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An inverse Problem for a parabolic System in an unbounded Guide

ABSTRACT. In this article we consider a two-by-two parabolic system defined on an unbounded guide with coefficients depending both on the space variable and on the time variable. The main aim of this paper is to obtain a stability result for the coefficients depending on the space variable. Using Carleman inequalities adapted for the guide, we obtain Hölder estimates of these coefficients in any finite portion of the guide with boundary measurements, given two sets of initial conditions.

KEY WORDS. inverse problems, Carleman inequalities, heat operator, system, unbounded guide

1 Introduction

Let ω be a bounded connex domain in $\mathbb{R}^{n-1}$, $n \geq 2$ with C^2 boundary. Denote $\Omega = \mathbb{R} \times \omega$ and $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$. We consider the following problem

$$\begin{cases} \partial_t u = \Delta u + \alpha\phi_1 u + \beta\phi_2 w + g_1 \ \text{ in } \ Q, \\ \partial_t w = \Delta w + \gamma\phi_3 u + \delta\phi_4 w + g_2 \ \text{ in } \ Q, \\ u(.,0) = a_1, \ w(.,0) = a_2 \text{ in } \ \Omega, \\ u = a_3, \ w = a_4 \text{ in } \ \Sigma, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are bounded coefficients defined on Ω such that

$$\alpha, \beta, \gamma, \delta \in \Lambda_1(M_0) = \{f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} \leq M_0\} \text{ for some } M_0 > 0,$$

and $\phi_1, \phi_2, \phi_3, \phi_4$ are bounded coefficients defined on $[0,T]$ such that for $i = 1, \cdots, 4$

$$\phi_i \in \Lambda_2(M_0) = \{f \in C^1([0,T]), f(\frac{T}{2}) \neq 0 \text{ and } \|f\|_{C^1([0,T])} \leq M_0\}.$$

The main problem is to estimate the coefficients $(\alpha, \beta, \gamma, \delta)$ from boundary observations of (u, w).

We will consider two sets of Cauchy and Dirichlet conditions A and B and denote

$$G=(g_1,g_2),\ A=(a_1,a_2,a_3,a_4),\ B=(b_1,b_2,b_3,b_4),\ \rho=(\alpha,\beta,\gamma,\delta,\phi_1,\phi_2,\phi_3,\phi_4),$$

$$\tilde{\rho_1}=(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta},\phi_1,\phi_2,\phi_3,\phi_4),\ \tilde{\rho_2}=(\alpha,\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta},\tilde{\phi_1},\phi_2,\phi_3,\phi_4),\ \tilde{\rho_3}=(\alpha,\tilde{\beta},\tilde{\gamma},\tilde{\delta},\phi_1,\phi_2,\phi_3,\phi_4). \tag{1.2}$$

Let two positive reals l, L be such that $l<L$. Denote

$$\Omega_L=(-L,L)\times\omega \text{ and } \Omega_l=(-l,l)\times\omega.$$

The first result of this paper gives a Hölder stability result (3.4) for the coefficients $\alpha,\beta,\gamma,\delta$ and is the following (see Theorem 3.1)

$$\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}$$

$$\leq K\Bigg(\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(w_A-\tilde{w}_A))|^2)\,d\sigma\,dt$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2+|\partial_\nu(\partial_t^k(w_B-\tilde{w}_B))|^2)\,d\sigma\,dt\Bigg)^\kappa$$

where K is a positive constant, $\kappa\in(0,1)$, γ_L is a part of the boundary (see (2.2)), and assuming that the hypothesis (3.3) is satisfied. We consider in the above result $V_A=(u_A,w_A)$ (resp. $\tilde{V}_A=(\widetilde{u}_A,\widetilde{w}_A)$) a solution of (1.1) associated with the coefficients (ρ,G,A) (resp. $(\tilde{\rho_1},G,A)$) and $V_B=(u_B,w_B)$ (resp. $\tilde{V}_B=(\widetilde{u}_B,\widetilde{w}_B)$) a solution of (1.1) associated with the coefficients (ρ,G,B) (resp. $(\tilde{\rho_1},G,B)$) where A is a set of Cauchy and Dirichlet conditions and B is a suitable change of initial and boundary conditions. The above result is an improvement of results obtained in [5] with different and less restrictive hypotheses but with two choices of Cauchy and Dirichlet conditions A and B. In abbreviated form we will call A and B the two sets of initial conditions. It is an improvement because on one hand the hypotheses, though quite differents, are easier to satisfy than in [5] and on the other hand there are no observation terms of the solutions (u,w) at a fixed time on the right-hand side of the estimate, such as $\|(u_A-\tilde{u}_A)(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}$ (see [5]). The idea of choosing two different sets of initial conditions can be found in [2] for a hyperbolic equation in a bounded domain (see also [6] for a hyperbolic system).
A consequence of the above result is given in Theorem 3.2 where the measurements are given for only one component (for example u) and is the following (see (3.6))

$$\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}$$

$$\leq K\Big(\|u_A-\tilde{u}_A\|^2_{H^2([0,T],H^2(\omega'\cap\Omega_L))}+\|u_A-\tilde{u}_A\|^2_{H^1([0,T],H^4(\omega'\cap\Omega_L))}$$

$$+\|u_B-\tilde{u}_B\|^2_{H^2([0,T],H^2(\omega'\cap\Omega_L))}+\|u_B-\tilde{u}_B\|^2_{H^1([0,T],H^4(\omega'\cap\Omega_L))}$$
$$\left.+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2)\,d\sigma\,dt\right)^\kappa$$

where $K>0$, $\kappa\in(0,1)$ and ω' is a neighborhood of γ_L, ω' being a subdomain of Ω such that $\gamma_L\subset\partial\omega'$, and assuming that $\alpha=\tilde{\alpha}$ and $\beta=\tilde{\beta}$ in ω'. We can relax the hypothesis that the coefficients α and β are supposed known in ω' when these coefficients are in $H^2(\Omega)$ and we obtain a similar result with the L^2-norms replaced by the H^2-norms for the coefficients α and β on the left-hand side of the above estimate and additional terms such as $\|(u_A-\tilde{u}_A)(.,\frac{T}{2})\|^2_{H^4(\Omega_L)}$ on the right-hand side of this estimate (see (3.7)).
The third result gives a Hölder result (3.10) for the coefficients $\phi_1,\beta,\gamma,\ \delta$ (assuming also that $\phi_i\in C^2([0,T])$) and is the following (see Theorem 3.3)

$$\sum_{i=0}^{2}\|\partial_t^i(\phi_1-\tilde{\phi}_1)\|^2_{L^2((0,T))}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}$$
$$\leq K\left(\sum_{k=0}^{1}(\|\partial_t^k(u_A-\tilde{u}_A)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)}+\|\partial_t^k(u_B-\tilde{u}_B)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)})\right.$$
$$+\|\partial_t^2(u_A-\tilde{u}_A)(\cdot,\frac{T}{2})\|^2_{L^2(\Omega_L)}+\|\partial_t^2(u_B-\tilde{u}_B)(\cdot,\frac{T}{2})\|^2_{L^2(\Omega_L)})+\|(w_A-\tilde{w}_A)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)}$$
$$+\|(w_B-\tilde{w}_B)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)}+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(w_A-\tilde{w}_A))|^2)\,d\sigma\,dt$$
$$\left.+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2+|\partial_\nu(\partial_t^k(w_B-\tilde{w}_B))|^2)\,d\sigma\,dt)\right)^\kappa$$

where K is still a positive constant, $\kappa\in(0,1)$, and $\tilde{\phi}_1$ belongs to a set of admissible coefficients (namely $\Lambda_3(M_3)$, see (3.8)). In the above case we denote $V_A=(u_A,w_A)$ (resp. $\tilde{V}_A=(\widetilde{u}_A,\widetilde{w}_A)$) a solution of (1.1) associated with (ρ,G,A) (resp. $(\tilde{\rho}_2,G,A)$) and $V_B=(u_B,w_B)$ (resp. $\tilde{V}_B=(\widetilde{u}_B,\widetilde{w}_B)$) a solution of (1.1) associated with (ρ,G,B) (resp. $(\tilde{\rho}_2,G,B)$). So this third result gives a determination of one coefficient depending on the time variable. Be careful that the meanings of $\tilde{V}_A$ and $\tilde{V}_B$ are not the same in Theorems 3.1 and 3.2 on one hand and Theorem 3.3 on the other hand.
Finally the fourth theorem gives a Hölder result (3.11) for the following reaction-diffusion system

$$\begin{cases}\partial_t u=\Delta u+\alpha\phi_1 u+\beta\phi_2 w+\Theta_1\cdot\nabla u+\Theta_2\cdot\nabla w+g_1 \ \text{ in }\ Q,\\ \partial_t w=\Delta w+\gamma\phi_3 u+\delta\phi_4 w+\Theta_3\cdot\nabla u+\Theta_4\cdot\nabla w+g_2 \ \text{ in }\ Q,\\ u(.,0)=a_1,\ w(.,0)=a_2 \text{ in }\ \Omega,\\ u=a_3,\ w=a_4 \text{ in }\ \Sigma,\end{cases}\tag{1.3}$$

where all the coefficients $\alpha, \beta, \gamma, \delta, \phi_1, \phi_2, \phi_3, \phi_4, \Theta_1, \Theta_2, \Theta_3, \Theta_4$ are bounded. We present here a result for the four coefficients $\beta, \gamma, \delta, \Theta_1$ (and assuming that Θ_1 has the form $\Theta_1 = \nabla\xi_1$). So denote now

$$\Theta = (\Theta_1, \cdots, \Theta_4), \quad \tilde{\Theta} = (\tilde{\Theta}_1, \Theta_2, \Theta_3, \Theta_4). \tag{1.4}$$

We get the following result

$$\begin{aligned}
&\|\beta - \tilde{\beta}\|^2_{L^2(\Omega_l)} + \|\gamma - \tilde{\gamma}\|^2_{L^2(\Omega_l)} + \|\delta - \tilde{\delta}\|^2_{L^2(\Omega_l)} + \|\Theta_1 - \tilde{\Theta}_1\|_{(L^2(\Omega_l))^n} \\
&\leq K \Bigg(\|(u_A - \tilde{u}_A)(., \frac{T}{2})\|^2_{H^3(\Omega_L)} + \|(u_B - \tilde{u}_B)(., \frac{T}{2})\|^2_{H^3(\Omega_L)} \\
&+ \int_{\gamma_L \times (0,T)} \sum_{k=0}^{1} (|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2) \, d\sigma \, dt \\
&+ \int_{\gamma_L \times (0,T)} \sum_{k=0}^{1} (|\partial_\nu(\partial_t^k(u_B - \tilde{u}_B))|^2 + |\partial_\nu(\partial_t^k(w_B - \tilde{w}_B))|^2) \, d\sigma \, dt \Bigg)^\kappa
\end{aligned}$$

where K is a positive constant, $\kappa \in (0, 1)$. This time we denote $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\widetilde{u}_A, \widetilde{w}_A)$) a solution of (1.3) associated with (ρ, G, A, Θ) (resp. $(\tilde{\rho}_3, G, A, \tilde{\Theta})$) and $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\widetilde{u}_B, \widetilde{w}_B)$) a solution of (1.3) associated with (ρ, G, B, Θ) (resp. $(\tilde{\rho}_3, G, B, \tilde{\Theta})$).

