Rostock. Math. Kolloq. **71**, 14 – 27 (2017/18)

Subject Classification (AMS)
03B05, 34D05, 34D20, 54C35, 54C50

Harry Poppe

Reference Stability for ODE

1 Introduction

We consider initial value problems for autonomous ODE, and we will study stability for these problems. The dignified definition of Ljapunow stability has two shortcomings. To overcome these difficulties we define the notion of reference stability. This notion has especially the advantage that we can characterize it topologically. We illustrate the procedere by simple examples.

2 Some simple but instructive examples

We consider the equations $\dot{x} = \pm x^n$, $n \geq 2$, $x(t_0) = x_0$. But these equations are autonomous and hence we let $t_0 = 0$.

These equations are of product type:

$$\dot{x} = g(t)h(x) = x^n \qquad (h(x) = \pm x^n)\,.$$

Since $h(x) = 0 \iff \pm x^n = 0 \iff x = 0$: $x \equiv 0$ is an equilibrium point of $\dot{x} = \pm x^n$, $x(0) = 0$. We will show that the zero solutions of our equations always are unique.

Proposition 2.1 *The zero solution of our equations always are unique.*

Proof: We have $h(x) = \pm x^n$, it is enough to consider $h(x) = x^n$.

Now

$$\int_y^1 \frac{1}{h(s)}\,ds = \int_y^1 s^{-n}\,ds = \left(\frac{1}{1-n}s^{1-n}\right)\Big|_y^1 = \frac{1}{1-n}\left(1 - \frac{1}{y^{n-1}}\right) \Longrightarrow$$
$$\left|\lim_{y\to 0}\left(\frac{1}{1-n}\right)\left(1-\frac{1}{y^{n-1}}\right)\right| = +\infty\,,$$

since $n \geq 2$. Thus by a well-known criterion (see [4]) $x \equiv 0$ is a unique solution.

(a) $\dot{x} = x^n$; for $x \neq 0$, $x_0 \neq 0$ we find:

$$\begin{aligned}
&\int_{x_0}^{x} s^{-n} ds = \int_0^t 1 ds = t\,, \\
&\frac{1}{1-n}\left(s^{1-u}\Big|_{x_0}^{x}\right) = t\,, \\
&x^{1-u} - x_0^{1-u} = (1-n)t \\
&x^{1-n} = x_0^{1-n} + (1-n)t
\end{aligned} \tag{2.1}$$

(b) $\dot{x} = -x^n$; we get

$$\begin{aligned}
&x^{1-n} = x_0^{1-n} + (1-n)(-t) \\
&x^{1-n} = x_0^{1-u} + (n-1)t
\end{aligned} \tag{2.2}$$

Example 2.2 (a) $n = 3$: $\dot{x} = x^3$, $x(0) = x_0$:

$x^{1-3} = x^{-2} = x_0^{-2} - 2t = \dfrac{1}{x_0^2} - 2t = \dfrac{1-2x_0^2 t}{x_0^2} \implies |x| = \dfrac{|x_0|}{\sqrt{1-2x_0^2 t}}$. We know that $x_0 \neq 0$ holds and hence x is defined on $\left(-\infty, \dfrac{1}{2x_0^2}\right)$.

We have two cases:

1. $x_0 > 0$, then by continuity and since $\dot{x} = x$ is autonomous: $\forall t \in \left(-\infty, \dfrac{1}{2x_0^2}\right)$: $x(t) > 0$, thus $|x| = x = \dfrac{x_0}{\sqrt{1-2x_0^2 t}}$.
2. $x_0 < 0$, by the same argument: $\forall t \in \left(-\infty, \dfrac{1}{1x_0^2}\right)$:
   $|x(t)| = -x(t) = \dfrac{-x_0}{\sqrt{1-2x_0^2 t}} \implies x = x(t) = \dfrac{x_0}{\sqrt{1-2x_0^2 t}}$,

and here $x_0 < 0 \implies x(t) < 0\,\forall t$.

Remarks 2.3 (a) Result concerning stability: $0 : \forall t \in [0, +\infty) : 0(t) = 0$ is defined on $[0, +\infty)$, but no other solution is defined on this interval.

(b) The sets of possible initial values of this equation are:

$$(0, +\infty), \quad (-\infty, 0)$$

(c) $\dfrac{1}{2x_0^2}$, $x_0 \neq 0$ is a pole:

$$\lim_{t \to \left(\frac{1}{2x_0^2}\right)} -\frac{x_0}{\sqrt{1-2x_0^2 t}} = \begin{cases} +\infty, & x_0 > 0 \\ -\infty, & x_0 < 0 \end{cases}$$

Example 2.4 (b) $n = 3$, $\dot{x} = -x^3$, $x(0) = x_0$, $x_0 \neq 0$. By (2.2) holds:

$$x^{1-n} = x_0^{1-n} + (n-1)t; n = 3 \Longrightarrow x^{-2} = x_0^{-2} + 2t,$$
$$x^{-2} = \frac{1 + 2x_0^2 t}{x_0^2} \Longrightarrow |x| = \frac{|x_0|}{\sqrt{1 + 2x_0^2 t}}\,,$$

and $\forall x_0$, $x_0 \neq 0 : \forall t \in [0, +\infty) : 1 + 2x_0^2 t > 0$.

