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Manfred Krüppel

Two-Scale Difference Equations with a Parameter and Power Sums related to Digital Sequences

ABSTRACT. This paper is a direct continuation of [19] concerning the representation of power sums related to digital sequences. Foundation is beside article [19] an existence theorem for differentiable solutions of certain two-scale difference equations with a parameter. By means of such solutions and a method developed in [19] we are able to give an explicit representation for general sums related to digital sequences. In particular, we give a summation formula for power sums of the sum of digits and incidentally, we find a new property of the Bernoulli polynomials.

KEY WORDS. Two-scale difference equations with a parameter, power sums of digital sums, Bernoulli polynomials, generating functions

1 Introduction

We consider a two-scale difference equation with a parameter $x \in X \subseteq \mathbb{R}$ of the form

$$\varphi\left(\frac{t}{p}, x\right) = \sum_{r=0}^{p-1} c_r(x)\varphi(t-r, x) \qquad (t \in \mathbb{R}) \tag{1.1}$$

with an integer $p > 1$ and real or complex coefficients $c_r(x)$ where $c_0(x)c_{p-1}(x) \neq 0$ and

$$\sum_{r=0}^{p-1} c_r(x) = 1 \qquad (x \in X). \tag{1.2}$$

It is known that for such $x \in X$ where $|c_r(x)| < 1$ for $r = 0, 1, \dots, p-1$ the equation (1.1) has a solution $\varphi(t, x)$ satisfying

$$\varphi(t, x) = 0 \quad \text{for} \quad t < 0, \qquad \varphi(t, x) = 1 \quad \text{for} \quad t > 1 \tag{1.3}$$

which is continuous with respect to t, cf. [18], see also [7], [11]. We show that if coefficients $c_r(x)$ are k-times differentiable then the solution $\varphi(t, x)$ is k-times differentiable with respect to x (Theorem 2.1, Theorem 2.2). This result is the base for the derivation of a formula for the sum of certain digital sequences where we use similar methods as in [19].

In particular, we investigate digital exponential sums (Section 3) and the digital power sums

$$S_k(N) := \sum_{n=0}^{N-1} s(n)^k \tag{1.4}$$

where $s(n)$ denotes the sum of digits of the integer n in the p-adic representation, i.e. if $n = \sum n_k p^k$ then $s(n) = \sum n_k$. In binary case $p = 2$ first Trollope [26] in 1968 has given an explicit expression for the sum $S_1(N)$. Delange [5] gave a simple proof and generalized the result to digits in arbitrary basis $p > 1$. The well-known Trollope-Delange formula reads

$$S_1(N) = \frac{1}{2} N \log_2 N + NG(\log_2 N) \tag{1.5}$$

where $G(u)$ is an 1-periodic continuous, nowhere differentiable function which can be expressed by means of the Takagi function. In 1994 the Trollope-Delange formula (1.5) was also proved in [6] by use of classical tools from analytic number theory, namely the Mellin-Perron formulae, see [6, Theorem 3.1, Remark 4.5].

For the basis $p = 2$ Coquet [3] in 1986 proved that

$$\frac{1}{N} S_2(N) = \left(\frac{\log_2 N}{2}\right)^2 + \log_2 N \; G_{2,1}(\log_2 N) + G_{2,0}(\log_2 N)$$

where $G_{2,1}(u)$, $G_{2,0}(u)$ are 1-periodic continuous functions and that for arbitrary integer $k > 1$ the power sum $S_k(N)$ can be represented in the form

$$\frac{1}{N} S_k(N) = \sum_{\ell=0}^{k} (\log_2 N)^\ell G_{k,\ell}(\log_2 N) \tag{1.6}$$

where $G_{k,\ell}(u)$ are 1-periodic functions, in particular $G_{k,k}(u) = \frac{1}{2^k}$. He also found certain recurrence relations between the functions $G_{k,\ell}$. In [22], by means of binomial measures a more explicit representation of $G_{k,\ell}$ was given and their continuity was proved, cf. also [23] and [15]. In [17] it was proved that the functions $G_{k,\ell}$ ($\ell = 0, 1, \ldots, k-1$) are nowhere differentiable. In 2012 Girgensohn [8] gives a new representation for $S_k(N)$ by use of functional equations and generating function techniques. If $q_k(t)$ is a sequence of polynomials given by $q_0(t) = 1$ and the recursion

$$q_{k+1}(t) = t(2q_k(t) - q_k(t-1)) \qquad (k \geq 0) \tag{1.7}$$

then it holds

$$S_k(N) = \sum_{\ell=0}^{k} \binom{k}{\ell} N 2^{-\ell} q_\ell(\log_2 N) f_{k-\ell}(x) \tag{1.8}$$

with certain 1-periodic continuous functions $f_k(x)$ and $x = \frac{N-p(N)}{p(N)}$ where $p(N) = 2^{[\log_2 N]}$ is the largest power of 2 less than or equal to N, cf. [8, Section 5].

For an arbitrary integer basis $p > 1$ in 2000 Muramoto et al. [20] have proved by means of multinomial measures that

$$\frac{1}{N} S_k(N) = \sum_{\ell=0}^{k} (\log_p N)^\ell H_{k,\ell}(\log_p N) \tag{1.9}$$

where $H_{k,\ell}(u)$ are 1-periodic continuous functions. In case $N = p^n$ it follows that $\frac{1}{p^n} S_k(p^n)$ can be represented as polynomial

$$\frac{1}{p^n} S_k(p^n) = \sum_{\ell=0}^{k} a_{k,\ell}\, n^\ell \tag{1.10}$$

with the coefficients $a_{k,\ell} = H_{k,\ell}(n) = H_{k,\ell}(0)$ since the functions $H_{k,\ell}(u)$ are 1-periodic. Certainly, the coefficients $a_{k,\ell}$ and also the polynomial

$$P_k(t) = \sum_{\ell=0}^{k} a_{k,\ell}\, t^\ell \tag{1.11}$$

depend on p. So equation (1.10) can be written in the form

$$\frac{1}{p^n} S_k(p^n) = P_k(n). \tag{1.12}$$

For the basis $p = 10$ the polynomials $P_k(n)$ were computed in [12] for $n = 1, 2, \ldots, 8$:

$$\begin{aligned}
\frac{1}{10^n} S_1(10^n) &= \frac{9}{2} n \\
\frac{1}{10^n} S_2(10^n) &= \frac{81}{4} n^2 + \frac{33}{4} n \\
\frac{1}{10^n} S_3(10^n) &= \frac{729}{8} n^3 + \frac{891}{8} n^2 \\
\frac{1}{10^n} S_4(10^n) &= \frac{6561}{16} n^4 + \frac{8019}{8} n^3 + \frac{3267}{16} n^2 - \frac{3333}{40} n \\
\frac{1}{10^n} S_5(10^n) &= \frac{59049}{32} n^5 + \frac{120285}{16} n^4 + \frac{147015}{32} n^3 - \frac{29997}{16} n^2 \\
\frac{1}{10^n} S_6(10^n) &= \frac{531441}{64} n^6 + \frac{3247695}{64} n^5 + \frac{3969405}{64} n^4 - \frac{1080783}{64} n^3 - \frac{329967}{32} n^2 + \frac{15873}{4} n \\
\frac{1}{10^n} S_7(10^n) &= \frac{4782969}{128} n^7 + \frac{40920957}{128} n^6 + \frac{83357505}{128} n^5 - \frac{56133}{128} n^4 - \frac{20787921}{64} n^3 + \frac{99999}{8} n^2 \\
\frac{1}{10^n} S_8(10^n) &= \frac{43046721}{256} n^8 + \frac{122762871}{64} n^7 + \frac{750217545}{128} n^6 + \frac{76284747}{32} n^5 - \frac{1372208607}{256} n^4 \\
&\quad + \frac{677777479}{64} n^3 + \frac{371092563}{320} n^2 - \frac{33333333}{80} n
\end{aligned}$$

Figure 1. The first polynomials $P_k(n)$ for the basis $p = 10$

These results were obtained as follows. If $f_k(x)$ is a sequence of functions defined by

$$f_0(x) = (1 + x + x^2 + \cdots + x^9)^n$$

and for $k \geq 1$ by

$$f_k(x) = x f'_{k-1}(x)$$

then it holds

$$S_k(10^n) = f_k(n)$$

cf. [4, Section 3]. So starting with $f_0(x)$ the polynomials in Figure 1 were computed by repeated differentiation, multiplication by x, and finally substitution $x = 1$, cf. [4, p. 342]. We see in Figure 1 that for odd $k > 1$ the linear term of $P_k(n)$ vanishes.

In this paper we give a new derivation of (1.9) as application of two-scale difference equations with a parameter (Theorem 4.3, Corollary 4.4). The main result can be written as

$$\sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{N} S_k(N) z^k = \left(\sum_{k=0}^{\infty} \frac{1}{k!} P_k(L) z^k \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} F_k(L) z^k \right) \qquad (z \in \mathbb{C}) \tag{1.13}$$

where $L = \log_p N$, where $P_k(t)$ are polynomials (1.11) satisfying (1.12), and where $F_0(u) = 1$ and $F_k(u)$ are 1-periodic continuous functions with $F_k(0) = 0$ for $k \geq 1$ (Theorem 6.11). In view of the Cauchy product relation (1.13) means that for $k \geq 0$ we have

$$\frac{1}{N} S_k(N) = \sum_{\ell=0}^{k} \binom{k}{\ell} P_\ell(L) F_{k-\ell}(L).$$

The polynomials $P_k(t)$ are given by their generating function

$$\sum_{k=0}^{\infty} \frac{1}{k!} P_k(t) z^k = \left(\frac{e^{pz} - 1}{p(e^z - 1)} \right)^t \qquad (z \in \mathbb{C}) \tag{1.14}$$

(Proposition 6.2), and the functions $F_k(u)$ are determined by the equation (1.1) with the coefficients

$$c_r(x) = \frac{e^{rx}}{1 + e^x + \cdots + e^{(p-1)x}} \qquad (r = 0, 1, \ldots, p-1) \tag{1.15}$$

in the following way: if equation (1.1) with (1.15) has the solution $\varphi(t, x)$ satisfying (1.3) and if $F(u, x)$ denotes the 1-periodic function with respect to u given by

$$F(u, x) := \frac{\varphi(p^u, x)}{(1 + e^x + \cdots + e^{(p-1)x})^u} \qquad (u \leq 0)$$

then

$$F_k(u) = \frac{\partial^k}{\partial x^k} F(u, x) \bigg|_{x=0}.$$

Moreover, the coefficients $a_{k,\ell} = a_{k,\ell}(p)$ of $P_k(t)$ are polynomials of degree at most k in p which are given by

$$a_{k,\ell}(p) = \frac{(-1)^k k!}{\ell!} \sum_{k_1+\cdots+k_\ell=k} \frac{k!}{k_1!\cdots k_\ell!} \left\{ \prod_{n=1}^{\ell} \frac{B_{k_n}}{k_n \cdot k_n!}(p^{k_n}-1) \right\} \tag{1.16}$$

where $k_1, \ldots, k_\ell$ are positive integers and where B_k denotes the Bernoulli numbers. In particular $a_{k,k}(p) = (\frac{p-1}{2})^k$ for $k \geq 0$, $a_{k,0}(p) = 0$ for $k \geq 1$ and

$$a_{k,1}(p) = \frac{(-1)^k B_k}{k}(p^k - 1) \qquad (k \geq 1)$$

(Proposition 5.4). Hence, for odd $k > 1$ we have $a_{k,1}(p) = 0$ which is the reason that for these k the linear term of $P_k(n)$ vanishes, cf. Figure 1.

