mathbb{C})$$

holds.

If we can show that the assumptions of the Stone-Weierstrass theorem 3.8 are fullfilled for $J(X)$ then

$$J(X) = \left(C_0((X^d\backslash\{0\},\tau_p),\mathbb{C}),\tau_{\|\cdot\|_{\sup}}\right) .$$

Now, by proposition 3.10 J is an isometry from X into $C_0((X^d\backslash\{0\},\tau_p),\mathbb{C})$ and thus $J(X)$ is closed in

$$\left(C_0((X^d\backslash\{0\},\tau_p),\mathbb{C}),\tau_{\|\cdot\|_{\sup}}\right)$$

Corollary 3.7 of [3] shows that $J(X)$ is selfadjoined too. J is injective and hence by [1], proposition 4.5, page 290, $J(X)$ separates the points of $X^d\backslash\{0\}$; now finally:

$$\forall h \in X^d\backslash\{0\} \Rightarrow h \neq 0 \Rightarrow \exists x \in X: \; h(x) \neq 0 \in \mathbb{C} ;$$

then $x \neq 0$ holds too; now, $\omega(x,\cdot) \in J(X)$ and $\omega(x,\cdot)(h) = h(x) \neq 0$. Thus the assumptions of the Stone-Weierstrass theorem are fulfilled.

4. This follows from 3. and from proposition 3.10.
5. X and $\left(C_0((X^d\backslash\{0\},\tau_p),\mathbb{C}),\tau_{\|\cdot\|_{\sup}}\right)$ are commutative, nonunital C^*-algebras and hence both spaces belong to the same class of spaces.

Corollary 3.12 *Equivalent are:*

(1) *X has the unit e*

(2) *$0 \in X^d$ is an isolated point of (X^d,τ_p)*

(3) *(X^d,τ_p) is compact (and Hausdorff)*

Proof. (1) $\Rightarrow$ (2): this assertion follows from [1], lemma 4.3, page 389

(2) $\Rightarrow$ (3): Since 0 is an isolated point then $(X^d\backslash\{0\})\cup\{0\}$ cannot be the one-point compactification of $(X^d\backslash\{0\},\tau_p)$ and thus $(X^d\backslash\{0\},\tau_p)$ is compact implying that (X^d,τ_p) is compact.

(3) $\Rightarrow$ (1): We have

$$X^{dd} = \left(C_b((X^d,\tau_p),\mathbb{C}),\tau_{\|\cdot\|_{\sup}}\right)$$

because (X^d,τ_p) is compact and Hausdorff.

Hence the constant function 1 is unit in X^{dd} implying by proposition 3.10 that $e := J^{-1}(1)$ is unit in X.

4 The embedding theorem of Kadison

4.1 The spaces X_{sa} and $S(X)$

Let X be an unital C^*-algebra. By X_{sa} we denote the set of all selfadjoined elements of X and by $S(X)$ we mean the set of states of X.

$X_{sa} \subseteq X$ is a real vector space and $(X_{sa}, \|\cdot\|)$ is real Banach subspace of X. The unit $e \in X$ belongs to X_{sa} : $e^* = e$. For instance:

$$x \in X \Rightarrow x^* \in X \Rightarrow x^*x \in X\,,$$

but $x^*x \in X_{sa}$ too: $(x^*x)^* = x^*x^{**} = x^*x$.

If X is commutative then of course X_{sa} is closed under multiplication.

4.2 The first and the second dual space of X_{sa}

According to our duality theory we define now the first dual space of X_{sa}.

e is the multiplicative unit in X and $e \in X_{sa}$. Hence we define:

Definition 4.1 $(X_{sa})^d = \{h:\ X_{sa} \to \mathbb{R} | h\ \textit{is linear, continuous and}\ h(e) = 1\}$

Remark 4.2

1. $(X_{sa})^d$ is not identical with the the Banachspace – dual

$$X'_{sa} = \{h:\ X_{sa} \to \mathbb{R}\,|\,h \text{ is linear and continuous}\}\,.$$

2. For $(X_{sa})^d$ does not hold:

$$h_1, h_2 \in (X_{sa})^d \Rightarrow h_1 + h_2 \in (X_{sa})^d :\ \text{ if } h_1 + h_2 \in (X_{sa})^d \text{ then } (h_1 + h_2)(e) = 1\,,$$

but otherwise:

$$(h_1 + h_2)(e) = h_1(e) + h_2(e) = 2\,,$$

a contradiction.

