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Weighted exponential Dichotomy of the Solutions of linear impulsive differential Equations in a Banach Space

ABSTRACT. In the paper a dependence is established between the ψ-exponential dichotomy of a homogeneous impulsive differential equation in a Banach space and the existence of ψ-bounded solution of the appropriate nonhomogeneous impulsive equation.

KEY WORDS AND PHRASES. Exponential dichotomy for impulsive differential equations, ψ-dichotomy, ψ-boundedness

1 Introduction

The impulsive differential equations are an adequate mathematical apparatus for simulation of numerous processes and phenomena in biology, physics, chemistry and control theory, e.t.c. which during their evolutionary development are subject to short time perturbations in the form of impulses. The qualitative investigation of these processes began with the work of Mil'man and Myshkis [17]. For the first time such equations were considered in an arbitrary Banach space in [2, 3, 18, 19].

The problem of ψ-boundedness and ψ-stability of the solutions of differential equations in finite dimensional Euclidean spaces, introduced for the first time by Akinyele [1] has been studied later by many authors. A beautiful explanation about the benefits of such a use of weighted stability and boundedness can be found for example in [15].

Inspired by the famous monographs of Coppel [6], Daleckii and Krein [7] as well as Massera and Schaeffer [16], where the important notion of exponential and ordinary dichotomy for ordinary differential equations is considered in details, Diamandescu [8]-[10] and Boi [4]-[5] introduced and studied the ψ-dichotomy for linear differential equations in finite dimensional Euclidean space, where ψ is a nonnegative continuous diagonal matrix function. The concept of ψ-dichotomy for arbitrary Banach spaces is introduced and studied in [11] and [12]. In this case $\psi(t)$ is an arbitrary bounded invertible linear operator for all t.

The goal of the present paper is to study such a weighted dichotomy for linear differential equations with impulse effect in arbitrary Banach spaces. We will establish a dependence between the ψ-exponential dichotomy of a homogeneous impulsive equation in a Banach space and the existence of a solution of the corresponding nonhomogeneous impulsive equation which is ψ-bounded on the semi-axis R_+.

The first investigation in this direction was made in [20] for the particular case of ψ-ordinary dichotomy.

It must be mentioned that in [13, 14] the attempt to introduce ψ-exponential dichotomy for impulsive differential equations in finite dimensional spaces is a real disaster - due to the meaningless use of the fundamental matrix there even the definitions are wrong.

2 Preliminaries

Let X be an arbitrary Banach space with norm $|.|$ and let $LB(X)$ be the space of all linear bounded operators acting in X with the norm $||.||$ and identity I. Denote $R_+ = [0, \infty)$.

We consider the nonhomogeneous impulsive equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t) \quad (t \neq t_n) \tag{1}$$

$$x(t_n + 0) = Q_n x(t_n) + h_n \quad (n = 1, 2, 3, ...) \tag{2}$$

where the operator valued function $A(.) : R_+ \to LB(X)$ and the function $f(.) : R_+ \to X$ are strongly measurable and Bochner integrable on the finite subintervals of R_+, $\{Q_n\}_{n=1}^{\infty}$ is a sequence of impulsive operators $Q_n \in LB(X)$ $(n = 1, 2, 3, ...)$, $T = \{t_n\}_{n=1}^{\infty}$ is a sequence of points on the semi-axis R_+ satisfying the condition

$$0 < t_1 < t_2 < ..., \ \lim_{n\to\infty} t_n = \infty$$

and $\{h_n\}_{n=1}^{\infty}$ is a sequence of elements $h_n \in X$. The corresponding homogeneous linear impulsive equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \quad (t \neq t_n) \tag{3}$$

$$x(t_n + 0) = Q_n x(t_n) \quad (t = 1, 2, 3, ...). \tag{4}$$

Definition 1 *By a solution of the impulsive equation (1), (2) (or (3), (4)) we shall call a function $x(t)$ which for $t \neq t_n$ satisfies equation (1) (or (3)), for $t = t_n$ satisfies condition (2) (or (4)) and is continuous from the left.*

It is known (see [18], [3]) that for the impulsive equation (3), (4) there exists an evolutionary Cauchy operator associating with any element $\xi \in X$ a solution $x(t)$ of the impulsive equation which satisfies the initial condition $x(s) = \xi$ $(0 \le s \le t < \infty)$.

