Rostock. Math. Kolloq. **72**, 11 – 33 (2019/20)

Subject Classification (AMS)
35J10

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Existence of solutions for a system involving the (2,q)-Laplacian operator in a bounded domain

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ABSTRACT. In this paper we study the existence of a non trivial weak solution for a system involving the Laplacian operator and the q-Laplacian operator in a bounded domain Ω of $\mathbb{R}^N$ with sufficiently smooth boundary.

KEY WORDS. (2,q)-Laplacian operator, system, existence of solutions

1 Introduction

We consider in this paper the following system for $i = 1, \cdots, m$,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = g_i(., u_1, \cdots, u_m) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases} \qquad (S, q, g)$$

where Ω is a bounded domain with sufficiently smooth boudary, $\Omega \subset \mathbb{R}^N$.
We recall that the q-Laplacian operator is defined by $\Delta_q\phi = div(|\nabla\phi|^{q-2}\nabla\phi)$ and we suppose $q > 2$ in the whole paper. We study the existence of a weak non-trivial solution $u = (u_1, \cdots, u_m) \in W$ for the system (S, q, g) where the variational space is $W = (W_0^{1,q}(\Omega))^m$, $W_0^{1,q}(\Omega)$ being the usual Sobolev space endowed with the norm $\|\phi\|_0^{1,q}(\Omega) = (\int_\Omega |\nabla\phi|^q)^{1/q}$. We also denote $H = (W_0^{1,2}(\Omega))^m$ and $\|.\|_W$, $\|.\|_H$, the norms on W and H ($\|u\|_W = (\sum_{i=1}^m \|u_i\|^q_{W_0^{1,q}(\Omega)})^{1/q}$).

We assume throughout all the paper that the bounded functions a_{ij}, w_i (for $i, j = 1, \cdots, m$) satisfy the following hypothesis

Assumption 1.1 **i)** *$a_{ij}, w_i \in L^\infty(\Omega)$, $a_{ii} \geq 0$, $w_i \geq 0$ a. e. on Ω.*

ii) *The matrix $A = (a_{ij})$ is symmetric and satisfies ${}^t\xi A\xi \geq 0$ for all ${}^t\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^m$.*

Note that the above Assumption 1.1ii) is satisfied when the matrix A is a positive definite one. Introduce now the following functionals for $u = (u_1, \cdots, u_m) \in W$

$$H_1(u) = \sum_{i=1}^{m} \int_\Omega (|\nabla u_i|^2 + a_{ii}u_i^2 + \sum_{j=1, i\neq j}^{m} a_{ij}u_j u_i), \tag{1.1}$$

and

$$H_2(u) = \sum_{i=1}^{m} \int_\Omega (|\nabla u_i|^q + w_i|u_i|^q). \tag{1.2}$$

Since A is symmetric then $H_1(u) = \sum_{i=1}^{m} \int_\Omega (|\nabla u_i|^2 + a_{ii}u_i^2 + 2\sum_{j=1,i<j}^{m} a_{ij}u_j u_i)$.
Note that $(H_1(u))^{1/2}$ and $(H_2(u))^{1/q}$ define norms on H and W equivalent to the norms $\|.\|_H$ and $\|.\|_W$ respectively.

We consider different cases for the functions g_i : in the second section we deal with $g_i(., u_1, \cdots, u_m) := h_i \in W^{-1,q'}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$. In the third section, we define $g_i(., u_1, \cdots, u_m) := m_i|u_i|^{q-2}u_i$ where the functions m_i are bounded and indefinite. In the fourth section we consider the case $g_i(., u_1, \cdots, u_m) := \lambda f_i|u_i|^{\gamma-2}u_i$ where the functions f_i are still bounded and undefinite, λ is a positive real parameter and the coefficient γ satisfies some hypotheses in which $\gamma < q$.

In each of the precedent cases, the system (S, q, g) will be rewritten under a variational form with $I(u)$ an adapted Euler functional defined in W and the existence of weak solutions for the system (S, q, g) will be equivalent to the existence of critical points for this functional I. In the second and third sections, we will mimimize the Euler functional I using either standard arguments (cf. Theorem I.1.2 in [18]) or the Moutain-Pass Theorem. In the third section, we will use the principal eigenvalue $\lambda_{1,q,\rho}$ of the q-Laplacian operator associated with a weight ρ whereas in the fourth section we will define a characteristic value λ_1^+ (see (4.7)).

Equations and systems with the p-Laplacian have been widely studied for the existence of solutions or the maximum and antimaximum principles (see for examples [3, 9–13], see also [14] for the fibering procedure). These last few years, equations with the (p,q)-Laplacian have been studied (see for examples [4, 6, 15, 19, 21] in a bounded domain and [5] in $\mathbb{R}^N$). Authors study the existence of solutions (sometimes the sign of these solutions and generalized eigenvalue problems) mainly by minimization of the energy functional either by standard arguments or the mountain-pass geometry, also by using the method of sub- and super-solutions. The case of the (2,q)-Laplacian arises in quantum physics (see [2]). A few systems with two equations have been studied (see for example [16] for a system with two equations, one with the p-Laplacian and the other one with the q-Laplacian ; see also [20] for a system of two equations with the (p,q)-Laplacian with critical nonlineariries) but as far as we know, there is no system with n equations for the (2,q)-Laplacian studied yet.

This paper is organised as follows: in section 2, we use standard arguments for minimizing the functional I when we consider the case where $g_i(.,u_1,\cdots,u_m) := h_i \in W^{-1,q'}(\Omega)$. In section 3 (in the case of $g_i(.,u_1,\cdots,u_m) := m_i|u_i|^{q-2}u_i$ and $q < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N > 2$ and $2^* = \infty$ if $N \leq 2$), first we recall some results of the existence of the principal eigenvalue for the q-Laplacian operator associated with a bounded weight (and the existence of a positive eigenfunction associated with). Then we use the Mountain-Pass Theorem in order to get the existence of a non-trivial solution for our system. Finally in section 4 (when $g_i(.,u_1,\cdots,u_m) := \lambda f_i|u_i|^{\gamma-2}u_i$ with $2 < \gamma < q$ and $\gamma < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N > 2$ and $2^* = \infty$ if $N \leq 2$), first we follow a method introduced by Cherfils-Il'Yasov in [7] for one equation involving the (p-q)-Laplacian operator to define a characteristic value λ_1^+. Then we get the existence of a non-trivial solution by means of global minimization of the Euler functional.

2 First case: $g_i(.,u_1,\cdots,u_m) := h_i \in W^{-1,q'}(\Omega)$

In this case the system (S,q,g) is rewritten under the following form

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

with $h_i \in W^{-1,q'}(\Omega)$ for each $i = 1,\cdots,m$. Recall that $-\Delta_q$ may be seen acting from $W_0^{1,q}(\Omega)$ into $W^{-1,q'}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ by

$$< -\Delta_q \phi, \psi >_{q',q} = \int_\Omega |\nabla\phi|^{q-2}\nabla\phi \cdot \nabla\psi \text{ for all } \phi,\psi \in W_0^{1,q}(\Omega)$$

(see [8, 17]) where $< .,. >_{q',q}$ denotes the duality mapping between $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$. Therefore the Euler functional is, for $u = (u_1,\cdots,u_m) \in W$,

$$I(u) = \frac{1}{2}H_1(u) + \frac{1}{q}H_2(u) - \sum_{i=1}^m < h_i, u_i >_{q',q} . \tag{2.2}$$

The result of the existence of solution for the system (2.1) is the following.

