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Ascoli-Arzela-Theory based on continuous convergence in an (almost) non-Hausdorff setting 2

1 Introduction

We start with the paper [1] and thus come back to continuous convergence and to the characterization of compactness with respect to this convergence structure for the space $C(X, Y)$ of continuous functions, where X and Y are topological spaces. More generally we can use for X, Y convergence spaces, as was done for instance in [11] and [15]. But in the first paper of this title X and Y were topological spaces and we will continue with this assumption.

What is the aim of our paper?

1. In the main theorems [15, (3.24), (3.27)], [1, 33] and corollary [11, 10] necessary and sufficient conditions were given to ensure that $H \subseteq C(X, Y)$ is relatively compact w. r. t. continuous convergence. Here, as a corollary, we characterize compactness of H.

2. In the papers [11], [1] not provided examples which show that the assumptions in our theorems (for instance that Y is Hausdorff) we cannot omit.

3. It is known for long time that the important notion of equicontinuity can be characterized using the canonical map as used in duality theory (embedding in a second dual) ([7], and [15, theorem 4.36]). In the paper [8] and especially in the book [12] this approach was extended to include even continuity too. But the two Ascoli-Arzela theorems ([12, (13.15), (13.21)]) based on this approach are not correct. We will show this by an instructive counter example. And we will give some comments for this situation.

2 Compactness in $(C(X,Y), c\text{-lim})$

We will use the following notion of relative compactness: Let X be a topological space, then $A \subset X$ is called relatively compact iff for each ultrafilter π on X,

$$A \in \pi \Longrightarrow \exists\, x \in X : \pi \longrightarrow x\,. \quad \text{(see [16], [3])}$$

We still need a lemma.

Lemma 2.1 *Let X be a topological space, Y a Hausdorff topological space; let η be a topology (lim a convergence structure) on $C(X,Y)$ with $\tau_p \leq \eta$ (τ_p-lim $\leq$ lim). If $H \subseteq C(X,Y)$ is η-compact, then H is τ_p-closed in Y^X.*

For a (simple) proof see lemma [3, theorem 3.1].

Now [1, theorem 33] states: If X, Y topological spaces, $H \subseteq C(X,Y)$, $H \neq \emptyset$ and we consider for H the two conditions:

(α) $\forall x \in X : H(x) = \{f(x) | f \in H\}$ is relatively compact.

(β) H is evenly continuous.

then the following holds:

1. Let X be Hausdorff; H relatively c-compact $\Longrightarrow (\alpha), (\beta)$.
2. Let X be a T_3-space, $(\alpha), (\beta) \Longrightarrow H$ is in $(C(X,Y), c\text{-lim})$ relatively compact.

Theorem 2.2 [Corollary of [1, theorem33]] *Let be X, Y topological spaces, $H \subseteq C(X,Y)$; for H we consider the conditions:*

(α) *$\forall x \in X : H(x)$ is relatively compact*

(β) *H is evenly continuous*

(γ) *H is in $Y^X \tau_p$-closed.*

Then hold:

1. Let Y be Hausdorff, H is c-lim-compact $\Longrightarrow (\alpha), (\beta), (\gamma)$.
2. $(\alpha), (\beta), (\gamma) \Longrightarrow H$ is in $(C(X,Y), c\text{-lim})$ compact.

Proof. 1. H c-lim-compact in $C(X,Y) \Longrightarrow H$ is c-lim-relatively compact; then follows $(\alpha), (\beta)$ by theorem 33. Now since $\tau_p \leq c$-lim holds in $C(X,Y)$, lemma 2.1 yields condition (γ) too.

2. By the Tychonoff theorem: $(\alpha) \Longrightarrow H$ is τ_p-relatively compact in Y^X, hence H is τ_p-compact in Y^X by (γ); $H \subseteq C(X,Y) \Longrightarrow H$ is in $C(X,Y)\tau_p$-compact too. Now let π be an ultrafilter on $C(X,Y)$ with $H \in \pi$: we find $g \in H : \pi \xrightarrow{\tau_p} g$, but then follows: $\pi \xrightarrow{c} g$ by (β) and by [1, theorem 31]. Hence H is in $C(X,Y)$ c-lim-compact.

3 Examples

For the construction of our examples we need a result of S. Mrowka which we found in [6] and a corollary of this result.