Lemma 1 ([3]) *Let the conditions $A(t), Q_n \in LB(X)$ hold, where $t \in R_+$ $(n = 1, 2, \dots)$. Then the evolutionary operator $V(t,s)$ $(0 \le s \le t < \infty)$ of the impulsive equation (3), (4) has the form*

$$V(t,s) = \begin{cases} V_0(t,s), \ t_n < s \le t \le t_{n+1} \\ V_0(t,t_n)\left(\prod_{j=n}^{k+1} Q_j V_0(t_j, t_{j-1})\right) Q_k V_0(t_k, s), \\ t_{k-1} < s \le t_k < t_n < t \le t_{n+1} \end{cases}$$

where $V_0(t,s)$ $(0 \le s \le t < \infty)$ is the evolutionary operator of equation (3).

The operator-valued function $V(t,s)$ satisfies the equalities

$$V(t,t) = I \ (0 \le t < \infty), \tag{5}$$

$$V(t,s) = V(t,\tau)V(\tau,s) \ (0 \le s \le \tau \le t < \infty). \tag{6}$$

Moreover, it is differentiable at the points $t \in (t_{j-1}, t_j]$ $(j = 1,2,3,\dots)$ and $s \in [t_{j-1}, t_j)$ $(j = 1,2,3,\dots)$, and it is

$$\frac{\mathrm{d}V(t,s)}{\mathrm{d}t} = A(t)V(t,s), \quad \frac{\mathrm{d}V(t,s)}{\mathrm{d}s} = V(t,s)A(s). \tag{7}$$

At the points t_n $(n = 1,2,3,\dots)$ the following equalities are staisfied:

$$V(t_n + 0, s) = Q_n V(t_n, s) \ (0 \le s \le t_n < \infty). \tag{8}$$

Lemma 2 ([3]) *Let the following conditions hold:*

1. $A(t), Q_n \in LB(X)$, where $t \in R_+ (n = 1, 2, \dots)$.

2. The operators Q_n have continuous inverses Q_n^{-1} $(n = 1,2,3,\dots)$.

Then the evolutionary operator $V(t,s)$ $(0 \le t, s < \infty)$ of the impulsive equation (3), (4) has the form

$$V(t,s) = \begin{cases} V_0(t,s), \ t_n < s, t \le t_{n+1} \\ V_0(t,t_n)\left(\prod_{j=n}^{k+1} Q_j V_0(t_j, t_{j-1})\right) Q_k V_0(t_k, s), \\ t_{k-1} < s \le t_k < t_n < t \le t_{n+1} \\ V_0(t,t_n)\left(\prod_{j=n}^{k-1} Q_j^{-1} V_0(t_j, t_{j+1})\right) Q_k^{-1} V_0(t_k, s), \\ t_{n-1} < t \le t_n < t_k < s \le t_{k+1} \end{cases}$$

where $V_0(t,s)$ $(0 \le s,t < \infty)$ *is the evolutionary operator of the equation* (3).

If the conditions of Lemma 2 are satisfied, then the following equalities hold:

$$V(t,s) = V^{-1}(s,t), \quad V(t,s) = V(t,\tau)V(\tau,s) \ (0 \le s,\tau,t < \infty), \tag{9}$$

$$V(t_n+0,s) = Q_n V(t_n,s) \ (0 \le s,t_n < \infty). \tag{10}$$

Let $RL(X)$ be the subspace of all invertible operators in $LB(X)$ whose inverse operators are bounded, too. Let $\psi(t) : R_+ \to RL(X)$ be a continuous operator-function with respect to $t \in R_+$.

