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Subdensity as a convenient concept for Bounded Topology

ABSTRACT. A *subdensity space* is a special case of a *density space*, which also occur under the name of *hypernear space* in [17]. Hence, most of *classical* spaces, like topological spaces, uniform spaces, proximity spaces, contiguity spaces or nearness spaces, respectively can be immediately described and studied in this *general* framework. Moreover, the more specific defined subdensity spaces allow us to consider and integrate the *fundamental* species of *b-topological* and *b-near spaces*, too, as presented and studied in [19]. In this paper it is shown that b-proximal spaces also can be involved, and b-topological spaces then have an *alternate* description by *different* corresponding subdensity spaces.

At last, we establish a one-to-one correspondence between suitable subdensity spaces and their related *strict* topological extensions [1]. This relationship generalizes the *one* of LODATO, studied by him in the realm of generalized proximity spaces [20].

KEY WORDS AND PHRASES. Bounded Topology; b-topological space; b-proximal space; strict topological extension

1 Basic Concepts

As usual $\underline{P}X$ denotes the power set of a set X, and we call $\mathcal{B}^X \subset \underline{P}X$ a *bornology* (on X) [8], if it possesses the following properties, i.e.

(b_0) $\emptyset \in \mathcal{B}^X$;

(b_1) $B_2 \subset B_1 \in \mathcal{B}^X$ imply $B_2 \in \mathcal{B}^X$;

(b_2) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;

(b_3) $B_1, B_2 \in \mathcal{B}^X$ imply $B_1 \cup B_2 \in \mathcal{B}^X$.

The elements of $\mathcal{B}^X$ are called *bounded sets*. Then, for bornologies $\mathcal{B}^X, \mathcal{B}^Y$ a function $f : X \longrightarrow Y$ is called *bi-bounded* iff f satisfies (bib$_1$) $f\mathcal{B}^X := \{f[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^Y$;

(bib$_2$) $f^{-1}\mathcal{B}^Y := \{f^{-1}[D] : D \in \mathcal{B}^Y\} \subset \mathcal{B}^X$.

Evidently, for corresponding power sets each map $f : X \longrightarrow Y$ is bi-bounded. As an *instructive* example we consider for sets X, Y as bornologies in each case the set of all *finite* subsets of those. Then, for each map $f : X \longrightarrow Y$ and some $B \in \mathcal{B}^X_{fi} := \{D \subset X : D$ is finite$\}$ we look at the power set on B and consider the restriction $f|_B$ of f on B. Then $f|_B$ is bi-bounded.

Then we make use of the following notations: For collections $\rho, \rho_1, \rho_2 \subset \underline{P}X$ we put:

$$\rho_2 << \rho_1 \text{ iff } \forall F_2 \in \rho_2 \exists F_1 \in \rho_1 \; F_1 \subset F_2;$$
$$\rho_1 \vee \rho_2 := \{F_1 \cup F_2 : F_1 \in \rho_1, F_2 \in \rho_2\};$$
$$\sec \rho := \{D \subset X : \forall F \in \rho D \cap F \neq \emptyset\}.$$

Definition 1.1 *We call a triple $(X, \mathcal{B}^X, N)$ consisting of a set X, bornology $\mathcal{B}^X$ and a function $N : \mathcal{B}^X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ an* episd-space *(shortly* esd-space*) iff the following axioms are satisfied:*

(esd$_1$) $\rho_2 << \rho_1 \in N(B), B \in \mathcal{B}^X, \rho_2 \subset \underline{P}X$ *imply* $\rho_2 \in N(B)$;

(esd$_2$) $B \in \mathcal{B}^X$ *implies* $\mathcal{B}^X \notin N(B) \neq \emptyset$;

(esd$_3$) $\rho \in N(\emptyset)$ *implies* $\rho = \emptyset$;

(esd$_4$) $x \in X$ *implies* $\{\{x\}\} \in N(\{x\})$;

(esd$_5$) $\emptyset \neq B_2 \subset B_1 \in \mathcal{B}^X$ *imply* $N(B_2) \subset N(B_1)$;

(esd$_6$) $\{cl_N(F) : F \in \rho\} \in N(B), \rho \subset \underline{P}X, B \in \mathcal{B}^X$ *imply* $\rho \in N(B)$, *where* $cl_N(F) := \{x \in X : \{F\} \in N(\{x\})\}$;

(esd$_7$) $\rho_1 \vee \rho_2 \in N(B), \rho_1, \rho_2 \subset \underline{P}X, B \in \mathcal{B}^X$ *imply* $\rho_1 \in N(B)$ *or* $\rho_2 \in N(B)$;

(esd$_8$) $B \in \mathcal{B}^X$ *implies* $cl_N(B) \in \mathcal{B}^X$;

(esd$_9$) $\rho \cap \mathcal{B}^X \in N(B), B \in \mathcal{B}^X \setminus \{\emptyset\}, \rho \subset \underline{P}X$ *imply* $\rho \in N(B)$.

If $\rho \in N(B)$ for some $B \in \mathcal{B}^X$, then we call ρ a B-collection *(in N). For esd-spaces $(X, \mathcal{B}^X, N)$, $(Y, \mathcal{B}^Y, M)$ a function $f : X \longrightarrow Y$ is called* bi-bounded sd-map *(shortly* bibsd-map *iff it satisfies* (bib$_1$), (bib$_2$) *and*

(sd) $B \in \mathcal{B}^X$ *and* $\rho \in N(B)$ *imply* $f\rho := \{f[F] : F \in \rho\} \in M(f[B])$.

We denote by ESD the corresponding category.

Remark 1.2 In a former paper [19] it was shown, that the category b-TOP of b-topological spaces and b-continuous maps as well as the category b-NEAR of b-nearness spaces and b-near maps can be *fully embedded* into ESD. In our following research we will establish a *further equivalent* description of b-topological spaces by means of *different* esd-spaces resulting into an alternate description of the category TOP, if the given bornology $\mathcal{B}^X$ of the considered esd-space is *saturated*, which means X is an element of $\mathcal{B}^X$. Moreover, we focus our attention on so called *b-proximal spaces* which also can be integrated into the above defined concept. Then, in a *natural* way, we will characterize those esd-spaces which can be *extended* to a certain topological one. In case of *saturation* this *new* established *connection* deliver us the well-known famous theorem of LODATO [20] up to isomorphism.

Definition 1.3 *For a set X let $\mathcal{B}^X$ be a bornology. A function $t : \mathcal{B}^X \longrightarrow \underline{P}X$ is called a* b-topological operator (b-topology) *(on $\mathcal{B}^X$) iff the following axioms are satisfied, i.e.*

(b-t$_1$) $B \in \mathcal{B}^X$ *implies* $t(B) \in \mathcal{B}^X$;

(b-t$_2$) $t(\emptyset) = \emptyset$;

(b-t$_3$) $B \in \mathcal{B}^X$ *implies* $B \subset t(B)$;

(b-t$_4$) $B_1 \subset B_2 \in \mathcal{B}^X$ *imply* $t(B_1) \subset t(B_2)$;

(b-t$_5$) $B \in \mathcal{B}^X$ *implies* $t(t(B)) \subset t(B)$;

(b-t$_6$) $B_1, B_2 \in \mathcal{B}^X$ *imply* $t(B_1 \cup B_2) \subset t(B_1) \subset t(B_2)$.

Then the triple $(X, \mathcal{B}^X, t)$ is called a b-topological space. *For b-topological spaces $(X, \mathcal{B}^X, t^X)$, $(Y, \mathcal{B}^Y, t^Y)$ a function $f : X \longrightarrow Y$ is called* b-continuous map *iff it is bi-bounded and satisfies the following condition, i.e.*

(cont) $B \in \mathcal{B}^X$ *implies* $f[t^X(B)] \subset t^Y(f[B])$.

We denote by b-TOP the corresponding category [19].

Example 1.4 For a set X let $\mathcal{B}_f^X$ be denote the set of *all* finite subsets of X. Thus, $\mathcal{B}_f^X$ defines a bornology on X. Then, for a fixed set $D \in \mathcal{B}_f^X$ we establish a b-topology $t^D : \mathcal{B}^X \longrightarrow \underline{P}X$ by setting $t^D(\emptyset) := \emptyset$ and $t^D(B) := B \cup D$, otherwise.

Remark 1.5 If $\mathcal{B}^X$ is saturated, then a b-topological space can be considered as topological space and vice versa. Moreover, if for bornologies $\mathcal{B}^X, \mathcal{B}^Y$ with saturated $\mathcal{B}^X f : X \longrightarrow Y$ is constant map, then f is *automatically* b-continuous.

Lemma 1.6 *For a b-topological space $(X, \mathcal{B}^X, t)$ we set: $N_t(\emptyset) := \{\emptyset\}$ and $N_t(B) := \{\rho \subset \underline{P}X : B \in \sec\{t(F) : F \in \rho \cap \mathcal{B}^X\}\}$, otherwise.*

Then $(X, \mathcal{B}^X, N_t)$ is an esd-space such that $t = cl_{N_t}$ (see also Chapter 2).