Hence $(X_{sa})^d$ is no vectorspace.

From remark 4.2, 2. we get: $(X_{sa})^d$ has the defect D according to 2.9. Hence by 2.10 the second dual space of X_{sa} reads:

Remark 4.3

$$(X_{sa})^{dd} = \left(C((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|}), \mu\right)\,,$$

where are: $\tau_{|\cdot|}$ the Euclidian topology and μ a topology for the space of continuous functions. μ still must be specified.

Remark 4.4 We don't know the properties of $((X_{sa})^d, \tau_p)$, especially we don't know wether or not, $((X_{sa}^d), \tau_p)$ is compact Hausdorff or not. But in exchange we find a Hausdorff and compact space as the next proposition will show.

Proposition 4.5 *The topological spaces* (X_{sa}^d, τ_p) *and* $(S(X), \tau_p)$ *are homeomorphic.*

For the proof we need a result, which we provide by the following proposition.

For the $\mathbb{C}^*$-algebra $\mathbb{C}$ we easily can prove the characterization of the convergence of a sequence (z_n), $\forall n \in \mathbb{N} : z_n \in \mathbb{C}$, $z \in \mathbb{C}$: let be $\forall n \in \mathbb{N} : \ z_n = x_n + iy_n, \ z = x + iy$.

Then holds:

$$z_n \to z \Leftrightarrow x_n \to x \text{ and } y_n \to y\,.$$

Somewhat more difficult to prove is the corresponding characterization in an arbitrary C^*-algebra.

Proposition 4.6 *Let X an unital $C*$-algebra, X_{sa} denotes the set of all selfadjoint elements of X. Let (x_n) be a sequence in X, $x \in X$. Convergence means norm-convergence. We write:*

$$x_n = a_n + ib_n, \ x = a + ib\,; \ \forall n : \ a_n, b_n \in X_{sa}, \ a, b \in X_{sa}\,.$$

Then holds: Equivalent are:

(1) $x_n \to x$

(2) $a_n \to a$ *and* $b_n \to b$

Proof. (2) $\to$ (1):

$$\begin{aligned} \|x_n - x\| &= \|(a_n - a) + i(b_n - b)\| \\ &\leq \|a_n - a\| + |i|\|b_n - b\| \\ &= \|a_n - a\| + \|b_n - b\| \to 0\,, \end{aligned}$$

hence $\|x_n - x\| \to 0$ too.

(1) $\to$ (2):

$$\forall n : \ a_n - a, \ b_n - b \in X_{sa}\,;$$

but then

$$(a_n - a)^2, \ (b_n - b)^2 \in X_{sa} \text{ and } (a_n - a)^2, \ (b_n - b)2$$

are positive.

Now, for instance

$$(b_n - b)^2 \leq (a_n - a)^2 + (b_n - b)^2\,,$$

since

$$\left[(a_n - a)^2 + (b_n - b)^2\right] - (b_n - b)^2 = (a_n - a)^2 \geq 0\,.$$

But

$$0 \leq (b_n - b)^2 \leq (a_n - a)^2 + (b_n - b)^2 \Rightarrow \|(b_n - b)^2\| \leq \|(a_n - a)^2 + (b_n - b)^2\|\,.$$

Otherwise: $x_n - x = (a_n - a) + i(b_n - b)$ yielding

$$\begin{aligned}\|x_n - x\|^2 &= \|[(a_n - a) + i(b_n - b)]^*[(a_n - a) + i(b_n - b)]\| \\ &= \|[(a_n - a) - i(b_n - b)][(a_n - a) + i(b_n - b)]\| \\ &= \|(a_n - a)^2 + (b_n - b)^2\|\end{aligned}$$

