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Studies on boundary value problems of singular fractional differential equations with impulse effects*

ABSTRACT. In this paper, we firstly prove existence and uniqueness of solutions of Cauchy problems for nonlinear fractional differential equations involving the Caputo fractional derivative, the Riemann-Liouville derivative, the Caputo type Hadamard derivative and the Riemann-Liouville type Hadamard fractional derivatives of order $\alpha \in [n-1, n)$ respectively by using Picard iterative technique under some suitable assumptions. Meanwhile, we get the iterative approximation solutions of these kind of Cauchy problems. Secondly we obtain exact expression of piecewise continuous solutions of the linear fractional differential equations. These results provide new methods to convert an impulsive fractional differential equation to a fractional integral equation. Thirdly, four classes of boundary value problems for singular fractional differential equations with impulse effects are proposed. Sufficient conditions are given for the existence of solutions of these problems. We allow the nonlinearity $p(t)f(t,x)$ in fractional differential equations to be singular at $t = 0, 1$. Finally, by establishing existence results on solvability of two class of impulsive boundary value problems of fractional differential equations, we make a comparison on impulsive boundary value problems for two kinds of fractional differential equations, one has a single starting point and the other one has multiple starting points. In order to avoid misleading the readers, a mistake in [Impulsive integral boundary value problems of the higher-order fractional differential equation with eigenvalue arguments, Adv. Differ. Equa. (2015) 2015: 382] is also corrected.

KEY WORDS. higher order singular fractional differential equation, iterative approximation solution, impulsive boundary value problem, Riemann-Liouville fractional derivative, Caputo fractional derivative, Riemann-Liouville type Hadamard fractional derivative, Caputo type Hadamard fractional derivative, fixed point theorem.

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Contents

1 Introduction

One knows that the fractional derivatives (Riemann-Liouville fractional derivative, Caputo fractional derivative and Hadamard fractional derivative and other type see [44]) are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory.

Fractional order differential equations are generalizations of integer order differential equations. Using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. Fractional differential equations have many applications see Chapter 10 in [71], books [45, 71, 76].

In recent years, there have been many results obtained on the existence and uniqueness of solutions of initial value problems or boundary value problems for nonlinear fractional differential equations, see [17, 19, 55, 65, 69, 70, 72, 83, 102, 109].

Dynamics of many evolutionary processes from various fields such as population dynamics, control theory, physics, biology, and medicine undergo abrupt changes at certain moments of time like earthquake, harvesting, shock, and so forth. These perturbations can be well approximated as instantaneous change of states or impulses. These processes are modeled by impulsive differential equations. In 1960, Milman and Myshkis introduced impulsive differential equations in their paper [63]. Based on their work, several monographs have been published by many authors like Samoilenko and Perestyuk [77], Lakshmikantham et al. [56], Bainov and Simeonov [22, 23], Bainov and Covachev [24], and Benchohra et al. [25].

Fractional differential equation was extended to impulsive fractional differential equations, since Agarwal and Benchohra published the first paper on the topic [3] in 2008. Since then many authors [1, 29, 32, 43, 47, 48, 52–54, 68, 72, 82, 83, 100] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems see [3, 10, 11, 49, 85], impulsive periodic boundary value problems see [18, 81, 93], impulsive initial value problems see [26, 31, 67, 78], two-point, three-point or multi-point impulsive boundary value problems see [9, 36, 84, 92, 106, 111], impulsive boundary value problems on infinite intervals see [105].

In [33], Feckan and Zhou pointed out that the formula of solutions for impulsive fractional differential equations in [2, 8, 14, 21] is incorrect and gave their correct formula. In [90, 92], the authors established a general framework to find the solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of the solutions to a kind of impulsive fractional differential equations respectively. In [88], the authors illustrated their comprehension for the counterexample in [33] and criticized the viewpoint in [33, 90, 92]. Next, in [34], Feckan et al. expounded for the counterexample in [33] and provided further five explanations in the paper.

In a fractional differential equation, there exist two cases concerning the derivatives: the first case is $D^\alpha = D^\alpha_{0+}$, i.e., the fractional derivative has a single start point $t = 0$. The other case is $D^\alpha = D^\alpha_{t_i^+}$, i.e., the fractional derivative has a multiple start points $t = t_i (i \in \mathbb{N}[0, m])$.

There have been many authors concerning the existence and uniqueness of solutions of boundary value problems of impulsive fractional differential equations with multiple start points $t = t_i (i \in \mathbb{N}[0, m])$.

Recently, Wang [80] consider the second case in which D^α has multiple start points, i.e., $D^\alpha = D^\alpha_{t_i^+}$. They studied the existence and uniqueness of solutions of the following initial value problem of the impulsive fractional differential equation

$$\begin{cases} {}^C D^\alpha_{t_i^+} u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u_j, j \in \mathbb{N}[0, n-1], \\ \Delta u^{(j)}(t_i)] = I_{ji}(u(t_i)), i \in \mathbb{N}[1, p], j \in [0, n-1], \end{cases} \tag{1.0.1}$$

where $\alpha \in (n-1, n)$ with n being a positive integer, ${}^C D^\alpha_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\mathbb{N}[a, b] = \{a, a+1, \cdots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_{ji} \in C(\mathbb{R}, \mathbb{R})(i \in \mathbb{N}[1, p], j \in \mathbb{N}[0, n-1])$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Henderson and Ouahab [41] studied the existence of solutions of the following initial value problems and periodic boundary value problems of impulsive fractional differential equations:

$$\begin{cases} {}^C D^\alpha_{t_i^+} u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u_j, j \in \mathbb{N}[0, 1], \\ u^{(j)}(t_i)] = I_{ji}(u(t_i)), i \in \mathbb{N}[1, p], j \in \mathbb{N}[0, 1], \end{cases}$$

and

$$\begin{cases} {}^C D^\alpha_{t_i^+} u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, p], \\ u^{(j)}(0) = u^{(j)}(b), j \in \mathbb{N}[0, 1], \\ u^{(j)}(t_i)] = I_{ji}(u(t_i)), i \in \mathbb{N}[1, p], j \in \mathbb{N}[0, 1], \end{cases}$$

where $\alpha \in (1, 2]$, $b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = b$, $f : [0, b] \times \mathbb{R} \to \mathbb{R}$, $I_{ji} : \mathbb{R} \to \mathbb{R}$ are continuous functions. Readers should also refer [89].

In [104], Zhao and Gong studied existence of positive solutions of the following nonlinear impulsive fractional differential equation with generalized periodic boundary value conditions

$$\begin{cases} {}^C D^q_{t_i^+} u(t) = f(t, u(t)), t \in (0, T] \setminus \{t_1, \cdots, t_p\}, \\ \Delta u(t_i)] = I_i(u(t_i)), i \in \mathbb{N}[1, p], \\ \Delta u'(t_i)] = J_i u(t_i)), i \in \mathbb{N}[1, p], \\ \alpha u(0) - \beta u(1) = 0, \ \alpha u'(0) - \beta u'(1) = 0, \end{cases} \tag{1.0.2}$$

where $q \in (1,2)$, ${}^C D^q_{t_i^+}$ represents the standard Caputo fractional derivatives of order q, $\alpha > \beta > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $\mathbb{N}[a,b] = \{a, a+1, \cdots, b\}$ with a, b being integers, $I_i, J_i \in C([0,+\infty),[0,+\infty))(i \in \mathbb{N}[1,p]$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Wang, Ahmad and Zhang [87] studied the existence and uniqueness of solutions of the following periodic boundary value problems for nonlinear impulsive fractional differential equation

$$\begin{cases} {}^C D^\alpha_{t_i^+} u(t) = f(t, u(t)), t \in (0,T] \setminus \{t_1, \cdots, t_p\}, \\ \Delta u(t_i)] = I_i(u(t_i)), i \in \mathbb{N}[1,p], \\ \Delta u'(t_i)] = I_i^*(u(t_i)), i \in \mathbb{N}[1,p], \\ u'(0) + (-1)^\theta u(T) = bu(T),\ u(0) + (-1)^\theta u(T) = 0, \end{cases} \tag{1.0.3}$$

where $\alpha \in (1,2)$, ${}^C D^\alpha_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\theta = 1, 2$, $\mathbb{N}[a,b] = \{a, a+1, \cdots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $I_i, I_i^* \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}[1,p]$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [9, 10, 99, 107], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^C D^\alpha_{t_i^+} x(t) = w(t) f(t, x(t), x'(t)), t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}[1,p], \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}[1,p], \\ ax(0) \pm bx'(0) = g_1(x),\ cx(1) + dx'(1) = g_2(x), \end{cases} \tag{1.0.4}$$

where $\alpha \in (1,2)$, ${}^C D^\alpha_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $a, b, c, d \geq 0$ with $ac + ad + bc \neq 0$, $\mathbb{N}[m,n] = \{m, m+1, \cdots, n\}$ with m, n being integers, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}[1,p]$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $w : [0,1] \to [0,+\infty)$ is a continuous function, $g_1, g_2 : PC(0,1] \to \mathbb{R}$ are two continuous functions.

In 2015, Zhou, Liu and Zhang [108] studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^C D^\alpha_{t_i^+} x(t) = \lambda x(t) + f(t, x(t), (Kx)(t), (Hx)(t)), t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}[1,p], \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}[1,p], \\ ax(0) - bx'(0) = x_0,\ cx(1) + dx'(1) = x_1, \end{cases} \tag{1.0.5}$$

where $\alpha \in (1,2)$, ${}^C D^\alpha_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $a \geq 0, b > 0, c \geq 0, d > 0$ with $\delta = ac+ad+bc \neq 0$, $\lambda > 0$, $x_0, x_1 \in \mathbb{R}$, $\mathbb{N}[m,n] = \{m, m+1, \cdots, n\}$ with m, n being integers, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}[1,p]$, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $(Hx)(t) = \int_0^1 h(t,s)x(s)ds$ and $(Kx)(t) = \int_0^t k(t,s)x(s)ds$.

In [50, 57], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^{C}D^{\alpha}_{t_i^+}x(t) = f(t, x(t)), t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}[1,p], \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}[1,p], \\ ax(0) + bx(1) = g_1(x),\ ax'(0) + bx'(1) = g_2(x), \end{cases}$$

where $\alpha \in (1,2)$, ${}^{C}D^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $a, b \in \mathbb{R}$ with $a \geq b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}[1,p]$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $g_1, g_2 : PC(0,1] \to \mathbb{R}$ are two continuous functions.

In [52], Liu and Li investigated the existence and uniqueness of solutions for the following nonlinear impulsive fractional differential equations

$$\begin{cases} {}^{C}D^{\alpha}_{t_i^+}u(t) = f(t, u(t), u'(t), u''(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,p], \\ u(0) = \lambda_1 u(T) + \xi_1 \int_0^T q_1(s, u(s), u'(s), u''(s))ds, \\ u'(0) = \lambda_2 u'(T) + \xi_2 \int_0^T q_2(s, u(s), u'(s), u''(s))ds, \\ u''(0) = \lambda_3 u''(T) + \xi_3 \int_0^T q_3(s, u(s), u'(s), u''(s))ds, \\ \Delta u(t_i)] = A_i(u(t_i)), i \in \mathbb{N}[1,p], \\ \Delta u'(t_i)] = B_i(u(t_i)), i \in \mathbb{N}[1,p], \\ \Delta u''(t_i)] = C_i(u(t_i)), i \in \mathbb{N}[1,p], \end{cases} \tag{1.0.6}$$

where $\alpha \in (2,3)$, ${}^{C}D^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\mathbb{N}[a,b] = \{a, a+1, \cdots, b\}$ with a, b being integers, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $\lambda_i, \xi_i \in \mathbb{R}(i = 1,2,3)$ are constants, $A_i, B_i, C_i \in C(\mathbb{R},\mathbb{R})(i \in \mathbb{N}[1,p]$, $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

Recently, in [20], to extend the problem for impulsive differential equation $u''(t) - \lambda u(t) = f(t, u(t)), u(0) = u(T) = 0, \Delta u'(t_i) = I_i(u(t_i)), i = 1, 2, \cdots, p$ to impulsive fractional differential equation, the authors studied the existence and the multiplicity of solutions for the Dirichlet's boundary value problem for impulsive fractional order differential equation

$$\begin{cases} {}^{C}D^{\alpha}_{T^-}({}^{C}D^{\alpha}_{0^+}x(t) + a(t)x(t) = \lambda f(t, x(t)), t \in [0,T], t \neq t_i, i \in \mathbb{N}[1,m], \\ \Delta {}^{C}D^{\alpha-1}_{T^-}({}^{C}D^{\alpha}_{0^+}x(t_i) = \mu I_i(x(t_i^-)), i \in \mathbb{N}[1,m],\ x(0) = x(T) = 0, \end{cases} \tag{1.0.7}$$

where $\alpha \in (1/2, 1]$, $\lambda, \mu > 0$ are constants, $\mathbb{N}[a,b] =: \{a, a+1, \cdots, b\}$ with $a \leq b$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_i : \mathbb{R} \to \mathbb{R}(i \in \mathbb{N}[1,m])$ are continuous functions, ${}^{C}D^{\alpha}_{0^+}$ (or ${}^{C}D^{\alpha}_{T^-}$) is the standard left (or right) Caputo fractional derivative of order α, $a \in C[0,T]$ and there exist constants $a_1, a_2 > 0$ such that $a_1 \leq a(t) \leq a_2$ for all $t \in [0,T]$, $\Delta x|_{t=t_i} = \lim_{t \to t_i^+} x(t) - \lim_{t \to t_i^-} x(t) = x(t_i^+) - x(t_i^-)$ and $x(t_i^+), x(t_i^-)$ represent the right and left limits of $x(t)$ at $t = t_i$ respectively, a, b, x_0 a constant with $a + b \neq 0$. One knows that the boundary condition $ax(0) + bx(T) = x_0$ becomes $x(0) - x(T) = \frac{x_0}{a}$ when $a + b = 0$, that is so called nonhomogeneous periodic type boundary condition.

For impulsive fractional differential equations whose derivatives have single start points $t = 0$, there has been few papers published. In [73], authors presented a new method to converting the impulsive fractional differential equation (with the Caputo fractional derivative) to an equivalent integral equation and established existence and uniqueness results for some boundary value problems of impulsive fractional differential equations involving the Caputo fractional derivatives with single start point. The existence and uniqueness of solutions of the following initial or boundary value problems were discussed in [73]:

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)),\ \Delta x'(t_i) = J_i(x(t_i)),\ i \in \mathbb{N}[1, p], \\ x(0) = x_0,\ x'(0) = x_1, \end{cases}$$

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)),\ \Delta x'(t_i) = J_i(x(t_i)),\ i \in \mathbb{N}[1, p], \\ x(0) + \phi(x) = x_0,\ x'(0) = x_1, \end{cases}$$

$$\begin{cases} {}^C D_{0^+}^{\beta} x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)),\ i \in \mathbb{N}[1, p], \\ ax(0) + bx(1) = 0, \end{cases}$$

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)),\ \Delta x'(t_i) = J_i(x(t_i)),\ i \in \mathbb{N}[1, p], \\ ax(0) - bx'(0) = x_0,\ cx(1) + dx'(1) = x_1, \end{cases}$$

and

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)),\ \Delta x'(t_i) = J_i(x(t_i)),\ i \in \mathbb{N}[1, p], \\ x(0) - ax(\xi) = x(1) - bx(\eta) = 0, \end{cases}$$

where $\alpha \in (1, 2], \beta \in (0, 1]$, $D_{0^+}^{*}$ is the Caputo fractional derivative with order $*$ and single start point $t = 0$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $I_i, J_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $a, b, c, d, x_0, x_1 \in \mathbb{R}$ are constants, $\phi : PC(0, 1] \to \mathbb{R}$ is a functional.

In [104], authors studied the existence of positive solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i)] = I_i(x(t_i)), i \in \mathbb{N}[1, p], \\ \Delta x'(t_i)] = J_i(x(t_i)), i \in \mathbb{N}[1, p], \\ ax(0) - bx(1) = 0,\ ax'(0) - bx'(1) = 0, \end{cases}$$

where $\alpha \in (1,2)$, ${}^{C}D_{0^+}^{\alpha}$ represents the standard Caputo fractional derivatives of order α, $a > b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R}, \mathbb{R})(i \in \mathbb{N}[1,p]$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

We firstly observed that in the above-mentioned work, the authors all require that the fractional derivatives are the Caputo type derivatives, the nonlinear term f and the impulse functions are continuous. It is easy to see that these conditions are very strongly restrictive and difficult to satisfy in applications. To the author's knowledge, there has been no paper published discussed the existence of solutions of boundary value problems of impulsive fractional differential equations involving other fractional derivatives such as the Riemann-Liouville fractional derivatives, Hadamard fractional derivatives.

Secondly, we known (see Theorem 3.1, Theorem 3.6 in [44]) under the assumption $f(t,x) \in L(0,1)$ for any $x \in \mathbb{R}$ that $x \in L(0,,1)$ satisfies a.e. the equation

$$D_{0^+}^{\alpha}x(t) = f(t, x(t)),\ a.e.,\ t \in (0,1),$$

$$D_{0^+}^{\alpha-j}x(0) = b_j,\ j \in \mathbb{N}[1, -[-\alpha]]$$

if and only if x satisfies

$$x(t) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha-j+1)} t^{\alpha-j} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds,\ a.e.,\ t \in [0,1].$$

When we consider the following initial value problem

$${}^{RL}D_{0^+}^{\frac{3}{2}}x(t) = t^{-\frac{1}{8}}(1-t)^{-\frac{9}{8}} =: f(t),\ a.e., t \in (0,1),\ \lim_{t \to 0^+} t^{\frac{1}{2}}x(t) = {}^{RL}D_{0^+}^{\frac{1}{2}}x(0) = 0. \quad \text{(IVP)}$$

Let

$$x_0(t) = \int_0^t \frac{(t-s)^{\frac{3}{2}-1}}{\Gamma(3/2)} f(s)ds = \int_0^t \frac{(t-s)^{\frac{3}{2}-1}}{\Gamma(3/2)} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}}ds, t \in (0,1).$$

Then

$${}^{RL}D_{0^+}^{\frac{1}{2}}x_0(t) = \int_0^t (t-s)^{\frac{3}{2}-\frac{1}{2}-1} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}}ds.$$

It is easy to check that

$$\int_0^t (t-s)^{\frac{3}{2}-1} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}}ds \le \int_0^t (t-s)^{-\frac{5}{8}} s^{-\frac{1}{8}} = t^{\frac{1}{4}}\mathbf{B}(3/8, 7/8), t \in [0,1],$$

$$\int_0^t (t-s)^{\frac{3}{2}-\frac{1}{2}-1} s^{-\frac{1}{8}}(1-s)^{-\frac{9}{8}}ds \le (1-t)^{-\frac{9}{8}} \int_0^t s^{-\frac{1}{8}} = (1-t)^{-\frac{9}{8}} \frac{8t^{\frac{7}{8}}}{7}, t \in [0,1).$$

So $\lim_{t\to 0^+} t^{\frac{1}{2}}x_0(t) = {}^{RL}D_{0^+}^{\frac{1}{2}}x_0(0) = 0$. Further more, we have

$$^{RL}D_{0^+}^{\frac{3}{2}}x_0(t) = \frac{1}{\Gamma(2-3/2)}\left[\int_0^t (t-s)^{2-3/2-1}\int_0^s \frac{(s-u)^{\frac{3}{2}-1}}{\Gamma(3/2)}f(u)duds\right]''$$

$$= \frac{1}{\Gamma(1/2)}\left[\int_0^t \int_u^t (t-s)^{-1/2}\frac{(s-u)^{1/2}}{\Gamma(3/2)}dsf(u)du\right]'' \text{ by interchanging the integral order}$$

$$= \frac{1}{\Gamma(1/2)}\left[\int_0^t (t-u)\int_0^1 (1-w)^{-1/2}\frac{w^{1/2}}{\Gamma(3/2)}dwf(u)du\right]'' \text{ by } \frac{s-u}{t-u} = w$$

$$= \left[\int_0^t (t-u)f(u)du\right]'' = f(t) \text{ by } \mathbf{B}(1/2,3/2) = \Gamma(1/2)\Gamma(3/2).$$

It follows that problem (IVP) has a unique solution x_0 in $C^0[0,1]$. However $f(t)$ is not integral on $[0,1]$ ($f \notin L(0,1)$). Hence it is interesting to investigate the existence and uniqueness of solutions of initial value problems of fractional differential equations under the assumptions the nonlinearity $f(t,x)$ is not integral for $x \in \mathbb{R}$.

In this paper, we firstly establish existence and uniqueness results for initial value problems of fractional differential equations under the assumption that the nonlinearity $f(t,x)$ is not integral for $x \in \mathbb{R}$. Then we obtain piecewise continuous solutions of fractional differential equations. Thirdly, we study the existence of solutions of four classes of impulsive boundary value problems of singular fractional differential equations. The first one is the impulsive mixed type integral boundary value problem as follows:

$$\begin{cases} {}^CD_{0^+}^{\beta}x(t) = p(t)f(t,x(t)), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m], \\ \lim_{t\to 0^+} x(t) = \int_0^1 \phi(s)G(s,x(s))ds, \\ x'(1) = \int_0^1 \psi(s)H(s,x(s))ds, \\ \Delta x(t_i) = I(t_i,x(t_i)), i \in \mathbb{N}[1,m], \\ \Delta x'(t_i) = J(t_i,x(t_i)), i \in \mathbb{N}[1,m], \end{cases} \tag{1.0.8}$$

where

(a) $1 < \beta < 2$, ${}^CD_{0^+}^{\beta}$ is the Caputo fractional derivative of order β with starting point 0,

(b) m is a positive integer, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, $\mathbb{N}[a,b] = \{a, a+1, a+2, \cdots, a+n\}$ with a, b being integers and $a \le b$,

(c) $\phi, \psi : (0,1) \to \mathbb{R}$ are measurable functions,

(d) $p : (0,1) \to \mathbb{R}$ is continuous and there exist numbers $k > 1-\beta$ and $l \in \max\{-\beta, -\beta - k, 0]$ such that $|p(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$,

(e) f, G, H defined on $(0,1] \times \mathbb{R}$ are **impulsive I-Carathéodory functions**, $I, J : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ is a **Discrete I-Carathéodory functions**.

The second one is the impulsive Dirichlet type integral boundary value problem as follows:

$$\begin{cases} {}^{RL}D_{0^+}^{\beta}x(t) = q(t)f(t,x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ \lim\limits_{t\to 0^+} t^{2-\beta}x(t) = \int_0^1 \phi(s)G(s,x(s))ds, \\ x(1) = \int_0^1 \psi(s)H(s,x(s))ds, \\ \lim\limits_{t\to t_i^+} (t-t_i)^{2-\beta}x(t) = I(t_i,x(t_i)), i \in \mathbb{N}[1,m], \\ \Delta^{RL}D_{0^+}^{\beta-1}x(t_i) = J(t_i,x(t_i)), i \in \mathbb{N}[1,m], \end{cases} \tag{1.0.9}$$

where

(f) $1 < \beta < 2$, ${}^{RL}D_{0+}^{\beta}$ is the Riemann-Liouville fractional derivative of order β with starting point 0, $n, t_i, \mathbb{N}[a,b]$ satisfies (b), $\phi, \psi : (0,1) \to \mathbb{R}$ satisfy (c),

(g) $q : (0,1) \to \mathbb{R}$ is continuous and there exist numbers $k > -1$ and $l \in \max\{-\beta, -\beta-k, 0]$ such that $|q(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$,

(h) f, G, H defined on $(0,1] \times \mathbb{R}$ are **impulsive II-Carathéodory functions**, $I, J : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ are **Discrete II-Carathéodory functions**.

We emphasize that much work on fractional boundary value problems involves either Riemann-Liouville or Caputo type fractional differential equations see [5–7, 11]. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [38], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Recent studies can be seen in [13, 15, 16].

Thirdly we study the following impulsive anti-periodic type integral boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{RLH}D_{1^+}^{\beta}x(t) = q(t)f(t,x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ \lim\limits_{t\to 1^+} (\log t)^{2-\beta}x(t) + x(e) = \int_0^1 \phi(s)G(s,x(s))ds, \\ \lim\limits_{t\to 1^+} {}^{RLH}D_{1^+}^{\beta-1}x(t) + {}^{RLH}D_{1^+}^{\beta-1}x(e) = \int_0^1 \psi(s)H(s,x(s))ds, \\ \lim\limits_{t\to t_i^+} (\log t - \log t_i)^{2-\beta}x(t) = I(t_i,x(t_i)), i \in \mathbb{N}[1,m], \\ \Delta^{RL}D_{1^+}^{\beta-1}x(t_i) = J(t_i,x(t_i)), i \in \mathbb{N}[1,m], \end{cases} \tag{1.0.10}$$

where

(i) $1 < \beta < 2$, ${}^{RLH}D_{1+}^{\beta}$ is the Hadamard fractional derivative of order β with starting point 1,

(j) m is a positive integer, $1 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = e$, $\phi, \psi : (1,e) \to \mathbb{R}$ are measurable functions, $q : (1,e) \to \mathbb{R}$ is continuous and satisfies that there exist numbers $k > -1$ and $l \in \max\{-\beta, -\beta - k, 0]$ such that $|q(t)| \le (\log t)^k (1 - \log t)^l$ for all $t \in (1,e)$,

(k) f, G, H defined on $(1,e] \times \mathbb{R}$ are **impulsive III-Carathéodory functions**, $I, J : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ are **Discrete III-Carathéodory functions**.

Finally we study the following impulsive Sturm-Liouville type integral boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{CH}D_{1^+}^{\beta} x(t) = p(t) f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ x(1) - \left(t\frac{d}{dt}\right) x(t)\Big|_{t=1} = \int_0^1 \phi(s) G(s, x(s)) ds, \\ x(e) + \left(t\frac{d}{dt}\right) x(t)\Big|_{t=e} = \int_0^1 \psi(s) H(s, x(s)) ds, \\ \lim_{t \to t_i^+} x(t) - x(t_i) = I(t_i, x(t_i)), i \in \mathbb{N}[1,m], \\ \lim_{t \to t_i^+} \left(t\frac{d}{dt}\right) x(t) - \left(t\frac{d}{dt}\right) x(t)\Big|_{t=t_i} = J(t_i, x(t_i)), i \in \mathbb{N}[1,m], \end{cases} \quad (1.0.11)$$

where

(l) $1 < \beta < 2$, ${}^{CH}D_{1^+}^{\beta}$ is the Caputo type Hadamard fractional derivative of order β with the starting point 1, $\left(t\frac{d}{dt}\right)^1 x(t) = tx'(t)$,

(m) $m, t_i, \mathbb{N}[a,b]$ satisfy (i), $\phi, \psi : (1,e) \to \mathbb{R}$ are measurable functions, $p : (1,e) \to \mathbb{R}$ is continuous and satisfies $p : (1,e) \to \mathbb{R}$ is continuous and satisfies that there exist numbers $k > 1 - \beta$ and $l \in \max\{-\beta, -\beta - k, 0]$ such that $|p(t)| \le (\log t)^k (1 - \log t)^l$ for all $t \in (1,e)$,

(n) f, G, H defined on $(1,e] \times \mathbb{R}$ are **impulsive IV-Carathéodory functions**, $I, J : \{t_i : i \in \mathbb{N}[1,m]\} \times \mathbb{R} \to \mathbb{R}$ are **Discrete IV-Carathéodory functions**.

A function $x : (0,1] \to \mathbb{R}$ is called a solution of BVP (1.0.9) (or BVP (1.0.8)) if $x|_{(t_i,t_{i+1}]}(i = 0,1, j \in \mathbb{N}[0,m])$ is continuous, the limits below exist $\lim_{t \to t_i^+} (t - t_i)^{2-\beta} x(t), i \in \mathbb{N}[0,m]$, (or $\lim_{t \to t_i^+} x(t)(i \in \mathbb{N}[0,m]$ and x satisfies the equations in (1.0.9) (or (1.0.8)).

A function $x : (1,e] \to \mathbb{R}$ is called a solution of BVP (1.0.11) (or BVP (1.0.10)) if $x|_{(t_i,t_{i+1}]}$ $(i \in \mathbb{N}[0,m])$ is continuous, the limits below exist $\lim_{t \to t_i^+} \left(\log \frac{t}{t_i}\right)^{2-\beta} x(t), i \in \mathbb{N}[0,m]$, (or $\lim_{t \to t_i^+} x(t)(i \in \mathbb{N}[0,m])$ and x satisfies equations in (1.0.11) (or (1.0.10)).

To get solutions of a boundary value problem of fractional differential equations, we firstly define a Banach space X, then we transform the boundary value problem into a integral equation and define a nonlinear operator T on X by using the integral equation obtained, finally, we prove that T has fixed point in X. The fixed points are just solutions of the boundary value problem. Three difficulties occur in known papers: one is how to transform the boundary value problem into a integral equation; the other one is how to define and prove a Banach space and the completely continuous property of the nonlinear operator defined; the third one is to choose a suitable fixed point theorem and impose suitable growth conditions on functions to get the fixed points of the operator.

To the best of the authors knowledge, no one has studied the existence of strong weak or weak solutions of BVP (1.0.i) ($i = 8, 9, 10, 11$). This paper fills this gap. Another purpose of this paper is to illustrate the similarity and difference of these three kinds of fractional differential equations. We obtain results on the existence of at least one solution for BVP (1.0.i) ($i = 8, 9, 10, 11$) respectively. For simplicity we only consider the left-sided operators here. The right-sided operators can be treated similarly. For clarity and brevity, we restrict our attention to BVPs with one impulse, the difference between the theory of one or an arbitrary number of impulses is quite similar.

The remainder of this paper is as follows: in Section 2, we present related definitions. In Section 3 some preliminary results are given (the first purpose is to establish existence and uniqueness of continuous solutions of four classes of initial value problems of non-linear fractional differential equations, the second purpose is to establish explicit expression of continuous solution of four classes of linear fractional differential equations (see Theorem 3.1.2, Theorem 3.2.2, Theorem 3.3.2 and Theorem 3.4.2 in Subsections 3.1, 3.2, 3.3 and 3.4 respectively). In Section 4, the purpose is to get exact expression of piecewise continuous solutions of four classes of linear fractional differential equations. In Section 5, we prove the main results for establishing existence results of solutions of (1.0.i) (i=8,9,10,11). In Sections 6, by establishing existence results on solvability of two class of impulsive boundary value problems of fractional differential equations, we make a comparison on two impulsive boundary value problems in which one has a single starting point and the other one has multiple starting points. In order to avoid misleading the readers, a mistake in a published paper is also corrected in this section.

2 Related definitions

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [44, 71, 76].

Let the Gamma function, Beta function and the classical Mittag-Leffler special function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \quad \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad \mathbf{E}_{\delta,\sigma}(x) = \sum_{\chi=0}^{\infty} \frac{x^\chi}{\Gamma(\chi\delta+\sigma)}$$

respectively for $\alpha > 0, p > 0, q > 0$. We note that $\mathbf{E}_{\delta,\delta}(x) > 0$ for all $x \in \mathbb{R}$ and $\mathbf{E}_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x > 0$ we have $\mathbf{E}_{\delta,\sigma}(-x) < \mathbf{E}_{\delta,\sigma}(0) = \frac{1}{\Gamma(\sigma)} < \mathbf{E}_{\delta,\sigma}(x)$.

Definition 2.1 [44] *Let $c \in \mathbb{R}$. The left Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (c, \infty) \to \mathbb{R}$ is given by*

$$I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} g(s) ds,$$

provided that the right-hand side exists.

Definition 2.2 [44] *Let $c \in \mathbb{R}$. The left Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $g : (c, +\infty) \to \mathbb{R}$ is given by*

$$^{RL}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha \in (n-1, n)$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.3 [44] *Let $c \in \mathbb{R}$. The left Caputo fractional derivative of order $\alpha > 0$ of a function $g : (c, +\infty) \to \mathbb{R}$ is given by*

$$^{C}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha \in (n-1, n)$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.4 [44] *Let $c > 0$. The left Hadamard fractional integral of order $\alpha > 0$ of a function $g : [c, +\infty) \to \mathbb{R}$ is given by*

$$^{H}I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s},$$

provided that the right-hand side exists.

Definition 2.5 [44] *Let $c > 0$. The left Riemann-Liouville type Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \to \mathbb{R}$ is given by*

$$^{RLH}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_c^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} g(s) \frac{ds}{s},$$

where $\alpha \in (n-1, n)$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.6 [42] *Let $c > 0$. The left Caputo type Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \to \mathbb{R}$ is given by*

$$^{CH}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n g(s) \frac{ds}{s},$$

where $\alpha \in (n-1, n)$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.7 *We call* $F:(0,1)\times\mathbb{R}\to\mathbb{R}$ *an* **impulsive I-Carathéodory function** *if it satisfies*

(i) $t\to F(t,u)$ *is measurable on* $(0,1)$ *for any* $u\in\mathbb{R}$,

(ii) $u\to F(t,u)$ *are continuous on* $\mathbb{R}$ *for all* $t\in(0,1)$,

(iii) *for each* $r>0$ *there exists* $M_r>0$ *such that*

$$|F(t,u)|\le M_r, t\in(0,1), |u|\le r.$$

Definition 2.8 *Let* $n-1<\alpha<n$ *and* $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$. *We call* $F:\bigcup_{i=0}^{m}(t_i,t_{i+1}]\times\mathbb{R}\to\mathbb{R}$ *an* **impulsive II-Carathéodory function** *if it satisfies*

(i) $t\to F(t,(t-t_i)^{\alpha-n}u)$ *is measurable on* $(t_i,t_{i+1}](i\in\mathbb{N}[0,m])$ *for any* $u\in\mathbb{R}$,

(ii) $u\to F(t,(t-t_i)^{\alpha-n}u)$ *are continuous on* $\mathbb{R}$ *for all* $t\in(t_i,t_{i+1}](i\in\mathbb{N}[0,m])$,

(iii) *for each* $r>0$ *there exists* $M_r>0$ *such that*

$$|F(t,(t-t_i)^{\alpha-n}u)|\le M_r, t\in(t_i,t_{i+1}], |u|\le r, (i\in\mathbb{N}[0,m]).$$

Definition 2.9 *Let* $n-1<\alpha<n$ *and* $1=t_0<t_1<\cdots<t_m<t_{m+1}=e$. *We call* $F:\bigcup_{i=0}^{m}(t_i,t_{i+1}]\times\mathbb{R}\to\mathbb{R}$ *an* **impulsive III-Carathéodory function** *if it satisfies*

(i) $t\to F\left(t,\left(\log\frac{t}{t_i}\right)^{\alpha-n}u\right)$ *is measurable on* $(t_i,t_{i+1}](i\in\mathbb{N}[0,m])$ *for any* $u\in\mathbb{R}$,

(ii) $u\to F\left(t,\left(\log\frac{t}{t_i}\right)^{\alpha-n}u\right)$ *are continuous on* $\mathbb{R}$ *for all* $t\in(t_i,t_{i+1}](i\in\mathbb{N}[0,m])$,

(iii) *for each* $r>0$ *there exists* $M_r>0$ *such that*

$$\left|F\left(t,\left(\log\frac{t}{t_i}\right)^{\alpha-n}u\right)\right|\le M_r, t\in(t_i,t_{i+1}], |u|\le r, (i\in\mathbb{N}[0,m]).$$

Definition 2.10 *We call* $F:(1,e)\times\mathbb{R}\to\mathbb{R}$ *an* **impulsive IV-Carathéodory function** *if it satisfies*

(i) $t\to F(t,u)$ *is measurable on* $(1,e)$ *for any* $u\in\mathbb{R}$,

(ii) $u\to F(t,u)$ *are continuous on* $\mathbb{R}$ *for all* $t\in(1,e)$,

(iii) *for each* $r>0$ *there exists* $M_r>0$ *such that*

$$|F(t,u)|\le M_r, t\in(1,e), |u|\le r.$$

Definition 2.11 *Let* $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$. *We call* $I:\{t_i:i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}$ *a* **discrete I-Carathéodory function** *if it satisfies*

(i) $u\to I(t_i,u)\,(i\in\mathbb{N}[1,m])$ *are continuous on* $\mathbb{R}$,

(ii) *for each* $r>0$ *there exists* $M_r>0$ *such that* $|I(t_i,u)|\le M_r, |u|\le r, i\in\mathbb{N}[1,m]$.

