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Jacqueline Fleckinger, Jesús Hernández, François de Thélin

Estimate of the validity interval for the Antimaximum Principle and application to a non-cooperative system

ABSTRACT. We are concerned with the sign of the solutions of non-cooperative systems when the parameter varies near a Principal eigenvalue of the system. With this aim we give precise estimates of the validity interval for the Antimaximum Principle for an equation and an example. We apply these results to a non-cooperative system. Finally a counterexample shows that our hypotheses are necessary. The Maximum Principle remains true only for a restricted positive cone.

KEY WORDS. Maximum Principle, Antimaximum Principle, Elliptic Equations and Systems, Non cooperative systems, Principal Eigenvalue.

1 Introduction

In this paper we use ideas concerning the Anti-Maximum Principle due to Clement and Peletier [5] and later to Arcoya Gámez [3] to obtain in Section 2 precise estimates concerning the validity interval for the Antimaximum Principle for one equation. An example shows that this estimate is sharp.

The Maximum Principle and then the Antimaximum Principle for the case of a single equation have been extensively studied later for cooperative elliptic systems (see the references ([1],[6],[7],[8],[10],[12]). The results in [10], are still valid for systems(with constant coefficients) involving the p-Laplacian. Some results for non-cooperative systems can be found *e.g.* in [4],[11]. Very general results concerning the Maximum Principle for equations and cooperative systems for different classes (classical, weak, very weak) of solutions were given by Amann in a long paper [2], in particular the Maximum Principle was shown to be equivalent to the positivity of the principal eigenvalue.

Here in Section 3, we consider a non-cooperative 2×2 system with constant coefficients depending on a real parameter μ having two real principal eigenvalues $\mu_1^- < \mu_1^+$. We obtain some theorems of Antimaximum Principle type concerning the behavior of different cones of couples of functions having positivity (or negativity) properties. We give several results of this type for values of $\mu_1^- < \mu$ but close to μ_1^- by combining the usual Maximum Principle and the results for the Antimaximum Principle in Section 2.

Finally a counterexample is given showing that the Maximum Principle does not hold in general for non cooperative systems, but a (partial, under an additional assumption) Maximum Principle for $\mu < \mu_1^-$ is also obtained.

2 Estimate of the validity interval for the Anti-maximum Principle

Let Ω be a smooth bounded domain in $I\!R^N$. We consider the following Dirichlet boundary value problem

$$-\Delta z \;=\; \mu z + \; h \;\text{ in } \Omega \;, \quad z = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where μ is a real parameter. We associate to (2.1) the eigenvalue problem

$$-\Delta \varphi \;=\; \lambda \varphi \;\text{ in } \Omega \;, \quad \varphi = 0 \text{ on } \partial\Omega. \tag{2.2}$$

We denote by λ_k, $k \in I\!N^*$ the eigenvalues $(0 < \lambda_1 < \lambda_2 \leq ...)$ and by φ_k a set of orthonormal associated eigenfunctions. We choose $\varphi_1 > 0$.

Hypothesis (H_0)**:** We write

$$h = \alpha \varphi_1 + h^{\perp} \tag{2.3}$$

where $\int_\Omega h^{\perp} \varphi_1 = 0$ and we assume $\alpha > 0$ and $h \in L^q$, $q > N$ if $N \geq 2$ and $q = 2$ if $N = 1$.

Theorem 1 *We assume (H_0) and $\lambda_1 < \mu \leq \Lambda < \lambda_2$. There exists a constant K depending only on Ω, Λ and q such that, for $\lambda_1 < \mu < \lambda_1 + \delta(h)$ with*

$$\delta(h) = \frac{K\alpha}{\|h^{\perp}\|_{L^q}}, \tag{2.4}$$

the solution z to (2.1) satisfies the Antimaximum Principle, that is

$$z < 0 \text{ in } \Omega; \quad \partial z/\partial\nu > 0 \text{ on } \partial\Omega, \tag{2.5}$$

where $\partial/\partial\nu$ denotes the outward normal derivative.

Remark 2.1 The Antimaximum Principle of Theorem 1, assuming $\alpha > 0$, is in the line of the version given by Arcoya- Gámez [3].