Definition 2 *A function* $u(.) : R_+ \to X$ *is said to be* ψ*-bounded on* R_+ *if* $\psi(t)u(t)$ *is bounded on* R_+.

Definition 3 *A function* $f(.) : R_+ \to X$ *is said to be* ψ*-integrally bounded on* R_+ *if it is measurable and there exists a positive constant* m *such that* $\int_t^{t+1} |\psi(\tau)f(\tau)|d\tau \le m$ *for all* $t \in R_+$.

Definition 4 *A sequence of points* $h = \{h_n\}_{n=1}^{\infty}$ *is said to be* ψ*-bounded on* R_+ *if* $\sup\limits_{n=1,2,3,\ldots} |\psi(t_n)h_n| < \infty, h_n \in X, t_n \in T (n = 1,2,3,\ldots)$.

Let $C_\psi(X,T)$ denote the space of all functions with values in X and ψ-bounded on R_+ which are continuous for $t \ne t_n$, have discontinuities of the first kind for $t = t_n$ and are continuous from the left which is a Banach space with the norm

$$|||f|||_{C_\psi} = \sup_{t \in R_+} |\psi(t)f(t)|.$$

Let $M_\psi(X,T)$ denote the Banach space of all functions with values in X and ψ-integrally bounded which are continuous for $t \ne t_n$, have discontinuities of the first kind for $t = t_n$ and are continuous from the left for $t = t_n$ with the norm

$$|||f|||_{M_\psi} = \sup_{t \in R_+} \int_t^{t+1} |\psi(s)f(s)|\mathrm{d}s.$$

Let $H_\psi(X,T)$ denote the space of all ψ-bounded sequences $h = \{h_n\}_{n=1}^{\infty}$ in X, i.e.

$$H_\psi(X,T) = \{h : \sup_{n=1,2,3,\ldots} |\psi(t_n)h_n| < \infty, h_n \in X, t_n \in T, n = 1,2,3,\ldots\}$$

with the norm

$$|||h|||_{H_\psi} = \sup_{n=1,2,3,\ldots} |\psi(t_n)h_n|.$$

Definition 5 *The homogeneous impulsive equation (3), (4) is said to be ψ-exponential dichotomous on R_+ if there exist a pair P_1 and $P_2 = I - P_1$ of mutually complementary projections in X and numbers $M, \delta > 0$ for which the inequalities*

$$||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)|| \leq Me^{-\delta(t-s)} \qquad (0 \leq s \leq t < \infty), \tag{11}$$

$$||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)|| \leq Me^{-\delta(s-t)} \qquad (0 \leq t \leq s < \infty) \tag{12}$$

hold, where $V(t) = V(t,0)$ and $V(t,s)$ $(0 \leq s,t < \infty)$ is the Cauchy evolutionary operator of the impulsive equation (3), (4).

The equation (3), (4) is said to have a ψ-ordinary dichotomy on R_+ if (11) and (12) hold with $\delta = 0$.

Lemma 3 *Equation (3), (4) has a ψ-exponential dichotomy on R_+ with positive constants ν_1 and ν_2 if and only if there exist a pair of mutually complementary projections P_1 and $P_2 = I - P_1$ and positive constants $M, \tilde{N}_1, \tilde{N}_2$ such that following inequalities are fulfilled:*

$$|\psi(t)V(t)P_1\xi| \leq \tilde{N}_1e^{-\nu_1(t-s)}|\psi(s)V(s)P_1\xi| \quad (\xi \in X, 0 \leq s \leq t), \tag{13}$$

$$|\psi(t)V(t)P_2\xi| \leq \tilde{N}_2e^{-\nu_2(s-t)}|\psi(s)V(s)P_2\xi| \quad (\xi \in X, 0 \leq t \leq s), \tag{14}$$

$$\|\psi(t)V(t)P_1V^{-1}(t)\psi^{-1}(t)\| \leq M \quad (t \geq 0). \tag{15}$$

The proof of the lemma is similar as the proof of Lemma 3.1 in [11] for equations without impulses and will be omitted.