Theorem 2.1 *Assume that Assumption 1.1 is satisfied and that $h_i \in W^{-1,q'}(\Omega)$ for each $i = 1,\cdots,m$. Then the system (2.1) has a unique solution.*

Proof. The functional $I : W \to \mathbb{R}$ defined by (2.2) is weakly lower semi-continuous by the compactness of the embedding of W to $(L^q(\Omega))^m$ and $(L^2(\Omega))^m$ and of class C^1 on W. Moreover this functional I is also coercive. Indeed by the Young's inequality we have

$$| < h_i, u_i >_{q',q} | \leq \|h_i\|_{W^{-1,q'}(\Omega)} \|u_i\|_{W_0^{1,q}(\Omega)} \leq \frac{1}{2q} \|u_i\|^q_{W_0^{1,q}(\Omega)} + C\|h_i\|^{q'}_{W^{-1,q'}(\Omega)}$$

with $C > 0$, C independent of u. And since $H_1(u) \geq 0$ and $H_2(u) \geq \|u\|_W$ we get that

$$I(u) \geq \frac{1}{2q}\|u\|_W - C\sum_{i=1}^{m} \|h_i\|^{q'}_{W^{-1,q'}(\Omega)}.$$

Therefore the functional I has a gobal minimizer (cf.[18, Theorem I.1.2]) and the system (2.1) has a solution.
Let us prove now the uniqueness of the solution. Suppose on the contrary that there exist two distinct solutions $u = (u_1, \cdots, u_m) \in W$ and $v = (v_1, \cdots, v_m) \in W$ for (2.1), so there exists k such that $u_k \neq v_k$. Since

$$(I'(u) - I'(v)) \cdot (u - v) = I'(u) \cdot u - I'(v) \cdot u - I'(u) \cdot v + I'(v) \cdot v = 0,$$

we have

$$\sum_{i=1}^{m} \int_\Omega |\nabla u_i|^2 + \sum_{i,j=1}^{m} \int_\Omega a_{ij} u_j u_i + \sum_{i=1}^{m} \int_\Omega (|\nabla u_i|^q + w_i |u_i|^q)$$

$$- \sum_{i=1}^{m} \int_\Omega \nabla v_i \cdot \nabla u_i - \sum_{i,j=1}^{m} \int_\Omega a_{ij} v_j u_i - \sum_{i=1}^{m} \int_\Omega (|\nabla v_i|^{q-2} \nabla v_i \cdot \nabla u_i + w_i |v_i|^{q-2} v_i u_i) = 0$$

and on the other hand

$$\sum_{i=1}^{m} \int_\Omega \nabla u_i \cdot \nabla v_i + \sum_{i,j=1}^{m} \int_\Omega a_{ij} u_j v_i + \sum_{i=1}^{m} \int_\Omega (|\nabla u_i|^{q-2} \nabla u_i \cdot \nabla v_i + w_i |u_i|^{q-2} u_i v_i)$$

$$- \sum_{i=1}^{m} \int_\Omega |\nabla v_i|^2 - \sum_{i,j=1}^{m} \int_\Omega a_{ij} v_j v_i - \sum_{i=1}^{m} \int_\Omega (|\nabla v_i|^q + w_i |v_i|^q) = 0.$$

So we get

$$\sum_{i=1}^{m} \int_\Omega \nabla u_i \cdot (\nabla u_i - \nabla v_i) + \sum_{i,j=1}^{m} \int_\Omega a_{ij} u_j (u_i - v_i) + \sum_{i=1}^{m} \int_\Omega |\nabla u_i|^{q-2} \nabla u_i \cdot (\nabla u_i - \nabla v_i)$$

$$+ \sum_{i=1}^{m} \int_\Omega w_i |u_i|^{q-2} u_i (u_i - v_i) - \sum_{i=1}^{m} \int_\Omega \nabla v_i \cdot (\nabla u_i - \nabla v_i) - \sum_{i,j=1}^{m} \int_\Omega a_{ij} v_j (u_i - v_i)$$

$$- \sum_{i=1}^{m} \int_\Omega |\nabla v_i|^{q-2} \nabla v_i \cdot (\nabla u_i - \nabla v_i) - \sum_{i=1}^{m} \int_\Omega w_i |v_i|^{q-2} v_i (u_i - v_i) = 0.$$

Thus

$$\sum_{i=1}^{m}\int_{\Omega}|\nabla u_i-\nabla v_i|^2+\sum_{i=1}^{m}\int_{\Omega}(|\nabla u_i|^{q-2}\nabla u_i-|\nabla v_i|^{q-2}\nabla v_i)\cdot(\nabla u_i-\nabla v_i)$$

$$+\sum_{i,j=1}^{m}\int_{\Omega}a_{ij}(u_j-v_j)(u_i-v_i)+\sum_{i=1}^{m}\int_{\Omega}w_i(|u_i|^{q-2}u_i-|v_i|^{q-2}v_i)(u_i-v_i)=0.$$

The last equality can be rewritten under the following form with the duality product $<.,.>_{q',q}$

$$\sum_{i=1}^{m}<-\Delta u_i+\Delta v_i,u_i-v_i>_{2,2}+\sum_{i=1}^{m}<-\Delta_q u_i+\Delta_q v_i,u_i-v_i>_{q',q}$$

$$+\sum_{i,j=1}^{m}<a_{ij}(u_j-v_j),u_i-v_i>_{2,2}+\sum_{i=1}^{m}<w_i(|u_i|^{q-2}u_i-|v_i|^{q-2}v_i),u_i-v_i>_{q',q}=0.$$

Moreover a consequence of the strict convexity of the spaces $W_0^{1,2}(\Omega)$ and $W_0^{1,q}(\Omega)$ is that the duality mappings $-\Delta$ and $-\Delta_q$ are strictly monotone. So from $u_k\neq v_k$ we get

$$<-\Delta u_k+\Delta v_k,u_k-v_k>_{2,2}>0,$$

and

$$<-\Delta_q u_k+\Delta_q v_k,u_k-v_k>_{q',q}\geq(\|u_k\|^{q-1}_{W_0^{1,q}(\Omega)}-\|v_k\|^{q-1}_{W_0^{1,q}(\Omega)})(\|u_k\|_{W_0^{1,q}(\Omega)}-\|v_k\|_{W_0^{1,q}(\Omega)})\geq 0$$

since $x\mapsto x^{q-1}$ is increasing on $[0,\infty)$ (and even $<-\Delta_q u_k+\Delta_q v_k,u_k-v_k>_{q',q}>0$ from [8, Proposition 1]).
Thus

$$\sum_{i=1}^{m}<-\Delta u_i+\Delta v_i,u_i-v_i>_{2,2}+\sum_{i=1}^{m}<-\Delta_q u_i+\Delta_q v_i,u_i-v_i>_{q',q}>0.$$

Furthermore, since the function $x\mapsto|x|^{q-2}x$ is increasing and $w_i\geq 0$, we have

$$\sum_{i=1}^{m}<w_i(|u_i|^{q-2}u_i-|v_i|^{q-2}v_i),u_i-v_i>_{q',q}\geq 0.$$

Finally from Assumption 1.1,

$$\sum_{i,j=1}^{m}<a_{ij}(u_j-v_j),u_i-v_i>_{2,2}\geq 0.$$

Therefore we get a contradiction. □

Remark: We can generalize Theorem 2.1 replacing the 2-Laplacian operator by the p-Laplacian with $2 < p < q$, that for the following system

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

and even for

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + b_i|u_i|^{p-2}u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

under the additional hypothesis that the bounded functions b_i, $i = 1, \cdots, m$ are non-negative.

3 Second case: $g_i(., u_1, \cdots, u_m) := m_i|u_i|^{q-2}u_i$

In this section we assume that

Assumption 3.1 *$q < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N > 2$ and $2^* = \infty$ if $N \leq 2$,*

and we rewrite the system (S, q, g) under the following form:
for $i = 1, \cdots, m$,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = m_i|u_i|^{q-2}u_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

Note that the decomposition with the weights $c_i := m_i - w_i$ does not necessarily coincide with the decomposition $c_i = c_{i+} - c_{i-}$ where $c_{i+} = \max(c_i, 0)$ and $c_{i-} = max(-c_i, 0)$. Define now for $u = (u_1, \cdots, u_m) \in W$ the functional

$$M(u) = \sum_{i=1}^m \int_\Omega m_i|u_i|^q. \tag{3.2}$$

The Euler functional associated with (3.1) is consequently for $u = (u_1, \cdots, u_m) \in W$,

$$I(u) = \frac{1}{2}H_1(u) + \frac{1}{q}H_2(u) - \frac{1}{q}M(u). \tag{3.3}$$

First let us recall the usual weighted eigenvalue problem for the q-Laplacian:

$$\begin{cases} -\Delta_q u = \lambda\rho|u|^{q-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.4}$$

with a bounded weight function ρ and a real parameter λ. It is said that λ is an eigenvalue of the q-Laplacian associated with the weight ρ if (3.4) has a non-trivial solution u which is called an eigenfunction associated with λ. It is well known (see [1]) that if the Lebesgue measure of $\{x \in \Omega, \rho(x) > 0\}$ is positive, then the first positive eigenvalue $\lambda_{1,q,\rho}$ of $-\Delta_q$ with weight function ρ is obtained by the Rayleight quotient

$$\lambda_{1,q,\rho} = \inf\{\frac{\int_\Omega |\nabla u|^q}{\int_\Omega \rho |u|^q}; u \in W_0^{1,q}(\Omega), \int_\Omega \rho |u|^q > 0\}. \tag{3.5}$$

Moreover, $\lambda_{1,q,\rho}$ has a positive eigenfunction $\phi_{1,q,\rho} \in C_0^{1,\alpha_q}(\overline{\Omega})$ (for some $\alpha_q \in (0,1)$). Assume in this section that

Assumption 3.2 **i)** *For all $i = 1, \cdots, m$, $m_i \in L^\infty(\Omega)$,*

ii) *For all $i = 1, \cdots, m$, the real 1 is not an eigenvalue of the q-Laplacian with the weight $m_i - w_i$.*

Assume also in this section that either Assumption 3.3 or Assumption 3.4 holds

Assumption 3.3 *There exists $k \in \{1, \cdots, m\}$ such that:*

$$meas\{x \in \Omega, (m_k - w_k)(x) > 0\} \neq 0 \text{ and } \lambda_{1,q,m_k - w_k} < 1.$$

Assumption 3.4 *There exist $k, l \in \{1, \cdots, m\}$, $k \neq l$ such that:*

$$meas\{x \in \Omega, (m_k - w_k)(x) > 0\} \neq 0 \text{ and } \lambda_{1,q,m_k - w_k} + \int_\Omega (w_l - m_l)|\phi_{1,q,m_k-w_k}|^q < 0$$

with ϕ_{1,q,m_k-w_k} the normalized eigenfunction associated with λ_{1,q,m_k-w_k}.