Proposition 3.1 *Let (X,τ) be a Hausdorff topologigal space, where of course τ means the system of all open sets of X; let $(A_i)_{i\in I}$ be a net in 2^X, 2^X is the set of closed sets of X. Then (A_i) has a subnet Kuratowski-Hausdorff-converging in 2^X.*

Proof. We know: a net(B_i) (from 2^X) converges iff $L_sB_i \subseteq L_iB_i$ holds, meaning:

$$\forall x \in X, \forall G \in \tau : x \in G \text{ and } G \cap B_i \neq \emptyset$$

for all i from a confinal set of I it follows that eventually $G \cap B_i \neq \emptyset$, since $\{G \in \tau | x \in G\}$ is a basis of the neighborhood filter $\underline{U}(x)$. Here by Ls, Li we denote the limit superior and limit inferior respectively.

Now we consider the two-point space $\{0,1\}$ provided with discrete topology.

$$\forall i \in I: \text{ let be } f_i \in \{0,1\}^\tau : \forall G \in \tau : f_i(G) = \begin{cases} 1, & A_i \cap G \neq \emptyset \\ 0, & A_i \cap G = \emptyset\,. \end{cases}$$

Obviously, the map $f_i \to A_i$ is injective.

$\{0,1\}^\tau$ with pointwise topology τ_p is compact by the Tychonoff theorem, and hence for (f_i) there exists a subnet (f_{i_k}) and a f from $\{0,1\}^\tau$ such that $f_{i_k} \xrightarrow{\tau_p} f$. Now we want to show $Ls(A_{i_k}) \subseteq Li(A_{i_k}) : \forall (x,G) \in X \times \tau : x \in Ls(A_{i_k})$ and $x \in G$: there exists a confinal subset

$$K_1 \subseteq K \text{ such that } \forall k \in K_1 : A_{i_k} \cap G \neq \emptyset$$

implying $\forall k \in K_1 : f_{i_k}(G) = 1$. We assume that $f(G) = 0$ holds, $\{0\}$ is open and for our net $(f_{i_k})_{k\in K}$ holds: $f_{i_k}(G) \longrightarrow f(G)$ implying eventually $f_{i_k}(G) = 0$, yielding a contradiction because K_1 is kofinal in K and $\forall k \in K_1 : f_{i_k}(G) = 1$.

Hence we have $f(G) = 1$; now $f_{i_k}(G) \longrightarrow f(G) = 1$ and $\{1\}$ is open implies: eventually $f_{i_k}(G) = 1$ and thus eventually $A_{i_k} \cap G \neq \emptyset$ showing that $x \in Li(A_{i_k})$.

Corollary 3.2 *Let X be a Hausdorff topological space, F the Sierpinski-space with open sets: $\emptyset, \{0\}, \{0,1\} = F$. Then $(C(X,F), c\text{-lim}) = C_c(X,F)$ is compact.*

Proof. Let χ_{A_i} be a net from $C(X,F)$, meaning that all A_i are closed sets in X, hence $\forall i \in I : A_i \in 2^X$. By the proposition 3.1 (A_i) has a subnet (A_{i_k}) converging to a set $A \in 2^X$. Hence we get $LsA_{i_k} = LiA_{i_k} = A$, $LsA_{i_k} = A$ shows:

$$\chi_{A_{i_k}} \xrightarrow{c} \chi_A \quad \text{in} \quad C(X,F)\,,$$

hence we found a subnet converging continuously to χ_A.

Thus $C_c(X,F)$ is compact.

At first we show that lemma 2.1 does not work if Y is not Hausdorff.

Example 3.3 Let be $X = \mathbb{R}$, the reals with Euclidian topology and F the Sierpinski-space. By corollary 3.2 $(C(\mathbb{R},F), c\text{-lim})$ is compact; the pointwise topology τ_p is splitting and thus $\tau_p\text{-lim} \leq c\text{-lim}$.

But by example [15, (2.16) (b)] $Cc(\mathbb{R},F)$ is not closed in F^X.

The basic result that for conjoining topologies the (relative) compactness of $H \subseteq C(X,Y)$ implies that H is evenly continuous is well-known ([10, chapt. 7, theorem 20]; [1, theorem 32]; [15, theorem 3.21]).

For a concrete formulation we take here [1, theorem 32]:

Let X be a topological space, Y a Hausdorff topological space and let $H \subseteq C(X,Y) \subseteq Y^X$. Let lim be a convergence structure for $C(X,Y)$ such that

1. H is in $(C(X,Y), \lim)$ relatively compact
2. lim is conjoining for $C(X,Y)$

Then H is evenly continuous.

In theorem [15, theorem (3.21)] X is a convergence space and Y is a Hausdorff pseudotopological convergence space.

Our next example shows that we cannot omit the assumption that Y is Hausdorff.