As above we get again

$$x_0 > 0 \Longrightarrow x = \frac{x_0}{\sqrt{1 + 2x_0^2 t}} > 0$$
$$x_0 < 0 \Longrightarrow x = \frac{x_0}{\sqrt{1 + 2x_0^2 t}} < 0$$

Remarks 2.5 1. Result concerning stability: $x \equiv 0$ is defined on $[0, +\infty)$ and all other solutions too.

2. The set of all possible initial values of this equation is $(-\infty, 0) \cup (0, +\infty)$.

More precisely we have

$$1 + 2x_0^2 t > 0 \Longleftrightarrow -\frac{1}{2x_0^2} < t, \text{ and } -\frac{1}{2x_0^2} < 0\,.$$

Remark 2.6 More example of ODE we consider to illustrate definitions or to apply the results of propositions.

3 Ljapunow – Stability

We consider an autonomous system of ordinary differential equations:

$$\dot{x} = f(x), \;\; f : G \to \mathbb{R}^n\,, \tag{3.1}$$

$G \subseteq \mathbb{R}^n$ is open and f is continuous. Let $t_0 \in \mathbb{R}$, $0 \leq t_0$ and let x be a solution, $x(t_0) = x_0 \in \mathbb{R}^n$ defined (at least) on $[t_0, +\infty)$.

We want to formulate that x is Ljapunow-stable in a precise way. But this is only possible, if we use the following definition:

Definition 3.1 *x is called Ljapunow-stable (L-stable) iff $\forall \varepsilon > 0 \; \exists \delta > 0$, $\delta = \delta(\varepsilon)$, $0 < \delta \leq \varepsilon : \forall y$, where y solves the initial value problem $\dot{y} = f(y)$, $y(t_0) = y_0 \in G$ (on some intervall of $\mathbb{R}$):*

$$\|y_0 - x_0\| < \delta \Longrightarrow \; (y \textit{ is defined on } [t_0, +\infty) \textit{ and } \; \forall t \geq t_0 : \|y(t) - x(t)\| < \varepsilon).$$

Remarks 3.2 1. The very definition of Ljapunow stability is in some sense unclear: several authors use definition 3.1, see for instance [1], [3], [4].

Other do not mention at all the domain of the (reference) solutions y in definition 3.1, see for instance [5], [6], [7].

2. Definition 3.1 has two serious shortcomings.

First shortcoming.
The two statements of the conclusion of the implication:

y is defined on $[t_0, +\infty)$, $\forall\, t \geq t_0 : \|y(t) - x(t)\| < \varepsilon$ are not independent:

by the Ljapunow definition of stability we find a family of implications: $\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) :$ $\|y(t_0) - x(t_0)\| < \delta \Longrightarrow$ (y exists on $[t_0, +\infty)$ and $\forall\, t \geq t_0 : \|y(t) - x(t)\| < \varepsilon$). This family depends on ε (and the associated $\delta(\varepsilon)$). Now we fix $\varepsilon = \overline{\varepsilon} > 0$ and we find $\delta = \delta(\overline{\varepsilon})$; indeed we now have one single implication: $\|y(t_0) - x(t_0)\| < \delta(\overline{\varepsilon}) \Longrightarrow$ (y is defined on $[t_0, +\infty)$ and $\forall\, t \geq t_0 : \|y(t) - x(t)\| < \overline{\varepsilon}$), in short: $A \Longrightarrow (B \wedge C)$.

But this implications is equivalent to

$$\neg(B \wedge C) \Longrightarrow \neg A\,, \text{ or}$$
$$\neg B \vee \neg C \Longrightarrow \neg A\,.$$

Now, if $\neg B$ is true, then there exists $t_1 \in (t_0, +\infty)$ such that y is not defined in $t_1 : y(t_1)$ does not exist.

If $\neg C$ is false, that is C is true, we have:

$$\forall\, t \geq t_0 : \|y(t) - x(t)\| < \overline{\varepsilon}\,,$$

which means $\|y(t) - x(t)\|$ is a (positive) real number and the assertion is: each of these numbers is smaller than $\overline{\varepsilon}$.

But here we find an error:

$$\|y(t_1) - x(t_1)\|$$

is no number, but a senseless symbol.

This senseless symbol also can occur if $\neg C$ is true. Then we find $t_2 \in (0, +\infty)$:

$$\|y(t_2) - x(t_2)\| \geq \overline{\varepsilon}\,.$$

and either $\|y(t_2) - x(t_2)\| \in \mathbb{R}$ or $\|y(t_2) - x(t_2)\|$ is a senseless symbol.

Second shortcoming.
If we have found a set of (explizite) solutions y, then we can often by the Ljapunow definition of stability easily, without starting to prove that x is stable or unstable, decide that the solution x is not stable. This we can conclude from the following proposition and its corollary.