Note that Assumption 3.4 is satisfed when $\lambda_{1,q,m_k-w_k}(m_k - w_k) + w_l - m_l < 0$ a. e. in Ω. Our aim is to study the existence of a weak solution for the system (3.1) by minimizing the functional I defined by (3.3). As in section 2, the functional I is weakly lower semi-continous on W but may be no more coercive so we cannot use standard arguments for minimizing I. First, we prove that any Palais-Smale sequence is bounded in W and has a strong convergent subsequence. Then we are able to apply the Mountain-Pass Lemma and Assumptions 3.3 or 3.4 allow us to get a non-trivial solution.

We say that $(u_n) \subset W$, $u_n = (u_{1n}, \cdots, u_{mn})$, is a Palais-Smale sequence if it satisfies the following conditions

$$|I(u_n)| \leq D \text{ for all } n \in \mathbb{N} \text{ and } \|I'(u_n)\|_{W^*} \to 0 \text{ as } n \to \infty \tag{3.6}$$

with some constant $D > 0$, W^* being the dual space of W.

Lemma 3.1 *Assume that Assumptions 1.1 and 3.2 are satisfied. If $(u_n) \subset W$, $u_n = (u_{1n}, \cdots, u_{mn})$, is a Palais-Smale sequence, then (u_n) is bounded in W.*

Proof. Let $(u_n) \subset W$, $u_n = (u_{1n}, \cdots, u_{mn})$, be a Palais-Smale sequence. We want to prove that $(\|u_n\|_W)_n$ is bounded or equivalently that $(H_2(u_n))_n$ is bounded. But

$$\frac{1}{q}H_2(u_n) = I(u_n) - \frac{1}{2}H_1(u_n) + \frac{1}{q}M(u_n) \le D + \frac{1}{q}M(u_n) \le D + C\|u_n\|^q_{(L^q(\Omega))^m} \tag{3.7}$$

with C a positive constant, C independent of u_n (since the functions m_i are bounded in the functional $M(u)$ defined by (3.2)). So it is sufficient to show that $(\|u_n\|_{(L^q(\Omega))^m})$ is bounded. We adapt ideas from [19]. Assume on the contrary that $\alpha_n := \|u_n\|_{(L^q(\Omega))^m} \to_{n\to\infty} \infty$ (for a subsequence) and denote $v_n = \frac{1}{\alpha_n}u_n = (v_{1n}, \cdots, v_{mn})$. From (3.7), we deduce that $(\|v_n\|_W)$ is bounded and from the compact embedding of W into $(L^q(\Omega))^m$ we get the existence of $v_0 = (v_{01}, \cdots, v_{0m}) \in W$ such that (v_n) converges to v_0, strongly in $(L^q(\Omega))^m$ and weakly in W (for a subsequence).

• Now we prove that (v_n) converges strongly to v_0 in W. Indeed by taking $\phi_n := \frac{1}{\alpha_n^{q-1}}(v_n - v_0)$, we obtain

$$I'(u_n).\phi_n = \frac{1}{\alpha_n^{q-1}}\sum_{i=1}^m \int_\Omega (\nabla u_{in}.\nabla(v_{in} - v_{0i}) + a_{ii}u_{in}(v_{in} - v_{0i}))$$

$$+\frac{1}{\alpha_n^{q-1}}\sum_{i=1}^m \int_\Omega (|\nabla u_{in}|^{q-2}\nabla u_{in}.\nabla(v_{in} - v_{0i}) + w_i|u_{in}|^{q-2}u_{in}(v_{in} - v_{0i}))$$

$$+\frac{1}{\alpha_n^{q-1}}\sum_{i,j;i\neq j}\int_\Omega a_{ij}u_{jn}(v_{in} - v_{0i}) - \frac{1}{\alpha_n^{q-1}}\sum_{i=1}^m \int_\Omega m_i|u_{in}|^{q-2}u_{in}(v_{in} - v_{0i}). \tag{3.8}$$

But $u_n = \alpha_n v_n$ so (3.8) becomes

$$I'(u_n).\phi_n = \frac{1}{\alpha_n^{q-2}}\sum_{i=1}^m \int_\Omega (\nabla v_{in}.\nabla(v_{in} - v_{0i}) + a_{ii}v_{in}(v_{in} - v_{0i}))$$

$$+\sum_{i=1}^m \int_\Omega (|\nabla v_{in}|^{q-2}\nabla v_{in}.\nabla(v_{in} - v_{0i}) + w_i|v_{in}|^{q-2}v_{in}(v_{in} - v_{0i}))$$

$$+\frac{1}{\alpha_n^{q-2}}\sum_{i,j;i\neq j}\int_\Omega a_{ij}v_{jn}(v_{in} - v_{0i}) - \sum_{i=1}^m \int_\Omega m_i|v_{in}|^{q-2}v_{in}(v_{in} - v_{0i}). \tag{3.9}$$

Note that $|I'(u_n).\phi_n| \le \|I'(u_n)\|_{W^*}\|\phi_n\|_W = \|I'(u_n)\|_{W^*}\frac{1}{\alpha_n^{q-1}}\|v_n - v_0\|_W$ so $I'(u_n).\phi_n \to_{n\to\infty} 0$ from (3.6), $\alpha_n \to_{n\to\infty} \infty$ and $(\|v_n\|_W)$ bounded. Moreover, since the functions a_{ij}, w_i, m_i are bounded there exists a positive constant, denoting C at each step, such that

$$|\int_\Omega a_{ij}v_{jn}(v_{in} - v_{0i}))| \le C\|v_{jn}\|_{L^2(\Omega)}\|v_{in} - v_{0i}\|_{L^2(\Omega)} \le C\|v_n\|_W\|v_n - v_0\|_{(L^q(\Omega))^m}$$

