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On the Square-Triangular Numbers and Balancing-Numbers

ABSTRACT. In 1770, Euler looked for positive integers n and m such that $n(n+1)/2 = m^2$. Integer solutions for this equation produce what he called square-triangular numbers. In this paper, we present a new explicit formula for this kind of numbers and establish a link with balancing numbers.

KEY WORDS. Triangular number, Square number, Square-triangular number, Balancing number

1 Introduction

A triangular number counts objects arranged in an equilateral triangle. The first five triangular numbers are 1, 3, 6, 10, 15, as shown in Figure 1. Let T_n denote the n^{th} triangular number, then T_n is equal to the sum of the n natural numbers from 1 to n, i.e.,

$$T_n = 1 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

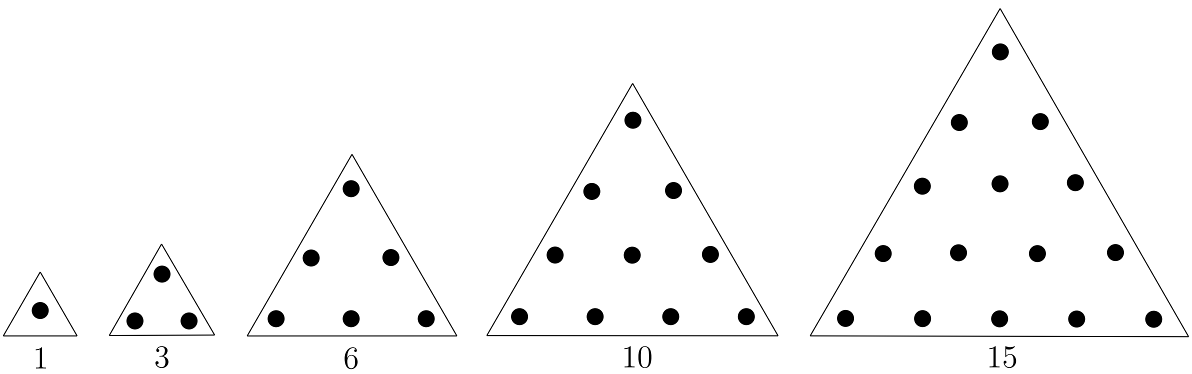

Figure 1: The first five triangular numbers

Similar considerations lead to square numbers which can be thought of as the numbers of objects that can be arranged in the shape of a square. The first five square numbers are 1, 4, 9, 16, 25, as shown in Figure 2. Let S_n denote the n^{th} square number, then we have

$$S_n = n^2.$$

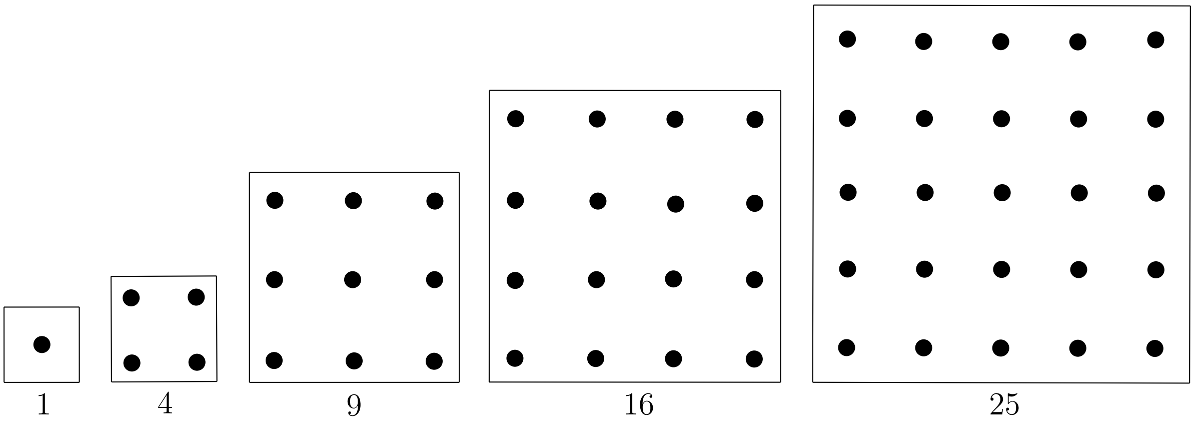

Figure 2: The first five square numbers

A square-triangular number is a number which is both a triangular and square number. The firsts non-trivial square-triangular number is 36, see Figure 3. A square-triangular number is a positive integer solution of the diophantine equation:

$$\frac{n(n+1)}{2} = m^2. \tag{1}$$

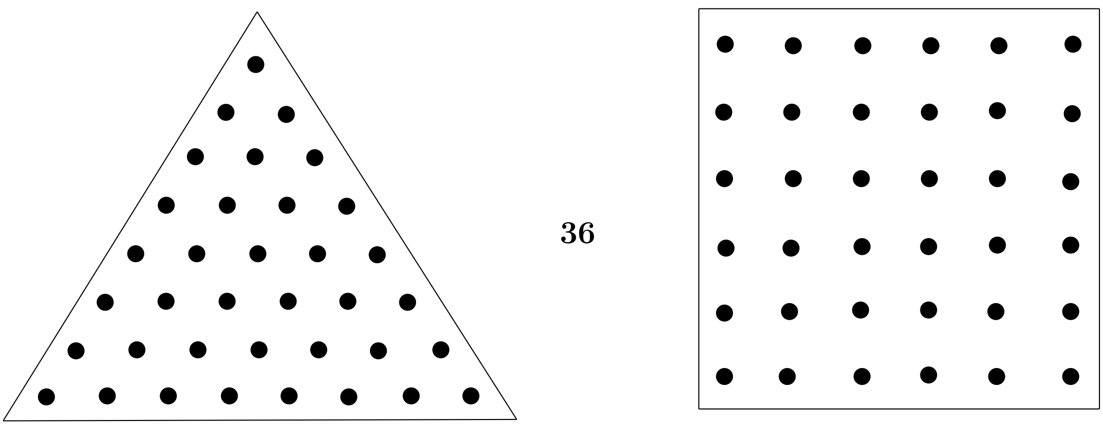

Figure 3: The first non-trivial square-triangular number

2 Main Results

Lemma 1 *(n,m) is a solution of Equation (1) if, and only if*

$$n=\sum_{i=0}^{k-1}\binom{2k}{2i+2}2^i \quad \textit{and} \quad m=\sum_{i=-1}^{k-2}\binom{2k}{2i+3}2^i, \ \textit{for } k\in\mathbb{N}^*.$$

Proof. From Equation (1), we have

$$n^2+n-2m^2=0. \tag{2}$$

Equation (2) can be rewritten as follows:

$$(2n+1)^2-2(2m)^2=1. \tag{3}$$

Letting $x=2n+1$ and $y=2m$, Equations (3) becomes the Pell equation:

$$x^2-2y^2=1. \tag{4}$$

It is well known, that the form x^2-2y^2 is irreducible over the field $\mathbb{Q}$ of rational numbers, but in the extension field $\mathbb{Q}(\sqrt{2})$ it can be factored as a product of linear factors $(x+y\sqrt{2})(x-y\sqrt{2})$. Using the norm concept for the extension field $\mathbb{Q}(\sqrt{2})$, Equation (4) can be written in the form:

$$N\left(x+y\sqrt{2}\right)=1. \tag{5}$$

It is easily checked that the set of all numbers of the form $x+y\sqrt{2}$, where x and y are integers, form a ring, which is denoted $\mathbb{Z}[\sqrt{2}]$. The subset of units of $\mathbb{Z}[\sqrt{2}]$, which we denote $\mathcal{U}$ forms a group. It is easy to show that $\alpha\in\mathcal{U}$ if and only if $N(\alpha)=\pm1$ [2]. Applying Dirichlet's Theorem, we can show that $\mathcal{U}=\{\pm\left(1+\sqrt{2}\right)^k,\ k\in\mathbb{Z}\}$.