Definition 2.12 *Let $n-1<\alpha<n$ and $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$. We call $I:\{t_i:i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}$ a* **discrete II-Carathéodory function** *if it satisfies*

(i) $u\to I\left(t_i,(t_i-t_{i-1})^{\alpha-2}u\right)(i\in\mathbb{N}[1,m])$ *are continuous on* $\mathbb{R}$*,*

(ii) *for each $r>0$ there exists $M_r>0$ such that*

$$\left|I\left(t_i,(t_i-t_{i-1}^{\alpha-2}u\right)\right|\le M_r, |u|\le r, i\in\mathbb{N}[1,m].$$

Definition 2.13 *Let $n-1\le\alpha<n$ and $1=t_0<t_1<\cdots<t_m<t_{m+1}=e$. We call $I:\{t_i:i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}$ a* **discrete III-Carathéodory function** *if it satisfies*

(i) $u\to I\left(t_i,(\log t_i-\log t_{i-1})^{\alpha-2}u\right)(i\in\mathbb{N}[1,m])$ *are continuous on* $\mathbb{R}$*,*

(ii) *for each $r>0$ there exists $M_r>0$ such that*

$$\left|I\left(t_i,(\log t_i-\log t_{i-1})^{\alpha-2}u\right)\right|\le M_r, |u|\le r, i\in\mathbb{N}[1,m].$$

Definition 2.14 *Let $1=t_0<t_1<\cdots<t_m<t_{m+1}=e$. We call $I:\{t_i:i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}$ a* **discrete IV-Carathéodory function** *if it satisfies*

(i) $u\to I(t_i,u)(i\in\mathbb{N}[1,m])$ *are continuous on* $\mathbb{R}$*,*

(ii) *for each $r>0$ there exists $M_r>0$ such that* $|I(t_i,u)|\le M_r, |u|\le r, i\in\mathbb{N}[1,m]$.

Definition 2.15 [64] *Let E and F be Banach spaces. A operator $T:E\to F$ is called a completely continuous operator if T is continuous and maps any bounded set into relatively compact set.*

Suppose that $n-1<\alpha<n$ and $0\le a<b$ are constants. The following Banach spaces are used:

(i) $C_0(a,b]$ *denote the set of all continuous functions on $(a,b]$ with the limit $\lim_{t\to a^+}x(t)$ existing, and the norm $||x||=\sup_{t\in(a,b]}|x(t)|$;*

(ii) $C_{n-\alpha}(a,b]$ *the set of all continuous functions on $(a,b]$ with the limit $\lim_{t\to a^+}(t-a)^{n-\alpha}x(t)$ existing, the norm $||x||_{n-\alpha}=\sup_{t\in(a,b]}(t-a)^{n-\alpha}|x(t)|$;*

(iii) $LC_{n-\alpha}(a,b]$ *denote the set of all continuous functions on $(a,b]$ with the limit $\lim_{t\to a^+}\left(\log\frac{t}{a}\right)^{n-\alpha}x(t)$ existing, and the norm $||x||=\sup_{t\in(a,b]}\left(\log\frac{t}{a}\right)^{n-\alpha}|x(t)|$.*

Let m be a positive integer and $\mathbb{N}[0,m]=\{0,1,2,\cdots,m\}$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$. The following Banach spaces are also used in this paper:

(iv)

$$P_mC_{n-\alpha}(0,1]=\left\{x:(0,1]\to\mathbb{R}:\ x|_{(t_i,t_{i+1}]}\in C_{n-\alpha}(t_i,t_{i+1}]:i\in\mathbb{N}[0,m]\ \right\}$$

with the norm

$$||x|| = ||x||_{P_mC_{n-\alpha}} = \max\left\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{n-\alpha}|x(t)| : i\in\mathbb{N}[0,m]\right\}.$$

(v)

$$P_mC_0(0,1] = \left\{x:(0,1]\to\mathbb{R}:\ x|_{(t_i,t_{i+1}]}\in C_0(t_i,t_{i+1}] : i\in\mathbb{N}[0,m]\ \right\}$$

with the norm

$$||x|| = ||x||_{P_mC_0(0,1]} = \max\left\{\sup_{t\in(t_i,t_{i+1}]}|x(t)| : i\in\mathbb{N}[0,m]\right\}.$$

Let m be a positive integer and $\mathbb{N}[0,m]=\{0,1,2,\cdots,m\}$, $1=t_0<t_1<\cdots<t_m<t_{m+1}=e$. We also use the Banach spaces

(vi)

$$LP_mC_{n-\alpha}(1,e] = \left\{x:(1,e]\to\mathbb{R}:\ \begin{array}{c} x|_{(t_i,t_{i+1}]}\in C(t_i,t_{i+1}], i\in\mathbb{N}[0,m], \\ \text{there exist the limits} \\ \lim\limits_{t\to t_i^+}\left(\log\frac{t}{t_i}\right)^{n-\alpha}x(t), i\in\mathbb{N}[0,m] \end{array}\right\}$$

with the norm

$$||x|| = ||x||_{LP_mC_{n-\alpha}} = \max\left\{\sup_{t\in(t_i,t_{i+1}]}\left(\log\frac{t}{t_i}\right)^{n-\alpha}|x(t)|, i\in\mathbb{N}[0,m]\right\}.$$

(vii)

$$LP_mC_0(1,e] = \left\{x:(1,e]\to\mathbb{R}:\ x|_{(t_i,t_{i+1}]}\in C_0(t_i,t_{i+1}], i\in\mathbb{N}[0,m]\ \right\}$$

with the norm

$$||x|| = ||x||_{LP_mC} = \max\left\{\sup_{t\in(t_i,t_{i+1}]}|x(t)|, i\in\mathbb{N}[0,m]\right\}.$$

3 Some preliminary results

Lakshmikantham et al. [58–61] investigated the basic theory of initial value problems for fractional differential equations involving Riemann-Liouville differential operators of order $q\in(0,1)$. The existence and uniqueness of solutions of the following initial value problems of fractional differential equations were discussed under the assumption that $f\in C_r[0,1]$. We will establish existence and uniqueness results for these problems under more weaker assumptions see **(A1)-(A4)**.

Suppose $n-1<\alpha<n$ and $\eta_j\in\mathbb{R}$ ($j\in\mathbb{N}[0,n-1]$). We will consider the following four classes of Cauchy problems for non-linear fractional differential equations:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t)=p_1(t)F_1(t,x(t)),\ t\in(0,1),\\ \lim\limits_{t\to0^+}x^{(j)}(t)=\eta_j, j\in\mathbb{N}[0,n-1],\end{cases}\tag{3.0.1}$$

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t)=p_2(t)F_2(t,x(t)),\ t\in(0,1),\\ \lim\limits_{t\to0^+}t^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\ \lim\limits_{t\to0^+}{}^{RL}D_{0^+}^{\alpha-j}x(t)=\eta_j, j\in\mathbb{N}[1,n-1],\end{cases}\tag{3.0.2}$$

$$\begin{cases} {}^{RLH}D_{0^+}^{\alpha}x(t)=p_3(t)F_3(t,x(t)),\ t\in(1,e),\\ \lim\limits_{t\to1^+}(\log t)^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\ \lim\limits_{t\to1^+}{}^{RLH}D_{1^+}^{\alpha-j}x(t)=\eta_j, j\in\mathbb{N}[1,n-1],\end{cases}\tag{3.0.3}$$

$$\begin{cases} {}^{CH}D_{0^+}^{\alpha}x(t)=p_4(t)F_4(t,x(t)),\ t\in(1,e),\\ \lim\limits_{t\to1^+}\left(t\frac{d}{dt}\right)^j x(t)=\eta_j, j\in\mathbb{N}[0,n-1].\end{cases}\tag{3.0.4}$$

where $\left(t\frac{d}{dt}\right)^j x(t)=t\frac{d\left(t\frac{d}{dt}\right)^{j-1}x(t)}{dt}$ for $j=2,3,\cdots$, $p_i(i=1,2,3,4)$ satisfy assumption **(Ai)** below, $F_i(i=1,2,3,4)$ is **i-Carathéodory function** ($i=\mathbf{I},\mathbf{II},\mathbf{III},\mathbf{IV}$ and satisfies the following assumption **(Hi)** ($i=1,2,3,4$).

(A1) there exists constants $k>-\alpha+n-1$, $l\le0$ with $l>\max\{-\alpha,-\alpha-k\}$ such that $|p_1(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$.

(A2) there exists constants $k>-1$, $l\le0$ with $l>\max\{-\alpha,-n-k\}$ such that $|p_2(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$.

(A3) there exists constants $k>-1$, $l\le0$ with $l>\max\{-\alpha,-n-k\}$ such that $|p_3(t)|\le (\log t)^k(1-\log t)^l$ for all $t\in(1,e)$.

(A4) there exists constants $k>-\alpha+n-1$, $l\le0$ with $l>\max\{-\alpha,-\alpha-k\}$ such that $|p_4(t)|\le(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$.

(H1) there exists a constant $L_1>0$ such that $|F_1(t,x_1)-F_1(t,x_2)|\le L_1|x_1-x_2|$ for all $t\in[0,1]]$ and $x_1,x_2\in\mathbb{R}$.

(H2) there exists a constant $L_2>0$ such that $|F_2(t,t^{\alpha-n}x_1)-F_2(t,t^{\alpha-n}x_2)|\le L_2|x_1-x_2|$ for all $t\in(0,1]$ and $x_1,x_2\in\mathbb{R}$.

(H3) there exists a constant $L_3>0$ such that $|F_3(t,(\log t)^{\alpha-n}x_1)-F_3(t,(\log t)^{\alpha-n}x_2)|\le L_3|x_1-x_2|$ for all $t\in(1,e]$ and $x_1,x_2\in\mathbb{R}$.

(H4) there exists a constant $L_4>0$ such that $|F_4(t,x_1)-F_4(t,x_2)|\le L_4|x_1-x_2|$ for all $t\in(1,e]$ and $x_1,x_2\in\mathbb{R}$.

3.1 Existence and uniqueness of solutions of (3.0.1)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{j=0}^{n-1} \frac{\eta_j t^j}{j!},\ t \in [0,1],$$

$$\phi_i(t) = \phi_0(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_{i-1}(s)) ds, t \in [0,1], i = 1, 2, \cdots.$$

Lemma 3.1.1 *Suppose that* **(A1), (H1)** *hold. Then* $\phi_i \in C[0,1]$.

Proof: One sees $\phi_0 \in C[0,1]$. Since F_1 is a **I-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1)} |F_1(t, \phi_0(t))| \leq \sup_{t \in (0,1), |x| \leq ||\phi_0||} |F_1(t,x)| < +\infty.$$

We have by using **(A1)**

$$\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_0(s)) ds \right| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds M_0$$

$$\leq M_0 \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds = M_0 t^{\alpha+k+l} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw$$

$$= M_0 t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \to 0 \text{ as } t \to 0^+,$$

implies ϕ_1 is continuous on $[0,1]$. So $\phi_1 \in C[0,1]$. By mathematical induction method, we can prove that $\phi_i \in C[0,1]$. □

Lemma 3.1.2 *Suppose that* **(A1), (H1)** *hold. Then* $\{\phi_i\}$ *is convergent uniformly on* $[0,1]$.

Proof: Since F_1 is a **I-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1), |x| \leq ||\phi_0||} |F_1(t,x)| < +\infty.$$

Then we have for $t \in [0,1]$ that

$$|\phi_1(t) - \phi_0(t)| = \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_0(s)) ds \right| \leq M_0 t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.$$

So **(H1)** implies that

$$|\phi_2(t)-\phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s)[F_1(s,\phi_1(s))-F_1(s,\phi_0(s))]ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l L_1|\phi_1(s)-\phi_0(s)|ds$$

$$\leq M_0L_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l s^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}ds$$

$$\leq M_0L_1\int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^{\alpha+2k+l}ds\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}$$

$$= M_0L_1t^{2\alpha+2k+2l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}.$$

Now suppose that

$$|\phi_j(t)-\phi_{j-1}(t)| \leq M_0L_1^{j-1}t^{j\alpha+jk+jl}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}.$$

We get

$$|\phi_{j+1}(t)-\phi_j(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s)[F_1(s,\phi_j(s))-F_1(s,\phi_{j-1}(s))]ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l L_1\left|\phi_j(s)-\phi_{j-1}(s)\right|ds$$

$$\leq M_0L_1^j\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l s^{j\alpha+jk+jl}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}ds$$

$$\leq M_0L_1^j\int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^{j\alpha+(j+1)k+jl}ds\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$= M_0L_1^jt^{(j+1)\alpha+(j+1)k+(j+1)l}\frac{\mathbf{B}(\alpha+l,j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k+il+1)}{\Gamma(\alpha)}$$

$$= M_0L_1^jt^{(j+1)\alpha+(j+1)k+(j+1)l}\prod_{i=0}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k+il+1)}{\Gamma(\alpha)}.$$

From the mathematical induction method, we get for every $i=1,2,\cdots$ that

$$|\phi_{i+1}(t)-\phi_i(t)| \leq M_0L_1^{i-1}t^{(i+1)\alpha+(i+1)k+(i+1)l}\prod_{j=0}^{i-1}\frac{\mathbf{B}(\alpha+l,j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}$$

$$\leq M_0L_1^{i-1}\prod_{j=0}^{i-1}\frac{\mathbf{B}(\alpha+l,j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}, t\in[0,1].$$

Consider

$$\sum_{i=1}^{+\infty} u_i = \sum_{i=1}^{+\infty} M_0 L_1^{i-1} \prod_{j=0}^{i-1} \frac{\mathbf{B}(\alpha+l, j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}.$$

One sees for sufficiently large n that

$$\frac{u_{i+1}}{u_i} = L_1 \frac{\mathbf{B}(\alpha+l,(i+1)\alpha+(i+1)k+(i+1)l)}{\Gamma(\alpha)} = L_1 \int_0^1 (1-x)^{\alpha+l-1} x^{(i+1)\alpha+(i+1)k+(i+1)l} dx$$

$$\leq L_1 \int_0^\delta (1-x)^{\alpha+l-1} x^{(i+1)\alpha+(i+1)k+(i+1)l} dx + L_1 \int_\delta^1 (1-x)^{\alpha+l-1} dx \text{ with } \delta \in (0,1)$$

$$\leq L_1 \int_0^\delta (1-x)^{\alpha+l-1} dx \delta^{(i+1)\alpha+(i+1)k+(i+1)l} + \frac{L_1}{\alpha+l} \delta^{\alpha+l}$$

$$\leq \frac{L_1}{\alpha+l} \delta^{(i+1)\alpha+(i+1)k+(i+1)l} + \frac{L_1}{\alpha+l} \delta^{\alpha+l}.$$

For any $\epsilon > 0$, it is easy to see that there exists $\delta \in (0,1)$ such that $\frac{L_1}{\alpha+l}\delta^{\alpha+l} < \frac{\epsilon}{2}$. For this δ, there exists an integer $N > 0$ sufficiently large such that $\frac{L_1}{\alpha+l}\delta^{(i+1)\alpha+(i+1)k+(i+1)l} < \frac{\epsilon}{2}$ for all $i > N$. So $0 < \frac{u_{i+1}}{u_i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $i > N$. It follows that $\lim_{i \to +\infty} \frac{u_{i+1}}{u_i} = 0$. Then $\sum_{i=1}^{+\infty} u_i$ is convergent. Hence

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_i(t) - \phi_{i-1}(t)] + \cdots, t \in [0,1]$$

is uniformly convergent. Then $\{\phi_i(t)\}$ is convergent uniformly on $[0,1]$. □

Lemma 3.1.3 *Suppose that* **(A1), (H1)** *hold. Then* $\phi(t) = \lim_{i \to +\infty} \phi_i(t)$ *defined on* $[0,1]$ *is a unique continuous solution of the integral equation*

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_1(s) F_1(s, x(s)) ds, t \in [0,1). \tag{3.1.1}$$

Proof: By $\phi(t) = \lim_{i \to +\infty} \phi_i(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $[0,1]$. From

$$\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_i(s)) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) F_1(s, \phi_j(s)) ds ds \right|$$

$$\leq L_1 ||\phi_i - \phi_j|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds$$

$$\leq L_1 ||\phi_i - \phi_j|| t^{\alpha+k+l} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}$$

$$\leq L_1 ||\phi_i - \phi_j|| \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \to 0 \text{ as } i, j \to +\infty,$$

we have

$$\phi(t)=\lim_{i\to+\infty}\phi_i(t)=\lim_{i\to+\infty}\left[\sum_{j=0}^{n-1}\frac{\eta_jt^j}{j!}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)F_1(s,\phi_{i-1}(s))ds\right]$$

$$=\sum_{j=0}^{n-1}\frac{\eta_jt^j}{j!}+\lim_{i\to+\infty}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)F_1(s,\phi_{i-1}(s))ds$$

$$=\sum_{j=0}^{n-1}\frac{\eta_jt^j}{j!}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)F_1(s,\phi(s))ds.$$

Then ϕ is a continuous solution of (3.1.1) defined on $[0,1]$.

Suppose that ψ defined on $[0,1]$ is also a solution of (3.1.1). Since F_1 is a **I-Carathéodory function**, we know

$$M_0=\sup_{t\in(0,1),|x|\le||\psi||}|F_1(t,x)|<+\infty.$$

Then

$$\psi(t)=\sum_{j=0}^{n-1}\frac{\eta_j}{j!}t^j+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)F_1(s,\psi(s))ds, t\in(0,1].$$

We need to prove that $\phi(t)\equiv\psi(t)$ on $[0,1]$. One sees that

$$|\psi(t)-\phi_0(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)F_1(s,\psi(s))ds\right|\le M_0t^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$|\psi(t)-\phi_1(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)[F_1(s,\psi(s))-F_1(s,\phi_0(s))]ds\right|$$

$$\le M_0L_1t^{2\alpha+2k+2l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}.$$

By mathematical induction method, we have

$$|\psi(t)-\phi_i(t)|\le M_0L_1^it^{i\alpha+ik+il}\prod_{j=0}^{i}\frac{\mathbf{B}(\alpha+l,j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}$$

$$\le M_0L_1^i\prod_{j=0}^{i}\frac{\mathbf{B}(\alpha+l,j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}, t\in[0,1].$$

Similarly we have

$$\lim_{m\to+\infty}M_0L_1^i\prod_{j=0}^{i}\frac{\mathbf{B}(\alpha+l,j\alpha+(j+1)k+jl+1)}{\Gamma(\alpha)}=0.$$

Then $\lim_{i\to+\infty}\phi_i(t)=\psi(t)$ uniformly on $[0,1]$. Then $\phi(t)\equiv\psi(t)$. Then (3.1.1) has a unique solution ϕ. The proof is completed. □

Theorem 3.1.1 *Suppose that* **(A1), (H1)** *hold. Then x is a solution of IVP (3.0.1) if and only if x is a solution of the integral equation (3.1.1).*

Proof: Suppose that x is a solution of IVP (3.0.1). Then $\lim_{t\to 0^+} x(t) = \eta_0$ and $||x|| = r < +\infty$. Since F_1 is a **I-Carathéodory function**, we know

$$M_r = \sup_{t\in(0,1),|x|\le r} |F_1(t,x)| < +\infty.$$

From **(A1)**, we have for $t \in [0,1)$

$$\left|\int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} p_1(s)F_1(s,x(s))ds\right| \le M_r \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} s^k(1-s)^l ds$$

$$\le M_r(1-t)^l \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} s^k ds = M_r(1-t)^l t^{\alpha+k-n+1} \int_0^1 \frac{(1-w)^{\alpha+l-n}}{\Gamma(\alpha-n+1)} w^k dw$$

$$= M_r(1-t)^l t^{\alpha+k-n+1} \frac{\mathbf{B}(\alpha-n+1,k+1)}{\Gamma(\alpha)}.$$

So $t \to \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} p_1(s)F_1(s,x(s))ds$ is continuous on $[0,1)$ by $k > -\alpha+n-1$, $l \le 0$ with $l > \max\{-\alpha, -\alpha-k\}$. It follows that

$$\lim_{t\to 0^+} \int_0^t \frac{(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)} p_1(s)F_1(s,x(s))ds = 0. \tag{3.1.2}$$

From ${}^C D_{0^+}^\alpha x(t) = p_1(t)F_1(t,x(t))$, a.e., $t \in (0,1)$, we have

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s)F_1(s,x(s))ds = I_{0^+}^\alpha p_1(t)F_1(t,x(t)) = I_{0^+}^\alpha\, {}^C D_{0^+}^\alpha x(t)$$

$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{1}{\Gamma(n-\alpha)} \int_0^s (s-w)^{-\alpha} x^{(n)}(w)dw\right) ds \text{ interchange the order of integrals}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^t \int_w^t (t-s)^{\alpha-1}(s-w)^{n-\alpha-1} ds x^{(n)}(w)dw \text{ use } \frac{s-w}{t-w} = u$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^t (t-u)^{n-1} \int_0^1 (1-u)^{\alpha-1} u^{n-\alpha-1} du x^{(n)}(w)dw \text{ by } \mathbf{B}(\alpha, n-\alpha) = \frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)}$$

$$= \frac{1}{(n-1)!} \int_0^t (t-u)^{n-1} x^{(n)}(w)dw$$

$$= \frac{1}{(n-1)!} \left[(t-u)^{n-1} x^{(n-1)}(w)\Big|_0^t + (n-1)\int_0^t (t-u)^{n-2} x^{(n-1)}(w)dw\right]$$

$$= \frac{\eta_{n-1}}{(n-1)!} t^{n-1} + \frac{1}{(n-2)!} \int_0^t (t-u)^{n-2} x^{(n-1)}(w)dw$$

$$= \cdots\cdots\cdots$$

$$= \sum_{j=1}^{n-1} \frac{\eta_j}{j!} t^j + \int_0^t x'(s)ds = x(t) - \sum_{j=0}^{n-1} \frac{\eta_j}{j!}.$$

It follows that x satisfies $x(t)=\sum\limits_{j=0}^{n-1}\frac{\eta_j}{j!}t^j+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)F_1(s,x(s))ds, t\in[0,1)$. Then $x\in C[0,1]$ is a solution of (3.1.1).

On the other hand, if x is a solution of (3.1.1). Then

$$x^{(\nu)}(t)=\sum_{j=\nu}^{n-1}\frac{\eta_j}{(j-\nu)!}t^{j-\nu}+\frac{1}{\Gamma(\alpha-\nu)}\int_0^t(t-s)^{\alpha-\nu-1}p_1(s)F_1(s,x(s))ds, t\in[0,1).$$

Together Lemma 3.1.1-Lemma 3.1.3 and (3.1.2), we have $x\in C[0,1]$ and $\lim\limits_{t\to 0^+}x^{(j)}(t)=\eta_j(j\in\mathbb{N}[0,n-1])$. Furthermore, we have

$${}^CD_{0^+}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}x^{(n)}(s)ds$$

$$=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}\left(\sum_{j=0}^{n-1}\frac{\eta_js^j}{j!}+\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}p_1(w)F_1(w,x(w))dw\right)^{(n)}ds$$

$$=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}p_1(w)F_1(w,x(w))dw\right)^{(n)}ds$$

$$=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}\frac{1}{\Gamma(\alpha-n+1)}\left(\int_0^s(s-w)^{\alpha-n}p_1(w)F_1(w,x(w))dw\right)'ds$$

$$=\frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n+1-\alpha)}\left[\int_0^t(t-s)^{n-\alpha}\left(\int_0^s(s-w)^{\alpha-n}p_1(w)F_1(w,x(w))dw\right)'ds\right]'$$

$$=\frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n+1-\alpha)}\left[(t-s)^{n-\alpha}\int_0^s(s-w)^{\alpha-n}p_1(w)F_1(w,x(w))dw\Big|_0^t\right.$$

$$\left.+(n-\alpha)\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-w)^{\alpha-n}p_1(w)F_1(w,x(w))dwds\right]'$$

$$=\frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\left[\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-w)^{\alpha-n}p_1(w)F_1(w,x(w))dwds\right]' \text{ by (3.1.2)}$$

$$=\frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\int_0^t\int_w^t(t-s)^{n-\alpha-1}(s-w)^{\alpha-n}dsp_1(w)F_1(w,x(w))dw\right]'$$

by changing the order of integrals

$$=\frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\left[\int_0^t\int_0^1(1-u)^{n-\alpha-1}u^{\alpha-n}dup_1(w)F_1(w,x(w))dw\right]' \text{ by } \frac{s-w}{t-w}=u$$

$$=\left[\int_0^tp_1(w)F_1(w,x(w))dw\right]' \text{ by } \mathbf{B}(n-\alpha,\alpha-n+1)=\Gamma(n-\alpha)\Gamma(\alpha-n+1)$$

$$=p_1(t)F_1(t,x(t)), a.e., t\in(0,1).$$

So $x \in C[0,1]$ is a solution of IVP (3.1.1). The proof is completed. □

Theorem 3.1.2 *Suppose that* **(A1), (H1)** *hold. Then*

(i) IVP (3.0.1) *has a unique solution.*

(ii) *Suppose that there exists constants* $k > -\alpha + n - 1$, $l \le 0$ *with* $l > \max\{-\alpha, -\alpha - k\}$, $M \ge 0$ *such that* $|F(t)| \le Mt^k(1-t)^l$ *for all* $t \in (0,1)$. *Then the following special problem*

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = +F(t), \ a.e., \ t \in (0,1), \\ \lim_{t \to 0^+} x^{(j)}(t) = \eta_j, j \in \mathbb{N}[0, n-1] \end{cases} \tag{3.1.3}$$

has a unique solution

$$x(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!} t^j + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds, \ t \in (0,1]. \tag{3.1.4}$$

Proof: **(i)** From Lemmas 3.1.1-Lemma 3.1.3, Theorem 3.1.1 implies that IVP (3.0.1) has a unique solution.

(ii) From the assumption and (i), we know that (3.1.3) has a unique solution. It is easy to see that

$$x(t) \equiv \sum_{j=0}^{n-1} \frac{\eta_j}{j!} t^j + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds$$

by replacing $p_1(t)F_1(t, x(t))$ with $F(t)$. Then x is a unique solution of (3.1.3). The proof is completed. □

3.2 Existence and uniqueness of solutions of (3.0.2)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha - v + 1)} t^{\alpha - v}, \ t \in (0,1],$$

$$\phi_i(t) = \phi_0(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_2(s) F_2(s, \phi_{i-1}(s)) ds, t \in (0,1], i = 1, 2, \cdots.$$

Lemma 3.2.1 *Suppose that* **(A2), (H2)** *hold. Then* $\phi_i \in C_{n-\alpha}[0,1]$.

Proof: It is easy to see that $\phi_0 \in C_{n-\alpha}[0,1]$. Since F_2 is a **II-Carathéodory function**, we know

$$M_0 = \sup_{t \in (0,1]} |F_2(t, \phi_0(t))| = \sup_{t \in (0,1]} |F_2(t, t^{\alpha-n} t^{n-\alpha} \phi_0(t))| \le \sup_{t \in (0,1], |x| \le ||\phi_0||} |F_2(t, t^{\alpha-n} x)| \le +\infty.$$

We have by $k > -1$, $l \leq 0$ with $l > \max\{-\alpha, -n-k\}$

$$t^{n-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi_0(s))ds\right|$$

$$\leq t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l M_0 ds + M_F t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l$$

$$\leq M_0 t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k ds$$

$$= M_0 t^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Then $t \to t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi_0(s))ds$ is continuous on $[0,1]$. We see $\phi_1 \in C_{n-\alpha}[0,1]$. By mathematical induction method, we can prove that $\phi_i \in C_{n-\alpha}[0,1]$. □

Lemma 3.2.2 *Suppose that* **(A2), (H2)** *hold. Then* $\{t \to t^{n-\alpha}\phi_i(t)\}$ *is convergent uniformly on* $[0,1]$.

Proof: Since F_2 is a **II-Carathéodory function**, we know

$$M_0 = \sup_{t\in(0,1]}|F_2(t,\phi_0(t))| = \sup_{t\in(0,1]}|F_2(t,t^{\alpha-n}t^{n-\alpha}\phi_0(t))| \leq \sup_{t\in(0,1],|x|\leq||\phi_0||}|F_2(t,t^{\alpha-n}x)| \leq +\infty.$$

We have for $t \in [0,1]$ similarly to Lemma 3.2.1 that

$$t^{n-\alpha}|\phi_1(t)-\phi_0(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi_0(s))ds\right| \leq M_0 t^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

So

$$t^{n-\alpha}|\phi_2(t)-\phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)[F_2(s,\phi_1(s))-F_2(s,\phi_0(s))]ds\right|$$

$$\leq t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l L_1|\phi_1(s)-\phi_0(s)|ds$$

$$\leq L_2 t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l s^{\alpha-n}[s^{n-\alpha}|\phi_1(s)-\phi_0(s)|]ds$$

$$\leq M_0 L_2 t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k s^{\alpha-n}s^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}ds$$

$$= M_0 L_2 t^{\alpha+n+2k+2l}\int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^{\alpha+2k+l}dw\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}$$

$$= M_0 L_2 t^{\alpha+n+2k+2l}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Furthermore

$$t^{n-\alpha}|\phi_3(t)-\phi_2(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)[F_2(s,\phi_2(s))-F_2(s,\phi_1(s))]ds\right|$$

$$\leq t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k+\alpha-n}(1-s)^l L_2|\phi_2(s))-\phi_1(s)|ds$$

$$\leq M_0L_2^2t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^{k+\alpha-n+\alpha+n+2k+2l}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}ds$$

$$\leq M_0L_2^2t^{3n+3k+3l}\int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^{2\alpha+3k+2l}dw\frac{\mathbf{B}(\alpha+l,n+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}$$

$$= M_0L_2^2t^{2\alpha+n+3k+3l}\frac{\mathbf{B}(\alpha+l,2\alpha+3k+2l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $i=1,2,\cdots$ that

$$t^{n-\alpha}|\phi_i(t)-\phi_{i-1}(t)|$$

$$\leq M_0L_2^{i-1}t^{(i-1)\alpha+n+ik+il}\frac{\mathbf{B}(\alpha+l,\alpha+k-n+1)}{\Gamma(\alpha)}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}$$

$$\leq M_0L_2^{i-1}\frac{\mathbf{B}(\alpha+l,\alpha+k-n+1)}{\Gamma(\alpha)}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}, t\in[0,1].$$

Similarly we can prove that

$$\sum_{i=1}^{+\infty}u_i = \sum_{i=1}^{+\infty}M_0L_2^{i-1}\frac{\mathbf{B}(\alpha+l,\alpha+k-n+1)}{\Gamma(\alpha)}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}$$

is convergent. Hence

$$t^{n-\alpha}\phi_0(t)+t^{n-\alpha}[\phi_1(t)-\phi_0(t)]+t^{n-\alpha}[\phi_2(t)-\phi_1(t)]+\cdots+t^{n-\alpha}[\phi_i(t)-\phi_{i-1}(t)]+\cdots, t\in[0,1]$$

is uniformly convergent. Then $\{t\to t^{n-\alpha}\phi_i(t)\}$ is convergent uniformly on $[0,1]$. □

Lemma 3.2.3 *Suppose that* **(A2), (H2)** *hold. Then* $\phi(t)=t^{\alpha-n}\lim_{i\to+\infty}t^{n-\alpha}\phi_i(t)$ *defined on* $(0,1)$ *is a unique continuous solution of the integral equation*

$$x(t)=\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,x(s)), t\in(0,1). \quad (3.2.1)$$

Proof: By $\lim_{i\to+\infty}t^{n-\alpha}\phi_i(t)=t^{n-\alpha}\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(0,1]$. From

$$t^{n-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi_p(s)) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi_q(s))ds\right|$$

$$\leq L_2 t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l|\phi_p(s)-\phi_q(s)|ds$$

$$\leq L_2||\phi_p-\phi_q||t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^{\alpha+k-n}ds$$

$$\leq L_2||\phi_p-\phi_q||t^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,\alpha+k-n+1)}{\Gamma(\alpha)}$$

$$\leq L_2||\phi_p-\phi_q||\frac{\mathbf{B}(\alpha+l,\alpha+k-n+1)}{\Gamma(\alpha)} \to 0 \text{ uniformly as } p,q\to+\infty,$$

we know

$$\phi(t) = t^{\alpha-n}\lim_{i\to+\infty} t^{n-\alpha}\phi_i(t)$$

$$= t^{\alpha-n}\lim_{i\to+\infty}\left[t^{n-\alpha}\sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v} + t^{n-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi_{i-1}(s))ds\right]$$

$$= \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v} + \lim_{i\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi_{i-1}(s))ds$$

$$= \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\phi(s))ds.$$

Then ϕ is a continuous solution of (3.2.1) defined on $(0,1]$.