Note that all our results imply uniqueness results. Up to our knowledge, there are few results concerning the simultaneous identification of more than one coefficient in each equation (see for examples [1, 2, 5, 6, 9, 10]) and note that in these papers the coefficients only depend on the space variable. Also notice that there are very few results where the measurements are given with only one component. Here the first and fourth theorems (Theorems 3.1 and 3.4) extend some results obtained in [5, Theorem 3.2] but with hypotheses (see (3.2) and (3.3)) less restrictive than in [5]. The second result (Theorem 3.2) gives a result for four coefficients depending on the space variable and with measurements of only one component. The third theorem (Theorem 3.3) also gives a result for four coefficients but one of each depending on the time variable. Furthermore, usually the papers investigate the case of bounded domains and give results with observations on a subdomain of the domain (see for example [1, 2, 10]). Here we present results with observations on a part of the boundary (see Theorems 3.1, 3.3, 3.4). Besides, because of our unbounded domain and our choice of weight functions (2.3), we will use cut-off functions in time and in the direction x_1 (see for example [12] where cut-off functions are removed but in a bounded domain). Finally, usually the results have observations terms with data of the solution at a fixed time (such as $\|(u_A - \tilde{u}_A)(., \frac{T}{2})\|^2_{H^2(\Omega_L)}$, see for example [5, 7, 8]). We have been able to remove them in Theorems 3.1, 3.2i) thanks to the properties of the weight functions. So the theorems presented here give stability results for four coefficients for a system defined on an unbounded

domain, with boundary measurements in Theorems 3.1, 3.3 and 3.4, measurements for only one component in Theorem 3.2, with a time variable coefficient in Theorem 3.3. These results extend previous results for one equation [7, 8] or for a system [5] defined on an unbounded guide. Last we recall that the method of Carleman estimates used for solving inverse problems has been initiated by [3].
This Paper is organized as folows: in Section 2, we recall the weight functions adapted for our unbounded domain and the Carleman estimate (2.6) as well as the crucial inequality (2.4) for our Hölder estimates. Then in Section 3 we state and prove our results.

2 Carleman estimate

Denote $Q_L = \Omega_L \times (0,T) = (-L,L) \times \omega \times (0,T)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $x' = (x_2, \cdots, x_n)$ and define the operator

$$A_0 u = \partial_t u - \Delta u.$$

Let $l > 0$, following [7] we are going to carry out special weight functions allowing us to avoid observations on the cross section of the wave guide in our inverse problem. For this we consider some positive real $L > l$ and we choose $\hat{a} = (a_1, a') \in \mathbb{R}^n \setminus \Omega$ such that if $\hat{d}(x) = |x' - a'|^2 - x_1^2$ for $x \in \Omega_L$, then

$$\hat{d} > 0 \text{ in } \Omega_L, \quad |\nabla \hat{d}| > 0 \text{ in } \overline{\Omega_L}. \tag{2.1}$$

Moreover we define

$$\Gamma_L = \{x \in \partial\Omega_L, < x - \hat{a}, \nu(x) > \geq 0\} \text{ and } \gamma_L = \Gamma_L \cap \partial\Omega. \tag{2.2}$$

Here $< .,. >$ denotes the usual scalar product in $\mathbb{R}^n$ and $\nu(x)$ is the outwards unit normal vector to $\partial\Omega_L$ at x. Notice that γ_L does not contain any cross section of the guide. From [14]-[15] we consider weight functions as follows: for $t \in (0,T)$, if $M_1 > \sup_{0<t<T}(t-T/2)^2 = (T/2)^2$,

$$\psi(x,t) = \hat{d}(x) - \left(t - \frac{T}{2}\right)^2 + M_1 \text{ and } \phi(x,t) = e^{\lambda\psi(x,t)}. \tag{2.3}$$

The constant $\lambda > 0$ will be set in Proposition 2.2 and is usually used as a large parameter in Carleman inequalities. Since we will not use it, we will consider λ fixed in the article. We recall from [7] and [8] the following result.

Proposition 2.1 *There exist $T > 0$, $L > l$, $\hat{a} \in \mathbb{R}^n \setminus \Omega_L$ and $\epsilon > 0$ such that (2.1) holds and, setting*

$$O_{L,\epsilon} = (\Omega_L \times ((0, 2\epsilon) \cup (T - 2\epsilon, T))) \cup (((-L, -L + 2\epsilon) \cup (L - 2\epsilon, L)) \times \omega \times (0,T)),$$

we have

$$d_1 < d_0 < d_2 \tag{2.4}$$

where

$$d_0 = \inf_{\Omega_l} \phi(\cdot, \theta), \qquad d_1 = \sup_{O_{L,\epsilon}} \phi, \qquad d_2 = \sup_{\overline{\Omega_L}} \phi(\cdot, \theta) \text{ and } \theta = \frac{T}{2}.$$

From now on and from simplicity we denote $\theta = \frac{T}{2}$ throughout the paper. These two above estimates (2.4) will be fruitful in Section 3 to solve our inverse problem. In the sequel C will be a generic positive constant. When needed, we will specify its dependency with respect to the different parameters. We will use the following notations: Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index with $\alpha_i \in \mathbb{N} \cup \{0\}$. We set $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and define

$$H^{2,1}(Q_L) = \{u \in L^2(Q_L), \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q_L), |\alpha| + 2\alpha_{n+1} \le 2\}$$

endowed with its norm

$$\|u\|^2_{H^{2,1}(Q_L)} = \sum_{|\alpha|+2\alpha_{n+1} \le 2} \|\partial_x^\alpha \partial_t^{\alpha_{n+1}} u\|^2_{L^2(Q_L)}.$$

We recall now a global Carleman-type estimate proved in [7, Proposition 4.2] or in [8, Proposition 3], based on a classical Carleman estimate (see Yamamoto [14, Theorem 7.3]). The key difference with the classical Carleman inequality in [14, Theorem 7.3] is to remove, on the cross-sections of Ω_L, the boundary condition and the observation. For that we need cut-off functions in time. On the other hand, to manage our infinite wave guide we also need to consider cut-off functions in space but only in the infinite direction x_1. These cut-off functions will induce additive terms coming from the commutator between the evolution operator and these cut-off functions. Let χ, η be C^∞ cut-off functions such that $\chi, \nabla\chi, \Delta\chi \in \Lambda_1(M_0)$, $0 \le \chi \le 1$, $0 \le \eta \le 1$,

$$\chi(x) = 0 \text{ if } x \in ((-\infty, -L+\epsilon) \cup (L-\epsilon, +\infty)) \times \omega),$$

$$\chi(x) = 1 \text{ if } x \in (-L+2\epsilon, L-2\epsilon) \times \omega,$$

$$\eta(t) = 0 \text{ if } t \in (0, \epsilon) \cup (T-\epsilon, T), \ \eta(t) = 1 \text{ if } t \in \times(2\epsilon, T-2\epsilon). \tag{2.5}$$

with ϵ defined in Proposition 2.1.

Proposition 2.2 [7, Proposition 4.2] *There exist a value of $\lambda > 0$ and positive constants s_0 and $C = C(\lambda, s_0)$ such that*

$$I(u) = \int_{Q_L} \left(\frac{1}{s\phi} (|\partial_t u|^2 + |\Delta u|^2) + s\phi\, |\nabla u|^2 + s^3\phi^3 |u|^2 \right) e^{2s\phi} dx\, dt$$

$$\le C\|e^{s\phi} A_0 u\|^2_{L^2(Q_L)} + Cs^3 e^{2sd_1} \|u\|^2_{H^{2,1}(Q_L)} + Cs \int_{\gamma_L \times (0,T)} |\partial_\nu u|^2 e^{2s\phi} d\sigma\, dt, \tag{2.6}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q_L)$ satisfying $u(.,0) = u(.,T) = 0$ in Ω_L, $u = 0$ on $(\partial\Omega \cap \partial\Omega_L) \times (0,T)$. We denote $\partial_\nu u = \nu \cdot \nabla u$ and recall that $A_0 u = \partial_t u - \Delta u$.

Since the method of Carleman estimates requires several time differentiations, we assume in the following that u, w (solution of (1.1) or (1.3)) belong to $\mathcal{H} = H^2([0,T], H^2(\Omega)) \cap W^{2,\infty}(\Omega \times (0,T))$ for Theorems 3.1, $\mathcal{H} = H^3([0,T], H^4(\Omega)) \cap W^{4,\infty}(\Omega \times (0,T))$ for Theorem 3.2, $\mathcal{H} = H^3([0,T], H^2(\Omega)) \cap W^{3,\infty}(\Omega \times (0,T))$ for Theorem 3.3, $\mathcal{H} = H^2([0,T], H^3(\Omega)) \cap W^{3,\infty}(\Omega \times (0,T))$ for Theorem 3.4, satisfying the a-priori bound

$$\|u\|_{\mathcal{H}} < M_2 \text{ and } \|w\|_{\mathcal{H}} < M_2 \text{ for given } M_2 > 0.$$

From now on, we use the notation $f(\theta) = f(.,\theta)$ for any function f defined on Q.

3 Inverse problem

3.1 Preliminary lemmas

From [11, Lemma 4.2], we derive the following result, also used in [7] or [5, Lemma 3.1].

Lemma 3.1 *There exist positive constants s_1 and C such that*

$$\int_{\Omega_L} e^{2s\phi(\theta)}(f(\theta))^2\, dx \le Cs \int_{Q_L} e^{2s\phi} f^2\, dx\, dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}(\partial_t f)^2\, dx\, dt$$

for all $s \ge s_1$ and $f \in H^1(0,T; L^2(\Omega_L))$.

For the sake of completeness, we recall its proof.

Proof. Consider η defined by (2.5) and any $w \in H^1(0,T;L^2(\Omega_L))$. Since $\eta(\theta) = 1$ and $\eta(0) = 0$, we have

$$\int_{\Omega_L} w(x,\theta)^2 dx = \int_{\Omega_L} (\eta(\theta) w(x,\theta))^2 dx = \int_{\Omega_L}\int_0^\theta \partial_t(\eta^2(t)|w(x,t)|^2) dt\, dx$$

$$= 2\int_0^\theta\int_{\Omega_L} \eta^2(t) w(x,t)\partial_t w(x,t) dx\, dt + 2\int_0^\theta\int_{\Omega_L} \eta(t)\partial_t\eta(t)|w(x,t)|^2 dx\, dt.$$

As $0 \le \eta \le 1$, using Young's inequality, it comes that for any $s > 0$,

$$\int_{\Omega_L} w(x,\theta)^2\, dx \le Cs\int_{Q_L} |w|^2 dx\, dt + \frac{C}{s}\int_{Q_L} |\partial_t w|^2 dx\, dt. \tag{3.1}$$

Then we can conclude replacing w by $e^{s\phi} f$ in (3.1). □

The following lemma will be only used for Theorem 3.4. It is a classical lemma for a first order partial differential operator but which necessites a strong positivity condition (3.2). This condition is nevertheless weaker than the one used in [8] or [5] (which was

$|\nabla \hat{d} \cdot \nabla \tilde{u}(\theta)| \geq R > 0$ in Ω_L). So we follow an idea developed in [13] for Lamé system in bounded domains, also used for example in [8] or in [5]. The lemma below will be used in the proof of Theorem 3.4 with $(v_1, \cdots, v_4) = (\tilde{w}_B(\theta), \tilde{u}_A(\theta), \tilde{w}_A(\theta), \tilde{u}_B(\theta))$. Recall that $\hat{d}$ is defined by (2.1).