Proof: Firstly, we have to verify that N_t is satisfying the axioms (esd$_1$) to (esd$_9$).

to (esd$_1$): $\rho_2 << \rho_1 \in N_t(B), \rho \subset \underline{P}X, B \in \mathcal{B}^X\setminus\{\emptyset\}$ and $F \in \rho_2 \cap \mathcal{B}^X$ imply the existence of $F_1 \in \rho_1$ with $F_1 \subset F_2$. Hence $F_1 \in \rho_1 \cap \mathcal{B}^X$ follows by applying (b$_1$), and $B \cap t(F_1) \neq \emptyset$ results by hypothesis. Consequently, $B \cap t(F_2) \neq \emptyset$ is valid according to (b-t$_4$), resulting into $\rho_2 \in N_t(B)$.

to (esd$_2$): Let $B \in \mathcal{B}^X$; in first case if $B = \emptyset$ we have $\emptyset \in N_t(B)$ by definition. In second case if $B \neq \emptyset$ we get $\{B\} \in N_t(B)$, since $B \cap t(B) \neq \emptyset$ is valid.

Further suppose $\mathcal{B}^X \in N_t(B)$, and without restriction $B \neq \emptyset$, otherwise $B = \emptyset$ contradicts. Then $B \in \sec\{t(F) : F \in B^X\}$ implies $B \cap t(\emptyset) \neq \emptyset$, which contradicts too. Hence $\mathcal{B}^X \notin N_t(B)$ follows.

to (esd$_3$): evident by definition of N_t.

to (esd$_4$): see especially proof of (esd$_2$).

to (esd$_5$): evident.

to (esd$_6$): For $\{cl_{N_t}(F) : F \in \rho\} \in N_t(B), \rho \subset \underline{P}X, B \in \mathcal{B}^X$ let $A \in \rho\cap\mathcal{B}^X$, we have to verify $B \cap t(A) \neq \emptyset$. Since $cl_{N_t}(A) \in \{cl_{N_t}(F) : F \in \rho\}$ we get $B \cap t(cl_{N_t}(A)) \neq \emptyset$ by hypothesis. Note, that $cl_{N_t}(A) \subset t(A) \in \mathcal{B}^X$ is valid. Consequently $B\cap t(t(A)) \neq \emptyset$ follows, and $B\cap t(A) \neq \emptyset$ results according to (b-t$_5$), showing our made assertion.

to (esd$_7$): $\rho_1 \vee \rho_2 \in N_t(B)$ and without restriction $B \neq \emptyset$ with $\rho_1 \neq \emptyset \neq \rho_2$ imply $B \in \sec\{t(F) : F \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X\}$. Now, let us suppose $\rho_1, \rho_2 \notin N_t(B)$. Hence there exists $F_1 \in \rho_1 \cap \mathcal{B}^X B \cap t(F_1) = \emptyset$ and $F_2 \in \rho_2 \cap \mathcal{B}^X\ B \cap t(F_2) = \emptyset$. But $F_1 \cup F_2 \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X$, since $\mathcal{B}^X$ is bornology and

$$\emptyset = (B \cap t(F_1)) \cup (B \cap t(F_2)) = B \cap (t(F_1) \cup t(F_2)) = B \cap t(F_1 \cup F_2)$$

according to (b-t$_4$) and (b-t$_6$), respectively which contradicts.

to (esd$_8$): evident.

to (esd$_9$): $B \in \mathcal{B}^X\setminus\{\emptyset\}$ and $\rho \cap \mathcal{B}^X \in N_t(B), \rho \subset \underline{P}X$ imply $B \in \sec\{t(F) : F \in (\rho \cap \mathcal{B}^X) \cap \mathcal{B}^X\}$, and $\rho \in N_t(B)$ results. To show the equality $t = cl_{N_t}$ is valid let without restriction $B \in \mathcal{B}^X\setminus\{\emptyset\}$, then $x \in cl_{N_t}(B)$ is equivalent to the statement $\{B\} \in N_t(\{x\})$, which is further equivalent to $\{x\} \in \sec\{t(F) : F \in B \cap \mathcal{B}^X\}$, at last resulting into the statement $x \in t(B)$ as equivalent to above.

□

Remark 1.7 As an interpretation of this Lemma we keep hold that every b-topological space is induced by a certain esd-space.

As a next step in our research we will introduce the concept of b-proximal spaces and related facts.

Definition 1.8 *For a bornology $\mathcal{B}^X$ a relation $\delta \subset \mathcal{B}^X \times \mathcal{B}^X$ is called* b-proximal*, and the triple $(X, \mathcal{B}^X, \delta)$ a* b-proximal space *iff δ satisfies the following conditions, i.e.*

(b-p$_1$) *$B \in \mathcal{B}^X$ implies $cl_\delta(B) \in \mathcal{B}^X$, where $cl_\delta(B) := \{x \in X : \{x\}\delta B\}$;*

(b-p$_2$) *$\emptyset\bar{\delta}D$ and $B\bar{\delta}\emptyset$ for each $B, D \in \mathcal{B}^X$;*

(b-p$_3$) *$B\delta(D_1 \cup D_2)$ iff $B\delta D_1$ or $B\delta D_2$ for each $B, D_1, D_2 \in \mathcal{B}^X$;*

(b-p$_4$) *$x \in X$ implies $\{x\}\delta\{x\}$;*

(b-p$_5$) *$B_1 \subset B \in \mathcal{B}^X$ and $B_1\delta D$ imply $B\delta D$ for each $D \in \mathcal{B}^X$;*

(b-p$_6$) *$B_1\delta D$ and $D \subset cl_\delta(B), B \in \mathcal{B}^X$ imply $B_1\delta B$.*

(Hereby, $\bar{\delta}$ denotes the negation *of δ). For b-proximal spaces $(X, \mathcal{B}^X, \delta)$, $(Y, \mathcal{B}^Y, \gamma)$ a function $f : X \longrightarrow Y$ is called* b-proximal map *iff f is bi-bounded and satisfies the following condition, i.e.*

(prox) *$B\delta D$ implies $f[B]\gamma f[D]$. We denote by b-PX the corresponding category.*

Remark 1.9 If $\mathcal{B}^X$ is saturated, then a b-proximal space $(X, \mathcal{B}^X, \delta)$ may be considered as a *generalized proximity space* and vice versa [14]. In *special* cases LEADER proximities as well as LODATO proximities then can be easily recovered.

Proposition 1.10 *For a b-topological space $(X, \mathcal{B}^X, t)$ we set: $B\delta_t D$ iff $B \cap t(D) \neq \emptyset$ for each $B, D \in \mathcal{B}^X$. Then $(X, \mathcal{B}^X, \delta_t)$ defines a b-proximal space which additionally is* additive *by satisfying*

(add) *$(B_1 \cup B_2)\delta D, B_1, B_2, D \in \mathcal{B}^X$ imply $B_1\delta D$ or $B_2\delta D$.*

Proof: straight forward. □

Definition 1.11 *A b-proximal space $(X, \mathcal{B}^X, \delta)$ is called* symmetric *iff in addition holds*

(s) *$B_1\delta B_2$ implies $B_2\delta B_1$ for each $B_1, B_2 \in \mathcal{B}^X$.*

Remark 1.12 Here, we only note that if $\mathcal{B}^X$ is saturated, then $(X, \mathcal{B}^X, \delta)$ can be *essentially* considered as a LODATO proximity space [20] and vice versa. We denote by b-SPX the corresponding full subcategory of b-PX.

2 b-TOP, b-PX and b-SPX as fully embedded subcategories of ESD

Now, firstly let us start with the objects of b-PX.