In this paper several times we need the Bernoulli polynomials $B_n(t)$ which are given by their generating function

$$\frac{ze^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} z^n \qquad (|z| < 2\pi) \tag{1.17}$$

and which have the form

$$B_n(t) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_\nu t^{n-\nu} \tag{1.18}$$

where $B_n = B_n(0)$ are the Bernoulli numbers. They satisfy the difference equations

$$B_n(t+1) - B_n(t) = nt^{n-1} \tag{1.19}$$

which imply the sum formula

$$\sum_{i=0}^{N-1} i^n = \tilde{B}_n(N) \tag{1.20}$$

with the modified Bernoulli polynomials

$$\tilde{B}_n(t) = \frac{1}{n+1}\{B_{n+1}(t) - B_{n+1}\} \tag{1.21}$$

of degree $n+1$ which have the generating function

$$\sum_{n=0}^{\infty} \frac{\tilde{B}_n(t)}{n!} z^n = \frac{e^{tz} - 1}{e^z - 1} \qquad (|z| < 2\pi). \tag{1.22}$$

Finally, let us mention a new property of the Bernoulli polynomials. We show that the polynomial $\frac{1}{p}\tilde{B}_k(p) - (\frac{p-1}{2})^k$ is divisible by $p+1$. For more details cf. Remark 5.8.

2 Functional equations with a parameter

As in the Introduction mentioned we consider the two-scale difference equation (1.1) with a parameter $x \in X \subseteq \mathbb{R}$ and coefficients $c_r(x)$ satisfying $c_0(x)c_{p-1}(x) \neq 0$ and

$$c_0(x) + \cdots + c_{p-1}(x) = 1 \qquad (x \in X). \tag{2.1}$$

We investigate solutions $\varphi(t,x) : \mathbb{R} \times X \mapsto \mathbb{R}$ satisfying the conditions

$$\varphi(t,x) = 0 \quad \text{for} \quad t < 0, \qquad \varphi(t,x) = 1 \quad \text{for} \quad t > 1 \tag{2.2}$$

and all $x \in X$. It is easy to see that if $\varphi(t,x)$ is a solution of the following system of equations

$$\varphi\left(\frac{r+t}{p}, x\right) = c_r(x)\varphi(t,x) + g_r(x) \qquad (t \in [0,1], \quad x \in X) \tag{2.3}$$

with

$$g_r(x) = \sum_{k=0}^{r-1} c_k(x) \tag{2.4}$$

so that $g_0(x) = 0$ and $g_p(x) = 1$ for all $x \in X$ then the function

$$\varphi_0(t,x) := \begin{cases} 0 & \text{for} \quad t < 0 \\ \varphi(t,x) & \text{for} \quad 0 \leq t \leq 1 \\ 1 & \text{for} \quad t > 1 \end{cases} \tag{2.5}$$

is a solution of (1.1). We are interested to solutions of (1.1) which are continuous with respect to t and differentiable with respect to x. The set of all such functions we denote by $\mathbf{D}$. If $\varphi(t,x)$ belongs to $\mathbf{D}$ then it follows by differentiation of (1.1) with respect to x that the partial derivative $\varphi_x(t,x) = \frac{\partial}{\partial x}\varphi(t,x)$ satisfies

$$\varphi_x\left(\frac{t}{p}, x\right) = \sum_{r=0}^{p-1} c_r(x)\varphi_x(t-r,x) + \Psi_1(t,x) \qquad (t \in \mathbb{R}, x \in X) \tag{2.6}$$

where

$$\Psi_1(t,x) = \sum_{r=0}^{p-1} c_r'(x)\varphi(t-r,x) \qquad (t \in \mathbb{R}, x \in X) \tag{2.7}$$

and by differentiation of (2.3) we get for $r \in \{0,1,\ldots,p-1\}$ the equations:

$$\varphi_x\left(\frac{r+t}{p}, x\right) = c_r(x)\varphi_x(t,x) + \psi_r(t,x) \qquad (t \in [0,1],\ x \in X) \tag{2.8}$$

where

$$\psi_r(t,x) = c_r'(x)\varphi(t,x) + g_r'(x). \tag{2.9}$$

Theorem 2.1 *Assume that $c_0(x), c_1(x), \ldots, c_{p-1}(x)$ $(x \in X)$ are differentiable functions with bounded derivatives and that*

$$L := \sup\{|c_0(x)|, \ldots, |c_{p-1}(x)| : x \in X\} < 1. \tag{2.10}$$

Then there exists exactly one solution $\varphi(t,x) \in \mathbf{D}$ of (1.1) satisfying (1.3). Moreover, there exists the partial derivative $\varphi_x(x,t)$ which satisfies (2.8) and which is continuous with respect to t.

Proof: First we determine the solution $\varphi(t,x)$ for $(t,x) \in [0,1] \times X$. For this we put $L' := \sup\{|c_0'(x)|, \ldots, |c_{p-1}'(x)| : x \in X\}$ and choose $\varepsilon > 0$ so small that $K := L + \varepsilon L' < 1$. Note that $\mathbf{D}$ is a Banach space with the norm

$$\|u\|_{\mathbf{D}} := \|u\|_\infty + \varepsilon \left\| \frac{\partial u}{\partial x} \right\|_\infty$$

where $\|u\|_\infty = \sup\{|u(x,t)| : (x,t) \in [0,1] \times X\}$, and that

$$\Omega := \{u \in \mathbf{D} : \ u(0,x) = 0, \ u(1,x) = 1, \ \forall x \in X\}$$

is a closed subset of $\mathbf{D}$. For $u \in \Omega$ we define an operator T for all $x \in X$ by $(Tu)(0,x) := 0$ and

$$(Tu)(t,x) := c_r(x)u(pt-r,x) + g_r(x) \qquad \text{for} \quad t \in \left(\frac{r}{p}, \frac{r+1}{p}\right]$$

where $r = 0, 1, \ldots, p-1$ so that in view of $u(1,x) = 1$ for all $x \in X$ and (2.4)

$$(Tu)\left(\frac{r+1}{p}, x\right) = c_r(x)u(1,x) + g_r(x) = c_r(x) + g_r(x) = g_{r+1}(x).$$

We show that T maps Ω into itself. At first we have $(Tu)(0,x) = 0$ and $(Tu)(1,x) = g_p(x) = 1$ by (2.4) and (2.1). Next, $(Tu)(t,x)$ is continuous with respect to t. This is clear at each point (t,x) with $t \in (\frac{r}{p}, \frac{r+1}{p})$ $(r = 0, \ldots, p-1)$ and left-hand continuous at $t = \frac{r+1}{p}$. But it is also right-hand continuous at $t = \frac{r+1}{p}$ with $r = 0, 1, \ldots, p-2$ since for $0 < h < 1$ we have

$$(Tu)\left(\frac{r+1+h}{p}, x\right) = c_{r+1}(x)u(1+h,x) + g_{r+1}(x)$$

which converges to $c_{r+1}(x)u(1,x) + g_{r+1}(x) = g_{r+2}(x) = (Tu)(\frac{r+1}{p}, x)$ as $h \to 0$.

Moreover, for $x \in X$ the partial derivative $\frac{\partial}{\partial x} Tu$ exists and is given for $(0,x)$ by $\frac{\partial}{\partial x}(Tu)(0,x) = 0$ and for (t,x) with $t \in (\frac{r}{p}, \frac{r+1}{p}]$ by

$$\frac{\partial}{\partial x}(Tu)(x,t) = c_r'(x)u(pt-r,x) + c_r(x)\frac{\partial}{\partial x}u(pt-r,x) + g_r'(x).$$

Hence, indeed T maps Ω into Ω. For $u_1, u_2 \in \Omega$ it holds

$$\|Tu_1 - Tu_2\|_\infty \le L\|u_1 - u_2\|_\infty$$

and

$$\left\|\frac{\partial}{\partial x}Tu_1 - \frac{\partial}{\partial x}Tu_2\right\|_\infty \le L'\|u_1 - u_2\|_\infty + L\left\|\frac{\partial}{\partial x}u_1 - \frac{\partial}{\partial x}u_2\right\|_\infty .$$

Therefore, and in view of $K = L + \varepsilon L'$ we have

$$\begin{aligned}
\|Tu_1 - Tu_2\|_{\mathbf{D}} &= \|Tu_1 - Tu_2\|_\infty + \varepsilon\left\|\frac{\partial}{\partial x}Tu_1 - \frac{\partial}{\partial x}Tu_2\right\|_\infty \\
&\le L\|u_1 - u_2\|_\infty + \varepsilon L'\|u_1 - u_2\|_\infty + \varepsilon L\left\|\frac{\partial}{\partial x}u_1 - \frac{\partial}{\partial x}u_2\right\|_\infty \\
&\le K\|u_1 - u_2\|_{\mathbf{D}}
\end{aligned}$$

By Banach's fixed point theorem there exists exactly one fixed point, i.e. (1.1) has exactly one solution $\varphi(t,x) \in \mathbf{D}$ which is defined for $(t,x) \in [0,1] \times X$ with $\varphi(0,x) = 0$ and $\varphi(1,x) = 1$ for all $x \in X$. Hence, if we continue $\varphi(t,x)$ by $\varphi(t,x) = 0$ for $t < 0$ and $\varphi(t,x) = 1$ for $t > 1$ then we get a solution $\varphi(t,x) : \mathbb{R} \mapsto \mathbb{R} \times X$ of (1.1) which is continuous with respect to t.

Next we show that the partial derivative $\varphi_x(t,x)$ is continuous with respect to $t \in [0,1]$. Differentiation of (2.3) with respect to x yields that $\varphi_x(t,x)$ satisfies the equations (2.8) where the functions (2.9) are continuous with respect to t. It follows by the result of Girgensohn [7, Theorem 1] that for each $x \in X$ the function $\varphi_x(t,x)$ is continuous with respect to $t \in [0,1]$.