Lemma 3.2 *Assume that the following assumption*

$$|v_1 \nabla \hat{d} \cdot \nabla v_2 - v_3 \nabla \hat{d} \cdot \nabla v_4| \geq R \text{ in } \Omega_L \text{ for some } R > 0 \tag{3.2}$$

holds. Consider the first order partial differential operator $Pf = v_1 \nabla f \cdot \nabla v_2 - v_3 \nabla f \cdot \nabla v_4$. *Then there exist positive constants* $s_1' > 0$ *and* $C > 0$ *such that for all* $s \geq s_1'$,

$$s^2 \int_{\Omega_L} e^{2s\phi(\theta)} f^2 \, dx \leq C \int_{\Omega_L} e^{2s\phi(\theta)} |Pf|^2 \, dx,$$

for all $f \in H_0^1(\Omega_L)$.

Proof. The proof follows [8] or [5]. Let $f \in H_0^1(\Omega_L)$. Denote $w = e^{s\phi(\theta)} f$ and $Qw = e^{s\phi(\theta)} P(e^{-s\phi(\theta)} w)$. So we get $Qw = Pw - s\lambda\phi(\theta) w (P\hat{d})$. Therefore we have

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2 \lambda^2 \int_{\Omega_L} (\phi(\theta))^2 w^2 (P\hat{d})^2 \, dx - 2s\lambda \int_{\Omega_L} \phi(\theta)(Pw) w (P\hat{d}) \, dx.$$

So

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2 \lambda^2 \int_{\Omega_L} (\phi(\theta))^2 w^2 (P\hat{d})^2 \, dx - s\lambda \int_{\Omega_L} \phi(\theta)(Pw^2)(P\hat{d}) \, dx.$$

Thus integrating by parts

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2 \lambda^2 \int_{\Omega_L} (\phi(\theta))^2 w^2 (P\hat{d})^2 \, dx + s\lambda \int_{\Omega_L} w^2 \nabla \cdot (\phi(\theta)(P\hat{d})(v_1 \nabla v_2 - v_3 \nabla v_4)) \, dx.$$

And we can conclude for s sufficiently large. □

3.2 Statements of results

3.2.1 First result Consider $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) a strong solution of (1.1) associated with (ρ, G, A) defined by (1.2) (resp. $(\tilde{\rho}_1, G, A)$) where A is a set of initial and boundary conditions. Consider also $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) a strong solution of (1.1) associated with (ρ, G, B) (resp. $(\tilde{\rho}_1, G, B)$) and where B is another set of initial and boundary conditions. Assume that all the coefficients $\alpha, \beta, \gamma, \delta, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, belong to $\Lambda_1(M_0)$ and all the coefficients ϕ_i to $\Lambda_2(M_0)$ (for $i = 1, \cdots, 4$).
Our main result is the following

Theorem 3.1 *Let $l > 0$. Let $T > 0$, $L > l$ and $\hat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Assume that*

$$|\tilde{u}_A(\cdot,\theta)\tilde{w}_B(\cdot,\theta) - \tilde{u}_B(\cdot,\theta)\tilde{w}_A(\cdot,\theta)| \geq R \text{ in } \Omega_L \text{ for some } R > 0. \tag{3.3}$$

Then there exists a sufficiently small number $\tau_0 > 0$ such that if $\tau \in (0, \tau_0)$,

$$\sum_{k=0}^{1} \int_{\gamma_L \times (0,T)} (|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2$$

$$+|\partial_\nu(\partial_t^k(u_B - \tilde{u}_B))|^2 + |\partial_\nu(\partial_t^k(w_B - \tilde{w}_B))|^2) d\sigma\, dt \leq \tau$$

then the following Hölder stability estimate holds

$$\|\alpha - \tilde{\alpha}\|^2_{L^2(\Omega_l)} + \|\beta - \tilde{\beta}\|^2_{L^2(\Omega_l)} + \|\gamma - \tilde{\gamma}\|^2_{L^2(\Omega_l)} + \|\delta - \tilde{\delta}\|^2_{L^2(\Omega_l)} \leq K\tau^\kappa \text{ for all } \tau \in (0, \tau_0). \tag{3.4}$$

Here, $K > 0$ and $\kappa \in (0, 1)$ are two constants depending on R, L, l, M_0, M_1, M_2, T and $\hat{a}$.

3.2.2 Second result As a consequence of Theorem 3.1, we can give a stability result with measurements of only one component. Theorem 3.2i) gives an estimate of the four coefficients $\alpha, \beta, \gamma, \delta \in L^2(\Omega)$ when $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$ in a neighborhood ω' of the boundary of interest γ_L. That means that these two coefficients α and β are supposed known in ω'. We relax this last hypothesis in Theorem 3.2ii) where an estimate of these four coefficients is given for $\alpha, \beta \in H^2(\Omega)$. Consider $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) a strong solution of (1.1) associated with (ρ, G, A) defined by (1.2) (resp. $(\tilde{\rho}_1, G, A)$). Consider also $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) a strong solution of (1.1) associated with (ρ, G, B) (resp. $(\tilde{\rho}_1, G, B)$). Assume that all the coefficients $\alpha, \beta, \gamma, \delta, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, belong to $\Lambda_1(M_0)$ and all the coefficients ϕ_i to $\Lambda_2(M_0)$ (for $i = 1, \cdots, 4$). For Theorem 3.2ii) we also suppose that $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in \Lambda'(M_0) = \{f \in H^2(\Omega),\ \|f\|_{H^2(\Omega)}\| \leq M_0\}$ and $\phi_i \in C^2([0, T])$.

Theorem 3.2 *Let $l > 0$. Let $T > 0$, $L > l$ and $\hat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Let ω' be a neighborhood of γ_L, $\omega' \subset \Omega_{L+\epsilon}$ such that $\gamma_L \subset \partial\omega'$, $\partial\omega'$ being C^2. Assume that the hypothesis(3.3) holds and that we also have*

$$|\beta\phi_2| \geq R > 0 \text{ in } Q_L. \tag{3.5}$$

i) *We suppose that $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$ in ω'.*

Then there exists a sufficiently small number $\tau_0 > 0$ such that if $\tau \in (0, \tau_0)$,

$$\|u_A - \tilde{u}_A\|^2_{H^2([0,T],H^2(\omega' \cap \Omega_L))} + \|u_A - \tilde{u}_A\|^2_{H^1([0,T],H^4(\omega' \cap \Omega_L))}$$

$$+\|u_B - \tilde{u}_B\|^2_{H^2([0,T],H^2(\omega' \cap \Omega_L))} + \|u_B - \tilde{u}_B\|^2_{H^1([0,T],H^4(\omega' \cap \Omega_L))}$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2)\ d\sigma\ dt\le\tau$$

then the following Hölder stability estimate holds

$$\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}\le K\tau^\kappa \text{ for all } \tau\in(0,\tau_0). \tag{3.6}$$

ii) *We suppose that* $\alpha,\beta,\tilde{\alpha},\tilde{\beta}\in H^2(\Omega)$.
Then there exists a sufficiently small number $\tau_0>0$ *such that if* $\tau\in(0,\tau_0)$,

$$\|(u_A-\tilde{u}_A)(\cdot,\theta)\|^2_{H^4(\Omega_L)}+\|(u_B-\tilde{u}_B)(\cdot,\theta)\|^2_{H^4(\Omega_L)}+\|u_A-\tilde{u}_A\|^2_{H^3([0,T],H^2(\omega'\cap\Omega_L))}$$

$$+\|u_A-\tilde{u}_A\|^2_{H^2([0,T],H^4(\omega'\cap\Omega_L))}+\|u_B-\tilde{u}_B\|^2_{H^3([0,T],H^2(\omega'\cap\Omega_L))}+\|u_B-\tilde{u}_B\|^2_{H^2([0,T],H^4(\omega'\cap\Omega_L))}$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2)\ d\sigma\ dt\le\tau$$

then the following Hölder stability estimate holds

$$\|\alpha-\tilde{\alpha}\|^2_{H^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{H^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}\le K\tau^\kappa \text{ for all } \tau\in(0,\tau_0). \tag{3.7}$$

Here, $K>0$ *and* $\kappa\in(0,1)$ *are two constants depending on* R, L, l, M_0, M_1, M_2, T, $\|g_0\|_{(C^1(\omega'))^n}$ *and* $\hat{a}$.

3.2.3 Third result

Now we present a result for the four coefficients $(\phi_1,\beta,\gamma,\delta)$. We consider here $V_A=(u_A,w_A)$ (resp. $\tilde{V}_A=(\tilde{u}_A,\tilde{w}_A)$) a strong solution of (1.1) associated with (ρ,G,A) defined by (1.2) (resp. $(\tilde{\rho}_2,G,A)$). Consider also $V_B=(u_B,w_B)$ (resp. $\tilde{V}_B=(\tilde{u}_B,\tilde{w}_B)$) a strong solution of (1.1) associated with (ρ,G,B) (resp. $(\tilde{\rho}_2,G,B)$). Assume that all the coefficients $\alpha,\beta,\gamma,\delta,\tilde{\beta},\tilde{\gamma},\tilde{\delta}$, belong to $\Lambda_1(M_0)$ and all the coefficients $\phi_i,\tilde{\phi}_1$ to $\Lambda_2(M_0)$ (for $i=1,\cdots,4$). Let the set of admissible coefficients

$$\Lambda_3(M_3)=\{f\in C^2([0,T]),|\partial_t^2(f-\phi_1)(t)|\le M_3|(f-\phi_1)(\theta)| \text{ for all } t\in[0,T]\} \tag{3.8}$$

with M_3 a positive constant.
Our result is the following.

Theorem 3.3 *Let* $l>0$. *Let* $T>0$, $L>l$ *and* $\hat{a}\in\mathbb{R}^n\setminus\Omega$ *satisfying the conditions of Proposition 2.1. We suppose that* $\tilde{\phi}_1\in\Lambda_3(M_3)$. *Assume that Assumption (3.3) holds and that*

$$|\alpha|\ge R>0 \text{ in } \Omega_L. \tag{3.9}$$

Then there exists a sufficiently small number $\tau_0 > 0$ *such that if* $\tau \in (0, \tau_0)$,

$$\sum_{k=0}^{1}(\|\partial_t^k(u_A - \tilde{u}_A)(\cdot,\theta)\|^2_{H^2(\Omega_L)} + \|\partial_t^k(u_B - \tilde{u}_B)(\cdot,\theta)\|^2_{H^2(\Omega_L)}) + \|\partial_t^2(u_A - \tilde{u}_A)(\cdot,\theta)\|^2_{L^2(\Omega_L)}$$

$$+\|\partial_t^2(u_B - \tilde{u}_B)(\cdot,\theta)\|^2_{L^2(\Omega_L)} + \|(w_A - \tilde{w}_A)(\cdot,\theta)\|^2_{H^2(\Omega_L)} + \|(w_B - \tilde{w}_B)(\cdot,\theta)\|^2_{H^2(\Omega_L)}$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2)\, d\sigma\, dt \leq \tau,$$

then the following Hölder stability estimate holds

$$\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}+\sum_{i=0}^{2}\|\partial_t^i(\phi_1-\tilde{\phi}_1)\|^2_{L^2(0,T)} \leq K\tau^\kappa \text{ for all } \tau \in (0,\tau_0). \tag{3.10}$$

Here, $K > 0$ *and* $\kappa \in (0, 1)$ *are two constants depending on* R, L, l, M_0, M_1, M_2, M_3, T, $\hat{a}$.