Proposition 3.3 *We consider the initial value problem (3.1) and let $x : [t_0, +\infty) \to \mathbb{R}^n$ be a solution. If y is another solution $y : (a, b) \to \mathbb{R}^n$, $t_0 \in (a, b)$, we denote by $D(y)$ the domain (a, b) of y. Now we assume that there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of solutions s. th. $y_n(t_0) \to x(t_0)$ and $\forall\, n \in \mathbb{N} : [t_0, +\infty) \nsubseteq D(y_n)$. Then x is not stable.*

Proof: We assume that x is stable: for $\varepsilon_0 = 1\ \exists\, \delta \in \mathbb{R},\ 0 < \delta \leq 1 : \forall\, y : \|y(t_0) - x(t_0)\| < \delta \Longrightarrow y$ is defined on $[t_0, +\infty)$ and $\forall\, t \geq t_0 : \|y(t) - x(t)\| < 1$; $\exists\, n_1 \in \mathbb{N} : y_{n_1}(t_0) \in U_\delta(x(t_0))$ and hence $\|y_{n_1}(t_0) - x(t_0)\| < \delta$. Thus y_{n_1} is defined on $[t_0, +\infty)$, yielding a constradiction since $[t_0, +\infty) \nsubseteq D(y_{n_1})$. Hence x is not stable.

Corollary 3.4 *Let S_0 be the set of all solutions of $\dot{y} = f(y)$, $y(t_0) = y_0 \in G$ which are not defined entirely on $(t_0, +\infty)$, hence $x \notin S_0$. Let S_0 be infinite and let $x(t_0)$ be a cluster point of $\{y(t_0) | y \in S_0\}$.*

Then x is not stable.

Example 3.5 We come back to example 2.2:

$$\dot{x} = x^3,\ x(0) = x_0 \in \mathbb{R}\,;$$

S_0 consists of all nontrivial solutions of the initial value problem and hence $\{y(0)|y \in S_0\} = (-\infty, 0) \cup (0, +\infty)$. Thus we can apply the corollary and since $x(0) = 0$ is a cluster point of $\{y(0)|y \in S_0\}$ we find that x is unstable.

But since we have no solution which we can compare with the zero function x on $[0, +\infty)$, the assertion "x is unstable" makes no sence.

4 The Reference-Stability

There exists a consequent and simple way out from the difficulties of the Ljapunow stability definition: we consider only the set of all solutions of the initial value problem which are defined (at least) on $[t_0, +\infty)$.

Definition 4.1 *Let x be defined on $[t_0, +\infty)$ and x is solution of the initial value problem (3.1)*

$$R = R(x) = \{y | y : [t_0, +\infty) \to \mathbb{R}^n,\ \dot{y} = f(y),\ y(t_0) = y_0 \in \mathbb{G} \text{ and } y \neq x\}\,;$$

$R(x)$ is called the set of reference solutions of the solution x. Of course, instead of $y : [t_0, +\infty) \to \mathbb{R}^n$ we can use: $[t_0, +\infty) \subseteq D(y)$.

Example 4.2 Let be $\dot{y} = f(y) = y$, $t_0 = 0$, $x : \forall\, t \geq 0 : x(t) = 0$, the zero solution: $x(0) = 0$. Then $R(x) = R(0) = \{y = y_0 e^t | y_0 \in \mathbb{R}\backslash\{0\}\}$ is the set of reference solutions of $x \equiv 0$.

Example 4.3 We consider example 2.4: $\dot{x} = -x^3$, $x(0) = x_0 \in \mathbb{R}$; then for the zero solution $x \equiv 0$, $x(0) = 0$, we find on

$$[0, +\infty) : R(x) = R(0) = \left\{ x = \frac{x_0}{\sqrt{1 + 2x_0^2 t}} \middle| x_0 \in \mathbb{R} \backslash \{0\} \right\} .$$

Definition 4.4 *We consider the initial value problem* (3.1) *and the solution* $x : [t_0, +\infty) \to \mathbb{R}^n$ *is to be investigated on stability; let* $R(x)$ *be the set of reference solutions of* x*; we assume:* $R(x) \neq \emptyset$*.* x *is called reference stable , R-stable, iff* $\forall\, \varepsilon > 0\, \exists\, \delta = \delta(\varepsilon)$, $0 < \delta \leq \varepsilon : \forall\, y \in R(x) :$

$$\|y(t_0) - x(t_0)\| < \delta \Longrightarrow \forall\, t \geq t_0 : \|y(t) - x(t)\| < \varepsilon\, .$$

Remarks 4.5

1. We emphasize insistently what was assumed in the definition: within reference stability we always assume $R(x) \neq \emptyset$. If $R(x) = \emptyset$ holds, we simply say that we have no stability problem. As an example for this situation we look at example 2.2: $\dot{x} = x^3$, $x(0) = 0$: here we have $R(x) = R(0) = \emptyset$.
2. As usual we still define: x is called to be asymptotically reference stable iff x is reference stable and $\exists\, \delta > 0 : \forall\, y \in R(x)$

$$\|y(t_0) - x(t_0)\| < \delta \Longrightarrow \lim_{t \to +\infty} \|y(t) - x(t)\| = 0$$

5 Topological characterization of the notion of reference stability

If x_0 is an equilibrium point of (3.1), then in [3] is defined: x_0 is called stable.iff for each neighborhood $V = V(x_0)$ there exists a neighborhood $U = U(x_0)$, $U \subseteq V$ and $U \subseteq G$ such that: for each solution y of (3.1), $y(t_0) = y_0 : y_0 \in U \Longrightarrow y$ is defined on $[t_0, +\infty)$ and $y([t_0, +\infty)) \subseteq V$.