Hence we get:

$$\|(b_n - b)^2\| \leq \|x_n - x\|^2\,;$$

$b_n - b \in X_{sa}$ and hence $b_n - b$ is normal $\forall n$, which gives us:

$$\|(b_n - b)^2\| = \|(b_n - b\|^2\,;$$

thus

$$\|b_n - b\|^2 \leq \|x_n - x\|^2 \Rightarrow \|b_n - b\| \leq \|x_n - x\| \text{ and } \|x_n - x\| \to 0 \Rightarrow \|b_n - b\| \to 0\,.$$

By this way we show

$$\|a_n - a\| \to 0$$

too.

Thus (1) $\Rightarrow$ (2) is proved too.

Proof of proposition 4.5

We define a map φ:

$$\varphi : S(X) \to (X_{sa})^d : \ \forall h \in S(X) : \ \varphi(h) = h|X_{sa}$$

Lemma 4.7 *φ is an injective and surjective map from $S(X)$ to $(X_{sa})^d$.*

Proof of the lemma.

At first we show:

$$\varphi(S(X)) \subseteq (X_{sa})^d : \ \forall h \in S(X) :$$

1. $\varphi(h) = h|X_{sa}$ is linear:

$$\forall x_1, x_2 \in X_{sa};\ \forall \alpha_1, \alpha_2 \in \mathbb{R} \Rightarrow \alpha_1 x_1 + \alpha_2 x_2 \in X_{sa}\,,$$

   but this linear combination is also an element of X, hence

$$h(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 h(x_1) + \alpha_2 h(x_2)\,,$$

   yielding:

$$(h|X_{sa})(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 (h|X_{sa})(x_1) + \alpha_2 (h|X_{sa})(x_2)$$

2. h continuous $\Rightarrow \varphi(h) = h|X_{sa}$ is continuous
3. $h \in S(X) \Rightarrow h(e) = 1$;

$$e \in X_{sa}:\ 1 = h(e) = (h|X_{sa})(e) \Rightarrow \varphi(h)(e) = (h|X_{sa})(e) = 1\,.$$

By 1, 2 and 3 we get:

$$h|X_{sa} \in (X_{sa})^d\,,$$

hence $\varphi(S(X)) \subseteq (X_{sa})^d$.

4. φ is injective:

   $\forall f, g \in S(X)$: let be

$$\varphi(f) = f|X_{sa} = g|X_{sa} = \varphi(g)$$

   We want to show: $\forall x \in X : f(x) = g(x)$, hence $f = g$:

   (a) $x \in X_{sa} : f(x) = (f|X_{sa})(x) = (g|X_{sa})(x) = g(x)$

   (b) $x \in X \backslash X_{sa} : x = x_1 + ix_2$, $x_1, x_2 \in X_{sa}$; f, g are linear on X:

$$f(x) = f(x_1) + if(x_2),\ g(x) = g(x_1) + ig(x_2)\,,$$

   but:

$$x_1, x_2 \in X_{sa} \Rightarrow f(x_1) = g(x_1),\ f(x_2) = g(x_2)$$

   showing $f(x) = g(x)$ and hence, finally $f = g$.

5. We show that φ is surjective too.

   $\forall h \in (X_{sa})^d$: we define the function $\tilde{h}$:

$$\forall x \in X :\ x = x_1 + ix_2,\ x_1, x_2 \in x_{sa}\,;$$
$$\tilde{h} : X \to \mathbb{C} :\ \tilde{h}(x) = h(x_1) + ih(x_2)$$

Lemma 4.8 *$\tilde{h} \in S(X)$ and $\varphi(\tilde{h}) = \tilde{h}|X_{sa} = h$*

Proof. (a) $\tilde{h}$ is linear.

