Rostock. Math. Kolloq. **72**, 49 – 71 (2019/20)

Subject Classification (MSC)
Primary 35K05, 35R30

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Applications of Carleman inequalities for a two-by-two parabolic system in an unbounded guide

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ABSTRACT. In this article we consider the inverse problem of determining some of the coefficients of a two-by-two parabolic system defined on an unbounded guide. Using an adapted Carleman estimate, we establish local stability results for at least two coefficients of this system in any finite portion of the guide. These estimates are obtained with data of the solution at a fixed time and boundary measurements for observations.

KEY WORDS. inverse problems, Carleman inequalities, heat operator, system, unbounded guide

1 Introduction

Let ω be a bounded connex domain in $\mathbb{R}^{n-1}$, $n \geq 2$ with C^2 boundary. Denote $\Omega := \mathbb{R} \times \omega$, $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. We consider the following system

$$\begin{cases} \partial_t u - \Delta u + au + bv = g_1 \text{ in } Q, \\ \partial_t v - \Delta v + cu + dv = g_2 \text{ in } Q, \\ u = h_1 \text{ and } v = h_2 \text{ on } \Sigma, \\ u(x,0) = u_0(x) \text{ and } v(x,0) = v_0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where a, b, c, d are bounded coefficients defined on Ω such that

$$a, b, c, d \in \Lambda(M_0) := \{f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} \leq M_0\} \text{ for some } M_0 > 0.$$

Our inverse problem is to estimate at least two coefficients between a, b, c, d from the data of the solution (u, v) at $T/2$ and the measurement of (u, v) on a part of the boundary.
We will consider (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a solution of (1.1) associated with (a, b, c, d, u_0, v_0, g_1, g_2, h_1, h_2) (resp. ($\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$, $\widetilde{u_0}$, $\widetilde{v_0}$, g_1, g_2, h_1, h_2)) and two positive reals l, L such that $l < L$. Denote

$$\Omega_L = (-L, L) \times \omega \text{ and } \Omega_l = (-l, l) \times \omega.$$

The first result of this paper gives a Hölder result (3.3) for the coefficients b and c in the case where $\widetilde{a} = a$, $\widetilde{d} = d$ and is the following (see Theorem 3.1)

$$\|b - \widetilde{b}\|^2_{L^2(\Omega_l)} + \|c - \widetilde{c}\|^2_{L^2(\Omega_l)} \le K \left(\|(u - \widetilde{u})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \|(v - \widetilde{v})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} \right.$$
$$\left. + \int_{\gamma_L \times (0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k(u - \tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v - \tilde{v}))|^2) \, d\sigma \, dt \right)^\kappa$$

where K is a positive constant, $\kappa \in (0,1)$, γ_L is a part of the boundary (see (2.2)), and assuming that the hypothesis (3.2) is satisfied.

The second result (3.15) of this paper is also a Hölder stability result for the four coefficients a, b, c, d (see Theorem 3.2)

$$\|a - \widetilde{a}\|^2_{L^2(\Omega_l)} + \|b - \widetilde{b}\|^2_{L^2(\Omega_l)} + \|c - \widetilde{c}\|^2_{L^2(\Omega_l)} + \|d - \widetilde{d}\|^2_{L^2(\Omega_l)}$$
$$\le K \left(\| \sum_{k=0}^{1} \partial_t^k (u - \widetilde{u})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \| \sum_{k=0}^{1} \partial_t^k (v - \widetilde{v})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} \right.$$
$$\left. + \int_{\gamma_L \times (0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k(u - \tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v - \tilde{v}))|^2) \, d\sigma \, dt \right)^\kappa$$

with stronger hypotheses (3.13) and (3.14) than those in Theorem 3.1 (see (3.2)).

The third theorem of this paper gives a Hölder stability result (3.34) (see Theorem 3.3) for the following reaction-diffusion system

$$\begin{cases} \partial_t u - \Delta u + au + bv + A_1 \cdot \nabla u + A_2 \cdot \nabla v = g_1 & \text{in } Q, \\ \partial_t v - \Delta v + cu + dv + A_3 \cdot \nabla u + A_4 \cdot \nabla v = g_2 & \text{in } Q, \\ u = h_1 \text{ and } v = h_2 & \text{on } \Sigma, \\ u(x,0) = u_0(x) \text{ and } v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where all the coefficients $a, b, c, d, A_1, A_2, A_3, A_4$ are bounded ($a, b, c, d \in \Lambda(M_0)$ and $A_1, A_2, A_3, A_4 \in \Lambda(M_0)^n \cap H^1(\Omega)^n$). We obtain a stability result for the coefficients b and A_3 (assuming A_3 has the form $A_3 = \nabla g$) with the same kind of observations in the right-hand side of (3.34) as we have obtained in (3.3) or (3.15). Assuming that the Assumptions (3.32) and (3.33) hold, we get the following result

$$\|b - \widetilde{b}\|^2_{L^2(\Omega_l)} + \|A_3 - \widetilde{A_3}\|^2_{(L^2(\Omega_l))^n}$$
$$\le K \left(\| \sum_{k=0}^{1} \partial_t^k (u - \widetilde{u})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \| \sum_{k=0}^{1} \partial_t^k (v - \widetilde{v})(., \frac{T}{2})\|^2_{H^3(\Omega_L)} \right.$$
$$\left. + \int_{\gamma_L \times (0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k(u - \tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v - \tilde{v}))|^2) \, d\sigma \, dt \right)^\kappa .$$

Of course each of these above stability results implies an uniqueness result.
Up to our knowledge, there are few results concerning the simultaneous identification of more than one coefficient in each equation (see for example [1] and also [5] where the authors give a stability result for the diffusion coefficient a and the potential b of the Schrödinger operator $i\partial_t q + a\Delta q + bq$). In previous papers, stability results have been obtained for parabolic systems but, as far as we know, these papers have investigated the case of bounded domains and have given results with observations on a subdomain of their domain ([1, 7]...). Furthermore, there is no result for a two-by-two parabolic system with only one observation on a part of the boundary and without any data of the solution at a fixed time even in a bounded domain. We will use here the global Carleman estimate (2.5) for one equation given in [3] based on a classical Carleman estimate given in [12, 13]. Our choice of weight functions is adapted for this unbounded domain but will give us Hölder, and not Lipschitz, estimates of the coefficients. Recall that the method using Carleman estimates for solving inverse problems has been initiated by [2]. Our results extend to a system previous results for one equation defined on an unbounded guide (see [3] for the heat operator $\partial_t u - \Delta u + qu$ and [4] for the heat operator $\partial_t u - \nabla \cdot (c\nabla u)$ where stability results are given either for the potential q or for the diffusion coefficient c).
This Paper is organized as follows. In section 2, we specify the weight functions used for our Carleman estimate (cf (2.1), (2.3)) and due to the particular symmetric form of these weight functions with respect to x_1 and $t - T/2$ we recall from [3] the inequality (2.4), crucial for our final estimates (3.3), (3.15) and (3.34). Then in section 3 we state and prove our stability results, first for the coefficients b, c, after for a, b, c, d and finally for b, A_3.

2 Carleman estimate

Denote $Q_L = \Omega_L \times (0,T) = (-L, L) \times \omega \times (0,T)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $x' = (x_2, \cdots, x_n)$ and define the operator

$$Au = \partial_t u - \Delta u.$$

Let $l > 0$, following [3] in this section, we consider some positive real $L > l$ and choose $a \in \mathbb{R}^n \setminus \Omega$ such that if

$$\tilde{d}(x) = |x' - a'|^2 - x_1^2 \text{ for } x \in \Omega_L, \text{ then } \tilde{d} > 0 \text{ in } \Omega_L,\ |\nabla \tilde{d}| > 0 \text{ in } \overline{\Omega_L}. \tag{2.1}$$

Moreover define

$$\Gamma_L = \{x \in \partial\Omega_L,\ < x - a, \nu(x) > \geq 0\} \text{ and } \gamma_L = \Gamma_L \cap \partial\Omega. \tag{2.2}$$

Here $< .,. >$ denotes the usual inner product in $\mathbb{R}^n$ and $\nu(x)$ the outward unit normal vector to $\partial\Omega_L$ at x. Notice that γ_L does not contain any cross section of the guide. From [12] we consider weight functions as follows: for $t \in (0,T)$, if $M_1 > \sup_{0<t<T}(t-T/2)^2 = (T/2)^2$,

$$\psi(x,t) = \tilde{d}(x) - \left(t - \frac{T}{2}\right)^2 + M_1, \quad \text{and } \phi(x,t) = e^{\lambda\psi(x,t)}. \tag{2.3}$$

The constant $\lambda > 0$ will be set in Proposition 2.2 and is usually used as a large parameter in Carleman inequalities. Since we will not use it, we will consider λ fixed in the article. We recall from [3] the following result.

