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Bénédicte Alziary, Jacqueline Fleckinger

Semi-linear cooperative elliptic systems involving Schrödinger operators: Groundstate positivity or negativity

ABSTRACT. We study here the behavior of the solutions to a 2×2 semi-linear cooperative system involving Schrödinger operators (considered in its variational form):

$$LU := (-\Delta + q(x))U = AU + \mu U + F(x, U) \quad \text{in} \quad \mathbb{R}^N$$

$$U(x)_{|x| \to \infty} \to 0$$

where q is a continuous positive potential tending to $+\infty$ at infinity; μ is a real parameter varying near the principal eigenvalue of the system; U is a column vector with components u_1 and u_2 and A is a square cooperative matrix with constant coefficient. F is a column vector with components f_1 and f_2 depending eventually on U.

1 Introduction

We study here the behaviour of the solutions to a 2×2 semi-linear cooperative system involving Schrödinger operators (considered in its variational form):

$$LU := (-\Delta + q(x))U = AU + \mu U + F(x, U) \quad \text{in} \quad \mathbb{R}^N$$

$$U(x)_{|x| \to \infty} \to 0$$

where q is a continuous positive potential tending to $+\infty$ at infinity; U is a column vector with components u_1 and u_2 and A is a square matrix with constant coefficients; moreover A is a cooperative matrix (which means that its coefficients outside the diagonal are non negative). F is a column vector with components f_1 and f_2 depending eventually on U. The real parameter μ varies near the principal eigenvalue of the system and plays a key role. According to its position it determines not only the sign of the solutions but also their position w.r.t. the groundstate.

Such systems have been intensively studied (very often for $\mu = 0$) and mainly for Dirichlet problems defined on bounded domains ([16], [17], [18], [21], [20], [25], [12], [4]). When the whole $\mathbb{R}^N$ is considered, as here, 2 cases are generally studied: either "Schrödinger systems" ([1], [2], [3], [7]), that is system involving Schrödinger operators, as here, or systems with a weight tending to 0 ([23], [6]). It is also possible to consider a combination of these 2 problems with a potential q and a weight g :

$$LU := (-\Delta + q(x))U = g(x)AU + \mu g(x)U + F(x, U) \text{ in } \mathbb{R}^N$$

as far as $\frac{g}{q}$ tends to 0 at infinity which is the condition for having some compactness and therefore a discrete spectrum.

The first results on Schrödinger systems, when F does not depend on U (linear systems) deal with cooperative systems and with the Maximum Principle (**MP**) that is:

"If the data F is non negative, $\neq 0$, then, any solution U is non negative".

As for the case of one equation, this Maximum Principle holds for a parameter $\mu < \Lambda^*$, where Λ^* is the principal eigenvalue of the system, which means that $LU - AU - \Lambda^* U = 0$ has a non zero solution which does not change sign.

For the classical case of an equation defined on a bounded domain with zero boundary conditions, $-\Delta u = \mu u + f(x)$, $f > 0$, Clément and Peletier [14] have shown that the solution u changes sign as soon as μ goes over λ_1, the first eigenvalue of the Dirichlet Laplacian defined on Ω. More precisely there exists a small positive δ, depending on f, such that for all $\mu \in (\lambda_1, \lambda_1 + \delta)$, $u < 0$. This phenomenon is known as "Anti-maximum Principle" (**AMP**).

In our present case, where we have no boundary, we have improved these results giving not only the sign of the solutions but also comparing the solutions with the groundstate (principal eigenfunction); it is what we call "groundstate positivity"(**GSP**) (resp. negativity) (resp. **GSN**). We extend in particular previous results established in [5] for linear systems to some semi-linear cooperative systems. For being not excessively technical, we limit our study to radial potentials and cooperative systems. Extensions to more general cases will appear somewhere else.

Our paper is organized as follows:

We recall first some previous results of the linear case that we use. Then we study a semi-linear equation. Finally we study a cooperative semi-linear system.

2 Linear Case: one equation

We shortly recall the case of a linear equation with a parameter μ varying near the principal eigenvalue of the operator.

$$(E) \qquad Lu := (-\Delta + q(x))u \;=\; \mu u + f(x) \text{ in } \mathbb{R}^N,$$

$$\lim_{|x|\to+\infty} u(x) \;=\; 0.$$

(H_q) q is a positive continuous potential tending to $+\infty$ at infinity.

We seek u in V where

$$V := \left\{ u \in L^2(\mathbb{R}^N)\, s.t.\, \|u\|_V = \left(\int |\nabla u|^2 + q(x)u^2 \right)^{1/2} < \infty \right\}.$$

If (H_q) is satisfied, the embedding of V into $L^2(\mathbb{R}^N)$ is compact (see *e.g.* [19], [15]). Hence L possesses an infinity of eigenvalues tending to $+\infty$:

$$0 < \lambda_1 < \lambda_2 \le .... \le \lambda_k \le ... ,\ \lambda_k \to +\infty \text{ as } k \to \infty.$$

Notation (Λ, ϕ)**:** We set from now on $\Lambda := \lambda_1$ the smallest one (which is positive and simple) and ϕ the associated eigenfunction, positive and with L^2-norm $\|\phi\| = 1$.