and therefore
$$\int_\Omega a_{ij}v_{jn}(v_{in}-v_{0i}))\to 0 \text{ as } n\to\infty. \tag{3.10}$$
By the same way, for $b_i=w_i$ or $b_i=m_i$,
$$|\int_\Omega b_i|v_{in}|^{q-2}v_{in}(v_{in}-v_{0i}))|\le C(\int_\Omega |v_{in}|^q)^{\frac{q-1}{q}}(\int_\Omega |v_{in}-v_{0i}|^q)^{1/q}\le C\|v_n\|_W^{q-1}\|v_n-v_0\|_{(L^q(\Omega))^m}$$
so
$$\int_\Omega b_i|v_{in}|^{q-2}v_{in}(v_{in}-v_{0i}))\to 0 \text{ as } n\to\infty. \tag{3.11}$$
Recall that $<.,.>_{q',q}$ is the duality product between $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$ with $\frac{1}{q}+\frac{1}{q'}=1$. From (3.9), (3.10), (3.11), we deduce that
$$\frac{1}{\alpha_n^{q-2}}\sum_{i=1}^m <-\Delta v_{in},v_{in}-v_{0i}>_{2,2}+\sum_{i=1}^m <-\Delta_q v_{in},v_{in}-v_{0i}>_{q',q}\to 0 \text{ as } n\to\infty. \tag{3.12}$$
Moreover we have (see also the proof of Theorem 2.1)
$$<-\Delta_q v_{in}+\Delta_q v_{0i},v_{in}-v_{0i}>_{q',q}\ge(\|v_{in}\|^{q-1}_{W_0^{1,q}(\Omega)}-\|v_{0i}\|^{q-1}_{W_0^{1,q}(\Omega)})(\|v_{in}\|_{W_0^{1,q}(\Omega)}-\|v_{0i}\|_{W_0^{1,q}(\Omega)})\ge 0 \tag{3.13}$$
and
$$<-\Delta v_{in}+\Delta v_{0i},v_{in}-v_{0i}>_{2,2}=\|v_{in}-v_{0i}\|^2_{W_0^{1,2}(\Omega)}\ge(\|v_{in}\|_{W_0^{1,2}(\Omega)}-\|v_{0i}\|_{W_0^{1,2}(\Omega)})^2. \tag{3.14}$$
From (3.13) and (3.14) we get
$$\begin{aligned}0\le&\sum_{i=1}^m(\|v_{in}\|^{q-1}_{W_0^{1,q}(\Omega)}-\|v_{0i}\|^{q-1}_{W_0^{1,q}(\Omega)})(\|v_{in}\|_{W_0^{1,q}(\Omega)}-\|v_{0i}\|_{W_0^{1,q}(\Omega)})\\&+\frac{1}{\alpha_n^{q-2}}\sum_{i=1}^m(\|v_{in}\|_{W_0^{1,2}(\Omega)}-\|v_{0i}\|_{W_0^{1,2}(\Omega)})^2\\\le&\frac{1}{\alpha_n^{q-2}}\sum_{i=1}^m<-\Delta v_{in},v_{in}-v_{0i}>_{2,2}+\sum_{i=1}^m<-\Delta_q v_{in},v_{in}-v_{0i}>_{q',q}\\&+\sum_{i=1}^m<\Delta_q v_{0i},v_{in}-v_{0i}>_{q',q}+\frac{1}{\alpha_n^{q-2}}\sum_{i=1}^m<\Delta v_{0i},v_{in}-v_{0i}>_{2,2}.\end{aligned}$$
Because the right-hand side of the above estimate tends to 0 as n tends to infinity (from (3.12) and the weak convergence of (v_n) to v_0 in W) we obtain that for $i=1,\cdots,m$, $\|v_{in}\|_{W_0^{1,q}(\Omega)}\to\|v_{0i}\|_{W_0^{1,q}(\Omega)}$ as $n\to\infty$ and therefore (v_n) strongly converges to v_0 in W.

• Finally, we prove that v_{0i} is a non-trivial solution of the eigenvalue problem of the q-Laplacian with weight m_i-w_i for at least one i.

Let $\phi=(\phi_1,\cdots,\phi_m)\in W$. Taking $\frac{1}{\alpha_n^{q-1}}\phi$ as a test function, since $u_n=\alpha_n v_n$, we have
$$I'(u_n).\frac{1}{\alpha_n^{q-1}}\phi=\frac{1}{\alpha_n^{q-2}}\sum_{i=1}^m\int_\Omega(\nabla v_{in}.\nabla\phi_i+\sum_{j=1}^m\int_\Omega a_{ij}v_{jn}\phi_i)$$

$$+\sum_{i=1}^{m}\int_{\Omega}(|\nabla v_{in}|^{q-2}\nabla v_{in}.\nabla\phi_i + w_i|v_{in}|^{q-2}v_{in}\phi_i - m_i|v_{in}|^{q-2}v_{in}\phi_i).$$

Letting $n\to\infty$, we see that for each $i=1,\cdots,m$,

$$\begin{cases} -\Delta_q v_{0i} + w_i|v_{0i}|^{q-2}v_{0i} = m_i|v_{0i}|^{q-2}v_{0i} \text{ in } \Omega \\ v_{0i} = 0 \text{ on } \partial\Omega \end{cases}. \tag{3.15}$$

Since $\|v_n\|_{(L^q(\Omega))^m} = 1$ and (v_n) converges strongly to v_0 in W we get that $\|v_0\|_W \geq 1$. Therefore there exists i such that v_{0i} is a weak solution to (3.15). This contradicts Assumption 3.2. □

Lemma 3.2 *Assume that Assumptions 1.1 and 3.2 are satisfied. If $(u_n)\subset W$, $u_n = (u_{1n},\cdots,u_{mn})$, is a Palais-Smale sequence, then (u_n) has a strong convergent subsequence in W.*

Proof. Let (u_n) be a Palais-Smale sequence in W, $u_n = (u_{1n},\cdots,u_{mn})$. By Lemma 3.1, the sequence (u_n) is bounded in W. From the compact embedding of $W^{1,q}(\Omega)$ into $L^q(\Omega)$ we get the existence of $u_0 = (u_{01},\cdots,u_{0m})\in W$ such that (u_n) converges to u_0 strongly in $(L^q(\Omega))^m$ and weakly in W (for a subsequence still denoted by (u_n)). We want to prove that $\|u_n\|_W \to \|u_0\|_W$ as $n\to\infty$ and we proceed as in the proof of Lemma 3.1.
Since $|I'(u_n).(u_n-u_0)| \leq \|I'(u_n)\|_{W^*}(\|u_n\|_W + \|u_0\|_W)$ we deduce that

$$I'(u_n).(u_n-u_0)\to 0 \text{ as } n\to\infty. \tag{3.16}$$

But

$$I'(u_n).(u_n-u_0) = \sum_{i=1}^{m}\int_{\Omega}(\nabla u_{in}\cdot\nabla(u_{in}-u_{0i}) + \sum_{j=1}^{m}a_{ij}u_{jn}(u_{in}-u_{0i}))$$

$$+\sum_{i=1}^{m}\int_{\Omega}(|\nabla u_{in}|^{q-2}\nabla u_{in}.\nabla(u_{in}-u_{0i}) + (w_i-m_i)|u_{in}|^{q-2}u_{in}(u_{in}-u_{0i})).$$

As in Lemma 3.1, denoting b_i either w_i or m_i, we have for $i,j=1,\cdots,m,$,

$$\int_{\Omega} b_i|u_{in}|^{q-2}u_{in}(u_{in}-u_{0i}))\to 0 \text{ as } n\to\infty, \int_{\Omega} a_{ij}u_{jn}(u_{in}-u_{0i}))\to 0 \text{ as } n\to\infty. \tag{3.17}$$

From (3.16) and (3.17), we get that

$$\sum_{i=1}^{m} < -\Delta u_{in}, u_{in}-u_{0i} >_{2,2} + \sum_{i=1}^{m} < -\Delta_q u_{in}, u_{in}-u_{0i} >_{q',q} \to 0 \text{ as } n\to\infty.$$

Moreover we have

$$0 \leq \sum_{i=1}^{m}(\|u_{in}\|^{q-1}_{W_0^{1,q}(\Omega)} - \|u_{0i}\|^{q-1}_{W_0^{1,q}(\Omega)})(\|u_{in}\|_{W_0^{1,q}(\Omega)} - \|u_{0i}\|_{W_0^{1,q}(\Omega)})$$

$$+\sum_{i=1}^{m}(\|u_{in}\|_{W_0^{1,2}(\Omega)}-\|u_{0i}\|_{W_0^{1,2}(\Omega)})^2$$

$$\leq \sum_{i=1}^{m} < -\Delta u_{in}, u_{in}-u_{0i} >_{2,2} + \sum_{i=1}^{m} < -\Delta_q u_{in}, u_{in}-u_{0i} >_{q',q}$$

$$+\sum_{i=1}^{m} < \Delta_q u_{0i}, u_{in}-u_{0i} >_{q',q} + \sum_{i=1}^{m} < \Delta u_i, u_{in}-u_{0i} >_{2,2} .$$

As in Lemma 3.1 we deduce that for $i=1,\cdots,m$, $\|u_{in}\|_{W_0^{1,q}(\Omega)} \to \|u_{0i}\|_{W_0^{1,q}(\Omega)}$ as $n\to\infty$ and therefore (u_n) strongly converges to u_0 in W. □

So we can state the main result of this section

Theorem 3.1 *Assume that Assumptions 1.1, 3.1 and 3.2 are satisfied. Assume also that either Assumption 3.3 or 3.4 holds. Then the system (3.1) has a non-trivial solution in W.*

Proof. The C^1-functional I satisfies the Palais-Smale conditions and $I(0)=0$.
• First, we claim that there exist positive constants $\rho^*>0$ and $\delta>0$ such that $I(u)\geq\delta$ for any $u=(u_1,\cdots,u_m)\in W$ satisfying $\|u\|_W=\rho^*$.
Let $u=(u_1,\cdots,u_m)\in W$. Put $\rho=\|u\|_W$ and note that $H_1(u)\geq\|u\|_H^2$ and $H_2(u)\geq\rho^q$.
Moreover, since $q<2^*$, for $i=1,\cdots,m$,

$$|\int_\Omega m_i|u_i|^q|\leq(\int_\Omega|m_i|^r)^{1/r}(\int_\Omega|u_i|^{qt})^{1/t} \text{ with } \frac{1}{r}+\frac{1}{t}=1 \text{ and } s:=qt<2^*.$$