It remains to show that $\varphi_x(0,x) = \varphi_x(1,x) = 0$ for all $x \in X$. According to (2.8) and (2.9), both with $t = 0$, we have

$$\varphi_x(0,x) = c_0(x)\varphi_x(0,x) + \psi_0(0,x)$$

where $\psi_0(0,x) = g_0'(x) = 0$, see (2.4). In view of $c_0(x) \ne 0$ and $c_0(x) \ne 1$, see (2.10), it follows $\varphi_x(0,x) = 0$. Moreover, (2.8) and (2.9) imply for $r = p-1$ and $t = 1$

$$\varphi_x(1,x) = c_{p-1}(x)\varphi_x(1,x) + \psi_{p-1}(x)$$

where

$$\begin{aligned}
\psi_{p-1}(x) &= c_{p-1}'(x)\varphi(1,x) + g_{p-1}'(x) \\
&= c_{p-1}'(x) + g_{p-1}'(x) \\
&= g_p'(x)
\end{aligned}$$

owing to (2.4) with $r = p$. But $g_p(x) = c_0(x) + \cdots + c_{p-1}(x) = 1$ for all $x \in X$ according to (2.1) so that $\psi_{p-1}(x) = g_p'(x) = 0$. It follows $\varphi_x(1,x) = c_{p-1}(x)\varphi_x(1,x)$ and hence $\varphi_x(1,x) = 0$ since $c_{p-1}(x) \ne 1$, see (2.10). □

Theorem 2.2 *Under the suppositions of Theorem 2.1 it holds that if the functions $c_0(x), \dots, c_{p-1}(x)$ are k-times differentiable then the solution $\varphi(t,x)$ of (1.1) is also k-times differentiable with respect to x and the k-th partial derivative*

$$\varphi_x^{(k)}(t,x) := \frac{\partial^k}{\partial x^k}\varphi(t,x) \tag{2.11}$$

is continuous with respect to t. For $k \geq 1$ we have $\varphi^{(k)}(t,x) = 0$ for $t \notin (0,1)$ and all $x \in X$, and

$$\varphi_x^{(k)}\left(\frac{t}{p}, x\right) = \sum_{r=0}^{p-1} c_r(x)\varphi_x^{(k)}(t-r,x) + \Psi_k(t,x) \qquad (t \in \mathbb{R}, x \in X) \tag{2.12}$$

where $\Psi_k(t,x)$ is recursively given by (2.7) and

$$\Psi_k(t,x) = \sum_{r=0}^{p-1} c_r'(x)\varphi_x^{(k-1)}(t-r,x) + \frac{\partial}{\partial x}\Psi_{k-1}(t,x). \tag{2.13}$$

Proof: The first part is a consequence of Theorem 2.1. Starting with (2.6) equation (2.12) with (2.13) can be proved by induction. If (2.12) with (2.13) is true for $k-1$ then by the product rule for the differentiation we get

$$\varphi_x^{(k)}\left(\frac{t}{p}, x\right) = \sum_{r=0}^{p-1} c_r(x)\varphi_x^{(k)}(t-r,x) + \sum_{r=0}^{p-1} c_r'(x)\varphi_x^{(k-1)}(t-r,x) + \frac{\partial}{\partial x}\Psi_{k-1}(t,x).$$

So the proof is complete. □

Next we use so-called Knopp function of the form

$$H(t) = \sum_{j=0}^{\infty} \frac{h(p^j t)}{p^j} \qquad (t \in \mathbb{R}) \tag{2.14}$$

with the generating, 1-periodic function $h(t)$ with $h(0) = 0$, cf. [11] or [16]. Obviously, (2.14) implies

$$H\left(\frac{t}{p}\right) = h\left(\frac{t}{p}\right) + \frac{1}{p}H(t) \qquad (t \in \mathbb{R}) \tag{2.15}$$

Conversely, if $H(.)$ satisfies (2.15) then $H(.)$ has the form (2.14).

Proposition 2.3 *Under the suppositions of Theorem 2.2 it holds that in case $c_r(0) = \frac{1}{p}$ for $r = 0, 1, \dots, p-1$ we have for $\varphi_x^{(k)}(t,0)$ with $k \geq 1$ the representation*

$$\varphi_x^{(k)}(t,0) = \sum_{j=0}^{\infty} \frac{1}{p^j} h_k(p^j t) \qquad (0 \leq t \leq 1)$$

where h_k is 1-periodic continuous function given by $h_k(t) = \Psi_k(t,0)$ for $0 \leq t < 1$.

Proof: Let $f(t)$ be a function of period 1 given by $f(t) = \varphi^{(k)}(t, 0)$ for $0 \le t \le 1$. The relation $\varphi^{(k)}(t, x) = 0$ for $k \ge 1$, all $t \notin (0, 1)$ and $x \in X$ implies $f(0) = f(1) = 1$. In view of $c_r(0) = \frac{1}{p}$ equation (2.12) for $x = 0$ and with $r + t$ in place of t yields

$$f\left(\frac{r+t}{p}\right) = \frac{1}{p} f(t) + \Psi_k\left(\frac{r+t}{p}, 0\right) \qquad (0 \le r \le p-1, 0 \le t \le 1)$$

and $f(0) = f(1) = 0$ implies $\Psi_k(0, 0) = \Psi(1, 0) = 0$, i.e. $h_k(0) = h_k(1) = 0$. It follows that

$$f(t) = \sum_{j=0}^{\infty} \frac{1}{p^j} h_k(p^j t) \qquad (t \in \mathbb{R}).$$

So the theorem is proved. □

3 Digital exponential sums

For integer $N \ge 1$ we investigate the digital exponential sum

$$S(N, x) := \sum_{n=0}^{N-1} e^{xs(n)} \qquad (x \in \mathbb{R}) \tag{3.1}$$

where $s(n)$ denotes the sum of digits of the integer n in the p-adic representation of n. For this we begin with a results of [19] concerning a formula for the sum

$$S(N) := \sum_{n=0}^{N-1} C_n \tag{3.2}$$

where C_n is an arbitrary sequence which is given by the p initial values $C_0 = 1, C_1, \dots, C_{p-1}$ such that

$$C := C_0 + \dots + C_{p-1} > 0 \tag{3.3}$$

and which satisfies the recurrence formula

$$C_{kp+r} = C_k C_r \qquad (k \in \mathbb{N}, r = 0, 1, \dots, p-1). \tag{3.4}$$

If the conditions (3.3) and (3.4) are fulfilled then the two-scale difference equation

$$\varphi\left(\frac{t}{p}\right) = \frac{1}{C} \sum_{r=0}^{p-1} C_r \varphi(t - r)$$

with C from (3.3) has in case $|C_r| < C$ a continuous solution $\varphi = \varphi_0$ satisfying $\varphi_0(t) = 0$ for $t < 0$ and $\varphi_0(t) = 1$ for $t > 1$, cf. [19], and it holds

Proposition 3.1 ([19]) *For $n \in \mathbb{N}$ the sum (3.2) can be represented as*

$$S(N) = N^{\alpha} F(\log_p N)$$

with $\alpha = \log_p C$ and an 1- periodic continuous function F which is given by

$$F(u) = \frac{\varphi_0(p^u)}{p^{\alpha u}} \qquad (u \leq 0).$$

Certainly, Proposition 3.1 is also valid if we consider a sequence $C_n = C_n(x)$ depending on a parameter x provided that the above conditions are fulfilled. In particular, for the sequence

$$C_n(x) := e^{xs(n)} \qquad (x \in \mathbb{R}) \tag{3.5}$$

we have

$$C(x) := C_0(x) + C_1(x) + \cdots + C_{p-1}(x) = 1 + e^x + \cdots + e^{(p-1)x} \tag{3.6}$$

since $s(n) = n$ for $n = 0, 1, \ldots, p-1$ and in view of $s(kp+r) = s(k) + s(r)$ for $k = 0, 1, 2, \ldots$ and $r = 0, 1, \ldots, p-1$. Hence, we have that

$$C_{kp+r}(x) = e^{xs(kp+r)} = e^{x(s(k)+s(r))} = C_k(x)C_r(x),$$

cf. (3.4). For $c_r(x) = \frac{C_r(x)}{C(x)}$ we have

$$c_0(x) + c_1(x) + \ldots + c_{p-1}(x) = 1$$

and $c_r(0) = \frac{1}{p}$ for $r \in \{0, 1, \ldots, p-1\}$. By Theorem 2.2 equation (1.1) with the actual coefficients $c_r(x)$, i.e.

$$\varphi\left(\frac{t}{p}, x\right) = \sum_{r=0}^{p-1} \frac{e^{rx}}{C(x)} \varphi(t-r, x) \qquad (t \in \mathbb{R}) \tag{3.7}$$

with $C(x)$ from (3.6) has a solution $\varphi_0(t, x)$ which is continuous with respect to t and arbitrary often differentiable with respect to $x \in X$.

According to Proposition 3.1 we have

Proposition 3.2 *For $N \in \mathbb{N}$ the sum $S(N, x)$ from (3.1) can be represented as*

$$S(N, x) = N^{\alpha(x)} F_0(\log_p N, x) \tag{3.8}$$

where

$$\alpha(x) = \log_p C(x) \tag{3.9}$$

with $C(x)$ from(3.6) and an 1-periodic function $F_0(u, x)$ with respect to u which is given by

$$F_0(u, x) = \frac{\varphi_0(p^u, x)}{p^{\alpha(x)u}} \qquad (u \leq 0, x \in \mathbb{R}) \tag{3.10}$$

which is continuous and 1-periodic with respect to u.

By Theorem 2.2 the function $F_0(u,x)$ is arbitrary often differentiable with respect to x and the k-th partial derivative

$$F_k(u,x) := \frac{\partial^k}{\partial x^k} F_0(u,x) \tag{3.11}$$

is 1-periodic with respect to u. So the functions

$$F_k(u) := F_k(u,0) \tag{3.12}$$

are continuous and 1-periodic.