We know that h is linear.

$$\forall x, y \in X,\ \forall \alpha, \beta \in \mathbb{C}:$$

We can write:

$$x = x_1 + ix_2, \qquad y = y_1 + iy_2$$
$$\alpha = \alpha_1 + i\alpha_2, \qquad \beta = \beta_1 + i\beta_2$$

Now we can compute $\alpha x + \beta y$ in X:

$$\begin{aligned}
\alpha x + \beta y &= (\alpha_1 + i\alpha_2)(x_1 + ix_2) + (\beta_1 + i\beta_2)(y_1 + iy_2) \\
&= \alpha_1 x_1 + i\alpha_2 x_1 + i\alpha_1 x_2 - \alpha_2 x_2 + \beta_1 y_1 + i\beta_2 y_1 + i\beta_1 y_2 - \beta_2 y_2 \\
&= \alpha_1 x_1 - \alpha_2 x_2 + \beta_1 y_1 - \beta_2 y_2 + i(\alpha_2 x_1 + \alpha_1 x_2 + \beta_2 y_1 + \beta_1 y_2)\,.
\end{aligned}$$

Then follows:

$$\begin{aligned}
\tilde{h}(\alpha x + \beta y) &= h(\alpha_1 x_1 - \alpha_2 x_2 + \beta_1 y_1 - \beta_2 y_2) + ih(\alpha_2 x_1 + \alpha_1 x_2 + \beta_2 y_1 + \beta_1 y_2) \\
&= \alpha_1 h(x_1) - \alpha_2 h(x_2) + \beta_1 h(y_1) - \beta_2 h(y) \\
&\quad + i\alpha_2 h(x_1) + i\alpha_1 h(x_2) + i\beta_2 h(y_1) + i\beta_1 h(y_2) \\
&= (\alpha_1 + i\alpha_2)h(x_1) + (i\alpha_1 + i^2\alpha_2)h(x_2) + \ldots \\
&= (\alpha_1 + i\alpha_2)h(x_1) + i(\alpha_1 + i\alpha_2)h(x_2) + \ldots \\
&= (\alpha_1 + i\alpha_2)(h(x_1) + ih(x_2) + \ldots \\
&= \alpha\tilde{h}(x) + \ldots\,;
\end{aligned}$$

hence h is linear:

$$\tilde{h}(\alpha x + \beta y) = \alpha\tilde{h}(x) + \beta\tilde{h}(y)\,,$$

(b) $\tilde{h}$ is continuous on X: let be (x_n) a sequence from X, $x \in X$ and $\|x_n - x\| \to 0$ for $n \to +\infty$; let further be:

$$\forall n:\ x_n = x_n^1 + ix_n^2,\ x = x_1 + ix_2$$

we want to show:

$$\tilde{h}(x_n) \to \tilde{h}(x):$$

by proposition 4.5 we get: $x_n \to x \Leftrightarrow x_n^1 \to x_1$ and $x_n^2 \to x_2$ yielding:

$$\tilde{h}(x_n) = h(x_n^1) + ih(x_n^2) \to h(x_1) + ih(x_2) = \tilde{h}(x)\,,$$

since h is continuous on $(X_{sa}, \|\cdot\|)$.

(c) $\tilde{h}|X_{sa} = h$: $\forall x \in X_{sa}$: $x = x + i \cdot 0$, hence:

$$(\tilde{h}|X_{sa})(x + i \cdot 0) = \tilde{h}(x + i \cdot 0) = h(x) + ih(0) = h(x)\,,$$

since $0 \in X_{sa}$ and h is linear yields: $h(0) = 0$

(d) $\tilde{h}(e) = 1$: $e \in X_{sa} \Rightarrow \tilde{h}(e) = (\tilde{h}|X_{sa})(e) = h(e) = 1$ by (c).

By a well-known theorem of the C^*-algebra-theory follows by (a), (b) and (d) that $\tilde{h}$ is positive, yielding by another theorem:

$$\|\tilde{h}\| = \tilde{h}(e)\,,$$

and hence we have $\|\tilde{h}\| = 1$ too.

Thus we have shown:

$$\tilde{h} \in S(X) \text{ and } \varphi(\tilde{h}) = h\,.$$

Hence indeed we got: $\varphi : S(X) \to (X_{sa})^d$ is injective and surjective.