Proposition 2.1 *There exists $T > 0$, $L > l$, $a \in \mathbb{R}^2 \setminus \Omega$ and $\tilde{\epsilon} > 0$ such that (2.1) holds and, setting*

$$O_{L,\tilde{\epsilon}} = (\Omega_L \times ((0, 2\tilde{\epsilon}) \cup (T - 2\tilde{\epsilon}, T))) \cup (((-L, -L + 2\tilde{\epsilon}) \cup (L - 2\tilde{\epsilon}, L)) \times \omega \times (0, T)),$$

we have

$$d_1 < d_0 < d_2 \tag{2.4}$$

where

$$d_0 = \inf_{\Omega_l} \phi(., \frac{T}{2}), \;\; d_1 = \sup_{O_{L,\tilde{\epsilon}}} \phi \text{ and } d_2 = \sup_{\overline{\Omega_L}} \phi(., \frac{T}{2}).$$

We will use the following notations: Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index with $\alpha_i \in \mathbb{N} \cup \{0\}$. We set $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and define

$$H^{2,1}(Q_L) = \{u \in L^2(Q_L), \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q_L), |\alpha| + 2\alpha_{n+1} \leq 2\}$$

endowed with its norm

$$\|u\|^2_{H^{2,1}(Q_L)} = \sum_{|\alpha|+2\alpha_{n+1} \leq 2} \|\partial_x^\alpha \partial_t^{\alpha_{n+1}} u\|^2_{L^2(Q_L)}.$$

We recall here a global Carleman-type estimate proved in [3], based on a classical Carleman estimate (see Yamamoto [12, Theorem 7.3]).

Proposition 2.2 *There exist a value of $\lambda > 0$ and positive constants s_0 and $C = C(\lambda, s_0)$ such that*

$$I(u) := \int_{Q_L} \left(\frac{1}{s\phi} (|\partial_t u|^2 + |\Delta u|^2) + s\phi \, |\nabla u|^2 + s^3 \phi^3 |u|^2 \right) e^{2s\phi} dx \, dt$$

$$\leq C \|e^{s\phi} A u\|^2_{L^2(Q_L)} + C s^3 e^{2sd_1} \|u\|^2_{H^{2,1}(Q_L)} + C s \int_{\gamma_L \times (0,T)} |\partial_\nu u|^2 e^{2s\phi} d\sigma \, dt, \tag{2.5}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q_L)$ satisfying $u(.,0) = u(.,T) = 0$ in Ω_L, $u = 0$ on $\partial\Omega_L \times (0,T)$. We denote $\partial_\nu u = \nu \cdot \nabla u$.

In fact the above Proposition 2.2 is still valid for a more general function u: we can replace the condition $u = 0$ on $\partial\Omega_L \times (0,T)$ in Proposition 2.2 by $u = 0$ on $(\partial\Omega \cap \partial\Omega_L) \times (0,T)$. Since the method of Carleman estimates requires several time differentiations, we assume in the following that u, v (resp. $\widetilde{u}, \widetilde{v}$) belong to $\mathcal{H} = H^3(0,T,H^3(\Omega))$ satisfying the a-priori bound

$$\|u\|_{\mathcal{H}} < M_2 \text{ and } \|v\|_{\mathcal{H}} < M_2 \text{ for given } M_2 > 0.$$

From now on, we use the notation $w(\frac{T}{2}) = w(., \frac{T}{2})$ for any function w.

3 Inverse problems

3.1 The first result

Consider here (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a strong solution of (1.1) associated with $(a,\ b,\ c,\ d,\ u_0,\ v_0,\ g_1,\ g_2,\ h_1,\ h_2)$ (resp. $(a,\ \widetilde{b},\ \widetilde{c},\ d,\ \widetilde{u_0},\ \widetilde{v_0},\ g_1,\ g_2,\ h_1,\ h_2)$). Assume that all the coefficients $a, b, c, d, \widetilde{b}, \widetilde{c}$ belong to $\Lambda(M_0)$. From [8, Lemma 4.2], we derive the following result, also used in [3]

Lemma 3.1 *There exist some positive constants C, s_1 such that*

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}|z(T/2)|^2 dx \le Cs\int_{Q_L} e^{2s\phi}\phi^2|z|^2 dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}|\partial_t z|^2 dx\ dt,$$

for all $s \ge s_1$ and $z \in H^1(0,T;L^2(\Omega_L))$.

For the sake of completeness, we recall its proof.

Proof. Consider η defined by (3.4) and any $w \in H^1(0,T;L^2(\Omega_L))$. Since $\eta(\frac{T}{2}) = 1$ and $\eta(0) = 0$, we have

$$\int_{\Omega_L} w(x,T/2)^2 dx = \int_{\Omega_L} (\eta(T/2)w(x,T/2))^2 dx = \int_{\Omega_L}\int_0^{T/2} \partial_t(\eta^2(t)|w(x,t)|^2) dt\ dx$$

$$= 2\int_0^{T/2}\int_{\Omega_L} \eta^2(t)w(x,t)\partial_t w(x,t) dx\ dt + 2\int_0^{T/2}\int_{\Omega_L} \eta(t)\partial_t\eta(t)|w(x,t)|^2 dx\ dt.$$

As $0 \le \eta \le 1$, using Young's inequality, it comes that for any $s > 0$,

$$\int_{\Omega_L} w(x,T/2)^2\ dx \le Cs\int_{Q_L} |w|^2 dx\ dt + \frac{C}{s}\int_{Q_L} |\partial_t w|^2 dx\ dt. \tag{3.1}$$

Then we can conclude replacing w by $e^{s\phi}z$ in (3.1). □

We can state our first main result for a two-by-to linear system which extend precedent results for one equation (see [3] and [4]). We do not follow here the proof of [1, Theorem 1.2] and rather use ideas from [3].

Theorem 3.1 *Let $l > 0$. Let $T > 0$, $L > l$ and $a \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. We make the following assumption*

$$|\widetilde{u}(., \frac{T}{2})| \geq R \text{ and } |\widetilde{v}(., \frac{T}{2})| \geq R \text{ in } \Omega_L \text{ for some } R > 0. \tag{3.2}$$

Then there exists a sufficiently small number δ_0 such that if $\delta \in (0, \delta_0)$,

$$\|(u - \widetilde{u})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \|(v - \widetilde{v})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \int_{\gamma_L \times (0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k(u - \tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v - \tilde{v}))|^2) d\sigma dt \leq \delta$$

then the following Hölder stability estimate holds

$$\|b - \widetilde{b}\|^2_{L^2(\Omega_l)} + \|c - \widetilde{c}\|^2_{L^2(\Omega_l)} \leq K\delta^\kappa \text{ for all } \delta \in (0, \delta_0). \tag{3.3}$$

Here, $K > 0$ and $\kappa \in (0, 1)$ are two constants depending on R, r, L, l, M_0, M_1, M_2, T and a.

Proof. Let χ, η be C^∞ cut-off functions defined by $\chi, \nabla\chi, \Delta\chi \in \Lambda(M_0)$, $0 \leq \chi \leq 1$, $0 \leq \eta \leq 1$,

$$\chi(x) = 0 \text{ if } x \in ((-\infty, -L + \tilde{\epsilon}) \cup (L - \tilde{\epsilon}, +\infty)) \times \omega),$$

$$\chi(x) = 1 \text{ if } x \in (-L + 2\tilde{\epsilon}, L - 2\tilde{\epsilon}) \times \omega,$$

$$\eta(t) = 0 \text{ if } t \in (0, \tilde{\epsilon}) \cup (T - \tilde{\epsilon}, T), \; \eta(t) = 1 \text{ if } t \in \times(2\tilde{\epsilon}, T - 2\tilde{\epsilon}). \tag{3.4}$$

Denote also

$$y = u - \widetilde{u}, \; y_0 = \chi\eta y, \; y_1 = \partial_t y_0, \; y_2 = \partial_t y_1, \; z = v - \widetilde{v}, \; z_0 = \chi\eta z, \; z_1 = \partial_t z_0 \text{ and } z_2 = \partial_t z_1.$$

Note that (y_0, z_0) satisfies

$$\begin{cases} \partial_t y_0 - \Delta y_0 + a y_0 + b z_0 = \rho_1 := (\widetilde{b} - b)\chi\eta\widetilde{v} + (\partial_t\eta)\chi y - (\Delta\chi)\eta y - 2\nabla\chi \cdot \nabla(\eta y) \text{ in } Q_L, \\ \partial_t z_0 - \Delta z_0 + c y_0 + d z_0 = \rho_2 := (\widetilde{c} - c)\chi\eta\widetilde{u} + (\partial_t\eta)\chi z - (\Delta\chi)\eta z - 2\nabla\chi \cdot \nabla(\eta z) \text{ in } Q_L, \\ y_0 = z_0 = 0 \text{ on } \partial\Omega_L \times (0, T). \end{cases} \tag{3.5}$$

and (y_1, z_1), (y_2, z_2) satisfy

$$\begin{cases} \partial_t y_1 - \Delta y_1 + a y_1 + b z_1 = \partial_t \rho_1 \text{ in } Q_L, \\ \partial_t z_1 - \Delta z_1 + c y_1 + d z_1 = \partial_t \rho_2 \text{ in } Q_L, \\ y_1 = z_1 = 0 \text{ on } \partial\Omega_L \times (0, T) \end{cases} \text{ and } \begin{cases} \partial_t y_2 - \Delta y_2 + a y_2 + b z_2 = \partial_t^2 \rho_1 \text{ in } Q_L, \\ \partial_t z_2 - \Delta z_2 + c y_2 + d z_2 = \partial_t^2 \rho_2 \text{ in } Q_L, \\ y_2 = z_2 = 0 \text{ on } \partial\Omega_L \times (0, T). \end{cases}$$