Lemma 2.1 *We assume $\lambda_1 < \mu \leq \Lambda < \lambda_2$ and $h \in L^q$, $q > N \geq 2$. We suppose that there exists a constant C_1 depending only on $\Omega, q,$ and Λ such that z satisfying (2.1) is such that*

$$\|z\|_{L^2} \leq C_1 \|h\|_{L^2}. \tag{2.6}$$

Then there exist constants C_2 and C_3, depending only on Ω, q and Λ such that

$$\|z\|_{\mathcal{C}^1} \leq C_2\|h\|_{L^q} \text{ and } \|z\|_{L^q} \leq C_3\|h\|_{L^q}. \tag{2.7}$$

Remark 2.2 Hypothesis (2.6) cannot hold, unless h is orthogonal to φ_1. Indeed, letting μ go to λ_1, (2.6) implies the existence of a solution to (2.1) with $\mu = \lambda_1$. Note that in the proof of Theorem 1, Lemma 2.1 is used for h (and hence z) orthogonal to φ_1 .

2.1 Proof of Lemma 2.1

All constants in this proof depend only on Ω, Λ and q.

Claim: $\|z\|_{L^q} \leq C_3\|h\|_{L^q}$.

If the claim is verified then, by regularity results for the Laplace operator combined with Sobolev imbeddings

$$\|z\|_{\mathcal{C}^1} \leq C_4\|z\|_{W^{2,q}} \leq C_5(\Lambda\|z\|_{L^q} + \|h\|_{L^q}). \tag{2.8}$$

From the claim and regularity results we deduce (2.7).

Proof of the claim:

- Step 1 We consider the sequence $p_j = 2 + \frac{8j}{N}$ for $j \in I\!N$. Observe that for any j, $W^{2,p_j} \hookrightarrow L^{p_{j+1}}$ and that there exists a constant $H(j)$ such that

$$\forall v \in W^{2,p_j},\ \|v\|_{L^{p_{j+1}}} \leq H(j)\|v\|_{W^{2,p_j}}. \tag{2.9}$$

The relation (2.9) is obvious if $2p_j \geq N$ and for $2p_j < N$ we have

$$\frac{Np_j}{N-2p_j} - p_{j+1} = \frac{2p_jp_{j+1} - 8}{N-2p_j} > 0$$

and the result follows by classical Sobolev imbedding.

- Step 2 We consider z satisfying (2.1). For $j = 0$, we derive from (2.6) and Hölder inequality that

$$\|z\|_{L^2} \leq C_5\|h\|_{L^q}. \tag{2.10}$$

By induction we assume that $z \in L^{p_j}$ with $p_j < q$ and that

$$\|z\|_{L^{p_j}} \leq K(j)\|h\|_{L^q}. \tag{2.11}$$

By Hölder inequality,

$$\|\mu z + h\|_{L^{p_j}} \leq \Lambda\|z\|_{L^{p_j}} + |\Omega|^{\frac{q-p_j}{qp_j}}\|h\|_{L^q}.$$

By regularity results for the Laplace operator:

$$\|z\|_{W^{2,p_j}} \; \leq \; C(j)(\Lambda\|z\|_{L^{p_j}} + |\Omega|^{\frac{q-p_j}{qp_j}}\|h\|_{L^q}) \leq C(j)(\Lambda K(j) + |\Omega|^{\frac{q-p_j}{qp_j}})\|h\|_{L^q}.$$

Using (2.9) the relation (2.11) holds for $j+1$ and the induction is proved.

- Step 3 Let J be such that $p_{J+1} \geq q > p_J$. After J iterations we get by (2.11)

$$\|z\|_{L^q} \; \leq \; C_6\|z\|_{L^{p_{J+1}}} \; \leq \; C_6 K(J+1)\|z\|_{W^{2,p}} \leq$$

$$C_7 K(J+1)\|\mu z + h\|_{L^{p_J}} \leq C_8(\Lambda\|h\|_{L^q} + \|h\|_{L^{p_J}}) \leq C_9\|h\|_{L^q},$$

which is the claim. □

2.2 Proof of Theorem 1

- Step 1: We prove the following inequality:

$$\|z^\perp\|_{\mathcal{C}^1} \; \leq \; C_2\|h^\perp\|_{L^q}. \tag{2.12}$$

We derive from (2.3)

$$z \; = \; \frac{\alpha}{\lambda_1 - \mu}\varphi_1 \; + \; z^\perp, \tag{2.13}$$

with $z^\perp$ solution of

$$-\Delta z^\perp \; = \; \mu z^\perp + \; h^\perp \; \text{ in } \Omega \; ; \;\; z^\perp = 0 \text{ on } \partial\Omega. \tag{2.14}$$