Suppose that ψ defined on $(0,1]$ is also a solution of (3.2.1). Since F_2 is a **II-Carathéodory function**, we know

$$M_0 = \sup_{t\in(0,1]}|F_2(t,\psi(t))| = \sup_{t\in(0,1]}|F_2(t,t^{\alpha-n}t^{n-\alpha}\psi(t))| \leq \sup_{t\in(0,1],|x|\leq||\psi||}|F_2(t,t^{\alpha-n}x)| \leq +\infty.$$

Then

$$\psi(t) = \sum_{v=1}^n \frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\psi(s))ds, t\in[0,1].$$

We need to prove that $\phi(t)\equiv\psi(t)$ on $(0,1]$. Then

$$t^{n-\alpha}|\psi(t)-\phi_0(t)| = t^{n-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,\psi(s))ds\right| \leq M_0 t^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$t^{n-\alpha}|\psi(t)-\phi_1(t)| = t^{n-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)[F_2(s,\psi(s))-F_2(s,\phi_0(s))]ds\right|$$

$$\leq M_0 L_2 t^{\alpha+n+2k+2l}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get

$$t^{n-\alpha}|\psi(t)-\phi_{i-1}(t)| = t^{n-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)[F_2(s,\psi(s))-F_2(s,\phi_{i-2}(s))]ds\right|$$

$$\le M_0L_2^{i-1}t^{(i-1)\alpha+n+ik+il}\frac{\mathbf{B}(\alpha+l_1,\alpha+k-n+1)}{\Gamma(\alpha)}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}$$

$$\le M_0L_2^{i-1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k-n+1)}{\Gamma(\alpha)}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}, t\in[0,1].$$

Hence

$$t^{n-\alpha}|\psi(t)-\phi_{i-1}(t)| \le M_0L_2^{i-1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k-n+1)}{\Gamma(\alpha)}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(j-1)l+1)}{\Gamma(\alpha)}, \text{for all } i=1,2,\cdots.$$

Similarly we have $\lim_{i\to+\infty} t^{n-\alpha}\phi_i(t) = t^{n-\alpha}\psi(t)$ uniformly on $(0,1]$. Then $\phi(t)\equiv\psi(t)$ on $(0,1]$. Then (3.2.1) has a unique solution ϕ. The proof is completed. □

Theorem 3.2.1 *Suppose that* **(A2), (H2)** *hold. Then $x\in C_{n-\alpha}(0,1]$ is a solution of IVP (3.0.2) if and only if $x\in C_{n-\alpha}(0,1]$ is a solution of the integral equation (3.2.1).*

Proof: Suppose that $x\in C_{n-\alpha}(0,1]$ is a solution of IVP (3.0.2). Then $t\to t^{n-\alpha}x(t)$ is continuous on $(0,1]$ by defining $t^{n-\alpha}x(t)|_{t=0} = \lim_{t\to0^+} t^{n-\alpha}x(t)$ and $||x|| = r < +\infty$. Since F_2 is a **II-Carathéodory function**, we know

$$M_r = \sup_{t\in(0,1]}|F_2(t,x(t))| = \sup_{t\in(0,1]}|F_2(t,t^{\alpha-n}t^{n-\alpha}x(t))| \le \sup_{t\in(0,1],|x|\le r}|F_2(t,t^{\alpha-n}x)| \le +\infty.$$

By $\frac{w}{s}=u$, we get

$$\lim_{s\to0^+}\int_0^s(s-w)^{n-\alpha-1}x(w)dw = \lim_{s\to0^+}\int_0^s(s-w)^{n-\alpha-1}w^{\alpha-n}w^{n-\alpha}x(w)dw$$

$$= \lim_{s\to0^+}\xi^{n-\alpha}x(\xi)\int_0^s(s-w)^{n-\alpha-1}w^{\alpha-n}dw \text{ by mean value theorem of integral, } \xi\in(0,s)$$

$$= \lim_{s\to0^+}\xi^{n-\alpha}x(\xi)\int_0^1(1-u)^{n-\alpha-1}u^{\alpha-n}du = \frac{\eta_n}{\Gamma(\alpha-n+1)}\mathbf{B}(n-\alpha,\alpha-n+1). \tag{3.2.2}$$

From (A2), we have similarly to Lemma 3.2.1 that

$$t^{n-\alpha}\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,x(s))ds\right| \le M_rt^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^lds \le M_rt^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

So $t\to t^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,x(s))ds$ is continuous on $[0,1]$ and

$$\lim_{t\to0^+}t^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,x(s))ds = 0. \tag{3.2.3}$$

We have from ${}^{RL}D_{0^+}^{\alpha}x(t)=p_2(t)F_2(t,x(t)),a.e.,t\in(0,1)$ that

$$\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,x(s))ds=I_{0^+}^{\alpha}p_2(t)F_2(t,x(t))=I_{0^+}^{\alpha}\,{}^{RL}D_{0^+}^{\alpha}x(t)$$

$$=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{1}{\Gamma(n-\alpha)}\left(\int_0^s(s-w)^{n-\alpha-1}x(w)dw\right)^{(n)}\right]ds$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}d\left(\int_0^s\frac{(s-w)^{n-\alpha-1}}{\Gamma(n-\alpha)}x(w)dw\right)^{(n-1)}$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}d\left({}^{RL}D_{0^+}^{\alpha-1}x(s)\right)$$

$$=\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\left({}^{RL}D_{0^+}^{\alpha-1}x(s)\right)\Big|_0^t+\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\left(\int_0^s(s-w)^{n-\alpha-1}x(w)dw\right)^{(n-1)}ds$$

$$=\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\left(\int_0^s(s-w)^{n-\alpha-1}x(w)dw\right)^{(n-1)}ds-\frac{\eta_1}{\Gamma(\alpha)}t^{\alpha-1}$$

$$=\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-2)}\int_0^t(t-s)^{\alpha-3}\left(\int_0^s(s-w)^{n-\alpha-1}x(w)dw\right)^{(n-2)}ds-\frac{\eta_1}{\Gamma(\alpha)}t^{\alpha-1}-\frac{\eta_2}{\Gamma(\alpha-1)}t^{\alpha-2}$$

$$=\cdots\cdots\cdots\cdots$$

$$=\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))}\int_0^t(t-s)^{\alpha-n}\left(\int_0^s(s-w)^{n-\alpha-1}x(w)dw\right)'ds-\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}$$

$$=\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-2))}\left[\int_0^t(t-s)^{\alpha-n+1}\left(\int_0^s(s-w)^{n-\alpha-1}x(w)dw\right)'ds\right]'-\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}$$

$$=\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-2))}\left[(t-s)^{\alpha-n+1}\left(\int_0^s(s-w)^{n-\alpha-1}x(w)dw\right)\Big|_0^t\right.$$

$$\left.+(\alpha-n+1)\int_0^t(t-s)^{\alpha-n}\int_0^s(s-w)^{n-\alpha-1}x(w)dwds\right]'-\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}$$

by using (3.1.10)

$$=\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))}\left[\int_0^t\int_u^t(t-s)^{\alpha-n}(s-w)^{n-\alpha-1}dsx(w)dw\right]'-\frac{\eta_n}{\Gamma(\alpha-n+1)}t^{\alpha-n}-\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}$$

$$=\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-(n-1))}\left[\int_0^t\int_0^1(1-w)^{\alpha-n}w^{n-\alpha-1}dwx(w)dw\right]'-\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}$$

$$=x(t)-\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}.$$

Then

$$x(t)=\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p_2(s)F_2(s,x(s))ds.$$

Then $x\in C_{n-\alpha}[0,1]$ is a solution of (3.2.1).

On the other hand, suppose that $x\in C_{n-\alpha}[0,1]$ is a solution of IVP (3.2.1). Then

$${}^{RL}D_{0^+}^{\alpha-\nu}x(t)=\sum_{v=1}^{\nu}\frac{\eta_v}{\Gamma(\nu-v+1)}t^{\nu-v}+\int_0^t\frac{(t-s)^{\nu-1}}{\Gamma(\nu)}p_2(s)F_2(s,x(s))ds.$$

So $\lim\limits_{t\to0^+}t^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)}$ and ${}^{RL}D_{0^+}^{\alpha-\nu}x(0)=\eta_\nu,\nu\in\mathbb{N}[1,n-1]$. Furthermore, we have

$${}^{RL}D_{0^+}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\int_0^t(t-s)^{n-\alpha-1}x(s)ds\right)^{(n)}$$

$$=\frac{1}{\Gamma(n-\alpha)}\left(\int_0^t(t-s)^{n-\alpha-1}\left(\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}s^{\alpha-v}+\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}p_2(u)F_2(u,x(u))du\right)ds\right)^{(n)}$$

$$=\frac{\left(\sum\limits_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}\int_0^t(t-s)^{n-\alpha-1}s^{\alpha-v}ds+\int_0^t(t-s)^{n-\alpha-1}\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}p_2(u)F_2(u,x(u))duds\right)^{(n)}}{\Gamma(n-\alpha)}$$

$$=\frac{1}{\Gamma(n-\alpha)}\left(\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{n-v}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-v}dw\right.$$

$$\left.+\int_0^t\int_u^t(t-s)^{n-\alpha-1}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dsp_2(u)F_2(u,x(u))du\right)^{(n)}$$

$$=\frac{1}{\Gamma(n-\alpha)}\left(\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}t^{n-v}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-v}dw\right.$$

$$\left.+\int_0^t(t-u)^{n-1}\int_0^1(1-w)^{n-\alpha-1}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwp_2(u)F_2(u,x(u))du\right)^{(n)}=p_2(t)F_2(t,x(t)).$$

So $x\in C_{n-\alpha}[0,1]$ is a solution of IVP (3.0.2). The proof is completed. □

Theorem 3.2.2 *Suppose that* **(A2), (H2)** *hold. Then*

(i) *IVP (3.0.2) has a unique solution.*

(ii) *If there exist constants $k>-1$, $l\le0$ with $l>\max\{-\alpha,-n-k\}$ and $M\ge0$ such that $|F(t)|\le Mt^k(1-t)^l$ for all $t\in(0,1)$, then following special problem*

$$\begin{cases}{}^{RL}D_{0^+}^{\alpha}x(t)=F(t),\ t\in(0,1],\\ \lim\limits_{t\to0^+}t^{n-\alpha}x(t)=\frac{\eta_n}{\Gamma(\alpha-n+1)},\\ \lim\limits_{t\to0^+}{}^{RL}D_{0^+}^{\alpha-j}x(t)=\eta_j,j\in\mathbb{N}[1,n-1]\end{cases}\tag{3.2.4}$$

has a unique solution

$$x(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds, \ t \in (0,1]. \tag{3.2.5}$$

Proof: (i) From Lemmas 3.2.1, 3.2.2 and 3.2.3, and Theorem 3.1.1, we see that IVP (3.0.2) has a unique solution.

(ii) From the assumptions mentioned and (i), we know (3.2.4) has a unique solution. We get

$$x(t) \equiv \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds$$

by replacing $p_2(t)F_2(t, x(t))$ with $F(t)$. Then we get x satisfies (3.2.5). The proof is completed. □

3.3 Existence and uniqueness of solutions of (3.0.3)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v}, \ t \in (1,e],$$

$$\phi_i(t) = \phi_0(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, \phi_{i-1}(s)) \frac{ds}{s}, t \in (1,e], i = 1, 2, \cdots .$$

Lemma 3.3.1 *Suppose that* **(A3), (H3)** *hold. Then* $\phi_i \in LC_{n-\alpha}(1,e]$.

Proof: In fact, we have $\phi_0 \in LC_{n-\alpha}[1,e]$. Since F_3 is a **III-Carathéodory function**, we have

$$M_0 = \sup_{t\in(1,e]} |F_3(t, x(t))| = \sup_{t\in(1,e]} |F_3(t, (\log t)^{\alpha-n} (\log t)^{n-\alpha} x(t))|$$

$$\leq \sup_{t\in(1,e], |x|\leq ||\phi_0||} |F_3(t, (\log t)^{\alpha-n} x)| \leq +\infty.$$

and

$$(\log t)^{n-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, \phi_0(s)) \frac{ds}{s} \right|$$

$$\leq M_0 (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^k (1 - \log s)^l \frac{ds}{s}$$

$$\leq M_0 (\log t)^{n-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l-1} (\log s)^k \frac{ds}{s}$$

$$= M_0 (\log t)^{n+k+l} \mathbf{B}(\alpha + l, k + 1) \to 0 \text{ as } t \to 0^+,$$

we know that

$$t \to \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s) F_3(s, \phi_0(s)) \frac{ds}{s}$$

is continuous on $(1,e]$ and $\lim\limits_{t\to 0^+}(\log t)^{n-\alpha}\phi_1(t)$ exists. Then $\phi_1\in LC_{n-\alpha}[1,e]$. By mathematical induction method, we can show $\phi_i\in LC_{n-\alpha}[1,e]$. □

Lemma 3.3.2 *Suppose that* **(A3), (H3)** *hold. Then* $\{t\to(\log t)^{n-\alpha}\phi_i(t)\}$ *is convergent uniformly on* $[1,e]$.

Proof: In fact we have for $t\in[1,e]$ similarly to Lemma 3.3.1

$$(\log t)^{n-\alpha}|\phi_1(t)-\phi_0(t)|=\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)F_3(s,\phi_0(s))\frac{ds}{s}\right|$$

$$\le M_0(\log t)^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

So

$$(\log t)^{n-\alpha}|\phi_2(t)-\phi_1(t)|=\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)[F_3(s,\phi_1(s))-F_3(s,\phi_0(s))]\frac{ds}{s}\right|$$

$$\le L_3\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^k|\phi_1(s)-\phi_0(s)|\frac{ds}{s}$$

$$\le L_3\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^k(\log s)^{\alpha-n}M_0(\log s)^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\frac{ds}{s}$$

$$\le M_0L_3\frac{1}{\Gamma(\alpha)}(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^{\alpha+2k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\frac{ds}{s}$$

$$=M_0L_3(\log t)^{\alpha+n+2k+2l}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $i=1,2,\cdots$ that

$$(\log t)^{n-\alpha}|\phi_i(t)-\phi_{i-1}(t)|$$

$$\le M_0L_3^{i-1}(\log t)^{(i-1)\alpha+n+ik+il}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)}$$

$$\le M_0L_3^{i-1}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)},t\in(1,e].$$

Similarly we can prove that

$$\sum_{i=1}^{+\infty}u_i=\sum_{i=1}^{+\infty}M_0L_3^{i-1}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)}$$

is convergent. Hence

$$(\log t)^{n-\alpha}\phi_0(t)+(\log t)^{n-\alpha}[\phi_1(t)-\phi_0(t)]+\cdots+(\log t)^{n-\alpha}[\phi_i(t)-\phi_{i-1}(t)]+\cdots,t\in(1,e]$$

is uniformly convergent. Then $\{t\to(\log t)^{n-\alpha}\phi_i(t)\}$ is convergent uniformly on $(1,e]$. □

Lemma 3.3.3 *Suppose that* **(A3), (H3)** *hold. Then* $\phi(t) = (\log t)^{\alpha-n} \lim_{i\to+\infty} (\log t)^{n-\alpha}\phi_i(t)$ *defined on* $(1, e)$ *is a unique continuous solution of the integral equation*

$$x(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s, x(s))\frac{ds}{s}, t \in (1, e). \quad (3.3.1)$$

Proof: By $\lim_{i\to+\infty}(\log t)^{n-\alpha}\phi_i(t) = (\log t)^{n-\alpha}\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(1, e]$. From

$$(\log t)^{n-\alpha}\left|\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s, \phi_p(s)\frac{ds}{s} - \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s, \phi_q(s))\frac{ds}{s}\right|$$

$$\le L_3||\phi_p - \phi_q||(\log t)^{n-\alpha}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l-1}(\log s)^k(\log s)^{\alpha-n}|\frac{ds}{s}$$

$$\le L_3||\phi_p - \phi_q||(\log t)^{n-\alpha}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l-1}(\log s)^{\alpha+k-n}\frac{ds}{s}$$

$$\le L_3||\phi_p - \phi_q||(\log t)^{\alpha+k+l}\mathbf{B}(\alpha + l, \alpha + k - n + 1)$$

$$\le L_3||\phi_p - \phi_q||\mathbf{B}(\alpha + l, \alpha + k - n + 1) \to 0 \text{ uniformly as } p, q \to +\infty,$$

we know that

$$\phi(t) = (\log t)^{\alpha-n}\lim_{i\to+\infty}(\log t)^{n-\alpha}\phi_i(t)$$

$$= \lim_{i\to+\infty}\left[\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s, \phi_{i-1}(s))\frac{ds}{s}\right]$$

$$= \sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \lim_{i\to+\infty}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s, \phi_{i-1}(s))\frac{ds}{s}$$

$$= \sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s, \phi(s))\frac{ds}{s}.$$

Then ϕ is a continuous solution of (3.3.1) defined on $(1, e]$.

Suppose that ψ defined on $(1, e]$ is also a solution of (3.3.1). Then

$$\psi(t) = \sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s, \psi(s))\frac{ds}{s}, t \in (1, e].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(1, e]$. Since F_3 is a **III-Carathéodory function**, we have

$$M_0 = \sup_{t\in(1,e]}|F_2(t, \psi(t))| = \sup_{t\in(1,e]}|F_3(t, (\log t)^{\alpha-n}(\log t)^{n-\alpha}\psi(t))|$$

$$\le \sup_{t\in(1,e],|x|\le||\psi||}|F_3(t, (\log t)^{\alpha-n}x)| \le +\infty.$$

Then

$$(\log t)^{n-\alpha}|\psi(t)-\phi_0(t)| = (\log t)^{n-\alpha}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p_3(s)F_3(s,\psi(s))\frac{ds}{s}\right|$$

$$\le M_0(\log t)^{n+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$(\log t)^{n-\alpha}|\psi(t)-\phi_1(t)| = (\log t)^{n-\alpha}\frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p_3(s)[F_3(s,\psi(s))-F_3(s,\phi_0(s))]\frac{ds}{s}\right|$$

$$\le M_0L_3(\log t)^{\alpha+n+2k+2l}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$(\log t)^{1-\alpha}|\psi(t)-\phi_{i-1}(t)|$$

$$= (\log t)^{n-\alpha}\frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p_3(s)[F_3(s,\psi(s))-F_3(s,\phi_{i-2}(s))]ds\right|$$

$$\le M_0L_3^{i-1}(\log t)^{(i-1)\alpha+n+ik+il}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)}$$

$$\le M_0L_3^{i-1}\prod_{j=1}^{i-1}\frac{\mathbf{B}(\alpha+l,(j-1)\alpha+jk+(i-1)l+1)}{\Gamma(\alpha)}, t\in(1,e],\ \text{for all } i=1,2,\cdots.$$

Similarly we have $\lim_{i\to+\infty}(\log t)^{n-\alpha}\phi_i(t)=(\log t)^{n-\alpha}\psi(t)$ uniformly on $(1,e]$. Then $\phi(t)\equiv\psi(t)$ on $(1,e]$. Then (3.2.1) has a unique solution ϕ. The proof is completed. □

Theorem 3.3.1 *Suppose that* **(A3), (H3)** *hold. Then x is a solution of IVP* (3.0.3) *if and only if $x\in LC_{n-\alpha}(1,e]$ is a solution of the integral equation* (3.3.1).

Proof: Suppose that x is a solution of IVP (3.0.3). Then $t\to(\log t)^{n-\alpha}x(t)$ is continuous on $[1,e]$ by defining $(\log t)^{n-\alpha}x(t)|_{t=1}=\lim_{t\to1^+}(\log t)^{n-\alpha}x(t)$ and $||x||=r<+\infty$. Since F_3 is a **III-Carathéodory function**, we know

$$M_r=\sup_{t\in(1,e]}|F_2(t,x(t))|=\sup_{t\in(1,e]}|F_3(t,(\log t)^{\alpha-n}(\log t)^{n-\alpha}x(t))|$$

$$\le \sup_{t\in(1,e],|x|\le r}|F_3(t,(\log t)^{\alpha-n}x)|\le+\infty.$$

So

$$\lim_{s\to 1^+}\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}=\lim_{s\to 1^+}\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}(\log w)^{\alpha-n}(\log w)^{n-\alpha}x(w)\frac{dw}{w}$$

$$=\lim_{s\to 1^+}(\log\xi)^{n-\alpha}x(\xi)\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}(\log w)^{\alpha-n}\frac{dw}{w}$$

by mean value theorem of integral, $\xi\in(1,s)$

$$=\lim_{s\to 1^+}(\log\xi)^{n-\alpha}x(\xi)\int_0^1(1-u)^{n-\alpha-1}u^{\alpha-n}du \text{ by } \frac{\log w}{\log s}=u$$

$$=\frac{\eta_n}{\Gamma(\alpha-n+1)}\mathbf{B}(n-\alpha,\alpha-n+1).$$

and for $v\in\mathbb{N}[1,n-1]$ we have

$$\lim_{t\to 1^+}\left(s\frac{d}{ds}\right)^{n-v}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)=\Gamma(n-v-(\alpha-v))\lim_{t\to 1^+}{}^{RLH}D_{0^+}^{\alpha-v}x(t)$$

$$=\Gamma(n-\alpha)\eta_v.$$

From (A3), we have

$$(\log t)^{n-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)F_3(s,x(s))\frac{ds}{s}\right|$$

$$\le(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}M_r(\log s)^k(1-\log s)^l\frac{ds}{s}$$

$$\le M_r(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^k\frac{ds}{s}=M_r(\log t)^{n+k+l}\mathbf{B}(\alpha+l,k+1).$$

So $t\to(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)F_3(s,x(s))\frac{ds}{s}$ is defined on $(1,e]$ and

$$\lim_{t\to 1^+}(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)F_3(s,x(s))\frac{ds}{s}=0. \quad (3.3.2)$$

Then $t\to(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)F_3(s,x(s))\frac{ds}{s}$ is continuous on $[1,e]$ by defining

$$(\log t)^{n-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)F_3(s,x(s))\frac{ds}{s}\Big|_{t=1}=0. \quad (3.3.3)$$

We have ${}^{H}I_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t) = {}^{H}I_{1^+}^{\alpha}p_3(t)F_3(t,x(t))$. So

$$\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_3(s)F_3(s,x(s))\frac{ds}{s} = {}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)] = {}^{H}I_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t)$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(s\frac{d}{ds}\right)^{n}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)\frac{ds}{s}$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}d\left[\left(s\frac{d}{ds}\right)^{n-1}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)\right]$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\left[\left(\log\frac{t}{s}\right)^{\alpha-1}\left(s\frac{d}{ds}\right)^{n-1}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)\Bigg|_1^t\right.$$

$$\left.+(\alpha-1)\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-2}\left(s\frac{d}{ds}\right)^{n-1}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)\frac{ds}{s}\right]$$

$$= -\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}(\log t)^{\alpha-1}\lim_{t\to1^+}\left(s\frac{d}{ds}\right)^{n-1}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)$$

$$+\frac{1}{\Gamma(\alpha-1)}\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-2}\left(s\frac{d}{ds}\right)^{n-1}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)\frac{ds}{s}$$

$$= -\frac{\eta_1}{\Gamma(\alpha)}(\log t)^{\alpha-1} \quad + \frac{1}{\Gamma(\alpha-1)}\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-2}\left(s\frac{d}{ds}\right)^{n-1}\left(\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right)\frac{ds}{s}$$

$$= \cdots\cdots\cdots$$

$$= -\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha-n+1)}\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-n}\left(\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\right)'ds$$

$$= -\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha-n+2)}\frac{1}{\Gamma(n-\alpha)}t\left[\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-n+1}\left(\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\right)'ds\right]'$$

$$= -\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)}t\left[\left(\log\frac{t}{s}\right)^{\alpha-n+1}\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\Bigg|_1^t\right.$$

$$\left.+(\alpha-n+1)\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-n}\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\frac{ds}{s}\right]'$$

$$= -\sum_{v=1}^{n-1}\frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)}t\left[\lim_{t\to1^+}(\log t)^{\alpha-n+1}\int_1^s\left(\log\frac{s}{w}\right)^{n-\alpha-1}x(w)\frac{dw}{w}\right.$$

$$\left.+(\alpha-n+1)\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{\alpha-n}\left(\log\frac{s}{w}\right)^{n-\alpha-1}\frac{ds}{s}x(w)\frac{dw}{w}\right]'$$

$$= -\sum_{v=1}^{n-1} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v} + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} t \left[\frac{\eta_n}{\Gamma(\alpha-n+1)} \mathbf{B}(n-\alpha, \alpha-n+1) (\log t)^{\alpha-n+1}\right.$$

$$\left. + (\alpha-n+1) \int_1^t \int_0^1 (1-u)^{\alpha-n} u^{n-\alpha-1} du x(w) \frac{dw}{w}\right]' = x(t) - \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log t)^{\alpha-v}.$$

Then $x \in LC_{n-\alpha}[1, e]$ is a solution of (3.3.1).

On the other hand, suppose that x is a solution of (3.3.1). Together with Lemmas 3.3.1, 3.3.2, 3.3.3 and the result at the beginning of the proof, it follows that $\lim_{t\to 1^+} (\log t)^{n-\alpha} x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}$ and ${}^{RLH}D_{1^+}^{\alpha-\nu} x(1) = \eta_\nu, \nu \in \mathbb{N}[1, n-1]$. Furthermore, we have

$${}^{RLH}D_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \left(\int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} x(s) \frac{ds}{s}\right)$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \left[\int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(\sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} (\log s)^{\alpha-v}\right.\right.$$

$$\left.\left. + \frac{1}{\Gamma(\alpha)} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1} p_3(w) F_3(w, x(w)) \frac{dw}{w}\right) \frac{ds}{s}\right]$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} (\log s)^{\alpha-v} \frac{ds}{s}$$

$$+ \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1} p_3(w) F_3(w, x(w)) \frac{dw}{w} \frac{ds}{s}$$

$$= \frac{1}{\Gamma(n-\alpha)} \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} \left(t\frac{d}{dt}\right)^n (\log t)^{n-v} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw$$

$$+ \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1} p_3(w) F_3(w, x(w)) \frac{dw}{w} \frac{ds}{s}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \int_u^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(\log \frac{s}{w}\right)^{\alpha-1} \frac{ds}{s} p_3(w) F_3(w, x(w)) \frac{dw}{w}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{w}\right)^{n-1} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-1} dw p_3(w) F_3(w, x(w)) \frac{dw}{w}$$

$$= \frac{1}{\Gamma(n)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{w}\right)^{n-1} p_3(w) F_3(w, x(w)) \frac{dw}{w} = p_3(t) F_3(t, x(t)).$$

So $x \in LC_{n-\alpha}(1, e]$ is a solution of IVP (3.0.3). The proof is completed. □

Theorem 3.3.2 *Suppose that* **(A3), (H3)** *hold. Then*

(i) *IVP (3.0.3) has a unique solution.*

(ii) *If there exist constants $k > -1$, $l \leq 0$ with $l > \max\{-\alpha, -n-k\}$ and $M \geq 0$ such that $|G(t)| \leq M(\log t)^k(1-\log t)^l$ for all $t \in (1,e)$, then following special problem*

$$\begin{cases} {}^{RLH}D_{1^+}^{\alpha}x(t) = G(t),\ t \in (1,e], \\ \lim\limits_{t\to 1^+} (\log t)^{n-\alpha}x(t) = \frac{\eta_n}{\Gamma(\alpha-n+1)}, \\ \lim\limits_{t\to 1^+} {}^{RLH}D_{1^+}^{\alpha-j}x(t) = \eta_j, j \in \mathbb{N}[1,n-1] \end{cases} \tag{3.3.4}$$

has a unique solution

$$x(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{u}\right)^{\alpha-1} G(s)\frac{ds}{s},\ t \in (1,e]. \tag{3.3.5}$$

Proof: **(i)** From Lemma 3.3.1, 3.3.2 and 3.3.3, IVP (3.0.3) has a unique solution.

(ii) From the assumption mentioned and (i), IVP (3.3.4) has a unique solution. We get

$$x(t) \equiv \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)}(\log t)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} G(s)\frac{ds}{s}.$$

By replacing $p_3(t)F_3(t,x(t))$ with $G(t)$. Then x satisfies (3.3.5). The proof is completed. □

3.4 Existence and uniqueness of solutions of (3.0.4)

Choose Picard function sequence as

$$\phi_0(t) = \sum_{j=0}^{n-1} \frac{\eta_j}{j!}(\log t)^j,\ t \in [1,e],$$

$$\phi_i(t) = \phi_0(t) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,\phi_{i-1}(s))\frac{ds}{s}, t \in [1,e], i = 1,2,\cdots.$$

Lemma 3.4.1 *Suppose that* **(A4), (H4)** *hold. Then $\phi_i \in C[1,e]$.*

Proof: On sees $\phi_0 \in C[1,e]$. Since F_4 is a **IV-Carathéodory function**, we know

$$M_0 = \sup_{t\in(1,e]} |F_4(t,\phi_0(t))| = \sup_{t\in(1,e]} \leq \sup_{t\in(1,e],|x|\leq||\phi_0||} |F_4(t,x)| \leq +\infty.$$

By

$$\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,\phi_0(s))\frac{ds}{s}\right|$$

$$\leq \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} M_0(\log s)^k(1-\log s)^l\frac{ds}{s}$$

$$\leq M_0\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l-1} (\log s)^k\frac{ds}{s}$$

$$= M_0(\log t)^{\alpha+k+l}\mathbf{B}(\alpha+l,k+1) \to 0 \text{ as } t \to 1^+,$$

we get that $\lim_{t\to 1^+}\phi_1(t)$ exists and ϕ_1 is continuous on $[1,e]$. Then $\phi_1\in C[1,e]$. By mathematical induction method, we see that $\phi_n\in C(1,e]$. □

Lemma 3.4.2 *Suppose that* **(A4), (H4)** *hold. Then ϕ_i is convergent uniformly on* $[1,e]$.

Proof: Let M_0 be defined in the proof of Lemma 3.4.1. In fact we have for $t\in[1,e]$ that

$$|\phi_1(t)-\phi_0(t)|=\left|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,\phi_0(s))\frac{ds}{s}\right|\le M_0(\log t)^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

So

$$|\phi_2(t)-\phi_1(t)|=\left|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)[F_4(s,\phi_1(s))-F_4(s,\phi_0(s))]\frac{ds}{s}\right|$$

$$\le\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^k(1-\log s)^lL_4|\phi_1(s)-\phi_0(s)|\frac{ds}{s}$$

$$\le M_0L_4\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^k(\log s)^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\frac{ds}{s}$$

$$=M_0L_4(\log t)^{2\alpha+2k+2l}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $i=1,2,\cdots$ that

$$|\phi_i(t)-\phi_{i-1}(t)|\le M_0L_4^{i-1}(\log t)^{i\alpha+ik+il}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha+l,(i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)}$$

$$\le M_0L_4^{i-1}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha+l,(i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)},t\in(1,e].$$

Similarly we can prove that

$$\sum_{i=1}^{+\infty}u_i=\sum_{i=1}^{+\infty}M_0L_4^{i-1}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha+l,(i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)}$$

is convergent. Hence

$$\phi_0(t)+[\phi_1(t)-\phi_0(t)]+[\phi_2(t)-\phi_1(t)]+\cdots+[\phi_i(t)-\phi_{i-1}(t)]+\cdots,t\in[1,e]$$

is uniformly convergent. Then $\{\phi_i(t)\}$ is convergent uniformly on $[1,e]$. □

Lemma 3.4.3 *Suppose that* **(A4), (H4)** *hold. Then $\phi(t)=\lim_{i\to+\infty}\phi_i(t)$ defined on $[1,e]$ is a unique continuous solution of the integral equation*

$$x(t)=\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,x(s))\frac{ds}{s}.\qquad(3.4.1)$$

Proof: By $\lim_{i\to+\infty}\phi_i(t)=\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $[1,e]$. From

$$\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,\phi_p(s))\frac{ds}{s}-\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,\phi_q(s))\frac{ds}{s}\right|$$

$$\leq L_4||\phi_p-\phi_q||\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^k(1-\log s)^l(\log s)^{\alpha-n}\frac{ds}{s}$$

$$\leq L_4||\phi_p-\phi_q||\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^{k+\alpha-n}\frac{ds}{s}$$

$$\leq L_4||\phi_p-\phi_q||(\log t)^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,\alpha+k-n+1)}{\Gamma(\alpha)}\to 0 \text{ uniformly as } p,q\to+\infty,$$

we know that

$$\phi(t)=\lim_{i\to\infty}\phi_i(t)=\lim_{i\to+\infty}\left[\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,\phi_{i-1}(s))\frac{ds}{s}\right]$$

$$=\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,\phi(s))\frac{ds}{s}.$$

Then ϕ is a continuous solution of (3.4.1) defined on $[1,e]$.

Suppose that ψ defined on $[1,e]$ is also a solution of (3.4.1). Then

$$\psi(t)=\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,\psi(s))\frac{ds}{s},t\in(1,e].$$

We need to prove that $\phi(t)\equiv\psi(t)$ on $(0,1]$. Since F_4 is a **IV-Carathéodory function**, we know

$$M_0=\sup_{t\in(1,e]}|F_4(t,\psi(t))|=\sup_{t\in(1,e]}\leq\sup_{t\in(1,e],|x|\leq||\psi||}|F_4(t,x)|\leq+\infty.$$

Now we have

$$|\psi(t)-\phi_0(t)|=\frac{1}{\Gamma(\alpha)}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,\psi(s))]\frac{ds}{s}\right|$$

$$\leq M_0(\log t)^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Furthermore, we have similarly to the proof of Lemma 3.4.2

$$|\psi(t)-\phi_1(t)|=\frac{1}{\Gamma(\alpha)}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)[F_4(s,\psi(s))-F_4(s,\phi_0(s))]\frac{ds}{s}\right|$$

$$\leq M_0L_4(\log t)^{2\alpha+2k+2l}\frac{\mathbf{B}(\alpha+l,\alpha+2k+l+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$|\psi(t)-\phi_{i-1}(t)| = \frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p_4(s)[F_4(s,\psi(s))-F_4(s,\phi_{i-2}(s))]\frac{ds}{s}\right|$$

$$M_0 L_4^{i-1}(\log t)^{i\alpha+ik+il}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha+l,(i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)}$$

$$\leq M_0 L_4^{i-1}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha+l,(i-1)\alpha+ik+(i-1)l+1)}{\Gamma(\alpha)}, t\in(1,e].$$

Similarly we have $\lim_{i\to+\infty}\phi_i(t)=\psi(t)$ uniformly on $[1,e]$. Then $\phi(t)\equiv\psi(t)$ on $[1,e]$. Then (3.4.1) has a unique solution ϕ. The proof is completed. □

Theorem 3.4.1 *Suppose that* **(A4), (H4)** *hold. Then $x\in C[1,e]$ is a solution of IVP* (3.0.4) *if and only if $x\in C[1,e]$ is a solution of the integral equation* (3.4.1).

Proof: Suppose that $x\in C(1,e]$ is a solution of IVP (3.0.4). Then x is continuous on $[1,e]$ by defining $x(t)|_{t=1}=\lim_{t\to1^+}x(t)$ and $||x||=r<+\infty$. Since F_4 is a **IV-Carathéodory function**, we know

$$M_r=\sup_{t\in(1,e]}|F_4(t,x(t))| = \sup_{t\in(1,e]} \leq \sup_{t\in(1,e],|x|\leq r}|F_4(t,x)|\leq+\infty.$$

One can see that

$$\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^k(1-\log s)^l\frac{ds}{s}$$

$$\leq\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^k\frac{ds}{s}\text{ by }\frac{\log s}{\log t}=u$$

$$=(\log t)^{\alpha+k+l}\int_0^1(1-u)^{\alpha+l-1}u^k du$$

$$\leq(\log t)^{\alpha+k+l}\int_0^1(1-u)^{\alpha+l-1}u^k du$$

$$=(\log t)^{\alpha+k+l}\mathbf{B}(\alpha+l,k+1).$$

From (A4), we have for $t\in[1,e]$ that

$$\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}p_4(s)F_4(s,x(s))\frac{ds}{s}\right|$$

$$\leq\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}M_r(\log s)^k(1-\log s)^l\frac{ds}{s}$$

$$\leq M_r\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l-1}(\log s)^k\frac{ds}{s}=M_r(\log t)^{\alpha+k+l}\mathbf{B}(\alpha+l,k+1).$$

So $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,x(s))\frac{ds}{s}$ is defined on $(1,e]$ and

$$\lim_{t\to 1^+} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,x(s))\frac{ds}{s} = 0. \tag{3.4.2}$$

Furthermore, we have $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,x(s))\frac{ds}{s}$ is continuous on $(1,e]$. So

$$t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,x(s))\frac{ds}{s}$$

is continuous on $[1,e]$ by defining

$$\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,x(s))\frac{ds}{s}\Big|_{t=1} = 0. \tag{3.4.3}$$

We have ${}^HI_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t) = {}^HI_{1^+}^{\alpha}p_4(t)F_4(t,x(t))]$. So

$$\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,x(s))\frac{ds}{s} = {}^HI_{1^+}^{\alpha}p_4(t)F_4(t,x(t))$$

$$= {}^HI_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t)$$

$$= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{n-\alpha-1}\left(w\frac{d}{dw}\right)^n x(w)\frac{dw}{w}\right]\frac{ds}{s}$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t\int_u^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left(\log \frac{s}{w}\right)^{n-\alpha-1}\frac{ds}{s}\left(w\frac{d}{dw}\right)^n x(w)\frac{dw}{w}$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_1^t \left(\log \frac{t}{w}\right)^{n-1}\int_0^1 (1-u)^{\alpha-1}u^{n-\alpha-1}du\left(w\frac{d}{dw}\right)^n x(w)\frac{dw}{w}$$

$$= \frac{1}{(n-1)!}\int_1^t \left(\log \frac{t}{w}\right)^{n-1}\left(w\frac{d}{dw}\right)^n x(w)\frac{dw}{w} = \int_1^t \left(\log \frac{t}{w}\right)^{n-1} d\left[\left(w\frac{d}{dw}\right)^{n-1}x(w)\right]$$

$$= \frac{1}{(n-1)!}\left(\log \frac{t}{w}\right)^{n-1}\left[\left(w\frac{d}{dw}\right)^{n-1}x(w)\right]\Big|_1^t + \frac{1}{(n-2)!}\int_1^t \left(\log \frac{t}{w}\right)^{n-2}\left[\left(w\frac{d}{dw}\right)^{n-1}x(w)\right]\frac{dw}{w}$$

$$= -\frac{\eta_{n-1}}{(n-1)!}(\log t)^{n-1} + \frac{1}{(n-2)!}\int_1^t \left(\log \frac{t}{w}\right)^{n-2}\left[\left(w\frac{d}{dw}\right)^{n-1}x(w)\right]\frac{dw}{w}$$

$$= \cdots\cdots\cdots$$

$$= -\sum_{j=1}^{n-2}\frac{\eta_{n-j}}{(n-j)!}(\log t)^{n-j} + \int_1^t x'(w)dw = x(t) - \sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j.$$

So

$$x(t) = \sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_4(s)F_4(s,x(s))\frac{ds}{s}.$$

Then $x \in C[1,e]$ is a solution of (3.4.1).