• First step: Applying (3.5) for $t = \frac{T}{2}$, if we denote

$$J := \int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2[|c-\widetilde{c}|^2|\widetilde{u}(\frac{T}{2})|^2 + |b-\widetilde{b}|^2|\widetilde{v}(\frac{T}{2})|^2]\ dx$$

then we get

$$J \le Ce^{2sd_2}F_0(\frac{T}{2}) + C\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}(|\partial_t y_0(\frac{T}{2})|^2 + |\partial_t z_0(\frac{T}{2})|^2)\ dx$$

with

$$F_0(T/2) = \|z_0(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^1(\Omega_L)} + \|y_0(T/2)\|^2_{H^2(\Omega_L)} + \|y(T/2)\|^2_{H^1(\Omega_L)}.$$

Note that

$$F_0(T/2) \le CF(T/2) \text{ with } F(T/2) = \|y(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^2(\Omega_L)}.$$

Moreover, since $\partial_t y_0 = y_1$, $\partial_t z_0 = z_1$ and $1 \le \phi$, using Lemma 3.1, we obtain

$$J \le Ce^{2sd_2}F(T/2) + Cs\int_{Q_L} e^{2s\phi}\phi^3(|y_1|^2+|z_1|^2)\ dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3(|y_2|^2+|z_2|^2)\ dx\ dt. \quad (3.6)$$

• Second step: Now we evaluate J with the Carleman inequalities (2.5) for y_i and z_i, $i = 1, 2$. Note that all the terms in $\|e^{s\phi}Ay_i\|^2_{L^2(Q_L)}$ or $\|e^{s\phi}Az_i\|^2_{L^2(Q_L)}$ with derivatives of χ or η will be bounded above by Ce^{2sd_1} with C a positive constant. Therefore, for s sufficiently large, there exists a positive constant C such that

$$\begin{aligned} I(y_i) + I(z_i) \le\ & C\int_{Q_L} e^{2s\phi}\chi^2[|c-\widetilde{c}|^2 + |b-\widetilde{b}|^2]\ dx\ dt + C\int_{Q_L} e^{2s\phi}(|y_i|^2 + |z_i|^2)\ dx\ dt + Ce^{2sd_1} \\ & + Cs^3e^{2sd_1}(\|y_i\|^2_{H^{2,1}(Q_L)} + \|z_i\|^2_{H^{2,1}(Q_L)}) + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt. \end{aligned}$$

Since $e^{2s\phi} \le e^{2s\phi(T/2)}$, we deduce that

$$\begin{aligned} I(y_i) + I(z_i) \le\ & C\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2 + |b-\widetilde{b}|^2]\ dx\ dt + Cs^3e^{2sd_1} \\ & + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt. \end{aligned}$$

Thus

$$\begin{aligned} s^3\int_{Q_L} e^{2s\phi}\phi^3(|y_i|^2 + |z_i|^2)\ dx\ dt \le\ & C\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2 + |b-\widetilde{b}|^2]\ dx \\ & + Cs^3e^{2sd_1} + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt. \end{aligned} \quad (3.7)$$

Therefore, from (3.6) and (3.7), we get for s sufficiently large

$$J \le Ce^{2sd_2}F(T/2) + \frac{C}{s^2}\left(s^3e^{2sd_1} + \int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2| + |b-\widetilde{b}|^2]\,dx\right.$$

$$\left. + s\int_{\gamma_L\times(0,T)} e^{2s\phi}\sum_{i=1}^{2}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\,d\sigma\,dt\right).$$

So we have

$$J \le Ce^{2sd_2}G(T/2) + Cse^{2sd_1} + \frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2| + |b-\widetilde{b}|^2]\,dx \tag{3.8}$$

with

$$G(T/2) = F(T/2) + \int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2 + |\partial_\nu\partial_t^k z|^2)\,d\sigma\,dt.$$

• Third and last step: In this step, we come back to the coefficients $b-\tilde{b}$ and $c-\tilde{c}$. First, from the hypothesis (3.2) we derive from (3.8), for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2 + |\widetilde{c}-c|^2)\,dx \le Ce^{2sd_2}G(T/2) + Cse^{2sd_1}. \tag{3.9}$$

Moreover, since $e^{2sd_0} \le e^{2s\phi(T/2)}$ in Ω_l and $\chi = 1$ in Ω_l, we deduce from (3.9) that

$$e^{2sd_0}(\|\widetilde{b}-b\|^2_{L^2(\Omega_l)} + \|\widetilde{c}-c\|^2_{L^2(\Omega_l)}) \le Ce^{2sd_2}G(T/2) + Cse^{2sd_1}.$$

This last inequality can be rewritten in the following form for s sufficiently large ($s \ge s_2$)

$$\|\widetilde{b}-b\|^2_{L^2(\Omega_l)} + \|\widetilde{c}-c\|^2_{L^2(\Omega_l)} \le C(e^{2s(d_2-d_0)}G(T/2) + se^{2s(d_1-d_0)}). \tag{3.10}$$

Note that if $G(T/2) = 0$, since (3.10) holds for any $s \ge s_2$ and $d_1 - d_0 < 0$ we get (3.3). Now if $G(T/2) \ne 0$, we recall from (2.4) that $d_1 - d_0 < 0$ and $d_2 - d_0 > 0$ and optimize (3.10) with respect to s. Indeed denote

$$f(s) = e^{2s(d_2-d_0)}G(T/2) + e^{2s(d_1-d_0)} \text{ and } g(s) = e^{2s(d_2-d_0)}G(T/2) + se^{2s(d_1-d_0)}.$$

We have $f(s) \sim g(s)$ at infinity. Moreover the function f has a minimum in

$$s_3 = \frac{1}{2(d_2-d_1)}\ln\left(\frac{d_0-d_1}{(d_2-d_0)G(T/2)}\right) \text{ and } f(s_3) = K'G(T/2)^\kappa$$

with $\kappa = \frac{d_0-d_1}{d_2-d_1}$ and $K' = (\frac{d_0-d_1}{d_2-d_0})^{\frac{d_2-d_0}{d_2-d_1}} + (\frac{d_0-d_1}{d_2-d_0})^{\frac{d_1-d_0}{d_2-d_0}}$. Finally the minimum s_3 is sufficiently large ($s_3 \ge s_2$) if the following condition $G(T/2) \le \delta_0$, with $\delta_0 = \frac{d_0-d_1}{(d_2-d_0)e^{2s_2(d_2-d_1)}}$, is satisfied. Then we get our result (3.3) and so we complete the proof of Theorem 3.1. □

Remark 4 • Note that the hypothesis (3.2) is quite usual (cf [1, 7] for a parabolic system in a bounded domain) and is removed in [1] by the control theory and in [7] by conditions on $a, \widetilde{b}, \widetilde{c}, d, \widetilde{u}_0, \widetilde{v}_0, h_1, h_2, g_1, g_2$. In some cases, one can also diagonalise the coupling matrix of the coefficients (see [6]) then use a parabolic positivity result (see [9, Theorem 13.5]) for the decoupling system. Of course we could obtain the same result as (3.3) for any coefficient in each equation of (1.1). But if we want to determine the coefficients b and d for example, we only have to assume that $|\widetilde{v}(., \frac{T}{2})| \geq R$ in Ω_L for some $R > 0$, instead of (3.2).
• In fact we can obtain in the right-hand side of (3.3) the term $\int_{\gamma_L\times(0,T)} \sum_{k=1}^{2} (|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2)\, d\sigma\, dt$ instead of $\int_{\gamma_L\times(0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2)\, d\sigma\, dt$ if we slightly modify d_1 (if we define $d_1 = \sup_{\overline{O_{L,\tilde{\varepsilon}}}} \phi$, the inequalities (2.4) still hold and all the terms inside the integrals on γ_L with derivatives of η are therefore bounded above by e^{2sd_1}).

3.2 The second result

Consider now (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a strong solution of (1.1) associated with $(a,\ b,\ c,\ d,\ u_0,$ $v_0,\ g_1,\ g_2,\ h_1,\ h_2)$ (resp. $(\widetilde{a},\ \widetilde{b},\ \widetilde{c},\ \widetilde{d},\ \widetilde{u}_0,\ \widetilde{v}_0,\ g_1,\ g_2,\ h_1,\ h_2)$). Assume that all the coefficients $a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ belong to $\Lambda(M_0)$. For our second main result, first we need the following lemma inspired from Klibanov and Timonov ([11]). Recall that χ and η are defined by (3.4).

Lemma 3.2 *There exists a positive constant C such that*

$$\int_{Q_L} e^{2s\phi}\phi\chi^2\eta^2 \left(\int_{T/2}^{t} f(\xi)\ d\xi\right)^2 dx\ dt \leq \frac{C}{s}\left(e^{2sd_1} + \int_{Q_L} e^{2s\phi}\chi^2\eta^2 f^2\ dx\ dt\right),$$

for all $s > 0$ and $f \in L^2(0, T, L^2(\Omega_L)) \cap L^\infty(Q_L)$.