It is classical (see *e.g.* [24]) that if $f \ge 0, \ne 0$, and $\mu < \Lambda$, there exists exactly one solution which is positive: the positivity is "improved", or in other words, the (strong) maximum principle **(MP)** is satisfied:

$$(MP) \qquad f \ge 0, \not\equiv 0 \;\Rightarrow\; u > 0.$$

Lately, as said above, another notion has been defined ([8], [10], [22]) the "groundstate positivity" (**GSP**) (resp. "negativity" (**GSN**)) which means that, there exists $k > 0$ such that the solution $u > k\phi$ (GSP) (resp. $u < -k\phi$ (GSN)).

We also say shortly "fundamental positivity" or "negativity", or also "ϕ-positivity" or "negativity". Indeed these properties are more precise than MP or AMP. But for proving them, it is necessary to have a potential growing fast enough, a potential with a super quadratic growth.

In [10] a class $\mathcal{P}$ of radial potentials is defined:

$$\mathcal{P} := \left\{ Q \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^*)/\exists R_0 > 0, Q' > 0\, a.e. \text{ on } [R_0, \infty),\ \int_{R_0}^{\infty} Q(r)^{-1/2} < \infty \right\}. \qquad (1)$$

The last inequality holds precisely if Q is growing sufficiently fast, indeed faster than r^2 (the harmonic oscillator). In this paper we consider only a radial potential $q \in \mathcal{P}$. Note that our proof is valid for more general potentials, in particular for perturbations of radial potential [9] or [10] . We assume here

(H_q') q is radial and is in $\mathcal{P}$

Remark 1 Note that since q is in $\mathcal{P}$ it satisfies (H_q).

On f we assume

(H_f^*) $f \in L^2(\mathbb{R}^N), \quad f^1 = \int f\phi > 0.$

For having more precise estimates on u, in particular the "groundstate negativity" **(GSN)**, we have to define another set X in which f varies, the set of "groundstate bounded functions":

$$X := \{h \in L^2(\mathbb{R}^N) : |h|/\phi \in L^\infty(\mathbb{R}^N)\}, \tag{2}$$

equipped with the norm $\|h\|_X = ess\sup_{\mathbb{R}^n}(|h|/\phi)$.

Theorem 1 *Assume (H_q') and (H_f^*), $f \in X$. For $\mu < \Lambda$ or $\Lambda < \mu < \lambda_2$ there exists $\delta > 0$ (defined below) depending on f and a positive constant C, depending on f such that if $0 < |\Lambda - \mu| < \delta$,*

$$\Lambda - \delta < \mu < \Lambda \;\Rightarrow\; u \geq \frac{C}{\Lambda - \mu}\phi > 0,$$

$$\Lambda < \mu < \Lambda + \delta \;\Rightarrow\; u \leq \frac{C}{\Lambda - \mu}\phi < 0.$$

Proof of Theorem 1 Decompose now u and f in (E) on ϕ and its orthogonal:

$$u = u^1\phi + u^\perp\,;\; f = f^1\phi + f^\perp;\; u^1 = \int u\phi,\; \int u^\perp\phi = \int f^\perp\phi, = 0;$$

we derive from Equation (E)

$$(L - \mu)u^1\phi = (\Lambda - \mu)u^1\phi = f^1\phi\,,\; Lu^\perp = \mu u^\perp + f^\perp. \tag{3}$$

Choose $\mu < \Lambda$ or $\Lambda < \mu < \lambda_2$. From the first equation we derive

$$u^1 = \frac{f^1}{(\Lambda - \mu)} \;\to\; \pm\infty \;\text{ as }\; (\Lambda - \mu) \to 0.$$

By use of Theorem 3.2 (c) in [9] or [10], we know that the restriction of the resolvent $(L-\mu)^{-1}$ to X is bounded from X into itself. The following lemma is a direct consequence of this result as it is shown in the proof of the Theorem 3.4 in [9].

Lemma 1 *There exists δ_0 small enough and there exists a constant c_0 (depending on δ_0) such that for all μ with $\Lambda - \delta_0 < \mu < \Lambda$ or $\Lambda < \mu < \Lambda + \delta_0 < \lambda_2$,*

$$-c_0\|f^\perp\|_X \le \|u^\perp\|_X \le c_0\|f^\perp\|_X.$$

Finally we take in account Lemma 1 and (3):

$$\|u^\perp\|_X \le c_0\|f^\perp\|_X \text{ and } u = \frac{f^1}{\Lambda-\mu}\phi + u^\perp;$$

for $|\Lambda-\mu| \to 0$, $\frac{f^1}{\Lambda-\mu}\phi \to \pm\infty$ when $u^\perp$ stays bounded. Hence, for $|\Lambda-\mu|$ small enough, more precisely for $|\Lambda-\mu| < \delta_1(f) := \frac{f^1}{c_0\|f^\perp\|_X}$, we have

$$\frac{f^1}{|\Lambda-\mu|} > c_0\|f^\perp\|_X.$$

We deduce that Theorem 1 is valid for $\delta := \min\{\delta_0, \delta_1(f)\}$.