Example 3.4 We use the same space as in example 3.3. Again we have a space $Y = F$ which is not Hausdorff. We have here $H = C(\mathbb{R},F)$ and the convergence structure lim for $C(\mathbb{R},F)$ is the continuous convergence: $\lim = c\text{-lim}$; c-lim is conjoining for $C(\mathbb{R},F)$ and $C_c(\mathbb{R},F)$ is compact. We will show that $C(\mathbb{R},F)$ is not evenly continuous. $\mathbb{R}$ and F are first countable spaces and hence by [15, theorem 3.18] we can use sequences instead of filters or nets to characterize even continuity: For $n \in \mathbb{N}$, $n \geq 1$ let $A_n = \left[\frac{1}{n}, 1\right] \subseteq \mathbb{R}$ and χ_{A_n} denotes, as usual, the characteristic function of A_n. For $0 \in \mathbb{R}$ we find $\chi_{A_n} \to 0 \in F$, $\frac{1}{n} \to 0 \in \mathbb{R}$.

Now we assume that $C(\mathbb{R},F)$ is evenly continuous; then follows $\chi_{A_n}\left(\frac{1}{n}\right) \to 0 \in F$; since $\{0\}$ is open in F there exists $n_0 \in \mathbb{N}$: $\forall n \geq n_0$: $\chi_{A_n}\left(\frac{1}{n}\right) = 0 \in F$, but $\forall n \in \mathbb{N}, n \geq 1 : \chi_{A_n}\left(\frac{1}{n}\right) = 1 \in F$, a contradiction.

Remark 3.5 Example 3.4 of course works for assertion 1 of theorem 2.2 too. Since $Cc(\mathbb{R}, F)$ also is relatively compact and c-lim is conjoining our example shows that we cannot omit in assertion 1 of [1, theorem 33] that Y is Hausdorff.

We consider a nice topological space Y, meaning that Y is at least Hausdorff and a topology τ for $C(X,Y)$. The fact that $H \subseteq C(X,Y)$ is τ-compact must not imply that H is evenly continuous if τ is not conjoining for $C(X,Y)$. We will explain this situation by an example. As concrete topologies τ we consider the pointwise topology τ_p and the uniform topology τ_u.

Example 3.6 We use an example from classical analysis of a sequence of functions from

$$C([0,1],\mathbb{R}) : \forall (n,x) \in (\mathbb{N} - \{0\}) \times [0,1] : f_n : f_n(x) = \frac{nx}{1+(nx)^2}, f_0 : \forall x \in [0,1] : f_0(x) = 0\,.$$

then holds:

1. $f_n \xrightarrow{\tau_p} f_0$
2. (f_n) does not converge uniformly to f_0

Proof. 1. $\forall n, n \geq 1 : f_n(0) = 0 \to 0 = f_0(0)$;
$\forall x \in (x,1] : \frac{(nx)^2}{1+(nx)^2} \leq 1 \Longrightarrow |f_n| = f_n = \frac{nx}{1+(nx)^2} \leq \frac{1}{x} \cdot \frac{1}{n} \to 0$, hence $|f_n(x) - f_0(x)| \to 0$ for $n \to +\infty$.

2. $\forall n \geq 1 : x = \frac{1}{n} \in (0,1]$ and $f_n\left(\frac{1}{n}\right) = \frac{1}{2}$. But then (f_n) cannot converges uniformly to f_0 on $[0,1]$.

Now let be $H = \{f_n | n \geq 1\} \cup \{f_0\} \subseteq C([0,1],\mathbb{R})$.

Then holds:

1. H is τ_p-compact
2. H is not evenly continuous
3. τ_p is not conjoining for $C([0,1],\mathbb{R})$.

Proof. 1. is obvious

2. (f_n) does not converge continuously to $f_0 : [0,1]$ is compact (and Hausdorff) implying that then c-lim $= \tau_u$-lim, yielding that$f_n \to f_0$ uniformly, a contradiction.

   If we assume that H is evenly continuous then $f_n \xrightarrow{\tau_p} f_0 \Longrightarrow f_n \xrightarrow{c} f_0$ by the basis [1, theorem 31], a contradiction.

3. If τ_p is conjoining then c-lim $\leq \tau_p$-lim since c-lim is splitting (and conjoining) for $C([0,1],\mathbb{R})$ implying $f_n \xrightarrow{\tau_p} f_0 \Longrightarrow f_n \xrightarrow{c} f_0$, a contradiction.

Finally, we will show that assertion 2 of theorem 2.2 is not true if condition (γ) is not fulfilled.

Example 3.7 Let be $X = Y = \mathbb{R}$ we consider the sequence (f_n), $n \in \mathbb{N}, n \geq 1 : f_n : f_n(x) = \frac{1}{n}x$, $\forall x \in \mathbb{R}$; again let be $f_0 : \forall x \in \mathbb{R} : f(x) = 0$, hence $\forall n \in \mathbb{N} : f_n \in C(\mathbb{R}, \mathbb{R})$.