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{Q_L} \phi\chi^2\eta^2 e^{2s\phi}\left(\int_{T/2}^{t} f(x,\xi)\ d\xi\right)^2 dx\ dt &\leq \int_{Q_L} \phi\chi^2\eta^2 e^{2s\phi}|t-\frac{T}{2}|\left|\int_{T/2}^{t} f(x,\xi)^2\ d\xi\right|\ dx\ dt\\ &\leq \int_{\Omega_L}\int_0^{T/2} \phi\chi^2\eta^2 e^{2s\phi}(\frac{T}{2}-t)\left|\int_{T/2}^{t} f(x,\xi)^2 d\xi\right| dxdt\\ &+ \int_{\Omega_L}\int_{T/2}^{T} \phi\chi^2\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^{t} f(x,\xi)^2 d\xi\right) dxdt.\end{aligned} \tag{3.11}$$

Note that

$$\partial_t(e^{2s\phi}) = -4s\lambda(t-\frac{T}{2})\phi e^{2s\phi}.$$

For the second integral of the right hand side of (3.11), since $\eta(T)=0$, by integration by parts we have

$$\int_{\Omega_L}\int_{T/2}^{T}\phi\chi^2\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^{t} f(x,\xi)^2 d\xi\right)\,dx\,dt$$

$$=-\frac{1}{4s\lambda}\int_{\Omega_L}\int_{T/2}^{T}\chi^2\eta^2\partial_t(e^{2s\phi})(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\,dx\,dt$$

$$=-\frac{1}{4s\lambda}\int_{\Omega_L}\left[\chi^2\eta^2 e^{2s\phi}(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\right]_{t=T/2}^{t=T}dx+\frac{1}{4s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta^2 f^2\,dx\,dt$$

$$+\frac{1}{2s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta\partial_t\eta(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\,dx\,dt$$

$$=\frac{1}{2s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta\partial_t\eta(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\,dx\,dt+\frac{1}{4s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta^2 f^2\,dx\,dt. \quad (3.12)$$

The first integral of (3.12) is bounded above by $\frac{C}{s}e^{2sd_1}$ due to the derivative of η. Therefore

$$\int_{\Omega_L}\int_{T/2}^{T}\phi\chi^2\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^{t} f(x,\xi)^2 d\xi\right)\,dx\,dt\le\frac{C}{s}\left(e^{2sd_1}+\int_{Q_L} e^{2s\phi}\chi^2\eta^2 f^2\,dx\,dt\right).$$

We obtain a similar result for the first integral of (3.11) and this concludes the proof of Lemma 3.2. □

Now we can state our second main result in view to obtain a stability estimate of the four coefficients of (1.1) with nearly the same observations that we obtained in Theorem 3.1 (see the right-hand sides of (3.3) and (3.15)).

Theorem 3.2 *Let $l>0$. Let $T>0$, $L>l$ and $a\in\mathbb{R}^n\setminus\Omega$ satisfying the conditions of Proposition 2.1. We make here the following assumptions*

$$|\widetilde{u}|\ge R \text{ and } |\partial_t(\frac{\widetilde{v}}{\widetilde{u}})|\ge R \text{ in } Q \text{ for some } R>0, \quad (3.13)$$

and

$$|\widetilde{v}|\ge R \text{ and } |\partial_t(\frac{\widetilde{u}}{\widetilde{v}})|\ge R \text{ in } Q \text{ for some } R>0. \quad (3.14)$$

Then there exists a sufficiently small number δ_0 such that if $\delta\in(0,\delta_0)$,

$$\|\sum_{k=0}^{1}\partial_t^k(u-\widetilde{u})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}+\|\sum_{k=0}^{1}\partial_t^k(v-\widetilde{v})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2+|\partial_\nu(\partial_t^k(v-\tilde{v}))|^2\,d\sigma\,dt\le\delta$$

then the following Hölder stability estimate holds

$$\|a-\widetilde{a}\|^2_{L^2(\Omega_l)}+\|b-\widetilde{b}\|^2_{L^2(\Omega_l)}+\|c-\widetilde{c}\|^2_{L^2(\Omega_l)}+\|d-\widetilde{d}\|^2_{L^2(\Omega_l)}\leq K\delta^{\kappa} \text{ for all } \delta\in(0,\delta_0). \tag{3.15}$$

Here, $K>0$ and $\kappa\in(0,1)$ are two constants depending on R, r, L, l, M_0, M_1, M_2, T and a.

Proof. As in Thereom 3.1 denote $y=u-\widetilde{u}$ and $z=v-\widetilde{v}$. Then (y,z) satisfies

$$\begin{cases} \partial_t y-\Delta y+ay+bz=(\widetilde{a}-a)\widetilde{u}+(\widetilde{b}-b)\widetilde{v} \text{ in } Q,\\ \partial_t z-\Delta z+cy+dz=(\widetilde{c}-c)\widetilde{u}+(\widetilde{d}-d)\widetilde{v} \text{ in } Q,\\ y=z=0 \text{ on } \Sigma. \end{cases}$$

• First step: Let $y_1=\frac{y}{\widetilde{u}}$ and $z_1=\frac{z}{\widetilde{u}}$. Then (y_1,z_1) satisfies

$$\begin{cases} \partial_t y_1-\Delta y_1+ay_1+bz_1=f_1+\widetilde{a}-a+(\widetilde{b}-b)\frac{\widetilde{v}}{\widetilde{u}} \text{ in } Q,\\ \partial_t z_1-\Delta z_1+cy_1+dz_1=f_2+\widetilde{c}-c+(\widetilde{d}-d)\frac{\widetilde{v}}{\widetilde{u}} \text{ in } Q,\\ y_1=z_1=0 \text{ on } \Sigma, \end{cases}$$

with $f_1:=\frac{1}{\widetilde{u}}(-y_1\partial_t\widetilde{u}+y_1\Delta\widetilde{u}+2\nabla y_1\cdot\nabla\widetilde{u})$ and $f_2:=\frac{1}{\widetilde{u}}(-z_1\partial_t\widetilde{u}+z_1\Delta\widetilde{u}+2\nabla z_1\cdot\nabla\widetilde{u})$. Denote now $y_2=\partial_t y_1$, $z_2=\partial_t z_1$, $y_3=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}y_2$ and $z_3=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}z_2$. Then

$$\begin{cases} \partial_t y_2-\Delta y_2+ay_2+bz_2=\partial_t f_1+(\widetilde{b}-b)\partial_t(\frac{\widetilde{v}}{\widetilde{u}}) \text{ in } Q,\\ \partial_t z_2-\Delta z_2+cy_2+dz_2=\partial_t f_2+(\widetilde{d}-d)\partial_t(\frac{\widetilde{v}}{\widetilde{u}}) \text{ in } Q,\\ y_2=z_2=0 \text{ on } \Sigma, \end{cases}$$

and

$$\begin{cases} \partial_t y_3-\Delta y_3+ay_3+bz_3=f_3+\widetilde{b}-b \text{ in } Q,\\ \partial_t z_3-\Delta z_3+cy_3+dz_3=f_4+\widetilde{d}-d \text{ in } Q,\\ y_3=z_3=0 \text{ on } \Sigma, \end{cases} \tag{3.16}$$

with

$$f_3:=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}\left(-y_3\partial_t^2(\frac{\widetilde{v}}{\widetilde{u}})+y_3\Delta(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+2\nabla y_3\cdot\nabla(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+\partial_t f_1\right)$$

and

$$f_4:=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}\left(-z_3\partial_t^2(\frac{\widetilde{v}}{\widetilde{u}})+z_3\Delta(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+2\nabla z_3\cdot\nabla(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+\partial_t f_2\right).$$

Finally let $y_4=\partial_t y_3$, $z_4=\partial_t z_3$, $y_5=\chi\eta y_4$ and $z_5=\chi\eta z_4$. Then

$$\begin{cases} \partial_t y_5-\Delta y_5+ay_5+bz_5=\chi\eta\partial_t f_3+f_5 \text{ in } Q_L,\\ \partial_t z_5-\Delta z_5+cy_5+dz_5=\chi\eta\partial_t f_4+f_6 \text{ in } Q_L, \end{cases} \tag{3.17}$$

with

$$f_5=(\partial_t\eta)\chi y_4-(\Delta\chi)\eta y_4-2\eta\nabla\chi\cdot\nabla y_4$$

and

$$f_6 = (\partial_t \eta)\chi z_4 - (\Delta\chi)\eta z_4 - 2\eta\nabla\chi\cdot\nabla z_4.$$

Due to the truncation functions, we can apply the Carleman estimates for y_5 and z_5 and now we estimate $I(y_5)+I(z_5)$ with (2.5). We have

$$\begin{aligned} I(y_5)+I(z_5) \le C\int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)dxdt + Cs^3e^{2sd_1} \\ + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_5|^2+|\partial_\nu z_5|^2)d\sigma dt. \end{aligned} \tag{3.18}$$

As in Thereom 3.1, all the terms in $\int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)\; dx\; dt$ with derivatives of η or χ will be bounded above by Ce^{2sd_1}. So since $\phi\ge 1$

$$\int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)\; dx\; dt \le C\int_{Q_L} e^{2s\phi}(y_5^2+z_5^2)\; dx\; dt + Ce^{2sd_1}$$

$$+C\int_{Q_L} e^{2s\phi}\chi^2\eta^2(|\partial_t f_3|^2+|\partial_t f_4|^2)\; dx\; dt$$

$$\le C\int_{Q_L} e^{2s\phi}(y_5^2+z_5^2)dxdt + Ce^{2sd_1} + C\int_{Q_L}\phi e^{2s\phi}\chi^2\eta^2\sum_{i=1}^{4}(y_i^2+|\nabla y_i|^2+z_i^2+|\nabla z_i|^2)dxdt. \tag{3.19}$$