On the other hand, if $x \in C[1,e]$ is a solution of (3.4.1), we get

$$ {}^{CH}D^{\nu}_{1^+}x(t) = \sum_{j=\nu}^{n-1} \frac{\eta_j}{(j-\nu)!} (\log t)^{j-\nu} + \frac{1}{\Gamma(\alpha-\nu)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\nu-1} p_4(s)F_4(s,x(s))\frac{ds}{s}. $$

Similar to (3.4.2), we have $\lim_{t\to 1^+} x(t) = \eta_0$ and ${}^{CH}D^{\nu}_{1^+}x(1) = \eta_\nu, \nu \in \mathbb{N}[1,n-1]$. Furthermore, we have for $t \in (1,e)$

$$ \left|\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-n} p_4(s)F_4(s,x(s))\frac{ds}{s}\right| \le \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-n} M_r(\log s)^k(1-\log s)^l \frac{ds}{s} $$

$$ \le M_r(1-\log t)^l \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-n} (\log s)^k \frac{ds}{s} $$

$$ = M_r(1-\log t)^l(\log t)^{\alpha-n+k+1}\mathbf{B}(\alpha-n+1,k+1) \to 0 \text{ as } t \to 1^+. $$

Then

$$ {}^{CH}D^{\alpha}_{1^+}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n x(s)\frac{ds}{s} $$

$$ = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n \left(\sum_{j=0}^{n-1} \frac{\eta_j}{j!}(\log s)^j + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} [B(u)x(u)+G(u)]\frac{du}{u}\right)\frac{ds}{s} $$

$$ = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n \sum_{j=0}^{n-1} \frac{\eta_j}{j!}(\log s)^j \frac{ds}{s} $$

$$ +\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n \left(\int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} p_4(u)F_4(u,x(u))\frac{du}{u}\right)\frac{ds}{s} $$

$$ = \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n \left(\int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} p_4(u)F_4(u,x(u))\frac{du}{u}\right)\frac{ds}{s} $$

$$ = \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^{n-1} \left(\int_1^s \left(\log \frac{s}{u}\right)^{\alpha-2} p_4(u)F_4(u,x(u))\frac{du}{u}\right)\frac{ds}{s} $$

$$ = \cdots\cdots\cdots $$

$$ = \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(\int_1^s \left(\log \frac{s}{u}\right)^{\alpha-n} p_4(u)F_4(u,x(u))\frac{du}{u}\right)' ds $$

$$ = \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha+1)}t \left[\int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha} \left(\int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} [B(u)x(u)+G(u)]\frac{du}{u}\right)' ds\right]' $$

$$ = \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha+1)}t \left[\left(\log \frac{t}{s}\right)^{n-\alpha} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-n} p_4(u)F_4(u,x(u))\frac{dw}{w}\Big|_1^t \right. $$

$$ \left. +(n-\alpha)\frac{1}{s}\int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \int_1^s \left(\log \frac{s}{w}\right)^{\alpha-n} p_4(w)F_4(w,x(w))\frac{dw}{w}ds\right]' $$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}t\left[\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\left(\log\frac{s}{w}\right)^{\alpha-n}\frac{ds}{s}p_4(w)F_4(w,x(w))\frac{dw}{w}\right]'$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}t\left[\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\left(\log\frac{s}{w}\right)^{\alpha-n}\frac{ds}{s}p_4(w)F_4(w,x(w))\frac{dw}{w}\right]'$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}t\left[\int_1^t\int_0^1(1-u)^{n-\alpha-1}u^{\alpha-n}du p_4(w)F_4(w,x(w))\frac{dw}{w}\right]' = p_4(t)F_4(t,x(t)).$$

So $x\in C[1,e]$ is a solution of IVP (3.0.4). The proof is completed. □

Theorem 3.4.2 *Suppose that* **(A4), (H4)** *hold. Then*

(i) *IVP (3.0.4) has a unique solution.*

(ii) *If there exist constants $k>-\alpha+n-1$, $l\le 0$ with $l>\max\{-\alpha,-\alpha-k\}$ and $M\ge 0$ such that $|G(t)|\le M(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$, then following special problem*

$$\begin{cases} {}^{CH}D_{0^+}^{\alpha}x(t)=G(t),\ t\in(1,e],\\ \lim_{t\to1^+}\left(t\frac{d}{dt}\right)^j x(t)=\eta_j, j\in\mathbb{N}[0,n-1]\end{cases}\tag{3.4.4}$$

has a unique solution

$$x(t)=\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s},\ t\in(1,e].\tag{3.4.5}$$

Proof: **(i)** From Lemmas 3.4.1, 3.4.2 and 3.4.3, Theorem 3.4.1, IVP (3.0.4) has a unique solution.

(ii) From the assumption mentioned and (i), IVP (3.4.4) has a unique solution. We get

$$x(t)\equiv\sum_{j=0}^{n-1}\frac{\eta_j}{j!}(\log t)^j+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

by replacing $p_4(t)F_4(t,x(t))$ with $G(t)$. Then x satisfies (3.4.5). The proof is completed. □

4 Exact piecewise continuous solutions of LFDEs

In this section, we present exact piecewise continuous solutions of the following linear fractional differential equations (LFDEs), respectively:

$${}^{C}D_{0^+}^{\alpha}x(t)=F_1(t), a.e.,\ t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m],\tag{4.0.1}$$

$${}^{RL}D_{0^+}^{\alpha}x(t)=F_2(t), a.e.,\ t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m],\tag{4.0.2}$$

$${}^{RLH}D_{0^+}^{\alpha}x(t)=F_3(t), a.e.,\ t\in(s_i,s_{i+1}], i\in\mathbb{N}[0,m],\tag{4.0.3}$$

and

$$ {}^{CH}D_{0^+}^{\alpha}x(t)=F_4(t),a.e.,\ t\in(s_i,s_{i+1}],i\in\mathbb{N}[0,m], \tag{4.0.4}$$

where $n-1<\alpha<n$, $\lambda\in\mathbb{R}$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$ in (3.2.1) and (3.2.2) and $1=t_0<t_1<\cdots<t_m<t_{m+1}=e$ in (3.2.3) and (3.2.4). We say that $x:(0,1]\to\mathbb{R}$ is a piecewise solution of (4.0.1) (or (4.0.2)) if $x\in P_mC(0,1]$ (or $P_mC_{n-\alpha}(0,1]$) and satisfies (4.0.1) or (4.0.2). We say that $x:(1,e]\to\mathbb{R}$ is a piecewise continuous solutions of (4.0.3) (or (4.0.4)) if $x\in LP_mC_{n-\alpha}(1,e]$, (or $LP_mC(1,e]$) and x satisfies all equations in (4.0.3) (or (4.0.4)).

(B1) there exists constants $k>-\alpha+n-1$, $l\le 0$ with $l>\max\{-\alpha,-\alpha-k\}$, $M\ge 0$ such that $|F_1(t)|\le Mt^k(1-t)^l$ for all $t\in(0,1)$.

(B2) there exists constants $k>-1$, $l\le 0$ with $l>\max\{-\alpha,-n-k\}$, $M\ge 0$ such that $|F_2(t)|\le t^k(1-t)^l$ for all $t\in(0,1)$.

(B3) there exists constants $k>-1$, $l\le 0$ with $l>\max\{-\alpha,-n-k\}$, $M\ge 0$ such that $|F_3(t)|\le(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$.

(B4) there exists constants $k>-\alpha+n-1$, $l\le 0$ with $l>\max\{-\alpha,-\alpha-k\}$, $M\ge 0$ such that $|F_4(t)|\le(\log t)^k(1-\log t)^l$ for all $t\in(1,e)$.

Theorem 4.0.1 *Suppose that* **(B1)** *holds. Then x is a piecewise solution of* (4.0.1) *if and only if there exist constants $c_{iv}(i\in\mathbb{N}[0,m],v\in\mathbb{N}[0,n-1])\in\mathbb{R}$ such that*

$$x(t)=\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}\frac{c_{\sigma v}}{v!}(t-t_\sigma)^v+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_1(s)ds,t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m]. \tag{4.0.5}$$

Proof: We find for $s\in(0,1)$

$$\begin{aligned}&\left|\int_0^s\frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)}F_1(u)du\right|\le\int_0^s\frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)}u^k(1-u)^ldu\\ &\le(1-s)^l\int_0^s\frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)}u^kdu=(1-s)^ls^{\alpha+k-n+1}\int_0^1\frac{(1-w)^{\alpha-n}}{\Gamma(\alpha-n+1)}w^kdw\\ &=(1-s)^ls^{\alpha+k-n+1}\frac{\mathbf{B}(\alpha-n+1,k+1)}{\Gamma(\alpha-n+1)}\to 0\text{ as }s\to 0^+.\end{aligned} \tag{4.0.6}$$

We also have for $t\in(t_i,t_{i+1}]$ that

$$\begin{aligned}&\left|\int_0^t(t-s)^{\alpha-1}F_1(s)ds\right|\le\int_0^t(t-s)^{\alpha-1}|F_1(s)|ds\le\int_0^t(t-s)^{\alpha-1}s^k(1-s)^lds\\ &\le\int_0^t(t-s)^{\alpha+l-1}s^kds=t^{\alpha+k+l}\int_0^1(1-w)^{\alpha+l-1}w^kdw=t^{\alpha+k+l}\mathbf{B}(\alpha+l,k+1).\end{aligned}$$

Then $\int_0^t(t-s)^{\alpha-1}F_1(s)ds$ is continuous on $[0,1]$.

If x is a solution of (4.0.5), then we know that $\lim_{t\to t_i^+} x(t)$ $(i \in \mathbb{N}[0,m])$ exist and $x \in P_mC(0,1]$. Now we prove that x satisfies differential equation in (4.0.1). In fact, for $t \in (t_0, t_1]$ we have by Theorem 3.1.2 ${}^CD_{0^+}^{\alpha} x(t) = F(t)$. For $t \in (t_i, t_{i+1}]$ $(i \in \mathbb{N}[1,m])$, we have by Definition 2.3

$$
\begin{aligned}
&{}^CD_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)ds \\
&= \frac{1}{\Gamma(n-\alpha)} \left[\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} x^{(n)}(s)ds + \int_{t_i}^t (t-s)^{n-\alpha-1} x^{(n)}(s)ds \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \left[\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} \left(\sum_{\sigma=0}^{j} \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!}(s-t_\sigma)^v \ + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u)du \right)^{(n)} ds \right. \\
&\left. + \int_{t_i}^t (t-s)^{n-\alpha-1} \left(\sum_{\sigma=0}^{i} \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!}(s-t_\sigma)^v \ + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u)du \right)^{(n)} ds \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u)du \right)^{(n)} ds \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u)du \right)' ds \\
&= \frac{1}{\Gamma(n-\alpha+1)} \left[\int_0^t (t-s)^{n-\alpha} \left(\int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u)du \right)' ds \right]' \\
&= \frac{1}{\Gamma(n-\alpha+1)} \left[(t-s)^{n-\alpha} \left(\int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u)du \right) \Big|_0^t \text{ using (4.0.6)} \right. \\
&\left. + (n-\alpha) \int_0^t (t-s)^{n-\alpha-1} \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u)duds \right]' \\
&= \frac{1}{\Gamma(n-\alpha)} \left[\int_0^t (t-s)^{n-\alpha-1} \int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u)duds \right]' \\
&= \frac{1}{\Gamma(n-\alpha)} \left[\int_0^t \int_u^t (t-s)^{n-\alpha-1} \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} ds F_1(u)du \right]' \\
&= \frac{1}{\Gamma(n-\alpha)} \left[\int_0^t \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-n}}{\Gamma(\alpha-n+1)} dw F_1(u)du \right]' \text{ by } \frac{s-u}{t-u} = w \\
&= F_1(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m].
\end{aligned}
$$

We have done that x satisfies (4.0.1) if x satisfies (4.0.5).

Now, we suppose that x is a solution of (4.0.1). We will prove that x satisfies (4.0.5) by mathematical induction method. Since x is continuous on $(t_i, t_{i+1}]$ and the limit $\lim_{t\to t_i^+} x(t)$ ($i \in \mathbb{N}[0,m]$) exists, then $x \in P_mC(0,1]$. For $t \in (t_0, t_1]$, we know from Theorem 3.1.2 that there exists $c_{0v} \in \mathbb{R}$ such that

$$x(t) = \sum_{v=0}^{n-1} \frac{c_{0v}}{v!} t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(s)ds, t \in (t_0, t_1].$$

Then (4.0.5) holds for $i = 0$. We suppose that (4.0.5) holds for all $i \in \mathbb{N}[0,j], j \le m-1$. We derive the expression of x on $(t_{j+1}, t_{j+2}]$. In order to get the expression of x on $(t_{j+1}, t_{j+2}]$, suppose that

$$x(t) = \Phi(t) + \sum_{\sigma=0}^{j} \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!}(t-t_\sigma)^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1(s)ds, t \in (t_{j+1}, t_{j+2}]. \tag{4.0.7}$$

By ${}^CD_{0^+}^\alpha x(t) = F_1(t), t \in (t_{j+1}, t_{j+2}]$, we get

$$F_1(t) = {}^CD_{0^+}^\alpha x(t) = \frac{1}{\mathbf{\Gamma}(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)ds$$

$$= \sum_{\rho=0}^{j} \int_{t_\rho}^{t_{\rho+1}} \frac{(t-s)^{n-\alpha-1}}{\mathbf{\Gamma}(n-\alpha)} \left(\sum_{\sigma=0}^{\rho} \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!}(s-t_\sigma)^v + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u)du \right)^{(n)} ds$$

$$+ \int_{t_{j+1}}^t \frac{(t-s)^{n-\alpha-1}}{\mathbf{\Gamma}(n-\alpha)} \left(\Phi(s) + \sum_{\sigma=0}^{j} \sum_{v=0}^{n-1} \frac{c_{\sigma v}}{v!}(s-t_\sigma)^v + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u)du \right)^{(n)} ds$$

$$= {}^CD_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\mathbf{\Gamma}(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} F_1(u)du \right)^{(n)} ds$$

$$= {}^CD_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\mathbf{\Gamma}(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u)du \right)' ds$$

$$= {}^CD_{t_{j+1}^+}^\alpha \Phi(t) + \frac{1}{\mathbf{\Gamma}(n-\alpha+1)} \left[\int_0^t (t-s)^{n-\alpha} \left(\int_0^s \frac{(s-u)^{\alpha-n}}{\Gamma(\alpha-n+1)} F_1(u)du \right)' ds \right]'.$$

By a similar computation, we get

$$F_1(t) = F_1(t) + {}^cD_{t_{j+1}^+}^\alpha \Phi(t).$$

It follows that ${}^CD_{t_{j+1}^+}^\alpha \Phi(t) = 0$ for all $t \in (t_{j+1}, t_{j+2}]$. By Theorem 3.1.2, we know that there exists $c_{j+1v} \in \mathbb{R}$ ($v \in \mathbb{N}[0, n-1]$) such that $\Phi(t) = \sum_{v=0}^{n-1} \frac{c_{j+1v}}{v!}(t-t_{j+1})^v$ for $t \in (t_{j+1}, t_{j+2}]$. Substituting Φ into (4.0.7), we get that (4.0.5) holds for $i = j+1$. By the mathematical induction method, we know that x satisfies (4.0.5) and $x|_{(t_i,t_{i+1}]}$ is continuous and $\lim_{t\to t_i^+} x(t)$ exists. The proof is completed. □

Remark 4.0.1 It is easy to see from (4.0.5) that x is a solution of (4.0.1) if and only if x satisfies that there exist numbers $C_{iv}(i \in \mathbb{N}[0,m], v \in \mathbb{N}[0,n-1]$ such that

$$x(t) = \sum_{v=0}^{n-1} C_{iv}t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_1(s)ds, t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m].$$

In [73], authors proved the similar result under a strong assumption that F_1 is continuous on $[0,1]$. So our result generalizes the result in [73]. One sees that (4.0.5) is more convenient for use. For example, when consider the following impulsive problem

$$^CD_{0^+}^{\alpha}x(t) = F_1(t), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m],$$

$$\Delta x^{(j)}(t_i) = a_{ij}, i \in \mathbb{N}[1,m], j \in \mathbb{N}[0,n-1],$$

$$x^{(j)}(0) = a_{0j}, j \in \mathbb{N}[0,n-1]$$

where $\alpha \in (n-1,n)$, F_1 satisfies **(B1)**. Then we can easily get by (4.0.5)

$$c_{0v} = a_{0v},\ v \in \mathbb{N}[0,n-1],\ \ c_{iv} = a_{iv},\ v \in \mathbb{N}[0,n-1], i \in \mathbb{N}[1,m].$$

So

$$x(t) = \sum_{\sigma=1}^{i}\sum_{v=0}^{n-1} \frac{a_{\sigma v}}{v!}(t-t_\sigma)^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_1(s)ds, t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m].$$

Theorem 4.0.2 *Suppose that* **(B2)** *holds. Then x is a solution of* (4.0.2) *if and only if there exist constants $c_{\sigma v}(\sigma \in \mathbb{N}[0,m], v \in \mathbb{N}[1,n]) \in \mathbb{R}$ such that*

$$x(t) = \sum_{\sigma=0}^{i}\sum_{v=1}^{n} \frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_2(s)ds, t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m]. \quad (4.0.8)$$

Proof: For $t \in (t_j,t_{j+1}]$ $(j \in \mathbb{N}[0,m])$, similarly to the beginning of the proof of Theorem 4.0.1 we know that

$$t^{n-\alpha}\left|\int_0^t (t-s)^{\alpha-1}F_2(s)ds\right| \le \int_0^t (t-s)^{\alpha-1}|F_2(s)|ds$$

$$\le t^{n-\alpha}\int_0^t (t-s)^{\alpha-1}s^k(1-s)^l ds \le t^{n-\alpha}\int_0^t (t-s)^{\alpha+l-1}s^k ds$$

$$= t^{n+k+l}\int_0^1 (1-w)^{\alpha+l-1}w^k dw = t^{n+k+l}\mathbf{B}(\alpha+l,k+1).$$

So $t^{n-\alpha}\int_0^t (t-s)^{\alpha-1}F_2(s)ds$ is continuous on $[0,1]$.

If x is a solution of (4.0.8), we have $x \in P_mC_{1-\alpha}[0,1]$. It follows for $t \in (t_i,t_{i+1}]$ and from

Definition 2.2 that

$$
\begin{aligned}
&{}^{RL}D_{0^+}^{\alpha}x(t)=\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\int_0^t(t-s)^{n-\alpha-1}x(s)ds\right]^{(n)}\\
&=\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(s-t_\sigma)^{\alpha-v}ds\right]^{(n)}\\
&+\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\int_{t_i}^t(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{i}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(s-t_\sigma)^{\alpha-v}ds\right]^{(n)}\\
&+\frac{1}{\Gamma(n-\alpha)}\left[\int_0^t(t-s)^{n-\alpha-1}\int_0^s\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}F_2(u)duds\right]^{(n)}\\
&=\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}(s-t_\sigma)^{\alpha-v}ds\right]^{(n)}\\
&+\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}\int_{t_i}^t(t-s)^{n-\alpha-1}(s-t_\sigma)^{\alpha-v}ds\right]^{(n)}\\
&+\frac{1}{\Gamma(n-\alpha)}\left[\int_0^t\int_u^t(t-s)^{n-\alpha-1}\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}dsF_2(u)du\right]^{(n)}\\
&=\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{j=0}^{i-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{n-v}\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}(1-w)^{n-\alpha-1}w^{\alpha-v}dw\right]^{(n)}\\
&+\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{n-\alpha-1}w^{\alpha-v}dw\right]^{(n)}\\
&+\frac{1}{\Gamma(n-\alpha)}\left[\int_0^t(t-u)^{n-1}\int_0^1(1-w)^{n-\alpha-1}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwF_2(u)du\right]^{(n)}\\
&=\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{\sigma=0}^{i-1}\sum_{j=\sigma}^{i-1}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{n-v}\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}(1-w)^{n-\alpha-1}w^{\alpha-v}dw\right]^{(n)}\\
&+\frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{n-\alpha-1}w^{\alpha-v}dw\right]^{(n)}\\
&+\left[\int_0^t\frac{(t-u)^{n-1}}{(n-1)!}F_2(u)du\right]^{(n)}
\end{aligned}
$$

$$= \frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{\sigma=0}^{i}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{n-v}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-v}dw\right]^{(n)}+F_2(t)$$

$$= F_2(t), t \in (t_i, t_{i+1}].$$

It follows that x is a solution of (4.0.2).

Now we prove that if x is a solution of (4.0.2), then x satisfies (4.0.8) and $x \in P_mC_{n-\alpha}[0,1]$ by mathematical induction method. By Theorem 3.2.2, we know that there exists a constant $c_{0v} \in \mathbb{R}(v \in \mathbb{N}[1,n]$ such that

$$x(t) = \sum_{v=1}^{n}\frac{c_{0v}}{\Gamma(\alpha-v+1)}t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_2(s)ds, t \in (t_0, t_1].$$

Hence (4.0.8) holds for $i=0$. Assume that (4.0.8) holds for $i = 0,1,2,\cdots,j \le m-1$, we will prove that (4.0.8) holds for $i = j+1$. Suppose that

$$x(t) = \Phi(t) + \sum_{\sigma=0}^{j}\sum_{v=1}^{n}\frac{c_{\sigma v}}{\Gamma(\alpha-v+1)}(t-t_\sigma)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_2(s)ds, t \in (t_{j+1}, t_{j+2}].$$

Then for $t \in (t_{j+1}, t_{j+2}]$ we have

$$F(t) = {}^{RL}D_{0^+}^{\alpha}x(t) = \frac{1}{\mathbf{\Gamma}(n-\alpha)}\left[\sum_{\rho=0}^{j}\int_{t_\rho}^{t_{\rho+1}}(t-s)^{n-\alpha-1}x(s)ds + \int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}x(s)ds\right]^{(n)}.$$

Similarly to the above discussion we can get

$$F(t) = {}^{RL}D_{0^+}^{\alpha}x(t) = F(t) + {}^{RL}D_{t_{j+1}^+}^{\alpha}\Phi(t).$$

So ${}^{RL}D_{t_{j+1}^+}^{\alpha}\Phi(t) = 0$ on $(t_{j+1}, t_{j+2}]$. Then Theorem 3.2.2 implies that there exists a constant $c_{j+1v} \in \mathbb{R}$ such that $\Phi(t) = \sum_{v=1}^{n}\frac{c_{j+1v}}{\Gamma(\alpha-v+1)}(t-t_{i+1})^{\alpha-v}(t-t_{j+1})^{\alpha}$ on $(t_{j+1}, t_{j+2}]$. Hence

$$x(t) = \sum_{\rho=0}^{j+1}\sum_{v=1}^{n}\frac{c_{\rho v}}{\Gamma(\alpha-v+1)}(t-t_\rho)^{\alpha-v} + \int_1^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F_2(s)ds, t \in (t_{j+1}, t_{j+2}].$$

By mathematical induction method, we know that (4.0.8) holds for $j \in \mathbb{N}[0-, m]$. The proof is completed. □

Theorem 4.0.3 *Suppose that* **(B3)** *holds. Then x is a solution of* (4.0.3) *if and only if there exist constants $c_{jv} \in \mathbb{R}$ $(j \in \mathbb{N}[0,m], v \in \mathbb{N}[1,n])$ such that*

$$x(t) = \sum_{j=0}^{i}\sum_{v=1}^{n}\frac{c_{jv}}{\Gamma(\alpha-v+1)}\left(\log\frac{t}{t_j}\right)^{\alpha-v} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}F_3(s)\frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m]. \tag{4.0.9}$$

Proof: The proof is similar to the proof of Theorem 4.0.2 and is omitted. □

Theorem 4.0.4 *Suppose that* **(B4)** *holds. Then x is a piecewise solution of* (4.0.4) *if and only if there exist constants $c_{jv} \in \mathbb{R}$ $(j \in \mathbb{N}[0,m], v \in \mathbb{N}[1,n])$ such that*

$$x(t) = \sum_{\rho=0}^{j}\sum_{v=0}^{n-1} \frac{c_{\rho v}}{v!}\left(\log\frac{t}{t_\rho}\right)^v + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} F_4(s)\frac{ds}{s}, t \in (t_j, t_{j+1}], j \in \mathbb{N}[0,m]. \quad (4.0.10)$$

Proof: The proof is similar to the proof of Theorem 4.0.1 and is omitted. □

5 Main results

In this section, we establish existence results for solutions of BVP (1.0.i) $(i = 8, 9, 10, 11)$ respectively. The following two well known fixed point theorems will be of use in the sections to follow. In particular, the Nonlinear Alternative ([28]: Theorem 5.1, p. 61) will be employed.

Lemma 5.0.1 [Nonlinear Alternative] *Let X be a normed space with C a convex subset of X. Let U be an open subset of C with $0 \in C$ and consider a compact map $H : \overline{U} \to C$. If*

$$u \neq \lambda H u \quad \text{for all } u \in \partial U \text{ and for all } \lambda \in [0,1]$$

then H has at least one fixed–point.

5.1 Solvability of BVP (1.0.8)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.8).

Lemma 5.1.1 *Suppose that $\sigma : (0,1) \to \mathbb{R}$ is continuous and satisfies that there exist numbers $k > 1 - \beta$ and $l \leq 0$ with $l > \max\{-\beta, -\beta - k\}$ such that $|\sigma(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$. The x is a solutions of*

$$\begin{cases} {}^C D_{0^+}^{\beta} x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ x(0) = a, \ \ x'(1) = b, \\ \Delta x(t_i) = I_i, \ \Delta x'(t_i) = J_i, \ i \in \mathbb{N}[1,m] \end{cases} \quad (5.1.1)$$

if and only if x *and*

$$x(t)=a+\left[b-\int_0^1\frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}\sigma(s)ds-\sum_{\sigma=1}^{m}J_\sigma\right]t$$
$$+\sum_{\sigma=1}^{i}[I_\sigma+(t-t_\sigma)J_\sigma]+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m]. \tag{5.1.2}$$

Proof: Let x be a solution of (5.1.1). We know by Theorem 4.0.1 that there exist numbers $c_{\sigma0}, c_{\sigma1}\in\mathbb{R}$ $(\sigma\in\mathbb{N}[0,m])$ such that

$$x(t)=\sum_{\sigma=0}^{i}[c_{\sigma0}+c_{\sigma1}(t-t_\sigma)]+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m]. \tag{5.1.3}$$

So

$$x'(t)=\sum_{\sigma=0}^{i}c_{\sigma1}+\int_0^t\frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\sigma(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m]. \tag{5.1.4}$$

By

$$\left|\int_0^t(t-s)^{\beta-1}\sigma(s)ds\right|\le\int_0^t(t-s)^{\beta-1}s^k(1-s)^l ds$$
$$\le\int_0^t(t-s)^{\beta+l-1}s^k ds=t^{\beta+k+l})\mathbf{B}(\beta+l,k+1)\to 0 \text{ as } t\to 0^+,$$

together with (5.1.3), (5.1.4) and the boundary conditions and the impulse assumption in (5.1.1) that

$$c_{00}=a,\quad c_{\sigma0}=I_\sigma,\quad c_{\sigma1}=J_\sigma,\quad \sigma\in\mathbb{N}[1,m],$$

$$\sum_{\sigma=0}^{m}c_{\sigma1}+\int_0^1\frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}\sigma(s)ds=b.$$

Then

$$c_{01}=b-\int_0^1\frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}\sigma(s)ds-\sum_{\sigma=1}^{m}J_\sigma. \tag{5.1.5}$$

Substituting $c_{\sigma0}, c_{\sigma1}(\sigma\in\mathbb{N}[0,m])$ into (5.1.3), we get (5.1.2) obviously.

On the other hand, if x satisfies (5.1.2), then $x|_{(t_i,t_{i+1}]}$ $(i\in\mathbb{N}[0,m])$ are continuous and the limits $\lim_{t\to t_i^+}x(t)$ $(i\in\mathbb{N}[0,m])$ exist. So $x\in P_mC(0,1]$. Using (5.1.5) and $c_{00}=a$, $c_{\sigma0}=I_\sigma+t_\sigma J_\sigma$, $c_{\sigma1}=J_\sigma$, $\sigma\in\mathbb{N}[1,m]$, we rewrite x by

$$x(t)=\sum_{\sigma=0}^{i}[c_{\sigma0}+c_{\sigma1}(t-t_\sigma)]+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,1].$$

Since σ is continuous on $(0,1)$ and $|\sigma(t)|\le t^k(1-t)^l$, one can show easily that x is continuous on $(t_i,t_{i+1}](i=0,1)$ and the limits $\lim_{t\to t_i^+}x(t)$ $(i\in\mathbb{N}[0,m])$ exist. So $x\in P_mC(0,1]$.

Furthermore, by direct computation, we have $x(0) = a, x'(1) = b$, $\lim_{t\to t_i^+} x(t) - x(t_i) = I_i$ and $\lim_{t\to t_i^+} x(t) - x(t_i) = J_i$. Furthermore, we have by Theorem 4.0.1

$$^{C}D_{0^+}^{\beta}x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

So x is a solution of (5.1.1). The proof is completed. □

Define the nonlinear operator Q on $P_mC(0, 1]$ by Qx for $x \in P_mC(0, 1]$ with

$$\begin{aligned}(Qx)(t) &= \int_0^1 \phi(s)G(s, x(s))ds \\ &+ \left[\int_0^1 \psi(s)H(s, x(s))ds - \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} p(s)f(s, x(s))ds - \sum_{\sigma=1}^{m} J(t_\sigma, x(t_\sigma))\right] t \\ &+ \sum_{\sigma=1}^{i} [I(t_\sigma, x(t_\sigma)) + (t - t_\sigma)J(t_\sigma, x(t_\sigma))] \\ &+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} p(s)f(s, x(s))ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].\end{aligned}$$

Lemma 5.1.2 *Suppose that* (a)-(e) *hold, and* f, G, H *are* **impulsive I-Carathéodory functions**, I, J **discrete I-Carathéodory functions**. *Then* $Q : P_mC(0, 1] \to P_mC(0, 1]$ *is well defined and is completely continuous,* $x \in P_mC(0, 1]$ *is a solution of BVP* (1.0.8) *if and only if* $x \in P_mC(0, 1]$ *is a fixed point of* Q.

Proof. The proof is similar to that of Theorem 3.1 in [85] and is omitted.

Theorem 5.1.1 *Suppose that* (a)-(e) *hold,* f, G, H *are* **impulsive I-Carathéodory functions**, I, J **discrete I-Carathéodory functions** *and*

(C1) *there exist nondecreasing functions* $M_f, M_g, M_h, M_I, M_J : [0, +\infty) \to [0, +\infty)$ *such that*

$$|f(t, x)| \le M_f(|x|), t \in (0, 1), x \in \mathbb{R},$$

$$|G(t, x)| \le M_G(|x|), t \in (0, 1), x \in \mathbb{R},$$

$$|H(t, x)| \le M_H(|x|), t \in (0, 1), x \in \mathbb{R},$$

$$|I(t_i, x)| \le M_I(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R},$$

$$|J(t_i, x)| \le M_J(|x|), i \in \mathbb{N}[1, m], x \in \mathbb{R}.$$

Then BVP (1.0.8) *has at least one solution if there exists* $r_0 > 0$ *such that*

$$||\phi||_1 M_G(r_0) + ||\psi||_1 M_H(r_0) + mM_I(r_0) + 2mM_J(r_0) \\ + \left[\frac{\mathbf{B}(\beta+l-1,k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(r_0) \le r_0. \tag{5.1.6}$$

Proof: From Lemma 5.1.1, Lemma 5.1.2 and the definition of Q, $x \in P_mC(0,1]$ is a solution of BVP (1.0.7) if and only if $x \in P_mC(0,1]$ is a fixed point of Q. Lemma 5.1.2 implies that Q is a completely continuous operator. From (C1), we have for $x \in P_mC(0,1]$ that

$$|f(t,x(t))| \le M_f(|x(t)|) \le M_f(||x||), t \in (0,1),$$

$$|G(t,x(t))| \le M_G(||x||), t \in (0,1),$$

$$|H(t,x(t))| \le M_H(||x||), t \in (0,1),$$

$$|I(t_i,x(t_i))| \le M_I(||x||), i \in \mathbb{N}[1,m],$$

$$|J(t_i,x(t_i))| \le M_J(||x||), i \in \mathbb{N}[1,m].$$

We consider the set $\Omega = \{x \in P_mC(0,1] : x = \lambda(Tx),\ \text{for some}\ \lambda \in [0,1)\}$. For $x \in \Omega$, we have for $t \in (t_i, t_{i+1}]$ that

$$|(Qx)(t)| \le ||\phi||_1 M_G(||x||) + ||\psi||_1 M_H(||x||) + \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} s^k (1-s)^l ds M_f(||x||) + mM_J(||x||)$$

$$+ mM_I(||x||) + mM_J(||x|) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s^k (1-s)^l ds M_f(||x||)$$

$$\le ||\phi||_1 M_G(||x||) + ||\psi||_1 M_H(||x||) + mM_I(||x||) + 2mM_J(||x||)$$

$$+ \left[\frac{\mathbf{B}(\beta+l-1,k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(||x||).$$

It follows that

$$||x|| = \lambda||Tx|| < ||Tx|| \le ||\phi||_1 M_G(||x||) + ||\psi||_1 M_H(||x||) + mM_I(||x||) + 2mM_J(||x||)$$

$$+ \left[\frac{\mathbf{B}(\beta+l-1,k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(||x||).$$

From (5.1.6), we choose $\Omega = \{x \in P_mC(0,1] : ||x|| < r_0\}$ and $C = X$. Then Ω is a open subset of C with $0 \in C$.

If there exists $x \in \partial\Omega$ such that $x = Qx$, then x is a solution of BVP (1.0.7). If there exist some $x \in \partial\Omega$ and $\lambda \in [0, 1)$ such that $x = \lambda Qx$, then

$$r_0 = ||x|| = \lambda||Qx|| < ||Qx|| \leq ||\phi||_1 M_G(r_0) + ||\psi||_1 M_H(r_0) + mM_I(r_0) + 2mM_J(r_0)$$

$$+ \left[\frac{\mathbf{B}(\beta+l-1,k+1)}{\Gamma(\beta-1)} + \frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(r_0) \leq r_0,$$

a contradiction. As a consequence of Lemma 5.0.1, we deduce that Q has a fixed point which is a solution of the problem BVP (1.0.8). The proof is completed. The proof of Theorem 5.1.1 is completed. □

Corollary 5.1.1 *Suppose that* (a)-(e)*, and* (C1) *hold. Then BVP* (1.0.8) *has at least one solution if*

$$\inf_{r\in(0,+\infty)} \frac{||\phi||_1 M_G(r)+||\psi||_1 M_H(r)+mM_I(r)+2mM_J(r)+\left[\frac{\mathbf{B}(\beta+l-1,k+1)}{\Gamma(\beta-1)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(r)}{r} < 1.$$

Proof: From the assumption, we know that there exists $r_0 > 0$ such that (5.1.6) holds. By Theorem 5.1.1, BVP (1.0.8) has at least one solution. The proof is omitted. □

Corollary 5.1.2 *Suppose that* (a)-(e)*, and* (C1) *hold. Then BVP* (1.0.8) *has at least one solution if*

$$\lim_{r\to+\infty} \frac{||\phi||_1 M_G(r)+||\psi||_1 M_H(r)+mM_I(r)+2mM_J(r)+\left[\frac{\mathbf{B}(\beta+l-1,k+1)}{\Gamma(\beta-1)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(r)}{r} = 0 \ \ or$$

$$\lim_{r\to 0} \frac{||\phi||_1 M_G(r)+||\psi||_1 M_H(r)+mM_I(r)+2mM_J(r)+\left[\frac{\mathbf{B}(\beta+l-1,k+1)}{\Gamma(\beta-1)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(r)}{r} = 0.$$

Proof: From the assumption, we know that there exists $r_0 > 0$ such that (5.1.6) holds. By Theorem 5.1.1, BVP (1.0.7) has at least one solution. The proof is omitted. □

5.2 Solvability of BVP (1.0.9)

In this subsection, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.9).