Remark 1 • Notice that the hypothesis $\tilde{\phi}_1 \in \Lambda_3(M_3)$ is satisfied when $\tilde{\phi}_1 \in C^2([0,T])$ is such that $\phi_1(\theta) \neq \tilde{\phi}_1(\theta)$ and $\frac{\sup_{t\in[0,T]}|\partial_t(\phi_1-\tilde{\phi}_1)(t)|}{|\phi_1(\theta)-\tilde{\phi}_1(\theta)|} \leq M_3$. Moreover note also that if $\tilde{\phi}_1 \in C^2([0,T])$ is such that $\phi_1(\theta) \neq \tilde{\phi}_1(\theta)$, then if we denote $f_1 = \phi_1 - \tilde{\phi}_1$, we have $f_1(\theta) \neq 0$. Therefore $t \mapsto |\frac{f_1(t)}{f_1(\theta)}|$ is bounded on $[0,T]$ so there exists a positive constant C_0 such that for all $t \in [0,T]$, $|f_1(t)| \leq C_0|f_1(\theta)|$. Similarly there exists a positive constant C_1 such that $|\partial_t f_1(t)| \leq C_1|f_1(\theta)|$ and there exists a positive constant C_2 such that $|\partial_t^2 f_1(t)| \leq C_2|f_1(\theta)|$. Note also that if $\tilde{\phi}_1 \in \Lambda_3(M_3)$ and $\tilde{\phi}_1(\theta) = \phi_1(\theta)$, then $\partial_t^2(\tilde{\phi}_1 - \phi_1) = 0$ in $[0,T]$. Therefore $\tilde{\phi}_1$ has the form $\tilde{\phi}_1(t) = \phi_1(t) + k(t-\theta)$ with k any real.
• Moreover if the function ϕ_1 is more regular, for example if $\phi_1 \in C^p([0,T])$ with $p \geq 2$, then Theorem 3.3 is still valid with a more generalized admissible set of coefficients $\Lambda_3'(M_3) = \{f \in C^p([0,T]), |\partial_t^p(f-\phi_1)(t)| \leq M_3|(f-\phi_1)(\theta)| \text{ for all } t \in [0,T]\}$. But in this case, because of our method, the observations terms at the fixed time θ on the right-hand side of the estimate (3.10) would demand more regularity.
• On the contrary, we can relax some of the observations terms on u (u_A and $\tilde{u}_A$) at θ on the right-hand side of (3.10) and only have $\|(u-\tilde{u})(\cdot,\theta)\|^2_{H^2(\Omega_L)}$ but for a more restrictive admissible set of coefficients $\Lambda_3''(M_3) = \{f \in C^2([0,T]), |\partial_t^i(f-\phi_1)(t)| \leq M_3|(f-\phi_1)(\theta)| \text{ for all } i = 0,1,2 \text{ and } t \in [0,T]\}$.

3.2.4 Fourth result

Finally, we consider the system (1.3). Consider $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) a strong solution of (1.3) associated with (ρ, G, A, Θ) defined by (1.2) and (1.4) (resp. $(\tilde{\rho}_3, G, A, \tilde{\Theta})$). Consider also $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) a strong solution of (1.3) associated with (ρ, G, B, Θ) (resp. $(\tilde{\rho}_3, G, B, \tilde{\Theta})$). Assume that all the coefficients $\alpha, \beta, \gamma, \delta, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, belong to $\Lambda_1(M_0)$ and all the coefficients ϕ_i to $\Lambda_2(M_0)$

(for $i=1,\cdots,4$). Moreover we suppose that $\Theta_i, \tilde{\Theta}_1$ belong to $(\Lambda_1(M_0))^n \cap (L^2(\Omega))^n$ (for $i=1,\cdots,4$) and there exist functions $\xi_1, \tilde{\xi}_1$ such that

$$\Theta_1 = \nabla\xi_1,\ \tilde{\Theta}_1 = \nabla\tilde{\xi}_1 \text{ in } \Omega.$$

Theorem 3.4 *Let $l>0$. Let $T>0$, $L>l$ and $\hat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Assume that Assumptions (3.2) and (3.3) are satisfied with $(v_1,\cdots,v_4) = (\tilde{w}_B(\cdot,\theta), \tilde{u}_A(\cdot,\theta), \tilde{w}_A(\cdot,\theta), \tilde{u}_B(\cdot,\theta))$.*
If $\xi_1 = \tilde{\xi}_1$ and $\Theta_1 = \tilde{\Theta}_1$ on $\partial\Omega \cap \partial\Omega_L$, then there exists a sufficiently small number $\tau_0 > 0$ such that if $\tau \in (0,\tau_0)$,

$$\sum_{k=0}^{1} \int_{\gamma_L \times (0,T)} (|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2 + |\partial_\nu(\partial_t^k(u_B - \tilde{u}_B))|^2$$

$$+|\partial_\nu(\partial_t^k(w_B - \tilde{w}_B))|^2) d\sigma\ dt + \|(u_A - \tilde{u}_A)(\cdot,\theta)\|^2_{H^3(\Omega_L)} + \|(u_B - \tilde{u}_B)(\cdot,\theta)\|^2_{H^3(\Omega_L)} \le \tau$$

then the following Hölder stability estimate holds

$$\|\beta - \tilde{\beta}\|^2_{L^2(\Omega_l)} + \|\gamma - \tilde{\gamma}\|^2_{L^2(\Omega_l)} + \|\delta - \tilde{\delta}\|^2_{L^2(\Omega_l)} + \|\Theta_1 - \tilde{\Theta}_1\|_{(L^2(\Omega_l))^n} \le K\tau^\kappa \tag{3.11}$$

for all $\tau \in (0,\tau_0)$.
Here, $K>0$ and $\kappa \in (0,1)$ are two constants depending on R, L, l, M_0, M_1, M_2, T and $\hat{a}$.

3.3 Proofs of theorems

3.3.1 Proof of Theorem 3.1 Let $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) be a solution of (1.1) associated with (ρ, G, A) (resp. $(\tilde{\rho}_1, G, A)$) and $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) be a solution of (1.1) associated with (ρ, G, B) (resp. $(\tilde{\rho}_1, G, B)$). We decompose the proof in several steps.

• First step:
Denote $V = (u,w) = V_A$, $\tilde{V} = (\tilde{u}, \tilde{w}) = \tilde{V}_A$ and

$$U = u - \tilde{u},\ W = w - \tilde{w},\ a = \alpha - \tilde{\alpha}.\ b = \beta - \tilde{\beta},\ c = \gamma - \tilde{\gamma},\ d = \delta - \tilde{\delta}. \tag{3.12}$$

Then (U,W) satisfy the following system

$$\begin{cases} \partial_t U = \Delta U + \alpha\phi_1 U + \beta\phi_2 W + a\phi_1\tilde{u} + b\phi_2\tilde{w} \text{ in } Q, \\ \partial_t W = \Delta W + \gamma\phi_3 U + \delta\phi_4 W + c\phi_3\tilde{u} + d\phi_4\tilde{w} \text{ in } Q, \\ U = W = 0 \text{ on } \Sigma. \end{cases} \tag{3.13}$$

Define

$$y_0 = \eta\chi U,\ z_0 = \eta\chi W,\ y_1 = \partial_t y_0,\ z_1 = \partial_t z_0 \tag{3.14}$$

We deduce that (y_i, z_i) for $i = 0, 1$ satisfy the following systems

$$\begin{cases} \partial_t y_0 = \Delta y_0 + \alpha\phi_1 y_0 + \beta\phi_2 z_0 + a\eta\chi\phi_1\tilde{u} + b\eta\chi\phi_2\tilde{w} + R_1 \text{ in } Q_L, \\ \partial_t z_0 = \Delta z_0 + \gamma\phi_3 y_0 + \delta\phi_4 z_0 + c\eta\chi\phi_3\tilde{u} + d\eta\chi\phi_4\tilde{w} + R_2 \text{ in } Q_L, \\ y_0 = z_0 = 0 \text{ on } \partial\Omega_L \times (0, T) \end{cases} \tag{3.15}$$

with

$$R_1 = -(\Delta\chi)\eta U - 2\eta\nabla\chi\cdot\nabla U + \chi\partial_t\eta U,\ R_2 = -(\Delta\chi)\eta W - 2\eta\nabla\chi\cdot\nabla W + \chi\partial_t\eta W.$$

We have

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha\phi_1 y_1 + \beta\phi_2 z_1 + R_3 \text{ in } Q_L, \\ \partial_t z_1 = \Delta z_1 + \gamma\phi_3 y_1 + \delta\phi_4 z_1 + R_4 \text{ in } Q_L, \\ y_1 = z_1 = 0 \text{ on } \partial\Omega_L \times (0, T), \end{cases} \tag{3.16}$$

with

$$R_3 = a\chi\partial_t(\eta\phi_1\tilde{u}) + b\chi\partial_t(\eta\phi_2\tilde{w}) + \partial_t R_1 + \alpha y_0\partial_t\phi_1 + \beta z_0\partial_t\phi_2,$$

$$R_4 = c\chi\partial_t(\eta\phi_3\tilde{u}) + d\chi\partial_t(\eta\phi_4\tilde{w}) + \partial_t R_2 + \gamma y_0\partial_t\phi_3 + \delta z_0\partial_t\phi_4.$$

• Second step: we estimate $\sum_{i=0}^{1}(I(y_i) + I(z_i))$ by the Carleman inequalities (2.6). Note that all the terms in $A_0 y_i$ or $A_0 z_i$ with derivatives of χ or η will be bounded above by Ce^{2sd_1} with C a positive constant (see Proposition 2.1 for the definitions of d_1 and d_2). Moreover all the terms such as $\int_{Q_L} e^{2s\phi} y_i^2\ dx\ dt$ on the right-and side of the estimates (2.6) will be absorbed by $I(y_i)$ for s sufficiently large. So we have for s sufficiently large,

$$\sum_{i=0}^{1}(I(y_i) + I(z_i)) \le C\int_{Q_L} e^{2s\phi}(a^2 + b^2 + c^2 + d^2)\chi^2\ dx\ dt + Cs^3 e^{2sd_1}$$

$$+Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}\sum_{i=0}^{1}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt.$$

Since $e^{2s\phi} \le e^{2s\phi(\theta)} \le e^{2sd_2}$ we get

$$\sum_{i=0}^{1}(I(y_i) + I(z_i)) \le C\int_{Q_L} e^{2s\phi}(a^2 + b^2 + c^2 + d^2)\chi^2\ dx\ dt + Cs^3 e^{2sd_1} + Cse^{2sd_2}F_0(\gamma_L) \tag{3.17}$$

with $F_0(\gamma_L) = \int_{\gamma_L\times(0,T)}\sum_{i=0}^{1}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt$.

• Third step: now we estimate $\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t^i f(\theta)|^2\ dx$ and $\int_{\Omega_L} e^{2s\phi(\theta)}|\Delta f(\theta)|^2\ dx$ for $f = y_0$ or $f = z_0$ and $i = 0, 1$. By Lemma 3.1, we have (since $\phi \ge 1$ and $\frac{1}{\phi} \ge \frac{1}{d_2}$)

$$\int_{\Omega_L} e^{2s\phi(\theta)}|y_0(\theta)|^2\ dx \le Cs\int_{Q_L} e^{2s\phi}y_0^2\ dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}y_1^2\ dx\ dt \le \frac{C}{s^2}(I(y_0) + I(y_1)),$$

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t y_0(\theta)|^2\ dx \le Cs\int_{Q_L} e^{2s\phi}y_1^2\ dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}|\partial_t y_1|^2\ dx\ dt \le CI(y_1),$$

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\Delta y_0(\theta)|^2\,dx \le Cs\int_{Q_L} e^{2s\phi}|\Delta y_0|^2\,dx\,dt+\frac{C}{s}\int_{Q_L} e^{2s\phi}|\Delta y_1|^2\,dx\,dt \le Cs^2(I(y_0)+I(y_1)).$$

Notice that the three above inequalities are satisfied replacing (y_0, y_1, y_2) by (z_0, z_1, z_2). Therefore

$$\int_{\Omega_L} e^{2s\phi(\theta)}(|y_0(\theta)|^2+|\partial_t y_0(\theta)|^2+|\Delta y_0(\theta)|^2+|z_0(\theta)|^2+|\partial_t z_0(\theta)|^2+|\Delta z_0(\theta)|^2)\,dx$$

$$\le Cs^2\sum_{i=0}^{1}(I(y_i)+I(z_i)).$$

So using (3.17) we deduce that

$$\int_{\Omega_L} e^{2s\phi(\theta)}(|y_0(\theta)|^2+|\partial_t y_0(\theta)|^2+|\Delta y_0(\theta)|^2+|z_0(\theta)|^2+|\partial_t z_0(\theta)|^2+|\Delta z_0(\theta)|^2)\,dx$$

$$\le Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\,dx\,dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_0(\gamma_L). \tag{3.18}$$

At last in this step, denote

$$R=(R_1,R_2,R_3,R_4). \tag{3.19}$$

• Fourth step: here we estimate $\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\,dx$.