Since $\chi\eta y_4 = y_5$ and $\chi\eta z_4 = z_5$, (3.19) implies

$$\int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)\; dx\; dt \le C\int_{Q_L} e^{2s\phi}(y_5^2+z_5^2+|\nabla y_5|^2+|\nabla z_5|^2)\; dx\; dt + Ce^{2sd_1}$$

$$+ C\int_{Q_L}\phi e^{2s\phi}\chi^2\eta^2\sum_{i=1}^{3}(y_i^2+|\nabla y_i|^2+z_i^2+|\nabla z_i|^2)\; dx\; dt. \tag{3.20}$$

From (3.18)-(3.20), we get for s sufficiently large

$$I(y_5)+I(z_5) \le Cs^3e^{2sd_1} + C\int_{Q_L}\phi e^{2s\phi}\chi^2\eta^2\sum_{i=1}^{3}(y_i^2+|\nabla y_i|^2+z_i^2+|\nabla z_i|^2)\; dx\; dt$$

$$+ Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_5|^2+|\partial_\nu z_5|^2)\; d\sigma\; dt. \tag{3.21}$$

Using now Lemma 3.2 we have

$$\int_{Q_L}\phi e^{2s\phi}\chi^2\eta^2 y_1^2\; dx\; dt = \int_{Q_L}\phi e^{2s\phi}\chi^2\eta^2\left(\int_{T/2}^{t}\partial_t y_1(\xi)d\xi + y_1(T/2)\right)^2\; dx\; dt$$

$$\le \frac{C}{s}e^{2sd_1} + \frac{C}{s}\int_{Q_L} e^{2s\phi}\chi^2\eta^2 y_2^2\; dx\; dt + C\int_{Q_L}\phi e^{2s\phi}\chi^2\eta^2 y_1(T/2)^2\; dx\; dt$$

$$\leq \frac{C}{s}e^{2sd_1} + \frac{C}{s}\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_3^2 \; dx \; dt + C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_1(T/2)^2 \; dx \; dt$$

$$\leq \frac{C}{s}e^{2sd_1}+\frac{C}{s}\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 \left(\int_{T/2}^{t} \partial_t y_3(\xi)d\xi + y_3(T/2)\right)^2 dx \; dt+C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_1(T/2)^2 dx \; dt$$

$$\leq \frac{C}{s}e^{2sd_1}+\frac{C}{s^2}\left(e^{2sd_1} + \int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_4^2 \; dx \; dt\right)+C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 (y_1(T/2)^2+y_3(T/2)^2) \; dx \; dt$$

$$\leq \frac{C}{s}e^{2sd_1} + \frac{C}{s^2}\int_{Q_L} e^{2s\phi} y_5^2 \; dx \; dt + Ce^{2sd_2}\int_{\Omega_L} (y_1(T/2)^2 + y_2(T/2)^2) \; dx. \tag{3.22}$$

Doing the same for $\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_i^2 \; dx \; dt$, $\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 z_i^2 \; dx \; dt$, $\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 |\nabla y_i|^2 \; dx \; dt$ and
$\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 |\nabla z_i|^2 \; dx \; dt$, for $i = 1,2,3$ we get from (3.21)-(3.22) and for s sufficienlty large

$$I(y_5) + I(z_5) \leq Cs^3 e^{2sd_1} + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_5|^2 + |\partial_\nu z_5|^2) \; d\sigma \; dt$$

$$+ Ce^{2sd_2}\int_{\Omega_L} \sum_{i=1}^{2}(y_i(T/2)^2 + z_i(T/2)^2 + |\nabla y_i(T/2)|^2 + |\nabla z_i(T/2)|^2) \; dx. \tag{3.23}$$

Note that (3.23) can be rewritten on the following form

$$I(y_5) + I(z_5) \leq Cs^3 e^{2sd_1} + Cse^{2sd_2}\int_{\gamma_L\times(0,T)} \sum_{k=0}^{2}(|\partial_\nu \partial_t^k y|^2 + |\partial_\nu \partial_t^k z|^2) \; d\sigma \; dt$$

$$+Ce^{2sd_2}\int_{\Omega_L} \sum_{k=0}^{1}(\partial_t^k y(T/2)^2 + \partial_t^k z(T/2)^2 + |\nabla \partial_t^k y(T/2)|^2 + |\nabla \partial_t^k z(T/2)|^2) \; dx$$

and so

$$I(y_5) + I(z_5) \leq Cs^3 e^{2sd_1} + Cse^{2sd_2} F_1(T/2) \tag{3.24}$$

with

$$F_1(T/2) = \int_{\gamma_L\times(0,T)} \sum_{k=0}^{2}(|\partial_\nu \partial_t^k y|^2+|\partial_\nu \partial_t^k z|^2) \; d\sigma \; dt+\sum_{k=0}^{1}(\|\partial_t^k y(T/2)\|^2_{H^1(\Omega_L)}+\|\partial_t^k z(T/2)\|^2_{H^1(\Omega_L)}).$$

• Second step: Now we evaluate (3.16) at $T/2$. We have

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b} - b|^2 + |\widetilde{d} - d|^2) \; dx \leq C\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\partial_t y_3(T/2)|^2 + |\partial_t z_3(T/2)|^2) \; dx$$
$$+ Ce^{2sd_2}F_2(T/2)$$

with

$$F_2(T/2) = \sum_{i=1}^{2}(\|y_i(T/2)\|^2_{H^2(\Omega_L)} + \|z_i(T/2)\|^2_{H^2(\Omega_L)}).$$

So, since $\eta(T/2)=1$,

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le C\int_{\Omega_L} e^{2s\phi(T/2)}(|y_5(T/2)|^2+|z_5(T/2)|^2)\,dx+Ce^{2sd_2}F_2(T/2). \tag{3.25}$$

Now let $\psi_1=e^{s\phi}y_5$ and $\psi_2=e^{s\phi}z_5$. Calculate $J_1=\int_{\Omega_L}\int_0^{T/2}\partial_t\psi_1(t)\psi_1(t)\,dx\,dt$ and $J_2=\int_{\Omega_L}\int_0^{T/2}\partial_t\psi_2(t)\psi_2(t)\,dx\,dt$. Since $\eta(0)=0$, we get

$$J_1=\frac{1}{2}\int_{\Omega_L}\psi_1(T/2)^2\,dx=\frac{1}{2}\int_{\Omega_L} e^{2s\phi(T/2)}y_5(T/2)^2\,dx \text{ and } J_2=\frac{1}{2}\int_{\Omega_L} e^{2s\phi(T/2)}z_5(T/2)^2\,dx.$$

Therefore (3.25) becomes

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le Ce^{2sd_2}F_2(T/2)$$

$$+C\int_{\Omega_L}\int_0^{T/2}\frac{1}{s}\partial_t\psi_1(t)s\psi_1(t)\,dx\,dt+C\int_{\Omega_L}\int_0^{T/2}\frac{1}{s}\partial_t\psi_2(t)s\psi_2(t)\,dx\,dt. \tag{3.26}$$

Using Young inequality, we deduce from (3.26)

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le \frac{C}{s}(I(y_5)+I(z_5))+Ce^{2sd_2}F_2(T/2). \tag{3.27}$$

From (3.24) and (3.27) we get

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le Cs^2e^{2sd_1}+Ce^{2sd_2}(F_1(T/2)+F_2(T/2)). \tag{3.28}$$

Proceeding as in Theorem 3.1, we obtain from (3.28)

$$\int_{\Omega_l}(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le Cs^2e^{2s(d_1-d_0)}+Ce^{2s(d_2-d_0)}F_3(T/2) \tag{3.29}$$

with

$$F_3(T/2)=\sum_{k=0}^{1}(\|\partial_t^k y(T/2)\|^2_{H^2(\Omega_L)}+\|\partial_t^k z(T/2)\|^2_{H^2(\Omega_L)})+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2+|\partial_\nu\partial_t^k z|^2)\,d\sigma\,dt.$$

Notice that in the first and second steps of this proof, we have only used the hypothesis (3.13).

• Third step: Finally using the hypothesis (3.14), we can proceed exactly as before and obtain

$$\int_{\Omega_l}(|\widetilde{a}-a|^2+|\widetilde{c}-c|^2)\,dx \le Cs^2e^{2s(d_1-d_0)}+Ce^{2s(d_2-d_0)}F_3(T/2). \tag{3.30}$$

From (3.29)-(3.30) we end the proof of Thereom 3.2. $\square$

Remark 5 • First note that our stability results (3.3) and (3.15) are obtained on Ω_l for the left-hand term while the observation data $G(T/2)$ and $F_3(T/2)$ are required on Ω_L for the right-hand term of (3.3), (3.15).

• Second we have used Lemma 3.2 instead of Lemma 3.1 in the proof of Theorem 3.2 in order to avoid a third derivative with respect to t in the observation terms. Indeed, if we no longer used Lemma 3.2 in the proof of Theorem 3.2, we could use a modified version of Lemma 3.1: applying (3.1) with $w = e^{s\phi}\chi\eta z$, we could obtain the following inequality

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2|z(T/2)|^2\,dx \le Cs\int_{Q_L} e^{2s\phi}\phi^2\chi^2\eta^2|z|^2\,dx\,dt + \frac{C}{s}e^{2sd_1} + \frac{C}{s}\int_{Q_L} e^{2s\phi}\chi^2\eta^2|\partial_t z|^2\,dx\,dt,$$

for all $z \in H^1(0,T;L^2(\Omega_L))$.