Lemma 5.2.1 *Suppose that $\sigma : (0, 1) \to \mathbb{R}$ is continuous and satisfies that there exist numbers $k > -1$ and $\max\{-\beta, -2-k\} < l \leq 0$ such that $|\sigma(t)| \leq t^k(1-t)^l$ for all $t \in (0, 1)$.*

The x is a solutions of

$$\begin{cases} {}^{RL}D_{0^+}^{\beta}x(t)=\sigma(t), t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m],\\ \lim\limits_{t\to0^+}t^{2-\beta}x(t)=a,\ \ x(1)=b,\\ \lim\limits_{t\to t_i^+}(t-t_i)^{2-\beta}x(t)=I_i, i\in\mathbb{N}[1,m],\\ \Delta^{RL}D_{0^+}^{\beta-1}x(t_i)=J_i, i\in\mathbb{N}[1,m],\end{cases}\tag{5.2.1}$$

if and only if x satisfies

$$x(t)=t^{\beta-1}\left[b-a-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds-\sum_{\sigma=1}^{m}\left(\frac{J_\sigma(1-t_\sigma)^{\beta-1}}{\Gamma(\beta)}+I_\sigma(1-t_\sigma)^{\beta-2}\right)\right]+at^{\beta-2}$$
$$+\sum_{\sigma=1}^{i}\left[\frac{(t-t_\sigma)^{\beta-1}}{\Gamma(\beta)}J_\sigma+(t-t_\sigma)^{\beta-2}I_\sigma\right]+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m].\tag{5.2.2}$$

Proof: Let x be a solution of (5.2.1). By Theorem 4.0.2, we know that there exist numbers $c_{\sigma0},c_{\sigma1}\in\mathbb{R}(\sigma\in\mathbb{N}[1,n])$ such that

$$x(t)=\sum_{\sigma=0}^{i}\left[\frac{c_{\sigma1}(t-t_\sigma)^{\beta-1}}{\Gamma(\beta)}+\frac{c_{\sigma2}(t-t_\sigma)^{\beta-2}}{\Gamma(\beta-1)}\right]+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m].\tag{5.2.3}$$

It follows that

$${}^{RL}D_{0^+}^{\beta-1}x(t)=\sum_{\sigma=0}^{i}c_{\sigma1}+\int_0^t\sigma(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m].\tag{5.2.4}$$

It follows from the boundary conditions and the impulse assumption in (3.3.1) that $c_{02}=\Gamma(\beta-1)a$, $c_{\sigma2}=\Gamma(\beta-1)I_\sigma(\sigma\in\mathbb{N}[1,m])$, $c_{\sigma1}=J_\sigma(\sigma\in\mathbb{N}[1,m])$ and

$$\sum_{\sigma=0}^{m}\left[\frac{c_{\sigma1}(1-t_\sigma)^{\beta-1}}{\Gamma(\beta)}+\frac{c_{\sigma2}(1-t_\sigma)^{\beta-2}}{\Gamma(\beta-1)}\right]+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds=b.$$

Then

$$c_{0,1}=\Gamma(\beta)\left[b-a-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\sigma(s)ds-\sum_{\sigma=1}^{m}\left(\frac{J_\sigma(1-t_\sigma)^{\beta-1}}{\Gamma(\beta)}+I_\sigma(1-t_\sigma)^{\beta-2}\right)\right].\tag{5.2.5}$$

Substituting $c_{\sigma1},c_{\sigma2}(\sigma\in\mathbb{N}[0,m])$ into (5.2.3), we get (5.2.2) obviously.

On the other hand, if x satisfies (3.3.2), then $x|_{(t_i,t_{i+1}]}(i\in\mathbb{N}[0,m])$ are continuous and the limits $\lim\limits_{t\to t_i^+}(t-t_i)^{2-\beta}x(t)(i\in\mathbb{N}[0,m])$ exist. So $x\in P_mC_{2-\beta}(0,1]$. Using (3.3.5) and $c_{02}=\Gamma(\beta-1)a$, $c_{\sigma2}=\Gamma(\beta-1)I_\sigma(\sigma\in\mathbb{N}[1,m])$, $c_{\sigma1}=J_\sigma(\sigma\in\mathbb{N}[1,m])$, we rewrite x by (5.2.3). Since σ is continuous on $(0,1)$ and $|\sigma(t)|\le t^k(1-t)^l$, one can show easily that x is continuous on $(t_i,t_{i+1}](i=0,1)$ and using the method at the beginning of the proof of this lemma, we know that both the limits $\lim_{t\to t_i^+}(t-t_i)^{2-\beta}x(t)(i\in\mathbb{N}[0,m])$ exist. So $x\in P_mC_{2-\beta}(0,1]$. Furthermore, by direct computation, we have $x(1)=b$, and $\lim_{t\to 0^+}t^{2-\beta}x(t)=a$. One have from Theorem 4.0.2 similarly that $D_{0^+}^{\beta}x(t)=\sigma(t)$ and for $t\in(t_i,t_{i+1}](i\in\mathbb{N}_0)$. So x is a solution of (5.2.1). The proof is completed. □

Define the nonlinear operator T on $P_mC_{2-\beta}(0,1]$ for $x\in P_mC_{2-\beta}(0,1]$ by

$$(Tx)(t)=t^{\beta-1}\Big[\int_0^1\psi(s)H(s,x(s))ds-\int_0^1\phi(s)G(s,x(s))ds-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q(s)f(s,x(s))ds$$

$$-\sum_{\sigma=1}^{m}\Big(\frac{J(t_\sigma,x(t_\sigma))(1-t_\sigma)^{\beta-1}}{\Gamma(\beta)}+I(t_\sigma,x(t_\sigma))(1-t_\sigma)^{\beta-2}\Big)\Big]+t^{\beta-2}\int_0^1\phi(s)G(s,x(s))ds$$

$$+\sum_{\sigma=1}^{i}\Big[\frac{(t-t_\sigma)^{\beta-1}}{\Gamma(\beta)}J(t_\sigma,x(t_\sigma))+(t-t_\sigma)^{\beta-2}I(t_\sigma,x(t_\sigma))\Big]$$

$$+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q(s)f(s,x(s))ds,t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m].$$

Lemma 5.2.2 *Suppose that* (a)-(e) *hold, and f,G,H are* **impulsive II-Carathéodory functions**, *I,J* **discrete II-Carathéodory functions**. *Then $T:P_mC_{2-\beta}(0,1]\to P_mC_{2-\beta}(0,1]$ is well defined and is completely continuous, $x\in P_mC_{2-\beta}(0,1]$ is a solution of BVP* (1.0.9) *if and only if $x\in P_mC_{2-\beta}(0,1]$ is a fixed point of T.*

Proof: The proof is similar to that of the proof of Lemma 3.1 in [81] and is omitted. □

Theorem 5.2.1 *Suppose that* (f)-(h), *f,G,H are* **impulsive II-Carateódory functions**, *I,J* **discrete II-Carathéodory functions** *and*

(C2) *there exist nondecreasing functions $M_f,M_g,M_h,M_I,M_J:[0,+\infty)\to[0,+\infty)$ such that*

$$|f(t,(t-t_i)^{\beta-2}x)|\le M_f(|x|),t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m],x\in\mathbb{R},$$

$$|G(t,(t-t_i)^{\beta-2}x)|\le M_G(|x|),t\in(t_i,t_{i1}],i\in\mathbb{N}[0,m],x\in\mathbb{R},$$

$$|H(t,(t-t_i)^{\beta-2}x)|\le M_H(|x|),t\in(t_i,t_{i1}],i\in\mathbb{N}[0,m],x\in\mathbb{R},$$

$$|I(t_i,(t_i-t_{i-1})^{\beta-2}x)|\le M_I(|x|),i\in\mathbb{N}[1,m],x\in\mathbb{R},$$

$$|J(t_i,(t_i-t_{i-1})^{\beta-2}x)|\le M_J(|x|),i\in\mathbb{N}[1,m],x\in\mathbb{R}.$$

Then BVP (1.0.9) has at least one solution if there exists a $r_0 > 0$ such that

$$||\psi||_1 M_H(r_0) + 2||\phi||_1 M_G(r_0) + \frac{2m}{\Gamma(\beta)} M_J(r_0) + \left[m + \sum_{\sigma=1}^{m}(1-t_\sigma)^{\beta-2}\right] M_I(r_0) + \left[\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(r_0) \le r_0. \tag{5.2.6}$$

Proof: From Lemma 5.2.1, the definition of T, $x \in P_mC_{2-\beta}(0,1]$ is a solution of BVP (1.0.8) if and only if $x \in P_mC_{2-\beta}(0,1]$ is a fixed point of T in $P_mC_{2-\beta}(0,1]$. Lemma 5.2.2 implies that T is a completely continuous operator. From (C2), we have for $x \in P_mC_{2-\beta}(0,1]$ that

$$|f(t,x(t))| = |f(t,(t-t_i)^{\beta-2}(t-t_i)^{2-\beta}x(t))|$$

$$\le M_f(|(t-t_i)^{2-\beta}x(t)|) \le M_f(||x||), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|G(t,x(t))| \le M_G(||x||), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|H(t,x(t))| \le M_H(||x||), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|I(t_i,x(t_i))| = |I(t_i,(t_i-t_{i-1})^{\beta-2}(t_i-t_{i-1})^{2-\beta}x(t_i))|$$

$$\le M_I((t_i-t_{i-1})^{2-\beta}|x(t_i)|) \le M_I(||x||), i \in \mathbb{N}[1,m],$$

$$|J(t_i,x(t_i))| \le M_J(||x||), i \in \mathbb{N}[1,m].$$

We consider the set $\Omega = \{x \in P_mC_{2-\beta}(0,1] : x = \lambda(Tx), \text{ for some } \lambda \in [0,1)\}$. For $x \in \Omega$, we have for $t \in (t_i,t_{i+1}]$

$$(t-t_i)^{2-\beta}|(Tx)(t)| \le (t-t_i)^{2-\beta}t^{\beta-1}\left[||\psi||_1 M_H(||x||) + ||\phi||_1 M_G(||x||)\right.$$

$$\left. + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} s^k(1-s)^l ds M_f(||x||) + \sum_{\sigma=1}^{m}\left(\frac{M_J(||x||)(1-t_\sigma)^{\beta-1}}{\Gamma(\beta)} + M_I(||x||)(1-t_\sigma)^{\beta-2}\right)\right]$$

$$+(t-t_i)^{2-\beta}t^{\beta-2}||\phi||_1 M_G(||x||) + (t-t_i)^{2-\beta}\sum_{\sigma=1}^{i}\left[\frac{(t-t_\sigma)^{\beta-1}}{\Gamma(\beta)}M_J(||x||) + (t-t_\sigma)^{\beta-2}M_I(||x||)\right]$$

$$+(t-t_i)^{2-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s^k(1-s)^l ds M_f(||x||)$$

$$\le ||\psi||_1 M_H(||x||) + 2||\phi||_1 M_G(||x||) + \frac{2m}{\Gamma(\beta)} M_J(||x||)$$

$$+ \left[m + \sum_{\sigma=1}^{m}(1-t_\sigma)^{\beta-2}\right] M_I(||x||) + \left[\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(||x||).$$

It follows that

$$||x|| = \lambda||Tx|| < ||Tx|| \le ||\psi||_1 M_H(||x||) + 2||\phi||_1 M_G(||x||) + \tfrac{2m}{\Gamma(\beta)} M_J(||x||)$$

$$+ \left[m + \sum_{\sigma=1}^{m} (1-t_\sigma)^{\beta-2}\right] M_I(||x||) + \left[\tfrac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)} + \tfrac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(||x||).$$

From (5.2.6), we choose $\Omega = \{x \in P_1C_{2-\beta}(0,1] : ||x|| \le r_0\}$. For $x \in \Omega$, we get $x \ne \lambda(Tx)$ for any $\lambda \in [0,1)$ and $x \in \partial\Omega$. In fact, if $x = \lambda(Tx)$ for some $\lambda \in [0,1)$ and $x \in \partial\Omega$, then

$$r_0 = ||x|| < ||\psi||_1 M_H(r_0) + 2||\phi||_1 M_G(r_0) + \tfrac{2m}{\Gamma(\beta)} M_J(r_0)$$

$$+ \left[m + \sum_{\sigma=1}^{m} (1-t_\sigma)^{\beta-2}\right] M_I(r_0) + \left[\tfrac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)} + \tfrac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(r_0) \le r_0,$$

which is a contradiction. As a consequence of Lemma 5.0.1, we deduce that T has a fixed point which is a solution of the problem BVP (1.0.9). The proof is completed. The proof of Theorem 5.2.1 is completed. □

Corollary 5.2.1 *Suppose that* (f)-(h), *and* (C2) *hold. Then BVP* (1.0.9) *has at least one solution if*

$$\inf_{r\in(0,+\infty)} \tfrac{1}{r}\Big[||\psi||_1 M_H(r) + 2||\phi||_1 M_G(r) + \tfrac{2m}{\Gamma(\beta)} M_J(r)$$

$$+ \left[m + \sum_{\sigma=1}^{m} (1-t_\sigma)^{\beta-2}\right] M_I(r) + \left[\tfrac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)} + \tfrac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(r)\Big] < 1.$$

Proof: From the assumption, we know that there exists $r_0 > 0$ such that (5.2.6) holds. By Theorem 4.0.1, BVP (1.0.9) has at least one solution. The proof is omitted. □

5.3 Solvability of BVP (1.0.10)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.10).

Lemma 5.3.1 *Suppose that* $\sigma : (0,1) \to \mathbb{R}$ *is continuous and there exist numbers* $k > -1$ *and* $\max\{-\beta, -2-k\} < l \le 0$ *such that* $|\sigma(t)| \le (\log t)^k (1-\log t)^l$ *for all* $t \in (1,e)$. *Then*

x is a solutions of

$$\begin{cases} {}^{RLH}D_{1^+}^{\beta}x(t)=\sigma(t), t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m],\\ \lim\limits_{t\to1^+}(\log t)^{2-\beta}x(t)+x(e)=a,\\ \lim\limits_{t\to1^+}{}^{RLH}D_{1^+}^{\beta-1}x(t)+{}^{RLH}D_{1^+}^{\beta-1}x(e)=b,\\ \lim\limits_{t\to t_i^+}(\log t-\log t_i)^{2-\beta}x(t)=I_i, \Delta^{RL}D_{1^+}^{\beta-1}x(t_i)=J_i, i\in\mathbb{N}[1,m],\\ \Delta^{RL}D_{1^+}^{\beta-1}x(t_i)=J_i, i\in\mathbb{N}[1,m], \end{cases} \tag{5.3.1}$$

if and only if x satisfies

$$\begin{aligned} x(t)=&\tfrac{1}{2\Gamma(\beta)}\left[b-\sum_{\sigma=1}^{m}J_\sigma-\int_1^e\sigma(s)\tfrac{ds}{s}\right](\log t)^{\beta-1}\\ &+\tfrac{1}{2}\left[a-\tfrac{1}{\Gamma(\beta)}\int_1^e\left(\log\tfrac{t}{s}\right)^{\beta-1}\sigma(s)\tfrac{ds}{s}-\sum_{\sigma=1}^{m}\left(\tfrac{J_\sigma}{\Gamma(\beta)}\left(\log\tfrac{e}{t_\sigma}\right)^{\beta-1}+I_\sigma\left(\log\tfrac{e}{t_\sigma}\right)^{\beta-2}\right)\right.\\ &\left.-\tfrac{1}{2\Gamma(\beta)}\left(b-\sum_{\sigma=1}^{m}J_\sigma-\int_1^e\sigma(s)\tfrac{ds}{s}\right)\right](\log t)^{\beta-2}+\tfrac{1}{\Gamma(\beta)}\sum_{\sigma=1}^{i}\left(\log\tfrac{t}{t_\sigma}\right)^{\beta-1}J_\sigma\\ &+\sum_{\sigma=1}^{i}\left(\log\tfrac{t}{t_\sigma}\right)^{\beta-2}I_\sigma+\tfrac{1}{\Gamma(\beta)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\beta-1}\sigma(s)\tfrac{ds}{s}, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m]. \end{aligned} \tag{5.3.2}$$

Proof: Let x be a solution of (5.3.1). We know from Theorem 4.0.3 that there exist numbers $c_{\sigma1},c_{\sigma2}\in\mathbb{R}$ such that

$$\begin{aligned} x(t)=&\sum_{\sigma=0}^{i}\left[\tfrac{c_{\sigma1}}{\Gamma(\beta)}\left(\log\tfrac{t}{t_\sigma}\right)^{\beta-1}+\tfrac{c_{\sigma2}}{\Gamma(\beta-1)}\left(\log\tfrac{t}{t_\sigma}\right)^{\beta-2}\right]\\ &+\tfrac{1}{\Gamma(\beta)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\beta-1}\sigma(s)\tfrac{ds}{s}, t\in(t_i,t_{i+1}], i\in\mathbb{N}[0,m]. \end{aligned} \tag{5.3.3}$$

By direct computation, we have for $t \in (t_i, t_{i+1}]$

$$^{RLH}D_{1^+}^{\beta-1}x(t) = \frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\int_1^t \left(\log\frac{t}{s}\right)^{1-\beta} x(s)\frac{ds}{s}$$

$$= \frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{v=0}^{i-1}\int_{t_v}^{t_{v+1}}\left(\log\frac{t}{s}\right)^{1-\beta}\left[\sum_{\sigma=0}^{v}\left(\frac{c_{\sigma 1}}{\Gamma(\beta)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-1} + \frac{c_{\sigma 2}}{\Gamma(\beta-1)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-2}\right)\right.$$

$$\left.+\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\right]\frac{ds}{s}$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\int_{t_i}^t\left(\log\frac{t}{s}\right)^{1-\beta}\left[\sum_{\sigma=0}^{i}\left(\frac{c_{\sigma 1}}{\Gamma(\beta)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-1} + \frac{c_{\sigma 2}}{\Gamma(\beta-1)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-2}\right)\right.$$

$$\left.+\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\right]\frac{ds}{s}$$

$$= \frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{v=0}^{i-1}\sum_{\sigma=0}^{v}\frac{c_{\sigma 1}}{\Gamma(\beta)}\int_{t_v}^{t_{v+1}}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-1}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{v=0}^{i-1}\sum_{\sigma=0}^{v}\frac{c_{\sigma 2}}{\Gamma(\beta-1)}\int_{t_v}^{t_{v+1}}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-2}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 1}}{\Gamma(\beta)}\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-1}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 2}}{\Gamma(\beta-1)}\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-2}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(\beta)}\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\int_1^t\left(\log\frac{t}{s}\right)^{1-\beta}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\frac{ds}{s}$$

changing the order of sums and integral, $\frac{\log s - \log t_\sigma}{\log t - \log t_\sigma} = w$ or $\frac{\log s - \log u}{\log t - \log u} = w$

$$= \frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i-1}\sum_{v=\sigma}^{i-1}\frac{c_{\sigma 1}}{\Gamma(\beta)}\int_{t_v}^{t_{v+1}}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-1}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i-1}\sum_{v=\sigma}^{i-1}\frac{c_{\sigma 2}}{\Gamma(\beta-1)}\int_{t_v}^{t_{v+1}}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-2}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 1}}{\Gamma(\beta)}\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-1}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 2}}{\Gamma(\beta-1)}\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{t_\sigma}\right)^{\beta-2}\frac{ds}{s}$$

$$+\frac{1}{\Gamma(\beta)}\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{1-\beta}\left(\log\frac{s}{u}\right)^{\beta-1}\frac{ds}{s}\sigma(u)\frac{du}{u}$$

$$=\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i-1}\frac{c_{\sigma 1}}{\Gamma(\beta)}\sum_{v=\sigma}^{i-1}\left(\log\frac{t}{t_\sigma}\right)\int_{\frac{\log t_v-\log t_\sigma}{\log t-\log t_\sigma}}^{\frac{\log t_{v+1}-\log t_\sigma}{\log t-\log t_\sigma}}(1-w)^{1-\beta}w^{\beta-1}dw$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i-1}\frac{c_{\sigma 2}}{\Gamma(\beta-1)}\sum_{v=\sigma}^{i-1}\int_{\frac{\log t_v-\log t_\sigma}{\log t-\log t_\sigma}}^{\frac{\log t_{v+1}-\log t_\sigma}{\log t-\log t_\sigma}}(1-w)^{1-\beta}w^{\beta-2}dw$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 1}}{\Gamma(\beta)}\left(\log\frac{t}{t_\sigma}\right)\int_{\frac{\log t_i-\log t_\sigma}{\log t-\log t_\sigma}}^{1}(1-w)^{1-\beta}w^{\beta-1}dw$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 2}}{\Gamma(\beta-1)}\int_{\frac{\log t_i-\log t_\sigma}{\log t-\log t_\sigma}}^{1}(1-w)^{1-\beta}w^{\beta-2}dw$$

$$+\frac{1}{\Gamma(\beta)}\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\int_1^t\left(\log\frac{t}{u}\right)\int_0^1(1-w)^{1-\beta}w^{\beta-1}dw\sigma(u)\frac{du}{u}$$

$$=\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 1}}{\Gamma(\beta)}\left(\log\frac{t}{t_\sigma}\right)\int_0^1(1-w)^{1-\beta}w^{\beta-1}dw$$

$$+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\sum_{\sigma=0}^{i}\frac{c_{\sigma 2}}{\Gamma(\beta-1)}\int_0^1(1-w)^{1-\beta}w^{\beta-2}dw$$

$$+\frac{1}{\Gamma(\beta)}\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)\int_1^t\left(\log\frac{t}{u}\right)\int_0^1(1-w)^{1-\beta}w^{\beta-1}dw\sigma(u)\frac{du}{u}$$

$$=\sum_{\sigma=0}^{i}c_{\sigma 1}+\int_1^t\sigma(u)\frac{du}{u},t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m].$$

We have

$$^{RLH}D_{1^+}^{\beta-1}x(t)=\sum_{\sigma=0}^{i}c_{\sigma 1}+\int_1^t\sigma(u)\frac{du}{u},t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m]. \tag{5.3.4}$$

One sees that

$$(\log t)^{2-\beta}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s}\right|\le(\log t)^{2-\beta}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}(\log s)^k\left(\log\frac{e}{s}\right)^l\frac{ds}{s}$$

$$\le(\log t)^{2-\beta}\int_1^t\left(\log\frac{t}{s}\right)^{\beta+l-1}(\log s)^k\frac{ds}{s}$$

$$=(\log t)^{2-\beta}(\log t)^{\beta+k+l}\int_0^1(1-w)^{\beta+l-1}w^kdw\to 0 \text{ as } t\to 1^+.$$

Similarly we have

$$\left|\int_1^t \sigma(s)\frac{ds}{s}\right| \leq \int_1^t (\log s)^k \left(\log \frac{e}{s}\right)^l \frac{ds}{s} \leq \int_1^t \left(\log \frac{t}{s}\right)^l (\log s)^k \frac{ds}{s}$$

$$= (\log t)^{k+l+1} \int_0^1 (1-w)^l w^k dw = (\log t)^{k+l+1}\mathbf{B}(l+1,k+1) \to 0 \text{ as } t \to 1^+.$$

It follows from (5.3.3), (5.3.4), the boundary conditions and the impulse assumption in (5.3.1) that

$$\frac{1}{\Gamma(\beta-1)}c_{02} + \sum_{\sigma=0}^{m}\left[\frac{c_{\sigma 1}}{\Gamma(\beta)}\left(\log \frac{e}{t_\sigma}\right)^{\beta-1} + \frac{c_{\sigma 2}}{\Gamma(\beta-1)}\left(\log \frac{e}{t_\sigma}\right)^{\beta-2}\right] + \frac{1}{\Gamma(\beta)}\int_1^e \left(\log \frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s} = a,$$

$$c_{01} + \sum_{\sigma=0}^{m} c_{\sigma 1} + \int_1^e \sigma(u)\frac{du}{u} = b.$$

and $c_{i2} = \Gamma(\beta-1)I_i(i \in \mathbb{N}[1,m])$, $c_{i1} = J_i(i \in \mathbb{N}[1,m])$. Then

$$c_{01} = \frac{1}{2}\left[b - \sum_{\sigma=1}^{m} J_\sigma - \int_1^e \sigma(s)\frac{ds}{s}\right],$$

$$c_{02} = \frac{\Gamma(\beta-1)}{2}\left[a - \frac{1}{\Gamma(\beta)}\int_1^e \left(\log \frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s} - \sum_{\sigma=1}^{m}\left(\frac{J_\sigma}{\Gamma(\beta)}\left(\log \frac{e}{t_\sigma}\right)^{\beta-1} + I_\sigma\left(\log \frac{e}{t_\sigma}\right)^{\beta-2}\right)\right.$$
$$\left. - \frac{1}{2\Gamma(\beta)}\left(b - \sum_{\sigma=1}^{m} J_\sigma - \int_1^e \sigma(s)\frac{ds}{s}\right)\right]. \tag{5.3.5}$$

Substituting $c_{\sigma v}(\sigma \in \mathbb{N}[0,m], v \in \mathbb{N}[1,2]$ into (5.3.3), we get (5.3.2) obviously.

On the other hand, if x satisfies (5.3.2), then $x|_{(t_i,t_{i+1}]}(i \in \mathbb{N}[0,m])$ are continuous and the limits $\lim_{t\to t_i^+}(\log t - \log t_i)^{2-\beta}x(t)(i \in \mathbb{N}[0,m]$ exist. So $x \in LP_mC_{2-\beta}(1,e]$. Furthermore, by direct computation, we have $\lim_{t\to 1^+}(\log t)^{2-\beta}x(t) + x(e) = a$, $\lim_{t\to 1^+} {}^{RLH}D_{1^+}^{\beta-1}x(t) + {}^{RLH}D_{1^+}^{\beta-1}x(e) = b$, $\lim_{t\to t_i^+}(\log t - \log t_i)^{2-\beta}x(t) = I_i, i \in \mathbb{N}[1,m]$ and $\Delta^{RL}D_{1^+}^{\beta-1}x(t_i) = J_i, i \in \mathbb{N}[1,m]$.

Using (5.3.5) and $c_{i2} = \Gamma(\beta-1)I_i(i \in \mathbb{N}[1,m])$, $c_{i1} = J_i(i \in \mathbb{N}[1,m])$, we rewrite x by (5.3.3).

One has similarly for $t \in (t_i, t_{i+1}]$

$$
\begin{aligned}
{}^{RLH}D_{1^+}^{\beta}x(t) &= \frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)^2\int_1^t\left(\log\frac{t}{s}\right)^{1-\beta}x(s)\frac{ds}{s}\\
&= \frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)^2\left[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\sum_{\sigma=0}^{i}\sum_{v=1}^{2}\frac{c_{\sigma v}}{\Gamma(\beta-v+1)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-v}\right.\right.\\
&\quad\left.+\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\right)\frac{ds}{s}\\
&\quad+\int_{t_j}^{t}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\sum_{\sigma=0}^{j}\sum_{v=1}^{2}\frac{c_{\sigma v}}{\Gamma(\beta-v+1)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-v}\right.\\
&\quad\left.\left.+\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\right)\frac{ds}{s}\right]\\
&= \frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)^2\left[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\sum_{\sigma=0}^{i}\sum_{v=1}^{2}\frac{c_{\sigma v}}{\Gamma(\beta-v+1)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-v}\right)\frac{ds}{s}\right]\\
&\quad+\frac{1}{\Gamma(2-\beta)}\left(t\frac{d}{dt}\right)^2\left[\int_{t_j}^{t}\left(\log\frac{t}{s}\right)^{1-\beta}\left(\sum_{\sigma=0}^{j}\sum_{v=1}^{2}\frac{c_{\sigma v}}{\Gamma(\beta-v+1)}\left(\log\frac{s}{t_\sigma}\right)^{\beta-v}\right)\frac{ds}{s}\right]\\
&\quad+\frac{1}{\Gamma(2-\beta)\Gamma(\beta)}\left(t\frac{d}{dt}\right)^2\left[\int_1^t\left(\log\frac{t}{s}\right)^{1-\beta}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\right)\frac{ds}{s}
\end{aligned}
$$

By using the method above, we have

$$
{}^{RLH}D_{1^+}^{\beta}x(t) = \sigma(t).
$$

So x is a solution of (5.3.1). The proof is completed. □

Define the nonlinear operator R on $LP_mC_{2-\beta}(1,e]$ for $x \in LP_mC_{2-\alpha}(1,e]$ by $(Rx)(t)$ by

$$
\begin{aligned}
(Rx)(t) &= \frac{1}{2\Gamma(\beta)}\left[\int_1^e\psi(s)H(s,x(s))ds - \sum_{\sigma=1}^{m}J(t_\sigma,x(t_\sigma)) - \int_1^e q(s)f(s,x(s))\frac{ds}{s}\right](\log t)^{\beta-1}\\
&\quad+\frac{1}{2}\left[\int_1^e\phi(s)G(s,x(s))ds - \frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{t}{s}\right)^{\beta-1}q(s)f(s,x(s))\frac{ds}{s}\right.\\
&\quad-\sum_{\sigma=1}^{m}\left(\frac{J(t_\sigma,x(t_\sigma))}{\Gamma(\beta)}\left(\log\frac{e}{t_\sigma}\right)^{\beta-1} + I(t_\sigma,x(t_\sigma))\left(\log\frac{e}{t_\sigma}\right)^{\beta-2}\right)\\
&\quad\left.-\frac{1}{2\Gamma(\beta)}\left(\int_1^e\psi(s)H(s,x(s))ds - \sum_{\sigma=1}^{m}J(t_\sigma,x(t_\sigma)) - \int_1^e q(s)f(s,x(s))\frac{ds}{s}\right)\right](\log t)^{\beta-2}
\end{aligned}
$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\sigma=1}^{i}\left(\log\frac{t}{t_\sigma}\right)^{\beta-1}J(t_\sigma,x(t_\sigma))+\sum_{\sigma=1}^{i}\left(\log\frac{t}{t_\sigma}\right)^{\beta-2}I(t_\sigma,x(t_\sigma))$$

$$+\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}q(s)f(s,x(s))\frac{ds}{s},t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m].$$

Lemma 5.3.2 *Suppose that* (i)-(k) *hold, and* f,G,H *are* **impulsive III-Carathéodory functions**, I,J **a discrete III-Carathéodory functions**. *Then* $R:LP_mC_{2-\beta}(1,e]\to LP_mC_{2-\alpha}(1,e]$ *is well defined and is completely continuous,* x *is a solution of BVP* (1.0.10) *if and only if* x *is a fixed point of* R *in* $LP_mC_{2-\alpha}(1,e]$.

Proof: The proof is similar to that of the proof of Lemma 3.3.2 and is omitted. □

Theorem 5.3.1 *Suppose that* (i), (j) *and* (k) *hold,* f,G,H *are* **impulsive III-Carathéodory functions**, I,J **a discrete III-Carathéodory functions** *and*

(C3) *there exist nondecreasing functions* $M_f,M_g,M_h,M_I,M_J:[0,+\infty)\to[0,+\infty)$ *such that*

$$\left|f\left(t,\left(\log\frac{t}{t_i}\right)^{\beta-2}x\right)\right|\le M_f(|x|),t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m],x\in\mathbb{R},$$

$$\left|G\left(t,\left(\log\frac{t}{t_i}\right)^{\beta-2}x\right)\right|\le M_g(|x|),t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m],x\in\mathbb{R},$$

$$\left|H\left(t,\left(\log\frac{t}{t_i}\right)^{\beta-2}x\right)\right|\le M_h(|x|),t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m],x\in\mathbb{R},$$

$$\left|I\left(t_i,\left(\log\frac{t_i}{t_{i-1}}\right)^{\beta-2}x\right)\right|\le M_I(|x|),i\in\mathbb{N}[1,m],x\in\mathbb{R},$$

$$\left|J\left(t_i,\left(\log\frac{t_i}{t_{i-1}}\right)^{\beta-2}x\right)\right|\le M_J(|x|),i\in\mathbb{N}[1,m],x\in\mathbb{R}.$$

Then BVP (1.0.10) *has at least one solution if there exists a constant* $r_0>0$ *such that*

$$\begin{aligned}&\left[\frac{||\psi||_1}{2\Gamma(\beta)}+\frac{||\psi||_1}{4\Gamma(\beta)}\right]M_H(r_0)+\frac{||\phi||_1}{2}M_G(r_0)\\&+\left[\frac{m}{2\Gamma(\beta)}+\frac{m}{2\Gamma(\beta)}+\frac{m}{4\Gamma(\beta)}+\frac{m}{\Gamma(\beta)}\right]M_J(r_0)+\left[\frac{1}{2}\sum_{\sigma=1}^{m}\left(\log\frac{e}{t_\sigma}\right)^{\beta-2}+m\right]M_I(r_0)\\&+\left[\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\beta)}+\frac{\mathbf{B}(\beta+l,k+1)}{2\Gamma(\beta)}+\frac{\mathbf{B}(l+1,k+1)}{4\Gamma(\beta)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(r_0)\le r_0.\end{aligned}\tag{5.3.6}$$

Proof: From Lemma 5.3.1 and the definition of R, $x \in LP_mC_{2-\beta}(1,e]$ is a solution of BVP (1.0.10) if and only if $x \in LP_mC_{2-\beta}(1,e]$ is a fixed point of R. Lemma 5.3.2 implies that R is a completely continuous operator.