$H = \{f_n | n \in \mathbb{N}, n \geq 1\}$; for fixed $x \in \mathbb{R} : H(x) = \left\{\frac{1}{n}x | n \in \mathbb{N}, n \geq 1\right\}$ is bounded and hence relatively compact in $Y = \mathbb{R}$. Thus condition (α) holds for H.

Moreover let be $x \in \mathbb{R}$;

$$\forall(\varepsilon, n) \in (0, +\infty) \times (\mathbb{N} - \{0\})$$

let be $\delta = \varepsilon$ and $y \in U_\delta(x) : |f_n(y) - f_n(x)| = |\frac{1}{n}y - \frac{1}{n}x| = \frac{1}{n}|y - x| \leq |y - x| < \varepsilon$; hence H is equicontinuous on $\mathbb{R}$ which implies that H is evenly continuous showing that condition (β) is fulfilled too.

We see at once that $f_n \xrightarrow{\tau_p} f_0$ and even $f_n \xrightarrow{c} f_0$ hold. $H \cup \{f_0\}$ is τ_p-compact; we have $f_0 \notin H$ and $f_m \neq f_n \forall(m, n) \in \mathbb{N} \times \mathbb{N}$, $m \neq n$, $m \geq 1$, $n \geq 1$, $f_n \xrightarrow{\tau_p} f_0$ in $C(\mathbb{R}, \mathbb{R}) \Longrightarrow f_n \xrightarrow{\tau_p} f_0$ in $\mathbb{R}^{\mathbb{R}}$: each τ_p-neighbourhood of f_0 in $\mathbb{R}^{\mathbb{R}}$ contains infinitively many functions f_n implying that f_0 is a τ_p-cluster point of H. Thus H is not τ_p-closed in $\mathbb{R}^{\mathbb{R}}$ and hence not τ_p-compact since $Y = \mathbb{R}$ is Hausdorff. $H \subseteq C(\mathbb{R}, \mathbb{R}) \Longrightarrow H$ is not τ_p-compact in $C(\mathbb{R}, \mathbb{R})$ implying that H is not c-lim-compact in $C(\mathbb{R}, \mathbb{R})$ since τ_p-lim $\leq c$-lim holds.

4 Duality and the Ascoli-Arzela theorems

In the introduction we mentioned that the equicontinuity of a subset $H \subseteq C(X, Y)$ can be characterized by embedding of X into a function space using the canonical map. In [8] this approach was extended to include even continuity and also the topological equicontinuity of Royden.

At length we find it in the book [12]. We want to consider here equicontinuity and even continuity. In [2], [4] and [5] R. Bartsch and I developed and studied a general duality system

$$(X, Y, X^d, X^{dd}, J : X \to X^{dd})$$

where X^d is the first dual space of X with respect to Y, X^{dd} is the second dual space of X w.r.t. Y and J denotes the canonical map as is known from classical duality examples.

And we can include these characterization of equicontinuity and even continuity into this general scheme:

Let X, Y be topological spaces and $H \subseteq C(X, Y)$. We can consider (H, τ_p) as the redefined first dual space of X w.r.t. Y according to [2, 4.3., p. 284]: $X^d = (H, \tau_p)$. by definition [2, 4.1.] we see that $H = X^d$ has no defect since in H there are no algebraic operations defined.

Hence by [2, definition 4.2.] and [4, definition 2.2.] the second dual space of X w. r. t. Y is $X^{dd} = C((H, \tau_p), Y)$. The canonical map

$$\begin{aligned} J &: X \to C((H, \tau_p), Y), \\ \forall x \in X &: Jx = \omega(x, \cdot)\,, \\ \omega(x, \cdot) &: (H, \tau_p) \to Y\,; \\ \forall h \in H &: \omega(x, \cdot)(h) = \omega(x, h) = h(x)\,. \end{aligned}$$

We now need the convergence structure of strict (strong) continuous convergence.

Generalizing a formulation, where sequences were used ([9]), in ([15, 2.25]) I defined:

Definition 4.1 *Let X, Y be topological spaces, Φ a filter in Y^X; we say that Φ converges strictly continuous to f, $\Phi \stackrel{\mathrm{str}\,c}{\longrightarrow} f$, iff for each $y \in Y$ and each filter φ on X : $f\varphi \to y \Longrightarrow \omega(\Phi \times \varphi) = \Phi(\varphi) \to y$.*

Remark 4.2

1. Of course, a net (f_i) from Y^X converges strictly continuous to $f \in Y^X$ iff for each $y \in Y$ and each net (x_k) from X holds: $f(x_k) \to y \Longrightarrow f_i(x_k) \to y$
2. $\mathrm{str}\,c$-lim is conjoining for $C(X, Y)$ since we see at once that c-lim $\leq \mathrm{str}\,c$-lim holds.
3. Strict continuous convergence has similar properties as of continuous convergence, especially $\mathrm{str}\,c$-lim is a pseudotopological convergence structure and if Y Hausdorff then $(C(X, Y), \mathrm{str}\,c\text{-lim})$ is Hausdorff too.
4. If X is compact and Hausdorff then c-lim $= \mathrm{str}\,c$-lim on $C(X, Y)$ (see [17], and also [13]).