Lemma 4.9 *$\varphi : (S(X), \tau_p) \to ((X_{sa})^d, \tau_p)$ is continuous.*

Proof. Let be (h_i) a net from $S(X)$, $h \in S(X)$ and $h_i \xrightarrow{\tau_p} h$; we want to show that $\varphi(h_i) \xrightarrow{\tau_p} \varphi(h)$ holds: $\forall x \in X_{sa}$, then $x \in X$ too and thus $h_i(x) \to h(x)$ in $\mathbb{R}$. Now,

$$\varphi(h_i)(x) = (h_i|X_{sa})(x) \to (h|X_{sa})(x)\,,$$

since $x \in X_{sa}$. Hence

$$\varphi(h_i) \xrightarrow{\tau_p} \varphi(h) \text{ in } (X_{sa})^d\,.$$

Finally we must still show:

Lemma 4.10 *$\varphi : (S(X), \tau_p) \to ((X_{sa})^d, \tau_p)$ is open:*

Proof. Let $G \subseteq S(X)$ be τ_p-open, we show: $\varphi(G)$ is τ_p-open in $(X_{sa})^d$: let be $h \in \varphi(G)$ and (h_k) a net from $(X_{sa})^d$ such that $h_k \xrightarrow{\tau_p} h$.

φ is bijective, hence there exists $g \in G$,

$$\forall k : g_k \in S(X) : \varphi(g) = h = g|X_{sa},\ \forall k : \varphi(g_k) = g_k|X_{sa} = h_k\,.$$

Now we want to show:

$$g_k \xrightarrow{\tau_p} g \text{ in } S(X) :$$

(a) $\forall x \in X_{sa}$: $g(x) = (g|X_{sa})(x) = \varphi(g)(x) = h(x)$; $\forall k$: $g_k(x) = h_k(x)$. Hence $h_k(x) \to h(x)$ meaning that holds:

$$g_k(x) \xrightarrow{\tau_p} g \text{ on } X_{sa}\,.$$

(b) $\forall x \in X \backslash X_{sa}: \ x = x_1 + ix_2,\ x_1, x_2 \in X_{sa}$; by (a) we get:

$$\begin{aligned} g_k(x_1) &= h_k(x_1) \to h(x_1) = g(x_1)\,, \\ g_k(x_2) &= h_k(x_2) \to h(x:2) = g(x_2) \\ &\Rightarrow g_k(x_1) + ig_k(x_2) \to g(x_1) + ig(x_2)\,. \end{aligned}$$

Now $g \in S(X)$ and $\forall k : g_k \in S(X)$ showing that these functions are linear:

$$\begin{aligned} x = x_1 + ix_2 \Rightarrow g(x) &= g(x_1) + ig(x_2), \\ \forall k : g_k(x) &= g_k(x_1) + ig_k(x_2) \end{aligned}$$

But then follows:

$g_k(x) \to g(x)$ on $X \backslash X_{sa}$, and thus from (a), (b) we get:

$$g_k(x) \to g(x),\ \forall x \in X,\ g_k \stackrel{\tau_p}{\to} g\,.$$

Since $g \in G$ and G is τ_p-open there exists k_o:

$\forall k \geq k_o : \ g_k \in G$ showing that holds:

$$\forall k \geq k_o : \varphi(g_k) = h_k \in \varphi(G)\,,$$

hence $\varphi(G)$ is τ-open in $(X_{sa})^d$.

Final *proof* of proposition 4.5. By lemma 4.7, 4.8, 4.9 and 4.10

$$\varphi : (S(X), \tau_p) \to ((X_{sa})^d, \tau_p)$$

is bijective, continuous and open yielding that φ is a topological map onto $(X_{sa})^d$ and thus $(S(X), \tau_p)$ and $((X_{sa})^d, \tau_p)$ are homeomorphic.

Corollary 4.11 *The first dual space of X_{sa} is a Hausdorff and compact topological space w. r. t. the pointwise topology τ_p.*

Proof. We know that the state space $(S(X), \tau_p)$ is a compact and Hausdorff space.