From (C3), we have for $x \in LP_mC_{2-\beta}(1,e]$ that

$$|f(t,x(t))| = \left|f\left(t, \left(\log \tfrac{t}{t_i}\right)^{\beta-2}\left(\log \tfrac{t}{t_i}\right)^{2-\beta} x(t)\right)\right|$$

$$\leq M_f\left(\left|\left(\log \tfrac{t}{t_i}\right)^{2-\beta} x(t)\right|\right) \leq M_f(||x||), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|G(t,x(t))| \leq M_G(||x||), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|H(t,x(t))| \leq M_H(||x||), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|I(t_i,x(t_i))| = \left|I\left(t_i, \left(\log \tfrac{t_i}{t_{i-1}}\right)^{\beta-2}\left(\log \tfrac{t_i}{t_{i-1}}\right)^{2-\beta} x(t)\right)\right|$$

$$\leq M_f\left(\left|\left(\log \tfrac{t_i}{t_{i-1}}\right)^{2-\beta} x(t)\right|\right) \leq M_G(||x||), i \in \mathbb{N}[1,m],$$

$$|I(t_i,x(t_i))| \leq M_H(||x||), i \in \mathbb{N}[1,m].$$

We consider the set $\Omega = \{x \in LP_mC_{2-\beta}(1,e] : x = \lambda(Rx),\ \text{for some}\ \lambda \in [0,1]\}$. For $x \in \Omega$, we have for $t \in (t_i, t_{i+1}]$ that

$$\left(\log \tfrac{t}{t_i}\right)^{2-\beta} |(Rx)(t)|$$

$$\leq \tfrac{1}{2\Gamma(\beta)}\left[||\psi||_1 M_H(||x||) + mM_J(||x||) + \textstyle\int_1^e (\log s)^l (1-\log s)^l \tfrac{ds}{s} M_f(||x||)\right] (\log t)^{\beta-1}\left(\log \tfrac{t}{t_i}\right)^{2-\beta}$$

$$+\tfrac{1}{2}\Big[||\phi||_1 M_G(||x||) + \tfrac{1}{\Gamma(\beta)}\textstyle\int_1^e \left(\log \tfrac{t}{s}\right)^{\beta-1}(\log s)^k (1-\log s)^l \tfrac{ds}{s} M_f(||x||)$$

$$+\sum_{\sigma=1}^{m}\left(\tfrac{1}{\Gamma(\beta)} M_J(||x||) + \left(\log \tfrac{e}{t_\sigma}\right)^{\beta-2} M_I(||x||)\right)$$

$$+\tfrac{1}{2\Gamma(\beta)}\left(||\psi||_1 M_H(||x||) + mM_J(||x||)\right.$$

$$\left.+\textstyle\int_1^e (\log s)^k (1-\log s)^l \tfrac{ds}{s} M_f(||x||)\right)\Big] (\log t)^{\beta-2}\left(\log \tfrac{t}{t_i}\right)^{2-\beta}$$

$$+\frac{1}{\Gamma(\beta)}\sum_{\sigma=1}^{i}\left(\log\frac{t}{t_i}\right)^{2-\beta}\left(\log\frac{t}{t_\sigma}\right)^{\beta-1}M_J(||x||)+\sum_{\sigma=1}^{i}\left(\log\frac{t}{t_i}\right)^{2-\beta}\left(\log\frac{t}{t_\sigma}\right)^{\beta-2}M_I(||x||)$$

$$+\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(||x||)$$

$$\leq\frac{||\psi||_1}{2\Gamma(\beta)}M_H(||x||)+\frac{m}{2\Gamma(\beta)}M_J(||x||)+\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\beta)}M_f(||x||)$$

$$+\frac{||\phi||_1}{2}M_G(||x||)+\frac{1}{2\Gamma(\beta)}\int_1^e\left(\log\frac{t}{s}\right)^{\beta+l-1}(\log s)^k\frac{ds}{s}M_f(||x||)$$

$$+\frac{m}{2\Gamma(\beta)}M_J(||x||)+\frac{1}{2}\sum_{\sigma=1}^{m}\left(\log\frac{e}{t_\sigma}\right)^{\beta-2}M_I(||x||)$$

$$+\frac{||\psi||_1}{4\Gamma(\beta)}M_H(||x||)+\frac{m}{4\Gamma(\beta)}M_J(||x||)+\frac{1}{4\Gamma(\beta)}\int_1^e(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(||x||)$$

$$+\frac{m}{\Gamma(\beta)}M_J(||x||)+mM_I(||x||)+\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta+l-1}(\log s)^k\frac{ds}{s}M_f(||x||)$$

$$\leq\left[\frac{||\psi||_1}{2\Gamma(\beta)}+\frac{||\psi||_1}{4\Gamma(\beta)}\right]M_H(||x||)+\frac{||\phi||_1}{2}M_G(||x||)$$

$$+\left[\frac{m}{2\Gamma(\beta)}+\frac{m}{2\Gamma(\beta)}+\frac{m}{4\Gamma(\beta)}+\frac{m}{\Gamma(\beta)}\right]M_J(||x||)+\left[\frac{1}{2}\sum_{\sigma=1}^{m}\left(\log\frac{e}{t_\sigma}\right)^{\beta-2}+m\right]M_I(||x||)$$

$$+\left[\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\beta)}+\frac{\mathbf{B}(\beta+l,k+1)}{2\Gamma(\beta)}+\frac{\mathbf{B}(l+1,k+1)}{4\Gamma(\beta)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(||x||).$$

It follows that

$$||x||=\lambda||Rx||<||Rx||\leq\left[\frac{||\psi||_1}{2\Gamma(\beta)}+\frac{||\psi||_1}{4\Gamma(\beta)}\right]M_H(||x||)+\frac{||\phi||_1}{2}M_G(||x||)$$

$$+\left[\frac{m}{2\Gamma(\beta)}+\frac{m}{2\Gamma(\beta)}+\frac{m}{4\Gamma(\beta)}+\frac{m}{\Gamma(\beta)}\right]M_J(||x||)+\left[\frac{1}{2}\sum_{\sigma=1}^{m}\left(\log\frac{e}{t_\sigma}\right)^{\beta-2}+m\right]M_I(||x||)$$

$$+\left[\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\beta)}+\frac{\mathbf{B}(\beta+l,k+1)}{2\Gamma(\beta)}+\frac{\mathbf{B}(l+1,k+1)}{4\Gamma(\beta)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(||x||).$$

From (5.3.6), we choose $\Omega=\{x\in LP_mC_{2-\beta}(1,e]:||x||\leq r_0\}$. For $x\in\partial\Omega$, we get $x\neq\lambda(Rx)$ for any $\lambda\in[0,1)$. In fact, if there exists $x\in\partial\Omega$ such that $x=\lambda(Rx)$ for some $\lambda\in[0,1)$.

Then

$$r_0 = ||x|| = \lambda||Rx|| < ||Rx|| \leq \left[\frac{||\psi||_1}{2\Gamma(\beta)} + \frac{||\psi||_1}{4\Gamma(\beta)}\right] M_H(r_0) + \frac{||\phi||_1}{2} M_G(r_0)$$

$$+ \left[\frac{m}{2\Gamma(\beta)} + \frac{m}{2\Gamma(\beta)} + \frac{m}{4\Gamma(\beta)} + \frac{m}{\Gamma(\beta)}\right] M_J(r_0) + \left[\frac{1}{2}\sum_{\sigma=1}^{m}\left(\log\frac{e}{t_\sigma}\right)^{\beta-2} + m\right] M_I(r_0)$$

$$+ \left[\frac{\mathbf{B}(l+1,k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l,k+1)}{2\Gamma(\beta)} + \frac{\mathbf{B}(l+1,k+1)}{4\Gamma(\beta)} + \frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right] M_f(r_0) \leq r_0,$$

a contradiction.

As a consequence of Lemma 5.0.1, we deduce that R has a fixed point which is a solution of the problem BVP (1.0.10). The proof is completed. The proof of Theorem 5.3.1 is completed. □

5.4 Solvability of BVP (1.0.11)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP (1.0.11).

Lemma 5.4.1 *Suppose that $\sigma : (0,1) \to \mathbb{R}$ is continuous and satisfies that there exist numbers $k > 1-\beta$ and $l \leq 0$ with $l > \max\{-\beta, -\beta-k\}$ such that $|\sigma(t)| \leq (\log t)^k(1-\log t)^l$ for all $t \in (1,e)$. The x is a solutions of*

$$\begin{cases} {}^{CH}D_{1^+}^{\beta}x(t) = \sigma(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ x(1) - \left(t\frac{d}{dt}\right)x(t)\Big|_{t=1} = a, \\ x(e) + \left(t\frac{d}{dt}\right)x(t)\Big|_{t=e} = b, \\ \lim_{t\to t_i^+} x(t) - x(t_i) = I_i, i \in \mathbb{N}[1,m], \\ \lim_{t\to t_i^+} \left(t\frac{d}{dt}\right)x(t) - \left(t\frac{d}{dt}\right)x(t)\Big|_{t=t_i} = J_i, i \in \mathbb{N}[1,m], \end{cases} \quad (5.4.1)$$

if and only if $x \in LP_mC(1,e]$ *and*

$$x(t) = \frac{1}{3}\left[b + 2a - \frac{1}{\Gamma(\beta)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta-1)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-2}\sigma(s)\frac{ds}{s}\right.$$
$$\left. - \sum_{\sigma=1}^{m} I_\sigma - \sum_{\sigma=1}^{m}\left(1+\log\frac{e}{t_\sigma}\right)J_\sigma\right]$$
$$+\frac{1}{3}\left[b - a - \frac{1}{\Gamma(\beta)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta-1)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-2}\sigma(s)\frac{ds}{s}\right.$$
$$\left. - \sum_{\sigma=1}^{m} I_\sigma - \sum_{\sigma=1}^{m}\left(1+\log\frac{e}{t_\sigma}\right)J_\sigma\right]\log t$$
$$+\sum_{\sigma=1}^{i}\left[I_\sigma + J_\sigma\left(\log\frac{t}{t_\sigma}\right)\right] + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m]. \tag{5.4.2}$$

Proof: Let x be a solution of (5.4.1). We know by Theorem 4.0.4 that there exist numbers $c_{\sigma 0}, c_{\sigma 1} \in \mathbb{R}(\sigma \in \mathbb{N}[0, n-1])$ such that

$$x(t) = \sum_{\sigma=0}^{i}\left[c_{\sigma 0} + c_{\sigma 1}\left(\log\frac{t}{t_\sigma}\right)\right] + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m]. \tag{5.4.3}$$

It follows that

$$\left(t\frac{d}{dt}\right)x(t) = \sum_{\sigma=0}^{i} c_{\sigma 1} + \frac{1}{\Gamma(\beta-1)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-2}\sigma(s)\frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m]. \tag{5.4.4}$$

It follows from (5.4.3), (5.4.4), the boundary conditions and the impulse assumption in (5.4.1) that $c_{\sigma 0} = I_\sigma$, $c_{\sigma 1} = J_\sigma$, $\sigma \in \mathbb{N}[1,m]$, $c_{00} - c_{01} = a$ and

$$\sum_{\sigma=0}^{m}\left[c_{\sigma 0} + c_{\sigma 1}\left(\log\frac{e}{t_\sigma}\right)\right] + \frac{1}{\Gamma(\beta)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s} + \sum_{\sigma=0}^{m} c_{\sigma 1} + \frac{1}{\Gamma(\beta-1)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-2}\sigma(s)\frac{ds}{s} = b.$$

Then

$$c_{00} = \frac{1}{3}\left[b + 2a - \frac{1}{\Gamma(\beta)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta-1)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-2}\sigma(s)\frac{ds}{s}\right.$$
$$\left. - \sum_{\sigma=1}^{m} I_\sigma - \sum_{\sigma=1}^{m}\left(1+\log\frac{e}{t_\sigma}\right)J_\sigma\right],$$
$$c_{01} = \frac{1}{3}\left[b - a - \frac{1}{\Gamma(\beta)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-1}\sigma(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta-1)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta-2}\sigma(s)\frac{ds}{s}\right.$$
$$\left. - \sum_{\sigma=1}^{m} I_\sigma - \sum_{\sigma=1}^{m}\left(1+\log\frac{e}{t_\sigma}\right)J_\sigma\right]. \tag{5.4.5}$$

Substituting $c_{\sigma 0}, c_{\sigma 1}(\sigma \in \mathbb{N}[0,m])$ into (5.4.3), we get (5.4.2) obviously.

On the other hand, if x satisfies (5.4.2), then $x|_{(t_i,t_{i+1}]}(i \in \mathbb{N}[0,m])$ are continuous and the limits $\lim_{t\to t_i^+} x(t)$. So $x \in LP_mC(1,e]$. Using (5.4.5), $c_{\sigma 0} = I_\sigma,\ \ c_{\sigma 1} = J_\sigma,\ \ \sigma \in \mathbb{N}[1,m]$, we rewrite x by (5.4.3). One have from Theorem 4.0.4 easily for $t \in (t_0,t_1]$ that ${}^{CH}D_{1^+}^{\beta}x(t) = \lambda x(t) + \sigma(t)$ and for $t \in (t_i,t_{i+1}]$ similarly to the proof of Theorem 4.0.1 that

$$
\begin{aligned}
{}^{CH}D_{1^+}^{\beta}x(t) &= \frac{1}{\Gamma(2-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{1-\alpha}\left(s\frac{d}{ds}\right)^2 x(s)\frac{ds}{s}\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{v=0}^{i-1}\int_{t_v}^{t_{v+1}}\left(\log\frac{t}{s}\right)^{1-\alpha}\left(s\frac{d}{ds}\right)^2\left[\sum_{\sigma=0}^{v}\left(c_{\sigma 0}+c_{\sigma 1}\left(\log\frac{s}{t_\sigma}\right)\right)\right.\\
&\quad\left.+\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\right]\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(2-\beta)}\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{1-\alpha}\left(s\frac{d}{ds}\right)^2\left[\sum_{\sigma=0}^{i}\left(c_{\sigma 0}+c_{\sigma 1}\left(\log\frac{s}{t_\sigma}\right)\right)\right.\\
&\quad\left.+\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{u}\right)^{\beta-1}\sigma(u)\frac{du}{u}\right]\frac{ds}{s} = \sigma(t).
\end{aligned}
$$

So x is a solution of (5.4.1). The proof is completed. □

Define the nonlinear operator J on $LP_mC(1,e]$ by (Jx) by

$$
\begin{aligned}
(Jx)(t) &= \frac{1}{3}\left[\int_1^e \psi(s)H(s,x(s))ds + 2\int_1^e \phi(s)G(s,x(s))ds - \frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-1}p(s)f(s,x(s))\frac{ds}{s}\right.\\
&\quad\left.-\frac{1}{\Gamma(\beta-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-2}p(s)f(s,x(s))\frac{ds}{s} - \sum_{\sigma=1}^{m}I(t_\sigma,x(t_\sigma)) - \sum_{\sigma=1}^{m}\left(1+\log\frac{e}{t_\sigma}\right)J(t_\sigma,x(t_\sigma))\right]\\
&\quad+\frac{1}{3}\left[\int_1^e \psi(s)H(s,x(s))ds - \int_1^e \phi(s)G(s,x(s))ds - \frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-1}p(s)f(s,x(s))\frac{ds}{s}\right.\\
&\quad\left.-\frac{1}{\Gamma(\beta-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-2}p(s)f(s,x(s))\frac{ds}{s} - \sum_{\sigma=1}^{m}I(t_\sigma,x(t_\sigma)) - \sum_{\sigma=1}^{m}\left(1+\log\frac{e}{t_\sigma}\right)J(t_\sigma,x(t_\sigma))\right]\log t\\
&\quad+\sum_{\sigma=1}^{i}\left[I(t_\sigma,x(t_\sigma)) + J(t_\sigma,x(t_\sigma))\left(\log\frac{t}{t_\sigma}\right)\right] + \frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}p(s)f(s,x(s))\frac{ds}{s},
\end{aligned}
$$

$t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m]$.

Lemma 5.4.2 *Suppose that* (g), (l), (m) *and* (n) *hold, and* f, G, H *are* **impulsive IV-Carathéodory functions**, I, J **discrete IV-Carathéodory functions**. *Then* R :

$LP_mC(1,e] \to LP_mC(1,e]$ is well defined and is completely continuous, x is a solution of BVP (1.0.11) if and only if x is a fixed point of J in $LP_mC(1,e]$.

Proof: The proof is similar to that of the proof of Theorem 3.3.2 and is omitted. □

Theorem 5.4.1 *Suppose that* (l)-(n) *hold, f, G, H are* **impulsive IV-Carathéodory functions**, *I, J* **discrete IV-Carathéodory functions** *and*

(C4) *there exist nondecreasing functions $M_f, M_g, M_h, M_I, M_J : [0,+\infty) \to [0,+\infty)$ such that*

$$|f(t,x)| \le M_f(|x|), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m], x \in \mathbb{R},$$

$$|G(t,x)| \le M_g(|x|), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m], x \in \mathbb{R},$$

$$|H(t,x)| \le M_h(|x|), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m], x \in \mathbb{R},$$

$$|I(t_i,x)| \le M_I(|x|), i \in \mathbb{N}[1,m], x \in \mathbb{R},$$

$$|J(t_i,x)| \le M_J(|x|), i \in \mathbb{N}[1,m], x \in \mathbb{R}.$$

Then BVP (1.0.11) has at least one solution if there exists a constant $r_0 > 0$ such that

$$\begin{aligned}&\tfrac{2||\psi||_1}{3}M_H(r_0) + ||\phi||_1 M_G(r_0) + \tfrac{5m}{3}M_I(r_0) + \tfrac{7m}{3}M_J(r_0)\\ &+\left[\tfrac{2\mathbf{B}(\beta+l,k+1)}{3\Gamma(\beta)} + \tfrac{2\mathbf{B}(\beta+l-1,k+1)}{3\Gamma(\beta-1)} + \tfrac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(r_0) \le r_0.\end{aligned} \tag{5.4.6}$$

Proof: From Lemma 5.4.1, and the definition of J, $x \in LP_mC(1,e]$ is a solution of BVP (1.0.11) if and only if $x \in LP_mC(1,e]$ is a fixed point of R. Lemma 5.4.2 implies that J is a completely continuous operator.

From (C4), we have for $x \in LP_mC(1,e]$ that

$$|f(t,x(t))| \le M_f(||x||), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|G(t,x(t))| \le M_f(||x||), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|H(t,x(t))| \le M_f(||x||), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|I(t_i,x(t_i))| \le M_I(||x||), i \in \mathbb{N}[1,m]$$

$$|J(t_i,x(t_i))| \le M_J(||x||), i \in \mathbb{N}[1,m].$$

We consider the set $\Omega=\{x\in LP_mC(1,e]: x=\lambda(Jx),\ \text{for some}\ \lambda\in[0,1)\}$. For $x\in\Omega$, we have for $t\in(t_i,t_{i+1}]$ that

$$|(Jx)(t)|\le$$

$$\frac{1}{3}\Big[||\psi||_1M_H(||x||)+2||\phi||_1M_G(||x||)+\frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-1}(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(||x||)$$

$$+\frac{1}{\Gamma(\beta-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-2}(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(||x||)+mM_I(||x||)+2mM_J(||x||)\Big]$$

$$+\frac{1}{3}\Big[||\psi||_1M_H(||x||)+||\phi||_1M_G(||x||)+\frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-1}(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(||x||)$$

$$+\frac{1}{\Gamma(\beta-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta-2}(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(||x||)+mM_I(||x||)+2mM_J(||x||)\Big]$$

$$+mM_I(||x||)+mM_J(||x||)+\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}(\log s)^k(1-\log s)^l\frac{ds}{s}M_f(||x||)$$

$$=\frac{2||\psi||_1}{3}M_H(||x||)+||\phi||_1M_G(||x||)+\frac{5m}{3}M_I(||x||)+\frac{7m}{3}M_J(||x||)$$

$$+\left[\frac{2\mathbf{B}(\beta+l,k+1)}{3\Gamma(\beta)}+\frac{2\mathbf{B}(\beta+l-1,k+1)}{3\Gamma(\beta-1)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(||x||).$$

It follows that

$$||x||=\lambda||Rx||<||Rx||\le\frac{2||\psi||_1}{3}M_H(||x||)+||\phi||_1M_G(||x||)+\frac{5m}{3}M_I(||x||)+\frac{7m}{3}M_J(||x||)$$

$$+\left[\frac{2\mathbf{B}(\beta+l,k+1)}{3\Gamma(\beta)}+\frac{2\mathbf{B}(\beta+l-1,k+1)}{3\Gamma(\beta-1)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(||x||).$$

From (5.4.6), we choose $\Omega=\{x\in LP_mC(1,e]:||x||\le r_0\}$. For $x\in\partial\Omega$, we get $x\ne\lambda(Jx)$ for any $\lambda\in[0,1)$. In fact, if there exists $x\in\partial\Omega$ such that $x=\lambda(Jx)$ for some $\lambda\in[0,1)$. Then

$$r_0=||x||=\lambda||Jx||<||Jx||\le\frac{2||\psi||_1}{3}M_H(r_0)+||\phi||_1M_G(r_0)+\frac{5m}{3}M_I(r_0)+\frac{7m}{3}M_J(r_0)$$

$$+\left[\frac{2\mathbf{B}(\beta+l,k+1)}{3\Gamma(\beta)}+\frac{2\mathbf{B}(\beta+l-1,k+1)}{3\Gamma(\beta-1)}+\frac{\mathbf{B}(\beta+l,k+1)}{\Gamma(\beta)}\right]M_f(r_0)\le r_0,$$

a contradiction.

As a consequence of Lemma 5.0.1, we deduce that J has a fixed point which is a solution of the problem BVP (1.0.11). The proof is completed. The proof of Theorem 5.4.1 is completed. □

6 Comparison results

In this section, we study the solvability of two class of impulsive boundary value problems of fractional differential equations for showing readers the differences between those equations one's fractional derivative has single start point and the other's has multiple starting points.

Example 6.0.1. Let $0 = t_0 < t_1 < t_2 = 1$. Consider the following problems:

$$\begin{cases} {}^CD^{\alpha}_{t_i^+}x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,1], \\ \Delta x(t_1) = I, \quad \Delta x'(t_1) = J, \\ x'(0) = x_0, \ x'(1) = x_1, \end{cases} \tag{6.0.1}$$

and

$$\begin{cases} {}^CD^{\alpha}_{0^+}x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,1], \\ \Delta x(t_1) = I, \quad \Delta x'(t_1) = J, \\ x'(0) = x_0, \ x'(1) = x_1. \end{cases} \tag{6.0.2}$$

It is easy to show that BVP (6.0.1) has solutions

$$x(t) = \begin{cases} c + x_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds, t \in [0, t_1], \\ c + I + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds - t_1 \left[x_1 - x_0 - \int_{t_1}^1 \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds\right] \\ +t\left[x_1 - \int_{t_1}^1 \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds\right] + \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds, t \in (t_1, 1] \end{cases}$$

if and only if

$$\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds + \int_{t_1}^1 \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds = x_1 - x_0 - J. \tag{6.0.3}$$

However, BVP (6.0.2) has solutions

$$x(t) = \begin{cases} c + x_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds, t \in [0, t_1], \\ c + I - t_1 J + (x_0 + J)t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds, t \in (t_1, 1] \end{cases}$$

if and only if

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds = x_1 - x_0 - J. \tag{6.0.4}$$

One sees that (6.0.3) is different from (6.0.4). □

Example 6.0.2. Let $0 = t_0 < t_1 < t_2 = 1$. Consider the following problems:

$$\begin{cases} {}^CD^{\alpha}_{t_i^+}x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,1], \\ \Delta x(t_1) = ax(t_1), \quad \Delta x'(t_1) = bx'(t_1), \\ x(0) = x_0, \ x'(1) = x_1, \end{cases} \tag{6.0.5}$$

and

$$\begin{cases} {}^CD^{\alpha}_{0^+}x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,1], \\ \Delta x(t_1) = ax(t_1), \quad \Delta x'(t_1) = bx'(t_1), \\ x(0) = x_0, \ x'(1) = x_1. \end{cases} \tag{6.0.6}$$

It is easy to show that

(i) BVP (6.0.5) has no solution if and only if $1+b=0$ and $x_1-\int_{t_1}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\neq 0$;

(ii) BVP (6.0.5) has a unique solution

$$x(t)=\begin{cases} x_0+\frac{t}{1+b}\left[x_1-\int_{t_1}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-(1+b)\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]\\ +\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in[0,t_1],\\ \frac{(1+a)t_1}{1+b}\left[x_1-\int_{t_1}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-(1+b)\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds+x_0\right.\\ \left.+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds\right]\\ +\left[x_1-\int_{t_1}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]t+\int_{t_1}^t\frac{t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in(t_1,1]\end{cases}$$

if and only if $1+b\neq 0$;

(iii) BVP (6.0.5) has infinitely many solutions

$$x(t)=\begin{cases} x_0+dt+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in[0,t_1],\\ (1+a)\left[t_1d+x_0+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds\right]+\left[x_1-\int_{t_1}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]t\\ +\int_{t_1}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in(t_1,1]\end{cases}$$

if and only if $1+b=0$ and $x_1-\int_{t_1}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds=0$;

(iv) BVP (6.0.6) has no solution if and only if $1+b=0$ and $x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds+\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\neq 0$;

(v) BVP (6.0.6) has a unique solution

$$x(t)=\begin{cases} x_0+\frac{t}{1+b}\left[x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-b\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]\\ +\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in[0,t_1],\\ (1+a)\left[x_0+\frac{t_1}{1+b}\left(x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-b\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right)\right.\\ \left.+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds\right]\\ -t_1\left[x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds\\ +\left[x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]t+\int_0^t\frac{t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in(t_1,1]\end{cases}$$

if and only if $1+b\neq 0$;

(vi) BVP (6.0.6) has infinitely many solutions

$$x(t)=\begin{cases} x_0+dt+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in[0,t_1],\\ (1+a)\left[x_0+dt_1+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds\right]-t_1\left[x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]\\ -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds\\ +\left[x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right]t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds, t\in(t_1,1]\end{cases}$$

if and only if $1+b=0$ and $x_1-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds+\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds=0$. □

Now, we consider the existence of solutions of the following nonlinear boundary value problems of fractional impulsive differential equations

$$\begin{cases} {}^CD^{\alpha}_{t_i^+}x(t)=p(t)f(t,x(t)),t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m],\\ \Delta x(t_i)=I(t_i,x(t_i)),\Delta x'(t_i)=J(t_i,x(t_i)),i\in\mathbb{N}[1,m],\\ ax(0)-bx'(0)=x_0,\ cx(1)+dx'(1)=x_1, \end{cases} \tag{6.0.7}$$

and

$$\begin{cases} {}^CD^{\alpha}_{0^+}x(t)=q(t)g(t,x(t)),t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m],\\ \Delta x(t_i)=I(t_i,x(t_i)),\Delta x'(t_i)=J(t_i,x(t_i)),i\in\mathbb{N}[1,m],\\ ax(0)-bx'(0)=x_0,\ cx(1)+dx'(1)=x_1, \end{cases} \tag{6.0.8}$$

where $\alpha\in(1,2)$, ${}^CD^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α with the start points $t_i,i\in\mathbb{N}[0,m]$, ${}^CD^{\alpha}_{0^+}$ represents the standard Caputo fractional derivatives of order α with the single start point 0, $a,b,c,d\in\mathbb{R}$ with $\delta=ac+ad+bc\neq0$, $x_0,x_1\in\mathbb{R}$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $I,J:\{t_i,i\in\mathbb{N}[1,m]\}\times\mathbb{R}\to\mathbb{R}(i\in\mathbb{N}[1,m]$, $f:\bigcup_{i=0}^{m}(t_i,t_{i+1})\times\mathbb{R}\to\mathbb{R}$, $g:(0,1)\times\mathbb{R}\to\mathbb{R}$, and p is defined on $\bigcup_{i=0}^{m}(t_i,t_{i+1})$, q is defined on $(0,1)$.

Lemma 6.0.1 *Suppose that $p:\bigcup_{i=0}^{m}(t_i,t_{i+1})\to\mathbb{R}$ is continuous and satisfies that there exist numbers $k>-1$ and $l>-1$ such that $|p(t)|\le(t-t_i)^k(t_{i+1}-t)^l$ for all $t\in(t_i,t_{i+1}),i\in\mathbb{N}[0,m]$. Then x is a solution of the following problem*

$$\begin{cases} {}^CD^{\alpha}_{t_i^+}x(t)=p(t),t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m],\\ \Delta x(t_i)=I_i,\ \Delta x'(t_i)=J_i,\ i\in\mathbb{N}[1,m],\\ ax(0)-bx'(0)=x_0,\ cx(1)+dx'(1)=x_1 \end{cases} \tag{6.0.9}$$

if and only if

$$x(t)=\begin{cases} c_0+d_0t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds,t\in[0,t_1],\\ c_0+\sum\limits_{j=1}^{i}I_j-\sum\limits_{j=1}^{i}t_jJ_j+\sum\limits_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds-\sum\limits_{j=1}^{i}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\\ +\left(d_0+\sum\limits_{j=1}^{i}J_j+\sum\limits_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right)t\\ +\int_{t_i}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds,t\in(t_i,t_{i+1}],i\in\mathbb{N}[1,m], \end{cases} \tag{6.0.10}$$

where

$$c_0=\frac{1}{\delta}\left[(c+d)x_0+bx_1-bc\sum_{j=1}^{m}I_j-b(c+d)\sum_{j=1}^{m}J_j+bc\sum_{j=1}^{m}t_jJ_j\right.$$

$$-bc\int_{t_m}^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds-bd\int_{t_m}^{1}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-bc\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds$$

$$\left.+bc\sum_{j=1}^{m}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-b(c+d)\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right],\qquad (6.0.11)$$

$$d_0=\frac{1}{\delta}\left[-cx_0+ax_1-ac\sum_{j=1}^{m}I_j-ac\sum_{j=1}^{m}t_jJ_j-a(c+d)\sum_{j=1}^{m}J_j\right.$$

$$-ac\int_{t_m}^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds-ad\int_{t_m}^{1}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-ac\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds$$

$$\left.+ac\sum_{j=1}^{m}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds-a(c+d)\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds\right].$$

Proof: Suppose that x is a solution of (6.0.9). From ${}^{C}D_{t_i^+}^{\alpha}x(t)=p(t),t\in(t_i,t_{i+1}]$, from Theorem 3.1.2, there exist numbers $c_i,d_i\in\mathbb{R}$ such that $x(t)=c_i+d_i(t-t_i)+\int_{t_i}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds,t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m]$. Then we have

$$ac_0-bd_0=x_0,$$

$$cc_m+(c+d)d_m=x_1-c\int_{t_m}^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds-d\int_{t_m}^{1}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds.\qquad (6.0.12)$$

Furthermore, we have

$$c_i-c_{i-1}+(d_i-d_{i-1})t_i=I_i+\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds,$$

$$d_i-d_{i-1}=J_i+\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds.$$

It follows that

$$c_i=c_0+\sum_{j=1}^{i}I_j-\sum_{j=1}^{i}t_jJ_j+\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds-\sum_{j=1}^{i}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds,$$

$$d_i=d_0+\sum_{j=1}^{i}J_j+\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds.$$

Then

$$c_m = c_0 + \sum_{j=1}^{m} I_j - \sum_{j=1}^{m} t_j J_j + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds - \sum_{j=1}^{m} t_j \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds,$$
$$d_m = d_0 + \sum_{j=1}^{m} J_j + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds. \tag{6.0.13}$$

Substituting (6.0.13) into (6.0.12), we get c_0 and d_0 defined by (6.0.11). Then x satisfies (6.0.10). On the other hand, we can prove that x is a solution of (6.0.9) if x satisfies (6.0.10). The proof is completed. □

Remark 6.0.1 Lemma 6.0.1 generalizes Lemma 2.3 in [108] since we allow p be singular at $t = t_i (i \in \mathbb{N}[0,m])$ while $p \in C[0,1]$ in [108]. We note that two examples are given at the end of [108]. In Example 4.1 and Example 4.2, the fractional derivative is $D_{0^+}^{\alpha}$ which has single starting point 0. So these examples are unsuitable.

Lemma 6.0.2 *Suppose that $q : (0,1) \to \mathbb{R}$ is continuous and satisfies that there exist numbers $k > -1$ and $l > -1$ such that $|q(t)| \le t^k (1-t)^l$ for all $t \in (0,1)$. Then x is a solution of the following problem*

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = p(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ \Delta x(t_i) = I_i, i \in \mathbb{N}[1,m], \\ \Delta x'(t_i) = J_i, i \in \mathbb{N}[1,m], \\ ax(0) - bx'(0) = x_0, \ cx(1) + dx'(1) = x_1 \end{cases} \tag{6.0.14}$$

if and only if

$$x(t) = \begin{cases} c_0 + d_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds, t \in [0, t_1], \\ c_0 + \sum_{j=1}^{i} I_j - \sum_{j=1}^{i} t_j J_j + \left(d_0 + \sum_{j=1}^{i} J_j \right) t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[1,m], \end{cases} \tag{6.0.15}$$

where

$$c_0 = \frac{1}{\delta} \left[(c+d)x_0 + bx_1 - bc \sum_{j=1}^{m} I_j + bc \sum_{j=1}^{m} t_j J_j - b(c+d) \sum_{j=1}^{m} J_j \right.$$
$$\left. -bc \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds - bd \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds \right],$$

$$d_0 = \frac{1}{\delta} \left[-cx_0 + ax_1 - ac \sum_{j=1}^{m} I_j + ac \sum_{j=1}^{m} t_j J_j - a(c+d) \sum_{j=1}^{m} J_j \right.$$
$$\left. -ac \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds - ad \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} p(s)ds \right].$$

Proof: From ${}^CD_{0^+}^\alpha x(t) = p(t), t \in (t_i, t_{i+1}]$, from **Remark 4.0.1**, there exist numbers $c_i, d_i \in \mathbb{R}$ such that $x(t) = c_i + d_i(t-t_i) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m]$. The remainder of the proof is similar to that of the proof of Lemma 6.0.1 and is omitted. □

Remark 6.0.2 In [103], the following problem was studied:

$$\begin{cases} {}^CD_{0^+}^q x(t) = w(t)g(t, x(t), x'(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ \Delta x(t_i) = I_i(x(t_i)), \Delta x'(t_i) = J_i(x(t_i)), i \in \mathbb{N}[1,m], \\ ax(0) - bx'(0) = 0, \ cx(1) + dx'(1) = 0, \end{cases} \tag{6.0.16}$$

where $q \in (1,2)$, ${}^CD_{0^+}^q$ represents the standard Caputo fractional derivatives of order q with the single start point 0, $a, b, c, d \in \mathbb{R}$ with $\delta = ac + ad + bc \neq 0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $I, J \in C(\mathbb{R}, \mathbb{R})(i \in \mathbb{N}[1,m]$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and w is continuous and nonnegative. We find that Theorem 2.1 in [103] is not correct. In fact, in the proof of Theorem 2.1, it claims that there exist b_0, b_1, c_0, c_1 such that

$$x(t) = \begin{cases} -b_0 - b_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds, t \in [0, t_1], \\ -c_0 - c_1 t + \int_{t_1}^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds, t \in (t_1, t_2]. \end{cases}$$

However, if x satisfies above equation, we have for $t \in (t_1, t_2]$

$$D_{0^+}^q x(t) = \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} x''(s)ds = \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} x''(s)ds$$

$$+\frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} x''(s)ds$$

$$= \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} \left[-b_0 - b_1 s + \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u)du\right]'' ds$$

$$+\frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} \left[-c_0 - c_1 s + \int_{t_1}^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u)du\right]'' ds$$

$$= \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} \left[\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u)du\right]'' ds + \frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} \left[\int_{t_1}^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u)du\right]'' ds$$

$$= \frac{1}{\Gamma(2-q)} \int_0^{t_1} (t-s)^{1-q} \left[\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u)du\right]' ds + \frac{1}{\Gamma(2-q)} \int_{t_1}^t (t-s)^{1-q} \left[\int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u)du\right]' ds$$

$$= \frac{1}{\Gamma(3-q)} \left[\int_0^{t_1} (t-s)^{2-q} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u)du\right)' ds\right]'$$

$$+\frac{1}{\Gamma(3-q)} \left[\int_{t_1}^t (t-s)^{2-q} \left(\int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \sigma(u)du\right)' ds\right]'$$

$$= \frac{1}{\Gamma(3-q)}\left[(t-s)^{2-q}\left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\sigma(u)du\right)\Big|_0^{t_1} + (2-q)\int_0^{t_1}(t-s)^{1-q}\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\sigma(u)duds\right]'$$

$$+\frac{1}{\Gamma(3-q)}\left[(t-s)^{2-q}\left(\int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\sigma(u)du\right)\Big|_{t_1}^{t} + (2-q)\int_{t_1}^{t}(t-s)^{1-q}\int_{t_1}^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\sigma(u)duds\right]'$$

$$= \frac{1}{\Gamma(3-q)}\left[(t-t_1)^{2-q}\int_0^{t_1} \frac{(s-u)^{q-2}}{\Gamma(q-1)}\sigma(u)du + (2-q)\int_0^{t_1}\int_u^t(t-s)^{1-q}\frac{(s-u)^{q-2}}{\Gamma(q-1)}ds\sigma(u)du\right]'$$

$$+\frac{1}{\Gamma(2-q)}\left[\int_{t_1}^t\int_u^t(t-s)^{1-q}\frac{(s-u)^{q-2}}{\Gamma(q-1)}ds\sigma(u)du\right]'$$

$$= \frac{1}{\Gamma(3-q)}\left[(t-t_1)^{2-q}\int_0^{t_1} \frac{(s-u)^{q-2}}{\Gamma(q-1)}\sigma(u)du + (2-q)\int_0^{t_1}\int_0^1(1-w)^{1-q}\frac{w^{q-2}}{\Gamma(q-1)}dw\sigma(u)du\right]'$$

$$+\frac{1}{\Gamma(2-q)}\left[\int_{t_1}^t\int_0^1(1-w)^{1-q}\frac{w^{q-2}}{\Gamma(q-1)}dw\sigma(u)du\right]'$$

$$= \sigma(t) + \frac{1}{\Gamma(2-q)}(t-t_1)^{1-q}\int_0^{t_1}\frac{(s-u)^{q-2}}{\Gamma(q-1)}\sigma(u)du \neq \sigma(t).$$