From the continous embedding of $W^{1,2}(\Omega)\subset L^s(\Omega)$ we deduce the existence of a positive constant C_1 such that $|\int_\Omega m_i|u_i|^q|\leq C_1\|u_i\|^q_{W^{1,2}(\Omega)}$. Thus

$$|M(u)|\leq C_1\|u\|_H^q$$

and

$$I(u)\geq\frac{1}{q}\rho^q+\frac{1}{2}\|u\|_H^2(1-\frac{2C_1}{q}\|u\|_H^{q-2}).$$

Recall also that there exists a positive constant $C_2>0$ such that $\|u\|_H\leq C_2\|u\|_W$ for all $u\in W$.
Therefore if $\rho\leq\rho^*:=\frac{1}{C_2}\left(\frac{q}{2C_1}\right)^{\frac{1}{q-2}}$, then $1-\frac{2C_1}{q}\|u\|_H^{q-2}\geq 1-\frac{2C_1}{q}(C_2\rho)^{q-2}\geq 0$ and

$$I(u)\geq\frac{1}{q}\rho^q:=\delta.$$

• Assume here that Assumption 3.3 is satisfied with $k=1$ for simplicity.
Let ϕ_{1,q,m_1-w_1} be the normalized eigenfunction associated with λ_{1,q,m_1-w_1} (i. e. be such that $\int_\Omega(m_1-w_1)|\phi_{1,q,m_1-w_1}|^q=1$, we may choose such ϕ_{1,q,m_1-w_1} because the equation (3.4) is

homogeneous). Denote $\Phi_q = (\phi_{1,q,m_1-w_1}, 0, \cdots, 0)$ and take R sufficiently large such that $\|R\Phi_q\|_W > \rho^*$. We have from (3.4) and (3.5)

$$I(R\Phi_q) = \frac{R^2}{2}H_1(\Phi_q) + \frac{R^q}{q}\int_\Omega (|\nabla\phi_{1,q,m_1-w_1}|^q + (w_1 - m_1)|\phi_{1,q,m_1-w_1}|^q)$$

$$= \frac{R^2}{2}H_1(\Phi_q) + \frac{R^q}{q}\int_\Omega (\lambda_{1,q,m_1-w_1}(m_1 - w_1) + w_1 - m_1)|\phi_{1,q,m_1-w_1}|^q.$$

So, since $\lambda_{1,q,m_1-w_1} < 1$,

$$I(R\Phi_q) = \frac{R^2}{2}H_1(\Phi_q) + \frac{R^q}{q}(\lambda_{1,q,m_1-w_1} - 1) < 0$$

for R sufficiently large.Therefore we can apply the mountain-pass theorem to deduce that I has a non-trivial critical point which is a non-trivial weak solution of the system (3.1).

• Assume now that Assumption 3.4 is satisfied with $k = 1$ and $l = 2$ for simplicity.

Denote again ϕ_{1,q,m_1-w_1} the normalized eigenfunction associated with λ_{1,q,m_1-w_1} such that $\int_\Omega (m_1 - w_1)|\phi_{1,q,m_1-w_1}|^q = 1$ and denote here $\Psi_q = (0, \phi_{1,q,m_1-w_1}, 0, \cdots, 0)$. Take R sufficiently large such that $\|R\Psi_q\|_W > \rho^*$. We have here

$$I(R\Psi_q) = \frac{R^2}{2}H_1(\Psi_q) + \frac{R^q}{q}\int_\Omega (|\nabla\phi_{1,q,m_1-w_1}|^q + (w_2 - m_2)|\phi_{1,q,m_1-w_1}|^q)$$

$$= \frac{R^2}{2}H_1(\Psi_q) + \frac{R^q}{q}\int_\Omega (\lambda_{1,q,m_1-w_1}(m_1 - w_1) + w_2 - m_2)|\phi_{1,q,m_1-w_1}|^q.$$

From Assumption 3.4, we get that $I(R\Psi_q) < 0$ for R sufficiently large.Therefore, as in the precedent case, we apply the mountain-pass theorem and deduce that I has a non-trivial critical point. □

Remark: As in section 2, we can generalize Theorem 3.1 replacing the 2-Laplacian operator by the p-Laplacian with $2 \le p < q$ for the following system

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + b_i|u_i|^{p-2}u_i + \lambda w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = \lambda m_i|u_i|^{q-2}u_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

under the additional hypotheses that the bounded functions b_i, $i = 1, \cdots, m$ are non-negative and λ is a real parameter. Then the hypothesis ii) in Assumption 3.2 is replaced by λ is not an eigenvalue of $-\Delta_q$ associated with $m_i - w_i$ for each i. Moreover the hypothesis $\lambda_{1,q,m_k-w_k} < 1$ in Assumption 3.3 is replaced by $\lambda_{1,q,m_k-w_k} < \lambda$.

4 Third case: $g_i(., u_1, \cdots, u_m) := \lambda f_i |u_i|^{\gamma-2} u_i$

In this section we rewrite the system (S, q, g) under the following form:
for $i = 1, \cdots, m$,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i |u_i|^{q-2} u_i + \sum_{j=1}^m a_{ij} u_j = \lambda f_i |u_i|^{\gamma-2} u_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume throughout all this section that the indefinite bounded functions f_i and the coefficients γ and q satisfy the following hypotheses

Assumption 4.1 **i)** $2 < \gamma < q$,

ii) $\gamma < 2^*$ *where* $2^* = \frac{2N}{N-2}$ *if* $2 < N$ *and* $2^* = \infty$ *if* $2 \geq N$,

iii) *For each* $i = 1, \cdots, m$, $f_i \in L^\infty(\Omega)$ *and* $meas\{x \in \Omega, f_i(x) > 0\} \neq 0$.

We also define the functionals

$$F(u) = \sum_{i=1}^m \int_\Omega f_i |u_i|^\gamma \tag{4.2}$$

and

$$I_\lambda(u) = \frac{1}{2} H_1(u) + \frac{1}{q} H_2(u) - \frac{\lambda}{\gamma} F(u) \tag{4.3}$$

where H_1 and H_2 are respectively defined by (1.1) and (1.2). We recall that we study here the existence of a weak non-trivial solution $u = (u_1, \cdots, u_m) \in W$ for the system (4.1) with respect to the real positive parameter λ and that the existence of weak solutions for the system (4.1) is equivalent to the existence of critical points for the Euler functional I_λ. The main result is the existence of a weak non-trivial solution for the system (4.1) associated with $\lambda > \lambda_1^+$ where λ_1^+ is defined by (4.7). For the first part of this section we follow a method developed by Cherfils-Il'Yasov in [7] for one equation with the (p,q)-Laplacian operator. This method is based on proving the existence of solution for $\lambda = \lambda_1^+$ then on applying the mountain-pass theorem for $\lambda > \lambda_1^+$. Although we also could apply the mountain-pass theorem for our case, we will use in fact standard arguments to minimize the functional I_λ.

In section 4.1 we present some preliminary results: we define λ_1^+ and we prove the existence of a solution for the system (4.1) for $\lambda = \lambda_1^+$. The section 4.2 is devoted to the main theorem of the existence of a solution for the system (4.1) associated with $\lambda > \lambda_1^+$.

4.1 Some preliminaries results

As in [7] we define for $\lambda > 0$, $t > 0$ and $u \in W$, $\tilde{I}_\lambda(t, u) = I_\lambda(tu)$.