Since

$$N\left(\left(1+\sqrt{2}\right)^k\right)=\left(N\left(1+\sqrt{2}\right)\right)^k=(-1)^k, \tag{6}$$

we obtain

$$N(\alpha)=1\Leftrightarrow\alpha=\left(1+\sqrt{2}\right)^{2k},\ k\in\mathbb{Z}. \tag{7}$$

Thus, all integral solutions of Equation (4) are given by:

$$\begin{aligned} x+\sqrt{2}y &= \left(1+\sqrt{2}\right)^{2k} \\ &= \sum_{i=0}^{2k}\binom{2k}{i}2^{i/2} \\ &= \left(\sum_{i=0}^{k}\binom{2k}{2i}2^i\right)+\sqrt{2}\left(\sum_{i=0}^{k-1}\binom{2k}{2i+1}2^i\right). \end{aligned} \tag{8}$$

We get, after identification

$$2n+1 = x = \sum_{i=0}^{k} \binom{2k}{2i} 2^i,$$

and

$$2m = y = \sum_{i=0}^{k-1} \binom{2k}{2i+1} 2^i.$$

Equivalently, we have

$$n = \sum_{i=0}^{k-1} \binom{2k}{2i+2} 2^i,$$

and

$$m = \sum_{i=-1}^{k-2} \binom{2k}{2i+3} 2^i.$$

This completes the proof. □

We have thus proved, via Lemma 2, the following theorem.

Theorem 2 *Let ST_n denotes the n^{th} square-triangular number. Then*

$$ST_n = S_m = T_k,$$

where

$$m = \sum_{i=-1}^{n-2} \binom{2n}{2i+3} 2^i \quad \textit{and} \quad k = \sum_{i=0}^{n-1} \binom{2n}{2i+2} 2^i.$$

3 A Link Between Square-Triangular Numbers and Balancing Numbers

Behera and Panda [1] introduced balancing numbers $m \in \mathbb{Z}^+$ as solutions of the equation:

$$1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r). \tag{9}$$

Theorem 3 *Let B_n be the n^{th} balancing number. Then*

$$ST_n = B_n^2.$$

Proof. By making the substitution $m+r=n$, with $n \geq m+1$, Equation (9) becomes

$$1+2+\cdots+(m-1) = (m+1)+(m+2)+\cdots+n. \tag{10}$$

Therefore

$$\begin{aligned}
m \text{ is a balancing number} \iff & 1+2+\cdots+(m-1) = (1+2+\cdots+n) - (1+2+\cdots+m) \\
\iff & \frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2} \\
\iff & \frac{m(m-1)}{2} + \frac{m(m+1)}{2} = \frac{n(n+1)}{2} \\
\iff & m^2 = \frac{n(n+1)}{2} \\
\iff & m^2 \text{ is a square-triangular number}
\end{aligned}$$

This completes the proof. □

Table 1 bellow summarizes the first ten square-triangular numbers with there associated triangular and balancing numbers, based on Theorem 2 and Theorem 3.

n	$N = \sum_{i=0}^{n-1} \binom{2n}{2i+2} 2^i$	$T_N = \frac{N(N+1)}{2}$	$B_n = \sum_{i=-1}^{n-2} \binom{2n}{2i+3} 2^i$	$ST_n = B_n^2$
1	1	1	1	1
2	8	36	6	36
3	49	1225	35	1225
4	288	41616	204	41616
5	1681	1413721	1189	1413721
6	9800	48024900	6930	48024900
7	57121	1631432881	40391	1631432881
8	332928	55420693056	235416	55420693056
9	1940449	1882672131025	1372105	1882672131025
10	11309768	63955431761796	7997214	63955431761796

Table 1: The first ten square-triangular numbers

4 Recurrence Relations for Square-Triangular Numbers

Theorem 4 *The sequence of square-triangular numbers $(ST_n)_n$ satisfies the recurrence relation:*

$$ST_n = 34ST_{n-1} - ST_{n-2} + 2, \; for \; n \geq 3,$$

with $ST_1 = 1$ and $ST_2 = 36$.