Hence Lemma 6.0.2 corrected Theorem 2.3 in [103]. □

We need the following assumption:

(D1) there exist nondecreasing functions $\phi_f \in L^1[0,1]$, $M_f, M_I, M_J : [0,+\infty) \to [0,+\infty)$, numbers $I_i, J_i \in \mathbb{R}$ such that

$$|f(t,x) - \phi_f(t)| \le M_f(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], x \in \mathbb{R},$$

$$|I(t_i,x) - I_i| \le M_I(|x|), i \in \mathbb{N}[1,m], x \in \mathbb{R},$$

$$|J(t_i,x) - J_i| \le M_J(|x|), i \in \mathbb{N}[1,m], x \in \mathbb{R}.$$

Denote

$$\Phi(t) = \begin{cases} \overline{c}_0 + \overline{d}_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\phi_f(s)ds, t \in [0,t_1], \\ \overline{c}_0 + \sum\limits_{j=1}^i I_j - \sum\limits_{j=1}^i t_j J_j + \sum\limits_{j=1}^i \int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\phi_f(s)ds \\ -\sum\limits_{j=1}^i t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds \\ +\left(\overline{d}_0 + \sum\limits_{j=1}^i J_j + \sum\limits_{j=1}^i\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds\right)t \\ +\int_{t_i}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\phi_f(s)ds, t \in (t_i,t_{i+1}], i \in \mathbb{N}[1,m], \end{cases}$$

where

$$\overline{c}_0=\frac{1}{\delta}\left[(c+d)x_0+bx_1-bc\sum_{j=1}^m I_j-b(c+d)\sum_{j=1}^m J_j+bc\sum_{j=1}^m t_jJ_j\right.$$

$$-bc\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\phi_f(s)ds-bd\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds$$

$$-bc\sum_{j=1}^m\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\phi_f(s)ds$$

$$\left.+bc\sum_{j=1}^m t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds-b(c+d)\sum_{j=1}^m\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds\right],$$

$$\overline{d}_0=\frac{1}{\delta}\left[-cx_0+ax_1-ac\sum_{j=1}^m I_j-ac\sum_{j=1}^m t_jJ_j-a(c+d)\sum_{j=1}^m J_j\right.$$

$$-ac\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\phi_f(s)ds-ad\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds$$

$$-ac\sum_{j=1}^m\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\phi_f(s)ds$$

$$\left.+ac\sum_{j=1}^m t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds-a(c+d)\sum_{j=1}^m\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)\phi_f(s)ds\right].$$

Theorem 6.0.1 *Suppose that* $p:\bigcup_{i=0}^m(t_i,t_{i+1})\to\mathbb{R}$ *is continuous and satisfies that there exist numbers* $k>-1$ *and* $l>-\alpha+1$ *such that* $|p(t)|\leq(t-t_i)^k(t_{i+1}-t)^l$ *for all* $t\in(t_i,t_{i+1}),i\in\mathbb{N}[0,m]$, f *is a* **impulsive I-Carathéodory function**, I,J **discrete I-Carathéodory functions** *and* (D1) *holds. Then BVP* (6.0.7) *has at least one solution if there exists* $r_0>0$ *such that*

$$\left[m+\frac{m|ac|}{|\delta|}+\frac{m|bc|}{|\delta|}\right]M_I(r_0+||\Phi||)+\left[2m+\frac{m[2|ac|+|ad|]}{|\delta|}+\frac{m[2|bc|+|bd|]}{|\delta|}\right]M_J(r_0+||\Phi||)$$

$$+\frac{2|\delta|+|ac|+|bc|}{|\delta|}\sum_{j=1}^{m+1}\frac{\mathbf{B}(\alpha+l,k+1)(t_j-t_{j-1})^{\alpha+k+l}}{\Gamma(\alpha)}M_f(r_0+||\Phi||)$$

$$+\frac{2|\delta|+2|ac|+2|bc|+|bd|+|ad|}{|\delta|}\sum_{j=1}^{m+1}\frac{\mathbf{B}(\alpha+l-1,k+1)(t_j-t_{j-1})^{\alpha+k+l-1}}{\Gamma(\alpha-1)}M_f(r_0+||\Phi||)\leq r_0. \tag{6.0.17}$$

Proof: Define the nonlinear operator T on $P_mC[0,1]$ by

$$(Tx)(t)=\begin{cases} c_0+d_0t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in[0,t_1],\\ c_0+\sum\limits_{j=1}^{i}I(t_j,x(t_j))-\sum\limits_{j=1}^{i}t_jJ(t_j,x(t_j))+\sum\limits_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\sum\limits_{j=1}^{i}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds\\ +\left(d_0+\sum\limits_{j=1}^{i}J(t_j,x(t_j))+\sum\limits_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds\right)t\\ +\int_{t_i}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[1,m],\end{cases}$$

where

$$c_0=\frac{1}{\delta}\left[(c+d)x_0+bx_1-bc\sum_{j=1}^{m}I(t_j,x(t_j))-b(c+d)\sum_{j=1}^{m}J(t_j,x(t_j))+bc\sum_{j=1}^{m}t_jJ(t_j,x(t_j))\right.$$
$$-bc\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-bd\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds$$
$$-bc\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds$$
$$\left.+bc\sum_{j=1}^{m}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds-b(c+d)\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds\right],$$

$$d_0=\frac{1}{\delta}\left[-cx_0+ax_1-ac\sum_{j=1}^{m}I(t_j,x(t_j))-ac\sum_{j=1}^{m}t_jJ(t_j,x(t_j))-a(c+d)\sum_{j=1}^{m}J(t_j,x(t_j))\right.$$
$$-ac\int_{t_m}^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-ad\int_{t_m}^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds$$
$$-ac\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds$$
$$\left.+ac\sum_{j=1}^{m}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds-a(c+d)\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)f(s,x(s))ds\right].$$

By Lemma 6.0.1, we have x is a solution of BVP (6.0.7) if and only if x is a fixed point of T in $P_mC[0,1]$. A standard proof shows that $T:P_mC[0,1]\to P_mC[0,1]$ is well defined and completely continuous.

Choose $\Omega=\{x\in P_mC[0,1]:||x-\Phi||\le r_0\}$. We will prove that $T\Omega\subset\Omega$. For $x\in\Omega$, we have $||x|| \le ||x-\Phi|| + ||\Phi|| \le r_0 + ||\Phi||$. Then (D1) implies

$$|f\,(t,x(t)) - \phi_f(t)| \le M_f(|x(t)|) \le M_f(||x||) \le M_f(r_0 + ||\Phi||), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m],$$

$$|I\,(t_i, x(t_i)) - I_i| \le M_I(r_0 + ||\Phi||), i \in \mathbb{N}[1,m],$$

$$|J\,(t_i, x(t_i)) - J_i| \le M_J(r_0 + ||\Phi||), i \in \mathbb{N}[1,m].$$

Then

$$|c_0 - \overline{c}_0| \le \frac{1}{|\delta|}\Bigg[|bc|\sum_{j=1}^{m}|I(t_j,x(t_j)) - I_j| + |b(c+d)|\sum_{j=1}^{m}|J(t_j,x(t_j)) - J_j|$$

$$+|bc|\sum_{j=1}^{m} t_j|J(t_j,x(t_j)) - J_j|$$

$$+|bc|\int_{t_m}^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s)) - \phi_f(s)|ds$$

$$+|bd|\int_{t_m}^{1}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|p(s)||f(s,x(s)) - \phi_f(s)|ds$$

$$+|bc|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s)) - \phi_f(s)|ds$$

$$+|bc|\sum_{j=1}^{m} t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}|p(s)||f(s,x(s)) - \phi_f(s)|ds$$

$$+|b(c+d)|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}|p(s)||f(s,x(s)) - \phi_f(s)|ds\Bigg]$$

$$\le \frac{1}{|\delta|}\left[m|bc|M_I(r_0 + ||\Phi||) + m[|bc| + |bd|]|M_J(r_0 + ||\Phi||) + m|bc|M_J(r_0 + ||\Phi||)\right.$$

$$+|bc|\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}(s-t_{j-1})^k(t_j-s)^l ds M_f(r_0 + ||\Phi||)$$

$$\left.+[2|bc| + |bd)|]\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}(s-t_{j-1})^k(t_j-s)^l ds M_f(r_0 + ||\Phi||)\right]$$

$$\le \frac{1}{|\delta|}\left[m|bc|M_I(r_0 + ||\Phi||) + m[2|bc| + |bd|]|M_J(r_0 + ||\Phi||)\right.$$

$$+|bc|\sum_{j=1}^{m+1}(t_j - t_{j-1})^{\alpha+k+l}\int_0^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw M_f(r_0 + ||\Phi||)$$

$$+[2|bc|+|bd)|]\sum_{j=1}^{m+1}(t_j-t_{j-1})^{\alpha+k+l-1}\int_0^1\frac{(1-w)^{\alpha+l-2}}{\Gamma(\alpha-1)}w^kdwM_f(r_0+||\Phi||)\Big]$$

$$=\frac{m|bc|}{|\delta|}M_I(r_0+||\Phi||)+\frac{m[2|bc|+|bd|]}{|\delta|}M_J(r_0+||\Phi||)$$

$$+\sum_{j=1}^{m+1}\left[\frac{|bc|}{|\delta|}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}(t_j-t_{j-1})^{\alpha+k+l}+\frac{[2|bc|+|bd)|]}{|\delta|}\frac{\mathbf{B}(\alpha+l-1,k+1)}{\Gamma(\alpha-1)}(t_j-t_{j-1})^{\alpha+k+l-1}\right]$$

$$M_f(r_0+||\Phi||).$$

Similarly we have

$$|d_0-\overline{d}_0|\le\frac{m|ac|}{|\delta|}M_I(r_0+||\Phi||)+\frac{m[2|ac|+|ad|]}{|\delta|}M_J(r_0+||\Phi||)$$

$$+\sum_{j=1}^{m+1}\left[\frac{|ac|}{|\delta|}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}(t_j-t_{j-1})^{\alpha+k+l}+\frac{2|ac|+|ad|}{|\delta|}\frac{\mathbf{B}(\alpha+l-1,k+1)}{\Gamma(\alpha-1)}(t_j-t_{j-1})^{\alpha+k+l-1}\right]$$

$$M_f(r_0+||\Phi||).$$

Use (6.0.17), we have for $t\in(t_i,t_{i+1}]$

$$|(Tx)(t)-\Phi(t)|\le\begin{cases}|c_0-\overline{c}_0|+|d_0-\overline{d}_0|+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s))-\phi_f(s)|ds, & i=0,\\ |c_0-\overline{c}_0|+|d_0-\overline{d}_0|+\sum_{j=1}^{i}|I(t_j,x(t_j))-I_j|+\sum_{j=1}^{i}t_j|J(t_j,x(t_j))-J_j| & \\ +\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s))-\phi_j(s)|ds & \\ +\sum_{j=1}^{i}t_j\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}|p(s)||f(s,x(s))-\phi_f(s)|ds & \\ +\Bigg(\sum_{j=1}^{i}|J(t_j,x(t_j))-J_j| & \\ +\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}|p(s)||f(s,x(s))\phi_f(s)|ds\Bigg)t & \\ +\int_{t_i}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s))-\phi_f(s)|ds, & i\in\mathbb{N}[1,m],\end{cases}$$

$$\le|c_0-\overline{c}_0|+|d_0-\overline{d}_0|$$

$$+\sum_{j=1}^{i}|I(t_j,x(t_j))-I_j|+2\sum_{j=1}^{i}|J(t_j,x(t_j))-J_j|$$

$$+2\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s))-\phi_j(s)|ds$$

$$+2\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}|p(s)||f(s,x(s))-\phi_f(s)|ds+\int_{t_i}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s))-\phi_f(s)|ds$$

$$\leq \sum_{j=1}^{i}|I(t_j,x(t_j))-I_j|+2\sum_{j=1}^{i}|J(t_j,x(t_j))-J_j|$$

$$+2\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)||f(s,x(s))-\phi_j(s)|ds$$

$$+2\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}|p(s)||f(s,x(s))-\phi_f(s)|ds+|c_0-\overline{c}_0|+|d_0-\overline{d}_0|$$

$$\leq mM_I(r_0+||\Phi||)+2mM_J(r_0+||\Phi||)$$

$$+2\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}(s-t_{j-1})^k(t_j-s)^l dsM_f(r_0+||\Phi||)$$

$$+2\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}(s-t_{j-1})^k(t_j-s)^l dsM_f(r_0+||\Phi||)+|c_0-\overline{c}_0|+|d_0-\overline{d}_0|$$

$$\leq mM_I(r_0+||\Phi||)+2mM_J(r_0+||\Phi||)$$

$$+2\sum_{j=1}^{m+1}(t_j-t_{j-1})^{\alpha+k+l}\int_0^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dwM_f(r_0+||\Phi||)$$

$$+2\sum_{j=1}^{m+1}(t_j-t_{j-1})^{\alpha+k+l-1}\int_0^1\frac{(1-w)^{\alpha+l-2}}{\Gamma(\alpha-1)}w^k dwM_f(r_0+||\Phi||)+|c_0-\overline{c}_0|+|d_0-\overline{d}_0|$$

$$\leq \frac{2|\delta|+|ac|+|bc|}{|\delta|}\sum_{j=1}^{m+1}\frac{\mathbf{B}(\alpha+l,k+1)(t_j-t_{j-1})^{\alpha+k+l}}{\Gamma(\alpha)}M_f(r_0+||\Phi||)$$

$$+\frac{2|\delta|+2|ac|+2|bc|+|bd|+|ad|}{|\delta|}\sum_{j=1}^{m+1}\frac{\mathbf{B}(\alpha+l-1,k+1)(t_j-t_{j-1})^{\alpha+k+l-1}}{\Gamma(\alpha-1)}M_f(r_0+||\Phi||)$$

$$+\left[m+\frac{m|ac|}{|\delta|}+\frac{m|bc|}{|\delta|}\right]M_I(r_0+||\Phi||)+\left[2m+\frac{m[2|ac|+|ad|]}{|\delta|}+\frac{m[2|bc|+|bd|]}{|\delta|}\right]M_J(r_0+||\Phi||)\leq r_0.$$

Hence $T\Omega\subset\Omega$. By Schauder's fixed point theorem see Theorem 2.2.1 in [35], T has a fixed point $x\in\Omega$ which is a solution of BVP (6.0.7). The proof is completed. □

Now suppose that

(D2) there exist nondecreasing functions $\phi_g \in L^1[0,1]$, $M_g, M_I, M_J : [0,+\infty) \to [0,+\infty)$, numbers $I_i, J_i \in \mathbb{R}$ such that

$$|g(t,x) - \phi_g(t)| \le M_g(|x|), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], x \in \mathbb{R},$$

$$|I(t_i,x) - I_i| \le M_I(|x|), i \in \mathbb{N}[1,m], x \in \mathbb{R},$$

$$|J(t_i,x) - J_i| \le M_J(|x|), i \in \mathbb{N}[1,m], x \in \mathbb{R}.$$

Denote

$$\Phi(t) = \begin{cases} c_0 + d_0 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)\phi_g(s)ds, t \in [0,t_1], \\ c_0 + \sum\limits_{j=1}^{i} I_j - \sum\limits_{j=1}^{i} t_j J_j + \left(d_0 + \sum\limits_{j=1}^{i} J_j \right) t \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)\phi_g(s)ds, t \in (t_i, t_{i+1}], i \in \mathbb{N}[1,m], \end{cases}$$

where

$$\overline{c}_0 = \frac{1}{\delta}\left[(c+d)x_0 + bx_1 - bc\sum_{j=1}^{m} I_j + bc\sum_{j=1}^{m} t_j J_j - b(c+d)\sum_{j=1}^{m} J_j \right.$$

$$\left. -bc\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)\phi_g(s)ds - bd\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} q(s)\phi_g(s)ds \right],$$

$$\overline{d}_0 = \frac{1}{\delta}\left[-cx_0 + ax_1 - ac\sum_{j=1}^{m} I_j + ac\sum_{j=1}^{m} t_j J_j - a(c+d)\sum_{j=1}^{m} J_j \right.$$

$$\left. -ac\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)\phi_g(s)ds - ad\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} q(s)\phi_g(s)ds \right].$$

Theorem 6.0.2 *Suppose that $q : (0,1) \to \mathbb{R}$ is continuous and satisfies that there exist numbers $k > -1$ and $l > -\alpha + 1$ such that $|q(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$, f is a* **impulsive I-Carathéodory function**, *I, J* **discrete I-Carathéodory functions** *and* (D2) *holds. Then BVP* (6.0.8) *has at least one solution if*

$$\begin{aligned} &\frac{|\delta|+m|ac|+m|bc|}{|\delta|} M_I(r_0 + ||\Phi||) + \frac{2m|bc|+2|ac|+|bd|+|ac|}{|\delta|} M_J(r_0 + ||\Phi||) \\ &+ \left[\frac{|\delta|+|ac|+|bc|}{|\delta|} \frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)} + \frac{|ad|+|bd|}{|\delta|} \frac{\mathbf{B}(\alpha+l-1,k+1)}{\Gamma(\alpha-1)} \right] M_g(r_0 + ||\Phi||) \le r_0. \end{aligned} \quad (6.0.18)$$

Proof: Define the nonlinear operator T on $P_mC[0,1]$ by

$$(Tx)(t)=\begin{cases} c_0+d_0t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)g(s,x(s))ds, t\in[0,t_1],\\ c_0+\sum\limits_{j=1}^{i}I(t_j,x(t_j))-\sum\limits_{j=1}^{i}t_jJ(t_j,x(t_j))+\left(d_0+\sum\limits_{j=1}^{i}J(t_j,x(t_j))\right)t\\ +\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)g(s,x(s))ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}[1,m],\end{cases}$$

where

$$c_0=\frac{1}{\delta}\left[(c+d)x_0+bx_1-bc\sum_{j=1}^{m}I(t_j,x(t_j))+bc\sum_{j=1}^{m}t_jJ(t_j,x(t_j))-b(c+d)\sum_{j=1}^{m}J(t_j,x(t_j))\right.$$

$$\left.-bc\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)g(s,x(s))ds-bd\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}q(s)g(s,x(s))ds\right],$$

$$d_0=\frac{1}{\delta}\left[-cx_0+ax_1-ac\sum_{j=1}^{m}I(t_j,x(t_j))+ac\sum_{j=1}^{m}t_jJ(t_j,x(t_j))-a(c+d)\sum_{j=1}^{m}J(t_j,x(t_j))\right.$$

$$\left.-ac\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)g(s,x(s))ds-ad\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}q(s)g(s,x(s))ds\right].$$

The remainder of the proof is similar to that of the proof of Theorem 6.0.1 and is omitted. □

Remark 6.0.3 In [101], Zhao studied the eigenvalue intervals and positive solutions of integral boundary value problem for the following higher-order nonlinear fractional differential equation with impulses

$$\begin{cases} -D_{0^+}^{\alpha}u(t)=\lambda a(t)f(t,u(t)), t\in(0,1)\setminus\{t_1,t_2,\cdots,t_m\},\\ \Delta u(t_k)=I_k(u(t_k)), k\in\mathbb{N}[1,m],\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ u(1)=\int_0^1u(s)dH(s),\end{cases}$$

where where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $n-1<\alpha<n$, $n\geq 3$, the number n is the smallest integer greater than or equal to α, the impulsive point sequence $\{t_i:i\in\mathbb{N}[1,m]\}$ satisfies $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $\Delta u(t_k)=u(t_k^)Cu(t_k^-)$, $u(t_k^-)k)=u(t_k)$, and $u(t_k^+)=\lim\limits_{h\to 0^+}u(t_k+h)$ and $u(t_kC)=\lim\limits_{t\to 0^-}u(t_k+h)$ represent the right and left-hand limits of $u(t)$ at $t=t_k$, respectively, $\lambda\geq 0$ is a parameter, $f\in C([0,1]\times[0,+\infty),[0,+\infty))$, $a\in C((0,1),[0,+\infty))$, $I_k\in C([0,+\infty),[0,\infty))$, the integral $\int_0^1u(s)dH(s)$ is the Riemann-Stieltjes integral with $H:[0,1]\to\mathbb{R}$. The following result was proved:

Result Z (Lemma 2.4 [101]). If $H:[0,1]\to\mathbb{R}$ is a function of bounded variation $\delta =: \int_0^1 s^{\alpha-1}dH(s)\neq \alpha-1$ and $h\in C([0,1])$, then the unique solution of

$$\begin{cases} -D_{0^+}^{\alpha}u(t)=h(t), t\in(0,1)\setminus\{t_1,t_2,\cdots,t_m\},\\ \Delta u(t_k)=I_k(u(t_k)), k=1,2,\cdots,m,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ u(1)=\int_0^1 u(s)dH(s), \end{cases} \tag{6.0.19}$$

is

$$u(t)=\int_0^1 G(t,s)h(s)ds+t^{\alpha-1}\sum_{t\le t_k<1}t_k^{1-\alpha}I_k(u(t_k)), t\in(0,1] \tag{6.0.20}$$

where

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1},\ s\in[0,t],\\ t^{\alpha-1}(1-s)^{\alpha-2},\ s\in[t,1], \end{cases}$$

$$G_2(t,s)=\frac{t^{\alpha-1}}{\alpha-1-\delta}\int_0^1 G_1(\tau,s)dH(\tau),\quad G(t,s)=G_1(t,s)+G_2(t,s).$$

Remark 6.0.4 Result Z is wrong. In fact, (6.0.20) can be re-written by

$$u(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+t^{\alpha-1}\left[\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}h(s)ds+\frac{1}{\alpha-1-\delta}\int_0^1 G_1\left[\int_0^1 G_1(\tau,s)h(s)ds\right]dH(\tau)\right.$$

$$\left.+\sum_{k=1}^{m}t_k^{1-\alpha}I_k(u(t_k))\right]-t^{\alpha-1}\sum_{j=1}^{k-1}t_j^{1-\alpha}I_j(u(t_j))$$

$$=: \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+A_0t^{\alpha-1}+t^{\alpha-1}\sum_{j=1}^{k-1}A_j, t\in(t_{k-1},t_k], k=0,1,\cdots,m.$$

Then by Definition 2.2 we have for $t\in(t_i,t_{i+1}]$ that

$$D_{0^+}^{\alpha}u(t)=\frac{\left[\int_0^t(t-s)^{n-\alpha-1}u(s)ds\right]^{(n)}}{\Gamma(n-\alpha)}=\frac{\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_\tau+1}(t-s)^{n-\alpha-1}u(s)ds+\int_{t_i}^t(t-s)^{n-\alpha-1}u(s)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_\tau+1}(t-s)^{n-\alpha-1}\left(\int_0^s\frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dv+A_0s^{\alpha-1}+s^{\alpha-1}\sum_{j=1}^{\tau}A_j\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_{t_i}^t(t-s)^{n-\alpha-1}\left(\int_0^s\frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dv+A_0s^{\alpha-1}+s^{\alpha-1}\sum_{j=1}^{i}A_j\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum\limits_{\tau=0}^{i-1}\sum\limits_{j=0}^{\tau} A_j \int_{t_\tau}^{t_\tau+1}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_0^t (t-s)^{n-\alpha-1}\int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dvds + \sum\limits_{j=0}^{i} A_j \int_{t_i}^{t}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum\limits_{j=0}^{i-1}\sum\limits_{\tau=j}^{i-1} A_j \int_{t_\tau}^{t_\tau+1}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_0^t \int_v^t (t-s)^{n-\alpha-1}\frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}dsh(v)dv + \sum\limits_{j=0}^{i} A_j \int_{t_i}^{t}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\int_0^t (t-v)^{n-1}\int_0^1 (1-w)^{n-\alpha-1}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwh(v)dv + \sum\limits_{j=0}^{i} A_j \int_{t_j}^{t}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\int_0^t (t-v)^{n-1}\int_0^1 (1-w)^{n-\alpha-1}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwh(v)dv + \sum\limits_{j=0}^{i} A_j t^{n-1}\int_{\frac{t_j}{t}}^{1}(1-w)^{n-\alpha-1}w^{\alpha-1}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= h(t) + \frac{\sum\limits_{j=0}^{i} A_j \left[t^{n-1}\int_{\frac{t_j}{t}}^{1}(1-w)^{n-\alpha-1}w^{\alpha-1}dw\right]^{(n)}}{\Gamma(n-\alpha)} \neq h(t) \text{ if } A_j \neq 0.$$

This shows us that Result Z is wrong.

We now correct Result Z. Involving Consider the following more general problem:

$$\begin{cases} -{}^C D_{0^+}^{\alpha} u(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta u^{(i)}(t_k) = I_i(t_k, u(t_k)), k \in \mathbb{N}[1, m],\ i \in \mathbb{N}[0, n-1], \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ u(1) = \int_0^1 u(s)dH(s). \end{cases} \tag{6.0.21}$$

Theorem 6.0.3 *Suppose that $\int_0^1 \frac{s^{n-1}}{(n-1)!}dH(s) \neq 1$. Then x is a solution of (6.0.21) if and only if*

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=1}^{k}\sum_{\nu=0}^{n-1} \frac{(t-t_j)^\nu}{\nu!} I_\nu(t_j, x(t_j))$$

$$+\frac{1}{1-\int_0^1 \frac{s^{n-1}}{(n-1)!}dH(s)} \left[\int_0^1 \frac{\alpha(1-s)^{\alpha-1}-(1-s)^\alpha}{\Gamma(\alpha+1)}h(s)ds - \sum_{j=1}^{m}\sum_{\nu=0}^{n-1} \frac{(1-t_j)^\nu}{\nu!} I_\nu(t_j, x(t_j))\right.$$

$$\left.+\sum_{j=1}^{m}\sum_{\nu=0}^{n-1} \int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!}dH(s) I_\nu(t_j, x(t_j))\right] \frac{t^{n-1}}{(n-1)!}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m].$$

Proof: Suppose x is a solution of (6.0.25). By $\alpha \in (n-1, n)$, using Theorem 4.0.2, we know from $-{}^C D_{0^+}^\alpha x(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m]$ that there exist constants $c_{\nu,j} \in \mathbb{R}$ such that

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=0}^{k}\sum_{\nu=0}^{n-1} c_{\nu,j}\frac{(t-t_j)^\nu}{\nu!}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m]. \tag{6.0.22}$$

It follows for $i = 1, 2, \cdots, n-1$ that

$$x^{(i)}(t) = -\int_0^t \frac{(t-s)^{\alpha-i-1}}{\Gamma(\alpha-i)}h(s)ds + \sum_{j=0}^{k}\sum_{\nu=i}^{n-1} c_{\nu,j}\frac{(t-t_j)^{\nu-i}}{(\nu-i)!}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m]. \tag{6.0.23}$$

(i) By $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$ and (6.22)-(6.23), we get $c_{0,0} = c_{1,0} = \cdots = c_{n-2,0} = 0$.

(ii) By $\Delta x^{(i)}(t_k) = I_i(t_k, x(t_k)), k \in \mathbb{N}[1, m],\ i \in \mathbb{N}[0, n-1]$ and (6.22)-(6.23), we get $c_{i,k} = I_i(t_k, x(t_k)), k = 1, 2, \cdots, m,\ i \in [0, n-1]$.

(iii) By $x(1) = \int_0^1 x(s)dH(s)$ and (6.22) and (i)-(ii), we get

$$-\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=0}^{m}\sum_{\nu=0}^{n-1} c_{\nu,j}\frac{(1-t_j)^\nu}{\nu!} = \int_0^1 x(s)dH(s) = \sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}} x(s)dH(s)$$

$$= \sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\left(-\int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dv + \sum_{j=0}^{\tau}\sum_{\nu=0}^{n-1} c_{\nu,j}\frac{(s-t_j)^\nu}{\nu!}\right)dH(s)$$

$$= -\sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dvds + \sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\sum_{j=0}^{\tau}\sum_{\nu=0}^{n-1} c_{\nu,j}\frac{(s-t_j)^\nu}{\nu!}dH(s)$$

$$= -\int_0^1\int_0^s \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dvds + \sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\sum_{j=0}^{\tau}\sum_{\nu=0}^{n-1} c_{\nu,j}\frac{(s-t_j)^\nu}{\nu!}dH(s)$$

$$= -\int_0^1\int_v^1 \frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}dsh(v)dv + \sum_{j=0}^{m}\sum_{\nu=0}^{n-1} c_{\nu,j}\int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!}dH(s)$$

$$= -\int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)}h(v)dv + \sum_{j=1}^{m}\sum_{\nu=0}^{n-1} I_\nu(t_j,x(t_j))\int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!}dH(s) + c_{n-1,0}\int_0^1 \frac{s^{n-1}}{(n-1)!}dH(s).$$

Hence

$$-\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=1}^{m}\sum_{\nu=0}^{n-1} I_\nu(t_j,x(t_j))\frac{(1-t_j)^\nu}{\nu!} + c_{n-1,0}$$

$$= -\int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)}h(v)dv + \sum_{j=1}^{m}\sum_{\nu=0}^{n-1} I_\nu(t_j,x(t_j))\int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!}dH(s) + c_{n-1,0}\int_0^1 \frac{s^{n-1}}{(n-1)!}dH(s).$$

It follows that

$$c_{n-1,0} = \frac{1}{1-\int_0^1 \frac{s^{n-1}}{(n-1)!}dH(s)}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - \sum_{j=1}^{m}\sum_{\nu=0}^{n-1}\frac{(1-t_j)^\nu}{\nu!}I_\nu(t_j,x(t_j))\right.$$
$$\left. -\int_0^1 \frac{(1-v)^\alpha}{\Gamma(\alpha+1)}h(v)dv + \sum_{j=1}^{m}\sum_{\nu=0}^{n-1}\int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!}dH(s)I_\nu(t_j,x(t_j))\right].$$

Substituting $c_{\nu,j}$ into (6.0.22), we get

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=1}^{k}\sum_{\nu=0}^{n-1}\frac{(t-t_j)^\nu}{\nu!}I_\nu(t_j,x(t_j))$$
$$+\frac{1}{1-\int_0^1 \frac{s^{n-1}}{(n-1)!}dH(s)}\left[\int_0^1 \frac{\alpha(1-s)^{\alpha-1}-(1-s)^\alpha}{\Gamma(\alpha+1)}h(s)ds - \sum_{j=1}^{m}\sum_{\nu=0}^{n-1}\frac{(1-t_j)^\nu}{\nu!}I_\nu(t_j,x(t_j))\right. \tag{6.0.24}$$
$$\left.+\sum_{j=1}^{m}\sum_{\nu=0}^{n-1}\int_{t_j}^1 \frac{(s-t_j)^\nu}{\nu!}dH(s)I_\nu(t_j,x(t_j))\right]\frac{t^{n-1}}{(n-1)!},$$

$$t \in (t_k,t_{k+1}], k \in \mathbb{N}[0,m].$$

On the other hand, if x satisfies (6.0.24), we can prove that x is a solution of (6.0.21) by direct computation. The proof is completed. □

Involving the Riemann-Liouville fractional derivative, we consider the following more general problem:

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m], \\ \Delta^{RL}D_{0^+}^{\alpha-i}x(t_k) = I_i(t_k, x(t_k)), k \in \mathbb{N}[1,m],\ i \in \mathbb{N}[1, n-1], \\ \Delta I_{0^+}^{n-\alpha}x(t_k) = I_n(t_k, x(t_k)), k \in \mathbb{N}[1,m], \\ I_{0^+}^{n-\alpha}x(0) = {}^{RL}D_{0^+}^{\alpha-1}x(0) = \cdots = {}^{RL}D_{0^+}^{\alpha-(n-2)}x(0) = 0,\ u(1) = \int_0^1 u(s)dH(s). \end{cases} \tag{6.0.25}$$

Theorem 6.0.4 *Suppose that $\int_0^1 s^{\alpha-(n-1)}dH(s) \neq 1$. Then x is a solution of (6.0.25) if and only if*

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=1}^{k}\sum_{\nu=1}^{n}\frac{(t-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}I_\nu(t_j, x(t_j))$$

$$+\frac{\Gamma(\alpha-n+2)}{1-\int_0^1 s^{\alpha-(n-1)}dH(s)}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - \sum_{j=1}^{m}\sum_{\nu=1}^{n} I_\nu(t_j, x(t_j))\frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right.$$

$$\left. -\int_0^1 \frac{(1-v)^{\alpha}}{\Gamma(\alpha+1)}h(v)dv + \sum_{j=1}^{m}\sum_{\nu=1}^{n-1} I_\nu(t_j, x(t_j))\int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)\right]\frac{t^{\alpha-(n-1)}}{\Gamma(\alpha-n+2)}, t \in (t_k, t_{k+1}],$$

$k \in \mathbb{N}[0,m]$.

Proof: Suppose x is a solution of (6.0.25). By $\alpha \in (n-1, n)$, using Theorem 4.0.1, we know from $-{}^{RL}D_{0^+}^{\alpha}x(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0,m]$ that there exist constants $c_{\nu,j} \in \mathbb{R}$ such that

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=0}^{k}\sum_{\nu=1}^{n} c_{\nu,j}\frac{(t-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0,m]. \tag{6.0.26}$$

It follows that

$$I_{0^+}^{n-\alpha}x(t) = -\int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)}h(s)ds + \sum_{j=0}^{k}\sum_{\nu=1}^{n} c_{\nu,j}\frac{(t-t_j)^{n-\nu}}{\Gamma(n-\nu+1)}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0,m] \tag{6.0.27}$$

and for $i = 1, 2, \cdots, n-1$ that

$${}^{RL}D_{0^+}^{\alpha-i}x(t) = -\int_0^t \frac{(t-s)^{i-1}}{\Gamma(i)}h(s)ds + \sum_{j=0}^{k}\sum_{\nu=1}^{i} c_{\nu,j}\frac{(t-t_j)^{i-\nu}}{\Gamma(i-\nu+1)}, t \in (t_k, t_{k+1}], k \in \mathbb{N}[0,m]. \tag{6.0.28}$$

(i) By $I_{0^+}^{n-\alpha}x(0) = {}^{RL}D_{0^+}^{\alpha-1}x(0) = \cdots = {}^{RL}D_{0^+}^{\alpha-(n-2)}x(0) = 0$ and (6.27) and (6.0.28), we get $c_{1,0} = c_{2,0} = \cdots = c_{n-2,0} = c_{n,0} = 0$.

(ii) By $\Delta^{RL}D_{0^+}^{\alpha-i}x(t_k)=I_i(t_k,x(t_k)),k\in\mathbb{N}[1,m],\ i\in\mathbb{N}[1,n-1]$ and (6.0.28), we get $c_{i,k}=I_i(t_k,x(t_k)),k\in\mathbb{N}[1,m],\ i\in\mathbb{N}[1,n-1]$.

(iii) By $\Delta I_{0^+}^{n-\alpha}x(t_k)=I_n(t_k,x(t_k)),k\in\mathbb{N}[1,m]$ and (6.27), we get $c_{n,k}=I_n(t_k,x(t_k)),k\in\mathbb{N}[1,m]$.