3 Semi-linear Schrödinger equation

We study now the case of a semi-linear equation. We first obtain bounds for the solutions, if they exist and then we show their existence via the method of "sub-super solutions". Finally, with additional assumptions, we prove the uniqueness of them.

Consider the semi-linear Schrödinger equation (SLSE)

$$(SLSE) \qquad Lu := (-\Delta + q(x))u = \mu u + f(x,u) \text{ in } \mathbb{R}^N,$$

$$\lim_{|x|\to+\infty} u(x) = 0.$$

We assume that the potential q satisfies (H_q') and we denote as above by (Λ, ϕ) the principal eigenpair with $\phi > 0$.

We work in $L^2(\mathbb{R}^N)$ and we consider the problem in its variational formulation. We seek u in V for a suitable f.

We assume that f satisfies :

(H_f) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function *i.e.* the function $f(\bullet, u)$ is Lebesgue measurable in $\mathbb{R}^N$, for every $u(x) \in \mathbb{R}$ and the function $f(x, \bullet)$ is continuous in $\mathbb{R}$ for almost every $x \in \mathbb{R}^N$. Moreover, f is such that

$$(i) \qquad \forall u \in L^2(\mathbb{R}^N), \quad f(.,u) \in L^2(\mathbb{R}^N),$$

$$(ii) \qquad \exists\, \kappa > 0 \quad s.t. \quad \forall u \in V, \quad f(x,u) \ge \kappa\phi(x) > 0$$

(iii) $\exists K > \kappa > 0 \quad s.t. \quad \forall u \in V, \ \ f(x,u) \le K\phi(x).$

Later we also suppose

(H'_f) $\forall x \in \mathbb{R}^N, \ u \to \dfrac{f(x,u)}{|u|}$ is strictly decreasing

Remark 2 Note that, by (ii) and (iii), for any $u \in V$, $f(.,u) \in X$ and hence the solutions, if they exist, are in X.

Let a parameter μ be given, with $|\mu - \Lambda|$ "small enough". In this section we prove groundstate positivity and negativity for the semi-linear Schrödinger equation.

Theorem 2 *If (H'_q) and (H_f) are satisfied , then there exists $\delta(f) > 0$ ($\delta = \delta(f) :=$ $\min\{\delta_0, \delta'_1(f) := \frac{\kappa}{c_0 K}\}$ where δ_0 and c_0 are given in Lemma 1) such that, for $0 < |\mu - \Lambda| < \delta$ there exists a solution u to $(SLES)$ such that*

$$\|u\|_X \le \frac{K}{|\Lambda - \mu|} + 2c_0 K.$$

Also

- for $\Lambda - \delta < \mu < \Lambda$, $u > \frac{\kappa}{\Lambda - \mu}\phi > 0$,

- for $\Lambda < \mu < \Lambda + \delta < \lambda_2$, $u < \frac{K}{\Lambda - \mu}\phi < 0$.

Moreover if (H'_f) is satisfied, the solution to $(SLSE)$ is unique.

Remark 3 If (ii) does not hold, for $\mu < \Lambda$, there exists a solution u such that

$$\|u\|_X \le \frac{K}{|\Lambda - \mu|} + 2c_0 K.$$

The existence is classical ($e.g.$ [3]) and the estimate follows from the proof below.

Proof of Theorem 2

We do the proof in 3 steps: first maximun and anti-maximum principles, secondly existence of the solution such that $u > \frac{\kappa}{\Lambda - \mu}\phi > 0$ for $\Lambda - \delta < \mu < \Lambda$ and such that $u < \frac{K}{\Lambda - \mu}\phi < 0$, for $\Lambda < \mu < \Lambda + \delta$, and thirdly the uniqueness.

Step 1. Maximun and anti-maximum principles

We prove the positivity or negativity of the solutions exactly as for the linear case, but, since f depends on u we have to show that δ (which depends on f in the linear case) is now uniform. This follows from hypotheses (ii) and (iii).

Let u be a solution to $Lu = \mu u + f(x,u)$. For this u, set

$$f^1(u) = \int f(x,u)\phi(x)dx\,,\ f^\perp(x,u) = f(x,u) - f^1(u)\phi(x).$$

Also $u^1 = \int u\phi(x)dx$ and $u^\perp = u - u^1\phi$.

Note that, always by (ii) and (iii), $0 < \kappa \le f^1(u) \le K$.