Lemma 2.1 *For a b-proximal space* $(X, \mathcal{B}^X, \delta)$ *we set:* $N_\delta(\emptyset) := \{\emptyset\}$ *and* $N_\delta(B) := \{\rho \subset \underline{P}X : \rho \cap \mathcal{B}^X \subset \delta(B)\}$, *where* $\delta(B) := \{D \in \mathcal{B}^X : B\delta D\}$, *otherwise. Then* $(X, \mathcal{B}^X, N_\delta)$ *is an esd-space.*

Proof: Straight forward. Here, we only will verify the validity of the axioms (esd$_6$), (esd$_7$) and (esd$_8$) in definition 1.1.

to (esd$_6$): For $\rho \subset \underline{P}X$ let $\{cl_{N_\delta}(F) : F \in \delta\} \in N_\delta(B)$, we have to verify $\rho \cap \mathcal{B}^X \subset \delta(B)$. $A \in \rho \cap \mathcal{B}^X$ implies $cl_{N_\delta}(A) \in \{cl_{N_\delta}(F) : F \in \rho\}$. Since $A \in \mathcal{B}^X$ we claim $cl_{N_\delta}(A) \subset cl_\delta(A)$, hence $cl_{N_\delta}(A) \in \mathcal{B}^X$. By hypothesis $cl_{N_\delta}(A) \in \delta(B)$ follows, showing that $B\delta cl_{N_\delta}(A) \subset cl_\delta(A)$ is valid. But δ is satisfying (b-p6), and $B\delta A$ results, hence $A \in \delta(B)$ follows.

to (esd$_7$): Without restriction let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\rho_1 \vee \rho_2 \in N_\delta(B)$, $\rho_1 \neq \emptyset \neq \rho_2$. If supposing $\rho_1, \rho_2 \notin N_\delta(B)$ we get $F_1, F_2 \notin \delta(B)$ for some $F_1 \in \rho_1 \cap \mathcal{B}^X$ and $F_2 \in \rho_2 \cap \mathcal{B}^X$. Hence $B\bar{\delta}F_1$ and $B\bar{\delta}F_2$ implying $B\bar{\delta}(F_1 \cup F_2)$ according to (b-p$_3$), note that $\mathcal{B}^X$ is bornology. But $F_1 \cup F_2 \in (\rho_1 \cup \rho_2) \cap \mathcal{B}^X$ leads us to a contradiction.

to (esd$_8$): $B \in \mathcal{B}^X$ implies $cl_\delta(B) \in \mathcal{B}^X$. We will show that $cl_{N_\delta}(B) \subset cl_\delta(B)$, then by (b$_1$) we get the desired result. $x \in cl_{N_\delta}(B)$ implies $\{B\} \in N_\delta(\{x\})$, hence $\{B\} \subset \delta(\{x\})$, and $\{x\}\delta B$ results, showing that $x \in cl_\delta(B)$ is valid.

□

Definition 2.2 *An esd-space* $(X, \mathcal{B}^X, N)$ *is called* conic *iff* N *satisfies the condition*

(con) $B \in \mathcal{B}^X$ *implies* $\bigcup\{\rho \subset \underline{P}X : \rho \in N(B)\} \in N(B)$.

Example 2.3 According to Lemma 1.6 we state that the esd-space $(X, \mathcal{B}^X, N_t)$ is conic.

Remark 2.4 Here, we note that the esd-space $(X, \mathcal{B}^X, N_\delta)$ is conic, too. But in *general* this property must not be *necessary* fulfilled, if, par example we look at the near subdensity spaces considered in [19].

Lemma 2.5 *For a conic esd-space* $(Y, \mathcal{B}^Y, M)$ *we put* $B\gamma_M D$ *iff* $\{D\} \in M(B)$ *for sets* $B, D \in \mathcal{B}^Y$. *Then* $(Y, \mathcal{B}^Y, \gamma_M)$ *is a b-proximal space such that* $N_{\gamma_M} = M$.

Proof: Straight forward. Here, we only will verify the validity of axiom (b-p6) in definition 1.8.

to (b-p6): $B_1\delta D$ and $D \subset cl_{\gamma_M}(B)$, $B \in \mathcal{B}^Y$ imply $\{D\} \in M(B_1)$, hence $\{cl_M(B)\} << \{cl_{\gamma_M}(B)\} << \{D\}$ follows, and $\{cl_M(B)\} \in M(B_1)$ is valid. We get $\{B\} \in M(B_1)$, according to (esd$_6$) which results in $B_1\gamma_M B$. It remains to prove the equality $N_{\gamma_M} = M$. Without restriction let $B \in \mathcal{B}^X\backslash\{\emptyset\}$ and $\rho \in N_{\gamma_M}(B)$, hence $\rho \cap \mathcal{B}^X \subset \gamma_M(B)$. Now, we will show that $\gamma_M(B) \subset \bigcup\{\sigma : \sigma \in M(B)\}$ holds. $D \in \gamma_M(B)$ implies $B\gamma_M D$, hence $\{D\} \in M(B)$ is valid with $D \in \{D\}$, and $D \in \bigcup\{\sigma : \sigma \in M(B)\}$ follows. Consequently, $\rho \cap \mathcal{B}^X \in M(B)$ can be deduced by applying (esd$_1$), resulting into $\rho \in M(B)$ according to (esd$_9$). The reverse case is easily to verify.

□

Theorem 2.6 *The full subcategory CON-ESD of ESD, whose objects are the conic esd-spaces is isomorphic to the category b-PX.*

Proof: Taking into account former results we further note that for a given b-proximal space $(X, \mathcal{B}^X, \delta)$ the equality $\gamma_{N_\delta} = \delta$ is valid. Moreover, we claim that for each b-proximal map f between b-proximal spaces f is bibsd-map between the corresponding esd-spaces and vice versa. □

Definition 2.7 *A conic esd- space* $(X, \mathcal{B}^X, N)$ *is called* proximal *iff* N *satisfies the condition*

(px) $B \in \mathcal{B}^X\backslash\{\emptyset\}$ *and* $\rho \in N(B)$ *imply* $\{B\} \in \bigcap\{N(F) : F \in \rho \cap \mathcal{B}^X\}$.

Remark 2.8 Here, we note that for a given symmetric b-proximal space $(X, \mathcal{B}^X, \delta)$ the corresponding esd-space $(X, \mathcal{B}^X, N_\delta)$ is proximal. Because for $B \in \mathcal{B}^X\backslash\{\emptyset\}$ and $\rho \in N_\delta(B)$ we have $\rho \cap \mathcal{B}^X \subset \delta(B)$. Then, $F \in \rho \cap \mathcal{B}^X$ implies $\{B\} \in N_\delta(F)$. Since by hypothesis $B\delta F$ is valid $F\delta B$ results, because δ is symmetric.

Corollary 2.9 *The full subcategory **PX-ESD** of **CON-ESD**, whose objects are the proximal esd-spaces is isomorphic to the category **b-SPX**.*

Proof: Here, we only note that for a given proximal esd-space the corresponding b-proximal space is symmetric. □

Proposition 2.10 *Every proximal esd-space* $(X, \mathcal{B}^X, N)$ *is* closed *by satisfying*

(clo) $B \in \mathcal{B}^X$ *implies* $N(cl_N(B)) = N(B)$.

Proof: Without restriction let $B \in \mathcal{B}^X\backslash\{\emptyset\}$ and $\rho \in N(cl_N(B))$, we will show that $\rho\cap\mathcal{B}^X \subset \cup\{\sigma : \sigma \in N(B)\}$ is valid. $F \in \rho \cap \mathcal{B}^X$ implies $\{cl_N(B)\} \in N(F)$, since $(X, \mathcal{B}^X, N)$ is proximal. Then $\{B\} \in N(F)$ follows by applying (esd$_6$), and $\{F\} \in N(B)$ results with respect to (px). Consequently, $F \in \cup\{\sigma : \sigma \in N(B)\}$ is valid, showing that $\rho \cap \mathcal{B}^X \in N(B)$, according to (esd$_1$). But this induce $\rho \in N(B)$ by applying (esd$_9$). The reverse inclusion then can be easily verified with respect to (esd$_5$). $\square$

Proposition 2.11 *Every proximal esd-space $(X, \mathcal{B}^X, N)$ is* linked *by satisfying*

(lik) $\rho \in N(B_1 \cup B_2), B_1, B_2 \in \mathcal{B}^X$ *imply* $\{F\} \in N(B_1) \cup N(B_2) \forall F \in \rho \cap \mathcal{B}^X$.

Proof: evident. $\square$

Definition 2.12 *A conic esd-space $(X, \mathcal{B}^X, N)$ is called* covered *iff N satisfies the condition*

(cov) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ *and* $\rho \in N(B)$ *imply* $B \in \sec\{cl_N(F) : F \in \rho \cap \mathcal{B}^X\}$.

Example 2.13 With respect to example 2.3 we note that $(X, \mathcal{B}^X, N_t)$ is a covered esd-space.