Moreover, if we did so, since we had to give up the end of the first step of the proof of Theorem 3.2, we'd rather follow the ideas of the proof of Theorem 3.1. Therefore, when in the second step we evaluated (3.16) for $t = T/2$, with the above inequality we would have to estimate $\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2|\partial_t y_3(T/2)|^2\,dx$ and $\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2|\partial_t z_3(T/2)|^2\,dx$; thus we could obtain $\int_{Q_L} e^{2s\phi}\chi^2\eta^2|\partial_t y_4|^2\,dx\,dt$ and $\int_{Q_L} e^{2s\phi}\chi^2\eta^2|\partial_t z_4|^2\,dx\,dt$ in the right-hand side of the estimates. Then we would have to apply the Carleman estimates for $\chi\eta y_4, \chi\eta z_4, \chi\eta\partial_t y_4, \chi\eta\partial_t z_4$ and so we would obtain a third derivative in time for the observation terms.

• Third the assumptions (3.13) and (3.14) are equivalent to $|\widetilde{u}| \ge R$, $|\widetilde{v}| \ge R$ and $\left|\det\left(\begin{pmatrix}\widetilde{u} & \partial_t\widetilde{u}\\ \widetilde{v} & \partial_t\widetilde{v}\end{pmatrix}\right)\right| \ge R$ with R a positive constant. For example, if $n=2$ and $\omega = (r_1, r_2)$ with $r_1 > 0$, let $\alpha(x_1)$ be a positive and bounded function in $C^2(\mathbb{R})$ such that $\min_{x_1\in\mathbb{R}}\alpha(x_1) > 2r_2^2$. Then $\widetilde{u}(x,t) = \alpha(x_1)t + x_2$ and $\widetilde{v}(x,t) = tx_2 + 1$ are solutions of the system (1.1) with $g_1 = g_2 = 0$, $\widetilde{a}(x) = \frac{\alpha''(x_1)+\alpha(x_1)x_2}{\alpha(x_1)-x_2^2}$, $\widetilde{b}(x) = \frac{-x_2\alpha''(x_1)-\alpha(x_1)^2}{\alpha(x_1)-x_2^2}$, $\widetilde{c}(x) = \frac{x_2^2}{\alpha(x_1)-x_2^2}$, $\widetilde{d}(x) = \frac{-x_2\alpha(x_1)}{\alpha(x_1)-x_2^2}$, and satisfy the conditions (3.13)-(3.14).

• Finally note that the above results remain valid for the system (1.2) when all the coefficients $a, b, c, d, A_1, A_2, A_3, A_4$ are bounded ($a, b, c, d \in \Lambda(M_0)$ and $A_1, A_2, A_3, A_4 \in (\Lambda(M_0))^n$). We obtain a stability result of at least two coefficients between a, b, c, d with the same observations in the right-hand sides of (3.3) or (3.15). In the next section we study the inverse problem of determining at least one of the coefficient A_1, A_2, A_3, A_4, for example A_3 if we assume that this coefficient has the form $A_3 = \nabla g$.

3.3 The third result

Consider now (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a strong solution of (1.2) associated with (a, b, c, d, A_1, A_2, A_3, A_4, u_0, v_0, g_1, g_2, h_1, h_2) (resp. (a, $\widetilde{b}$, c, d, A_1, A_2, $\widetilde{A_3}$, A_4, $\widetilde{u_0}$, $\widetilde{v_0}$, g_1, g_2, h_1, h_2)). Assume that all the coefficients a, b, c, d belong to $\Lambda(M_0)$, $A_1, A_2, A_3, A_4, \widetilde{A_3}$ belong to $(\Lambda(M_0))^n \cap (H^1(\Omega))^n$ and that there exist functions $g, \widetilde{g}$ such that

$$A_3 = \nabla g,\ \ \widetilde{A_3} = \nabla\widetilde{g} \text{ in } \Omega. \tag{3.31}$$

The Assumption (3.31) implies conditions on $A_3, \widetilde{A_3}$: if ${}^tA_3 = (c_1, \cdots, c_n)$, it means that for all $i, j = 1, \cdots, n$, $\partial_{x_i} c_j = \partial_{x_j} c_i$, in other words $rot(A_3) = 0$ if $n = 3$.
Now following an idea developed in [10] for Lamé system in bounded domains, also used for example in [4], we obtain the following result

Lemma 3.3 *Assume that the following assumption*

$$|\nabla d \cdot \nabla \widetilde{u}(T/2)| \geq R \text{ in } \Omega_L \text{ for some } R > 0 \tag{3.32}$$

holds. Consider the first order partial differential operator $Pf = \nabla f \cdot \nabla \widetilde{u}(T/2)$. Then there exist positive constants $s_4 > 0$ and $C > 0$ such that for all $s \geq s_4$,

$$s^2 \int_{\Omega_L} e^{2s\phi(T/2)} |f|^2 \, dx \leq C \int_{\Omega_L} e^{2s\phi(T/2)} |Pf|^2 \, dx,$$

for all $f \in H_0^1(\Omega_L)$.

Proof. The proof follows [4]. Let $f \in H_0^1(\Omega_L)$. Denote $w = e^{s\phi(T/2)} f$ and $Qw = e^{s\phi(T/2)} P(e^{-s\phi(T/2)} w)$. So we get $Qw = Pw - sw\nabla\phi(T/2) \cdot \nabla \widetilde{u}(T/2)$. Therefore we have

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2 \int_{\Omega_L} w^2 |\nabla\phi(T/2) \cdot \nabla \widetilde{u}(T/2)|^2 \, dx - 2s \int_{\Omega_L} (Pw) w (\nabla\phi(T/2) \cdot \nabla \widetilde{u}(T/2)) \, dx$$

$$\begin{aligned} \int_{\Omega_L} |Qw|^2 \, dx &\geq s^2 \lambda^2 \int_{\Omega_L} w^2 (\phi(T/2))^2 |\nabla d \cdot \nabla \widetilde{u}(T/2)|^2 \, dx \\ &-2s\lambda \int_{\Omega_L} (\nabla w \cdot \nabla \widetilde{u}(T/2)) w \phi(T/2) (\nabla d \cdot \nabla \widetilde{u}(T/2)) \, dx. \end{aligned}$$

So

$$\begin{aligned} \int_{\Omega_L} |Qw|^2 \, dx &\geq s^2 \lambda^2 \int_{\Omega_L} w^2 (\phi(T/2))^2 |\nabla d \cdot \nabla \widetilde{u}(T/2)|^2 \, dx \\ &-s\lambda \int_{\Omega_L} \phi(T/2) (\nabla d \cdot \nabla \widetilde{u}(T/2)) (\nabla(w^2) \cdot \nabla \widetilde{u}(T/2)) \, dx. \end{aligned}$$

Thus integrating by parts

$$\begin{aligned} \int_{\Omega_L} |Qw|^2 \, dx &\geq s^2 \lambda^2 \int_{\Omega_L} w^2 (\phi(T/2))^2 |\nabla d \cdot \nabla \widetilde{u}(T/2)|^2 \, dx \\ &+s\lambda \int_{\Omega_L} w^2 \nabla \cdot (\phi(T/2)(\nabla d \cdot \nabla \widetilde{u}(T/2)) \nabla \widetilde{u}(T/2)) \, dx \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_L} e^{2s\phi(T/2)} |Pf|^2 \, dx = \int_{\Omega_L} |Qw|^2 \, dx &\geq s^2 \lambda^2 \int_{\Omega_L} e^{2s\phi(T/2)} f^2 (\phi(T/2))^2 |\nabla d \cdot \nabla \widetilde{u}(T/2)|^2 \, dx \\ &+s\lambda \int_{\Omega_L} e^{2s\phi(T/2)} f^2 \nabla \cdot (\phi(T/2)(\nabla d \cdot \nabla \widetilde{u}(T/2)) \nabla \widetilde{u}(T/2)) \, dx. \end{aligned}$$

And we can conclude for s sufficiently large. □

The strong positivity assumption (3.32) is frequently involved in inverse problems and is removed in [4] for one equation by the construction of an adapted control. Now we state the third result.

Theorem 3.3 *Let $l > 0$. Let $T > 0$, $L > l$ and $a \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Assume that Assumptions (3.31) and (3.32) hold. We also make the following hypothesis*

$$|\widetilde{v}(., \frac{T}{2})| \geq R \text{ in } \Omega_L \text{ for some } R > 0. \tag{3.33}$$

If $g = \widetilde{g}$ and $A_3 = \widetilde{A_3}$ on $\partial\Omega \cap \partial\Omega_L$, then there exists a sufficiently small number δ_0 such that if $\delta \in (0, \delta_0)$,

$$\|\sum_{k=0}^{1} \partial_t^k (u - \widetilde{u})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \|\sum_{k=0}^{1} \partial_t^k (v - \widetilde{v})(., \frac{T}{2})\|^2_{H^3(\Omega_L)}$$

$$+ \int_{\gamma_L \times (0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k (u - \tilde{u}))|^2 + |\partial_\nu(\partial_t^k (v - \tilde{v}))|^2 \, d\sigma \, dt \leq \delta$$

then the following Hölder stability estimate holds

$$\|b - \widetilde{b}\|^2_{L^2(\Omega_l)} + \|A_3 - \widetilde{A_3}\|^2_{(L^2(\Omega_l))^n} \leq K\delta^\kappa \text{ for all } \delta \in (0, \delta_0). \tag{3.34}$$

Here, $K > 0$ and $\kappa \in (0, 1)$ are two constants depending on R, r, L, l, M_0, M_1, M_2, T and a.