With this decomposition, reporting in $(SLSE)$, we obtain 2 equations:

$$(L-\mu)u^1\phi = (\Lambda-\mu)u^1\phi = f^1\phi\,,\ \ Lu^\perp = \mu u^\perp + f^\perp.$$

Choose $\mu < \Lambda$ or $\Lambda < \mu < \lambda_2$. From the first equation we derive

$$u^1 = \frac{f^1}{(\Lambda-\mu)} \to \pm\infty\, as\, (\Lambda-\mu) \to 0.$$

Now we proceed exactly as for the linear case. By use of Theorem 3.2 (c) in [9] or [10], we know that the restriction of the resolvent $(L-\mu)^{-1}$ to X is bounded from X into itself. So by (iii) and by Lemma 1 there exists a δ_0 small enough and there exists a constant c_0 (depending on δ_0) such that for all μ with $|\Lambda-\mu| < \delta_0$,

$$\|u^\perp\|_X \le c_0\|f^\perp(x,u)\|_X \le c_0\|f(x,u) - f^1(u)\phi(x)\|_X \le 2c_0K.$$

Write now

$$u = \frac{f^1(u)}{\Lambda-\mu}\phi + u^\perp$$

Hence $\|u\|_X \le \frac{f^1(u)}{|\Lambda-\mu|} + \|u^\perp\|_X \le \frac{K}{|\Lambda-\mu|} + 2c_0K$. For $|\Lambda-\mu| \to 0$, $\frac{f^1}{\Lambda-\mu}\phi \to \pm\infty$ when $u^\perp$ stays bounded. For $|\Lambda-\mu|$ small enough, that is here $|\Lambda-\mu| < \delta_1'(f) := \frac{\kappa}{2c_0K}$, we get (since $f^1 > 0$)

$$\frac{f^1}{|\Lambda-\mu|} \ge \frac{\kappa}{|\Lambda-\mu|} > 2c_0K \ge c_0\|f^\perp\|_X.$$

Finally Maximum and anti-maximum principles are valid for
$\delta(f) := \min\{\delta_0, \delta_1'(f)\}$.

Step 2. Existence of solutions

We prove the existence of solutions by Schauder fixed point theory; for this purpose we need some classical elements: a set $\mathcal{K}^\pm$ constructed with the help of sub-super solutions and a compact operator T acting in $\mathcal{K}^\pm$ such that $\mathcal{K}^\pm$ stays invariant by T: $T(\mathcal{K}^\pm) \subset \mathcal{K}^\pm$.

1: "Sub-super solution" :

- Case $\Lambda - \delta < \mu < \Lambda$.

Obviously, by (ii), $u_0 = \frac{\kappa}{\Lambda-\mu}\phi > 0$ is a subsolution:

$$L(u-u_0) = \mu(u-u_0) + f - (\Lambda-\mu)u_0 = \mu(u-u_0) + f - \kappa\phi$$

and by (ii) and GSP, $u - u_0 \geq 0$.

Analogously $u^0 = \frac{K}{\Lambda-\mu}\phi > 0$ (K given in (iii)) is a supersolution :

$$Lu^0 = \frac{\Lambda}{\Lambda-\mu}K\phi = \Lambda u^0 = \mu u^0 + (\Lambda-\mu)u^0.$$

Remark 4 The sub- and supersolutions tend to $+\infty$ as $\mu \nearrow \Lambda$.

- Case $\Lambda < \mu < \Lambda + \delta < \lambda_2$. $v^0 = \frac{\kappa}{\Lambda-\mu}\phi < 0$ is a supersolution. Indeed

$$L(v^0 - u) = \mu(v^0-u) + \kappa\phi - f$$

and by (H_f) and the anti-maximum $0 > v^0 \geq u$.

Analogously, $v_0 = \frac{K}{\Lambda-\mu}\phi < 0$ is a subsolution.

Remark 5 The sub- and supersolutions tend to $-\infty$ as $\mu \searrow \Lambda$.

Remark 6 Obviously, $u_0 < u^0$ for $\Lambda - \delta < \mu < \Lambda$ (resp. $v_0 < v^0$ for $\Lambda < \mu < \Lambda + \delta$).

2: The operator T

We define $T : u \in L^2 \longrightarrow w = Tu \in V$, where $w \in X$ is the unique solution to $Lw = \mu w + f(x,u)$.

3: The invariant set $\mathcal{K}^+ := [u_0, u^0]$ for $\Lambda - \delta < \mu < \Lambda$ (resp. $\mathcal{K}^- := [v_0, v^0]$ for $\Lambda < \mu < \Lambda + \delta$).

If $\mu < \Lambda$, by the maximum principle and the hypothesis (iii) , $u \leq u^0$ implies $w \leq u^0$. Indeed,

$$L(u^0 - w) = \mu(u^0 - w) + (\Lambda-\mu)u^0 - f(x,u) = \mu(u^0-w) + K\phi - f(x,u);$$

since, by (iii), $K\phi - f(x,u) \geq 0$, we apply the maximum principle and hence $w \leq u^0$. The 3 other cases lead to analogous calculation.