Lemma 2.14 *For a covered esd-space $(X, \mathcal{B}^X, N)$ the restriction of cl_M on $\mathcal{B}^X$, denoted by cl_M^b is a b-topology on $\mathcal{B}^X$ such that $N_{cl_M^b} = M$.*

Proof: Firstly, we only will verify the validity of the axioms (b-t$_5$) and (b-t$_6$), respectively in definition 1.3. Then, the remaining is clear.

to (b-t$_5$): $x \in cl_M^b(cl_M^b(B)), B \in \mathcal{B}^X$ imply $\{cl_M^b(B)\} \in M(\{x\})$, hence $\{cl_M(B)\} \in M(\{x\})$ is valid, and $\{B\} \in M(\{x\})$ results, according to (esd$_6$). But then $x \in cl_M(B) = cl_M^b(B)$ follows.

to (b-t$_6$): $B_1, B_2 \in \mathcal{B}^X$ and without restriction let $B_1 \neq \emptyset \neq B_2 \cdot x \in cl_M^b(B_1 \cup B_2)$ implies $\{B_1 \cup B_2\} \in M(\{x\})$, by paying attention to the fact that $\mathcal{B}^X$ is bornology. Since $\{B_1\} \vee \{B_2\} = \{B_1 \cup B_2\}$, we get $\{B_1\} \in M(\{x\})$ or $\{B_2\} \in M(\{x\})$ by applying (esd$_7$), resulting into $x \in cl_M^b(B_1) \cup cl_M^b(B_2)$. In showing the equality $N_{cl_M^b} = M$ let without restriction $B \in \mathcal{B}^X \backslash \{\emptyset\}$. $\rho \in N_{cl_M^b}(B)$ implies $B \in \sec\{cl_M^b(F) : F \in \rho \cap \mathcal{B}^X\}$, which is the same as $B \in \sec\{cl_M(F) : F \in \rho \cap \mathcal{B}^X\}$. Since $(X, \mathcal{B}^X, M)$ is conic, we know that $\bigcup\{\sigma : \sigma \in M(B)\} \in M(B)$. Thus, it remains to verify $\rho \cap \mathcal{B}^X \subset \cup\{\sigma : \sigma \in M(B)\}$, because then $\rho \cap \mathcal{B}^X \in M(B)$ follows, according to (esd$_1$), and $\rho \in M(B)$ is valid by applying (esd$_9$). $F \in \rho \cap \mathcal{B}^X$ implies $B \cap cl_M(F) \neq \emptyset$, hence $x \in cl_M(F)$ for some $x \in B$. Consequently, $\{F\} \in M(\{x\}) \subset M(B)$ follows, showing that $F \in \bigcup\{\sigma : \sigma \in M(B)\}$, which put an end of this. Then, the reverse inclusion is easily to verify.

$\square$

Theorem 2.15 *The full subcategory* ***COV-ESD*** *of* ***CON-ESD****, whose objects are the covered esd-spaces is isomorphic to the category* ***b-TOP****.*

Proof: Taking into account former results we further note that for each b-continuous map f between b-topological spaces f is bibsd-map between the corresponding esd-spaces and vice versa. $\square$

Theorem 2.16 *The category* ***CON-ESD*** *is bireflective in* ***ESD****.*

Proof: For an esd-space $(X, \mathcal{B}^X, N)$ we set: $N^C(\emptyset) := \{\emptyset\}$ and $N^C(B) := \{\mathcal{A} \subset \underline{P}X : \{cl_N(A) : A \in \mathcal{A} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}\}$, otherwise. Then $(X, \mathcal{B}^X, N^C)$ is conic esd-space, and $\underline{1}_X : (X, \mathcal{B}^X, N) \longrightarrow (X, \mathcal{B}^X, N^C)$ is bibsd-map. In the following we only will verify the validity of the axioms (esd_6), (esd_7) in definition 1.1 and that of axiom (con) in definition 2.2. Then the remaining statements are obvious.

to (esd_6): $\{cl_{N^C}(A) : A \in \mathcal{A}\} \in N^C(B)$, $B \in \mathcal{B}^X \backslash \{\emptyset\}, \mathcal{A} \subset \underline{P}X$ imply $\{cl_N(F) : F \in \{cl_{N^C}(A) : A \in \mathcal{A}\} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$. We will show that $\{cl_N(A) : A \in \mathcal{A} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$. $A \in \mathcal{A} \cap \mathcal{B}^X$ implies $cl_N(cl_{N^C}(A)) \in \bigcup\{\rho : \rho \in N(B)\}$, since $cl_{N^C}(A) \in \mathcal{B}^X$. Further we have the inclusion $cl_{N^C}(A) \subset cl_N(A)$ is valid: $x \in cl_{N^C}(A)$ implies $\{A\} \in N^C(\{x\})$, hence $cl_N(A) \in \rho$ for some $\rho \in N(\{x\})$. $\{cl_N(A)\} \in N(\{x\})$ holds by applying (esd_1), and $\{A\} \in N(\{x\})$ results according to (esd_6), hence $x \in cl_N(A)$ follows. By hypothesis $cl_N(cl_{N^C}(A)) \in \sigma$ for some $\sigma \in N(B)$, and $\{cl_N(A)\} \in N(B)$ follows by applying (esd_6), again. Consequently our assertion holds.

to (esd_7): Let $\mathcal{A}_1 \vee \mathcal{A}_2 \in N^C(B)$ and without restriction $B \in \mathcal{B}^X \backslash \{\emptyset\}$ with $\mathcal{A}_1 \neq \emptyset \neq \mathcal{A}_2$. Then $\{cl_N(A) : A \in (\mathcal{A}_1 \vee \mathcal{A}_2) \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$ follows. If supposing $\mathcal{A}_1, \mathcal{A}_2 \notin N^C(B)$ we can choose $A_1 \in \mathcal{A}_1 \cap \mathcal{B}^X$ with $cl_N(A_1) \notin \bigcup\{\rho : \rho \in N(B)\}$ and $A_2 \in \mathcal{A}_2 \cap \mathcal{B}^X$ with $cl_N(A_2) \notin \bigcup\{\rho : \rho \in N(B)\}$. Consequently, $A_1 \cup A_2 \in (\mathcal{A}_1 \vee \mathcal{A}_2) \cap \mathcal{B}^X$ follows, since $\mathcal{B}^X$ is bornology. By hypothesis $cl_N(A_1 \cup A_2) \in \mathcal{A}$ for some $\mathcal{A} \in N(B)$, hence $\{cl_N(A_1 \cup A_2)\} \in N(B)$ is valid. But $\{cl_N(A_1)\} \vee \{cl_N(A_2)\} = \{cl_N(A_1 \cup A_2)\}$ is holding, and consequently $\{cl_N(A_1)\} \in N(B)$ or $\{cl_N(A_2)\} \in N(B)$ follows by applying (esd_7) which contradicts.

to (con): Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$. We have to verify $\bigcup\{\mathcal{A} : \mathcal{A} \in N^C(B)\} \in N^C(B)$, which means that $\{cl_N(F) : F \in \bigcup\{\mathcal{A} : \mathcal{A} \in N^C(B)\} \cap \mathcal{B}^X\} \subset \bigcup\{\rho : \rho \in N(B)\}$. Now, let $cl_N(F)$ be given for $F \in \bigcup\{\mathcal{A} : \mathcal{A} \in N^C(B)\} \cap \mathcal{B}^X$ hence $F \in \mathcal{A}$ for some $\mathcal{A} \in N^C(B)$. By hypothesis there exists $\rho \in N(B)$ with $cl_N(F) \in \rho'$, and $cl_N(F) \in \bigcup\{\rho : \rho \in N(B)\}$ results. Now, let $(Y, \mathcal{B}^Y, M)$ be a conic esd-space and $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$ be a bibsd-map, we have to show $f:(X,\mathcal{B}^X,N^C)\longrightarrow(Y,\mathcal{B}^Y,M)$ is bibsd-map, too. Since by hypothesis f is bi-bounded, we will now verify the validity of axiom (sd) in definition 1.1.

to (sd): Without restriction let $B\in\mathcal{B}^X\backslash\{\emptyset\}$ and $\mathcal{A}\in N^C(B)$, hence by definition $\{cl_N(A):A\in\mathcal{A}\cap\mathcal{B}^X\}\subset\bigcup\{\rho:\rho\in N(B)\}$ is valid. It suffices to show $f\mathcal{A}\cap\mathcal{B}^Y\in M(f[B])$. Therefore its being enough to verify the validity of the inclusion $f\mathcal{A}\cap\mathcal{B}^Y\subset\bigcup\{\mathcal{M}:\mathcal{M}\in M(f[B])\}$. $D\in f\mathcal{A}\cap\mathcal{B}^Y$ implies $D=f[A]$ for some $A\in\mathcal{A}$. Then $A\subset f^{-1}[f[A]]=f^{-1}[D]\in\mathcal{B}^X$, and $A\in\mathcal{B}^X$ follows. Hence $cl_N(A)\in\rho$ for some $\rho\in N(B)$ by hypothesis. Consequently, $f\rho\in M(f[B])$ follows with $f[cl_N(A)]\in f\rho$. Since $cl_M(f[A])\supset f[cl_N(A)]$ we get $\{cl_M(f[A])\}\in M(f[B])$, and $\{D\}=\{f[A]\}\in M(f[B])$ results, according to (esd_6). But then $f\mathcal{A}\cap\mathcal{B}^Y\in M(f[A])$ is valid, since by hypothesis $(Y,\mathcal{B}^Y,M)$ is conic, and at last $f\mathcal{A}\in M(f[B])$ can be deduced by applying (esd_9).