We choose now the two sets of conditions A and B and consider V_A, $\tilde{V_A}$, V_B and $\tilde{V_B}$. From now on, each function f defined in the precedent steps is denoted either f_A or f_B when it is related either by the conditions A or B. Denote now $F_{0A}(\gamma_L)=F_0(\gamma_L)$ associated with $(V_A,\tilde{V_A})$, and $F_{0B}(\gamma_L)=F_0(\gamma_L)$ associated with $(V_B,\tilde{V_B})$ (see (3.17) in the second step):

$$F_{0A}(\gamma_L)=\int_{\gamma_L\times(0,T)}\sum_{i=0}^{1}(|\partial_\nu y_{iA}|^2+|\partial_\nu z_{iA}|^2)\,d\sigma\,dt,\ F_{0B}(\gamma_L)=$$

$$\int_{\gamma_L\times(0,T)}\sum_{i=0}^{1}(|\partial_\nu y_{iB}|^2+|\partial_\nu z_{iB}|^2)\,d\sigma\,dt.$$

Let R_A be defined by (3.19) for $(V_A,\tilde{V_A})$ (resp. R_B for $(V_B,\tilde{V_B})$). Multiplying the first equation of (3.15) written for y_{0A} by $\tilde{w}_B$ and the first equation of (3.15) written for y_{0B} by $\tilde{w}_A$ and subtracting, we eliminate the term in b and we get

$$a\eta\chi\phi_1(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)=\tilde{w}_B(\partial_t y_{0A}-\Delta y_{0A}-\alpha\phi_1 y_{0A}-\beta\phi_2 z_{0A}-R_{1A})$$

$$-\tilde{w}_A(\partial_t y_{0B}-\Delta y_{0B}-\alpha\phi_1 y_{0B}-\beta\phi_2 z_{0B}-R_{1B}). \tag{3.20}$$

By hypothesis (3.3), applying (3.20) for $t=\theta$, since $\eta=1$ in a neighborhood of θ we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}a^2\chi^2(\phi_1(\theta))^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}\left(|\partial_t y_{0A}(\theta)|^2+|\partial_t y_{0B}(\theta)|^2+|\Delta y_{0A}(\theta)|^2+|\Delta y_{0B}(\theta)|^2\right.$$

$$+|y_{0A}(\theta)|^2+|z_{0A}(\theta)|^2+|y_{0B}(\theta)|^2+|z_{0B}(\theta)|^2\big)\ dx+Ce^{2sd_1}.$$

But $\phi_1 \in \Lambda_2(M_0)$. So from (3.18) applied for $y_{0A}, y_{0B}, z_{0A}, z_{0B}$ we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}a^2\chi^2\ dx \le Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L) \quad (3.21)$$

with $F_1(\gamma_L) = F_{0A}(\gamma_L) + F_{0B}(\gamma_L)$. Similarly we can replace a by b on the left-hand side of (3.21), still using (3.15) for y_{0A} and y_{0B}. Indeed

$$-b\eta\chi\phi_2(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) = \tilde{u}_B(\partial_t y_{0A} - \Delta y_{0A} - \alpha\phi_1 y_{0A} - \beta\phi_2 z_{0A} - R_{1A})$$

$$-\tilde{u}_A(\partial_t y_{0B} - \Delta y_{0B} - \alpha\phi_1 y_{0B} - \beta\phi_2 z_{0B} - R_{1B}).$$

So we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2)\chi^2\ dx \le Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L). \quad (3.22)$$

We do the same to obtain c and d using this time (3.15) for z_{0A} and z_{0B} and the hypothesis (3.3). Therefore

$$\int_{\Omega_L} e^{2s\phi(\theta)}(c^2+d^2)\chi^2\ dx \le Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L). \quad (3.23)$$

Adding (3.22) and (3.23), we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt \le$$
$$Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt + Cs^5e^{2sd_1} + Cs^3e^{2sd_2}F_1(\gamma_L).$$

Now we proceed as in [2, 11, 12] in order to prove that $s^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt$ can be absorbed by the left-hand side of the above estimate for s sufficiently large ($s \ge s_2$). Indeed

$$s^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt = \int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\Big(\int_0^T s^2e^{2s(\phi-\phi(\theta))}\ dt\Big)\ dx.$$

But $\phi - \phi(\theta) = -e^{\lambda(\hat{d}+M_1)}(1-e^{-\lambda(t-\theta)^2})$ and there exists a positive constant C such that $\phi - \phi(\theta) \le -C(1-e^{-\lambda(t-\theta)^2})$. Therefore $\int_0^T s^2e^{2s(\phi-\phi(\theta))}\ dt \le \int_0^T s^2e^{-2sC(1-e^{-\lambda(t-\theta)^2})}\ dt$ uniformly in x. Moreover by the Lebesgue convergence theorem, we have

$$\int_0^T s^2e^{-2sC(1-e^{-\lambda(t-\theta)^2})}\ dt \to 0 \text{ as } s\to\infty.$$

Thus for s sufficiently large, we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\ dx \leq Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L).$$

Since $e^{2sd_0} \leq e^{2s\phi(\theta)}$ in Ω_l and $\chi = 1$ in Ω_l, we deduce that

$$e^{2sd_0}(\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}) \leq Cs^3(e^{2sd_2}F_1(\gamma_L)+s^2e^{2sd_1})$$

which can be rewritten

$$\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)} \leq Cs^3(e^{2s(d_2-d_0)}F_1(\gamma_L)+s^2e^{2s(d_1-d_0)}). \tag{3.24}$$

As $d_1 - d_0 < 0$ and $d_2 - d_0 > 0$, we can optimize the above inequality with respect to s (see for example [5, 7, 8]). Indeed, note that if $F_1(\gamma_L) = 0$, since (3.24) holds for any $s \geq s_2$ and $d_1 - d_0 < 0$ we get (3.4). Now if $F_1(\gamma_L) \neq 0$ is sufficiently small ($F_1(\gamma_L) < \frac{d_0-d_1}{d_2-d_0}$), we optimize (3.24) with respect to s. Indeed denote

$$f(s) = e^{2s(d_2-d_0)}F_1(\gamma_L)+e^{2s(d_1-d_0)} \text{ and } g(s) = e^{2s(d_2-d_0)}F_1(\gamma_L)+s^2e^{2s(d_1-d_0)}.$$

We have $f(s) \sim g(s)$ at infinity. Moreover the function f has a minimum in

$$s_3 = \frac{1}{2(d_2-d_1)}\ln(\frac{d_0-d_1}{(d_2-d_0)F_1(\gamma_L)}) \text{ and } f(s_3) = K'F_1(\gamma_L)^{\kappa}$$

with $\kappa = \frac{d_0-d_1}{d_2-d_1}$ and $K' = (\frac{d_0-d_1}{d_2-d_0})^{\frac{d_2-d_0}{d_2-d_1}} + (\frac{d_0-d_1}{d_2-d_0})^{\frac{d_1-d_0}{d_2-d_0}}$. Finally the minimum s_3 is sufficiently large ($s_3 \geq s_2$) if the following condition $F_1(\gamma_L) \leq \tau_0$, with $\tau_0 = \frac{d_0-d_1}{(d_2-d_0)e^{2s_2(d_2-d_1)}}$, is satisfied. So we conclude for Theorem 3.1.

3.3.2 Proof of Theorem 3.2 We keep the notations of the proof of Theorem 3.1. In this theorem, we want to remove all the observation terms on w obtained in Theorem 3.1 and express them in terms of u. So we look at the terms $\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt$ for $i = 0, 1$ appearing in step 2 of Theorem 3.1. Recall that $z_i = 0$ outside $\Omega_{L-\epsilon}$ and $\gamma_L \subset \partial\omega'$.
As in [4, Lemma 2] we choose $g_0 \in C^2(\overline{\omega'}, \mathbb{R}^n)$ such that $g_0 = \nu$ on the C^2-boundary $\partial\omega'$ where ν is the normal vector to $\partial\omega'$. We have by integration by parts for any integer $i = 0, 1$,

$$\int_{\omega'\times(0,T)} e^{2s\phi}\Delta z_i\ g_0\cdot\nabla z_i\ dx\ dt = -\int_{\omega'\times(0,T)} \nabla(e^{2s\phi}g_0\cdot\nabla z_i)\cdot\nabla z_i\ dx\ dt$$
$$+\int_{\partial\omega'\times(0,T)} e^{2s\phi}g_0\cdot\nabla z_i\ \partial_\nu z_i\ d\sigma\ dt.$$

So

$$\int_{\omega'\times(0,T)} e^{2s\phi}\Delta z_i\ g_0\cdot\nabla z_i\ dx\ dt = -\int_{\omega'\times(0,T)} \nabla(e^{2s\phi}g_0\cdot\nabla z_i)\cdot\nabla z_i\ dx\ dt$$

$$+\int_{\partial\omega'\times(0,T)} e^{2s\phi}\ |\partial_\nu z_i|^2\ d\sigma\ dt.$$

and we get

$$\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt \le Cs\int_{(\omega'\cap\Omega_L)\times(0,T)} e^{2s\phi}(|\nabla z_i|^2+|\Delta z_i|^2)\ dx\ dt. \tag{3.25}$$

From the first equation in (3.15) we have

$$\beta\phi_2 z_0 = \partial_t y_0 - \Delta y_0 - \alpha\phi_1 y_0 - a\eta\chi\phi_1\tilde{u} - b\eta\chi\phi_2\tilde{w} - R_1 \text{ in } Q_L. \tag{3.26}$$

By the same way, from (3.16) we have

$$\beta\phi_2 z_1 = \partial_t y_1 - \Delta y_1 - \alpha\phi_1 y_1 - R_3 \text{ in } Q_L. \tag{3.27}$$

i) First assume that $a = b = 0$ in ω'. From hypothesis (3.5), (3.25)-(3.27) we get

$$\sum_{i=0}^{1}\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt \le Cs\sum_{i=0}^{1}\int_{(\omega'\cap\Omega_L)\times(0,T)} e^{2s\phi}(|\nabla\partial_t y_i|^2+|\nabla(\Delta y_i)|^2+|\nabla y_i|^2+|y_i|^2$$

$$+|\Delta\partial_t y_i|^2+|\Delta(\Delta y_i)|^2+|\Delta y_i|^2)\ dx\ dt + Cse^{2sd_1}.$$