4: T is compact in X.

First note that $\mathcal{K}^+ \subset X$ (resp. $\mathcal{K}^- \subset X$). $Lw - \mu w = f(x,u)$ can also be written $w = (L-\mu I)^{-1} f(x,u) = T(u)$. Since by [10], [9], the resolvent $R(\mu) := (L-\mu I)^{-1}$ is compact in X for $\mu \in (\Lambda-\delta, \Lambda)$ or $(\Lambda, \Lambda+\delta)$, and since $F : u \to f(x,u)$ is continuous, $T = R(\mu)F$ is compact.

We deduce from Schauder fixed point theory that there exists a solution to $(SLSE)$ in $\mathcal{K}^+$, (resp. in $\mathcal{K}^-$).

Step 3. Uniqueness

For proving uniqueness we follow [13], p. 57. First we assume not only (H_f) but also (H'_f). Assume that u and v are two solutions:

$$Lu = \mu u + f(x, u)\ ,\ \ Lv = \mu v + f(x, v)$$

The solutions are in X and we have shown that $u, v > u_0 > 0$ for $\Lambda - \delta < \mu < \Lambda$ (resp. $u, v < v^0 < 0$ for $\Lambda < \mu < \Lambda + \delta$). Hence we can write

$$\frac{Lu}{u} = \mu + \frac{f(x, u)}{u}\ ;\ \frac{Lv}{v} = \mu + \frac{f(x, v)}{v}.$$

By subtraction $q(x)$ and μ disappear. Multiply by $u^2 - v^2$ and integrate.

$$\int \left[\frac{-\Delta u}{u} + \frac{\Delta v}{v}\right][u^2 - v^2] \ = \ \int \left[\frac{f(x, u)}{u} - \frac{f(x, v)}{v}\right][u^2 - v^2];$$

the last term is non positive by (H'_f).

We transform exactly as in [13] the first term.

$$\int \left[\frac{-\Delta u}{u} + \frac{\Delta v}{v}\right][u^2 - v^2] \ = \ \int \left|\nabla u - \frac{u}{v}\nabla v\right|^2 + \left|\nabla v - \frac{v}{u}\nabla u\right|^2 =$$

$$\int \left|v\nabla(\frac{u}{v})\right|^2 + \left|u\nabla(\frac{v}{u})\right|^2 \geq 0; \qquad (7)$$

therefore both terms are equal to 0 and

$$u^2 - v^2 = 0 \ \Rightarrow\ u = v\, a.e.;$$

by regularity, $u = v$.

4 Semi-linear cooperative system

We extend here to a class of semi-linear systems previous results shown in [5] where linear systems of the form $LU = \mu U + AU + F(x)$ are studied.

We study for $a > 0$, $b > 0$, $c > 0$

$$(S) \qquad \left\{ \begin{array}{rcl} Lu_1 &=& (\mu + a)u_1 + bu_2 + f_1(x, u_1) \\ Lu_2 &=& cu_1 + (\mu + d)u_2 + f_2(x, u_2) \end{array} \right. \quad in\ \mathbb{R}^N, .$$

$$u_1(x), u_2(x)_{|x|\to\infty} \to 0.$$

We write shortly $LU = \mu U + AU + F(x,U)$, where A is the cooperative matrix with components a, b, c, d:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Notation (ξ_1, Y)**:** Denote ξ_1 the largest eigenvalue of A (the other one being denoted by ξ_2); Y is the eigenvector associated with ξ_1:

$$AY = \xi_1 Y.$$

$$\xi_1 = \frac{a + d + \sqrt{(a-d)^2 + 4bc}}{2}.$$

An easy calculation shows that $(L - A)(Y\phi) = (\Lambda - \xi_1)Y\phi$; moreover here $Y\phi$ is with components which do not change sign: we choose both components of Y positive:

$$y_1 = b > 0\,,\; y_2 = \frac{d - a + \sqrt{(a-d)^2 + 4bc}}{2} > 0.$$

Notation Λ^***:** $\Lambda^* := \Lambda - \xi_1$ is the principal eigenvalue of System (S) with associated eigenvector $Y\phi$:

$$(L - A)(Y\phi) = (\Lambda - \xi_1)Y\phi = \Lambda^* Y\phi.$$

Hypotheses We assume

(H_A) A is a 2×2 cooperative matrix with positive coefficients outside the diagonal.