By the variational characterization of λ_2:

$$\lambda_2 \int_\Omega |z^\perp|^2 \; \leq \; \int_\Omega |\nabla z^\perp|^2 \; = \; \mu \int_\Omega |z^\perp|^2 \; + \; \int_\Omega z^\perp h^\perp.$$

Hence

$$\|z^\perp\|_{L^2} \; \leq \; \frac{1}{\lambda_2 - \Lambda}\|h^\perp\|_{L^2}.$$

By Lemma 2.1, we derive (2.12).

- Step 2: *Close to the boundary:*

We show now that on the boundary $\frac{\partial z}{\partial \nu}(x) > 0$. and near the boundary $z < 0$.

Since $\partial\varphi_1/\partial\nu < 0$ on $\partial\Omega$, we set

$$A := min_{\partial\Omega}|\partial\varphi_1/\partial\nu| > 0. \tag{2.15}$$

By a continuity argument there exists $\varepsilon > 0$ such that

$$dist(x, \partial\Omega) < \varepsilon \;\; \Rightarrow \;\; \partial\varphi_1/\partial\nu(x) \leq -A/2. \tag{2.16}$$

Hence by (2.12) to (2.16) , for any $x \in \Omega$ such that $dist(x, \partial\Omega) < \varepsilon$, and if

$$0 < \mu - \lambda_1 < \frac{\alpha A}{4C_2\|h^\perp\|_{L^q}},$$

we have

$$\frac{\partial z}{\partial \nu}(x) = \frac{\alpha}{\lambda_1 - \mu}\frac{\partial \varphi_1}{\partial \nu}(x) + \frac{\partial z^\perp}{\partial \nu}(x) \geq \frac{\alpha}{\lambda_1 - \mu}\frac{\partial \varphi_1}{\partial \nu}(x) - C_2\|h^\perp\|_{L^q},$$

hence

$$\frac{\partial z}{\partial \nu}(x) \geq \frac{\alpha}{2(\lambda_1 - \mu)}\frac{\partial \varphi_1}{\partial \nu}(x) > 0. \tag{2.17}$$

Therefore $\frac{\partial z}{\partial \nu}(x) > 0$ on $\partial\Omega$. Moreover since $z = \varphi_1 = 0$ on $\partial\Omega$, we deduce from (2.17) that, for $x \in \Omega$ with $dist(x, \partial\Omega) < \varepsilon' \leq \varepsilon/2$ (ε' small enough),

$$z(x) \leq \frac{\alpha}{2(\lambda_1 - \mu)}\varphi_1(x) < 0,$$

where ε' does not depend on μ.

- Step 3: *Inside* Ω:

We consider now $\Omega_{\varepsilon'} := \{x \in \Omega,\ dist(x, \partial\Omega) > \varepsilon'\}$. Set

$$B := \min_{\Omega_{\varepsilon'}} \varphi_1(x) > 0.$$

We have in $\Omega_{\varepsilon'}$ by (2.12) and (2.13)

$$z(x) = \frac{\alpha}{\lambda_1 - \mu}\varphi_1(x) + z^\perp(x) \leq \frac{\alpha}{\lambda_1 - \mu}B + C_2\|h^\perp\|_{L^q} < 0$$

if we choose

$$\mu - \lambda_1 < \frac{\alpha \min(B, A/2)}{C_2\|h^\perp\|_{L^q}}.$$

We derive now Theorem 1. □

2.3 An example

Let $N = 1$, $\Omega =]0, 1[$ and $h = h_1\varphi_1 + h_2\varphi_2$ with $h_1 > 0$, $h_2 > 0$. We note that