□

Theorem 2.17 *The category **COV-ESD** is bicoreflective in **CON-ESD**.*

Proof: For a conic esd-space $(X,\mathcal{B}^X,N)$ we set: $N^{CV}(\emptyset):=\{\emptyset\}$ and $N^{CV}(B):=\{\rho\subset\underline{P}X: B\in\sec\{cl_N(F):F\in\rho\cap\mathcal{B}^X\}\}$, otherwise. Then $(X,\mathcal{B}^X,N^{CV})$ is a covered esd-space, and $\underline{1}_X:(X,\mathcal{B}^X,N^{CV})\longrightarrow(X,\mathcal{B}^X,N)$ is bibsd-map. It is straight forward to verify that $(X,\mathcal{B}^X,N^{CV})$ is a covered esd-space. In showing that $\underline{1}_X$ is bibsd-map let $\rho\in N^{CV}(B)$ and without restriction $B\in\mathcal{B}^X\backslash\{\emptyset\}$. Consequently, $B\in\sec\{cl_N(F):F\in\rho\cap\mathcal{B}^X\}$ holds by definition of N^{CV}. Now, we will verify that $\rho\cap\mathcal{B}^X$ is a subset of $\bigcup\{\mathcal{A}:\mathcal{A}\in N(B)\}$. $F\in\rho\cap\mathcal{B}^X$ implies the existence of an element $x\in B$ with $x\in cl_N(F)$. Hence $\{F\}\in N(\{x\})\subset N(B)$ follows, showing that $F\in\bigcup\{\mathcal{A}:\mathcal{A}\in N(B)\}$ is valid. Now, let $(Y,\mathcal{B}^Y,M)$ be a covered esd-space and $f:(Y,\mathcal{B}^Y,M)\longrightarrow(X,\mathcal{B}^X,N)$ be a bibsd-map, we have to show $f:(Y,\mathcal{B}^Y,M)\longrightarrow(X,\mathcal{B}^X,N^{CV})$ is bibsd-map, too. Since by hypothesis f is bi-bounded we will verify the validity of axiom (sd) in definition 1.1. Without restriction let $B\in\mathcal{B}^Y\backslash\{\emptyset\}$ and $\rho\in M(B)$, hence $B\in\sec\{cl_M(F):F\in\rho\cap\mathcal{B}^Y\}$. For $A\in f\rho\cap\mathcal{B}^X$ we have $A=f[F]$ for some $F\in\rho$ with $f^{-1}[A]\in\mathcal{B}^Y$, since f is bi-bounded. Consequently, $F\in\mathcal{B}^Y$ is valid, and we can choose $y\in cl_M(F)$ for some $y\in B$ by hypothesis. But f also satisfies (sd) in definition 1.1, hence $f(y)\in cl_N(A)\cap f[B]$ results, concluding the proof. □

3 Topological extensions and related esd-spaces

We will now consider the problem for finding a *one-to-one correspondence* between certain topological extensions and their related esd-spaces. This question arises from a problem formulated by LODATO in 1966 as follows:

He asked for an axiomatization of the following binary nearness relation on the power set of a set X: there exists an embedding of X into a topological space Y such that subsets A and B are *near* in X iff their closures meet in Y.

Now, we will generalize and solve this problem for esd-spaces, *involving* also LODATO's result as a special case. At first, we define the category ***BTEXT*** of so-called *bornotopological extensions* – shortly btop-extensions – and related morphisms (see also [19]).

Definition 3.1 *Objects of **BTEXT** are triples* $E := (e, \mathcal{B}^X, Y)$, *where* $X := (X, t_X)$, $Y := (Y, t_Y)$ *are topological spaces (given by closure operators* t_X *respectively* t_Y*) with bornology* $\mathcal{B}^X$, *so that iff* $B \in \mathcal{B}^X$ *then* $t_X(B) \in \mathcal{B}^X$ *also holds.*

$e : X \longrightarrow Y$ *is a function satisfying the following conditions:*

(btx$_1$) $B \in \mathcal{B}^X$ *implies* $t_X(B) = e^{-1}[t_y(e[B])]$, *where* e^{-1} *denotes the* inverse *image under* e*;*

(btx$_1$) $t_Y(e[X]) = Y$, *which means that the image of* X *under* e *is* dense *in* Y.

Morphisms *in **BTEXT** have the form* $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$, *where* $f : X \longrightarrow X'$ $g : Y \longrightarrow Y'$ *are continuous maps such that* f *is bi-bounded, and the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{e'} & Y' \end{array} .$$

If $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ *and* $(f', g') : (e', \mathcal{B}^{X'}, Y') \longrightarrow (e'', \mathcal{B}^{X''}, Y'')$ *are **BTEXT**-morphisms, then they can be* composed *according to the rule* $(f', g') \circ (f, g) := (f' \circ f, g' \circ g) : (e, \mathcal{B}^X, Y) \longrightarrow (e'', \mathcal{B}^{X''}, Y'')$, *where "$\circ$" denotes the* composition *of maps.*

Remark 3.2 Observe, that axiom (btx$_1$) in this definition is *automatically* satisfied if $e : X \longrightarrow Y$ is a *topological embedding*. Moreover, we admit an *ordinary* bornology $\mathcal{B}^X$, which need *not* be necessary *coincide* with the power set $\underline{P}X$.

Definition 3.3 *We call such an extension* $E := (e, \mathcal{B}^X, Y)$

(i) strict *iff* E *satisfies the condition*

(st) $\{t_Y(e[A]) : A \subset X\}$ *forms a base for the* closed *subsets of* Y *[1];*

(ii) symmetric *iff* E *satisfies the condition*

(sy) $x \in X$ *and* $y \in t_Y(\{e(x)\})$ *imply* $e(x) \in t_Y(\{y\})$ *[3].*

Example 3.4 For a symmetric bornotopological extension $E := (e, \mathcal{B}^X, Y)$ we consider the triple $(X, \mathcal{B}^X, N^e)$, where N^e is defined by setting:

$N^e(\emptyset) := \{\emptyset\}$ and
$N^e(B) := \{\rho \subset \underline{P}X : t_Y(e[B]) \in \sec\{t_Y(e[F]) : F \in \rho \cap \mathcal{B}^X\}\}$, otherwise.

Then $(X, \mathcal{B}^X, N^e)$ is a proximal esd-space such that for each $B \in \mathcal{B}^X$ $cl_{N^e}(B) = t_X(B)$.

Proof: Firstly, we will verify the above cited equality. Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$.

to "$\subset$": $x \in cl_{N^e}(B)$ implies $\{B\} \in N^e(\{x\})$, hence $t_Y(\{e(x)\}) \cap t_Y(e[B]) \neq \emptyset$. Then we can choose $y \in t_Y(e[B])$ with $y \in t_Y(\{e(x)\})$. Since by hypothesis E is symmetric, we get $e(x) \in t_Y(\{y\})$. But then $e(x) \in t_Y(e[B])$ is valid, because t is topological. Consequently, $x \in t_X(B)$ follows by applying (btx_1) in definition 3.1.

to "$\supset$": $x \in t_X(B)$ implies $e(x) \in t_Y(e[B])$ according to (btx_1), hence $\{B\} \in N^e(\{x\})$ follows, resulting into $x \in cl_{N^e}(B)$. Further, we only will verify the validity of the axioms (esd_6) and (esd_7), respectively. Then the remaining statements are clear.

to (esd_6): $\{cl_{N^e}(F) : F \in \rho\} \in N^e(B), B \in \mathcal{B}^X \backslash \{\emptyset\}, \rho \subset \underline{P}X$ imply $t_Y(e[B]) \in \sec\{t_Y(e[A]) : A \in \{cl_{N^e}(F) : F \in \rho\} \cap \mathcal{B}^X\}$. Then $F' \in \rho \cap \mathcal{B}^X$ implies $cl_{N^e}(F') \in \{cl_{N^e}(F) : F \in \rho\} \cap \mathcal{B}^X$, because $cl_{N^e}(F') = t_X(F') \in \mathcal{B}^X$ by definition 3.1. By hypothesis $t_Y(e[B]) \cap t_Y(e[t_X(F')]) \neq \emptyset$ follows. But $e[t_X(F')] \subset t_Y(e[F'])$ holds by applying (btx_1), and $t_Y(e[t_X(F')]) \subset t_Y(e[F'])$ can be deduced, since t_Y is topological, resulting into $\rho \in N^e(B)$.

to (esd_7): Let $\rho_1 \vee \rho_2 \in N^e(B)$ and without restriction $\rho_1 \neq \emptyset \neq \rho_2, B \neq \emptyset$. By definition we get $t_Y(e[B]) \in \sec\{t_Y(e[F]) : F \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X\}$. If supposing $\rho_1, \rho_2 \notin N^e(B)$. Then we can choose $F_1 \in \rho_1 \cap \mathcal{B}^X$ with $t_Y(e[B]) \cap t_Y(e[F_1]) = \emptyset$ and $F_2 \in \rho_2 \cap \mathcal{B}^X$ with $t_Y(e[B]) \cap t_Y(e[F_2]) = \emptyset$. Hence $F_1 \cup F_2 \in (\rho_1 \vee \rho_2) \cap \mathcal{B}^X$, since $\mathcal{B}^X$ is bornology. Consequently, $t_Y(e[B]) \cap t_Y(e[F_1 \cup F_2]) \neq \emptyset$ results. On the other hand we have $\emptyset = t_Y(e[B]) \cap (t_Y(e[F_1]) \cup t_Y(e[F_2])) = t_Y(e[B]) \cap t_Y(e[F_1] \cup e[F_2]) = t_Y(e[B]) \cap t_Y(e[F_1 \cup F_2])$, which contradicts.