Proposition 3.3 *We have*

$$F_0(u) = 1. \tag{3.13}$$

Proof: For $x=0$ equation (3.7) takes the form

$$\varphi\left(\frac{t}{p},0\right) = \sum_{r=0}^{p-1} \frac{1}{p}\varphi(t-r,0) \qquad (t\in\mathbb{R})$$

with the unique solution $\varphi(t,0)=t$ for $0\le t\le 1$, cf. [18]. Further, owing to (3.9) we get

$$\alpha(0) = \log_p C(0) = \log_p p = 1$$

and hence for $u \le 0$

$$F_0(u,0) = \frac{\varphi_0(p^u,0)}{p^{\alpha(0)u}} = \frac{p^u}{p^u} = 1 \qquad (u\le 0).$$

The periodicity of $F_0(u)$ implies $F_0(u)=1$ for all real u. □

4 Power sums of the sum of digits

Now, for integer $N\ge 1$ we investigate the power sums of the sum of digits

$$S_k(N) = \sum_{n=0}^{N-1} s(n)^k \tag{4.1}$$

where $k\ge 0$ is an integer and use that

$$\left.\frac{\partial^k}{\partial x^k} S(N,x)\right|_{x=0} = \left.\sum_{n=0}^{N-1} s(n)^k e^{xs(n)}\right|_{x=0} = S_k(N).$$

So according to Proposition 3.2 we have

Proposition 4.1 *For $N \in \mathbb{N}$ the power sum (4.1) can be represented as*

$$S_k(N) = \frac{\partial^k}{\partial x^k} N^{\alpha(x)} F_0(\log_p N, x)\bigg|_{x=0} \tag{4.2}$$

As abbreviation we put

$$c(x) = \frac{C'(x)}{C(x)} = \frac{e^x + 2e^{2x} + \cdots + (p-1)e^{(p-1)x}}{1 + e^x + \cdots + e^{(p-1)x}} \tag{4.3}$$

with $C(x)$ from (3.6). In particular

$$c(0) = \frac{1}{p} S_1(p) = \frac{p-1}{2}. \tag{4.4}$$

For $k \in \mathbb{N}$ and $\ell = 0, \ldots, k$ we introduce functions $c_{k,\ell}(x)$ as follows:

$$\left.\begin{array}{lcll} c_{k,k}(x) & = & c(x)^k & \text{for} \quad k \geq 0 \\ c_{k,0}(x) & = & 0 & \text{for} \quad k \geq 1 \\ c_{k+1,\ell}(x) & = & c_{k,\ell-1}(x)c(x) + c'_{k,\ell}(x) & \text{for} \quad k \geq 1, \;\; 1 \leq \ell \leq k \end{array}\right\}. \tag{4.5}$$

So $c_{0,0}(x) = 1$ in view of $c(x) \neq 0$. It is easy to see that for $k \geq 1$ it holds

$$c_{k,1}(x) = c^{(k-1)}(x), \qquad c_{k,k-1}(x) = \frac{k(k-1)}{2} c(x)^{k-2} c'(x). \tag{4.6}$$

Lemma 4.2 *For integer $k \geq 0$ we have*

$$\frac{d^k}{dx^k} N^{\alpha(x)} = N^{\alpha(x)} \sum_{\ell=0}^{k} c_{k,\ell}(x) (\log_p N)^\ell. \tag{4.7}$$

Proof: Formula (4.7) is true for $k = 0$ since $c_{0,0}(x) = 1$. In view of $\alpha(x) = \log_p C(x)$ we get

$$\frac{d}{dx} N^{\alpha(x)} = N^{\alpha(x)} \log N \frac{1}{\log p} \frac{C'(x)}{C(x)} = N^{\alpha(x)} c(x) \log_p N$$

i.e. for $k = 1$ formula (4.7) is true, too. Assume (4.7) is true for a fixed k. Then we get

$$\frac{d^{k+1}}{dx^{k+1}} N^{\alpha(x)} = N^{\alpha(x)} \sum_{\ell=0}^{k} c_{k,\ell}(x) c(x) (\log_p N)^{\ell+1} + N^{\alpha(x)} \sum_{\ell=0}^{k} c'_{k,\ell}(x) (\log_p N)^\ell$$

which in view of (4.5) and $c_{k,k}(x)c(x) = c(x)^{k+1} = c_{k+1,k+1}(x)$ yields the assertion. $\square$

Theorem 4.3 *For integer $N \geq 0$ we have*

$$\sum_{n<N} s(n)^k e^{xs(n)} = N^{\alpha(x)} \sum_{\ell=0}^{k} (\log_p N)^\ell H_\ell(\log_p N, x) \tag{4.8}$$

with $\alpha(x) = \log_p C(x)$ and

$$H_\ell(u,x) = \sum_{\kappa=0}^{k-\ell} \binom{k}{\kappa} c_{k-\kappa,\ell}(x) F_\kappa(u,x) \tag{4.9}$$

with $F_k(u,x)$ from (3.11).

Proof: We use (3.1) and (3.8) with $\alpha = \alpha(x) = \log_p C(x)$. By Leibniz's formula and Lemma 4.2 we have with $u = \log_p N$

$$\begin{aligned} \frac{\partial^k}{\partial x^k} N^\alpha F_0(u,x) &= \sum_{\kappa=0}^{k} \binom{k}{\kappa} \frac{\partial^\kappa}{\partial x^\kappa} N^\alpha \frac{\partial^{k-\kappa}}{\partial x^{k-\kappa}} F_0(u,x) \\ &= N^\alpha \sum_{\kappa=0}^{k} \binom{k}{\kappa} \sum_{\ell=0}^{\kappa} u^\ell c_{\kappa,\ell}(x) F_{k-\kappa}(u,x) \\ &= N^\alpha \sum_{\ell=0}^{k} \sum_{\kappa=\ell}^{k} \binom{k}{\kappa} u^\ell c_{\kappa,\ell}(x) F_{k-\kappa}(u,x). \end{aligned}$$

Replacing κ by $k-\kappa$ we get

$$\frac{\partial^k}{\partial x^k} N^\alpha F_0(u,x) = N^\alpha \sum_{\ell=0}^{k} \sum_{\kappa=0}^{k-\ell} \binom{k}{\kappa} u^\ell c_{k-\kappa,\ell}(x) F_\kappa(u,x)$$

which in view of (3.1) and (3.8) yields (4.8) with (4.9). □

Corollary 4.4 *For the power sum (4.1) we have*

$$\frac{1}{N} S_k(N) = \sum_{\ell=0}^{k} (\log_p N)^\ell H_\ell(\log_p N) \tag{4.10}$$

where $H_\ell(u) = H_\ell(u,0)$ from (4.9), i.e.

$$H_\ell(u) = \sum_{\kappa=0}^{k-\ell} \binom{k}{\kappa} c_{k-\kappa,\ell} F_\kappa(u) \tag{4.11}$$

with

$$c_{k,\ell} := c_{k,\ell}(0) \tag{4.12}$$

and $F_k(u)$ from (3.12).

Lemma 4.5 *For integer $k \geq 0$ we have*

$$c^{(k)}(0) = \frac{(-1)^{k+1} B_{k+1}}{k+1}(p^{k+1} - 1) \tag{4.13}$$

with the Bernoulli numbers B_{k+1}.

Proof: From (4.3) we have by Leibniz's formula

$$c^{(k)}(x) = \sum_{n=0}^{k} \binom{k}{n} C^{(n+1)}(x) \left(\frac{1}{C(x)}\right)^{(k-n)}. \tag{4.14}$$

In order to compute $c^{(k)}(0)$ first note that

$$C^{(n+1)}(0) = 1 + 2^{n+1} + \ldots + (p-1)^{n+1} = \tilde{B}_{n+1}(p), \tag{4.15}$$

cf. (1.20). Moreover, in view of

$$C(x) = 1 + e^x + \cdots + e^{(p-1)x} = \frac{e^{px} - 1}{e^x - 1}$$

and

$$\frac{1}{C(x)} = \frac{e^x - 1}{e^{px} - 1} = \frac{e^{\frac{1}{p}(px)} - 1}{e^{px} - 1} = \sum_{n=0}^{\infty} \frac{\tilde{B}_n(\frac{1}{p})}{n!}(px)^n \qquad \left(|x| < \frac{2\pi}{p}\right),$$

cf. (1.22) with $z = px$ and $t = \frac{1}{p}$, we find

$$\left(\frac{1}{C(x)}\right)^{(n)}\Bigg|_{x=0} = p^n \tilde{B}_n\left(\frac{1}{p}\right). \tag{4.16}$$

From (4.14), (4.15) and (4.16) we get

$$c^{(k)}(0) = \sum_{n=0}^{k} \binom{k}{n} \tilde{B}_{n+1}(p) p^{k-n} \tilde{B}_{k-n}\left(\frac{1}{p}\right).$$

In view of the Cauchy product of two power series we see that

$$\sum_{k=0}^{\infty} \frac{c^{(k)}(0)}{k!} z^k = \sum_{n=0}^{\infty} \frac{\tilde{B}_{n+1}(p)}{n!} z^n \sum_{n=0}^{\infty} \frac{p^n \tilde{B}_n(\frac{1}{p})}{n!} z^n \tag{4.17}$$

which in view of (1.22) is convergent for $|z| < \frac{2\pi}{p}$. Moreover, from (1.22) we find

$$\sum_{n=0}^{\infty} \frac{\tilde{B}_{n+1}(p)}{n!} z^n = \frac{d}{dz}\left(\frac{e^{pz} - 1}{e^z - 1}\right) = \frac{pe^{pz}(e^z - 1) - (e^{pz} - 1)e^z}{(e^z - 1)^2},$$

$$\sum_{n=0}^{\infty} \frac{\tilde{B}_n(\frac{1}{p})}{n!}(pz)^n = \frac{e^z - 1}{e^{pz} - 1},$$

and by (4.17) we get

$$\sum_{k=0}^{\infty} \frac{c^{(k)}(0)}{k!} z^k = \frac{pe^{pz}}{e^{pz}-1} - \frac{e^z}{e^z-1} \qquad (z \neq 0, |z| < 2\pi).$$

According to (1.17) we have for $z \neq 0$ and $|z| < 2\pi$

$$\frac{e^z}{e^z-1} = \frac{1}{1-e^{-z}} = -\sum_{n=0}^{\infty} \frac{B_n}{n!}(-z)^{n-1}$$

and

$$\frac{pe^{pz}}{e^{pz}-1} = -\sum_{n=0}^{\infty} \frac{pB_n}{n!}(-pz)^{n-1}.$$

Hence,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{c^{(k)}(0)}{k!} z^k &= \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!}(p^n-1) z^{n-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!}(p^{k+1}-1) z^k \end{aligned}$$

with $k = n-1$. This implies (4.13). □

Remark 4.6 Note that in case $p = 2$ we have $c(x) = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$ so that in view of

$$\frac{1}{1+e^x} = \frac{1}{2}\left(1 - \tanh\left(\frac{x}{2}\right)\right)$$

we get $c^{(k)}(0) = \frac{(-1)^{k+1}B_{k+1}}{k+1}(2^{k+1}-1)$.

5 Specific power sums

We begin with a formula for the digital power sum

$$S_k(pN) = \sum_{n<pN} s(n)^k. \tag{5.1}$$

Proposition 5.1 *For the sums (5.1) we have*

$$S_k(pN) = \sum_{\ell=0}^{k} \binom{k}{\ell} S_{k-\ell}(p) S_\ell(N) \tag{5.2}$$

where $S_0(N) = N$.