$$\varphi_1(x) - s\varphi_2(x) = sin\pi x(1 - 2scos\pi x) > 0 \tag{2.18}$$

in Ω implies $s \leq 1/2$. For this example, taking $\mu = \lambda_1 + \varepsilon, \varepsilon > 0$, we have:

$$z = \frac{h_1}{\lambda_1 - \mu}\varphi_1 + \frac{h_2}{\lambda_2 - \mu}\varphi_2 = -\frac{h_1}{\varepsilon}\left(\varphi_1 - \frac{\varepsilon h_2}{h_1(\lambda_2 - \lambda_1 - \varepsilon)}\varphi_2\right).$$

If the Antimaximum Principle holds, $z < 0$ in Ω, and by (2.18), we have

$$\frac{\varepsilon h_2}{h_1(\lambda_2 - \lambda_1 - \varepsilon)} \leq \frac{1}{2},$$

hence

$$\varepsilon \leq \frac{h_1(\lambda_2 - \lambda_1)}{2h_2(1 + \frac{h_1}{2h_2})} \leq \frac{h_1(\lambda_2 - \lambda_1)}{2h_2}.$$

We obtain an estimate of $\delta(h)$ similar to that in Theorem 1.

3 A non-cooperative system

Now we will consider the 2×2 non-cooperative system depending on a real parameter μ:

$$-\Delta u \;=\; au \,+\, bv \,+\, \mu u \,+\, f \;\text{ in } \Omega, \tag{S_1}$$

$$-\Delta v \;=\; cu \,+\, dv \,+\, \mu v \,+\, g \;\text{ in } \Omega, \tag{S_2}$$

$$u = v = 0 \text{ on } \partial\Omega. \tag{S_3}$$

or shortly

$$-\Delta U = AU + \mu U + F \text{ in}\,\Omega\,,\ U = 0 \,\text{on}\, \partial\Omega. \tag{S}$$

Hypothesis (H_1) We assume $b > 0\,, c < 0,$ and

$$D := (a-d)^2 + 4bc > 0. \tag{3.1}$$

3.1 Eigenvalues of the system

As usual we say that μ is an eigenvalue of System (S) if $(S_1) - (S_3)$ has a non trivial solution $U = (u, v) \neq 0$ for $F \equiv 0$ and we say that μ is a principal eigenvalue of System (S) if there exists $U = (u, v)$ with $u > 0, v > 0$ solution to (S) with $F \equiv 0$.

Notice that, since (S) is not cooperative, it is not necessarily true that there is a lowest principal eigenvalue μ_1 and that the Maximum Principle holds if and only if $\mu_1 > 0$ (Amann [2]).

We seek solutions $u = p\varphi_1$, $v = q\varphi_1$ to the eigenvalue problem where, as above, (λ_1, φ_1) is the principal eigenpair for $-\Delta$ with Dirichlet boundary conditions.

Principal eigenvalues correspond to solutions with $p, q > 0$. The associated linear system is

$$(a + \mu - \lambda_1)p \,+\, bq \;=\; 0,$$

$$cp \,+\, (d + \mu - \lambda_1)q \;=\; 0,$$

and it follows from (H_1) that $(a + \mu - \lambda_1)$ and $(d + \mu - \lambda_1)$ should have opposite signs. We should have

$$Det(A + (\mu - \lambda_1)I) = (a + \mu - \lambda_1)(d + \mu - \lambda_1) - bc = 0,$$

which implies by (H_1) that the condition on signs is satisfied and this whatever the sign of μ could be. (Notice that $D > 0$ implies that both roots are real and that $D = 0$ gives a real double root).

We have then shown directly that our system has (at least) two principal eigenvalues. Their signs will depend on the coefficients. If, for example, $a < \lambda_1$, $d < \lambda_1$, the largest one is positive. We will denote the two principal eigenvalues by μ_1^- and μ_1^+ where

$$\mu_1^- \;:=\; \lambda_1 - \xi_1 \;<\; \mu_1^+ \;:=\; \lambda_1 - \xi_2, \tag{3.2}$$

where the eigenvalues of Matrix A are:

$$\xi_1 = \frac{a+d+\sqrt{D}}{2} > \xi_2 = \frac{a+d-\sqrt{D}}{2}.$$

Remark 3.1 Usually the Maximum Principle holds if and only if the first eigenvalue is positive. Here by replacing $-\Delta$ by $-\Delta + K$ with $K > 0$ large enough we may get $\mu_1^- > 0$. Nevertheless the Maximum Principle needs an additional condition (see Theorem 4 and its remark).