Proof. It is well known that the sequence of balancing numbers satisfies the following recurrence relations [1]:

$$B_{n+1} = 6B_n - B_{n-1}, \tag{11}$$

and

$$B_n^2 - B_{n+1}B_{n-1} = 1\cdot \tag{12}$$

Hence

$$\begin{aligned} B_n^2 &= (6B_{n-1} - B_{n-2})^2 \\ &= 36B_{n-1}^2 - 12B_{n-1}B_{n-2} + B_{n-2}^2\cdot \end{aligned}$$

From Equation (11), we get

$$\begin{aligned} B_n^2 &= 36B_{n-1}^2 - 12\left(\frac{B_n + B_{n-2}}{6}\right)B_{n-2} + B_{n-2}^2 \\ &= 36B_{n-1}^2 - 2B_nB_{n-2} - B_{n-2}^2 \\ &= 34B_{n-1}^2 - 2\left(B_nB_{n-2} - B_{n-1}^2\right) - B_{n-2}^2\cdot \end{aligned}$$

From Equation (12), we get

$$B_n^2 = 34B_{n-1}^2 - B_{n-2}^2 + 2\cdot$$

This completes the proof according to Theorem 3. □

5 Generating Function for Square-Triangular Numbers

In this section, we present the generating function based on some relations on balancing numbers.

Theorem 5 *The generating function of ST_n is:*

$$f(x) = \frac{x(1+x)}{(1-x)(x^2-34x+1)}\cdot$$

Proof. Let $f(x) = \sum_{n\geq 1} ST_n\, x^n$. Then

$$34xf(x) = \sum_{n\geq 2} 34ST_{n-1}\, x^n,$$

and

$$x^2 f(x) = \sum_{n\geq 3} ST_{n-2}\ x^n \cdot$$

Therefore

$$\begin{aligned} 34xf(x) - x^2 f(x) &= 34x^2 + \sum_{n\geq 3}(34ST_{n-1} - ST_{n-2})\ x^n \\ &= 34x^2 + \sum_{n\geq 3}(34ST_{n-1} - ST_{n-2} + 2)\ x^n - 2\sum_{n\geq 3} x^n \cdot \end{aligned}$$

By Theorem 6, we have

$$\begin{aligned} 34xf(x) - x^2 f(x) &= 34x^2 + \sum_{n\geq 3} ST_n\ x^n - 2\left(\frac{1}{1-x} - 1 - x - x^2\right) \\ &= 34x^2 + \left(f(x) - x - 36x^2\right) - 2\left(\frac{1}{1-x} - 1 - x - x^2\right) \\ &= f(x) - \frac{x(1+x)}{1-x} \cdot \end{aligned}$$

Hence

$$(1 - 34x + x^2) f(x) = \frac{x(1+x)}{1-x} \cdot$$

This completes the proof. □

By using the generating function we can have the following equivalent explicit formula for the sequence of square-triangular numbers $(ST_n)_n$ that may be convenient to include.

Theorem 6 *For $n \geq 1$, we have*

$$ST_n = \frac{\left(17 + 12\sqrt{2}\right)^n + \left(17 - 12\sqrt{2}\right)^n - 2}{32} \cdot$$

Proof. From expanding the generating function of ST_n in partial fractions, we obtain

$$f\,(x) = \frac{1}{16\,(x-1)} + \frac{12\sqrt{2} - 17}{32\left(12\sqrt{2} - 17 + x\right)} + \frac{12\sqrt{2} + 17}{32\left(12\sqrt{2} + 17 - x\right)} \cdot$$

Therefore

$$\begin{aligned} f\,(x) &= -\frac{1}{16}\sum_{n\geq 0} x^n + \frac{1}{32}\sum_{n\geq 0}\frac{(-x)^n}{\left(-17 + 12\sqrt{2}\right)^n} + \frac{1}{32}\sum_{n\geq 0}\frac{x^n}{\left(17 + 12\sqrt{2}\right)^n} \\ &= -\frac{1}{16}\sum_{n\geq 0} x^n + \frac{1}{32}\sum_{n\geq 0}\left(17 + 12\sqrt{2}\right)^n x^n + \frac{1}{32}\sum_{n\geq 0}\left(17 - 12\sqrt{2}\right)^n x^n \cdot \end{aligned}$$

Then

$$ST_n = -\frac{1}{16} + \frac{1}{32}\left(17 + 12\sqrt{2}\right)^n + \frac{1}{32}\left(17 - 12\sqrt{2}\right)^n \cdot$$

Hence, the result follows. □

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