We come now back to the second dual space 4.3 of X_{sa}:

$$\begin{aligned} (X_{sa})^{dd} &= \big(C(((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|})), \mu\big) \\ &= \big(C_b(((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|})), \mu\big) \\ &= \big(C_b((S(X), \tau_p), (\mathbb{R}, \tau_{|\cdot|})), \mu\big)\ , \end{aligned}$$

where $C_b(\cdot, \cdot)$ of course means the space of bounded and continuous real functions.

Then for μ we can choose the sup-norm and hence the uniform topology.

We state now:

1. $(X_{sa}, \|\cdot\|)$ (and $S(X), \|\cdot\|$) and $(\mathbb{R}, |\cdot|)$ both are real Banach spaces.

   X_{sa} and $\mathbb{R}$ both have a multiplicative unit. Hence both spaces belong to the same class of spaces.

2. $(X_{sa}, \|\cdot\|)$ and $((X_{sa})^{dd}, \|\cdot\|_{\text{sup}})$ are real Banach spaces; both have a multiplicative unit. $X_{sa}, (X_{sa})^{dd}$ belong to the same class of spaces.

We still need a lemma.

Lemma 4.12

$$\forall x \in X_{sa} : \|x\| \in \sigma(x)\,.$$

Proof. 1. $0 \in X_{sa}$, but 0^{-1} does not exist and hence $\|0\| = 0 \in \sigma(0)$.

2. $x \in X_{sa}$ and $x \neq 0$; $x \in X_{sa} \Rightarrow \sigma(x) \subseteq \mathbb{R}$ and x is normal and thus:

$$r(x) = s = \sup\left\{|x|\middle|\lambda \in \mathbb{R} \text{ and } \lambda \in \sigma(x)\right\} = \|x\|\,.$$

   $\|x\| > 0 \Rightarrow \exists$ sequence $(\lambda_n):\ \forall n:\ \lambda_n \in (\sigma(x), \|x\|)$ such that $\lambda_n \to s$.

   $\sigma(x)$ is Hausdorff and compact and thus $\sigma(x)$ is sequentially compact and Hausdorff too since $(X, \|\cdot\|)$ is a metric space. Thus we find a subsequence (λ_{n_k}) of (λ_n) and $\lambda \in \sigma(x)$, $\lambda > 0 : \lambda_{n_k} \to \lambda$, but also $\lambda_{n_k} \to s = \|x\|$, implying $\lambda = \|x\| \in \sigma(x)$.

4.3 Proof of the Kadison embedding theorem

Theorem 4.13 *Let X be an unital C^*-algebra. Then holds:*

1. $J:\ (X_{sa}, \|\cdot\|) \to ((X_{sa})^{dd}, \|\cdot\|_{sup}) = \left(C_b((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|}), \tau_{\|\cdot\|_{sup}}\right)$ *is an isometric and isomorphic map onto* $(X_{sa})^{dd}$

2. $J(X_{sa})$ *separates the points of* $(X_{sa})^d$

3. $J(X_{sa})$ *is a closed subspace of* $(X_{sa})^{dd}$

Proof. By corollary 4.1 of [1], p. 284 we get: $J(X_{sa}) \subseteq (X_{sa})^{dd}$. Now

$$(X_{sa})^d \subseteq X'_{sa} = \{h:\ X_{sa} \to \mathbb{R}|\text{ his linear and } h \text{ is continuous}\}$$
$$\forall h \in X^d_{sa}\ \exists g \in S(X):\ \varphi(g) = g|X_{sa} = h\,;$$

hence

$$\|h\| = \|g|X_{sa}\| = \sup\{|g(x)|\big|x \in x_{sa} \text{ and}$$
$$\|x\| \leq 1\} \leq \sup\{|g(x)|\big|\ x \in X \text{ and } \|x\| \leq 1\} = \|g\| = 1$$

and thus $\|h\| \leq 1$.