In [8] the author considers only unique solutions and thus he can assign to each initial value $y(t_0)$ the solution y, $y(t_0) \to y$ and he assums that all y belong to the Banachspace $C_b([t_0, +\infty), \mathbb{R}^n)$ of all bounded continuous functions on $[t_0, +\infty)$ equipped with the sup-norm. Now he remarks that stability of a solution x is equivalent to the continuity of the map $y(t_0) \to y$ at the point $x(t_0)$. (See remark on page 137 of [8]). But the author has no precise domain of his map and the bounded continuous functions are not enough, since one wants for instance to consider instability too.

Best suited for topological characterization of stability is the notion of reference stability (see section 4).

Before we study such characterizations we will provide some facts from elementary general topology.

For topological spaces the notion of a neighborhood is important. Often a topology on a set is defined by open sets. But we also can start with neighborhoods. For a proof of the following propositions see [2], [9].

Proposition 5.1 *Let X be a set and for each $x \in X$ there exists a nonempty family $\underline{B}(x)$ of such subsets of X s. th. $B = (\underline{B}(x))_{x\in X}$ has the properties:*

(a) $B \in \underline{B}(x) \Longrightarrow x \in B$

(b) $B_1,\ B_2 \in \underline{B}(x)\,\exists\, B_3 \in \underline{B}(x) : B_3 \subseteq B_1 \cap B_2$

(c) $\forall\, V \in \underline{B}(x)\,\exists\, B \in \underline{B}(x)\,\forall\, y \in B\,\exists\, W \in \underline{B}(y) : W \subseteq V$

$G \subseteq X$ is called open iff

$$\forall\, x \in G\,\exists\, B \in \underline{B}(x) : B \subseteq G\,.$$

Then $\tau = \{G \subseteq X | G \text{ open}\}$ is a topology on X and τ is uniquely determined by the system $B = (\underline{B}(x))_{x\in X}$.

Moreover $\forall\, x \in X : \underline{B}(x)$ is a base of the τ-neighborhoodsystem $\underline{U}(x)$.

Hence we say that the base system B generates the topology τ.

Corollary 5.2 *Let be τ_1, τ_2 topologies on X which are generated by the base neighborhood systems $(\underline{B}^1(x))_{x\in X}$, $(\underline{B}^2(x))_{x\in X}$.*

If holds: $\forall\, x \in X\,\forall\, B_1 \in \underline{B}^1(x)\,\exists\, B_2 \in \underline{B}^2(x) : B_2 \subseteq B_1$ then we find: $\tau_1 \subseteq \tau_2$.

Proof: $\forall\, G \in \tau_1$, hence G is open w. r. t. τ_1 and we want to show that G is τ_2-open too: $\forall\, z \in G : G \in \tau_1 \Longrightarrow \exists B_1 \in \underline{B}^1(z) : z \in B_1 \subseteq G$; by assumption there exists $B_2 \in \underline{B}^2(z)$ s. th. $B_2(z) \subseteq B_1(z) \Longrightarrow z \in B_2 \subseteq G$ and hence G is open w. r. t, $\tau_2 : G \in \tau_2$.

Now we are looking for suitable topologies on $C([t_0, +\infty), \mathbb{R}^n)$; $[t_0, +\infty)$ (with Euclidian topology) is a locally compact Hausdorff space. Thus the compact-open topology for $C([t_0, +\infty), \mathbb{R}^u)$ has many open sets. But for applications to characterize stability we need "uniform topologies".

Remark 5.3 Algebraic operations in $C([t_0, +\infty), \mathbb{R}^n)$ and in $C([t_0, +\infty), \mathbb{R})$ we can define pointwise; we consider these spaces as vector spaces over $\mathbb{R}$.

Definition 5.4 *Let $M \subseteq C([t_0, +\infty), \mathbb{R})$, $M \neq 0$ and all functions from M are positive: $\forall\, (\alpha, t) \in M \times [t_0, +\infty) : \alpha(t) > 0$; now for $f \in C([t_0, +\infty), \mathbb{R}^u)$ we define α-neighborhoods of $f : B_\alpha(f) = \{g \in C([t_0, +\infty), \mathbb{R}^u) | \forall\, t \in [t_0, +\infty) : \|g(t) - f(t)\| < \alpha(t)\}$.*

Which properties M must have such that $B = (B_\alpha(f))_{(\alpha,f)\in M\times C([t_0,+\infty),\mathbb{R}^n)}$ is a base neighborhood system (see proposition 5.1).

Proposition 5.5 *We assume that holds:*

(1) $\alpha \in M \Longrightarrow \frac{1}{2}\alpha \in M$

(2) $\alpha, \beta \in M \Longrightarrow \min\{\alpha, \beta\} \in M$.

Then B is a base neighborhood system.

Proof: At first we remark that $\frac{1}{2}\alpha$, $\min\{\alpha, \beta\}$ are positive continuous functions. We will show that B fulfills the base neighborhood systems axioms (a), (b), (c) of proposition 5.1.