(iv) By $x(1)=\int_0^1 x(s)dH(s)$ and (6.26) and (i)-(iii), we get

$$-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\sum_{j=0}^{m}\sum_{\nu=}^{n}c_{\nu,j}\frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}=\int_0^1 x(s)dH(s)=\sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}x(s)dH(s)$$

$$=\sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\left(-\int_0^s\frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dv+\sum_{j=0}^{\tau}\sum_{\nu=1}^{n}c_{\nu,j}\frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)dH(s)$$

$$=-\sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\int_0^s\frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dvds+\sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\sum_{j=0}^{\tau}\sum_{\nu=1}^{n}c_{\nu,j}\frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)$$

$$=-\int_0^1\int_0^s\frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}h(v)dvds+\sum_{\tau=0}^{m}\int_{t_\tau}^{t_{\tau+1}}\sum_{j=0}^{\tau}\sum_{\nu=1}^{n}c_{\nu,j}\frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)$$

$$=-\int_0^1\int_v^1\frac{(s-v)^{\alpha-1}}{\Gamma(\alpha)}dsh(v)dv+\sum_{j=0}^{m}\sum_{\nu=1}^{n}c_{\nu,j}\int_{t_j}^1\frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)$$

$$=-\int_0^1\frac{(1-v)^{\alpha}}{\Gamma(\alpha+1)}h(v)dv+\sum_{j=1}^{m}\sum_{\nu=1}^{n-1}I_\nu(t_j,x(t_j))\int_{t_j}^1\frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)+c_{n-1,0}\int_0^1\frac{s^{\alpha-(n-1)}}{\Gamma(\alpha-(n-1)+1)}dH(s).$$

Hence

$$-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\sum_{j=1}^{m}\sum_{\nu=1}^{n}I_\nu(t_j,x(t_j))\frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}+\frac{c_{n-1,0}}{\Gamma(\alpha-n+2)}$$

$$=-\int_0^1\frac{(1-v)^{\alpha}}{\Gamma(\alpha+1)}h(v)dv+\sum_{j=1}^{m}\sum_{\nu=1}^{n-1}I_\nu(t_j,x(t_j))\int_{t_j}^1\frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)$$

$$+c_{n-1,0}\int_0^1\frac{s^{\alpha-(n-1)}}{\Gamma(\alpha-n+2)}dH(s).$$

It follows that

$$c_{n-1,0}=\frac{\Gamma(\alpha-n+2)}{1-\int_0^1 s^{\alpha-(n-1)}dH(s)}\left[\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-\sum_{j=1}^{m}\sum_{\nu=1}^{n}I_\nu(t_j,x(t_j))\frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right.$$

$$\left.-\int_0^1\frac{(1-v)^{\alpha}}{\Gamma(\alpha+1)}h(v)dv+\sum_{j=1}^{m}\sum_{\nu=1}^{n-1}I_\nu(t_j,x(t_j))\int_{t_j}^1\frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)\right].$$

Substituting $c_{\nu,j}$ into (6.0.26), we get

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=1}^{k}\sum_{\nu=1}^{n} \frac{(t-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} I_\nu(t_j, x(t_j))$$

$$+\frac{\Gamma(\alpha-n+2)}{1-\int_0^1 s^{\alpha-(n-1)}dH(s)} \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - \sum_{j=1}^{m}\sum_{\nu=1}^{n} I_\nu(t_j, x(t_j))\frac{(1-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right. \tag{6.0.29}$$

$$\left. -\int_0^1 \frac{(1-v)^{\alpha}}{\Gamma(\alpha+1)}h(v)dv + \sum_{j=1}^{m}\sum_{\nu=1}^{n-1} I_\nu(t_j, x(t_j)) \int_{t_j}^1 \frac{(s-t_j)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)\right] \frac{t^{\alpha-(n-1)}}{\Gamma(\alpha-n+2)},$$

$t \in (t_k, t_{k+1}], k \in \mathbb{N}[0, m]$.

On the other hand, if x satisfies (6.0.29), we can prove that x is a solution of (6.0.25) by direct computation. The proof is completed. □

Remark 6.0.5 In [80], Wang studied the the existence and uniqueness of solutions of the following impulsive problems:

$$\begin{cases} {}^C D_{t_i^+}^q x(t) = f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], \\ \Delta x^{(j)}(t_i) = I_{j,i}(x(t_i)), i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, q-1], \\ x^{(j)}(0) = x_j, j \in \mathbb{N}[0, q-1], \end{cases} \tag{6.0.30}$$

where ${}^C D_{t_i^+}^q x$ is the standard Caputo fractional derivative of order $q \in (n-1, n)$ with the starting point t_i. The equivalent integral equation of IVP (6.0.30) was obtained in [80] see Lemma 2.2 [80].

Suppose that x is a solution of (6.0.30). By Theorem 3.1.2, there exist constants $c_{i,\nu} \in \mathbb{R}$ ($\nu \in \mathbb{N}[0, q-1], i \in \mathbb{N}[0, m]$) such that

$$x(t) = \int_{t_i}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds + \sum_{\nu=0}^{q-1} c_{i,\nu}\frac{(t-t_i)^{\nu}}{\Gamma(\nu+1)},\ t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m].$$

Then

$$x^{(j)}(t) = \int_{t_i}^t \frac{(t-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s))ds + \sum_{\nu=j}^{q-1} c_{i,\nu}\frac{(t-t_i)^{\nu-j}}{\Gamma(\nu-j+1)},\ t \in (t_i, t_{i+1}], i \in \mathbb{N}[0, m], j \in \mathbb{N}[1, q-1].$$

By $x^{(j)}(0) = x_j, j \in \mathbb{N}[0, q-1]$, we get $c_{0,j} = x_j(j \in \mathbb{N}[0, q-1]$.

By $\Delta x^{(j)}(t_i) = I_{j,i}(x(t_i)), i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, q-1]$, we get

$$c_{i,j} - \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s, x(s))ds + \sum_{\nu=j}^{q-1} c_{i-1,\nu}\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}\right]$$

$$= I_{j,i}(x(t_i)), i \in \mathbb{N}[1, m], j \in \mathbb{N}[0, q-1].$$

It follows that

$$c_{i-1,\nu}=\sum_{\chi=\nu}^{q-1}c_{i-2,\chi}\frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)}+\int_{t_{i-2}}^{t_{i-1}}\frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)}f(s,x(s))ds+I_{\nu,i-1}(x(t_{i-1}))$$

and

$$c_{i,j}=\sum_{\nu=j}^{q-1}c_{i-1,\nu}\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}+\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)}f(s,x(s))ds+I_{j,i}(x(t_i))$$

$$=\sum_{\nu=j}^{q-1}\left[\sum_{\chi=\nu}^{q-1}c_{i-2,\chi}\frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)}+\int_{t_{i-2}}^{t_{i-1}}\frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)}f(s,x(s))ds+I_{\nu,i-1}(x(t_{i-1}))\right]\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}$$

$$+\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)}f(s,x(s))ds+I_{j,i}(x(t_i))$$

$$=\sum_{\nu=j}^{q-1}\sum_{\chi=\nu}^{q-1}c_{i-2,\chi}\frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)}\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}+\sum_{\nu=j}^{q-1}\int_{t_{i-2}}^{t_{i-1}}\frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)}f(s,x(s))ds\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}$$

$$+\sum_{\nu=j}^{q-1}I_{\nu,i-1}(x(t_{i-1}))\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}+\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)}f(s,x(s))ds+I_{j,i}(x(t_i)).$$

By $\sum_{\nu=j}^{\chi}\frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)}\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}=\frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)}$, we have

$$c_{i,j}=\sum_{\chi=j}^{q-1}c_{i-2,\chi}\sum_{\nu=j}^{\chi}\frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)}\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}+\sum_{\nu=j}^{q-1}\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}I_{\nu,i-1}(x(t_{i-1}))+I_{j,i}(x(t_i))$$

$$+\sum_{\nu=j}^{q-1}\int_{t_{i-2}}^{t_{i-1}}\frac{(t_{i-1}-s)^{q-\nu-1}}{\Gamma(q-\nu)}f(s,x(s))ds\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}+\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)}f(s,x(s))ds$$

$$=\sum_{\chi=j}^{q-1}c_{i-2,\chi}\frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)}+\sum_{\nu=j}^{q-1}\frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)}I_{\nu,i-1}(x(t_{i-1}))+I_{j,i}(x(t_i))$$

$$+\int_{t_{i-2}}^{t_{i-1}}\frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)}f(s,x(s))ds+\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)}f(s,x(s))ds$$

$$= \sum_{\chi_1=j}^{q-1} c_{i-3,\chi_1} \sum_{\chi=j}^{\chi_1} \frac{(t_{i-2}-t_{i-3})^{\chi_1-\chi}}{\Gamma(\chi_1-\chi+1)} \frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)}$$

$$+ \sum_{\chi=j}^{q-1} \sum_{\nu=j}^{\chi} \frac{(t_{i-1}-t_{i-2})^{\chi-\nu}}{\Gamma(\chi-\nu+1)} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\chi,i-2}(x(t_{i-2})) + \sum_{\nu=j}^{q-1} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\nu,i-1}(x(t_{i-1})) + I_{j,i}(x(t_i))$$

$$+ \int_{t_{i-3}}^{t_{i-2}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds + \int_{t_{i-2}}^{t_{i-1}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds$$

$$+ \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds$$

$$= \sum_{\chi_1=j}^{q-1} c_{i-3,\chi_1} \frac{(t_i-t_{i-3})^{\chi_1-j}}{\Gamma(\chi_1-j+1)} + \sum_{\chi=j}^{q-1} \frac{(t_i-t_{i-2})^{\chi-j}}{\Gamma(\chi-j+1)} I_{\chi,i-2}(x(t_{i-2})) + \sum_{\nu=j}^{q-1} \frac{(t_i-t_{i-1})^{\nu-j}}{\Gamma(\nu-j+1)} I_{\nu,i-1}(x(t_{i-1}))$$

$$+ I_{j,i}(x(t_i)) + \int_{t_{i-3}}^{t_{i-2}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds$$

$$+ \int_{t_{i-2}}^{t_{i-1}} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds$$

$$= \cdots\cdots\cdots$$

$$= \sum_{\chi=j}^{q-1} c_{0,\chi_1} \frac{t_i^{\chi-j}}{\Gamma(\chi-j+1)} + \sum_{\nu=1}^{i-1} \sum_{\chi=j}^{q-1} \frac{(t_i-t_\nu)^{\chi-j}}{\Gamma(\chi-j+1)} I_{\chi,\nu}(x(t_\nu)) + I_{j,i}(x(t_i))$$

$$+ \int_0^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds.$$

Hence

$$x(t) = \int_{t_i}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \sum_{j=0}^{q-1} \left[\sum_{\chi=j}^{q-1} c_{0,\chi_1} \frac{t_i^{\chi-j}}{\Gamma(\chi-j+1)} + \int_0^{t_i} \frac{(t_i-s)^{q-j-1}}{\Gamma(q-j)} f(s,x(s))ds \right.$$

$$\left. + \sum_{\nu=1}^{i-1} \sum_{\chi=j}^{q-1} \frac{(t_i-t_\nu)^{\chi-j}}{\Gamma(\chi-j+1)} I_{\chi,\nu}(x(t_\nu)) + I_{j,i}(x(t_i)) \right] \frac{(t-t_i)^j}{\Gamma(j+1)}.\ t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m]. \tag{6.0.31}$$

On the other hand, we can prove that x is a solution of (6.0.30) if x satisfies (6.0.31). We note that Lemma 2.2 in [80] is different from our result. □

We can investigate the following similar problem for impulsive fractional differential equation which involves a single starting point 0:

$$\begin{cases} {}^{C}D_{0^+}^{q} x(t) = f(t,x(t)), t \in (t_i,t_{i+1}], i \in \mathbb{N}[0,m], \\ \Delta x^{(j)}(t_i) = I_{j,i}(x(t_i)), i \in \mathbb{N}[1,m], j \in \mathbb{N}[0,q-1], \\ x^{(j)}(0) = x_j, j \in \mathbb{N}[0,q-1]. \end{cases} \tag{6.0.32}$$

In fact, suppose that x is a solution of IVP(6.0.32). By Theorem 4.0.1, there exist constants $c_{j,\nu}\in\mathbb{R}(j\in\mathbb{N}[0,m],\nu\in\mathbb{N}[0,n-1])$ such that

$$x(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\sum_{j=0}^{i}\sum_{\nu=0}^{n-1}c_{j,\nu}\frac{(t-t_j)^{\nu}}{\Gamma(\nu+1)},t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m].$$

Then

$$x^{(k)}(t)=\int_0^t\frac{(t-s)^{q-k-1}}{\Gamma(q-k)}f(s,x(s))ds+\sum_{j=0}^{i}\sum_{\nu=k}^{n-1}c_{j,\nu}\frac{(t-t_j)^{\nu-k}}{\Gamma(\nu-k+1)},t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m].$$

By $\Delta x^{(k)}(t_i)=I_{k,i}(x(t_i)),i\in\mathbb{N}[1,m],k\in\mathbb{N}[0,q-1]$, we get $c_{i,k}=I_{k,i}(x(t_i)),i\in\mathbb{N}[1,m]$, $k\in\mathbb{N}[0,q-1]$.

By $x^{(k)}(0)=x_k,k\in\mathbb{N}[0,q-1]$, we get $c_{0,k}=x_k(k\in\mathbb{N}[0,q-1])$.

Hence

$$\begin{aligned}x(t)&=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\sum_{\nu=0}^{n-1}x_\nu\frac{t^{\nu}}{\Gamma(\nu+1)}\\&+\sum_{j=1}^{i}\sum_{\nu=0}^{q-1}I_{\nu,j}(x(t_j))\frac{(t-t_j)^{\nu}}{\Gamma(\nu+1)},t\in(t_i,t_{i+1}],i\in\mathbb{N}[0,m].\end{aligned}\tag{6.0.33}$$

On the other hand, if x satisfies (6.0.33), we can prove that x is a solution of (6.0.32). Then (6.0.32) is equivalent to (6.0.33). The proof is completed. □

References

[1] **Arara, A., Benchohra, M., Hamidi, N.**, and **Nieto, J. J. :** *Fractional order differential equations on an unbounded domain.* Nonl. Anal. 72 (2010), 580 – 586

[2] **Agarwal, R. P., Benchohra, M.**, and **Hamani, S. :** *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions.* Acta Appl. Math. 109 (2010), 973 – 1033

[3] **Agarwal, R. P., Benchohra, M.**, and **Slimani, B. A. :** *Existence results for differential equations with fractional order and impulses.* Mem. Differential Equations Math. Phys. 44 (2008), 1 – 21

[4] **Bashir, A.** , and **Ntouyas, S. K. :** *A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations.* Frac. Calcu. Appl. Anal. 17(2014), 348 – 360

[5] **Bashir, A.**, and **Ntouyas, S. K. :** *On Hadamard fractional integro-differential boundary value problems.* J. Appl. Math. Comput. 47(2015), 119 – 131

[6] **Bashir, A.**, and **Ntouyas, S. K. :** *On three-point Hadamard-type fractional boundary value problems.* Int. Electron. J. Pure Appl. Math. 8 (4) (2014), 31 – 42

[7] **Bashir, A., Ntouyas, S. K.**, and **Alsaedi, A. :** *New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions.* Bound. Value Probl. 2013 (2013), 275

[8] **Bashir, A.**, and **Sivasundaram, S. :** *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations.* Nonlinear Anal. Hybrid Syst. 3 (2009), 251 – 258

[9] **Bashir, A.**, and **Sivasundaram, S. :** *Existence of solutions for impulsive integral boundary value problems of fractional order.* Nonlinear Anal. Hybrid Syst. 4 (2010), 134 – 141

[10] **Bashir, A.**, and **Nieto, J. J. :** *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory.* Topological Methods in Nonlinear Analysis 35 (2010), 295 – 304

[11] **Bashir, A.**, and **Nieto, J. J. :** *Existence of solutions for impulsive anti-periodic boundary value problems of fractional order.* Taiwanese Journal of Mathematics 15 (3) (2011), 981 – 993

[12] **Bai, C. :** *Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative.* J. Math. Anal. Appl. 384 (2011), 211 – 231

[13] **Butzer, P. L., Kilbas, A. A.**, and **Trujillo, J. J. :** *Compositions of hadamard-type fractional integration operators and the semigroup property.* Journal of Mathematical Analysis and Applications 269 (2002) 387 – 400

[14] **Balachandran, K.**, and **Kiruthika :** *Existence of solutions of abstract fractional impulsive semilinear evolution equations.* Electron. J. Qual. Theory Differ. Equ. 4 (2010), 1 – 12

[15] **Butzer, P. L., Kilbas, A. A.**, and **Trujillo, J. J. :** *Fractional calculus in the mellin setting and hadamard-type fractional integrals.* Journal of Mathematical Analysis and Applications, 269(2002), 1 – 27

[16] **Butzer, P. L., Kilbas, A. A.**, and **Trujillo, J. J. :** *Mellin transform analysis and integration by parts for hadamard-type fractional integrals.* Journal of Mathematical Analysis and Applications, 270(2002), 1 – 15

[17] **Belmekki, M., Nieto, Juan J.**, and **Rodriguez-Lopez, R. :** *Existence of periodic solution for a nonlinear fractional differential equation.* Boundary Value Problems 2009 (2009), Article ID 324561, DOI: 10.1155/2009/324561

[18] **Belmekki, M., Nieto, J. J.**, and **Rodriguez-Lopez, R. :** *Existence of solution to a periodic boundary value problem for a nonlinear impulsive fractional differential equation.* Electronic Journal of Qualitative Theory of Differential Equations 2014, No. 16, 1 – 27

[19] **Benchohra, M., Graef, J.**, and **Hamani, S. :** *Existence results for boundary value problems with nonlinear frational differential equations, Applicable Analysis 87 (2008), 851 – 863.*

[20] **Bonanno, G., Rodriguez-López, R.**, and **Tersian, S. :** *Existence of solutions to boundary value problem for impulsive fractional differential equations.* Fractional Calculus and Applied Analysis, 17 (3) (2014), 717 – 744

[21] **Benchohra, M.**, and **Seba, D. :** *Impulsive fractional differential equations in Banach spaces.* Electron. J. Qual. Theory Differ. Equ. (8) (2009), 1 – 14 (Special Edition I)

[22] **Bainov, D. D.**, and **Simeonov, P. S. :** *Systems with Impulsive Effects.* Horwood, Chichister, UK, 1989

[23] **Bainov, D. D.**, and **Simeonov, P. S. :** *Impulsive Differential Equations: Periodic Solutions and ItsApplications.* Longman Scientific and Technical Group, Harlow, UK, 1993

[24] **Bainov, D. D.**, and **Covachev, V. :** *Impulsive Differential Equations with a Small Parameter.* vol. 24 of Series on Advances in Mathematics for Applied Sciences, World Scientific, River Edge, NJ, USA, 1994

[25] **Benchohra, M., Henderson, J.**, and **Ntonyas, S. K. :** *Impulsive Differential Equations and Inclusions.* vol. 2, Hindawi Publishing Corporation, New York, NY, USA, 2006

[26] **Benchohra, M.**, and **Slimani, B. A. :** *Impulsive fractional differential equations.* Electron. J. Differential Equations 2009 (10), 1 – 11

[27] **Diethelm, K. :** *The analysis of fractional differential equations, Lecture notes in mathematics.* edited by J. M. M. Cachan etc., Springer-Verlag Berlin Heidelberg 2010

[28] **Dugundji, J.**, and **Granas, A. :** *Fixed point theory.* I. Monografie Matematyczne [Mathematical Monographs], 61. Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1982

[29] **Dehghant, R.**, and **Ghanbari, K. :** *Triple positive solutions for boundary value problem of a nonlinear fractional differential equation, Bulletin of the Iranian Mathematical Society, 33 (2007), 1 – 14.*

[30] **Delbosco, D.**, and **Rodino, L. :** *Existence and uniqueness for a nonlinear fractional differential equation.* Journal of Mathematical Analysis and Applications, vol. 204, no. 2, pp. 609 – 625, 1996

[31] **Dabas, J., Chauhan, A.**, and **Kumar, M. :** *Existence of the Mild Solutions for Impulsive Fractional Equations with Infinite Delay.* International Journal of Differential Equations Volume 2011, Article ID 793023, 20 pages

[32] **Ergoren, H.**, and **Kilicman, A. :** *Some Existence Results for Impulsive Nonlinear Fractional Differential Equations with Closed Boundary Conditions.* Abstract and Applied Analysis, Volume 2012, Article ID 387629, 15 pages

[33] **Feckan, M., Zhou, Y.**, and **Wang, J. :** *On the concept and existence of solution for impulsive fractional differential equations, Commun Nonlinear Sci Numer Simulat 17 (2012), 3050 – 3060.*

[34] **Feckan, M., Zhou, Y.**, and **Wang, J. R. :** *Response to "Comments on the concept of existence of solution for impulsive fractional differential equations [Commun Nonlinear Sci Numer Simul 2014; 19:401 – 3.]".* Commun. Nonlinear Sci. Numer. Simul. 19 (12) (2014), 4213 – 4215

[35] **Ge, W. :** *Boundary value problems of nonlinear ordinary differential equations.* Science Press, Beijing, 2007

[36] **Guo, T.**, and **Jiang, W. :** *Impulsive problems for fractional differential equations with boundary value conditions.* Computers and Mathematics with Applications 64 (2012), 3281 – 3291

[37] **Gao, Z., Liu, Y.**, and **Liu, G. :** *Existence and Uniqueness of Solutions to Impulsive Fractional Integro-Differential Equations with Nonlocal Conditions.* Applied Mathematics, 2013, 4, 859 – 863

[38] **Hadamard, J. :** *Essai sur létude des fonctions donnes par leur développement de Taylor.* Journal de Mathématiques Pures et Appliquées 4e Série, vol. 8, pp. 101 – 186, 1892

[39] **Hannaa, L. A-H.**, and **Luchko, Yu. F. :** *Operational calculus for the Caputo-type fractional Erdélyi-Kober derivative and its applications.* Integral Transforms and Special Functions, 25 (5)(2014), 359 – 373

[40] **Hu, Z.**, and **Liu, W. :** *Solvability of a Coupled System of Fractional Differential Equations with Periodic Boundary Conditions at Resonance.* Ukrainian Mathematical Journal 65 (2014) 1619 – 1633

[41] **Henderson, J.**, and **Ouahab, A. :** *Impusive differential inclusions with fractioanl order.* Comput. Math. Appl. 59 (2010), 1191 – 1226

[42] **Jarad, F., Abdeljawad, T.**, and **Baleanu, D. :** *Caputo-type modification of the Hadamard fractional derivatives.* Advances in Difference Equations 2012, 2012: 142

[43] **Kaufmann, E.**, and **Mboumi, E. :** *Positive solutions of a boundary value problem for a nonlinear fractional differential equation.* Electronic Journal of Qualitative Theory of Differential Equations 3 (2008), 1 – 11

[44] **Kilbas, A. A., Srivastava, H. M.**, and **Trujillo, J. J. :** *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies 204. Elsevier Science B. V., Amsterdam, 2006

[45] **Kilbas, A. A., Marzan, S. A.** , and **Tityura, A. A. :** *Hadamard type fractional integrals and derivatives and differential equations of fractional order (Russian).* Dokl. Akad. Nauk. 389 (6)(2003), 734 – 738. Translated in Dolk. Math. 67 (2)(2003), 263 – 267

[46] **Karaca, I. Y.**, and **Tokmak, F. :** *Existence of Solutions for Nonlinear Impulsive Fractional Differential Equations with* $p-$*Laplacian Operator.* Mathematical Problems in Engineering Volume 2014 (2014), Article ID 692703, 11 pages

[47] **Liu, Y. :** *Positive solutions for singular FDES.* U. P. B. Sci. Series A, 73 (2011) 89 – 100

[48] **Liu, Y. :** *Solvability of multi-point boundary value problems for multiple term Riemann-Liouville fractional differential equations.* Comput. Math. Appl. 64 (4)(2012), 413 – 431

[49] **Liu, Y.**, and **Ahmad, B. :** *A Study of Impulsive Multiterm Fractional Differential Equations with Single and Multiple Base Points and Applications.* The Scientific World Journal Volume 2014, Article ID 194346, 28 pages

[50] **Li, X., Chen, F.**, and **Li, X. :** *Generalized anti-periodic boundary value problems of impulsive fractional differential equations.* Communications in Nonlinear Science and Numerical Simulation, vol. 18, no. 1, pp. 28 – 41, 2013

[51] **Luchko, Y.**, and **Trujillo, Juan J. :** *Caputo-type modification of the Erdélyi-Kober fractional derivative.* Fractional Calculus and Applied Analysis 10 (3)(2007), 249 – 267

[52] **Liu, Z.**, and **Li, X. :** *Existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations.* Communications in Nonlinear Science and Numerical Simulation 18 (6), 2013, 1362 – 1373

[53] **Liu, Z., Lu, L.**, and **Szanto, I. :** *Existence of solutions for fractional impulsive differential equations with p-Laplacian operator.* Acta Mathematica Hungarica 141 (3) 2013, 203 – 219

[54] **Lizama, C.**, and **Poblite, V. :** *Periodic solutions of fractional differential equations with delays.* J. Evol. Equs. 11 (2011), 57 – 70

[55] **Lizama, C.**, and **Poblite, V. :** *Regularity of mild solutions of fractional order differential equations.* Appl. Math. Comput. 224 (2013), 803 – 816

[56] **Lakshmikantham, V., Bainov, D. D.**, and **Simeonov, P. S. :** *Theory of Impulsive Differential Equations.* vol. 6 of Series in Modern Applied Mathematics, World Scientific, Teaneck, NJ, USA, 1989

[57] **Li, P.**, and **Shang, H. :** *Impulsive problems for fractional differential equations with nonlocal boundary value conditions.* Abstract and Applied Analysis, 2014 (2014), Article ID 510808, 13 pages

[58] **Lakshmikantham, V.**, and **Vatsala, A. S. :** *Theory of fractional differential inequalities and applications.* Communications in Applied Analysis, vol. 11, no. 3 – 4, pp. 395 – 402, 2007

[59] **Lakshmikantham, V.**, and **Vatsala, A. S. :** *Basic theory of fractional differential equations.* Nonlinear Analysis: Theory, Methods and Applications, vol. 69, no. 8, pp. 2677 – 2682, 2008

[60] **Lakshmikantham, V.**, and **Vatsala, A. S. :** *General uniqueness and monotone iterative technique for fractional differential equations.* Applied Mathematics Letters, vol. 21, no. 8, pp. 828 – 834, 2008

[61] **Lakshmikantham, V. :** *Theory of fractional functional differential equations.* Nonlinear Analysis: Theory, Methods and Applications, vol. 69, no. 10, pp. 3337 – 3343, 2008

[62] **Mardanov, M. J., Mahmudov, N. I.**, and **Sharifov, Y. A. :** *Existence and uniqueness theorems for impulsive fractional differential equations with the two-point and integral boundary conditions.* The Scientific World Journal, Volume 2014, Article ID 918730, 8 pages

[63] **Milman, V. D.**, and **Myskis, A. D. :** *On the stability of motion in the presence of impulses.* Siberial Mathematical Journal, vol. 1, pp. 233 – 237, 1960

[64] **Mawhin, J. :** *Topological degree methods in nonlinear boundary value problems.* In: NSFCBMS Regional Conference Series in Math., American Math. Soc. Providence, RI, 1979

[65] **Miller, K. S.**, and **Samko, S. G. :** *Completely monotonic functions.* Integr. Transf. Spec. Funct. 12 (2001), 389 – 402

[66] **Miller, K. S.**, and **Ross, B. :** *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley, New York, 1993

[67] **Mophou, G. M. :** *Existence and uniqueness of mild solutions to impulsive fractional differential equations.* Nonlinear Anal. 72 (2010), 1604 – 1615

[68] **Nakhushev, A. M. :** *The Sturm-Liouville Problem for a Second Order Ordinary Differential equations with fractional derivatives in the lower terms.* Dokl. Akad. Nauk SSSR 234 (1977), 308 – 311

[69] **Nieto, J. J. :** *Maximum principles for fractional differential equations derived from Mittag-Leffler functions.* Applied Mathematics Letters 23 (2010), 1248 – 1251

[70] **Nieto, J. J. :** *Comparison results for periodic boundary value problems of fractional differential equations.* Fractional Differential Equations 1 (2011), 99 – 104

[71] **Podlubny, I. :** *Frational Differential Equations.* Mathematics in Science and Engineering, Academic Press, San Diego, USA, 1999

[72] **Rida, S. Z., El-Sherbiny, H. M.**, and **Arafa, A. :** *On the solution of the fractional nonlinear Schrodinger equation.* Physics Letters A 372 (2008), 553 – 558

[73] **Rehman, M.**, and **Eloe, P. W. :** *Existence and uniqueness of solutions for impulsive fractional differential equations.* Applied Mathematics and Computation 224 (2013), 422 – 431

[74] **Rehman, M.**, and **Khan, R. :** *A note on boundaryvalueproblems for a coupled system of fractional differential equations.* Computers and Mathematics with Applications 61 (2011), 2630 – 2637

[75] **Sun, J. :** *Nonlinear Functional Analysis and Its Application.* Science Press, Beijing (in Chinese) (2008)

[76] **Samko, S. G., Kilbas, A. A.**, and **Marichev, O. I. :** *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, Yverdon, 1993

[77] **Samoilenko, A. M.**, and **Perestyuk, N. A. :** *Differential Equations with Impulses.* Viska Scola, Kiev, Ukraine, 1987

[78] **Tian, Y.**, and **Bai, Z. :** *Existence results for the three-point impulsive boundary value problem involving fractional differential equations.* Computers and Mathematics with Applications 59 (2010), 2601 – 2609

[79] **Wang, X. :** *Impulsive boundary value problem for nonlinear differential equations of fractional order.* Comput. Math. Appl. 62 (5) (2011), 2383 – 2391

[80] **Wang, X. :** *Existence of solutions for nonlinear impulsive higher order fractional differential equations.* Electronic Journal of Qualitative Theory of Differential Equations 80 (2011), 1 – 12

[81] **Wang, X.**, and **Bai, C. :** *Periodic boundary value problems for nonlinear impulsive fractional differential equations.* Electronic Journal of Qualitative Theory and Differential Equations 3 (2011), 1 – 15

[82] **Wei, Z., Dong, W.**, and **Che, J. :** *Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative.* Nonlinear Analysis: Theory, Methods and Applications 73 (2010), 3232 – 3238

[83] **Wei, Z.**, and **Dong, W. :** *Periodic boundary value problems for Riemann-Liouville fractional differential equations.* Electronic Journal of Qualitative Theory of Differential Equations 87 (2011), 1 – 13

[84] **Wang, X.**, and **Chen, H. :** *Nonlocal Boundary Value Problem for Impulsive Differential Equations of Fractional Order.* Advances in Difference Equations 2011, 2011:404917

[85] **Wang, G., Ahmad, B.**, and **Zhang, L. :** *Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order.* Nonlinear Analysis 74 (2011), 792 – 804

[86] **Wang, G., Liu, S., Agawarl, R. P.**, and **Zhang, L. :** *Positive solutions on integral boundary value problem involving Riemann-Liouville fractional derivative.* Journal of Fractional Calculus and Applications **4 (2)** (2013), 312 – 321

[87] **Wang, G., Ahmad, B.**, and **Zhang, L. :** *On impulsive boundary value problems of fractional differential equations with irregular boundary conditions.* Abstract and Applied Analysis, Volume 2012, Article ID 356132, 18 pages

[88] **Wang, G., Ahmad, B., Zhang, L.**, and **Nieto, J. J. :** *Comments on the concept of existence of solution for impulsive fractional differential equations.* Commun. Nonlinear Sci. Numer. Simul. 19 (12) (2014), 401 – 403

[89] **Wang, G., Ahmad, B., Zhang, L.**, and **Nieto, J. J. :** *Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions.* Comput. Math. Appl. 62 (2011), 1389 – 1397

[90] **Wang, J. R., Li, X.**, and **Wei, W. :** *On the natural solution of an impulsive fractional differential equation of order $q \in (1,2)$.* Commun. Nonlinear Sci. Numer. Simul. 17 (2012), 4384 – 4394

[91] **Wang, J., Xiang, H.**, and **Liu, Z. :** *Positive Solution to Nonzero Boundary Values Problem for a Coupled System of Nonlinear Fractional Differential Equations.* International Journal of Differential Equations, 2010 (2010), Article ID 186928, 12 pages, DOI: 10.1155/2010/186928

[92] **Wang, J. R., Zhou, Y.**, and **Feckan, M. :** *On recent developments in the theory of boundary value problems for impulsive fractional differential equations.* Comput. Math. Appl. 64 (2012), 3008 – 3020

[93] **Wang, J.**, and **Zhou, Y. :** *On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives.* Applied Mathematics Letters 39 (2015), 85 – 90

[94] **Xu, Y.**, and **Liu, X. :** *Some boundary value problems of fractional differential equations with fractional impulsive conditions.* Journal of Computational Analysis and Applications 19 (1) (2015), 426

[95] **Yang, W. :** *Positive solutions for singular coupled integral boundary value problems of nonlinear Hadamard fractional differential equations.* J. Nonlinear Sci. Appl. 8 (2015), 110 – 129

[96] **Yang, L.**, and **Chen, H. :** *Nonlocal boundary value problem for impulsive differential equations of fractional order.* Advances in Difference Equations, vol. 2011, Article ID 404917, 2011

[97] **Yang, A.**, and **Ge, W. :** *Positive solutions for boundary value problems of N-dimension nonlinear fractional differential systems.* Boundary Value Problems, 2008, article ID 437453, DOI: 10.1155/2008/437453

[98] **Ye, H., Gao, J.**, and **Ding, Y. :** *A generalized Gronwall inequality and its application to a fractional differential equation.* J. Math. Anal. Appl. 238 (2007), 1075 – 81

[99] **Yang, X.**, and **Liu, Y. :** *Picard iterative processes for initial value problems of singular fractional differential equations.* Advances in Difference Equations 2014, 2014: 102

[100] **Zhang, S. :** *The existence of a positive solution for a nonlinear fractional differential equation.* J. Math. Anal. Appl. 252 (2000), 804 – 812

[101] **Zhao, K. :** *Impulsive integral boundary value problems of the higher-order fractional differential equation with eigenvalue arguments.* Adv. Differ. Equa. (2015) 2015: 382

[102] **Zhang, S. :** *Positive solutions for boundary-value problems of nonlinear fractional differential equation.* Electron. J. Diff. Eqns. 36 (2006), 1 – 12

[103] **Zhou, J.**, and **Feng, M. :** *Green's function for Sturm-Liouville-type boundary value problems of fractional order impulsive differential equations and its application.* Boundary Value Problems 2014, 2014: 69

[104] **Zhao, K.**, and **Gong, P. :** *Positive solutions for impulsive fractional differential equations with generalized periodic boundary value conditions.* Advances in Difference Equations, 2014, 2014: 255

[105] **Zhao, X.**, and **Ge, W. :** *Some results for fractional impulsive boundary value problems on infinite intervals.* Applications of Mathematics 56 (4) (2011), 371 – 387

[106] **Zhang, X., Huang, X.**, and **Liu, Z. :** *The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay.* Nonlinear Analysis: Hybrid Systems 4 (2010), 775 – 781

[107] **Zhou, W.**, and **Liu, X. :** *Existence of solution to a class of boundary value problem for impulsive fractional differential equations.* Adv. Differ. Equ. 2014, (2014): 12

[108] **Zhou, W., Liu, X.**, and **Zhang, J. :** *Some new existence and uniqueness results of solutions to semilinear impulsive fractional integro-differential equations.* Advances in Difference Equations 2015, 2015: 38

[109] **Zhao, Y., Sun, S., Han, Z.**, and **Zhang, M. :** *Positive solutions for boundary value problems of nonlinear fractional differential equations.* Applied Mathematics and Computation 217 (2011), 6950 – 6958

[110] **Zhang, X., Zhu, C.**, and **Wu, Z. :** *Solvability for a coupled system of fractional differential equations with impulses at resonance.* Boundary Value Problems 2013, 2013: 80

[111] **Zhang, X., Zhang, X.**, and **Zhang, M. :** *On the concept of general solution for impulsive differential equations of fractional order* $q \in (0, 1)$. Applied Mathematics and Computation 247 (2014), 72 – 89

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