Definition 6 *The homogeneous impulsive equation (3), (4) is said to have a ψ-bounded growth on R_+ if for some fixed $l > 0$ there exists a constant $c \geq 1$ such that every solution $x(t)$ of (3), (4) satisfies*

$$|\psi(t)x(t)| \leq c|\psi(s)x(s)| \quad (0 \leq s \leq t \leq s + l). \tag{16}$$

Lemma 4 *Equation (3), (4) has ψ-bounded growth on R_+ if and only if there exist positive constants $K \geq 1$ and $\alpha > 0$ such that*

$$\|\psi(t)V(t)V^{-1}(s)\psi^{-1}(s)\| \leq Ke^{\alpha(t-s)} \quad (0 \leq s \leq t). \tag{17}$$

The proof of the lemma is similar as the proof of Lemma 3.2 in [11] for equations without impulses and will be omitted.

Remark 1 It is easy to see that the condition for ψ-bounded growth (and for bounded growth) of (3), (4) is independent of the choice of l. Hence we will use the Definition 6 with fixed $l = 1$.

Lemma 5 *If (3), (4) has ψ-bounded growth on R_+, then (15) is a consequence of (13) and (14).*

The proof of the lemma is similar as the proof of Lemma 3.5 in [11] for equations without impulses and will be omitted.

3. Main results

We shall say that condition (H) is satisfied if the following conditions hold:

H1. $A(t), Q_n \in LB(X)$, where $t \in R_+ (n = 1, 2, 3, \dots)$.

H2. $Q_n \in RL(X)$ $(n = 1, 2, 3, \dots)$.

H3. $\psi(t) : R_+ \to RL(X)$ is a continuous operator-function with respect to $t \in R_+$.

Theorem 2 *Let us assume the following:*

1. Condition (H) is satisfied.

2. Equation (3), (4) is ψ-exponential dichotomous.

3. There exist a number $l > 0$ and a positive integer λ such that each interval on R_+ with length l contains not more than λ points of the sequence T.

Then for any function $f \in C_\psi(X, T)$ and any sequence $h \in H_\psi(X, T)$ there exists a solution of the nonhomogeneous equation (1), (2) which is ψ-bounded on R_+.

Proof. Consider the function

$$\begin{aligned}\tilde{x}(t) = &\int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s \\ &+ \sum_{t_j<t} \psi(t)V(t)P_1V^{-1}(t_j+0)h_j - \sum_{t_j\geq t} \psi(t)V(t)P_2V^{-1}(t_j+0)h_j\end{aligned} \tag{18}$$

In order to prove the boundedness of $\tilde{x}(t)$ we shall estimate the norms of the summands in (18). By (11) and (12) we have

$$\begin{aligned}&|\int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s| = \\ &\quad = |\int_0^t \psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\ &\quad \leq \int_0^t ||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s \\ &\quad \leq Me^{-\delta t}\int_0^t e^{\delta s}\mathrm{d}s\ |||f|||_{C_\psi} \leq \frac{M}{\delta}|||f|||_{C_\psi}\end{aligned} \tag{19}$$

and

$$
\begin{aligned}
&|\int_t^{\infty} \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s| \\
&\quad = |\int_t^{\infty} \psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\
&\quad \leq \int_t^{\infty} ||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s \\
&\quad \leq Me^{\delta t}\int_t^{\infty} e^{-\delta s}\mathrm{d}s\ |||f|||_{C_\psi} \leq \frac{M}{\delta}|||f|||_{C_\psi}.
\end{aligned}
\tag{20}
$$