(a) $\forall (\alpha, f) : f \in B_\alpha(f)$, since $\forall t \in [t_0, +\infty)$:

$$\|(f(t) - f(t)\| = 0 < \alpha(t)$$

(b) $\forall f \in C([t_0, +\infty), \mathbb{R}^n)$
$\forall \alpha_1, \alpha_2 \in M$: let $\beta = \min\{\alpha_1, \alpha_2\}$, then $B_\beta(f) \subseteq B_{\alpha_1}(f) \cap B_{\alpha_2}(f)$, since

$$\forall t \geq t_0 : \min\{\alpha_1(t), \alpha_2(t)\} \leq \alpha_1(t),\ \min\{\alpha_1(t), \alpha_2(t)\} \leq \alpha_2(t)$$

(c) $\forall B_\alpha(f) \in (B_\beta(f))_{\beta\in M} : \alpha \in M \Longrightarrow \frac{1}{2}\alpha \in M \Longrightarrow B_{\frac{\alpha}{2}}(f) \in (B_\beta(f))_{\beta\in M}; \forall g \in B_{\frac{\alpha}{2}}(f)$: we will show that $B_{\frac{\alpha}{2}}(g) \subseteq B_{\frac{\alpha}{2}}(f)$ holds: $\forall (h, t) \in B_{\frac{\alpha}{2}}(g) \times [t_0, +\infty)$:

$$\begin{aligned}\|h(t) - f(t)\| &= \|h(t) - g(t) + g(t) - f(t)\| \leq \|h(t) - g(t)\| + \|g(t) - f(t)\| \\ &< \frac{1}{2}\alpha(t) + \frac{1}{2}\alpha(t) = \alpha(t)\,,\end{aligned}$$

hence $h \in B_\alpha(f)$.

Remark 5.6 If M fulfills (1), (2) then the α-base neighborhood system generates an unique topology $\tau = \tau_M$ for $C([t_0, +\infty), \mathbb{R}^u)$.

Lemma 5.7 *$M_1, M_2 \subseteq C([0, +\infty), \mathbb{R})$, M_1, M_2 generate the topologies τ_1, τ_2 respectively. Then holds:*

$$M_1 \subseteq M_2 \Longrightarrow \tau_1 \subseteq \tau_2$$

Proof: We show that the identity map id: $(C([t, +\infty), \mathbb{R}^u), \tau_2) \to (C([t_0, +\infty), \mathbb{R}^u), \tau_1)$ is continuous: let $G \in \tau_1$ be open $\Longrightarrow \mathrm{id}^{-1}(G) = G$; $G \in \tau_1 \Longrightarrow \forall h \in G\ \exists \alpha \in M_1 : B(h) = \{g \in C([t_0, +\infty), \mathbb{R}^n) | \forall t \geq t_0 : \|g(t) - h(t)\| < \alpha(t)\} \subseteq G$. But $\alpha \in M_1 \Longrightarrow \alpha \in M2$ and hence $G \in \tau_2$.

Definition 5.8 *Now we consider examples of the generating set $M \subseteq C([t_0, +\infty), \mathbb{R})$ and the corresponding topologies:*

1. *By ε we mean now the constant function:*

$$\varepsilon : \forall\, t \in [t_0, +\infty) : \varepsilon(t) = \varepsilon,\ \ M_\varepsilon = \{\varepsilon | \varepsilon > 0\}$$

   Of course:

$$\varepsilon > 0 \Longrightarrow \frac{1}{2}\varepsilon > 0;\ \ \varepsilon_1, \varepsilon_2 \in M_\varepsilon \Longrightarrow \min\{\varepsilon_1, \varepsilon_2\} \in M_\varepsilon$$

   As is well known, $B = (B_\varepsilon(f))_{(\varepsilon,f)\in M_\varepsilon \times C([t_0,+\infty),\mathbb{R}^n)}$ generates the uniform topology τ_u on $C([t_0, +\infty), \mathbb{R}^n)$

2. *We consider a subset of $M_\varepsilon : \forall\, n \in \mathbb{N},\ n \geq 1 : \varepsilon_n = \frac{1}{n}$: the constant functions now have the value $\frac{1}{n}$; $M_{(\frac{1}{n})} = \{\frac{1}{n} | n \in \mathbb{N}, n \geq 1\}$. $M_{(\frac{1}{n})}$ generates a topology:*

$$\frac{1}{n} \in M_{(\frac{1}{n})} \Longrightarrow \frac{1}{2}\frac{1}{n} = \frac{1}{2n} \in M_{(\frac{1}{n})},\quad \min\left\{\frac{1}{n}, \frac{1}{m}\right\} \in M_{(\frac{1}{n})}\,. \tag{5.1}$$

3. *M_c, the symbol c, means: converging to zero; $M_c = \{\alpha \in M | \lim_{t\to+\infty} \alpha(t) = 0\}$. We denote the topology generated by M_c on $C([t_0, +\infty), \mathbb{R}^n)$ by τ_{pc}: positive – converging topology. Clearly:*

$$\alpha \in M_c \Longrightarrow \frac{1}{2}\alpha \in M_c, \alpha_1, \alpha_2 \in M_c \Longrightarrow \forall\, t \geq t_0 : 0 < \min\{\alpha_1(t), \alpha_2(t)\} \leq \alpha_1(t)$$
$$(and\ \leq \alpha_2(t))$$

   and thus $\lim\limits_{t\to+\infty} \min\{\alpha_1(t), \alpha_2(t)\} = 0$ *showing* $\min\{\alpha_1, \alpha_2\} \in M_c$.

4. *$M_a = \{\alpha \in C([t_0, +\infty), \mathbb{R}) | \forall\, t \geq t_0 : \alpha(t) > 0\}$; thus a means "all". Of course:*

$$\alpha \in M_a \Longrightarrow \frac{1}{2}\alpha \in M_a, \alpha_1, \alpha_2 \in M_a \Longrightarrow \min\{\alpha_1, \alpha_2\} \in M_a\,.$$

The topology generated by M_a we denote by τ_m, since this topology was used by Marston Morse; τ_m first was defined by E. Hewitt, it is also called Whitney – or fine topology.

