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Results on partial Derivatives of the incomplete Beta Function

ABSTRACT. The incomplete Beta function $B(a, b; x)$ is defined by

$$B(a, b; x) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt\,,$$

for $a, b > 0$ and $0 < x < 1$. This definition was extended to negative integer values of a and b by Özçağ et al. Partial derivatives of the incomplete Beta function $B(a, b; x)$ for negative integer values of a and b were then evaluated. In the following, it is proved that

$$B_{0,1}(-1, 1; x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x} - 1$$

and

$$nB_{0,1}(-n, 1; x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x^n} - n^{-1} + \sum_{i=1}^{n-1}\frac{x^{-i}}{i}\,,$$

for $n = 2, 3, \ldots$, where

$$\frac{\partial^{m+n}}{\partial a^m \partial b^n} B(a, b; x) = B_{m,n}(a, b; x)\,.$$

Further results are also given.

KEY WORDS. Beta function, incomplete Beta function, neutrix, neutrix limit

1 INTRODUCTION

In a change of notation, the incomplete Beta function $B(a, b; x)$ is defined by

$$B(a, b; x) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \qquad a, b > 0, \quad 0 < x < 1$$

see Özçağ et al [6].

The following definitions were given by van der Corput [4].

Definition 1.1 *A neutrix N is defined as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further, if for some $\nu \in N$, $\nu(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.*

Definition 1.2 *Let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function in N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to b and we write $\mathrm{N}{-}\lim_{\xi \to b} f(\xi) = c$.*

Note that if f tends to c in the normal sense as ξ tends to b, then it converges to c in the neutrix sense.

Now let N be the neutrix having domain $N' = (0, x) \quad (0 < x < 1)$ and range N'' the real numbers, with the negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \quad \ln^r \epsilon \qquad (\lambda < 0, \quad r = 1, 2, \ldots)$$

and all functions which converge to zero in the normal sense as ϵ tends to zero.

It was proved, see Özçağ et al. [6] and [7] that

$$B(a, b; x) = \mathrm{N}{-}\lim_{\epsilon \to 0} \int_\epsilon^x t^{a-1}(1-t)^{b-1}\, dt$$

for all values of a and b and in general

$$\begin{aligned} \frac{\partial^{m+n}}{\partial a^m \partial b^n} B(a, b; x) &= B_{m,n}(a, b; x) \\ &= \mathrm{N}{-}\lim_{\epsilon \to 0} \int_\epsilon^x t^{a-1} \ln^m t\, (1-t)^{b-1} \ln^n(1-t)\, dt \end{aligned}$$

for $m, n = 0, 1, 2, \ldots$ and all values of a and b.

Note that $B_{m,n}(a, b; x)$ is not necessarily equal to $B_{m,n}(b, a; x)$.

Note also that if $a > 0$, then

$$B_{m,n}(a, b; x) = \lim_{\epsilon \to 0} \int_\epsilon^x t^{a-1} \ln^m t\, (1-t)^{b-1} \ln^n(1-t)\, dt$$

for $m, n = 0, 1, 2, \ldots$.

The following results were proved in [2]:

$$B(0, 0; x) = \ln \frac{x}{1-x}, \tag{1}$$

$$B(n, 0; x) = -\ln(1-x) - \sum_{i=1}^{n-1} \frac{x^i}{i}, \quad n = 1, 2, \ldots, \tag{2}$$

the sum being empty when $n = 1$ and

$$B(-n, 0; x) = \ln \frac{x}{1-x} - \sum_{i=1}^{n} \frac{x^{-i}}{i}, \quad n = 1, 2, \dots . \tag{3}$$

2 MAIN RESULTS

We now prove the following theorem:

Theorem 2.1

$$B(0, -n; x) = \ln \frac{x}{1-x} + \sum_{i=1}^{n} \frac{(1-x)^{-i}}{i} - \phi(n)\,, \tag{4}$$

for $n = 1, 2, \dots,$ where

$$\phi(n) = \sum_{i=1}^{n} \frac{1}{i}$$

is the $n-th$ harmonic number.