Analogously having in mind also the conditions 3 and H3 we obtain for the next summands

$$
\begin{aligned}
&|\sum_{t_j<t}\psi(t)V(t)P_1V^{-1}(t_j+0)h_j| \\
&\quad = |\sum_{t_j<t}\psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j+0)h_j| \\
&\quad = |\sum_{t_j<t}\psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j)h_j| \\
&\quad \leq \sum_{t_j<t}||\psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)||\ |\psi(t_j)h_j| \\
&\quad \leq M\left(\sum_{t_j<t}e^{\delta(t_j-t)}\right)|||h|||_{H_\psi} \leq \frac{M\lambda}{1-e^{-\delta l}}\ |||h|||_{H_\psi}
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
&|\sum_{t\leq t_j}\psi(t)V(t)P_2V^{-1}(t_j+0)h_j| \\
&\quad = |\sum_{t\leq t_j}\psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j+0)h_j| \\
&\quad = |\sum_{t\leq t_j}\psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j)h_j| \\
&\quad \leq \sum_{t\leq t_j}||\psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)||\ |\psi(t_j)h_j| \\
&\quad \leq M\left(\sum_{t\leq t_j}e^{\delta(t-t_j)}\right)|||h|||_{H_\psi} \leq \frac{M\lambda}{1-e^{-\delta l}}\ |||h|||_{H_\psi}.
\end{aligned}
\tag{22}
$$

From (18) - (22) it follows that $\tilde{x}(t)$ is bounded on R_+ and satisfies for $t\in R_+$ the inequality

$$|\tilde{x}(t)| \leq \frac{2M}{\delta}|||f|||_{C_\psi} + \frac{2M\lambda}{1-e^{-\delta l}}\ |||h|||_{H_\psi}$$

Let be $x(t)=\psi^{-1}(t)\tilde{x}(t)$. Obviously $x(t)$ is ψ-bounded on R_+. It is immediately verified that the function $x(t)$ is continuous for $t\neq t_n$ and that the limit values $x(t_n+0)$ $(n=1,2,...)$

exist. We shall show that the function $x(t)$ satisfies the impulsive equation (1), (2) using the equalities (7) and (10).

We differentiate $x(t)$ by $t \neq t_n$ and get

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= A(t)\int_0^t V(t)P_1V^{-1}(s)f(s)\mathrm{d}s + V(t)P_1V^{-1}(t)f(t)\\ &+ V(t)P_2V^{-1}(t)f(t) - A(t)\int_t^\infty V(t)P_2V^{-1}(s)f(s)\mathrm{d}s\\ &+ \sum_{t_j<t} A(t)V(t)P_1V^{-1}(t_j+0)h_j - \sum_{t_j\geq t} A(t)V(t)P_2V^{-1}(t_j+0)h_j\\ &= A(t)x(t) + V(t)P_1V^{-1}(t)f(t) + V(t)P_2V^{-1}(t)f(t)\\ &= A(t)x(t) + f(t).\end{aligned}$$

Analogously we obtain for $t = t_n \quad (n = 1, 2, ...)$ taking into account (10)

$$\begin{aligned}&x(t_n+0)\\ &\quad= \int_0^{t_n} V(t_n+0)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_{t_n}^\infty V(t_n+0)P_2V^{-1}(s)f(s)\mathrm{d}s\\ &\quad+ \sum_{t_j\leq t_n} V(t_n+0)P_1V^{-1}(t_j+0)h_j - \sum_{t_j>t_n} V(t_n+0)P_2V^{-1}(t_j+0)h_j\\ &\quad= Q_n\int_0^{t_n} V(t_n)P_1V^{-1}(s)f(s)\mathrm{d}s - Q_n\int_{t_n}^\infty V(t_n)P_2V^{-1}(s)f(s)\mathrm{d}s\\ &\quad+ Q_n\sum_{t_j<t_n} V(t_n)P_1V^{-1}(t_j+0)h_j - Q_n\sum_{t_j\geq t_n} V(t_n)P_1V^{-1}(t_j+0)h_j\\ &\quad+ V(t_n+0)P_1V^{-1}(t_n+0)h_n + V(t_n+0)P_2V^{-1}(t_n+0)h_n\\ &\quad= Q_nx(t_n) + h_n.\end{aligned}$$

Hence the function $x(t)$ is a ψ-bounded solution of the nonhomogeneous impulsive equation (1), (2) on R_+. Theorem 2 is proved. □

Remark 3 Theorem 2 still holds, if the condition $f \in C_\psi(X, T)$ is replaced by the weaker condition $f \in M_\psi(X, T)$.