So

$$\sum_{i=0}^{1}\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt \le Cse^{2sd_1}+Cse^{2sd_2}G_0(\omega')$$

with $G_0(\omega') = \|y_0\|^2_{H^1(0,T,H^4(\omega'\cap\Omega_L))} + \|y_0\|^2_{H^2(0,T,H^2(\omega'\cap\Omega_L))}$.
Therefore (3.17) is still valid with $sF_0(\gamma_L)$ replaced by $s^2G_1(\gamma_L) = s^2\int_{\gamma_L\times(0,T)}$ $\sum_{i=0}^{1}|\partial_\nu y_i|^2\ d\sigma\ dt + s^2G_0(\omega')$. Thus we follow the proof of Theorem 3.1 substituting $F_0(\gamma_L)$ by $G_1(\gamma_L)$. The rest of the proof (steps 3 and 4) remains unchanged.

ii) Here we suppose that $\alpha,\beta,\tilde{\alpha},\tilde{\beta}\in H^2(\Omega)$. We will need to differentiate y_0 and z_0 twice with respect to t (in order to get (3.35)) and we have

$$\begin{cases} \partial_t y_2 = \Delta y_2 + \alpha\phi_1 y_2 + \beta\phi_2 z_2 + \partial_t R_3 + \alpha\partial_t\phi_1 y_1 + \beta\partial_t\phi_2 z_1 \ \text{ in } \ Q_L, \\ \partial_t z_2 = \Delta z_2 + \gamma\phi_3 y_2 + \delta\phi_4 z_2 + \partial_t R_4 + \gamma\partial_t\phi_3 y_1 + \delta\partial_t\phi_4 z_1 \ \text{ in } \ Q_L, \\ y_2 = z_2 = 0 \text{ on } \partial\Omega_L\times(0,T). \end{cases} \tag{3.28}$$

Therefore

$$\beta\phi_2 z_2 = \partial_t y_2 - \Delta y_2 - \alpha\phi_1 y_2 - \partial_t R_3 - \alpha\partial_t\phi_1 y_1 - \beta\partial_t\phi_2 z_1 \text{ in } Q_L. \tag{3.29}$$

Notice that we can take $\sum_{k=0}^{2}\int_{\gamma_L\times(0,T)}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(w_A-\tilde{w}_A))|^2+\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2+|\partial_\nu(\partial_t^k(w_B-\tilde{w}_B))|^2)d\sigma\ dt$ as observation terms in (3.4). So we apply (3.25) for

$i = 0, 1, 2$.
From (3.25)-(3.29) we get

$$\sum_{i=0}^{2}\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\,d\sigma\,dt \le Cs\sum_{i=0}^{2}\int_{(\omega'\cap\Omega_L)\times(0,T)} e^{2s\phi}(|\nabla\partial_t y_i|^2+|\nabla(\Delta y_i)|^2+|\nabla y_i|^2+|y_i|^2$$
$$+|\Delta\partial_t y_i|^2+|\Delta(\Delta y_i)|^2+|\Delta y_i|^2$$
$$+(a^2+b^2)\chi^2+|\nabla(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(a\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt+Cse^{2sd_1}.$$

So

$$\sum_{i=0}^{2}\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\,d\sigma\,dt \le Cse^{2sd_2}\tilde G_0(\omega')+Cse^{2sd_1}$$
$$+Cs\int_{Q_L} e^{2s\phi}((a^2+b^2)\chi^2+|\nabla(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(a\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt$$

with $\tilde G_0(\omega') = \|y_0\|^2_{H^2(0,T,H^4(\omega'\cap\Omega_L))} + \|y_0\|^2_{H^3(0,T,H^2(\omega'\cap\Omega_L))}$.
Thus the estimate (3.17) becomes

$$\sum_{i=0}^{2}(I(y_i)+I(z_i)) \le Cs^3e^{2sd_1}+Cs^2e^{2sd_2}\tilde G_1(\gamma_L)$$
$$+Cs^2\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt \quad (3.30)$$

with $\tilde G_1(\gamma_L) = \int_{\gamma_L\times(0,T)}\sum_{i=0}^{2}|\partial_\nu y_i|^2\,d\sigma\,dt + \tilde G_0(\omega')$.
As in the third step of Theorem 3.1 when we get (3.18), by Lemma 3.1 we have

$$\sum_{i=0}^{1}\int_{\Omega_L} e^{2s\phi(\theta)}(|y_i(\theta)|^2+|\nabla y_i(\theta)|^2+|\Delta y_i(\theta)|^2+|z_i(\theta)|^2+|\nabla z_i(\theta)|^2+|\Delta z_i(\theta)|^2)\,dx$$
$$\le Cs^2\sum_{i=0}^{2}(I(y_i)+I(z_i)).$$

So from (3.30)

$$\sum_{i=0}^{1}\int_{\Omega_L} e^{2s\phi(\theta)}(|y_i(\theta)|^2+|\nabla y_i(\theta)|^2+|\Delta y_i(\theta)|^2+|z_i(\theta)|^2+|\nabla z_i(\theta)|^2+|\Delta z_i(\theta)|^2)\,dx$$
$$\le Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt$$
$$+Cs^5e^{2sd_1}+Cs^4e^{2sd_2}\tilde G_1(\gamma_L). \quad (3.31)$$

Now we estimate $\int_{\Omega_L} e^{2s\phi(\theta)}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx$ as in the fourth step of Theorem 3.1. We consider two sets of initial conditions A and B and the corresponding solutions $V_A, \tilde{V}_A, V_B, \tilde{V}_B$ of (1.1). As in (3.20)-(3.23) we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}(|\partial_t y_{0A}(\theta)|^2+|\partial_t y_{0B}(\theta)|^2+|\Delta y_{0A}(\theta)|^2$$
$$+|\Delta y_{0B}(\theta)|^2+|y_{0A}(\theta)|^2+|y_{0B}(\theta)|^2+|\partial_t z_{0A}(\theta)|^2+|\partial_t z_{0B}(\theta)|^2+|\Delta z_{0A}(\theta)|^2$$
$$+|\Delta z_{0B}(\theta)|^2+|z_{0A}(\theta)|^2+|z_{0B}(\theta)|^2)\,dx+Ce^{2sd_1}.$$

So from (3.31) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\,dx \le Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)$$
$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2))\,dx\,dt \quad (3.32)$$

with $G_2(\gamma_L)=\tilde{G}_{1A}(\gamma_L)+\tilde{G}_{1B}(\gamma_L)$.
We apply the same ideas for $\nabla(a\chi), \nabla(b\chi), \Delta(a\chi), \Delta(b\chi)$.
For any integer $1\le i\le n$, taking the space derivative with respect to x_i in (3.20), we obtain

$$\partial_{x_i}(a\chi)\eta\phi_1(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)+a\eta\chi\phi_1\partial_{x_i}(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)$$
$$=\partial_{x_i}\left(\tilde{w}_B(\partial_t y_{0A}-\Delta y_{0A}-\alpha\phi_1 y_{0A}-\beta\phi_2 z_{0A}-R_{1A})\right)$$
$$-\partial_{x_i}\left(\tilde{w}_A(\partial_t y_{0B}-\Delta y_{0B}-\alpha\phi_1 y_{0B}-\beta\phi_2 z_{0B}-R_{1B}\right). \quad (3.33)$$

Therefore by hypothesis (3.3) we deduce that

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(a\chi)|^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}(a\chi)^2\,dx+Ce^{2sd_1}$$
$$+\int_{\Omega_L} e^{2s\phi(\theta)}(|\nabla\partial_t y_{0A}(\theta)|^2+|\nabla\Delta y_{0A}(\theta)|^2+|\nabla y_{0A}(\theta)|^2+|\nabla z_{0A}(\theta)|^2$$
$$+|\nabla\partial_t y_{0B}(\theta)|^2+|\nabla\Delta y_{0B}(\theta)|^2+|\nabla y_{0B}(\theta)|^2+|\nabla z_{0B}(\theta)|^2)\,dx.$$

From (3.31)-(3.32) we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(a\chi)|^2\,dx \le Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)+Ce^{2sd_2}(\|y_{0A}(\theta)\|^2_{H^3(\Omega_L)}+\|y_{0B}(\theta)\|^2_{H^3(\Omega_L)})$$
$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt. \quad (3.34)$$

Taking again the space derivative with respect to x_i in (3.33) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\Delta(a\chi)|^2\,dx \le Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)+Ce^{2sd_2}(\|y_{0A}(\theta)\|_{H^4(\Omega_L)}+\|y_{0B}(\theta)\|_{H^4(\Omega_L)})$$

$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt. \quad (3.35)$$

Similarly for b, so from (3.32),(3.34),(3.35) we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx$$
$$\leq Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)+Ce^{2sd_2}(\|y_{0A}(\theta)\|_{H^4(\Omega_L)}+\|y_{0B}(\theta)\|_{H^4(\Omega_L)})$$
$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt.$$

As in the proof of Theorem 3.1 (see the fourth step) we can absorb the last term of the above estimate by the left-hand side so we deduce that for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\theta)}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx$$
$$\leq Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_3(\gamma_L)$$

with $G_3(\gamma_L)=G_2(\gamma_L)+\|y_{0A}(\theta)\|_{H^4(\Omega_L)}+\|y_{0B}(\theta)\|_{H^4(\Omega_L)}$ and we conclude as for Theorem 3.1.

3.3.3 Proof of Theorem 3.3

Let $V_A=(u_A,w_A)$ (resp. $\tilde V_A=(\widetilde u_A,\widetilde w_A)$) be a solution of (1.1) associated with (ρ,G,A) (resp. $(\tilde\rho_2,G,A)$) and $V_B=(u_B,w_B)$ (resp. $\tilde V_B=(\widetilde u_B,\widetilde w_B)$) be a solution of (1.1) associated with (ρ,G,B) (resp. $(\tilde\rho_2,G,B)$). As for Theorems 3.1 and 3.2 we decompose the proof in several steps.

• First step: We keep the notations of (3.12)

$$V=(u,w)=V_A,\ \tilde V=(\tilde u,\tilde w)=\tilde V_A,\ U=u-\tilde u,\ W=w-\tilde w, b=\beta-\tilde\beta,\ c=\gamma-\tilde\gamma,\ d=\delta-\tilde\delta.$$

and now define

$$f_1=\phi_1-\tilde\phi_1.$$

We still define (see (3.14)) (for $i=0,1,2$)

$$y_0=\eta\chi U,\ z_0=\eta\chi W,\ y_i=\partial_t^i y_0,\ z_i=\partial_t^i z_0.$$

The systems (3.13), (3.15), (3.16) become

$$\begin{cases}\partial_t U=\Delta U+\alpha\phi_1 U+\beta\phi_2 W+\alpha f_1\tilde u+b\phi_2\tilde w \text{ in } Q,\\ \partial_t W=\Delta W+\gamma\phi_3 U+\delta\phi_4 W+c\phi_3\tilde u+d\phi_4\tilde w \text{ in } Q,\\ U=W=0 \text{ in } \Sigma,\end{cases}$$

and (y_i,z_i) for $i=0,1$ satisfy the following systems

$$\begin{cases}\partial_t y_0=\Delta y_0+\alpha\phi_1 y_0+\beta\phi_2 z_0+\alpha f_1\eta\chi\tilde u+b\phi_2\eta\chi\tilde w+S_1 \text{ in } Q_L,\\ \partial_t z_0=\Delta z_0+\gamma\phi_3 y_0+\delta\phi_4 z_0+c\phi_3\eta\chi\tilde u+d\phi_4\eta\chi\tilde w+S_2 \text{ in } Q_L,\\ y_0=z_0=0 \text{ on } \partial\Omega_L\times(0,T)\end{cases} \quad (3.36)$$

with

$$S_1 = R_1 = -(\Delta\chi)\eta U - 2\eta\nabla\chi\cdot\nabla U + \chi\partial_t\eta U,\ S_2 = R_2 = -(\Delta\chi)\eta W - 2\eta\nabla\chi\cdot\nabla W + \chi\partial_t\eta W.$$