Proof: Write $n = pm + r$ with $0 \le r \le p-1$ and $0 \le m \le N-1$, we get in view of $s(pm+r) = s(m) + s(r)$ that

$$\begin{aligned}\sum_{n<pN} s(n)^k &= \sum_{r=0}^{p-1} \sum_{m<N} \{s(m)+s(r)\}^k \\ &= \sum_{r=0}^{p-1} \sum_{m<N} \sum_{\kappa=0}^{k} \binom{k}{\kappa} s(m)^\kappa s(r)^{k-\kappa} \\ &= \sum_{\kappa=0}^{m} \binom{k}{\kappa} \sum_{r=0}^{p-1} s(r)^{k-\kappa} \sum_{m<N} s(m)^\kappa.\end{aligned}$$

This yields the assertion. □

For $N = p^n$ we get from (4.10)

$$\frac{1}{p^n} S_k(p^n) = \sum_{\ell=0}^{k} a_{k,\ell}\, n^\ell \tag{5.3}$$

with coefficients $a_{k,\ell}$ depending on p which owing to (4.11) are given by

$$a_{k,\ell} = \sum_{\kappa=0}^{k-\kappa} \binom{k}{\kappa} c_{k-\kappa,\ell} F_\kappa(0), \tag{5.4}$$

cf. [4]. In particular for $n = 1$ we have $S_k(p) = \tilde{B}_k(p)$ with the modified Bernoulli polynomials $\tilde{B}_k(\cdot)$, cf. (1.21), and (5.3) implies

$$\sum_{\ell=0}^{k} a_{k,\ell}(p) = \frac{1}{p} \tilde{B}_k(p). \tag{5.5}$$

Lemma 5.2 *For the coefficients $a_{k,\ell} = a_{k,\ell}(p)$ we have the relation*

$$\sum_{\nu=\kappa+1}^{k} \binom{\nu}{\kappa} a_{k,\nu} = \sum_{\ell=\kappa}^{k-1} \binom{k}{\ell} \frac{1}{p} S_{k-\ell}(p) a_{\ell,\kappa} \tag{5.6}$$

Proof: We use (5.2) with $N = p^n$. By (5.3) we have

$$\begin{aligned}\frac{1}{p^{n+1}} S_k(p^{n+1}) &= \sum_{\ell=0}^{k} a_{k,\ell}(n+1)^\ell \\ &= \sum_{\ell=0}^{k} a_{k,\ell} \sum_{\kappa=0}^{\ell} \binom{\ell}{\kappa} n^\kappa \\ &= \sum_{\kappa=0}^{k} \sum_{\ell=\kappa}^{m} a_{k,\kappa} \binom{\ell}{\kappa} n^\kappa\end{aligned}$$

and

$$\begin{aligned}\sum_{\kappa=0}^{k}\binom{k}{\kappa}\frac{1}{p}S_{k-\kappa}(p)\frac{1}{p^n}S_\kappa(p^n) &= \sum_{\kappa=0}^{k}\binom{k}{\kappa}\frac{1}{p}S_{k-\kappa}(p)\sum_{k=0}^{\kappa}a_{\ell,\kappa}n^\kappa\\ &= \sum_{\kappa=0}^{k}\sum_{\ell=\kappa}^{k}\binom{k}{\ell}\frac{1}{p}S_{k-\ell}(p)a_{\ell,\kappa}n^\kappa.\end{aligned}$$

According to (5.2) we get

$$\sum_{\nu=\kappa}^{k}\binom{\nu}{\kappa}a_{k,\nu}=\sum_{\ell=\kappa}^{k}\binom{k}{\ell}\frac{1}{p}S_{k-\ell}(p)a_{\ell,\kappa}$$

which yields (5.6). □

We already know from Proposition 3.3 that $F_0(u)=1$ for $u\in\mathbb{R}$.

Proposition 5.3 *For the values $F_k(0)$ with $k\geq 1$ we have*

$$F_k(0)=0 \qquad (k\geq 1). \tag{5.7}$$

For all $k\in\mathbb{N}_0$ and $\ell=0,1,\ldots,k$ we have $a_{k,\ell}=c_{k,\ell}$. In particular $a_{k,k}=(\frac{p-1}{2})^k$ for $k\geq 0$ and $a_{k,0}=0$ for $k\geq 1$. The further numbers $a_{k,\ell}$ are uniquely determined by

$$\ell a_{k,\ell}(p)=\sum_{\mu=\ell-1}^{k-1}\binom{k}{\mu}\frac{1}{p}S_{k-\mu}(p)a_{\mu,\ell-1}(p)-\sum_{\nu=\ell+1}^{k}\binom{\nu}{\ell-1}a_{k,\nu}(p). \tag{5.8}$$

Proof: From (5.4) we get $a_{k,k}=c_{k,k}F_0(0)=(\frac{p-1}{2})^k$ for $k\geq 0$ according to (4.5), (4.4) and $F_0(.)=1$. Formula (5.8) follows from (5.6). If $a_{k',\ell'}$ are given for $0\leq k'<k$, $0\leq\ell'\leq k'$ and for $k'=k$, $\ell<\ell'\leq k$ then $a_{k,\ell}$ is determined by (5.8).

Next we show that $a_{k,0}=0$ for $k\geq 1$. At first we get from (5.3) with (5.4) in case $k=1$ that

$$\frac{1}{p^n}S_1(p^n)=a_{1,0}+a_{1,1}n=a_{1,0}+\frac{1}{p}S_1(p)n$$

since $a_{1,1}=\frac{1}{p}S_1(p)$ and $n=1$ implies $a_{1,0}=0$. Now, equation (5.4) for $\kappa=0$ yields

$$\sum_{\nu=1}^{m}a_{k,\nu}=\sum_{\ell=0}^{k-1}\binom{k}{\kappa}\frac{1}{p}S_{k-\ell}(p)a_{\ell,0}.$$

Using (5.5) it follows

$$\frac{1}{p}S_k(p)-a_{k,0}=\frac{1}{p}S_k(p)+\sum_{\ell=1}^{k-1}\binom{k}{\ell}\frac{1}{p}S_{k-\ell}(p)a_{\ell,0}$$

and

$$a_{k,0}=-\sum_{\ell=1}^{k-1}\binom{k}{\ell}\frac{1}{p}S_{k-\ell}(p)a_{\ell,0}.$$

Now, $a_{1,0}=0$ implies $a_{k,0}=0$ for $k\geq 1$.

Finally we show that $a_{k,\ell} = c_{k,\ell}$. Equation (5.4) for $\ell = 0$ yields

$$a_{k,0} = \sum_{\kappa=0}^{k} \binom{k}{k} c_{k-\kappa,0} F_\kappa(0) = c_{0,0} F_k(0) = F_k(0),$$

i.e. $F_0(0) = 1$ and $F_k(0) = 0$ for $k \geq 1$. Now, equation (5.4) yields

$$a_{k,\ell} = \sum_{j=0}^{k} \binom{k}{j} c_{k-j,\ell} F_j(0) = c_{k,\ell} F_0(0) = c_{k,\ell}$$

and the proposition is proved completely. □

We already know that $a_{k,0} = 0$ for $k \geq 1$ and

$$a_{k,k} = \left(\frac{p-1}{2}\right)^k \qquad (k \geq 0). \tag{5.9}$$

The first $a_{k,\ell} = a_{k,\ell}(p)$ are computed by means of (5.8)

$$\begin{array}{lllll}
a_{1,1} = \frac{p-1}{2} & & & & \\
a_{2,1} = \frac{p^2-1}{12} & a_{2,2} = (\frac{p-1}{2})^2 & & & \\
a_{3,1} = 0 & a_{3,2} = \frac{(p-1)^2(p+1)}{8} & a_{3,3} = (\frac{p-1}{2})^3 & & \\
a_{4,1} = -\frac{p^4-1}{120} & a_{4,2} = \frac{(p-1)^2(p+1)^2}{48} & a_{4,3} = \frac{(p-1)^3(p+1)}{8} & a_{4,4} = (\frac{p-1}{2})^4 & \\
a_{5,1} = 0 & a_{5,2} = -\frac{(p-1)^2(p+1)(p^2+1)}{48} & a_{5,3} = \frac{5(p-1)^3(p+1)^2}{96} & a_{5,4} = \frac{5(p-1)^4(p+1)}{48} & a_{5,5} = (\frac{p-1}{2})^5
\end{array}$$

Figure 2. The first numbers $a_{k,\ell}$

In the following we need the

Proposition 5.4 *For $k \geq 1$ we have*

$$a_{k,1}(p) = \frac{(-1)^k B_k}{k}(p^k - 1) \tag{5.10}$$

with the Bernoulli numbers B_k, and

$$a_{k,k-1}(p) = \binom{k}{2} \left(\frac{p-1}{2}\right)^{k-1} \frac{p+1}{6}. \tag{5.11}$$

Proof: We use $a_{k,\ell}(p) = c_{k,\ell}(0)$, cf. Proposition 5.3, and both relations in (4.6), i.e. $c_{k,1}(x) = c^{(k-1)}(x)$ and $c_{k,k-1}(x) = \binom{k}{2} c(x)^{k-2} c'(x)$ with $c(x)$ from (4.3). First we compute $a_{k,1} = a_{k,1}(p)$. Now, $a_{k,1}(p) = c^{(k-1)}(0)$ so that (5.10) follows from Lemma 4.5.

We know that $a_{k,k-1} = \binom{k}{2} c^{k-2}(0) c'(0)$ where $c(0) = \frac{1}{p}(1 + 2 + \cdots + (p-1)) = \frac{p-1}{2}$, i.e, $a_{k,k-1} = \binom{k}{2}(\frac{p-1}{2})^{k-2} c'(0)$ where $c'(0)$ is independent of k. In particular, $a_{2,1} = c'(0)$. From (5.10) we know that $a_{2,1} = \frac{p^2-1}{12}$ and it follows (5.11). $\square$

Proposition 5.5 *For $k \geq 2$ and $1 \leq \ell < k$ we have that $a_{k,\ell}(p)$ are polynomials in p of degree at most k with $a_{k,\ell}(-1) = 0$. Moreover,*

$$a_{k,\ell}(p) = \left(\frac{p-1}{2}\right)^{\ell} \tilde{a}_{k,\ell}(p) \tag{5.12}$$

where $\tilde{a}_{k,\ell}(p)$ are polynomials in p of degree at most $k - \ell$ which are given by $\tilde{a}_{k,k}(p) = 1$ and

$$\ell \tilde{a}_{k,\ell}(p) = \sum_{\mu=\ell-1}^{k-1} \binom{k}{\mu} \frac{2\tilde{B}_{k-\mu}(p)}{p(p-1)} \tilde{a}_{\mu,\ell-1}(p) - \sum_{\nu=\ell+1}^{k} \binom{\nu}{\ell-1} \left(\frac{p-1}{2}\right)^{\nu-\ell} \tilde{a}_{k,\nu}(p). \tag{5.13}$$

Proof: From (5.4) we get $a_{k,k} = c_{k,k} F_0(0) = (\frac{p-1}{2})^k$ for $k \geq 0$ and $a_{k,0} = c_{0,0} F_k(0) = F_k(0)$ for $k \geq 1$. Assume that $a_{k',\ell'}(p)$ are given polynomials with $deg\, a_{k',\ell'}(p) \leq k'$ if $0 \leq k' < k$, $0 \leq \ell' \leq k'$ and if $k' = k$, $\ell < \ell' \leq k$. Then $a_{k,\ell}(p)$ is determined by (5.8) and $deg\, a_{k,\ell}(p) \leq k$.