(H_F) : $f_1, f_2 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are Caratheodory function *i.e.* the functions $f_1(\bullet, u_1)$ or $f_2(\bullet, u_2)$ are Lebesgue measurable in $\mathbb{R}^N$, for every $u_1(x)$ or $u_2(x)$ in $\mathbb{R}$ and the functions $f_1(x, \bullet)$, $f_2(x, \bullet)$ are continuous in $\mathbb{R}$ for almost every $x \in \mathbb{R}^N$. Moreover, f_1, f_2 are such that

$$(i) \qquad \forall u_1, u_2 \in L^2(\mathbb{R}^N),\ f_1(x,u_1), f_2(x,u_2) \in L^2(R^N),$$

$$(ii) \qquad \exists\,\kappa > 0\ s.t.\ f_1(x,u_1), f_2(x,u_2) \geq \kappa\phi(x)\ \forall u_1, u_2 \in L^2(\mathbb{R}^N),$$

$$(iii) \qquad \exists K > \kappa > 0\ s.t.\ f_1(x,u_1), f_2(x,u_2) \leq K\phi(x)\ \forall u_1, u_2 \in L^2(\mathbb{R}^N).$$

(H'_F) : $\frac{f_1(x,u_1)}{|u_1|}$ and $\frac{f_2(x,u_2)}{|u_2|}$ are decreasing w.r.t. u_1 and u_2.

We introduce 2 sets :

$$\mathcal{K_S}^+ := \left\{ (u_1, u_2) \in X^2 \,/\, u_1 \in \left(\frac{\kappa y_1 \phi}{\max(y_1,y_2)(\Lambda^* - \mu)}\,, \frac{K y_1 \phi}{\min(y_1,y_2)(\Lambda^* - \mu)} \right), \right.$$

$$\left. u_2 \in \left(\frac{\kappa y_2 \phi}{\max(y_1,y_2)(\Lambda^* - \mu)}\,, \frac{K y_2 \phi}{\min(y_1,y_2)(\Lambda^* - \mu)} \right) \right\}$$

for $\mu < \Lambda^*$, and

$$\mathcal{K}_{\mathcal{S}}{}^{-} := \left\{ (u_1, u_2) \in X^2 \,/\, u_1 \in \left(\frac{K y_1 \phi}{\min(y_1, y_2)(\Lambda^* - \mu)}, \frac{\kappa y_1 \phi}{\max(y_1, y_2)(\Lambda^* - \mu)} \right), \right.$$
$$\left. u_2 \in \left(\frac{K y_2 \phi}{\min(y_1, y_2)(\Lambda^* - \mu)}, \frac{\kappa y_2 \phi}{\max(y_1, y_2)(\Lambda^* - \mu)} \right) \right\}$$

for $\Lambda^* < \mu$.

Theorem 3 *If (H_A) and (H_F) are satisfied there exists $\delta > 0$, depending on f_1 and f_2 such that if $\Lambda^* - \delta < \mu < \Lambda^*$ (resp. $\Lambda^* < \mu < \Lambda^* + \delta$), (with $\delta < min\{\frac{\xi_2 - \xi_1}{2}, \lambda_2 - \Lambda\}$) System (S) has a solution which is in $\mathcal{K}_{\mathcal{S}}^{+}$, (resp. in $\mathcal{K}_{\mathcal{S}}^{-}$). Moreover, if (H_F') is satisfied, the solution is unique.*

Proof of Theorem 3 We use of course the results above as well as previous results for linear systems obtained in [5] where Theorem 3 is shown for suitable assumptions on f_1 and f_2 (independent on u).

1. Maximun and anti-maximum principles

We diagonalize System(S) thanks to the change of basis matrix P, and we get a system of 2 equations. Here

$$P = \begin{pmatrix} b & b \\ \xi_1 - a & \xi_2 - a \end{pmatrix}, \; P^{-1} = \frac{1}{b(\xi_1 - \xi_2)} \begin{pmatrix} a - \xi_2 & b \\ \xi_1 - a & -b \end{pmatrix},$$

Set

$$D := P^{-1}AP = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}; \; U = PV\,; \; G := P^{-1}F. \tag{12}$$

We obtain

$$LV = DV + \mu V + G \tag{13}$$

which is a system of 2 equations (with obvious notation):

$$Lv_1 = (\xi_1 + \mu)v_1 + g_1(u_1, u_2);$$

$$Lv_2 = (\xi_2 + \mu)v_2 + g_2(u_1, u_2).$$

Note that g_1 and g_2 are in X.

The second equation, where the parameter $\xi_2 + \mu$ stays away (below) from Λ, has a ϕ bounded solution v_2. Concerning the first equation, we apply Theorem 2 above. We compute g_1, g_2 and get (*ii'*) $$\exists\,\kappa' > 0 \; s.t. \; g_1(x,u_1,u_2) \;\geq\; \kappa'\phi(x) \;\forall u_1, u_2 \;\in\; L^2(\mathbb{R}^N),$$