3.2 Main Theorems

3.2.1 The case $\boldsymbol{\mu_1^- < \mu < \mu_1^+}$

We assume in this subsection that the parameter μ satisfies:

(H_2) $\quad \mu_1^- \;<\; \mu \;<\; \mu_1^+$.

Theorem 2 *Assume* (H_1), (H_2), *and*

(H_3)
$$d < a,$$

(H_4)
$$f \geq 0,\ g \geq 0,\ f, g \not\equiv 0, f, g \in L^q,\ q > N\, if\, N \geq 2\,;\ q = 2\, if\, N = 1.$$

Then there exists $\delta > 0$, *independent of* μ, *such that if*

(H_5)
$$\mu < \mu_1^- + \delta,$$

we get

$$u < 0,\ v > 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} > 0,\ \frac{\partial v}{\partial \nu} < 0\ on\ \partial\Omega.$$

Remark 3.2 If in the theorem above we reverse signs of f, g, u, v that is $f \leq 0$, $g \leq 0$, $f, g \not\equiv 0$, then for μ satisfying (H_5), we get

$$u > 0,\ v < 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} > 0\ on\ \partial\Omega.$$

Note that the counterexample in subsection (3.3) shows that for f, g of opposite sign($fg < 0$), u or v may change sign.

Theorem 3 *Assume* (H_1), (H_2), *and*

(H_3') $$a < d,$$

(H_4') $$f \leq 0,\ g \geq 0,\ f,g \not\equiv 0\,,\ f,g \in L^q,\ q > N\, if\, N \geq 2\,;\ q = 2\, if\, N = 1.$$

Then there exists $\delta > 0$, *independent of* μ, *such that if*

(H_5) $$\mu < \mu_1^- + \delta,$$

we obtain

$$u < 0,\ v < 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} > 0,\ \frac{\partial v}{\partial \nu} > 0\ on\ \partial\Omega.$$

Remark 3.3 If in the theorem above we reverse signs of f, g, u, v that is $f \geq 0$, $g \leq 0$, $f, g \not\equiv 0$, then for μ satisfying (H_5), we get

$$u > 0,\ v > 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} < 0\ on\ \partial\Omega.$$

Note that, by the changes used in the proof of the theorem above, the counterexample in subsection (3.3) shows that for f, g with same sign ($fg > 0$), u or v may change sign.

3.2.2 The case $\mu < \mu_1^-$

We assume in this Section that the parameter μ satisfies:

(H_2') $\mu < \mu_1^-$.

Theorem 4 *Assume* (H_1), (H_2'), *and*

(H_3') $$a < d,$$

(H_4'') $$f \geq 0,\ g \geq 0,\ f, g \not\equiv 0,\ f, g \in L^2.$$

Assume also $t^*g - f \geq 0$, $t^*g - f \not\equiv 0$ *with*

$$t^* = \frac{d - a + \sqrt{D}}{-2c}.$$

Then

$$u > 0,\ v > 0\ in\ \Omega;\ \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} < 0\ on\ \partial\Omega.$$

Remark 3.4 As above we can reverse signs of f, g, u, v .