Now we come to the characterizations of even/equi-continuity as already announced.

Proposition 4.3 *Let X, Y be topological spaces, $H \subseteq C(X, Y)$; equivalent are:*

(1) $J : X \to (C((H, \tau_p), Y), \mathrm{str}\,c\text{-lim})$ *is continuous*

(2) H *is evenly continuous*

Proof. (1)$\Longrightarrow$(2): $\forall (x, y) \in X \times Y$, for each net (x_k) in X s. th. $x_k \to x$, for each net (h_i) from H s. th. $h_i(x) \to y$ we want to show: $h_i(x_k) \to y$.

Now by (1) $x_k \to x \Longrightarrow Jx_k \to Jx$, meaning that $\omega(x_k, \cdot) \stackrel{\mathrm{str}\,c}{\longrightarrow} \omega(x, \cdot)$.

$$\forall k \in K : \omega(x_k, \cdot) : (H, \tau_p) \to Y$$

is continuous and $\omega(x, \cdot) : (H, \tau_p) \to Y$ is continuous by [2, lemma 4.1., (1)] and hence $\omega(x_k, \cdot), \omega(x, \cdot) \in C((H, \tau_p), Y)$.

By the definition of strict continuous convergence and since we know that $h_i(x) \to y$, which means $\omega(x,\cdot)(h_i) \to y$ we get at once:

$$\omega(x_k,\cdot)(h_i) = h_i(x_k) \to y\,.$$

Hence H is evenly continuous.

(2)$\Longrightarrow$(1): $\forall (x,y) \in X \times Y : \forall (x_k), (x_k)$ net from X s.th. $x_k \to x$, we will show: $Jx_k \to Jx$ w.r.t. str c-lim: $\omega(x_k,\cdot) \xrightarrow{\text{str}\,c} \omega(x,\cdot)$: let (h_i) be a net from H such that $\omega(x,\cdot)(h_i) \to y$, hence $h_i(x) \to y$; now by (2): $x_k \to x$ and $h_i(x) \to y \Longrightarrow h_i(x_k) \to y$, meaning $\omega(x_k,\cdot)(h_i) \to y$. Thus $\omega(x_k,\cdot) \xrightarrow{\text{str}\,c} \omega(x,\cdot)$ yielding that J is continuous.

Remark 4.4 Proposition 4.3 was proved in [12, theorem (13.16)]. But instead of strict continuity here was used the notion of Pettis-convergence:

[12, (13.7) Definition]. A net (f_i) from $H \subseteq C(X,Y)$ Pettis converges to f if for each $y \in Y$ and each neighborhood V of y there is a neighborhood W of y such that eventually $f_i(f^{-1}(W)) \subseteq V$.

But in [17] was shown that the two convergence structures are equivalent.

The following proposition was proved in [12, theorem (13.12)]. Our proof is somewhat more clear.

Proposition 4.5 *Let X be a topological and $(Y,\mathcal{A})$ an uniform space; let be $H \subseteq C(X,Y)$.*

Equivalent are:

(1) *H is equicontinuous*

(2) *$J : X \to (C((H,\tau_p),Y),\tau_u)$ is continuous*

Proof. (1)$\Longrightarrow$(2): $((x_k),x), (x_k)$ a net in $X, x \in X$; we want to show:

$$x_k \to x \Longrightarrow Jx_k = \omega(x_k,\cdot) \to \omega(x,\cdot) = Jx$$

w.r.t. the uniform topology $\tau_u : \forall V \in \mathcal{A}$, for (V,x) by (1) there exists a neighborhood

$$U \in \underline{U}(x) : \forall (y,h) \in U \times H : (h(y),h(x)) = (\omega(y,\cdot)(h), \omega(x,\cdot)(h)) \in V;$$
$$\exists k_0 \in K : \forall k \geq k_0 : x_k \in U\,.$$

Now we have:

$$\forall (k,h) \in \{k \in K | k \geq k_0\} \times H : x_k \in U \Longrightarrow (h(x_k),h(x)) = (\omega(x_k,h),\ w(x,h)) \in V$$

showing that $\omega(x_k,\cdot) \xrightarrow{\tau_u} \omega(x,\cdot)$ holds.