$\square$

Definition 3.5 *For a proximal esd-space $(X, \mathcal{B}^X, N)$ and for $B \in \mathcal{B}^X \sigma \subset \underline{P}X$ is called* B-bunch *in N iff σ satisfies the following conditions:*

(bun_1) $\emptyset \notin \sigma$;

(bun_2) $F_1 \cup F_2 \in \sigma$ *iff* $F_1 \in \sigma$ *or* $F_2 \in \sigma$;

(bun_3) $B \in \sigma \in N(B)$;

(bun_4) $A \in \sigma$ *and* $A \subset cl_N(F) : F \in \mathcal{B}^X$ *imply* $F \in \sigma$;

(bun_5) $A \in \sigma \cap \mathcal{B}^X$ *implies* $\{A\} \in \bigcap\{N(F) : F \in \sigma \cap \mathcal{B}^X\}$.

Proposition 3.6 *For a proximal esd-space $(X, \mathcal{B}^X, N)$ and for $B \in \mathcal{B}^X \setminus \{\emptyset\}$ with $x \in B$ $x_N := \{A \subset X : x \in cl_N(A)\}$ is a B-bunch in N. Moreover, x_N is maximal element in $N(\{x\}) \setminus \{\emptyset\}$, ordered by inclusion.*

Proof: Evidently, x_N is satisfying (bun_1) and (bun_2). $B \in x_N$, since $\{B\} << \{\{x\}\} \in N(\{x\}) \subset N(B)$ and (esd_6) are holding.

to (bun_4): $A \in x_N$ and $A \subset cl_N(F), F \in \mathcal{B}^X$ imply $x \in cl_N(A)$, hence $x \in cl_N(F)$ follows, showing that $F \in x_N$ is valid.

to (bun_5): $A \in x_N \cap \mathcal{B}^X$ and $F \in x_N \cap \mathcal{B}^X$ imply $\{A\} \in N(\{x\}) \subset N(cl_N(F)) = N(F)$, according to proposition 2.10.

Now, let $\sigma \in N(\{x\}) \setminus \{\emptyset\}$ with $x_N \subset \sigma$. For $F \in \sigma$ we have $\{F\} \in N(\{x\})$, and $x \in cl_N(F)$ follows, showing that $\sigma = x_N$ holds. □

Definition 3.7 *A proximal esd-space $(X, \mathcal{B}^X, N)$ is called a* bunch space *iff N satisfies the condition*

(bun) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ *and* $\rho \in N(B)$ *imply* $\forall F \in \rho \cap \mathcal{B}^X \exists$ *B-bunch* σ *in* N *with* $F \in \sigma$.

Proposition 3.8 *The esd-space $(X, \mathcal{B}^X, N^e)$ is a bunch space.*

Proof: For $B \in \mathcal{B}^X \setminus \{\emptyset\}, \rho \in N^e(B)$ let $F \in \rho \cap \mathcal{B}^X$, hence by definition $t_Y(e[B]) \cap t_Y(e[F]) \neq \emptyset$ holds, so that we can choose $y_F \in t_Y(e[B]) \cap t_Y(e[F])$. Now, we put $t(y_F) := \{A \subset X : y_F \in t_Y(e[A])\}$, hence $F \in t(y_F) \cdot t(y_F)$ is a B-bunch in N^e, since $\emptyset \notin t(y_F)$, and for $A_1 \cup A_2 \in t(y_F)$ we have $y_F \in t_Y(A_1 \cup A_2) = t_Y(A_1) \cup t_Y(A_2)$, showing that $A_1 \in t(y_F)$ or $A_2 \in t(y_F)$ is valid. If $A_1 \in t(y_F)$ and $A_1 \subset A_2 \subset X$, then $y_F \in t_y(e[A_1])$ is valid with $t_Y(e[A_1]) \subset t_Y(e[A_2])$, and consequently $y_F \in t_Y(e[A_2])$ follows, resulting into $A_2 \in t(y_F)$. By definition $B \in t(y_F)$ holds, and $t(y_F) \in N^e(B)$, because for $A \in t(y_F) \cap \mathcal{B}^X$ we have $y_F \in t_Y(e[A]) \cap t_Y(e[B])$. Now, let $A \in t(y_F)$ and $A \subset cl_{N^e}(F), F \in \mathcal{B}^X$, hence $y_F \in t_y(e[A]) \subset t_Y(e[cl_{N^e}(F)]) = t_Y(e[t_X(F)]) \subset t_Y(e[F])$ follows by applying (btx_1). Consequently, $F \in t(y_F)$ results. At last let $A \in t(y_F) \cap \mathcal{B}^X$ and $F \in t(y_F) \cap \mathcal{B}^X$, then $\{A\} \in N^e(F)$ follows, because $y_F \in t_Y(e[A]) \cap t_Y(e[F])$ is valid. The above arguments are showing that $(X, \mathcal{B}^X, N^e)$ is bunch space. □

Convention 3.9 By ***SYBTEXT*** we denote the full subcategory of ***BTEXT***, whose objects are the symmetric btop-extensions and by ***BUN*** the full subcategory of ***PX-ESD*** whose objects are the bunch spaces.

Theorem 3.10 *Let H : **SYBTEXT** $\longrightarrow$ **BUN** be defined by*

(a) *for a **SYBTEXT**-object $E := (e, \mathcal{B}^X, Y)$ we put $H(E) := (X, \mathcal{B}^X, N^e)$;*

(b) *for a* ***BTEXT****-morphism* $(f,g) : E \longrightarrow E'$ *we put* $H(f,g) := f$.

Then H : ***SYBTEXT*** $\longrightarrow$ ***BUN*** *is a functor.*

Proof: We already know that the image of H lies in ***BUN***. Now, let $(f,g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ be a ***BTEXT***-morphism: it has to be shown that f is bibsd-map.

By hypothesis f is bi-bounded. Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\rho \in N^e(B)$, we have to verify that $f\rho \in N^{e'}(f[B])$. For showing this statement let $A \in f\rho \cap \mathcal{B}^{X'}$, then we claim $t_{Y'}(e'[f[B]]) \cap t_{Y'}(e'[A]) \neq \emptyset$, which would prove our assertion. We have $A \in \mathcal{B}^{X'}$ with $A = f[F]$ for some $F \in \rho$. By hypothesis we get $t_Y(e[B]) \cap t_Y(e[F]) \neq \emptyset$. Note, that F is also an element of $\mathcal{B}^X$, since $F \subset f^{-1}[f[F]] = f^{-1}[A] \in \mathcal{B}^X$ is valid, and f is bi-bounded. Now, we can choose an element $y \in t_Y(e[B]) \cap t_Y(e[F])$. Consequently, $g(y) \in g[t_Y(e[B])] \cap g[t_Y(e[F])]$ follows.

But the proposed diagram (see 3.1) commutes so that $t_{Y'}(g[e[B]]) = t_{Y'}(e'[f[B]])$ and $t_{Y'}(e'[A]) = t_{Y'}(g[e[F]] = t_{Y'}(e'[f[F]])$ are valid, which put an end of this. Evidently, H fulfills the remaining properties for being a functor. $\square$

4 Strict bornotopological extensions

In the previous section we have found a functor H from ***SYBTEXT*** to ***BUN***. Now, we are going to introduce a related one in the *opposite* direction.

Lemma 4.1 *Let* $(X, \mathcal{B}^X, N)$ *be a proximal esd-space. We set:* $X^b := \{\sigma \subset \underline{P}X : \sigma$ *is* B*-bunch in* N *for some* $B \in \mathcal{B}^X \backslash \{\emptyset\}\}$*, and for each* $A^b \subset X^b$ *we put:* $t_{X^b}(A^b) := \{\sigma \in X^b : \triangle A^b \subset \sigma\}$*, where* $\triangle A^b := \{F \in \mathcal{B}^X : \forall \sigma \in A^b F \in \sigma\}$*. (By* convention *$\triangle A^b = \mathcal{B}^X$ if $A^b = \emptyset$). Then $t_{X^b} : \underline{P}X^b \longrightarrow \underline{P}X^b$ is a topological closure operator.*