Lemma 4.1 *Assume that Assumptions 1.1, 4.1 i), 4.1 iii) are satisfied. For given u in $W, u \neq 0$ such that $F(u) \neq 0$, the unique solution $(t(u), \lambda(u))$ of the system* $\begin{cases} \frac{\partial}{\partial t}\tilde{I}_\lambda(t,u) = 0 \\ \frac{\partial^2}{\partial t^2}\tilde{I}_\lambda(t,u) = 0 \end{cases}$ *is given by*

$$t(u) = \left(\frac{\gamma-2}{q-\gamma}\right)^{\frac{1}{q-2}} \left(\frac{H_1(u)}{H_2(u)}\right)^{\frac{1}{q-2}} > 0, \quad \lambda(u) = C_{q,\gamma}\frac{H_1(u)^\alpha H_2(u)^{1-\alpha}}{F(u)} \tag{4.4}$$

with

$$\alpha = \frac{q-\gamma}{q-2}, \quad C_{q,\gamma} = \frac{q-2}{(q-\gamma)^\alpha(\gamma-2)^{1-\alpha}}. \tag{4.5}$$

Proof. The system (S) $\begin{cases} \frac{\partial}{\partial t}\tilde{I}_\lambda(t,u) = 0 \\ \frac{\partial^2}{\partial t^2}\tilde{I}_\lambda(t,u) = 0 \end{cases}$ is equivalent to the system

$$\begin{cases} tH_1(u) + t^{q-1}H_2(u) - \lambda t^{\gamma-1}F(u) = 0 \\ H_1(u) + (q-1)t^{q-2}H_2(u) - \lambda(\gamma-1)t^{\gamma-2}F(u) = 0 \end{cases}$$

and to the following system

$$\begin{cases} H_1(u) + t^{q-2}H_2(u) - \lambda t^{\gamma-2}F(u) = 0 \\ H_1(u) + (q-1)t^{q-2}H_2(u) - \lambda(\gamma-1)t^{\gamma-2}F(u) = 0 \end{cases}.$$

Therefore

$$(q-2)t^{q-2}H_2(u) - \lambda(\gamma-2)t^{\gamma-2}F(u) = 0. \tag{4.6}$$

Note that the system (S) is not solvable in the case where $u \in W$, $u \neq 0$ satisfies $F(u) = 0$ (since if $u \neq 0$, then $H_2(u) \neq 0$ and from (4.6) we deduce $F(u) \neq 0$).
We deduce that

$$\lambda = \frac{(q-2)t^{q-2}H_2(u)}{(\gamma-2)t^{\gamma-2}F(u)}.$$

Replacing λ by $\frac{(q-2)t^{q-2}H_2(u)}{(\gamma-2)t^{\gamma-2}F(u)}$ in $H_1(u) + t^{q-2}H_2(u) - \lambda t^{\gamma-2}F(u) = 0$, we get that $t^{q-2} = (\frac{\gamma-2}{q-\gamma})\frac{H_1(u)}{H_2(u)}$. And we obtain (4.4) associated with (4.5). □

Thus we can define the following characteristic points (recall that F is defined by (4.2))

$$\Lambda_1^+ = \inf\{\lambda(u), u \in W, F(u) > 0\} \text{ and } \lambda_1^+ = \frac{\gamma}{2^\alpha q^{1-\alpha}}\Lambda_1^+. \tag{4.7}$$

Lemma 4.2 *Assume that Assumptions 1.1 and 4.1 are satisfied.*
We have $0 < \Lambda_1^+ < \lambda_1^+$.

Proof. Let $u = (u_1, \cdots, u_m) \in W$ be such that $F(u) > 0$.
First from $\gamma < 2^*$, let (t, l) be such that $\gamma < t < 2^*$ and $\frac{1}{l} + \frac{\gamma}{t} = 1$. Since $W_0^{1,2}(\Omega) \subset L^t(\Omega)$ with a continuous embedding and since the functions f_i are bounded, there exist positive constants still denoting C at each step and depending on some Sobolev constants, such that for $i = 1, \cdots, m$

$$|\int_\Omega f_i|u_i|^\gamma| \leq (\int_\Omega |f_i|^l)^{1/l}(\int_\Omega |u_i|^t)^{\gamma/t} \leq C\|u_i\|^\gamma_{L^t(\Omega)} \leq C\|u_i\|^\gamma_{W_0^{1,2}(\Omega)}.$$

Then

$$F(u) \leq CH_1(u)^{\gamma/2}.$$

By the same way, from $\gamma < q$, let $s = \frac{q}{\gamma}$ and r be such that $\frac{1}{s} + \frac{1}{r} = 1$.
Then we have

$$|\int_\Omega f_i|u_i|^\gamma| \leq m(\int_\Omega |f_i|^r)^{1/r}(\int_\Omega |u_i|^{\gamma s})^{1/s} \leq C\|u_i\|^\gamma_{L^q(\Omega)} \leq C\|u_i\|^\gamma_{W_0^{1,q}(\Omega)}$$

and

$$F(u) \leq CH_2(u)^{\gamma/q}.$$

Therefore there exists a positive constant C', independent of u, such that

$$\lambda(u) = C_{q,\gamma}\frac{H_1(u)^\alpha H_2(u)^{1-\alpha}}{F(u)} \geq C'C_{q,\gamma}\frac{F(u)^{\frac{2\alpha}{\gamma}}F(u)^{\frac{q(1-\alpha)}{\gamma}}}{F(u)} = C'C_{q,\gamma}$$

since $\frac{2\alpha}{\gamma} + \frac{q(1-\alpha)}{\gamma} = 1$. Thus $\Lambda_1^+ > 0$.

Finally we prove that $\Lambda_1^+ < \lambda_1^+$.
Indeed note that $\lambda_1^+ > \Lambda_1^+ \Leftrightarrow \frac{\gamma}{2^\alpha q^{1-\alpha}}\Lambda_1^+ > \Lambda_1^+ \Leftrightarrow (\frac{\gamma}{2})^{q-2} > (\frac{q}{2})^{\gamma-2}$.
Denote $\mu = \frac{q-2}{2} > 0$ and $\eta = \frac{\gamma-2}{2} > 0$. Since $2 < \gamma < q$ we have $\mu > \eta$. Moreover the function f defined by $f(x) = (1+x)^{1/x}$, is strictly decreasing on $(0, \infty)$. Then $(1+\mu)^{1/\mu} < (1+\eta)^{1/\eta}$. And we get that $(\frac{q}{2})^{\gamma-2} < (\frac{\gamma}{2})^{q-2}$. So $\Lambda_1^+ < \lambda_1^+$. □

We obtain now the following result that will enable us to get the existence of a non-trivial solution for the system (4.1) associated with λ_1^+.

Proposition 4.1 *Assume that Assumptions 1.1 and 4.1 are satisfied. Assume that* $u = (u_1, \cdots, u_m) \in W$ *satisfies* $F(u) \neq 0$ *and* $\lambda'(u) = 0$ *(i.e.* u *is a critical point of* $\lambda(u)$*). Then* $\tilde{u} = (\tilde{u_1}, \cdots, \tilde{u_m}) \in W$ *is a non-trivial solution of the system (4.1) associated with* $\tilde{\lambda} = \frac{\gamma}{2^\alpha q^{1-\alpha}}\lambda(u)$ *where for all* $i = 1, \cdots, m$, $\tilde{u_i} = \frac{1}{s}u_i$ *and* $\frac{1}{s} = (\frac{q}{2})^{\frac{1}{q-2}}t(u) > 0$. *Moreover* $I_{\tilde{\lambda}}(\tilde{u}) = 0$.

Proof. Let $u = (u_1, \cdots, u_m) \in W$ which satisfies $F(u) \neq 0$ and $\lambda'(u) = 0$.
For all test function ϕ, we have

$$\frac{\partial \lambda}{\partial u_1}(u) \cdot \phi = 0.$$

So

$$2C_{q,\gamma}\alpha(H_1(u))^{\alpha-1}(H_2(u))^{1-\alpha}(F(u))^{-1}\int_\Omega(\nabla u_1 \cdot \nabla\phi + a_{11}u_1\phi + \sum_{j=2}^m a_{1j}u_j\phi)$$
$$+qC_{q,\gamma}(1-\alpha)(H_1(u))^{\alpha}(H_2(u))^{-\alpha}(F(u))^{-1}\int_\Omega(|\nabla u_1|^{q-2}\nabla u_1 \cdot \nabla\phi + w_1|u_1|^{q-2}u_1\phi)$$
$$-\gamma C_{q,\gamma}\alpha(H_1(u))^{\alpha}(H_2(u))^{1-\alpha}(F(u))^{-2}\int_\Omega f_1|u_1|^{\gamma-2}u_1\phi = 0.$$

And

$$2C_{q,\gamma}\alpha\left(\frac{H_1(u)}{H_2(u)}\right)^{\alpha-1}\int_\Omega(\nabla u_1 \cdot \nabla\phi + a_{11}u_1\phi + \sum_{j=2}^m a_{1j}u_j\phi)$$
$$+qC_{q,\gamma}(1-\alpha)\left(\frac{H_1(u)}{H_2(u)}\right)^{\alpha}\int_\Omega(|\nabla u_1|^{q-2}\nabla u_1 \cdot \nabla\phi + w_1|u_1|^{q-2}u_1\phi)$$
$$-\lambda(u)\gamma\int_\Omega f_1|u_1|^{\gamma-2}u_1\phi = 0.$$