(*iii'*) $$\exists K' > \kappa' > 0 \; s.t. \; g_1(x,u_1,u_2),\, |g_2(x,u_1,u_2)| \;\leq\; K'\phi(x) \;\forall u_1, u_2 \in L^2(\mathbb{R}^N),$$

where κ' and K' are 2 positive constants depending on κ, K and on the coefficients of A. This follows from $\xi_1 - \xi_2 > 0$ and $(a-\xi_2) = \frac{a-d}{2} + \frac{\sqrt{(a-d)^2+4bc}}{2}$ with $(a-d)^2 + 4bc > (a-d)^2$, so that

$$g_1 = \frac{1}{\xi_1 - \xi_2}[(a-\xi_2)f_1 + bf_2] > \kappa'\phi > 0.$$

Analoguously we have $g_1 < K'\phi$. Therefore Theorem 2 holds here with $\delta = \min(\delta_0, \frac{\kappa'}{c_0 K'}, \frac{\xi_1-\xi_2}{2})$. Finally we deduce from the maximum principle for $\Lambda^* - \delta < \mu < \Lambda^*$ that $v_1 > \frac{\kappa'}{\Lambda^*-\mu}\phi > 0$.

If $\Lambda^* < \mu < \Lambda^* + \delta$, reasoning similarly, we deduce $v_1 < \frac{K'}{\Lambda^*-\mu}\phi < 0$. As $\mu \to \Lambda^*$, v_1 tends to ∞ when v_2 stays bounded. Indeed, by Remark 3,

$$\|v_2\|_X \leq \frac{K'}{|\Lambda - \xi_2 - \mu|} + 2c_0K' < \frac{2K'}{\xi_1 - \xi_2} + 2c_0K';$$

the last inequality follows from $\delta < \frac{\xi_1-\xi_2}{2}$.

Now we go back to $U = PV$.

$$u_1 = av_1 + bv_2\,,\; u_2 = (\xi_1 - a)v_1 + (\xi_2 - a)v_2.$$

Combining the estimates above on v_1 and v_2, we conclude that, as $|\Lambda^* - \mu| \to 0$, there exists δ^*, depending only on L, A, κ, K such that as $\mu \nearrow \Lambda^*$, u_1 has the sign of $a > 0$ and $u_2 > 0$. If $\mu \searrow \Lambda^*$, u_1 has the sign of $-a < 0$ and $u_2 < 0$.

2. Existence of the solution in $\mathcal{K}^+_{\mathcal{S}}$, (resp. in $\mathcal{K}^-_{\mathcal{S}}$)

Sub-supersolutions:

1. Case $\Lambda^* - \delta^* < \mu < \Lambda^*$. Recall that Y has positive components y_1 and y_2, and the principal eigenvector $\Phi = Y\phi$ satisfies

$$L\Phi - \mu\Phi - A\Phi = (\Lambda^* - \mu)\Phi.$$

Inspired by the case of one equation, we seek a subsolution U_0 of the form $cY\Phi$.

$$L(U - U_0) = A(U - U_0) + \mu(U - U_0) + (F(x,U) - (\Lambda^* - \mu)c\Phi).$$

For c such that $F(x,U) - (\Lambda^* - \mu)cY\phi(x) > 0$, for $\mu < \Lambda^*$, we get $U - U_0 > 0$ by the maximum principle. Finally, since $F(x,U) - \dfrac{\kappa}{\max(y_1,y_2)}Y\phi > 0$, a subsolution is

$$U_0 = \frac{\kappa}{\max(y_1,y_2)}\frac{1}{(\Lambda^* - \mu)}Y\phi.$$

Analogously $U^0 = \dfrac{K}{\min(y_1, y_2)(\Lambda^* - \mu)} Y\phi$ is a supersolution.

2. Case $\Lambda^* < \mu < \Lambda^* + \delta^*$. We have similar results with change of sign and replacing K by κ.

$$V_0 = \frac{K}{\min(y_1, y_2)(\Lambda^* - \mu)} Y\phi$$

$$V^0 = \frac{\kappa}{\max(y_1, y_2)} \frac{1}{(\Lambda^* - \mu)} Y\phi$$

The operator T: We define $T : (u_1, u_2) \longrightarrow (w_1, w_2)$ where (w_1, w_2) is the solution to the linear system

$$(S') \qquad \begin{cases} Lw_1 &= (a+\mu)w_1 + bw_2 + f_1(x, u_1) \\ Lw_2 &= cw_1 + (d+\mu)w_2 + f_2(x, u_2) \end{cases} \quad in\ \mathbb{R}^N, .$$

$$w_1(x), w_2(x)_{|x| \to \infty} \to 0.$$

The rectangle: If $(u_1, u_2) \in \mathcal{K}^+_{\mathcal{S}}$ for $\Lambda^* - \delta^* < \mu < \Lambda^*$ (resp. $(u_1, u_2) \in \mathcal{K}^-_{\mathcal{S}}$ for $\Lambda^* < \mu < \Lambda^* + \delta^*$) then $(w_1, w_2) \in \mathcal{K}^+_{\mathcal{S}}$ (resp $\mathcal{K}^-_{\mathcal{S}}$). Indeed, for $\Lambda^* - \delta^* \mu < \Lambda^*$, this can be written with obvious notations

$$L(W - U_0) = (\mu + A)(W - U_0) + F;$$

for $\mu < \Lambda^*$, since F has non negative components, $F \not\equiv 0$, then $W - U_0 > 0$. Analogously, we obtain the supersolution $U^0 - W > 0$.