3.3 Counterexample: $a > d$

We consider the system in 1 dimension

$$-u'' \;=\; 4u \;+\; v \;+\; \mu u \;+\; f \;\text{ in } I :=]0;\pi[,$$

$$-v'' \;=\; -u \;+\; v \;+\; \mu v \;+\; g \;\text{ in } I,$$

$$u(0) = u(\pi) = v(0) = v(\pi) = 0.$$

$\lambda_1 = 1$ and $\lambda_2 = 4$; $\varphi_1 = \sin x$, $\varphi_2 = \sin 2x$. We compute $\mu_1^- = 1 - \frac{5+\sqrt{5}}{2}$. Choose $f = \varphi_1 - \frac{1}{2}\varphi_2 \geq 0$ and $g = kf$ with $k \neq 0$ to be determined later. We obtain

$$u = u_1\varphi_1 + u_2\varphi_2 \text{ and } v = v_1\varphi_1 + v_2\varphi_2,$$

where

$$u_1 = \frac{k-\mu}{\mu^2+3\mu+1}\;,\; u_2 = \frac{\mu-k-3}{2(\mu^2-3\mu+1)},$$

1/ Choosing $\mu = -3 < \mu_1^-$, we get $v_1 = -1$ and $v_2 = \frac{1-3k}{38}$. Therefore

$$-v = \varphi_1 + \frac{3k-1}{38}\varphi_2,$$

and for $\frac{3k-1}{38} > \frac{1}{2}$, v changes sign. Hence Maximum Principle does not hold.

2/ Choosing $\mu_1^- < \mu = \mu_1^- + \epsilon$, $k = \mu_1^- + \epsilon^2$, we have

$$\frac{u_2}{u_1} = \left(\frac{\mu-k-3}{k-\mu}\right)\left(\frac{\mu^2+3\mu+1}{2(\mu^2-3\mu+1)}\right) = \left(\frac{3+\epsilon}{\epsilon}\right)\left(\frac{\sqrt{5}-\epsilon}{(9+3\sqrt{5})-(6+\sqrt{5})\epsilon+\epsilon^2}\right).$$

So that $\frac{u_2}{u_1} \to \infty$ as $\epsilon \to 0$. Hence for these $f > 0$, $g < 0$, u changes sign. □

3.4 Proofs of the main results

3.4.1 Some computations and associate equation

In the following we introduce

$$\gamma_1 \;=\; \frac{1}{2}(a+d+2\mu-\sqrt{D}) \;=\; \lambda_1+\mu-\mu_1^+; \tag{3.3}$$

$$\gamma_2 = \frac{1}{2}(a+d+2\mu+\sqrt{D}) \;=\; \lambda_1+\mu-\mu_1^-, \tag{3.4}$$

and some auxiliary results used in the proofs of our results.

Lemma 3.1 *We have*

(L1) $$\mu < \mu_1^+ \Leftrightarrow \gamma_1 < \lambda_1.$$

(L2) $$\mu_1^- < \mu \Leftrightarrow \lambda_1 < \gamma_2.$$

(L3) $$\sqrt{D} < a - d \Leftrightarrow d + \mu < \gamma_1 < \gamma_2 < a + \mu.$$

(L4) $$\sqrt{D} < d - a \Leftrightarrow a + \mu < \gamma_1 < \gamma_2 < d + \mu.$$

(L5) $$\mu < \mu_1^+ + \delta \Leftrightarrow \gamma_1 < \lambda_1 + \delta.$$

(L6) $$\mu < \mu_1^- + \delta \Leftrightarrow \gamma_2 < \lambda_1 + \delta.$$

3.4.2 Proofs of Theorems 2 and 3

Proof of Theorem 2, $a > d$:

We introduce now

$$w = u + tv, \tag{3.5}$$

with

$$t = \frac{a - d + \sqrt{D}}{-2c} = \frac{2b}{a - d - \sqrt{D}} \tag{3.6}$$

so that

$$-\Delta w = \gamma_1 w + f + tg \, in \, \Omega; \tag{3.7}$$

$$w|_{\partial\Omega} = 0.$$

We remark that

$$t = \frac{b}{\gamma_1 - d - \mu} = \frac{b}{a + \mu - \gamma_2} = \frac{\gamma_1 - a - \mu}{c} = \frac{d + \mu - \gamma_2}{c}. \tag{3.8}$$