(2)$\Longrightarrow$(1): $\forall (x,V) \in X \times \mathcal{A}$, $(H,V) = \{(p,q) \in C(H,Y) \times C(H,Y) \,|\, \forall h \in H : (p(h), q(h) \in V\}$; now $\omega(x,\cdot) \in C((H,\tau_p),Y)$; we consider

$$(H,V)(\omega(x,\cdot)) = \{p \in C(H,Y) | \forall h \in H : (p(h), \omega(x,\cdot)(h)) = (p(h), h(x)) \in V\}$$

is a τ_u-neighborhood of $\omega(x,\cdot)$. Hence $\exists U \in \underline{U}(x) : J(U) \subseteq (H,V)(\omega(x,\cdot))$ by (2) showing that holds:

$$\forall (y,h) \in \underline{U}(x) \times H \Longrightarrow (h(y), h(x)) = (\omega(y,\cdot)(h), \omega(x,\cdot)(h)) \in V\,,$$

since $\omega(y,\cdot) \in (H,V)(\omega(x,\cdot)$. But this means that H is equicontinuous.

A conjoining topology or convergence structure can be defined (or characterized) by the continuity of the evaluation map ω. And if we consider the definition of continuous convergence then it is nearby that a conjoining convergence structure also can be characterized in a suitable way using the embedding into the second dual.

This is our next result.

Proposition 4.6 *Let X, Y be topological spaces, let $H \subseteq C(X,Y)$ and let* $\lim$ *be a convergence structure on H (maybe also* $\lim$ *is defined on $C(X,Y)$ s. th. $(H, \lim)$ is a convergence space). We assume that τ_p-$\lim \leq \lim$ holds. Then are equivalent:*

(1) $\lim$ *is conjoining for H*

(2) $J : X \to (C((H,\lim),Y), c\text{-}\lim)$ *is continuous*

Proof. We know that lim is conjoining for H iff $\omega = \omega(\cdot,\cdot) : X \times (H, \lim) \to Y$ is continuous.

(1)$\Longrightarrow$(2): $\forall (x,(x_k)), x \in X, (x_k)$ a net from X s. th. $x_k \to x$. We will show:

$$Jx_k \xrightarrow{c} Jx, \text{ hence } \omega(x_{x_k},\cdot) \xrightarrow{c} \omega(x,\cdot)\,.$$

Since τ_p-$\lim \leq \lim$ holds:

$$\forall k \in K, \forall x \in X : \omega(x_k,\cdot), \omega(x,\cdot) \in C((H,\lim),Y)\,.$$

Let (h_i) a net from $H, h \in H$ and $h_i \xrightarrow{\lim} h$; now

$$x_k \to x, h_i \xrightarrow{\lim} h \Longrightarrow \omega(x_k, h_i) \to \omega(x,h)$$

since ω is continuous, hence

$$h_i(x_k) \to h(x) \Longrightarrow \omega(x_k,\cdot)(h_i) \to \omega(x,\cdot)(h)$$

showing that $Jx_k \xrightarrow{c} Jx$ wich means: J is continuous.

(2)$\Longrightarrow$(1): Let be (x_k) a net from X, $x_k \to x \in X$, (h_i) a net from H s. th. $h_i \overset{\lim}{\longrightarrow} h \in H$; by (2): $x_k \to x \Longrightarrow \omega(x_k, \cdot) \overset{c}{\longrightarrow} \omega(x, \cdot)$; but then

$$h_i \overset{\lim}{\longrightarrow} h \Longrightarrow \omega(x_k, \cdot)(h_i) \longrightarrow \omega(x, \cdot)(h) \Longrightarrow \omega(x_k, h_i) \longrightarrow \omega(x, h)$$

yielding that lim is conjoining for H.

Corollary 4.7 *We use the assumptions of proposition 4.6*

1. *Let* $\lim = c\text{-}\lim$ *for* $C(X,Y)$*; since* $c\text{-}\lim$ *is conjoining for* $C(X,Y)$ *and hence for* $H \subseteq C(X,Y)$ *too we get:*

$$J : X \to (C((H, c\text{-}\lim), Y), c\text{-}\lim)$$

   is continuous

Remark: *For* $H = C(X,Y)$ *this result was shown in* [11, *theorem* 3., 1.]

2. $\lim = \operatorname{str} c\text{-}\lim$ *is conjoining and hence we get:*

$$J : X \to (C((H, \operatorname{str} c\text{-}\lim), Y), c\text{-}\lim)$$

   is continuous.