Proof. In the case $f \in M_\psi(X, T)$ the estimates (19) and (20) can be replaced by the following estimates

$$
\begin{aligned}
&|\int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s| \\
&\quad = |\int_0^t \psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\
&\quad \leq \int_0^t ||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s \\
&\quad \leq M\int_0^t e^{-\delta(t-s)}\ |\psi(s)f(s)|\mathrm{d}s \leq M|||f|||_{M_\psi}\sum_{k=0}^{\infty} e^{-\delta k} \\
&\quad \leq \frac{M}{1-e^{-\delta}}|||f|||_{M_\psi},
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
&|\int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s| \\
&\quad = |\int_t^\infty \psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s| \\
&\quad \leq \int_t^\infty ||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s \\
&\quad \leq M\int_t^\infty e^{-\delta(s-t)}\ |\psi(s)f(s)|\mathrm{d}s \leq M|||f|||_{M_\psi}\sum_{k=0}^{\infty} e^{-\delta k} \\
&\quad \leq \frac{M}{1-e^{-\delta}}|||f|||_{M_\psi}.
\end{aligned} \tag{24}
$$

□

Remark 4 Theorem 2 obviously holds without condition 3 if we consider inhomogeneous equations with $h = 0$. In this case the ψ-bounded solutions lie in the subspace $C^0_\psi(X,T)$ of the space $C_\psi(X,T)$ which consists of the functions satisfying the condition

$$
x(t_n + 0) = Q_n x(t_n) \quad (n = 1, 2, 3, \dots). \tag{25}
$$

Let X_1 be the linear manifold of all $\xi \in X$ for which the functions $V(t)\xi$ $(t \in R_+)$ are ψ-bounded.

For our next main result we will need the following lemma.

Lemma 6 ([20]) *Assume the following:*

1. Condition (H) is satisfied.

2. $B_\psi(X)$ is an arbitrary Banach space of functions $f(.) : R_+ \to X$ and for any function $f \in B_\psi(X)$ the nonhomogeneous equation (1), (2) has at least one ψ-bounded on R_+ solution $x \in C_\psi(X,T)$.

3. The set X_1 is a complementary subspace of X and X_2 is a complement of it ($X_1+X_2 = X$).

Then to each function $f(t) \in B_\psi(X)$ there corresponds a unique solution $x(t)$ which is ψ-bounded on R_+ and starts from X_2, i.e. $x(0) \in X_2$.

This solution satisfies the estimate

$$|||x|||_{C_\psi} \leq k|||f|||_{B_\psi}, \tag{26}$$

where $k > 0$ is a constant not depending on f.

Now we are ready for our second main result - a theorem, which is like an inverse of Theorem 2.

Theorem 5 *Let us assume the following:*

1. Condition (H) is satisfied.

2. The homogeneous impulsive equation (3), (4) has a ψ-bounded growth on R_+.

3. The linear manifold

$$X_1 = \{\xi \in X : \sup_{0 \leq t < \infty} |\psi(t)V(t)\xi| < \infty\} \tag{27}$$

is a complementary subspace (i.e. there exists a subspace X_2 of X for which $X = X_1 + X_2$).

4. For each function $f \in C_\psi(X,T)$ the nonhomogeneous impulsive equation (1), (2) for $h = \{h_n\}_{n=1}^{\infty} = 0$ has at least one solution belonging to the subspace $C_\psi^0(X,T)$.