Define $\tilde{u_i} = \frac{1}{s}u_i$ for $i = 1, \cdots, m$, $s > 0$ and $H(u) = \frac{H_1(u)}{H_2(u)}$. Then

$$2C_{q,\gamma}\alpha(H(u))^{\alpha-1}s\int_\Omega(\nabla\tilde{u_1} \cdot \nabla\phi + a_{11}\tilde{u_1}\phi + \sum_{j=2}^m a_{1j}\tilde{u_j}\phi)$$
$$+C_{q,\gamma}(1-\alpha)(H(u))^{\alpha}qs^{q-1}\int_\Omega(|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1} \cdot \nabla\phi + w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\lambda(u)\gamma s^{\gamma-1}\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi = 0.$$

And equivalently

$$2C_{q,\gamma}\alpha(H(u))^{\alpha-1}s^{2-\gamma}\int_\Omega(\nabla\tilde{u_1} \cdot \nabla\phi + a_{11}\tilde{u_1}\phi + \sum_{j=2}^m a_{1j}\tilde{u_j}\phi)$$
$$+C_{q,\gamma}(1-\alpha)(H(u))^{\alpha}qs^{q-\gamma}\int_\Omega(|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1} \cdot \nabla\phi + w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\lambda(u)\gamma\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi = 0.$$

Multiplying this last equation by $\frac{1}{2^\alpha q^{1-\alpha}}$ and denoting $\tilde{\lambda} = \frac{\gamma}{2^\alpha q^{1-\alpha}}\lambda(u)$ we get

$$\left(\frac{2(q-\gamma)}{q(\gamma-2)H(u)}\right)^{\frac{\gamma-2}{q-2}} s^{2-\gamma}\int_\Omega(\nabla\tilde{u_1} \cdot \nabla\phi + a_{11}\tilde{u_1}\phi + \sum_{j=2}^m a_{1j}\tilde{u_j}\phi)$$

$$+\left(\frac{q(\gamma-2)H(u)}{2(q-\gamma)}\right)^{\frac{q-\gamma}{q-2}} s^{q-\gamma}\int_\Omega(|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1}\cdot\nabla\phi+w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\tilde{\lambda}\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi=0.$$

Choosing $s=\left(\frac{2(q-\gamma)}{q(\gamma-2)H(u)}\right)^{\frac{1}{q-2}}$ we obtain

$$\int_\Omega(\nabla\tilde{u_1}\cdot\nabla\phi+a_{11}\tilde{u_1}\phi+\sum_{j=2}^m a_{1j}\tilde{u_j}\phi)+\int_\Omega(|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1}\cdot\nabla\phi+w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\tilde{\lambda}\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi=0.$$

Doing the same for $\tilde{u_i}$, $i=2,\cdots,m$, we get that $\tilde{u}=(\tilde{u_1},\cdots,\tilde{u_m})$ is a weak solution of (4.1) associated with $\tilde{\lambda}$.

Now we prove that $I_{\tilde{\lambda}}(\tilde{u})=0$.
Recall that $\tilde{u}=\frac{1}{s}u$ and $\tilde{\lambda}=\frac{\gamma}{2^\alpha q^{1-\alpha}}\lambda(u)$. Then we have

$$I_{\tilde{\lambda}}(\tilde{u})=\frac{H_1(u)}{2s^2}+\frac{H_2(u)}{qs^q}-\frac{C_{q,\gamma}H_1(u)^\alpha H_2(u)^{1-\alpha}}{2^\alpha q^{1-\alpha}s^\gamma}.$$

Denoting $r=\frac{C_{q,\gamma}H_1(u)^\alpha H_2(u)^{1-\alpha}}{2^\alpha q^{1-\alpha}s^\gamma}$, since $\frac{1}{s}=(\frac{q}{2})^{\frac{1}{q-2}}t(u)>0$ and $\alpha=\frac{q-\gamma}{q-2}$ we get

$$I_{\tilde{\lambda}}(\tilde{u})=r\left[\frac{1}{C_{q,\gamma}}\left(\frac{qH_1(u)}{2H_2(u)}\right)^{1-\alpha}\left(\frac{q}{2}\right)^{\alpha-1}(t(u))^{2-\gamma}+\frac{1}{C_{q,\gamma}}\left(\frac{2H_2(u)}{qH_1(u)}\right)^{\alpha}\left(\frac{q}{2}\right)^{\alpha}(t(u))^{q-\gamma}-1\right].$$

But $t(u)=\left(\frac{\gamma-2}{q-\gamma}\right)^{\frac{1}{q-2}}\left(\frac{H_1(u)}{H_2(u)}\right)^{\frac{1}{q-2}}>0$ and $C_{q,\gamma}=\frac{q-2}{(q-\gamma)^\alpha(\gamma-2)^{1-\alpha}}$ so

$$I_{\tilde{\lambda}}(\tilde{u})=r\left[\frac{q-\gamma}{q-2}+\frac{\gamma-2}{q-2}-1\right]=0.$$

□

Proposition 4.2 *Assume that Assumptions 1.1 and 4.1 are satisfied and* $0<\lambda<\Lambda_1^+$. *Then the system* (4.1) *has no non-trivial solution in* W *associated with* λ.

Proof. Assume that $0<\lambda<\Lambda_1^+$. Assume also that the system (4.1) has a non-trivial solution $u=(u_1,\cdots,u_m)\in W$ associated with λ. Then we have
$H_1(u)+H_2(u)=\lambda F(u)$. Note that this is impossible if $F(u)\le 0$.

Therefore assume now that $F(u) > 0$.
Recall that $\tilde{I}_\lambda(t,v) = I_\lambda(tv) = \frac{t^2}{2}H_1(v) + \frac{t^q}{q}H_2(v) - \frac{\lambda t^\gamma}{\gamma}F(v)$ for all $t > 0$ and $v \in W$. We have $\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) = tH_1(v) + t^{q-1}H_2(v) - \lambda t^{\gamma-1}F(v)$ and in particular, since u is a weak solution of (4.1), note that

$$\frac{\partial}{\partial t}\tilde{I}_\lambda(\|u\|, \frac{1}{\|u\|}u) = I'_\lambda(u) \cdot \frac{1}{\|u\|}u = 0.$$

Moreover we have $\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) = t^{\gamma-1}R_\lambda(t,v)$ with

$$R_\lambda(t,v) = t^{2-\gamma}H_1(v) + t^{q-\gamma}H_2(v) - \lambda F(v).$$

Let $v \in W$ be such that $v \neq 0$ and $F(v) > 0$. Note that from Lemma 4.1 we have $R_{\lambda(v)}(t(v), v) = 0$.
Moreover we can prove that $R_\lambda(t,v) \geq R_\lambda(t(v), v)$ for all $t > 0$.
Indeed let $f(t) = t^{2-\gamma}H_1(v) + t^{q-\gamma}H_2(v)$. The function f admits a global minimum on $t(v)$ on $(0,\infty)$ so $f(t) \geq f(t(v)) = \left(\frac{H_1(v)}{\alpha}\right)^\alpha \left(\frac{H_2(v)}{1-\alpha}\right)^{1-\alpha} > 0$. Therefore $R_\lambda(t,v) \geq R_\lambda(t(v), v)$ for all $t > 0$.
Finally since $\lambda < \Lambda_1^+ \leq \lambda(v)$, we get that $R_\lambda(t,v) > R_{\lambda(v)}(t,v)$ for all $t > 0$. Thus $R_\lambda(t,v) \geq R_\lambda(t(v), v) > R_{\lambda(v)}(t(v), v) = 0$ and $\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) = \frac{\partial}{\partial t}I_\lambda(tv) > 0$ for all $t > 0$.
So, choosing $t = \|u\|$ and $v = \frac{1}{\|u\|}u$, we get a contradiction since $\frac{\partial}{\partial t}\tilde{I}_\lambda(\|u\|, \frac{1}{\|u\|}u) = 0$. □

Now we obtain a minimizer for Λ_1^+.