As we have hoped, we can show: $\tau_u = \tau_{(\frac{1}{n})}$.

Proposition 5.9 *On $C([t_0, +\infty), \mathbb{R}^n)$ holds $\tau_u = \tau_{(\frac{1}{n})}$.*

Proof: $M_{(\frac{1}{n})} \subseteq M_\varepsilon \Longrightarrow \tau_{(\frac{1}{n})} \subseteq \tau_u$ by lemma 5.7. By corollary 5.2 we find $\tau_u \subseteq \tau_{(\frac{1}{n})}$.

Corollary 5.10 *$(C([t_0, +\infty), \mathbb{R}^u)\tau_u)$ is a topological A_1-space. Hence we can use sequences instead of nets or filter.*

Proposition 5.11 *Moreover we have: $\tau_u \leq \tau_{pc}$.*

Proof: $\forall\,(f,\varepsilon) \in C([t_0,+\infty),\mathbb{R}^n) \times (0,+\infty) : B_\varepsilon(f) \in \tau_u$; let $h = \frac{\varepsilon}{2}e^{-t}$, $t \in [t_0,+\infty)$; since $0 \leq t_0$ we get for $0 \leq t_0 \leq t : e^{-t} \leq 1 \Longrightarrow \frac{\varepsilon}{2}e^{-t} \leq \frac{\varepsilon}{2} < \varepsilon$, thus showing that $B_h(f) \subseteq B_\varepsilon(f)$ holds and $B_h(f) \in \tau_{pc}$. Hence by corollary 5.2 $\tau_u \subseteq \tau_{pc}$.

Corollary 5.12 *For our topologies τ_u, τ_{pc}, τ_m holds:*

$$\tau_u \leq \tau_{pc} \leq \tau_m$$

Now we come to the main point of this section: stability as continuity.

As already remarked the basic idea of stability is nothing else then the continuity of a natural map into the space of continuous functions. Using reference stability we can define this map in a clear and exact way:

We consider the initial value problem (3.1). Let x be a solution which is defined on $[t_0,+\infty)$ and $R(x)$ be the set of reference solutions of x (definition 4.1).

Let $\tilde{R}(x) = R(x) \cup \{x\}$ and we assume that all solution of $\tilde{R}(x)$ are unique; moreover V_{t_0} (V means "value")$= \{y(t_0) | y \in \tilde{R}(x)\}$, $V_{t_0} \subseteq G \subseteq \mathbb{R}^n$ and for V_{t_0} we consider the Euclidian topology of $\mathbb{R}^n$, which can be generated by an arbitrary compatible norm of $\mathbb{R}^n$. Then the map F is well defined:

$$F : V_{t_0} \to C([t_0,+\infty),\mathbb{R}^n) : \forall\, y(t_0) \in V_{t_0} : F(y(t_0)) = y$$

$C([t_0,+\infty),\mathbb{R}^n)$ we provide with the uniform topology τ_u.

Remark 5.13 Since of course some $y \in \tilde{R}(x)$ may be unbounded we use $C([t_0,+\infty),\mathbb{R}^n)$ and not the space $C_b([t_0,+\infty),\mathbb{R}^n)$ of bounded continuous maps.

Now using the generation of τ_u by base ε-neighborhoods (see 5.8, 1.) and the characterization of the continuity of a map by (base) neighbourhoods it is not hard to prove the assertion of the following theorem:

Theorem 5.14 *Equivalent are:*

(1) *x is reference stable*

(2) *the map $F : V_{t_0} \to (C([t_0,+\infty),\mathbb{R}^n),\tau_u)$ is continuous in $x(t_0)$.*

Application of theorem 5.14 to concrete examples. We consider again Example 2.4: $\dot{x} = -x^3$, $t_0 = 0$, $x(0) = x_0$; on $[0,+\infty)$ are defined:

the zero solution 0 and the set of reference solutions

$$R(0) = \left\{ x = \frac{x_0}{\sqrt{1+2x_0^2 t}} | x_0 \in \mathbb{R}\backslash\{0\} \right\}, \text{ hence } \tilde{R}(0) = \left\{ x = \frac{x_0}{\sqrt{1+2x_0^2 t}} | x_0 \in \mathbb{R} \right\};$$

for $x_0 = 0$ we obtain the zero function:

$$\frac{0}{\sqrt{1}} = 0\,, \quad V_{t_0} = V_0 = \left\{x(0) = x_0 | x_0 \in \tilde{R}(0)\right\} = \mathbb{R}\,.$$

Of course the solutions of $R(0)$ are unique solutions and by proposition 2.1 $x \equiv 0$ is unique. Thus all elements of $\tilde{R}(0)$ are unique solutions and we can apply theorem 5.14:

Now we show the continuity of the map F: we can use convergence too and especially we can use sequences here:

let (x_0^n) be a sequence from V_0 s. th. $(x_0^u) \to 0(0) = 0$. We will show:

$$F(x_0^n) = (x(t;0,x_0))_n \to F(0(0)) = 0 \text{ uniformly on } [0,+\infty)\,:$$

$$\forall\, t \geq 0,\ x_0 \neq 0,\ 1 \leq 1 + 2x_0^2 t \Longrightarrow 1 \leq \sqrt{1+2x_0^2 t} \Longrightarrow 0 < \frac{1}{\sqrt{1+2x_0^2 t}} \leq 1$$
$$\Longrightarrow 0 < \frac{|x_0|}{\sqrt{1+2x_0^2 t}} \leq |x_0|\,.$$

Hence $|F(x_0^n) - 0| = |F(x_0^u)| \leq |x_0^n|$, but $x_0^u \to 0 \Longrightarrow |x_0^u| \to 0 \Longrightarrow F(x_0^u) \to 0$ uniformly on $[0,+\infty)$, since $|x_0^u|$ does not depend on t.