Then the impulsive equation (3), (4) is ψ-exponential dichotomous.

Proof. Let $x(t)$ be a nontrivial ψ-bounded solution of the impulsive equation (3), (4) with initial value $x(0) \in X_1$. Set

$$y(t) = x(t) \int_0^t \chi(\tau)|\psi(\tau)x(\tau)|^{-1} \mathrm{d}\tau,$$

where

$$\chi(t) = \begin{cases} 1 : & 0 \leq t \leq t_0 + \tau \\ 1 - (t - t_0 - \tau) : & t_0 + \tau < t \leq t_0 + \tau + 1 \\ 0 : & t_0 + \tau + 1 \leq t \end{cases}$$

It is not hard to check that the function $y(t)$ is a solution of the nonhomogeneous impulsive equation (1), (2) for $h = 0$ and for

$$f(t) = \chi(t)\frac{x(t)}{|\psi(t)x(t)|}.$$

Obviously $f \in C_\psi(X,T)$ and $|||f|||_{C_\psi} = 1$. But $y(0) = 0 \in X_2$, and applying Lemma 6 it follows

$$|||y|||_{C_\psi} = \sup_{t \in R_+} |\psi(t)y(t)| \le k|||f|||_{C_\psi} = k$$

from (26). Hence

$$|\psi(t)y(t)| = |\psi(t)x(t)| \int_0^t \chi(s)|\psi(s)x(s)|^{-1}\mathrm{d}s \le k \quad (t \in R_+).$$

By $t = t_0 + \tau$ we obtain the inequality

$$|\psi(t_0+\tau)y(t_0+\tau)| = |\psi(t_0+\tau)x(t_0+\tau)| \int_0^{t_0+\tau} |\psi(s)x(s)|^{-1}\mathrm{d}s \le k. \tag{28}$$

Let consider the function

$$\varphi(t) = \int_0^t |\psi(s)x(s)|^{-1}\mathrm{d}s.$$

From (28) it follows

$$\frac{\varphi'(t_0+\tau)}{\varphi(t_0+\tau)} \ge \frac{1}{k}.$$

After integrating the inequality with respect to τ on $[1,\tau]$ this implies the estimate

$$\varphi(t_0+\tau) \ge \varphi(t_0+1)e^{\frac{(\tau-1)}{k}} \quad (\tau \ge 1). \tag{29}$$

From condition 2 of the theorem it follows for $s \in [t_0, t_0+1]$ that there exists a constant $c > 1$ such that

$$|\psi(s)x(s)| \le c|\psi(t_0)x(t_0)|$$

and that is why

$$\varphi(t_0+1) = \int_{t_0}^{t_0+1} |\psi(s)x(s)|^{-1}\mathrm{d}s \ge c^{-1}|\psi(t_0)x(t_0)|^{-1}.$$

From here, taking into account the estimates (28) and (29) we obtain for $\tau \ge 1$ the relation

$$|\psi(t_0+\tau)x(t_0+\tau)| \le \frac{k}{\varphi(t_0+\tau)} \le \frac{ke^{-\frac{\tau-1}{k}}}{\varphi(t_0+1)} \le kce^{\frac{1}{k}}e^{-\frac{\tau}{k}}|\psi(t_0)x(t_0)|.$$

For $\tau \le 1$ we have

$$|\psi(t_0+\tau)x(t_0+\tau)| \le c|\psi(t_0)x(t_0)| \le ce^{\frac{1-\tau}{k}}|\psi(t_0)x(t_0)|.$$

Hence we obtain the estimate

$$|\psi(t)x(t)| \le Ne^{-\nu(t-t_0)}|\psi(t_0)x(t_0)|, \tag{30}$$

where $\nu = \frac{1}{k}$ and $N = \max\{ce^{\frac{1}{k}}, kce^{\frac{1}{k}}\}$, i.e. the inequality (13).