Proposition 4.3 *Assume that Assumptions 1.1 and 4.1 are satisfied. There exists $v = (v_1, \cdots, v_m) \in W$ such that $\lambda(v) = \Lambda_1^+$.*

Proof. First note that $\lambda(tu) = \lambda(u)$ for all $t > 0$ and $u \in W$.
Define $\tilde{t}(u) = \frac{1}{((H_1(u))^\alpha (H_2(u))^{1-\alpha})^{\frac{1}{\gamma}}}$ for $u \in W \setminus \{0\}$ and note that

$$(H_1(\tilde{t}(u)u)^\alpha (H_2(\tilde{t}(u)))^{1-\alpha} = 1.$$

Therefore we can derive that

$$\Lambda_1^+ = \inf\{\lambda(u), u \in W \text{ such that } F(u) > 0 \text{ and } H_1(u)^\alpha H_2(u)^{1-\alpha} = 1\}.$$

Then we consider a minimizing sequence (v_n) of Λ_1^+.
We have $\gamma = 2\alpha + q(1-\alpha)$, so

$$\|v_n\|_H^\gamma = \|v_n\|_H^{2\alpha} \ \|v_n\|_H^{q(1-\alpha)}$$

and since $W \subset H$ with a continuous embedding, there exists a positive constant C such that

$$\|v_n\|_H^\gamma \leq C\|v_n\|_H^{2\alpha} \ \|v_n\|_W^{q(1-\alpha)}.$$

But H_1 and H_2 are equivalent norms respectively in H and W so we get that

$$\|v_n\|_H^\gamma \leq C(H_1(v_n))^\alpha (H_2(v_n))^{1-\alpha} = C$$

(for a positive constant C). We deduce that (v_n) is a bounded sequence in H. By the compact embedding $W_0^{1,2}(\Omega) \subset L^\gamma(\Omega)$ (for $\gamma < 2^*$), we get the existence of $v = (v_1, \cdots, v_m) \in H$ such that (v_n) converges to v, strongly in $(L^\gamma(\Omega))^m$ and weakly in H (for a subsequence).

Afterwards we prove that $F(v) > 0$, $v \in W$ and since H_1 and H_2 are weakly lower semi-continous in H and W respectively, we get that $\lambda(v) = \Lambda_1^+$.

Indeed, since F is a continuous function and $F(v_n) > 0$, $F(v_n) \to_{n\to\infty} F(v)$, we have $F(v) \geq 0$. Moreover, if $F(v) = 0$, then $\lambda(v_n) = \frac{C_{q,\gamma}}{F(v_n)} \to_{n\to\infty} \infty$. This contradicts $\lambda(v_n) \to_{n\to\infty} \Lambda_1^+$. So $F(v) > 0$ and $v \neq 0$.

Now we prove that $v \in W$. Recall that (v_n) is a bounded sequence in H and that (v_n) converges to $v \neq 0$ strongly in $(L^\gamma(\Omega))^m$. So there exists a positive constant C' such that $\|v_n\|_{(L^\gamma(\Omega))^m} \geq C' > 0$ for n large enough. Therefore, from the continuous embedding $H \subset (L^\gamma(\Omega))^m$, we get that $\|v_n\|_H \geq C' > 0$ for n large enough.
Finally from $\|v_n\|_H \geq C' > 0$ and $\|v_n\|_H^{2\alpha} \|v_n\|_W^{(1-\alpha)q} \leq C$ we obtain that (v_n) is a bounded sequence in W. Therefore (v_n) admits a subsequence, still denoted (v_n) such that (v_n) converges to v strongly in $(L^\gamma(\Omega))^2$ and weakly in W. Thus $v \in W$.

Finally we prove that $\lambda(v) = \Lambda_1^+$.
From the weakly semi-continuousness of H_1 and H_2 respectively on H and W we have

$$H_1(v) \leq \liminf H_1(v_n) \text{ and } H_2(v) \leq \liminf H_2(v_n).$$

But $\lambda(v_n) = \frac{C_{q,\gamma}}{F(v_n)} = \frac{C_{q,\gamma}(H_1(v_n))^\alpha (H_2(v_n))^{1-\alpha}}{F(v_n)} \to_{n\to\infty} \Lambda_1^+$. Passing to the limit inf as n tends to ∞ we get that $\Lambda_1^+ \geq \frac{C_{q,\gamma}(H_1(v))^\alpha (H_2(v))^{1-\alpha}}{F(v)} = \lambda(v)$. We deduce that

$$\lambda(v) = \Lambda_1^+.$$

This concludes the proof. □

Contrary to [7], we are not able to prove that the minimizer v is non-negative because of the coupling terms $a_{ij}v_jv_i$ in $H_1(v)$. Finally combining Propositions 4.1 and 4.3, since v (defined by Proposition 4.3) is a critical point of $\lambda(u)$, we derive the existence of a non-trivial weak solution $u^+ = (u_1^+, \cdots, u_m^+)$ for the system (4.1) associated with λ_1^+. This is the following result

Proposition 4.4 *Assume that Assumptions 1.1 and 4.1 are satisfied.*
There exists $u^+ = (u_1^+, \cdots, u_m^+) \in W$ *a non-trivial solution for the system* (4.1) *associated with* λ_1^+. *Moreover* $I_{\lambda_1^+}(u^+) = 0$ *and* $F(u^+) > 0$.

Proof. From Proposition 4.3 we have $\lambda(v) = \Lambda_1^+ = \inf\{\lambda(u), u \in W, F(u) > 0\}$. Thus v is a critical point of the function λ on W. From Proposition 4.1 we derive that there exists a non-trivial solution $u^+ = (u_1^+, \cdots, u_m^+)$ of system (4.1) associated with $\frac{\gamma}{2^\alpha q^{1-\alpha}}\lambda(v) = \lambda_1^+$ where for all $i = 1, \cdots, m$, $u_i^+ = \frac{1}{s}v_i$ and $\frac{1}{s} = (\frac{q}{2})^{\frac{1}{q-2}}t(v) > 0$. Moreover from Proposition 4.1, $I_{\lambda_1^+}(u^+) = 0$ and from Proposition 4.3, $F(u^+) = \frac{1}{s^\gamma}F(v) > 0$. □

4.2 Main result

Theorem 4.1 *Assume that Assumptions 1.1 and 4.1 are satisfied. If* $\lambda > \lambda_1^+$, *then the system* (4.1), *associated with* λ, *admits a non-trivial solution in* W.

Proof. Even if we could follow [7] for proving this result using the mountain-pass theorem, we use here standard arguments by global minimization of the C^1-functional I_λ. Note that I_λ is weakly lower semi-continuous by the compact embedding of W into $(L^q(\Omega))^m$ and $(L^2(\Omega))^m$. Moreover I_λ is coercive: indeed for any $u \in W$,

$$I_\lambda(u) \geq \frac{1}{q}H_2(u) - \frac{\lambda}{\gamma}F(u).$$

Since $|F(u)| \leq C\|u\|_W^\gamma$ with C a positive constant, we get that

$$I_\lambda(u) \geq \frac{1}{q}\|u\|_W^q \left(1 - \frac{\lambda C q}{\gamma}\|u\|_W^{\gamma-q}\right).$$

Thus I_λ is coercive. Furthermore from Proposition 4.4, we have $I_{\lambda_1^+}(u^+) = 0$ and $F(u^+) > 0$. Finally from the hypothesis $\lambda > \lambda_1^+$, we get that $I_\lambda(u^+) < I_{\lambda_1^+}(u^+) = 0$. Therefore we deduce that I_λ has a non-trivial critical point which is a non-trivial weak solution of the system (4.1) associated with λ. □

Remarks: We can get the same results for a larger class of coefficients, assuming that $a_{ij}, w_i, f_i \in L^r(\Omega)$ for some $r > 1$ as in [7]. But we have not been able to adapt this method for a system with a (p,q)-Laplacian operator (with $p \neq 2$) and even for a non-symmetric system with a (2,q)-Laplacian operator. However in the particular case where the matrix A is not symmetric and has the following form: $A = (a_{ij})$ with $a_{j1} = Ka_{1j}$ for $j = 2, \cdots, m$ for some positive constant $K > 0$ (K independent of j) and $a_{ij} = a_{ji}$ for $i, j \geq 2$, we can generalize all the above results. Indeed we introduce the diagonal matrix $D = (d_{ij})$ with

$d_{11} = K$, $d_{ii} = 1$ for $i = 2, \cdots, m$ and $d_{ij} = 0$ if $i \neq j$. We replace the functionals H_1, H_2 and F (defined before by (1.1),(1.2),(4.2)) by

$$H_2(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega (|\nabla u_i|^q + w_i |u_i|^q), \; F(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega f_i |u_i|^\gamma,$$

$$H_1(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega (|\nabla u_i|^2 + a_{ii} u_i^2 + \sum_{j=1, i\neq j}^{m} a_{ij} u_j u_i),$$

$$H_1(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega |\nabla u_i|^2 + \int_\Omega {}^t U D A U \text{ with} {}^t U = (u_1, \cdots, u_m).$$

Therefore if we assume that the matrix DA satisfies the following hypothesis ${}^t\xi DA\xi \geq 0$ for all ${}^t\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^m$, we still derive that the Euler functional I_λ defined by (4.3) (with the new functionals H_1, H_2 and F) is associated with the system (4.1) and the existence of weak solutions for the system (4.1) is equivalent to the existence of critical points for I_λ. Finally due to the coupling term of system (4.1), note that we just obtain the existence of a non-trivial solution in Theorem 4.1 contrary to [7] where a non-negative solution is obtained.

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received: May 05, 2019

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