Proof. As in Theorem 3.1 denote

$$y = u - \widetilde{u},\ y_0 = \chi\eta y,\ y_1 = \partial_t y_0,\ y_2 = \partial_t y_1,\ z = v - \widetilde{v},\ z_0 = \chi\eta z,\ z_1 = \partial_t z_0 \text{ and } z_2 = \partial_t z_1.$$

Then (y_0, z_0) satisfies

$$\begin{cases} \partial_t y_0 - \Delta y_0 + a y_0 + b z_0 + A_1 \cdot \nabla y_0 + A_2 \cdot \nabla z_0 = \xi_1 \text{ in } Q_L, \\ \partial_t z_0 - \Delta z_0 + c y_0 + d z_0 + A_3 \cdot \nabla y_0 + A_4 \cdot \nabla z_0 = \xi_2 \text{ in } Q_L, \\ y_0 = z_0 = 0 \text{ on } \partial\Omega_L \times (0, T) \end{cases} \tag{3.35}$$

with

$$\xi_1 := \chi\eta(\widetilde{b} - b)\widetilde{v} + (\partial_t\eta)\chi y - (\Delta\chi)\eta y - 2\nabla\chi \cdot \nabla(\eta y) + \eta y A_1 \cdot \nabla\chi + \eta z A_2 \cdot \nabla\chi$$

and

$$\xi_2 := \chi\eta(\widetilde{A_3} - A_3) \cdot \nabla\widetilde{u} + (\partial_t\eta)\chi z - (\Delta\chi)\eta z - 2\nabla\chi \cdot \nabla(\eta z) + \eta y A_3 \cdot \nabla\chi + \eta z A_4 \cdot \nabla\chi.$$

Then

$$\xi_2 = \eta\nabla(\chi(\widetilde{g} - g)) \cdot \nabla\widetilde{u} - \eta(\widetilde{g} - g)\nabla\chi \cdot \nabla\widetilde{u} + (\partial_t\eta)\chi z - (\Delta\chi)\eta z - 2\nabla\chi \cdot \nabla(\eta z) + \eta y A_3 \cdot \nabla\chi + \eta z A_4 \cdot \nabla\chi.$$

• First step: We evaluate (3.35) for $t = \frac{T}{2}$ and we get

$$\partial_t y_0(T/2) - \Delta y_0(T/2) + a y_0(T/2) + b z_0(T/2) + A_1 \cdot \nabla y_0(T/2) + A_2 \cdot \nabla z_0(T/2)$$

$$= \chi(\widetilde{b} - b)\widetilde{v}(T/2) - (\Delta\chi) y(T/2) - 2\nabla\chi \cdot \nabla y(T/2) + y(T/2) A_1 \cdot \nabla\chi + z(T/2) A_2 \cdot \nabla\chi \quad (3.36)$$

and

$$\begin{aligned} &\partial_t z_0(T/2) - \Delta z_0(T/2) + c y_0(T/2) + d z_0(T/2) + A_3 \cdot \nabla y_0(T/2) + A_4 \cdot \nabla z_0(T/2) \\ &= P(\chi(\widetilde{g} - g)) - (\widetilde{g} - g)\nabla\chi \cdot \nabla\widetilde{u}(T/2) - (\Delta\chi) z(T/2) - 2\nabla\chi \cdot \nabla z(T/2) + y(T/2) A_3 \cdot \nabla\chi \\ &\qquad + z(T/2) A_4 \cdot \nabla\chi \end{aligned} \quad (3.37)$$

with P the operator defined in Lemma 3.3. From (3.36) we have

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} \chi^2 |b - \widetilde{b}|^2 |\widetilde{v}(\frac{T}{2})|^2 \, dx \leq C \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} |\partial_t y_0(\frac{T}{2})|^2 \, dx$$

$$+ C e^{2sd_2} (\|z_0(T/2)\|^2_{H^1(\Omega_L)} + \|y_0(T/2)\|^2_{H^2(\Omega_L)} + \|y(T/2)\|^2_{H^1(\Omega_L)} + \|z(T/2)\|^2_{L^2(\Omega_L)}).$$

So

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} \chi^2 |b - \widetilde{b}|^2 |\widetilde{v}(\frac{T}{2})|^2 \, dx \leq C e^{2sd_2} F_1(T/2) + C \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} |\partial_t y_0(\frac{T}{2})|^2 \, dx$$

with

$$F_1(T/2) = \|y(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^1(\Omega_L)}.$$

Then, applying Lemma 3.1 we get

$$\begin{aligned} \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} \chi^2 |b - \widetilde{b}|^2 |\widetilde{v}(\frac{T}{2})|^2 \, dx &\leq C e^{2sd_2} F_1(T/2) + C s \int_{Q_L} e^{2s\phi} \phi^3 |y_1|^2 \, dx \, dt \\ &\quad + \frac{C}{s} \int_{Q_L} e^{2s\phi} \phi^3 |y_2|^2 \, dx \, dt. \end{aligned} \quad (3.38)$$

Moreover using Lemma 3.3 for (3.37) we have

$$s^2 \int_{\Omega_L} e^{2s\phi(T/2)} \chi^2 (\widetilde{g} - g)^2 \, dx \leq C \int_{\Omega_L} e^{2s\phi(T/2)} |P(\chi(\widetilde{g} - g))|^2 \, dx$$

$$\leq C e^{2sd_1} + C \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} |\partial_t z_0(\frac{T}{2})|^2 \, dx$$

$$+ C e^{2sd_2} (\|z_0(T/2)\|^2_{H^2(\Omega_L)} + \|y_0(T/2)\|^2_{H^1(\Omega_L)} + \|y(T/2)\|^2_{L^2(\Omega_L)} + \|z(T/2)\|^2_{H^1(\Omega_L)}).$$

Applying again Lemma 3.1 we get

$$s^2 \int_{\Omega_L} e^{2s\phi(T/2)} \chi^2 (\widetilde{g} - g)^2 dx \leq C e^{2sd_1} + C e^{2sd_2} F_2(T/2) + C s \int_{Q_L} e^{2s\phi} \phi^3 |z_1|^2 \, dx \, dt$$

$$+\frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3|z_2|^2dxdt \tag{3.39}$$

with

$$F_2(T/2) = \|y(T/2)\|^2_{H^1(\Omega_L)} + \|z(T/2)\|^2_{H^2(\Omega_L)}.$$

From (3.38)-(3.39) we obtain

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2|b-\widetilde{b}|^2|\widetilde{v}(\frac{T}{2})|^2\ dx + \int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(\widetilde{g}-g)^2\ dx$$

$$\leq \frac{C}{s^2}e^{2sd_1} + Ce^{2sd_2}F_3(T/2) + Cs\int_{Q_L} e^{2s\phi}\phi^3(|y_1|^2+|z_1|^2)dxdt + \frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3(|y_2|^2+|z_2|^2)dxdt \tag{3.40}$$

with

$$F_3(T/2) = \|y(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^2(\Omega_L)}.$$

Using now Assumption (3.33), we get from (3.40) and for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2((b-\widetilde{b})^2 + (\widetilde{g}-g)^2)\ dx \leq \frac{C}{s^2}e^{2sd_1} + Ce^{2sd_2}F_3(T/2)$$

$$+ Cs\int_{Q_L} e^{2s\phi}\phi^3(|y_1|^2+|z_1|^2)\ dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3(|y_2|^2+|z_2|^2)\ dx\ dt. \tag{3.41}$$

• Second step: As in Theorem 3.1, now we use the Carleman inequalities (2.5) for y_i and z_i, $i=1,2$. Recall that $\phi \leq \phi(T/2)$ so we get for s sufficiently large

$$I(y_i) + I(z_i) \leq C\int_{\Omega_L} e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2 + \chi^2|b-\widetilde{b}|^2)\ dx + Cs^3e^{2sd_1}$$

$$+Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt.$$

Thus

$$s^3\int_{Q_L} e^{2s\phi}\phi^3(|y_i|^2+|z_i|^2)\ dx\ dt \leq C\int_{\Omega_L} e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2 + \chi^2|b-\widetilde{b}|^2)\ dx$$

$$+ Cs^3e^{2sd_1} + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt. \tag{3.42}$$

Therefore, from (3.41) and (3.42), we get for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2((b-\widetilde{b})^2 + (\widetilde{g}-g)^2)\ dx \leq Ce^{2sd_2}F_3(T/2) + Cse^{2sd_1}$$

$$+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2+\chi^2|b-\widetilde{b}|^2)\ dx + \frac{C}{s}\int_{\gamma_L\times(0,T)} e^{2s\phi}\sum_{i=1}^{2}(|\partial_\nu y_i|^2+|\partial_\nu z_i|^2)\ d\sigma\ dt.$$

Thus we have for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2((b-\widetilde{b})^2+(\widetilde{g}-g)^2)\,dx \le Ce^{2sd_2}F_4(T/2)+Cse^{2sd_1}+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(T/2)}|\nabla(\chi(g-\widetilde{g}))|^2|\,dx \tag{3.43}$$

with

$$F_4(T/2) = F_3(T/2) + \int_{\gamma_L\times(0,T)} \sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2 + |\partial_\nu\partial_t^k z|^2)\,d\sigma\,dt.$$