Proof. We have

$$\begin{aligned}
\int_{\epsilon}^{x} t^{-1}(1-t)^{-n-1}\,dt &= \int_{1-x}^{1-\epsilon} t^{-n-1}(1-t)^{-1}\,dt \\
&= \int_{1-x}^{1-\epsilon} \left[(1-t)^{-1} + \sum_{i=1}^{n+1} t^{-i} \right] dt \\
&= \ln x - \ln \epsilon - \ln(1-x) + \ln(1-\epsilon) - \sum_{i=1}^{n} \frac{1}{i} \left[\frac{1}{(1-\epsilon)^{i}} - \frac{1}{(1-x)^{i}} \right]
\end{aligned}$$

and it follows that

$$\begin{aligned}
B(0, -n; x) &= \mathrm{N}\!-\!\lim_{\epsilon \to 0} \int_{\epsilon}^{x} t^{-1}(1-t)^{-n-1}\,dt \\
&= \ln \frac{x}{1-x} - \mathrm{N}\!-\!\lim_{\epsilon \to 0} [\ln \epsilon - \ln(1-\epsilon)] - \lim_{\epsilon \to 0} \sum_{i=1}^{n} \frac{1}{i} \left[\frac{1}{(1-\epsilon)^{i}} - \frac{1}{(1-x)^{i}} \right] \\
&= \ln \frac{x}{1-x} - \sum_{i=1}^{n} \frac{1}{i} \left[1 - \frac{1}{(1-x)^{i}} \right].
\end{aligned}$$

Equation (4) follows. □

Equation (4) corrects a result given in [6].

Theorem 2.2

$$B_{1,0}(1,-1;x) = (1-x)^{-1}\ln x - \ln\frac{x}{1-x} \tag{5}$$

and

$$nB_{1,0}(1,-n;x) = (1-x)^{-n}\ln x - \ln\frac{x}{1-x} - \sum_{i=1}^{n-1}\frac{(1-x)^{-i}}{i} + \phi(n-1) \tag{6}$$

for $n = 2,3,\ldots$.

Proof. We have

$$\begin{aligned} n\int_{\epsilon}^{x} \ln t(1-t)^{-n-1}\,dt &= \int_{\epsilon}^{x}\ln t\,d(1-t)^{-n} \\ &= (1-x)^{-n}\ln x - (1-\epsilon)^{-n}\ln\epsilon - \int_{\epsilon}^{x} t^{-1}(1-t)^{-n}\,dt \end{aligned}$$

and it follows that

$$\begin{aligned} nB_{1,0}(1,-n;x) &= \operatorname*{N-lim}_{\epsilon\to 0} n\int_{\epsilon}^{x}\ln t(1-t)^{-n-1}\,dt \\ &= (1-x)^{-n}\ln x - \operatorname*{N-lim}_{\epsilon\to 0}(1-\epsilon)^{-n}\ln\epsilon - \operatorname*{N-lim}_{\epsilon\to 0}\int_{\epsilon}^{x} t^{-1}(1-t)^{-n}\,dt \\ &= (1-x)^{-n}\ln x - B(0,-n+1;x)\,. \end{aligned}$$

Equation (5) now follows on using equation (1) and equation (6) follows on using equation (4) for $n = 2,3,\ldots$. □

Theorem 2.3

$$B_{0,1}(-1,1;x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x} - 1 \tag{7}$$

and

$$nB_{0,1}(-n,1;x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x^n} - n^{-1} + \sum_{i=1}^{n-1}\frac{x^{-i}}{i}\,, \tag{8}$$

for $n = 2,3,\ldots$.