We show that $a_{k,\ell}(-1) = 0$ for all $k \geq 2$ and $\ell = 0, \ldots, k-1$. This is true for $a_{k,0} = 0$ and $a_{k,k-1}$ according to (5.11) with $k \geq 2$. Assume that for $k \geq 2$ we have $a_{k',\ell'}(-1) = 0$ if $0 \leq k' < k$, $0 \leq \ell' \leq k' - 1$ and if $k' = k$, $\ell < \ell' \leq k - 1$. Then from (5.8) we get in view of $a_{\ell-1,\ell-1}(-1) = (-1)^{\ell-1}$ and $a_{k,k}(-1) = (-1)^k$, cf. (5.9), that

$$\begin{aligned} \ell a_{k,\ell}(-1) &= \binom{k}{\ell-1}(-1)\tilde{B}_{k-\ell+1}(-1)(-1)^{\ell-1} - \binom{k}{\ell-1}(-1)^k \\ &= \binom{k}{\ell-1}(-1)^{\ell}\left\{\tilde{B}_{k-\ell+1}(-1) - (-1)^{k-\ell}\right\} \\ &= 0 \end{aligned}$$

since $\tilde{B}_n(-1) = (-1)^{n-1}$ which follows from (1.19) with $t = -1$ and (1.21) where $B_n = B_n(0)$.

Next we show (5.12) which is true for all $a_{k,k}$ and $a_{k,0}$, $k = 0, 1, 2, \ldots$ since $a_{k,k} = (\frac{p-1}{2})^k$ and $a_{k,0} = 0$ for $k \geq 1$. Assume (5.12) is true for all $a_{k',\ell'}$ with $0 \leq k' < k$ and $0 \leq \ell' \leq k'$ as well as for $a_{k,\ell'}$ with $\ell < \ell' \leq k$. Then by division of (5.8) with $(\frac{p-1}{2})^{\ell}$ we get (5.13) which implies that indeed $\tilde{a}_{k,\ell}(p)$ is a polynomial in p and the supposition is proved by induction. $\square$

Proposition 5.6 *For the power sum (4.1) with $N \in \mathbb{N}$ and $L = \log_p N$ we have*

$$\frac{1}{N} S_k(N) = \left(\frac{p-1}{2} L\right)^k + \sum_{\ell=0}^{k-1} \left(\frac{p-1}{2} L\right)^\ell \sum_{\kappa=0}^{k-\ell} \binom{k}{\kappa} \tilde{a}_{k-\kappa,\ell}(p) F_\kappa(L) \tag{5.14}$$

where $\tilde{a}_{k,\ell}(p)$ are polynomials in p of degree at most $k-\ell$, given by $\tilde{a}_{k,k}(p) = 1$ and the recursion (5.13).

Proof: We use Corollary 4.4, $F_0(\cdot) = 1$ and $c_{k,\ell} = a_{k,\ell}(p) = (\frac{p-1}{2})^\ell\, \tilde{a}_{k,\ell}(p)$. So we get

$$\begin{aligned}\frac{1}{N} S_k(N) &= \sum_{\ell=0}^{k} L^\ell \sum_{\kappa=0}^{k-\ell} \binom{k}{\kappa} \left(\frac{p-1}{2}\right)^\ell \tilde{a}_{k-\kappa,\ell}(p) F_\kappa(L) \\ &= \left(\frac{p-1}{2} L\right)^k + \sum_{\ell=0}^{k-1} \left(\frac{p-1}{2} L\right)^\ell \sum_{\kappa=0}^{k-\ell} \binom{k}{\kappa} \tilde{a}_{k-\kappa,\ell}(p) F_\kappa(L)\end{aligned}$$

and (5.14) is proved. □

Remark 5.7 In view of (5.11) and (5.12) formula (5.14) yields for arbitrary integer k the asymptotic relation

$$\frac{1}{N} S_k(N) = \left(\frac{p-1}{2} L\right)^k + \left(\frac{p-1}{2} L\right)^{k-1} \left\{ \binom{k}{2} \frac{p+1}{6} + k F_1(L) \right\} + o(L^{k-1}) \tag{5.15}$$

as $N \to \infty$. In case $k = 1$ we get from (5.14) the formula of Trollope-Delange

$$\frac{1}{N} S_1(N) = \frac{p-1}{2} L + F_1(L), \tag{5.16}$$

in case $k = 2$

$$\frac{1}{N} S_2(N) = \left(\frac{p-1}{2} L\right)^2 + \frac{p-1}{2} L \left\{ \frac{p+1}{6} + 2F_1(L) \right\} + F_2(L) \tag{5.17}$$

which for $p = 2$ is known by Coquet, cf. [3], and in case $k = 3$

$$\begin{aligned}\frac{1}{N} S_3(N) &= \left(\frac{p-1}{2} L\right)^3 + \left(\frac{p-1}{2} L\right)^2 \left\{ \frac{p+1}{2} + 3F_1(L) \right\} \\ &\quad + \frac{p-1}{2} L \left\{ \frac{p+1}{3} F_1(L) + 3F_2(L) \right\} + F_3(L),\end{aligned}$$

cf. also [22] for $p = 2$ or [17, Theorem 6.3]. In case $k = 4$ we get

$$\begin{aligned}\frac{1}{N} S_4(N) &= \left(\frac{p-1}{2} L\right)^4 + \left(\frac{p-1}{2} L\right)^3 \{p + 1 + 4F_1(L)\} \\ &\quad + \left(\frac{p-1}{2} L\right)^2 \left\{ \frac{p+1}{12} + 2(p+1)F_1(L) + 6F_2(L) \right\} \\ &\quad + \frac{p-1}{2} L \left\{ -\frac{p+1}{60} + (p+1)F_2(L) + 4F_3(L) \right\} + F_4(L).\end{aligned}$$

Remark 5.8 In case $N = p$ we have $L = 1$, and in view of $F_0(1) = 1$ and $F_k(1) = 0$ for $k > 0$ as well as $S_k(p) = 1^k + 2^k + \cdots + (p-1)^k = \tilde{B}_k(p)$, cf. (1.20), we get from (5.14) that

$$\frac{1}{p}\tilde{B}_k(p) = \left(\frac{p-1}{2}\right)^k + \sum_{\ell=0}^{k-1}\left(\frac{p-1}{2}\right)^\ell \tilde{a}_{k,\ell}(p) \tag{5.18}$$

cf. also (5.5) and (5.12). Because $\tilde{a}_{k,\ell}(-1) = 0$ for $\ell < k$ we have that $\frac{1}{p}\tilde{B}_k(p) - (\frac{p-1}{2})^k$ is divisible by $p+1$. Hence $\frac{1}{n+1}\tilde{B}_k(n+1) - (\frac{n}{2})^k = \frac{1}{n+1}(1^k + 2^k + \cdots + n^k) - (\frac{n}{2})^k$ is divisible by $n+2$. So, in particular, for $k = 1, 2, \ldots$ we have

$$\begin{aligned}
\tfrac{1}{n+1}\sum_{i=1}^{n} i - \tfrac{n}{2} &= 0\\
\tfrac{1}{n+1}\sum_{i=1}^{n} i^2 - (\tfrac{n}{2})^2 &= \tfrac{1}{12}n(n+2)\\
\tfrac{1}{n+1}\sum_{i=1}^{n} i^3 - (\tfrac{n}{2})^3 &= \tfrac{1}{8}n^2(n+2)\\
\tfrac{1}{n+1}\sum_{i=1}^{n} i^4 - (\tfrac{n}{2})^4 &= \tfrac{1}{240}n(n+2)(33n^2+6n-4)\\
\tfrac{1}{n+1}\sum_{i=1}^{n} i^5 - (\tfrac{n}{2})^5 &= \tfrac{1}{96}n^2(n+2)(13n^2+6n-4)
\end{aligned}$$

and so on.

6 Power series and generating functions

We start with the power sums $S_k(p^n)$, cf. (4.1) with $N = p^n$.

Proposition 6.1 *For $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{\infty}\frac{1}{k!}S_k(p^n)z^k = \left(\frac{e^{pz}-1}{e^z-1}\right)^n \qquad (z \in \mathbb{C}). \tag{6.1}$$

Proof: We prove (6.1) by induction on n. In case $n = 1$ we use (1.20) and (1.22) so that

$$\sum_{k=0}^{\infty}\frac{1}{k!}S_k(p)z^k = \sum_{k=0}^{\infty}\frac{1}{k!}\tilde{B}_k(p)z^m = \frac{e^{pz}-1}{e^z-1}$$

where we have convergence for all $z \in \mathbb{C}$ in view of

$$\frac{e^{pz}-1}{e^z-1} = 1 + e^z + \cdots + e^{(p-1)z}.$$

Assume (6.1) is true for a certain $n \geq 0$. Then we have in view of Proposition 5.1 with $N = p^n$ and the Cauchy product of two power series

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{k!}S_k(p^{n+1})z^k &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}S_m(p^n)z^k\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!}S_k(p)z^k\right) = \left(\frac{e^{pz}-1}{e^z-1}\right)^n\frac{e^{pz}-1}{e^z-1}\\
&= \left(\frac{e^{pz}-1}{e^z-1}\right)^{n+1}. \qquad \square
\end{aligned}$$