Analogously we consider the case if the solution $x(t)$ of the impulsive equation (3), (4) has an initial value $x(0) \in X_2$. Then we will consider the function

$$\tilde{y}(t) = x(t) \int_t^\infty \chi(s)|\psi(s)x(s)|^{-1} \mathrm{d}s$$

instead of $y(t)$. It is easy to check that the function $\tilde{y}(t)$ is a solution of the nonhomogeneous impulsive equation (1), (2) for $h = 0$ and for

$$\tilde{f}(t) = -\chi(t)\frac{x(t)}{|\psi(t)x(t)|}.$$

The solution $\tilde{y}(t)$ is ψ-bounded because $\tilde{y}(t) = 0$ for $t \geq t_0 + \tau + 1$. But $\tilde{y}(0) \in X_2$ and obviously $\tilde{f} \in C_\psi(X, T)$. Now we can apply Lemma 6, and from (26), taking into account that $|||\tilde{f}|||_{C_\psi} = 1$, it follows

$$|\psi(t)\tilde{y}(t)| = |\psi(t)x(t)| \int_t^\infty \chi(s)|\psi(s)x(s)|^{-1} \mathrm{d}s \leq k|||\tilde{f}|||_{C_\psi} = k.$$

By $\tau \to \infty$ we find the inequality

$$\int_t^\infty |\psi(s)x(s)|^{-1} \mathrm{d}s \leq k|\psi(t)x(t)|^{-1}. \tag{31}$$

Setting

$$\tilde{\varphi}(t) = \int_t^\infty |\psi(s)x(s)|^{-1} \mathrm{d}s$$

we obtain

$$\tilde{\varphi}'(t) \leq \frac{1}{k}\tilde{\varphi}(t).$$

By integration the estimate

$$\tilde{\varphi}(t) \leq \tilde{\varphi}(t_0)e^{\frac{t-t_0}{k}} \tag{32}$$

follows. Now let $\tau \geq t$. From $x(\tau) = V(\tau)V^{-1}x(t)$ it arises

$$\psi(\tau)x(\tau) = \psi(\tau)V(\tau)V^{-1}\psi^{-1}(t)\psi(t)x(t)$$

and

$$|\psi(\tau)x(\tau)| = ||\psi(\tau)V(\tau)V^{-1}\psi^{-1}(t)||\ |\psi(t)x(t)|.$$

Condition 2 of the theorem and Lemma 4 imply that there exist constants $K \geq 1, \alpha > 0$ such that

$$|\psi(\tau)x(\tau)| = Ke^{\alpha(\tau-t)}|\psi(t)x(t)|.$$

Then

$$\begin{aligned}|\psi(t)x(t)|\tilde{\varphi}(t) &= |\psi(t)x(t)| \int_t^\infty |\psi(s)x(s)|^{-1} \mathrm{d}s \\ &\geq \int_t^\infty |\psi(s)x(s)|\frac{e^{-\alpha(s-t)}}{K}|\psi(s)x(s)|^{-1} \mathrm{d}s = \frac{1}{K}\int_t^\infty e^{-\alpha(s-t)} \mathrm{d}s = \frac{1}{K\alpha}.\end{aligned}$$

Having in mind (31) and (32) it follows

$$|\psi(t)x(t)| \geq \frac{(K\alpha)^{-1}}{\tilde{\varphi}(t)} \geq \frac{(K\alpha)^{-1}}{\tilde{\varphi}(t_0)} e^{\frac{1}{k}(t-t_0)} \geq \frac{(K\alpha)^{-1}}{k} e^{\frac{1}{k}(t-t_0)} |\psi(t_0)x(t_0)|.$$

This inequality is from the same type as the desired estimate (14). From condition 2 of the theorem and Lemma 5 and Lemma 3 it follows that the impulsive equation (3), (4) is ψ-exponential dichotomous. Hence Theorem 5 is proved. □

Remark 6 Theorem 5 holds without condition 3 if the space X is finite dimensional.

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