As already mentioned in our text [1, theorem 32] provides a necessary compactness criterion: for each conjoining topology or convergence structure: the compactness of $H \subseteq C(X,Y)$ implies that H is evenly continuous. But conversely we can't obtain a smooth sufficient criterion for an arbitrary conjoining convergence structure: We have a simple, but fundamental fact: pointwise convergence plus even continuity equals continuous convergence but not more. (see for instance [1, theorem 31]). And continuous convergence is the smallest conjoining convergence structure for $C(X,Y)$. Already in a paper from 1971 ([14, theorem 1]) I proved a necessary and sufficient compactness criterion for conjoining convergence structures. This criterion shows that one can't go beyond c-lim. With some slight improvements the original theorem reads as follows:

Theorem 4.8 *Let* X, Y *be topological spaces and* Y *is Hausdorff; let* $H \subseteq C(X,Y)$ *and* lim *be a pseudotopological convergence structure for* $C(X,Y)$*. We assume that* lim *is a conjoining convergence structure for* $C(X,Y)$*. Equivalent are:*

(1) H *is* lim*-compact*

(2) (α) $\forall x \in X$*:* $H(x)$ *is relatively compact*

   (β) H *is evenly continuous*

   (γ) H *is* τ_p*-closed in* Y^X

   (δ) $\lim = c\text{-}\lim$ *on* H

Proof. (1)$\Longrightarrow$(2): Since lim is conjoining for $C(X,Y)$ we have c-lim $\leq$ lim and hence H is also c-lim-compact. But then follow $(\alpha), (\beta), (\gamma)$ by theorem 2.2. We have c-lim $\leq$ lim on H; now let be: $\forall(\psi, f)$, ψ ultrafilter on $C(X,Y)$, $f \in H$; let be $H \in \psi$ and $\psi \xrightarrow{c\text{-lim}} f$; $(H, \lim)$ is compact and hence $\psi \xrightarrow{\lim} g \in H$ and thus $\psi \xrightarrow{c\text{-lim}} g$.

Y is Hausdorff by assumption and thus $(C(X,Y), c\text{-lim})$ is Hausdorff too implying: $g = f$. But then we see: $\psi \xrightarrow{c\text{-lim}} f \Longrightarrow \psi \xrightarrow{\lim} f$.since c-lim and lim are pseudotopological convergence spaces we get: lim $\leq c$-lim and hence lim $= c$-lim on H.

(2)$\Longrightarrow$(1): Theorem 2.2 shows $(\alpha), (\beta) and (\gamma) \Longrightarrow H$ is c-lim compact in $C(X,Y)$; now $(H, c\text{-lim})$ compact and $(H, \lim) = (H, c\text{-lim})$ by (δ) implies that H is lim-compact too.

Concluding we will consider the two Ascoli-Arzela theorems in [12] (as announced in the introduction), where we (partially), use our notations:

Theorem [12, (13.15)] *Let X be a regular space and Y a uniform space. Then $H \subseteq C(X,Y)$ is compact w. r. t. a jointly continuous topology η if and only if*

(a) *H is η-closed*

(b) *$H(x)$ has compact closures for each $x \in X$*

(c) *the natural map*

$$J : X \to (C((H, \tau_p), Y), \tau_U)$$

is continuous.

By proposition 4.5 we know that condition (c) is equivalent to H being equicontinuous.

Now theorem 4.8 shows that in general (a), (b) and (c) of (13.15) do not imply the compactness of H for each conjoining topology η for $C(X,Y)$ (or for H). For instance, if X is not compact in general τ_u-lim is strictly stronger than c-lim. Look at our example 4.9. Thus the sufficient assertion of theorem (13.15) is wrong. Quite analogously we find that [12, theorem (13.21)] is not correct too.

Here we have even continuity instead of equicontinuity.

We come now to our last example.

Example 4.9 We consider again example 3.6. Now let be

$$H \subseteq C(\mathbb{R}, \mathbb{R}),\ H = \left\{ f_n : \forall x \in \mathbb{R} : f_n(x) = \frac{1}{n} x | n \in \mathbb{N} \right\} \cup \{f_0\} = \left\{ \frac{1}{n} x | n \geq 1 \right\} \cup \{f_0\},$$

where f_0 is the zerofunction on $\mathbb{R}$. We show that hold:

(1) H is equicontinuous and hence evenly continuous.