Now we consider the polynomials

$$P_k(t) := \sum_{\ell=0}^{k} a_{k,\ell}(p)t^\ell \tag{6.2}$$

with the coefficients $a_{k,\ell}(p)$ given by (5.8). We remember that in particular, $a_{k,k} = (\frac{p-1}{2})^k$ and that $\frac{1}{p^n}S_k(p^n) = P_k(n)$, cf. (5.3). According to Figure 2 the first polynomials $P_k(t)$ read:

$$\begin{aligned}
P_0(t) &= 1 \\
P_1(t) &= \tfrac{p-1}{2}t \\
P_2(t) &= \tfrac{p^2-1}{12}t + (\tfrac{p-1}{2})^2t^2 \\
P_3(t) &= \tfrac{(p-1)^2(p+1)}{8}t^2 + (\tfrac{p-1}{2})^3t^3 \\
P_4(t) &= -\tfrac{p^4-1}{120}t + \tfrac{(p-1)^2(p+1)^2}{48}t^2 + \tfrac{(p-1)^3(p+1)}{8}t^3 + (\tfrac{p-1}{2})^4t^4 \\
P_5(t) &= -\tfrac{(p-1)^2(p+1)(p^2+1)}{48}t^2 + \tfrac{5(p-1)^3(p+1)^2}{96}t^3 + \tfrac{5(p-1)^4(p+1)}{48}t^4 + (\tfrac{p-1}{2})^5t^5
\end{aligned}$$

Figure 3. The first polynomials $P_k(t)$

Proposition 6.2 *The polynomials (6.2) have the generating function*

$$\sum_{k=0}^{\infty} \frac{1}{k!}P_k(t)z^k = \left(\frac{e^{pz}-1}{p(e^z-1)}\right)^t \qquad (z \in \mathbb{C}) \tag{6.3}$$

and starting with $P_0(t) = 1$ they satisfy the recursions

$$P_k(t) = t\sum_{\ell=1}^{k} \binom{k-1}{\ell-1} P_{k-\ell}(t)a_{\ell,1}(p) \tag{6.4}$$

where

$$a_{\ell,1}(p) = \frac{(-1)^\ell B_\ell}{\ell}(p^\ell - 1)$$

with the Bernoulli numbers B_ℓ, cf. (5.10).

Proof: According to $\frac{1}{p^n}S_k(p^n) = P_k(n)$, cf. (5.3), and (6.1) we have for $n \in \mathbb{N}$

$$\sum_{k=0}^{\infty} \frac{1}{k!}P_k(n)z^k = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{p^n}S_k(p^n)z^k = \left(\frac{e^{pz}-1}{p(e^z-1)}\right)^n \qquad (z \in \mathbb{C}) \tag{6.5}$$

so that (6.3) is true for $t = n \in \mathbb{N}$. We show that for all $t \in \mathbb{C}$

$$\left(\frac{e^{pz}-1}{p(e^z-1)}\right)^t = \sum_{k=0}^{\infty} \frac{1}{k!}Q_k(t)z^k \qquad (z \in \mathbb{C}) \tag{6.6}$$

where $Q_k(t)$ are polynomials with respect to t of degree k. For fix p and t we put

$$f(z) := \left(\frac{e^{pz}-1}{p(e^z-1)}\right)^t = (g(z))^t \tag{6.7}$$

with

$$g(z) = \frac{e^{pz}-1}{p(e^z-1)} = \frac{1}{p}(1+e^z+\cdots+e^{(p-1)z}) \qquad (z \in \mathbb{C})$$

and we have $Q_k(t) = f^{(k)}(z)|_{z=0}$, in particular $Q_0(t) = f(0) = 1$. Formula (6.7) yields $\log f(z) = t \log g(z)$. Hence $f'(z) = tf(z)\frac{g'(z)}{g(z)}$ and by the product rule of Leibniz we get

$$f^{(k+1)}(z) = t\sum_{\ell=0}^{k}\binom{k}{\ell} f^{(k-\ell)}(z)\left(\frac{g'(z)}{g(z)}\right)^{(\ell)}.$$

Note that $\frac{g'(z)}{g(z)} = c(z)$ with $c(z)$ from (4.3) and by Lemma 4.5 we have

$$\left.\left(\frac{g'(z)}{g(z)}\right)^{(\ell)}\right|_{z=0} = \frac{(-1)^{\ell+1}B_{\ell+1}}{\ell+1}(p^{\ell+1}-1)$$

so that

$$Q_{k+1}(t) = t\sum_{\ell=0}^{k}\binom{k}{\ell} Q_{k-\ell}(t)\frac{(-1)^{\ell+1}B_{\ell+1}}{\ell+1}(p^{\ell+1}-1). \tag{6.8}$$

It follows by induction on k that $Q_k(t)$ are polynomials with respect to t of degree k. We know that $Q_0(t) = 1$ and owing to (6.8) we get $Q_1(t) = tQ_0(t)B_1(p-1) = t(-\frac{1}{2})(p-1)$. Assume that $Q_k(t)$ with fixed $k \geq 1$ is a polynomial of degree k then (6.8) implies that Q_{k+1} is a polynomial of degree $k+1$. Finally, $Q_k(t) = P_k(t)$ for all t since $Q_k(n) = P_k(n)$ for all integer $n \geq 1$ according to (6.5) and (6.6). We get (6.4) from (6.8) if we replace Q by P as well as $k+1$ by k and $\ell+1$ by ℓ. $\square$

Remark 6.3 **1.** A consequence of (6.3) is the following additions theorem

$$P_k(s+t) = \sum_{\ell=0}^{k}\binom{k}{\ell} P_\ell(s)P_{k-\ell}(t). \tag{6.9}$$

2. Formula (6.3) with $n=1$ yields in view of (1.21) the values for $P_k(1)$, namely

$$P_k(1) = \frac{1}{p}\tilde{B}_k(p) \qquad (k = 0, 1, 2, \ldots). \tag{6.10}$$

For the values $P_k(-1)$ we have

$$P_k(-1) = p^{k+1}\tilde{B}_k\left(\frac{1}{p}\right) \qquad (k = 0, 1, 2, \ldots) \tag{6.11}$$

which follows from

$$\left(\frac{e^{pz}-1}{p(e^z-1)}\right)^{-1} = p\frac{e^{\frac{1}{p}(pz)}-1}{(e^{pz}-1)} = p\sum_{k=0}^{\infty}\frac{\tilde{B}_k(\frac{1}{p})}{k!}(pz)^k = \sum_{k=0}^{\infty}\frac{p^{k+1}\tilde{B}_k(\frac{1}{p})}{k!}z^k$$

and (6.3) with $n=-1$. We remember that we have by (5.3)

$$P_k(n) = \frac{1}{p^n}S_k(p^n) \qquad (n=0,1,2,\ldots), \tag{6.12}$$

in particular, $P_0(n)=0$ and $P_k(1)=\frac{1}{p}S_k(p)=\frac{1}{p}\tilde{B}_k(p)$, cf. (5.5).

Proposition 6.4 *In case $p=2$ the polynomials $P_k(t)$ satisfy the recursion*

$$P_0(t)=1 \qquad \textit{and} \qquad P_{k+1}(t) = t\left(P_k(t)-\frac{1}{2}P_k(t-1)\right) \quad \textit{for} \quad k\geq 0.$$

Proof: From (6.3) with $p=2$ we get

$$\sum_{k=0}^{\infty}\frac{P_k(t)}{k!}z^k = \left(\frac{e^{2z}-1}{2(e^z-1)}\right)^t = \left(\frac{e^z+1}{2}\right)^t.$$

Note that

$$\sum_{k=0}^{\infty}\frac{P_{k+1}(t)}{k!}z^k = \sum_{k=1}^{\infty}\frac{P_k(t)}{(k-1)!}z^{k-1} = \frac{d}{dz}\left(\sum_{k=0}^{\infty}\frac{P_k(t)}{k!}z^k\right)$$

and

$$\frac{d}{dz}\left(\frac{e^z+1}{2}\right)^t = t\left(\frac{e^z+1}{2}\right)^{t-1}\frac{1}{2}e^z.$$

Further,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{P_k(t)-\frac{1}{2}P_k(t-1)}{k!}z^k &= \left(\frac{e^z+1}{2}\right)^t - \frac{1}{2}\left(\frac{e^z+1}{2}\right)^{t-1} \\ &= \left(\frac{e^z+1}{2}\right)^{t-1}\frac{1}{2}e^z\end{aligned}$$

Compare of coefficients implies assertion. □

Remark 6.5 By equating coefficients of t^ℓ in relation (6.4) we find a new recursion for the polynomials $a_{k,\ell}(p)$, namely

$$a_{k,\ell}(p) = \sum_{j=1}^{k-\ell+1}\binom{k-1}{j-1}a_{k-j,\ell-1}(p)a_{j,1}(p), \tag{6.13}$$

cf. (5.8).

Now for integer $\ell \geq 0$ we introduce the generating functions of $a_{k,\ell} = a_{k,\ell}(p)$ by

$$G_\ell(z) := \sum_{k=0}^{\infty} \frac{a_{k,\ell}}{k!} z^k. \tag{6.14}$$

Note that $G_0(z) = 1$ since $a_{0,0} = 1$ and $a_{k,0} = 0$ for $k \geq 1$ and that for $\ell \geq 1$

$$G_\ell(z) = \sum_{k=0}^{\infty} \frac{a_{k+1,\ell}}{(k+1)!} z^{k+1} \tag{6.15}$$

since $a_{0,\ell} = 0$, i.e. $G_\ell(0) = 0$. In particular

$$G_1(z) = \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k \cdot k!} (p^k - 1) z^k \tag{6.16}$$

with the Bernoulli numbers B_k, cf. (5.10).

Proposition 6.6 *For $\ell \geq 1$ we have*

$$G_\ell(z) = \frac{1}{\ell!} G_1(z)^\ell \tag{6.17}$$

with $G_1(z)$ from (6.16).

Proof: Let be $\ell \geq 1$. Relation (6.13) with $k+1$ instead of k and $\ell+1$ instead of ℓ can be written as

$$\begin{aligned} a_{k+1,\ell+1} &= \sum_{j=1}^{k-\ell+1} \binom{k}{j-1} a_{k+1-j,\ell} a_{j,1} \\ &= \sum_{i=0}^{k} \binom{k}{i} a_{k-i,\ell} a_{i+1,1} \end{aligned}$$

with $i = j - 1$ where we have used that $a_{k-i,\ell} = 0$ for $i \geq k - \ell + 1$ in view of $a_{m,n} = 0$ for $m < n$. So we have after multiplication with z^k

$$\frac{a_{k+1,\ell+1}}{k!} z^k = \sum_{i=0}^{k} \frac{a_{k-i,\ell}}{(k-i)!} z^{k-i} \frac{a_{i+1,1}}{i!} z^i$$

and summation over k yields in view of the Cauchy product and the relations

$$\sum_{k=0}^{\infty} \frac{a_{k+1,\ell+1}}{k!} z^k = \left(\sum_{k=0}^{\infty} \frac{a_{k+1,\ell+1}}{(k+1)!} z^{k+1} \right)' = G'_{\ell+1}(z)$$

and

$$\sum_{i=0}^{\infty} \frac{a_{i+1,1}}{i!} z^i = \left(\sum_{i=0}^{\infty} \frac{a_{i+1,1}}{(i+1)!} z^{i+1} \right)' = G'_1(z)$$

that

$$G'_{\ell+1}(z) = G_\ell(z)G'_1(z)$$

which is also valid for $\ell = 0$ since $G_0(z) = 1$.