Proof: Firstly, we note that $t_{X^b}(\emptyset) = \emptyset$, since $\emptyset \notin \sigma$ for each $\sigma \in X^b$. Now, let A^b be a subset of X^b and consider $\sigma \in A^b$. Then $F \in \triangle A^b$ implies $F \in \sigma$, hence $A^b \subset t_{X^b}(A^b)$ is valid. If$A_1^b \subset A_2^b$, then $\triangle A_2^b \subset \triangle_1^b$ implying $t_{X^b}(A_1^b) \subset t_{X^b}(A_2^b)$. For arbitrary subsets $A_1^b, A_2^b \subset X^b$ we consider an element $\sigma \in X^b$ such that $\sigma \notin t_{X^b}(A_1^b) \cup t_{X^b}(A_2^b)$. Then we get $\triangle A_1^b \not\subset \sigma$ and $\triangle A_2^b \not\subset \sigma$. We can choose $F_1 \in \triangle A_1^b$ with $F_1 \notin \sigma$ and $F_2 \in \triangle A_2^b$ with $F_2 \notin \sigma$. By (bun$_2$) we get $F_1 \cup F_2 \notin \sigma$. On the other hand $F_1 \cup F_2 \in \mathcal{B}^X$, since $\mathcal{B}^X$ is bornology, and $F_1 \cup F_2 \in \triangle A_1^b \cap \triangle A_2^b \subset \triangle(A_1^b \cup A_2^b)$ imply $\sigma \notin t_{X^b}(A_1^b \cup A_2^b)$. At last, let σ be an element of $t_{X^b}(t_{X^b}(A^b))$, $A^b \subset X^b$, and suppose $\sigma \notin t_{X^b}(A^b)$. We can choose $F \in \triangle A^b$, with $F \notin \sigma$. By assumption we have $\triangle t_{X^b}(A^b) \subset \sigma$, hence $F \notin \triangle t_{X^b}(A^b)$. Consequently, there exists $\sigma_1 \in t_{X^b}(A^b)$ with $F \notin \sigma_1$. But this implies $\triangle A^b \subset \sigma_1$, and $F \in \sigma_1$ results, which contradicts. $\square$

Theorem 4.2 *For proximal esd-spaces* $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ *let* $f : X \longrightarrow Y$ *be a bibsd-map. Define a function* $f^b : X^b \longrightarrow Y^b$ *by setting for each* $\sigma \in X^b$: $f^b(\sigma) := \{D \subset Y : f^{-1}[cl_M(D)] \in \sigma\}$. *Then the following statements are valid:*

(1) f^b *is a continuous map from* (X^b, t_{X^b}) *to* (Y^b, t_{Y^b});

(2) *the composites* $f^b \circ e_X$ *and* $e_Y \circ f$ *coincide, where* $e_X : X \longrightarrow X^b$ *denotes that function which assigns the* $\{x\}$*-bunch* x_N *to each* $x \in X$.

Proof: First, let σ be a B-bunch in N. We will show that $f^b(\sigma)$ is a $f[B]$-bunch in M. It is easy to verify that $f^b(\sigma)$ satisfies the conditions (bun_1) and (bun_2), respectively (see 3.4). In order to establish (bun_3) we observe that $B \in \sigma \in N(B)$ is valid by hypothesis. Since $cl_M(f[B]) \supset f[B]$ we have $f^{-1}[cl_M(f[B])] \supset f^{-1}[f[B]] \supset B$. Then $f[B] \in f^b(\sigma)$ results by applying (bun_1). In showing $f^b(\sigma) \in M(f[B])$, we will verify that $\{cl_M(D) : D \in f^b(\sigma)\} << f\sigma$ (note, that f is satisfying (sd) in definition 1.1). For any $D \in f^b(\sigma)$ we have $f^{-1}[cl_M(D)] \in \sigma$, hence $cl_M(D) \supset f[f^{-1}[cl_M(D)]] \in f\sigma$. By applying ($esd_6$) we obtain the desired result. Now, let $D \in f^b(\sigma)$ and $D \subset cl_M(F), F \in \mathcal{B}^Y$. We have to show that $f^{-1}[cl_M(F)] \in \sigma$. By hypothesis $f^{-1}[cl_M(D)] \in \sigma$ is valid. $f^{-1}[cl_M(F)] \in \mathcal{B}^X$ holds by applying (esd_8) and since f is bi-bounded. Consequently, $f^{-1}[cl_M(D)] \subset cl_N(f^{-1}[cl_M(D)]) \subset cl_N(f^{-1}[cl_M(F)])$ follows, leading us to the desired result by applying (bun_4) for σ. At last let $D \in f^b(\sigma) \cap \mathcal{B}^Y$. For $F \in f^b(\sigma) \cap \mathcal{B}^Y$ we have to show that $\{D\} \in M(F)$ is valid. Since M is proximal, therefore it suffices to prove $\{F\} \in M(D)$. By hypothesis $f^{-1}[cl_M(D)] \in \sigma \cap \mathcal{B}^X$, note that f is bi-bounded. On the other hand if $F \in f^b(\sigma) \cap \mathcal{B}^Y$ we also have $f^{-1}[cl_M(F)] \in \sigma \cap \mathcal{B}^X$. But σ satisfies (bun_5), hence $\{f^{-1}[cl_M(F)]\} \in N(f^{-1}[cl_M(D)])$ is valid. Consequently, $\{cl_M(F)\} \in M(cl_M(D))$ follows, since f satisfies (sd) and by applying (esd_5). But then $\{F\} \in M(D)$ results according to (esd_6) and proposition 2.10. Taking all these facts into account we conclude that $f^b(\sigma)$ defines a $f[B]$-bunch in M, and thus $f^b(\sigma) \in Y^b$ is valid.

to (1): Let $A^b \subset X^b, \sigma \in t_{X^b}(A^b)$ and suppose $f(\sigma) \notin t_{Y^b}(f^b[A^b])$. Then $\triangle f^b[A^b] \not\subset f^b(\sigma)$, hence $D \notin f^b(\sigma)$ for some $D \in \triangle f^b[A^b]$, which means $f^{-1}[cl_M(D)] \notin \sigma$. But $\triangle A^b \subset \sigma$ implies $f^{-1}[cl_M(D)] \notin \sigma_1$ for some $\sigma_1 \in A^b$. Consequently, $D \notin f^b(\sigma_1)$ results, which contradicts, because $D \in \triangle f^b[A^b]$ is valid.

to (2): Now, let x be an element of X. We will prove the validity of the equation $f^b(e_X(x)) = e_Y(f(x))$. To this end let $D \in e_Y(f(x))$. Then $f(x) \in cl_M(D)$ follows, and $x \in f^{-1}[cl_M(D)]$ is valid. Consequently, $f^{-1}[cl_M(D)] \in x_N = e_X(x)$ holds, and $D \in f^b(e_X(x))$ results, proving the inclusion $e_Y(f(x)) \subset f^b(e_X(x))$. Conversely, we note that $\{cl_M(D) : D \in f^b(e_X(x))\} << fe_X(x) \in M(\{f(x)\})$, since by supposition f satisfies (sd). But $e_Y(f(x))$ is maximal in $M(\{f(x)\}) \setminus \{\emptyset\}$, and thus we obtain the desired result.

□

Theorem 4.3 *We obtain a functor G : **BUN** to **SYBTEXT** by setting:*

(a) *$G(X, \mathcal{B}^X, N) := (e_X, \mathcal{B}^X, X^b)$ for any bunch space $(X, \mathcal{B}^X, N)$ with $X := (X, cl_N)$ and $X^b := (X^b, t_{X^b})$;*

(b) *$G(f) := (f, f^b)$ for any bibsd-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$.*

Proof: With respect to (esd$_6$), cl_N is topological closure operator, and by Lemma 4.1 this also holds for t_{X^b}. Therefore we get topological spaces with bornology $\mathcal{B}^X$, and $e_X : X \longrightarrow X^b$ is a map according to theorem 4.2. Moreover, e_X is a function satisfying (btx$_1$) and (btx$_2$), respectively.

To establish (btx$_1$) let $B \in \mathcal{B}^X$ and suppose $x \in cl_N(B)$. Then we get $\triangle e_X[B] \subset x_N$, hence $e_X(x) \in t_{X^b}(e_X[B])$, which means $x \in e_X^{-1}[t_{X^b}(e_X[B])$. Conversely, let x be an element of $e_X^{-1}[t_{X^b}(e_X[B])]$. Then by definition we have $\triangle e_X[B] \subset x_N$. Since $B \in \triangle e_X[B]$ we get $x \in cl_N(B)$. To establish (btx$_2$) let $\sigma \in X^b$ and suppose $\sigma \notin t_{X^b}(e_X[X])$. By definition we get $\triangle e_X[X] \not\subset \sigma$, so that there exists a set $F \in \triangle e_X[X]$ with $F \notin \sigma$. But then $X \subset cl_N(F)$ follows. Since $B \in \sigma$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$ we get $B \subset cl_N(F)$, hence $F \in \sigma$, because σ is satisfying (bun$_4$). But this contradicts, and $\sigma \in t_{X^b}(e_X[X])$ is valid. Moreover, we have that f and f^b are continuous maps (see also theorem 4.2), and the diagram

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "f"'] & X^b \arrow[d, "f^b"] \\ Y \arrow[r, "e_Y"] & Y^b \end{tikzcd} \quad \text{commutes.}$$