Thus from theorem 5.14 follows that the zero solution 0 is reference stable.

Remarks 5.15 1. Using the continuity – arguments we were able to avoid any epsilontics.

2. Let $\delta = 1 : \forall\, x_0 \in \mathbb{R}$, $x_0 \neq 0$, $|x_0 - 0(0)| = |x_0| < 1$ we get

$$\lim_{t\to+\infty} |x(t)| = \lim_{t\to+\infty} \frac{|x_0|}{\sqrt{1+2x_0^2 t}} = 0\,,$$

hence the zero solution 0 is even asymptotically reference stable.

Example 5.16 We consider the equation $\dot{x} = x^2$, $x(0) = x_0$.

For $x_0 \neq 0$ by (2.1) we find for the solutions:

$$x^{1-n} = x_0^{1-n} + (1-n)t$$

and

$$n = 2 \Longrightarrow x^{-1} = x_0^{-1} - t \Longrightarrow x = \frac{1}{x_0^{-1} - t} = \frac{x_0}{1 - x_0 t}\,;\ x_0 \neq 0\,:$$

Since we look for solutions which are defined at least on $[0,+\infty)$, we find here:

$$\frac{1}{x_0} \notin (0,+\infty) \Longleftrightarrow x_0 < 0 \Longleftrightarrow [0,+\infty) \subseteq \left(\frac{1}{x_0}, +\infty\right) \Longleftrightarrow [0,+\infty) \subseteq D(x(t;0,x_0))$$

and equivalently:

$$\frac{1}{x_0} \in (0,+\infty) \Longleftrightarrow x_0 > 0 \Longleftrightarrow [0,+\infty) \not\subseteq D(x(t;0,x_0))\,.$$

Now we study the stability of the zero solution $x \equiv 0$ on $[0,+\infty)$.

$\{x_0 \big| [0,+\infty) \not\subseteq D(x(t;0,x_0)) = \{x_0 | x_0 > 0\} = (0,+\infty)\}$. But $0 = 0(0)$ is clusterpoint of $(0,+\infty)$ yielding by corollary 3.4 that $x \equiv 0$ is unstable in the sense of Ljapunow.

Since we know that only for $x(0) = x_0 < 0$ holds: $[0,+\infty) \subseteq D(x(t;0,x_0))$ we get as set of all reference solution of $x \equiv 0$:

$$R(0) = \left\{x = \frac{x_0}{1-x_0t} \Big| x_0 < 0\right\} \Longrightarrow \tilde{R}(0) = \left\{x = \frac{x_0}{1-x_0t} \Big| x_0 \le 0\right\};$$
$$V_0 = \left\{x(0) \big| x \in \tilde{R}(0)\right\} = (-\infty, 0]\,.$$

We show that $x \equiv 0$ is reference stable: By the same arguments as above we find too: all elements of $\tilde{R}(0)$ are unique solutions.

Now:

$$x_0 < 0 \Longrightarrow -x_0 > 0 \Longrightarrow -tx_0 \ge 0,\ \text{since } t \ge 0;$$
$$-tx_0 \ge 0 \Longrightarrow 1 - tx_0 \ge 1 \Longrightarrow \frac{1}{1-x_0t} \le 1 \Longrightarrow \frac{|x_0|}{1-x_0t} \le |x_0|\,;$$

now let (x_0^n) be a sequence from $V_0\backslash\{0\}$, $x_0^n = (x(t;0,x_0))_n$;

$$|F(x_0^n) - 0(0)| = |F(x_0^n)| = \left|\frac{x_0^n}{1-x_0^n t}\right| \le |x_0^n|\,;$$
$$(x_0^n) \to 0 \Longrightarrow |x_0^n| \to 0 \Longrightarrow F(x_0^n) \to 0 = 0(0) = F(0) : (F(x_0^n))_n$$

converges uniformly on $[0,+\infty)$ to $F(0)$, yielding that F is continuous in $0(0) = 0$. Hence by theorem 5.14 $0 = x \equiv 0$ is reference stable.

$$\forall\, x_0 < 0 : |x_0| < 1 \Longrightarrow \lim_{t\to+\infty} |x(t;0,x_0)| = \lim_{t\to+\infty} \frac{|x_0|}{1-x_0t} = 0\,,$$

meaning that 0 is asymptotically reference stable.

We need a simple lemma.