Now we can prove (6.17) by induction on ℓ. Obviously, it is true for $\ell = 1$. Assume that it is true for an integer $\ell \geq 1$. Then we get

$$\begin{aligned} G'_{\ell+1}(z) &= G_\ell(z)G'_1(z) \\ &= \frac{1}{\ell!}G_1(z)^\ell G'_1(z) \\ &= \frac{1}{(\ell+1)!}\frac{d}{dz}G_1(z)^{\ell+1}. \end{aligned}$$

So $G_{\ell+1}(z) = \frac{1}{(\ell+1)!}G_1(z) + c$ where $c = 0$ in view of $G_1(0) = 0$ and $G_{\ell+1}(0) = 0$. □

Proposition 6.7 *The polynomials $a_{k,\ell}(p)$ have the explicit representation*

$$a_{k,\ell}(p) = \frac{(-1)^k k!}{\ell!} \sum_{k_1+\cdots+k_\ell=k} \frac{k!}{k_1!\cdots k_\ell!}\left(\prod_{n=1}^{\ell} \frac{B_{k_n}}{k_n \cdot k_n!}(p^{k_n}-1)\right) \tag{6.18}$$

where $k_1, \ldots, k_\ell$ are positive integers and where B_k are the Bernoulli numbers. Moreover,

$$a_{k,\ell}\left(\frac{1}{p}\right) = \frac{(-1)^\ell}{p^k} a_{k,\ell}(p). \tag{6.19}$$

Proof: By Proposition 6.6 we have in view of (6.14) and (6.16)

$$\sum_{k=0}^{\infty} \frac{a_{k,\ell}(p)}{k!} z^k = \frac{1}{\ell!}\left(\sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k \cdot k!}(p^k-1)z^k\right)^\ell. \tag{6.20}$$

Applying the multimonial theorem (cf. H. Hall [10], Combinatorical theory Wiley (1986)) we get for the right-hand side of (6.20) with positive integers k_i

$$\frac{1}{\ell!} \sum_{k_1+\ldots+k_\ell=k} \frac{k!}{k_1!\cdots k_\ell!}\left(\prod_{n=1}^{\ell} \frac{(-1)^{k_n} B_{k_n}}{k_n \cdot k_n!}(p^{k_n}-1)z^{k_1+\cdots+k_\ell}\right)$$

which in view of $(-1)^{k_1}\cdots(-1)^{k_\ell} = (-1)^k$ if $k_1 + \cdots + k_\ell = k$ is equal to

$$\frac{1}{\ell!}\sum_{k=\ell}^{\infty}(-1)^k \sum_{k_1+\cdots+k_\ell=k} \frac{k!}{k_1!\cdots k_\ell!}\left(\prod_{n=1}^{\ell} \frac{B_{k_n}}{k_n \cdot k_n!}(p^{k_n}-1)\right) z^k$$

Now comparing coefficients of z^k in (6.20) yields (6.18).

From (6.18) with $\frac{1}{p}$ instead of p we get in view of

$$\begin{aligned}\sum_{k_1+\cdots+k_\ell=k}\left(\prod_{n=1}^{\ell}\frac{B_{k_n}}{k_n\cdot k_n!}\left(\frac{1}{p^{k_n}}-1\right)\right) &= \frac{1}{p^k}\sum_{k_1+\cdots+k_\ell=k}\left(\prod_{n=1}^{\ell}\frac{B_{k_n}}{k_n\cdot k_n!}(1-p^{k_n})\right)\\ &= \frac{(-1)^\ell}{p^k}\sum_{k_1+\cdots+k_\ell=k}\left(\prod_{n=1}^{\ell}\frac{B_{k_n}}{k_n\cdot k_n!}(p^{k_n}-1)\right)\end{aligned}$$

the relation (6.19). □

Remark 6.8 Let $A_{k,\ell}$ be the main coefficient of the polynomial $a_{k,\ell}(p)$, that means

$$a_{k,\ell}(p) = A_{k,\ell}p^k + o(p^k) \qquad (k\to\infty). \tag{6.21}$$

Then from (6.18) we see that

$$A_{k,\ell} = (-1)^\ell a_{k,\ell}(0)$$

and (5.5) implies in view of (1.21) and (1.18) that

$$\sum_{\ell=0}^{k} A_{k,\ell} = \frac{1}{k+1} \tag{6.22}$$

and that

$$\sum_{\ell=0}^{k} a_{k,\ell}(0) = \frac{B_k}{k+1} \tag{6.23}$$

with the Bernoulli numbers B_k, see Figure 2.

We know already from Proposition 5.5 that for $k\ge 2$ and $1\le \ell < k$ the polynomials $a_{k,\ell}(p)$ are divisible by $p+1$.

Proposition 6.9 *For integer $\ell\ge 1$ and $r\ge 1$ the polynomial $a_{\ell+2r,\ell}(p)$ is divisible by $(p+1)^2$, see Figure 2.*

Proof: We write short $a_{k,\ell}$ for $a_{k,\ell}(p)$ and use induction on ℓ. The assertion is true for $\ell = 1$ since $a_{2r+1,1}(p) = 0$ according to (5.10) and $B_{2r+1} = 0$. Assume that for a fixed $\ell\ge 1$ the polynomials $a_{\ell+2r,\ell}(p)$ for all $r\ge 1$ are divisible by $(p+1)^2$. By (6.13) with $\ell+1+2r$ instead of k and $\ell+1$ instead of ℓ we get for arbitrary integer $r\ge 1$

$$\begin{aligned}a_{\ell+1+2r,\ell+1}(p) &= \sum_{j=1}^{2r+1}\binom{\ell+2r}{j-1}a_{\ell+1+2r-j,\ell}(p)a_{j,1}(p)\\ &= a_{\ell+2r,\ell}a_{1,1}+\binom{\ell+2r}{1}a_{\ell+2r-1,\ell}a_{2,1}+\cdots+\binom{\ell+2r}{2r}a_{\ell,\ell}a_{2r+1,1}.\end{aligned}$$

By induction assumption the first product $a_{\ell+2r,\ell}a_{1,1}$ is divisible by $(p+1)^2$. The last product $a_{\ell,\ell}a_{2r+1,1} = 0$ since $a_{2r+1,1} = 0$ for $r \geq 1$. Moreover, all another products $a_{\ell+2r-1,\ell}a_{2,1}, \dots, a_{\ell+1,\ell}a_{2r,1}$ are also divisible by $(p+1)^2$ since each of the both factors (polynomials) is divisible by $p+1$ according to Proposition 5.5. Consequently, $a_{\ell+1+2r,\ell+1}(p)$ is divisible by $(p+1)^2$ and the assertion is proved by induction. □

Remark 6.10 Comparison with recursion (1.7) and formula (1.8) yields $q_k(t) = 2^k P_k(t)$ for the polynomials $q_k(t)$ introduced in [8].

Theorem 6.11 *For $N \in \mathbb{N}$ and $L = \log_p N$ we have*

$$\sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{N} S_k(N) z^k = \left(\sum_{k=0}^{\infty} \frac{1}{k!} P_k(L) z^k\right)\left(\sum_{k=0}^{\infty} \frac{1}{k!} F_k(L) z^k\right) \qquad (z \in \mathbb{C}) \tag{6.24}$$

with the polynomials (6.2) and

$$\sum_{k=0}^{\infty} \frac{1}{k!} F_k(L) z^k = \left(\sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{N} S_k(N) z^k\right)\left(\sum_{k=0}^{\infty} \frac{1}{k!} P_k(-L) z^k\right) \qquad (z \in \mathbb{C}). \tag{6.25}$$

Proof: For the power sum (4.1) we have by Corollary 4.4 in view of $c_{k,\ell} = a_{k,\ell}(p)$, cf. Proposition 5.3, that

$$\begin{aligned}\frac{1}{N} S_k(N) &= \sum_{\ell=0}^{k} L^\ell \sum_{\kappa=0}^{k-\ell} \binom{k}{\kappa} a_{k-\kappa,\ell} F_\kappa(L) \\ &= \sum_{\kappa=0}^{k} \sum_{\ell=0}^{k-\kappa} \binom{k}{\kappa} a_{k-\kappa,\ell} L^\ell F_\kappa(L)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{k!}\frac{1}{N} S_k(N) &= \sum_{\kappa=0}^{k} \frac{1}{(k-\kappa)!} \sum_{\ell=0}^{k-\kappa} a_{k-\kappa,\ell}(p) L^\ell \frac{1}{\kappa!} F_\kappa(L) \\ &= \sum_{\kappa=0}^{k} \frac{1}{(k-\kappa)!} P_{k-\kappa}(L) \frac{1}{\kappa!} F_\kappa(L).\end{aligned}$$

In view of the Cauchy product of two power series we get (6.24) with the polynomials (6.2). If we replace t by $-t$ in (6.24) and in (6.2) we see that (6.24) implies (6.25). □

Remark 6.12 **1.** For $N \in \mathbb{N}$ and $L = \log_p N$ we have by Theorem 6.11

$$\frac{1}{N} S_k(N) = \sum_{\ell=0}^{k} \binom{k}{\ell} P_k(L) F_{k-\ell}(L) \tag{6.26}$$

and

$$F_k(L) = \sum_{\ell=0}^{k} \binom{k}{\ell} P_\ell(-L) \frac{1}{N} S_{k-\ell}(N). \tag{6.27}$$

For the case $p = 2$ one can find in [8] a similarly representation of $S_k(N)$ by means of generating functions.

2. Up to now we only know about the 1-periodic functions $F_k(u)$ with $k \geq 1$ that $F_k(0) = 0$. By means of (6.27) we are able to compute the values $F_k(u)$ for $u = \log_p N$ if $N \leq p$ since for these N the sums $S_{k-\ell}(N)$ are the usual power sums

$$\sum_{n=0}^{N-1} n^{k-\ell} = \tilde{B}_{k-\ell}(N)$$

cf. (1.20). According to (6.27) we get for $u = u_N := \log_p N$ with $N \leq p$ that

$$F_k(u_N) = \sum_{\ell=0}^{k} \binom{k}{\ell} P_\ell(-u_N) \frac{1}{N} \tilde{B}_{k-\ell}(N). \tag{6.28}$$

where $\tilde{B}_n(\cdot)$ are the generalized Bernoulli polynomials, cf. (1.21).

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Author:

Manfred Krüppel
Institut für Mathematik,
Universität Rostock,
Ulmenstr. 69,
D-18057 Rostock,
Germany

e-mail: manfred.krueppel@uni-rostock.de