We argue exactly as for one equation: $\mathcal{K}^+_{\mathcal{S}}$ or $\mathcal{K}^-_{\mathcal{S}}$ is invariant by T and $LW = (A+\mu)W + F(x, U)$ can be written $W = (L - A - \mu I)^{-1} \hat{F}(x, u) = T(U)$. Since by [10], [9], the resolvent $R(\mu) := (L - \mu I)^{-1}$ is compact in X for $\mu \in (\Lambda^* - \delta^*, \Lambda^*)$ or $(\Lambda^*, \Lambda^* + \delta^*)$, and since $\hat{F} : u \to F(x, u)$ is continuous, $T = R(\mu)\hat{F}$ is compact.

We apply the fixed point theorem. There exists a solution U.

3. Uniqueness

We assume now (H'_F). assume there are 2 positive solutions (u_1, u_2) and (v_1, v_2) to (S); for the first equation we have $Lu_1 = (\mu + a)u_1 + bu_2 + f_1(x, u_1)$ and $Lv_1 = (\mu + a)v_1 + bv_2 + f_1(x, v_2)$. Since we are in $\mathcal{K}^+$ (resp. $\mathcal{K}^-$), divide by bu_1 the first equation and by bv_1 the second one and subtract:

$$\frac{-\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} = \frac{u_2}{u_1} - \frac{v_2}{v_1} + \frac{f_1(x, u_1)}{bu_1} - \frac{f_1(x, v_1)}{bv_1}. \tag{14}$$

Exactly as in [13] multiply by $(u_1^2 - v_1^2)$ and integrate; hence

$$\int \left(\frac{-\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} \right) (u_1^2 - v_1^2) = \int \left(\frac{u_2}{u_1} - \frac{v_2}{v_1} + \frac{f_1(x, u_1)}{bu_1} - \frac{f_1(x, v_1)}{bv_1} \right) (u_1^2 - v_1^2).$$

The first terme is non-negative by (7):

$$\int \left(\frac{-\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} \right) (u_1^2 - v_1^2) > 0.$$

Then do exactly the same calculus with the second equation in (S) and add these two lines: we derive from (14) that $T_1 = T_2$ with

$$T_1 = \int \left(\frac{-\Delta u_1}{bu_1} + \frac{\Delta v_1}{bv_1} \right) (u_1^2 - v_1^2) + \int \left(\frac{-\Delta u_2}{cu_2} + \frac{\Delta v_2}{cv_2} \right) (u_2^2 - v_2^2).$$

$$T_2 = \int \left(\frac{u_2}{u_1} - \frac{v_2}{v_1} + \frac{f_1(x,u_1)}{bu_1} - \frac{f_1(x,v_1)}{bv_1} \right) (u_1^2 - v_1^2) + \int \left(\frac{u_1}{u_2} - \frac{v_1}{v_2} + \frac{f_2(x,u_2)}{cu_2} - \frac{f_2(x,v_2)}{cv_2} \right) (u_2^2 - v_2^2).$$

Of course the 1st term T_1 is non-negative by (7). By (H'_F),

$$\int \left(\frac{f(x,u_1)}{bu_1} - \frac{f_1(x,v_1)}{bv_1} \right) (u_1^2 - v_1^2) + \int \left(\frac{f_2(x,u_2)}{cu_1} - \frac{f_2(x,v_2)}{cv_1} \right) (u_2^2 - v_2^2) < 0.$$

We develop what is left and get

$$\int \left(\frac{u_2}{u_1} - \frac{v_2}{v_1} \right) (u_1^2 - v_1^2) + \int \left(\frac{u_1}{u_2} - \frac{v_1}{v_2} \right) (u_2^2 - v_2^2) = - \int \left(\sqrt{\frac{u_2 v_1^2}{u_1}} - \sqrt{\frac{u_1 v_2^2}{u_2}} \right)^2 - \int \left(\sqrt{\frac{v_2 u_1^2}{v_1}} - \sqrt{\frac{v_1 u_2^2}{v_2}} \right)^2 < 0$$

Hence $T_1 = T_2 = 0$ and $u_1 = v_1, u_2 = v_2$. The solution is unique.

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Authors:

Bénédicte Alziary
Institut de Mathématiques
de Toulouse CeReMath, TSE,
Univ. Toulouse 1,

e-mail: alziary@ut-capitole.fr

Jacqueline Fleckinger
Institut de Mathématiques
de Toulouse CeReMath,
Univ. Toulouse 1,
address: 21 allée de Brienne
31042 Toulouse Cedex

e-mail: jfleckinger@gmail.com