Lemma 5.17 *Let be (h_n) a sequence from $C([0,+\infty),\mathbb{R}^n)$ and let $h \in ([0,+\infty),\mathbb{R}^n)$ be bounded: $\forall\, t \in [0,+\infty)$ $\|h(t)\| \le a$, $a \in \mathbb{R}$, $a > 0$. If (h_n) converges uniformly to h on $[0,+\infty)$ then almost all members of the sequence (h_n) are bounded too.*

Proof: $h_n \xrightarrow{\tau_u} h \Longrightarrow \exists\, n_1 \in \mathbb{N} : \forall\, (t,n) \in [0,+\infty) \times \{n \in \mathbb{N} | n \geq n_1\}$: $\|h_n(t) - h(t)\| < 1$; now

$$\|h_n(t)\| - \|h(t)\| \leq \|h_n(t) - h(t)\| \Longrightarrow \|h_n(t)\| \leq \|h_n(t) - h(t)\| + \|h(t)\| < 1 + a\,,$$

hence h_n is bounded $\forall\, n \geq n_1$.

We will apply this lemma and consider example 4.2: $\dot{x} = x$, $x(0) = x_0$; $x \equiv 0$ on $[0,+\infty)$ is solution: $0(0) = 0$.

$$R(0) = \{x_0 e^t | x_0 \in \mathbb{R} \backslash \{0\}\} \Longrightarrow \tilde{R}(0) = \{x_0 e^t | x_0 \in \mathbb{R}\}, V_0 = \{x(0) = x_0 | x \in \tilde{R}(0)\}\,.$$

$x \equiv 0$ is not reference stable.

Proof: We consider the sequence $(x_0^n) = \left(\frac{1}{n}\right)$ from V_0; $\frac{1}{n} \to 0$ but all $F\left(\frac{1}{n}\right) = x\left(t; 0, \frac{1}{n}\right) = \frac{1}{n} e^t$ are unbounded and hence by the lemma 5.17 $F\left(\frac{1}{n}\right)$ does not converges uniformly to 0.

Thus by theorem 5.14 $x \equiv 0$ is not reference stable.

We still consider the positive – converging topology τ_{pc}.

Proposition 5.18 *Under the assumptions of theorem 5.14 holds:*

If the solution x is τ_{pc}-stable then x is asymptotically reference stable.

Proof: We consider the map

$$F : V_{t_0} \to C([t_0,+\infty),\mathbb{R}^n),\ \ x(t_0) \in V_{t_0} \Longrightarrow F(x(t_0)) = x \in C([t_0,+\infty),\mathbb{R}^n)$$

and x is τ_{pc}-stable means that

$$F : V_{t_0} \to (C([t_0,+\infty),\mathbb{R}^n),\tau_{pc})$$

is continuous in $x(t_0)$; 5.11 shows that $\tau_u \subseteq \tau_{pc}$, yielding that $F : V_{t_0} \to (C([t_0,+\infty),\mathbb{R}^n),\tau_u)$ is continuous in $x(t_0)$ too. Hence by theorem 5.14 x is R-stable.

Let $\alpha \in M_c$: $\forall\, t \in [t_0,+\infty)$: $\alpha(t) > 0$ and $\lim_{t\to\infty} \alpha(t) = 0$. By the τ_{pc}-continuity of F in $x(t_0)$ we find $\delta > 0$, $\delta = \delta(\alpha) : \forall\, y(t_0) \in U_\delta(x(t_0)) \cap V_{t_0} \Longrightarrow F(y(t_0)) = y \in U_\alpha(x(t_0)) \Longrightarrow \forall\, t \geq t_0$:

$$\|y(t) - x(t)\| < \alpha(t),\ \ \alpha(t) \to 0 \Longrightarrow \|y(t) - x(t)\| \to 0$$

for $t \to +\infty$, showing that x is asymptotically R-stable, since we finally have: $\forall\, y \in R(x)$: $\|y(t_0) - x(t_0)\| < \delta \Longrightarrow \|y(t) - x(t)\| \to 0$ for $t \to +\infty$.

Remark 5.19 If we want to define the basics of reference stability by means of topologies for the function space $C([0,+\infty),\mathbb{R}^n)$, then we can proceed:

1. reference stability by the uniform topology τ_u
2. asymptotic reference stability by τ_{pc}.

References

[1] **Barreira, L.**, and **Valls, C. :** *Ordinary Differential Equations.* Amer. Math. Soc., vol. 137, 2012

[2] **Engelking, R. :** *General Topology.* Heldermann Verlag Berlin, 1989

[3] **Hirsch, M. W., Smale, S.**, and **Devaney, R. L. :** *Differential Equations, Dynamical Systems, and an Introduction to Chaos.* Sec. Ed., Academic Press, 2004

[4] **Kabelka, B. :** *Gewöhnliche Differentialgleichungen.* http://fsmath.at/~bkabelka/math/analysis/diff-gl/16.htm

[5] **Krasnov, M. L., Kislyov, A. I.**, and **Makarenko, G. I. :** *A Book of Problems in Ordinary Differential Equations.* Mir Publishers Moscow, 1981

[6] **Rouche, N., Habets, P.**, and **Laloy, M. :** *Stability Theory by Ljapunows Direct Method.* Springer Verlag, 1973

[7] **Verhulst, F. :** *Nonlinear Differential Equation and Dynamical Systems.* Springer Verlag, 1996

[8] **Vidyasagar, M. :** *Nonlinear Systems Analysis.* Sec. Edit., Prentice Hall, 1993

[9] **Willard, S. :** *General Topology.* Addison-Wesley Publ. Comp., 2004

received: April 28, 2017

Author:

Harry Poppe
Universität Rostock
Institut für Mathematik
e-mail: harry.poppe@uni-rostock.de