(2) $H(x)$ is compact for each $x \in \mathbb{R}$

(3) H is τ_p-compact

(4) H is in $\mathbb{R}^{\mathbb{R}}\tau_p$-closed

(5) H is c-lim-compact

(6) H is τ_{co}-compact

(7) H is τ_u-closed in $C(\mathbb{R},\mathbb{R})$

(8) H is not τ_u-compact

Proof. (1) For example 3.6 we showed that $H-\{f_0\}$ is equicontinuous, we show in the same manner that H is equicontinuous:

$$\forall (x,y)\in\mathbb{R}\times\mathbb{R},\ \forall n\geq 1: |f_n(x)-f_n(y)| = \frac{1}{n}|x-y|;$$

$$\forall\varepsilon>0\ \exists\delta=\delta(\varepsilon)>0,\ \delta:=\varepsilon:\forall(x,y)\in\mathbb{R}\times\mathbb{R}: |x-y|<\delta\Longrightarrow\frac{1}{n}|x-y|\leq\frac{\delta}{n}\leq\varepsilon\,;$$

but also

$$|x-y|<\delta\Longrightarrow|f_0(x)-f_0(y)|=|0-0|\leq\varepsilon\,.$$

Thus H is uniformly equicontinuous and hence equicontinuous and evenly continuous.

(2) $\forall x\in\mathbb{R}: H(x)=\{f_n(x)|n\in\mathbb{N}\}$ is homeomorph to the compact set

$$\left\{\frac{1}{n}|n\in\mathbb{N},\,n\geq 1\right\}\cup\{0\}\subseteq\mathbb{R}=Y\,.$$

(3) $\forall x\in\mathbb{R}: f_n(x)=\frac{x}{n}\to 0$ showing $f_u\xrightarrow{\tau_p}f_0$ and hence $H=\{f_n|n\in\mathbb{N}\backslash\{0\}\}\cup\{f_0\}$ is τ_p-compact.

(4) H is τ_p-compact in $C(\mathbb{R},\mathbb{R})\Longrightarrow H$ is τ_p-compact in $\mathbb{R}^{\mathbb{R}}$; $(\mathbb{R}^{\mathbb{R}},\tau_p)$ is Hausdorff $\Longrightarrow H$ is in $\mathbb{R}^{\mathbb{R}}\tau_p$-closed.

(5) By theorem 2.2 from (1), (2) and (4) follows that H is in $C(\mathbb{R},\mathbb{R})$ c-lim-compact.

(6) $\mathbb{R}=X$ is locally compact and Hausdorff and thus τ_{co}-lim $=c$-lim, where τ_{co} is the compact-open topology. Then $(C(\mathbb{R},\mathbb{R}),c\text{-lim})$ is a topological space.

(7) The uniform topology τ_u in $\mathbb{R}^{\mathbb{R}}$ can be defined by the use of neighborhoods. And then we see that τ_u is first countable. Hence we can work with sequences.

We assume that H has a τ_u-accumulation point $g\in C(\mathbb{R},\mathbb{R})$; $g\notin H\Longrightarrow g\neq f_0$ on $\mathbb{R}$. Then there exists a sequence (f_n) from H s.th. $f_n\xrightarrow{\tau_u}g$; then holds $f_n\xrightarrow{\tau_p}g$ too. Otherwise $\forall n\in\mathbb{N}: f_n\in H$ and (f_n) cannot be a constant sequence. Hence we find a subsequence (f_{n_k}) s.th. $f_{n_k}\xrightarrow{\tau_p}f_0$ implying that $f_{n_k}\xrightarrow{\tau_p}g$; but then $g=f_0$ because $(C(\mathbb{R},\mathbb{R}),\tau_p)$ is Hausdorff; $g=f_0$ yields a contradiction.

Thus H is τ_u-closed.

(8) We assume that H is τ_u-compact; since H consists of a sequence there exists a subsequence (g_{n_k}) of (f_n) and a $g \in H$ s. th. $g_{n_k} \xrightarrow{\tau_u} g$ yielding $g_{n_k} \xrightarrow{\tau_p} g$ too. But then we know from the proof of (7) that $g = f_0$ holds.

Now $\{g_{n_k} | k \in \mathbb{N}\}$ is an infinite set of unbounded functions on $\mathbb{R}$ showing that $g_{n_k} \xrightarrow{\tau_u} f_0$ is not possible, a contradiction. Hence H is not τ_u-compact.

Remarks 1. Here we have again an concrete example which shows that in general does not hold: (f_n) is converging pointwise, (f_n) is equicontinuous implies that $f_n)$ converges uniformly.

2. What is the result of example 4.9?

The uniform topology τ_u (for $C(\mathbb{R}, \mathbb{R})$) is conjoining. By assertions (1), (2), (7) of 4.9 we see that the assumptions of [12, theorem (13.15)] are fullfilled.

Thus this theorem asserts that H is τ_u-compact, but this contradicts assertion (8) of 4.9 which states that H is not τ_u-compact.

Since H is evenly continuous too our example also works for [12, theorem (13.21)] yielding that the sufficient assertion of this theorem also is wrong.

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