Finally, this establishes that the composition of bibsd-maps is preserved by G. In showing $(e_X, \mathcal{B}^X, X^b)$ is symmetric, let x be an element of X such that $\sigma \in t_{X^b}(\{e_X(x)\})$. We have to prove $x_N \in t_{X^b}(\{\sigma\})$. By hypothesis we have $x_N \cap \mathcal{B} \subset \sigma$ and must show that $\triangle\{\sigma\} \subset x_N$. To this end let $F \in \triangle\{\sigma\}$, hence $F \in \sigma \cap \mathcal{B}^X$ follows. We already know that $\{x\} \in \sigma$ is valid, and consequently $\{F\} \in N(\{x\})$ follows by applying (bun$_5$). But this implies $x \in cl_N(F)$, and $F \in x_N$ results. At last we will show that the image of G also is contained in ***ST-SYBTEXT*** the full subcategory of ***SYBTEXT***, whose objects are the *strict* symmetric bornotopological extensions. □

Corollary 4.4 *The image of G is contained in **ST-SYBTEXT**.*

Proof: Consider $\sigma \notin X^b$ and let A^b be closed in X^b with $\sigma \notin A^b$. Then $\sigma \notin t_{X^b}(A^b)$, hence $\triangle A^b \not\subset \sigma$. We can find some $F \in \triangle A^b$ such that $F \notin \sigma$. Now, for each $\sigma_1 \in A^b$ we have $F \in \sigma_1$, which implies $\triangle e_X[F] \subset \sigma_1$, because $D \in \triangle e_X[F]$ implies $F \subset cl_N(D)$ with $D \in \mathcal{B}^X$, and σ_1 satisfies (bun$_4$). Therefore we conclude $\sigma_1 \in t_{X^b}(e_X[F])$, and $A^b \subset t_{X^b}(e_X[F])$

results. On the other hand, since $F \notin \sigma$ we have $\triangle e_X[F] \not\subset \sigma$, hence $\sigma \notin t_{X^b}(e_X[F])$, and $t_{X^b}(e_X[F]) \subset A^b$ results, which put an end of this. $\square$

Theorem 4.5 *Let $H : \textbf{SYBTEXT} \longrightarrow \textbf{BUN}$ and $G : \textbf{BUN} \longrightarrow \textbf{SYBTEXT}$ be the above defined functors. For each object $(X, \mathcal{B}^X, N)$ of $\textbf{BUN}$ let $t_{(\mathcal{B}^X,N)}$ denote the identity map $id_X : H(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$. Then $t : H \circ G \longrightarrow 1_{\textbf{BUN}}$ is* natural equivalence *from $H \circ G$ to the identity functor $1_{\textbf{BUN}}$, i.e. $id_X : H(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$ is bibsd-map in* both *directions for each object $(X, \mathcal{B}^X, N)$, and the following diagram commutes for each bibsd-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$:*

$$\begin{array}{ccc} H(G(X,\mathcal{B}^X,N)) & \xrightarrow{id_X} & (X,\mathcal{B}^X,N) \\ {\scriptstyle H(G(f))}\downarrow & & \downarrow{\scriptstyle f} \\ H(G(Y,\mathcal{B}^Y,M)) & \xrightarrow{id_Y} & (Y,\mathcal{B}^Y,M) \end{array}$$

Proof: The commutativity of the diagram is obvious, because of $H(G(f)) = f$. It remains to prove that $id_X : H(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$ is bibsd-map in *both* directions. Since $H(G(X, \mathcal{B}^X, N)) = (X, \mathcal{B}^X, N^{ex})$ by definition of G respectively H, it suffices to show that for each $B \in \mathcal{B}^X \setminus \{\emptyset\}$ we have $N^{ex}(B) \subset N(B) \subset N^{ex}(B)$. To this end assume $\rho \in N^{ex}(B), B \neq \emptyset$. Then $t_{X^b}(e_X[B]) \in \sec\{t_{X^b}(F) : F \in \rho \cap \mathcal{B}^X\}$. Now, we will show that $\rho \cap \mathcal{B}^X$ is subset of $\bigcup\{\mathcal{A} : \mathcal{A} \in N(B)\}$. Note, that $(X, \mathcal{B}^X, N)$ is conic by assumption. $F \in \rho \cap \mathcal{B}^X$ implies the existence of $\sigma \in t_{X^b}(B) \cap t_{X^b}(F)$, hence $\triangle e_X[B], \triangle e_X[F] \subset \sigma$ are valid. Consequently, $B, F \in \sigma \cap \mathcal{B}^X$ result, and $\{F\} \in N(B)$ follows, since σ satisfies (bun_5). Consequently, $F \in \bigcup\{\mathcal{A} : \mathcal{A} \in N(B)\}$ is valid, showing that $\rho \cap \mathcal{B}^X \in N(B)$. But then $\rho \in N(B)$ follows by applying (esd_9). Conversely, let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\rho \in N(B)$. We have to verify $t_{X^b}(e_X[B]) \in \sec\{t_{X^b}(F) : F \in \rho \cap \mathcal{B}^X\} \cdot F \in \rho \cap \mathcal{B}^X$ implies the existence of a B-bunch σ in N with $F \in \sigma$, according to (bun). Now, we claim that the following statements are valid, i.e.

(a) $\sigma \in t_{X^b}(e_X[B])$;

(b) $\sigma \in t_{X^b}(e_X[F])$.

to (a): We have to check that the inclusion $\triangle e_X[B] \subset \sigma$ is valid. $A \in \triangle e_X[B]$ implies $B \subset cl_N(A)$. Since $B \in \sigma$ we get $cl_N(A) \in \sigma$, and $A \in \sigma$ results, according to (bun_4). Note, that $A \in \mathcal{B}^X$ by definition.

to (b): We must show that the inclusion $\triangle e_X[F] \subset \sigma$ is valid. But by hypothesis we know that $F \in \sigma$ holds, hence this proving is as above.

$\square$

Corollary 4.6 *For a btop-T_1 extension $E := (e, \mathcal{B}^X, Y)$, where e is topological embedding and Y T_1-space, then $(X, \mathcal{B}^X, N^e)$ is separated by satisfying*

(sep) $x, z \in X$ *and* $\{\{z\}\} \in N^e(\{x\})$ *imply* $x = z$.

Proof: For $x, z \in X$ with $\{\{z\}\} \in N^e(\{x\})$ there exists $y \in t_Y(\{e(x)\}) \cap t_Y(\{e(z)\})$. By hypothesis $e(x) = y = e(z)$ follows, and $x = z$ results, because e is injective. $\square$

Corollary 4.7 *For a separated proximal esd-space $(X, \mathcal{B}^X, N)$ the function $e_X : X \longrightarrow X^b$ is injective.*

Proof: For $x, z \in X$ let $e_X(x) = e_X(z)$, hence $z \in cl_N(\{x\})$, and $\{\{x\}\} \in N(\{z\})$ follows. By hypothesis $x = z$ results. $\square$

Remark 4.8 In making the main theorem of this paper more *transparent* we state that a proximal esd-space $(X, \mathcal{B}^X, N)$ is a bunch space iff it can be considered as subspace of a topological space Y, such that the B-collections in N for non-empty bounded sets B are characterized by the fact that their closures of bounded members in Y meet the closure of B in Y. In case if $\mathcal{B}^X$ is *saturated*, then proximal esd-spaces *essentially* coincide with LODATO proximity spaces up to isomorphism. Hence the main theorem generalizes the one of LODATO, presented by him in the past and where symmetric generalized proximities are playing an important role, especially those arising from a family of bunches on a set X.

Diagram of used categories

$$
\begin{tikzcd}
 & \boldsymbol{DEN} = \boldsymbol{HN} & & \\
 & \boldsymbol{SN} \arrow[u] & & \\
 & \boldsymbol{ESD} \arrow[u] & & \\
 & \boldsymbol{CON\text{-}ESD} \arrow[u] & & \boldsymbol{N\text{-}SD} \simeq \boldsymbol{b\text{-}NEAR} \arrow[ull] \\
\boldsymbol{b\text{-}TOP} \simeq \boldsymbol{COV\text{-}ESD} \arrow[ur] & & \boldsymbol{PX\text{-}ESD} \simeq \boldsymbol{b\text{-}SPX} \arrow[ul] & \boldsymbol{NEAR} \arrow[u] \\
\boldsymbol{TOP} \arrow[u] & & \boldsymbol{BUN} \arrow[u] \arrow[d, leftrightarrow, "1{,}1\mathrm{rel}"'] & \boldsymbol{LO\text{-}PROX} \arrow[u] \arrow[ul] \\
 & & \odot \arrow[r] & \boldsymbol{ST\text{-}SYBTEXT}
\end{tikzcd}
$$

legend

$\odot \longrightarrow \odot$ subcategory of	
$\boldsymbol{DEN}$	$\hat{=}$ density spaces
$\boldsymbol{HN}$	$\hat{=}$ hypernear spaces
$\boldsymbol{SN}$	$\hat{=}$ supernear spaces

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