Proof. We have

$$\begin{aligned} n\int_{\epsilon}^{x} t^{-n-1}\ln(1-t)\,dt &= -\int_{\epsilon}^{x}\ln(1-t)\,dt^{-n} \\ &= \epsilon^{-n}\ln(1-\epsilon) - x^{-n}\ln(1-x) - \int_{\epsilon}^{x} t^{-n}(1-t)^{-1}\,dt \end{aligned}$$

and it follows that

$$nB_{0,1}(-n,1;x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0} n\int_\epsilon^x t^{-n-1}\ln(1-t)\,dt$$
$$= -n^{-1} - x^{-n}\ln(1-x) - B(-n+1,0;x)\,.$$

Equation (7) now follows on using equation (1) and equation (8) follows on using equation (3) for $n = 2, 3, \ldots$. □

Theorem 2.4

$$B_{0,1}(-n,r+1;x) = \sum_{i=0}^{r}\frac{(-1)^{i-1}}{n-i}\binom{r}{i}\left[x^{i-n}\ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1}\frac{x^{-k}}{k} + \frac{1}{n-i}\right] \tag{9}$$

for $n = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots, n-1$, the sum $\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}$ being empty when $i = n-1$,

$$B_{0,1}(-n,n+1;x) = \sum_{i=0}^{n-1}\frac{(-1)^{i-1}}{n-i}\binom{n}{i}\left[x^{i-n}\ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1}\frac{x^{-k}}{k} + \frac{1}{n-i}\right] - (-1)^n\sum_{i=1}^{\infty}\frac{x^i}{i^2}\,, \tag{10}$$

for $n = 1, 2, \ldots$, the sum $\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}$ being empty when $i = n-1$ and

$$B_{0,1}(-n,r+1;x) = \sum_{i=0}^{n-1}\frac{(-1)^{i-1}}{n-i}\binom{r}{i}\left[x^{i-n}\ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1}\frac{x^{-k}}{k} + \frac{1}{n-i}\right] + (-1)^n\binom{r}{n}\sum_{i=1}^{\infty}\frac{x^i}{i^2}$$
$$+ \sum_{i=n+1}^{r}\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x) - \ln(1-x) - \sum_{k=1}^{i-n}\frac{x^k}{k}\right] \tag{11}$$

for $n = 1, 2, \ldots$ and $r = n+1, n+2, \ldots$.

Proof. Integrating by parts, we have

$$\int_\epsilon^x t^{-n-1}(1-t)^r\ln(1-t)\,dt = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\int_\epsilon^x t^{i-n-1}\ln(1-t)\,dt$$
$$= \sum_{i=0}^{r}\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x) - \epsilon^{i-n}\ln(1-\epsilon) + \int_\epsilon^x t^{i-n}(1-t)^{-1}\,dt\right], \tag{12}$$

for $r = 0, 1, 2, \ldots, n-1$.

Since

$$\epsilon^{i-n}\ln(1-\epsilon) = -\sum_{j=1}^{\infty}\frac{\epsilon^{i+j-n}}{j},$$

it follows that

$$\mathrm{N}\!-\!\lim_{\epsilon\to 0}\epsilon^{i-n}\ln(1-\epsilon) = \begin{cases} -(n-i)^{-1}, & 0 \le i \le n-1, \\ 0, & i \ge n. \end{cases} \tag{13}$$

It now follows from equations (12) and (13) that

$$\begin{aligned} B_{0,1}(-n, r+1; x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_{\epsilon}^{x} t^{-n-1}\ln(1-t)(1-t)^{r}\,dt \\ &= \sum_{i=0}^{r}\frac{(-1)^{i}}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x) + B(-n+i+1, 0; x) - \frac{1}{i-n}\right] \\ &= \sum_{i=0}^{r}\frac{(-1)^{i-1}}{n-i}\binom{r}{i}\left[x^{i-n}\ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1}\frac{x^{-k}}{k} + \frac{1}{n-i}\right], \end{aligned}$$

on using equation (3), proving equation (9).