We have

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha\phi_1 y_1 + \beta\phi_2 z_1 + S_3 \ \text{ in } \ Q_L, \\ \partial_t z_1 = \Delta z_1 + \gamma\phi_3 y_1 + \delta\phi_4 z_1 + S_4 \ \text{ in } \ Q_L, \\ y_1 = z_1 = 0 \text{ on } \partial\Omega_L \times (0,T), \end{cases}$$

with

$$S_3 = \partial_t\left(\alpha f_1\eta\chi\tilde{u} + b\phi_2\eta\chi\tilde{w}\right) + \partial_t S_1 + \alpha y_0\partial_t\phi_1 + \beta z_0\partial_t\phi_2,$$

$$S_4 = R_4 = \partial_t\left(c\phi_3\eta\chi\tilde{u} + d\phi_4\eta\chi\tilde{w}\right) + \partial_t S_2 + \gamma y_0\partial_t\phi_3 + \delta z_0\partial_t\phi_4.$$

We also have

$$\begin{cases} \partial_t y_2 = \Delta y_2 + \alpha\phi_1 y_2 + \beta\phi_2 z_2 + \partial_t S_3 + \alpha\partial_t\phi_1 y_1 + \beta\partial_t\phi_2 z_1 \ \text{ in } \ Q_L, \\ \partial_t z_2 = \Delta z_2 + \gamma\phi_3 y_2 + \delta\phi_4 z_2 + \partial_t S_4 + \gamma\partial_t\phi_3 y_1 + \delta\partial_t\phi_4 z_1 \ \text{ in } \ Q_L, \\ y_2 = z_2 = 0 \text{ on } \partial\Omega_L \times (0,T). \end{cases}$$

- In the second step we estimate $\sum_{i=0}^{2}(I(y_i)+I(z_i))$ as in Theorem 3.1 and we get

$$\sum_{i=0}^{2}(I(y_i)+I(z_i)) \le C\int_{Q_L} e^{2s\phi}(b^2+c^2+d^2)\chi^2\,dx\,dt + C\int_{Q_L} e^{2s\phi}\chi^2\Big(\sum_{i=0}^{2}(\partial_t^i f_1)^2\Big)\,dx\,dt$$
$$+Cs^3e^{2sd_1} + Cse^{2sd_2}\tilde{F}_0(\gamma_L) \tag{3.37}$$

with $\tilde{F}_0(\gamma_L) = \int_{\gamma_L\times(0,T)}\sum_{i=0}^{2}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\,d\sigma\,dt$ (nearly same definition as before since (3.17)).

Now following the proof of Theorem 3.1 we look at (3.18) in this context. First note that because of the fourth step of this proof, we can no longer use the estimates of the Laplacian terms in (3.18) and contrary to Theorems 3.1, 3.2, 3.4, we have to take care of the powers of s on the right-hand sides of our estimates. In fact we could only look at the estimate of $\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z_0(\theta)|^2\,dx$ but because of the remarks given just after the proof of this theorem, we will keep more terms. So we will not estimate $\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z_0(\theta)|^2\,dx$ as in Theorems 3.1, 3.2, 3.4 (see the third step of Theorem 3.1) and for that, we need to differentiate twice y_0 and z_0 with respect to t. Thus

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z_0(\theta)|^2\,dx \le Cs\int_{Q_L} e^{2s\phi}|z_1|^2\,dx\,dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}|z_2|^2 \le \frac{C}{s^2}(I(z_1)+I(z_2)).$$

So we have (coming from Lemma 3.1 as in (3.18)) and by (3.37)

$$\int_{\Omega_L} e^{2s\phi(\theta)}(y_0(\theta)|^2 + |\partial_t y_0(\theta)|^2 + |z_0(\theta)|^2 + |\partial_t z_0(\theta)|^2)\,dx \le \frac{C}{s^2}\sum_{i=0}^{2}(I(y_i)+I(z_i))$$

$$\leq \frac{C}{s^2}\int_{Q_L} e^{2s\phi}(b^2+c^2+d^2)\chi^2\;dx\;dt+\frac{C}{s^2}\int_{Q_L} e^{2s\phi}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\;dx\;dt+Cse^{2sd_1}+\frac{C}{s}e^{2sd_2}\tilde{F}_0(\gamma_L).$$

Since $\phi \leq \phi(\theta)$ we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}(|y_0(\theta)|^2+|\partial_t y_0(\theta)|^2+|z_0(\theta)|^2+|\partial_t z_0(\theta)|^2)\;dx \leq \frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\;dx$$

$$+\frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\;dx\;dt+Cse^{2sd_1}+\frac{C}{s}e^{2sd_2}\tilde{F}_0(\gamma_L). \tag{3.38}$$

• Third step: here we estimate $\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(b^2+c^2+d^2)\;dx$ as in Theorem 3.1 with two different sets of conditions A and B. We recall that each function f precendently defined is denoted either f_A or f_B when it is related either by the conditions A or B.
For the coefficient b we can write from the first equation of (3.36)

$$-b\eta\chi\phi_2(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)=\tilde{u}_B(\partial_t y_{0A}-\Delta y_{0A}-\alpha\phi_1 y_{0A}-\beta\phi_2 z_{0A}-\alpha f_1\eta\chi\tilde{u}_A-S_{1A})$$

$$-\tilde{u}_A(\partial_t y_{0B}-\Delta y_{0B}-\alpha\phi_1 y_{0B}-\beta\phi_2 z_{0B}-\alpha f_1\eta\chi\tilde{u}_B-S_{1B}).$$

Note that the terms in f_1 disappear in the above equality. For the coefficients c and d we use the second equation of (3.36) and proceed as in Theorem 3.1. Indeed, for example for c, we have

$$c\eta\chi\phi_3(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)=\tilde{w}_B(\partial_t z_{0A}-\Delta z_{0A}-\gamma\phi_3 y_{0A}-\delta\phi_4 z_{0A}-S_{2A})$$

$$-\tilde{w}_A(\partial_t z_{0B}-\Delta z_{0B}-\gamma\phi_3 y_{0B}-\delta\phi_4 z_{0B}-S_{2B}).$$

Therefore by hypothesis (3.3) and (3.38) we obtain for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\;dx \leq \frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\;dx\;dt+Cse^{2sd_1}+Ce^{2sd_2}F_2(\theta) \tag{3.39}$$

with $F_2(\theta)=\tilde{F}_{0A}(\gamma_L)+\tilde{F}_{0B}(\gamma_L)+\|\Delta y_{0A}(\theta)\|_{L^2(\Omega_L)}+\|\Delta y_{0B}(\theta)\|_{L^2(\Omega_L)}+\|\Delta z_{0A}(\theta)\|_{L^2(\Omega_L)}+\|\Delta z_{0B}(\theta)\|_{L^2(\Omega_L)}$.
• Fourth step: we estimate now $\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\;dx\;dt$. Here again we use the two different sets of coefficients A and B. From (3.36) for y_{0A} and y_{0B}, we get

$$\alpha\eta\chi f_1(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)=\tilde{w}_B(\partial_t y_{0A}-\Delta y_{0A}-\alpha\phi_1 y_{0A}-\beta\phi_2 z_{0A}-S_{1A})$$

$$-\tilde{w}_A(\partial_t y_{0B}-\Delta y_{0B}-\alpha\phi_1 y_{0B}-\beta\phi_2 z_{0B}-S_{1B}). \tag{3.40}$$

Applying (3.40) for $t=\theta$, by hypotheses (3.3) and (3.9), using again (3.38) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(f_1(\theta))^2\;dx \leq \frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\;dx\;dt$$

$$+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx + Cse^{2sd_1} + Ce^{2sd_2}F_2(\theta). \tag{3.41}$$

Deriving now (3.40) with respect to t, we have

$$(\partial_t f_1)\alpha\eta(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) + f_1\partial_t(\alpha\eta\chi(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A)) =$$
$$\partial_t(\tilde{w}_B(\partial_t y_{0A} - \Delta y_{0A} - \alpha\phi_1 y_{0A} - \beta\phi_2 z_{0A} - S_{1A}) - \tilde{w}_A(\partial_t y_{0B} - \Delta y_{0B} - \alpha\phi_1 y_{0B} - \beta\phi_2 z_{0B} - S_{1B}))$$

and evaluating this last equation at $t=\theta$, still by hypotheses (3.3) and (3.9), we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(\partial_t f_1(\theta))^2\ dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(f_1(\theta))^2\ dx$$
$$+C\int_{\Omega_L} e^{2s\phi(\theta)}\sum_{i=0}^{1}(|\partial_t^i z_{0A}(\theta)|^2 + |\partial_t^i z_{0B}(\theta)|^2) + Ce^{2sd_2}F_3(\theta) \tag{3.42}$$

with

$$F_3(\theta) = \sum_{k=0}^{2}(\|\partial_t^k y_{0A}(\theta)\|^2_{L^2(\Omega_L)} + \|\partial_t^k y_{0B}(\theta)\|^2_{L^2(\Omega_L)})$$
$$+\sum_{k=0}^{1}(\|\partial_t^k \Delta y_{0A}(\theta)\|^2_{L^2(\Omega_L)} + \|\partial_t^k \Delta y_{0B}(\theta)\|^2_{L^2(\Omega_L)}).$$

From (3.38), (3.41) and (3.42) we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2((f_1(\theta))^2 + (\partial_t f_1(\theta))^2)\ dx \le \frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt$$
$$+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx + Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta) \tag{3.43}$$

with $F_4(\theta) = F_2(\theta) + F_3(\theta)$.
Moreover by Taylor's formula, we have

$$f_1(t) = f_1(\theta) + \partial_t f_1(\theta)(t-\theta) + \partial_t^2 f_1(c_\theta)\frac{(t-\theta)^2}{2} \text{ and } \partial_t f_1(t) = \partial_t f_1(\theta) + \partial_t^2 f_1(c'_\theta)(t-\theta)$$

with $c_\theta, c'_\theta \in [0,T]$. Therefore, since $\tilde{\phi_1} \in \Lambda_3(M_3)$ the admissible set of coefficients, we get

$$\sum_{i=0}^{2}(\partial_t^i f_1)^2 \le C(f_1(\theta))^2 + (\partial_t f_1(\theta))^2),$$

so from (3.43) we deduce that for s sufficiently large

$$\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt \le \frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx + Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta). \tag{3.44}$$

• Fifth and last step: now addding (3.39) and (3.44) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\,dx + \int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\,dx\,dt \le Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta).$$

So

$$\int_{\Omega_l} e^{2s\phi(\theta)}(b^2+c^2+d^2)\,dx + \int_{\Omega_l\times(0,T)} e^{2s\phi(\theta)}(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\,dx\,dt \le Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta)$$

and we conclude as for Theorem 3.1 by optimizing the above inequality with respect to s.