• Third step: We apply the same ideas for $\nabla(\chi(\widetilde{g}-g))$. For any integer $1 \le i \le n$, taking the space derivative with respect to x_i in (3.37), we obtain

$$\begin{aligned}\partial_t\partial_{x_i} z_0(T/2) &- \Delta\partial_{x_i} z_0(T/2) + \partial_{x_i}\left(cy_0(T/2) + dz_0(T/2) + A_3\cdot\nabla y_0(T/2) + A_4\cdot\nabla z_0(T/2)\right)\\ &= P(\partial_{x_i}(\chi(\widetilde{g}-g))) + \nabla(\chi(\widetilde{g}-g))\cdot\nabla(\partial_{x_i}\widetilde{u}(T/2)) - \partial_{x_i}((\widetilde{g}-g)\nabla\chi\cdot\nabla\widetilde{u}(T/2))\\ &- \partial_{x_i}((\Delta\chi)z(T/2) - 2\nabla\chi\cdot\nabla z(T/2) + y(T/2)A_3\cdot\nabla\chi + z(T/2)A_4\cdot\nabla\chi).\end{aligned} \tag{3.44}$$

We can apply again Lemma 3.3: there exists a positive constant C such that for s sufficiently large,

$$s^2\int_{\Omega_L} e^{2s\phi(T/2)}\partial_{x_i}(\chi(\widetilde{g}-g))^2\,dx \le C\int_{\Omega_L} e^{2s\phi(T/2)}(P(\partial_{x_i}(\chi(\widetilde{g}-g))))^2\,dx.$$

Thus, using (3.44) we obtain

$$\begin{aligned}s^2\int_{\Omega_L} e^{2s\phi(T/2)}(\partial_{x_i}(\chi(\widetilde{g}-g)))^2\,dx &\le Ce^{2sd_2}F_5(T/2)+Ce^{2sd_1}+C\int_{\Omega_L} e^{2s\phi(T/2)}|\partial_{x_i}z_1(T/2)|^2\,dx\\ &+C\int_{\Omega_L} e^{2s\phi(T/2)}|\nabla(\chi(g-\widetilde{g}))|^2|\,dx\end{aligned}$$

with $F_5(T/2) = \|z(T/2)\|^2_{H^3(\Omega_L)} + \|y(T/2)\|^2_{H^2(\Omega_L)}$. So using Lemma 3.1 we get

$$\begin{aligned}s^2\int_{\Omega_L} e^{2s\phi(T/2)}(\partial_{x_i}(\chi(\widetilde{g}-g)))^2\,dx &\le Ce^{2sd_2}F_5(T/2)+Ce^{2sd_1}+C\int_{\Omega_L} e^{2s\phi(T/2)}|\nabla(\chi(g-\widetilde{g}))|^2|\,dx\\ &+ Cs\int_{Q_L} e^{2s\phi}(\partial_{x_i}z_1)^2\,dx\,dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}(\partial_{x_i}z_2)^2\,dx\,dt.\end{aligned} \tag{3.45}$$

Moreover by the Carleman inequality (2.5), we have for $j = 1, 2$,

$$\begin{aligned}s\int_{Q_L} e^{2s\phi}(z_j^2 + |\nabla z_j|^2)\,dx\,dt &\le C\int_{Q_L} e^{2s\phi}|Az_j|^2\,dx\,dt + Cs^3e^{2sd_1}\|z_j\|^2_{H^{2,1}(Q_L)}\\ &+Cs\int_{\gamma_L\times(0,T)} |\partial_\nu z_j|^2 e^{2s\phi}\,d\sigma\,dt.\end{aligned}$$

Thus

$$s\int_{Q_L} e^{2s\phi}(z_j^2+|\nabla z_j|^2)\,dx\,dt \le C\int_{Q_L} e^{2s\phi}(y_j^2+|\nabla y_j|^2+z_j^2+|\nabla z_j|^2)\,dx\,dt+\int_{Q_L} e^{2s\phi}|\nabla(\chi(\widetilde{g}-g))|^2\,dx\,dt$$

$$+Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}|\partial_\nu z_j|^2e^{2s\phi}\,d\sigma\,dt. \tag{3.46}$$

By the same way we obtain

$$s\int_{Q_L}e^{2s\phi}(y_j^2+|\nabla y_j|^2)\,dx\,dt\le C\int_{Q_L}e^{2s\phi}(y_j^2+|\nabla y_j|^2+z_j^2+|\nabla z_j|^2)\,dx\,dt+\int_{Q_L}e^{2s\phi}(\chi(\widetilde{b}-b))^2\,dx\,dt$$

$$+Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}|\partial_\nu y_j|^2e^{2s\phi}\,d\sigma\,dt. \tag{3.47}$$

From (3.46) and (3.47) we deduce

$$s\int_{Q_L}e^{2s\phi}(z_j^2+y_j^2+|\nabla z_j|^2+|\nabla y_j|^2)dxdt\le C\int_{Q_L}e^{2s\phi}(y_j^2+|\nabla y_j|^2+z_j^2+|\nabla z_j|^2)dxdt+Cs^3e^{2sd_1}$$

$$+C\int_{Q_L}e^{2s\phi}(|\nabla(\chi(\widetilde{g}-g))|^2+(\chi(\widetilde{b}-b))^2)\,dx\,dt+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt. \tag{3.48}$$

Since $\phi\le\phi(T/2)$, (3.48) implies for s sufficiently large

$$s\int_{Q_L}e^{2s\phi}(z_j^2+y_j^2+|\nabla z_j|^2+|\nabla y_j|^2)\,dx\,dt\le Cs^3e^{2sd_1}$$

$$+C\int_{\Omega_L}e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2+(\chi(\widetilde{b}-b))^2)\,dx+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt$$

and so

$$s\int_{Q_L}e^{2s\phi}\sum_{j=1}^2(|\nabla z_j|^2+|\nabla y_j|^2)\,dx\,dt\le s\int_{Q_L}e^{2s\phi}\sum_{j=1}^2(z_j^2+y_j^2+|\nabla z_j|^2+|\nabla y_j|^2)\,dx\,dt$$

$$\le Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}\sum_{j=1}^2(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt$$

$$+C\int_{\Omega_L}e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2+(\chi(\widetilde{b}-b))^2)\,dx. \tag{3.49}$$

Using inequalities (3.45) for $1\le i\le n$ and (3.49), we get

$$s^2\int_{\Omega_L}e^{2s\phi(T/2)}|\nabla(\chi(\widetilde{g}-g))|^2\,dx\le Ce^{2sd_2}F_5(T/2)+C\int_{\Omega_L}e^{2s\phi(T/2)}[|\nabla(\chi(g-\widetilde{g}))|^2|+|\chi(b-\widetilde{b})|^2]\,dx$$

$$+Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}\sum_{j=1}^2(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt.$$

Therefore for s sufficienlty large

$$s^2\int_{\Omega_L}e^{2s\phi(T/2)}|\nabla(\chi(\widetilde{g}-g))|^2\,dx\le Ce^{2sd_2}F_5(T/2)+C\int_{\Omega_L}e^{2s\phi(T/2)}(\chi(b-\widetilde{b}))^2\,dx$$

$$+ Cs^3 e^{2sd_1} + Cs \int_{\gamma_L \times (0,T)} e^{2s\phi} \sum_{j=1}^{2} (|\partial_\nu z_j|^2 + |\partial_\nu y_j|^2) \, d\sigma \, dt. \tag{3.50}$$

• Fourth step: Now we gather (3.43) and (3.50) and we get for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(T/2)} \chi^2 (|\widetilde{b} - b|^2 + |\widetilde{g} - g|^2 + |\nabla(\chi(\widetilde{g} - g))|^2) \, dx \leq C e^{2sd_2} F_6(T/2) + Cse^{2sd_1}, \tag{3.51}$$

with $F_6(T/2) = F_4(T/2) + F_5(T/2)$. Moreover, since $e^{2sd_0} \leq e^{2s\phi(T/2)}$ in Ω_l and $\chi = 1$ in Ω_l, we deduce that

$$\|\widetilde{b} - b\|^2_{L^2(\Omega_l)} + \|\widetilde{g} - g\|^2_{H^1(\Omega_l)} \leq C(e^{2s(d_2 - d_0)} F_6(T/2) + se^{2s(d_1 - d_0)}).$$

This concludes the proof of Theorem 3.3. □

Remark 6 In Theorem 3.3 we have presented the case of determining the coefficients b and A_3. Of course we could obtain similar results for at least two coefficients between a, b, c, d, A_1, A_2, A_3, A_4. If we want to determine A_1 and A_3, we only have to assume that Assumption (3.32) holds intead of (3.32)-(3.33). If we want to estimate the coefficients A_2 and A_3, we still have to assume the hypothesis (3.32) satisfied but in this case, we should also assume that the following hypothesis

$$|\nabla d \cdot \nabla \widetilde{v}(T/2)| \geq R \text{ in } \Omega_L \text{ for some } R > 0$$

holds. Note also that the last item of Remark 4 still holds for (3.34). To conclude, if we would like to determine more than two coefficients, we could procede with the same method used in Theorem 3.2.

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received: October 16, 2020

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