But then we can apply proposition 4.3, p. 287 of [1]. At first we get:

$$\forall x \in X_{sa}: \ \|J(x)\|_{\sup} \leq \|x\| \,.$$

Moreover we have:

$\forall x \in X_{sa}$, $x \neq 0$, by lemma 4.12 we know: either $\|x\| \in \sigma(x)$ or $-\|x\| \in \sigma(x)$. Let us consider $-\|x\|$: there exists $h \in S(X): h(x) = -\|x\|$; but $x \in X_{sa} \Longrightarrow h(x) = h\big|X_{sa}(x)$ showing that $h|X_{sa} \in X_{sa}^d$ and $h|X_{sa}(x) = -\|x\|$, implying $|h|X_{sa}(x)| = |-\|x\|| = \|x\|$ and hence $\|x\| \leq |h|X_{sa}|$. Of course this last result we get also if $\|x\| \in \sigma(x)$. This implies by the above mentioned proposition that holds $\|x\| \leq \|J(x)\|_{\sup}$. Hence we have:

$$\forall x \in X_{sa}: \ \|J(x)\|_{\sup} = \|x\| \,,$$

yielding that $J: X_{sa} \to J(X_{sa})$ is an isometric map.

Now J is then an injective map onto $J(X_{sa})$ and thus the homomorphy theorem 4.4, p. 284 of [1] shows that J is an isomorphic map for real Banach spaces too, meaning that point 1. of our theorem is proved, but only for $J: X_{sa} \to J(X_{sa})$.

Proposition 4.3 of [1] shows also 2.. $J(X_{sa})$ separates the points of $(X_{sa})^d$.

Since J is an isometric map J is an uniform isomorphy too, yielding that $J(X_{sa})$ is a complete subspace of $(X_{sa}^{dd}, \|\cdot\|)$ since X_{sa} is complete.

Thus we proved 3.:

$J(X_{sa})$ is a closed subspace of $(X_{sa})^{dd}$.

Concluding we find:

$$e \in X_{sa} \Rightarrow \omega(e,\cdot) \in (X_{sa})^{dd}, \text{ but: } \forall h \in (X_{sa})^d: \ \omega(e,\cdot)(h) = h(e) = 1\,,$$

showing that the constant function $\omega(e,\cdot) \equiv 1$ belongs to $J(X_{sa})$.

But this result together with assertions 2., 3. shows that $J(X_{sa}) = (X_{sa})^{dd}$ by the theorem of Stone-Weierstrass.

Now our proof is complete.

Concluding remarks We consider our basic assumptions 2.3, 2.5 and 2.11:

(1) X and Y belong to the same class of spaces

(2) Y always has a topology

(3) X and X^{dd} are in the same class of spaces

As we have shown in our text the general procedure runs as follows:

We start with the space X and want to define the second dual space of X and to embedd X into X^{dd} using the canonical map J. To do so we must choose a suitable space Y such that (1) is fulfilled. Then we can define the first dual space X^d of X with respect to Y, where (2) holds. According to the properties of X^d we are able to define the second dual space X^{dd} of X w. r. t. Y such that (3) is fulfilled and $J: \ X \to X^{dd}$ embedds X into or onto X^{dd}.

References

[1] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space, I.* In: Theory and applications of proximity, nearness and uniformity. Caserta: Dipartimento di Matematica, Seconda Università di Napoli, Quaderni di Matematica 22; Rome: Aracne 2009, 275 – 297

[2] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space, II.* Int. J. Pure, Appl. Math. **84**, 2013, 651 – 667

[3] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space, III.* N. Z. J. **46**, 2016, 1 – 8

[4] **Folland, G. B. :** *A course in Abstarct Harmonic Analysis.* CRC Press 1985

[5] **Kadison, R. V.**, and **Ringrose, J. R. :** *Fundamentals of the Theory of Operator Algebras, Volumes I, II, III, IV.* Amer. Math. Soc.1997

[6] **Poppe, H. :** *Compactness in General Function Spaces.* Deutscher Verlag der Wissenschaften, Berlin 1974

[7] **Salamon, D. A. :** *Measure and Integration.* EMS 2016

[8] **Zhu, K. H. :** *An Introduction to Operator Algebras.* CRC Press 1993

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