For the case $r = n$, equation (12) has to be replaced by the equation

$$\begin{aligned} \int_{\epsilon}^{x} t^{-n-1}(1-t)^{n}\ln(1-t)\,dt &= \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}\int_{\epsilon}^{x} t^{i-n-1}\ln(1-t)\,dt \\ &= \sum_{i=0}^{n-1}\frac{(-1)^{i}}{i-n}\binom{n}{i}\left[x^{i-n}\ln(1-x) - \epsilon^{i-n}\ln(1-\epsilon) + \int_{\epsilon}^{x} t^{i-n}(1-t)^{-1}\,dt\right] \\ &\quad + (-1)^{n}\int_{\epsilon}^{x} t^{-1}\ln(1-t)\,dt\,. \end{aligned} \tag{14}$$

It now follows from equations (13) and (14) that

$$\begin{aligned} B_{0,1}(-n, n+1; x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_{\epsilon}^{x} t^{-n-1}\ln(1-t)(1-t)^{n}\,dt \\ &= \sum_{i=0}^{n-1}\frac{(-1)^{i}}{i-n}\binom{n}{i}\left[x^{i-n}\ln(1-x) + B(-n+i+1, 0; x) - \frac{1}{i-n}\right] \\ &\quad + (-1)^{n}B_{0,1}(0, 1; x)\,. \end{aligned} \tag{15}$$

Now

$$\int_{\epsilon}^{x} t^{-1}\ln(1-t)\,dt = -\sum_{i=1}^{\infty}\int_{\epsilon}^{x}\frac{t^{i-1}}{i}\,dt = -\sum_{i=1}^{\infty}\frac{x^{i}-\epsilon^{i}}{i^{2}}$$

and so

$$\mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-1}\ln(1-t)\,dt = -\sum_{i=1}^\infty \frac{x^i}{i^2} = B_{0,1}(0,1;x)\,. \tag{16}$$

It now follows from equations (15) and (16) that

$$\begin{aligned} B_{0,1}(-n,n+1;x) = \sum_{i=0}^{n-1}\frac{(-1)^{i-1}}{n-i}\binom{n}{i}\Bigg[x^{i-n}\ln(1-x)+\ln\frac{x}{1-x} \\ -\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}+\frac{1}{n-i}\Bigg] - (-1)^n\sum_{i=1}^\infty\frac{x^i}{i^2}\,, \end{aligned}$$

proving equation (10).

When $r>n$, equation (12) has to be replaced by

$$\begin{aligned} &\int_\epsilon^x t^{-n-1}(1-t)^r\ln(1-t)\,dt = \sum_{i=0}^r(-1)^i\binom{r}{i}\int_\epsilon^x t^{i-n-1}\ln(1-t)\,dt \\ &= \sum_{i=0}^{n-1}\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x)-\epsilon^{i-n}\ln(1-\epsilon)+\int_\epsilon^x t^{i-n}(1-t)^{-1}\,dt\right] \\ &\quad+(-1)^n\binom{r}{n}\int_\epsilon^x t^{-1}\ln(1-t)\,dt \\ &\quad+\sum_{i=n+1}^r\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x)-\epsilon^{i-n}\ln(1-\epsilon)+\int_\epsilon^x t^{i-n}(1-t)^{-1}\,dt\right]. \end{aligned} \tag{17}$$

It now follows from equations (16) and (17) that

$$\begin{aligned} &B_{0,1}(-n,r+1;x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-n-1}(1-t)^r\ln(1-t)\,dt \\ &= \sum_{i=0}^{n-1}\frac{(-1)^{i-1}}{n-i}\binom{r}{i}\left[x^{i-n}\ln(1-x)+\ln\frac{x}{1-x}-\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}+\frac{1}{n-i}\right] \\ &\quad-(-1)^n\binom{r}{n}\sum_{i=1}^\infty\frac{x^i}{i^2}+\sum_{i=n+1}^r\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x)-\ln(1-x)-\sum_{k=1}^{i-n}\frac{x^k}{k}\right], \end{aligned}$$

since it was proved in [6] that

$$B(n,0;x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{n-1}(1-t)^{-1}\,dt = -\ln(1-x)-\sum_{k=1}^{n-1}\frac{x^k}{k}$$

for $n=1,2,\ldots$. Equation (11) is now proved. □

For further related results see [1], [2], [3] and [5].

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