Remark 2 • If the admissible set of coefficients is $\Lambda_3'(M_3)$ (thus less restrictive than $\Lambda_3(M_3)$), then we would have to derive $p-1$ times (3.40) with respect to t and that would demand more regularity for the observation terms on u.
• On the contrary if the admissible set of coefficients is $\Lambda_3''(M_3)$, so more restrictive than $\Lambda_3(M_3)$ (or if $\tilde{\phi}_1 \in C^2([0,T])$ is such that $\tilde{\phi}_1(\theta) \neq \phi_1(\theta)$ and $\frac{\sup_{t\in[0,T]}|\partial_t^i(\phi_1-\tilde{\phi}_1)(t)|}{|\phi_1(\theta)-\tilde{\phi}_1(\theta)|} \le M_3$ for $i=0,1,2$), then we can drop (3.42) and (3.43) in the above proof. Therefore the result remains valid without $F_3(\theta)$ and so $F_4(\theta) = F_2(\theta)$. Thus the observations terms on u are only $\|(u_A-\tilde{u_A})(\cdot,\theta)\|^2_{H^2(\Omega_L)}$ and $\|(u_B-\tilde{u_B})(\cdot,\theta)\|^2_{H^2(\Omega_L)}$.

3.3.4 Proof of Theorem 3.4 Here again we follow the method described before. Let $V_A=(u_A,w_A)$ (resp. $\tilde{V}_A=(\tilde{u}_A,\tilde{w}_A)$) be a strong solution of (1.3) associated with (ρ,G,A,Θ) defined by (1.2) and (1.4) (resp. $(\tilde{\rho}_3,G,A,\tilde{\Theta})$). Consider also $V_B=(u_B,w_B)$ (resp. $\tilde{V}_B=(\tilde{u}_B,\tilde{w}_B)$) a strong solution of (1.3) associated with (ρ,G,B,Θ) (resp. $(\tilde{\rho}_3,G,B,\tilde{\Theta})$).
• As before, in a first step we define

$$V=(u,w)=V_A,\ \tilde{V}=(\tilde{u},\tilde{w})=\tilde{V}_A,\ U=u-\tilde{u},\ W=w-\tilde{w}, b=\beta-\tilde{\beta},\ c=\gamma-\tilde{\gamma},\ d=\delta-\tilde{\delta}$$

and also

$$H=\Theta_1-\tilde{\Theta}_1=\nabla h \text{ with } h=\xi_1-\tilde{\xi}_1.$$

Recall that for $i=0,1$,

$$y_0=\eta\chi U,\ z_0=\eta\chi W,\ y_1=\partial_i y_0,\ z_1=\partial_t z_0.$$

Then

$$\begin{cases} \partial_t y_0 = \Delta y_0 + \alpha\phi_1 y_0 + \beta\phi_2 z_0 + \Theta_1\cdot\nabla y_0 + \Theta_2\cdot\nabla z_0 + b\eta\chi\phi_2\tilde{w} + \eta\nabla(\chi h)\cdot\nabla\tilde{u} + T_1 \text{ in } Q_L,\\ \partial_t z_0 = \Delta z_0 + \gamma\phi_3 y_0 + \delta\phi_4 z_0 + \Theta_3\cdot\nabla y_0 + \Theta_4\cdot\nabla z_0 + c\eta\chi\phi_3\tilde{u} + d\eta\chi\phi_4\tilde{w} + T_2 \text{ in } Q_L,\\ y_0 = z_0 = 0 \text{ on } \partial\Omega_L\times(0,T) \end{cases} \tag{3.45}$$

with

$$T_1 = (\partial_t \eta)\chi U - (\Delta\chi)\eta U - 2\nabla\chi \cdot \nabla(\eta U) - \eta U \Theta_1 \cdot \nabla\chi - \eta W \Theta_2 \cdot \nabla\chi - \eta h \nabla\tilde{u} \cdot \nabla\chi$$

$$T_2 = (\partial_t \eta)\chi W - (\Delta\chi)\eta W - 2\nabla\chi \cdot \nabla(\eta W) - \eta U \Theta_3 \cdot \nabla\chi - \eta W \Theta_4 \cdot \nabla\chi.$$

And

$$\begin{cases} \partial_t y_1 & = \Delta y_1 + \alpha\phi_1 y_1 + \beta\phi_2 z_1 + \Theta_1 \cdot \nabla y_1 + \Theta_2 \cdot \nabla z_1 + b\eta\chi\partial_t(\phi_2\tilde{w}) + \eta\nabla(\chi h)\cdot\nabla\partial_t\tilde{u} + T_3 \\ & \quad \text{in } Q_L, \\ \partial_t z_1 & = \Delta z_1 + \gamma\phi_3 y_1 + \delta\phi_4 z_1 + \Theta_3 \cdot \nabla y_1 + \Theta_4 \cdot \nabla z_1 + c\eta\chi\partial_t(\phi_3\tilde{u}) + d\eta\chi\partial_t(\phi_4\tilde{w}) + T_4 \\ & \quad \text{in } Q_L, \\ y_1 & = z_1 = 0 \text{ on } \partial\Omega_L \times (0,T) \end{cases}$$

with

$$T_3 = \alpha y_0 \partial_t\phi_1 + \beta z_0 \partial_t \phi_2 + \partial_t\eta(b\chi\phi_2\tilde{w} + \nabla(\chi h)\cdot\nabla\tilde{u}) + \partial_t T_1,$$

$$T_4 = \gamma y_0 \partial_t\phi_3 + \delta z_0 \partial_t\phi_4 + \partial_t\eta(c\chi\phi_3\tilde{u} + d\chi\phi_4\tilde{w}) + \partial_t T_2.$$

Thus we obtain

$$\sum_{i=0}^{1}(I(y_i)+I(z_i)) \le C\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2)\ dx\ dt + Cs^3 e^{2sd_1}$$

$$+Cs\sum_{i=0}^{1}\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt.$$

We deduce that (see the third step of Theorem 3.1)

$$\sum_{i=0}^{1}\int_{\Omega_L} e^{2s\phi(\theta)}(|y_i(\theta)|^2+|\nabla y_i(\theta)|^2+|z_i(\theta)|^2+|\nabla z_i(\theta)|^2)\ dx + \int_{\Omega_L} e^{2s\phi(\theta)}(|\Delta y_0(\theta)|^2+|\Delta z_0(\theta)|^2)\ dx$$

$$\le Cs^2\sum_{i=0}^{1}(I(y_i)+I(z_i))$$

$$\le Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2+|\nabla(\chi h)|^2)\ dx\ dt + Cs^5 e^{2sd_1} + Cs^3 e^{2sd_2}F_0(\gamma_L) \tag{3.46}$$

with $F_0(\gamma_L)$ defined by (3.17).

• In a second step we consider the solutions of (1.3) associated with two different sets of initial conditions A and B and we recall that each function f precendently defined is denoted either f_A or f_B when it is related either by the conditions A or B. As in the fourth step of Theorem 3.1 we have a similar estimate to (3.23) for the coefficients c and d. Indeed, writing (3.45) for z_{0A} and z_{0B}, by the hypothesis (3.3) and from (3.46) we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}(c^2+d^2)\chi^2\ dx \le$$

$$Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2+|\nabla(\chi h)|^2)\ dx\ dt + Cs^5e^{2sd_1} + Cs^3e^{2sd_2}F_1(\gamma_L) \tag{3.47}$$

with $F_1(\gamma_L)$ defined by (3.21). Now we eliminate b in (3.45) in order to estimate the coefficient h and we evaluate at $t=\theta$. We use here the partial differential operator P defined in Lemma 3.2.

$$P(\chi h) = \tilde{w}_B(\theta)\nabla(\chi h)\cdot\nabla\tilde{u}_A(\theta) - \tilde{w}_A(\theta)\nabla(\chi h)\cdot\nabla\tilde{u}_B(\theta)$$

$$\begin{aligned} P(\chi h) = \tilde{w}_B(\theta)[&\partial_t y_{0A}(\theta) - \Delta y_{0A}(\theta) - \alpha\phi_1 y_{0A}(\theta) - \beta\phi_2 z_{0A}(\theta) \\ &- \Theta_1\cdot\nabla y_{0A}(\theta) - \Theta_2\cdot\nabla z_{0A}(\theta) - T_{1A}(\theta)] \\ -\tilde{w}_A(\theta)[&\partial_t y_{0B}(\theta) - \Delta y_{0B}(\theta) - \alpha\phi_1 y_{0B}(\theta) - \beta\phi_2 z_{0B}(\theta) - \Theta_1\cdot\nabla y_{0B}(\theta) - \Theta_2\cdot\nabla z_{0B}(\theta) - T_{1B}(\theta)]. \end{aligned} \tag{3.48}$$

From Lemma 3.2 we have

$$s^2\int_{\Omega_L} e^{2s\phi(\theta)}(\partial_{x_i}(h\chi))^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}|P(\partial_{x_i}(\chi h))|^2\,dx.$$

So taking the space derivative with respect to x_i (for $i=1,\cdots,n$) in (3.48), from (3.46) we get that

$$\begin{aligned} s^2\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx + \\ Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2)\,dx\,dt \\ + Ce^{2sd_2}(\|y_{0A}(\theta)\|^2_{H^3(\Omega_L)} + \|y_{0B}(\theta)\|^2_{H^3(\Omega_L)}) + Cs^5e^{2sd_1} + Cs^3e^{2sd_2}F_1(\gamma_L) \end{aligned}$$

and for s sufficiently large,

$$\begin{aligned} s^2\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx \le Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2)\,dx\,dt \\ + Cs^5e^{2sd_1} + Cs^3e^{2sd_2}F_5(\theta) \end{aligned} \tag{3.49}$$

with $F_5(\theta) = F_1(\gamma_L) + \|y_{0A}(\theta)\|^2_{H^3(\Omega_L)} + \|y_{0B}(\theta)\|^2_{H^3(\Omega_L)}$. Now we look at the coefficient b. We also use (3.45) for y_{0A} and y_{0B}

$$\begin{aligned} -b\eta\chi\phi_2(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) = \tilde{u}_B(\partial_t y_{0A} - \Delta y_{0A} - \alpha\phi_1 y_{0A} - \beta\phi_2 z_{0A} - \Theta_1\cdot\nabla y_{0A} - \Theta_2\cdot\nabla z_{0A} \\ -\eta\nabla(\chi h)\cdot\nabla\tilde{u}_A - T_{1A}) - \tilde{u}_A(\partial_t y_{0B} - \Delta y_{0B} - \alpha\phi_1 y_{0B} - \beta\phi_2 z_{0B} - \Theta_1\cdot\nabla y_{0B} \\ -\Theta_2\cdot\nabla z_{0B} - \eta\nabla(\chi h)\cdot\nabla\tilde{u}_B - T_{1B}). \end{aligned} \tag{3.50}$$

Therefore, evaluating (3.50) at $t=\theta$, still using hypothesis (3.3), from (3.46) we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}b^2\chi^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx$$

$$+Cs^2 \int_{Q_L} e^{2s\phi}((b^2 + c^2 + d^2)\chi^2 + |\nabla(\chi h)|^2) \, dx \, dt + Cs^5 e^{2sd_1} + Cs^3 e^{2sd_2} F_1(\gamma_L). \quad (3.51)$$

Thus from (3.49)-(3.51) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}(b\chi)^2 \, dx \le Cs^2 \int_{Q_L} e^{2s\phi}((b^2 + c^2 + d^2)\chi^2 + |\nabla(\chi h)|^2) \, dx \, dt$$
$$+ Cs^5 e^{2sd_1} + Cs^3 e^{2sd_2} F_5(\theta). \quad (3.52)$$

Finally adding (3.47), (3.49), (3.52), as in the proof of Theorem 3.1 we can neglect $s^2 \int_{Q_L} e^{2s\phi}((b^2 + c^2 + d^2)\chi^2 + |\nabla(\chi h)|^2) \, dx \, dt$ by the left-hand side so we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}((b^2 + c^2 + d^2)\chi^2 + |\nabla(\chi h)|^2) \le Cs^5 e^{2sd_1} + Cs^3 e^{2sd_2} F_5(\theta)$$

and we conclude as in Theorem 3.1.

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