Note first that Hypothesis (H_3) implies $t > 0$ and $a - d > \sqrt{D}$. By (H_2), (H_4), and $(L1)$ in Lemma 3.1, $\gamma_1 < \lambda_1$, and we apply the Maximum Principle which gives $w > 0$ on Ω and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$. We compute

$$a + \mu - \frac{b}{t} = a + d + 2\mu - \gamma_1 = \gamma_2, \tag{3.9}$$

and since $v = (w - u)/t$, we derive

$$-\Delta u = (a + \mu - \frac{b}{t})u + \frac{b}{t} w + f = \gamma_2 u + \frac{b}{t} w + f,$$

where $\frac{b}{t}w + f > 0$. From (H_5) and $(L6)$, $\gamma_2 \leq \lambda_1 + \delta_1$, where

$$\delta_1 := \delta(\frac{b}{t}w + f), \tag{3.10}$$

we deduce from the Antimaximum Principle that $u < 0$ on Ω and $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$. Hence $cu + g > 0$.

Now (H_2), (L_1) and (L_3) imply $d + \mu < \gamma_1 < \lambda_1$ and the Maximum Principle applied to (S_2) gives $v > 0$ on Ω and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$.

We apply now Section 1 to estimate δ_1.

$$h := \frac{b}{t}w + f = (\gamma_1 - d - \mu)w + f = \sigma\varphi_1 + h^{\perp}. \tag{3.11}$$

First we compute σ:

Set $f = \alpha\varphi_1 + f^{\perp}$, $g = \beta\varphi_1 + g^{\perp}$, $w = \kappa\varphi_1 + w^{\perp}$. Since

$$-\Delta w = \gamma_1 w + f + \frac{b}{\gamma_1 - d - \mu} g,$$

we calculate:

$$\sigma = \alpha + (\gamma_1 - d - \mu)\kappa = \alpha \frac{\lambda_1 - d - \mu}{\lambda_1 - \gamma_1} + \beta \frac{b}{\lambda_1 - \gamma_1}.$$

Now we estimate $\|h^{\perp}\|_{L^2}$.

$$-\Delta w^{\perp} = \gamma_1 w^{\perp} + f^{\perp} + \frac{b}{\gamma_1 - d - \mu} g^{\perp}.$$

The variational characterization of λ_2 gives

$$(\lambda_2 - \gamma_1)\|w^{\perp}\|_{L^2} \leq \|f^{\perp}\|_{L^2} + \frac{b}{\gamma_1 - d - \mu}\|g^{\perp}\|_{L^2}.$$

We derive from (3.11)

$$\|h^{\perp}\|_{L^2} \leq \|f^{\perp}\|_{L^2} + (\gamma_1 - d - \mu)\|w^{\perp}\|_{L^2} \leq \frac{\lambda_2 - d - \mu}{\lambda_2 - \gamma_1}\|f^{\perp}\|_{L^2} + \frac{b}{\lambda_2 - \gamma_1}\|g^{\perp}\|_{L^2}.$$

Reasoning as in Lemma 2.1, we show that there exists a constant C_3 such that

$$\|h^{\perp}\|_{L^q} \leq C_3 \left(\frac{\lambda_2 - d - \mu}{\lambda_2 - \gamma_1}\|f^{\perp}\|_{L^q} + \frac{b}{\lambda_2 - \gamma_1}\|g^{\perp}\|_{L^q} \right). \tag{3.12}$$

In fact for proving (3.12) we use the same sequence than that in Lemma 2.1 and we show by induction that

$$\|z^{\perp}\|_{L^{p_j}} \leq K(j)\left(\|f^{\perp}\|_{L^q} + \|g^{\perp}\|_{L^q}\right).$$

Now we apply the Antimaximum Principle to the equation

$$-\Delta u = \gamma_2 u + h.$$

This is possible since by $(L6)$ in Lemma 3.1, $\lambda_1 < \gamma_2 < \lambda_1 + \delta_2 = \lambda_1 + \delta(h)$ where, as in Theorem 1, $\delta(h) = \frac{K\sigma}{\|h^\perp\|_{L^q}}$.

Moreover we notice that $\lambda_1 - \gamma_1 = \mu_1^+ - \mu \le \mu_1^+ - \mu_1^-$ and therefore, since $\alpha > 0$ and $\beta > 0$ by (H_4),

$$\sigma = \alpha \frac{\lambda_1 - d - \mu}{\lambda_1 - \gamma_1} + \beta \frac{b}{\lambda_1 - \gamma_1} \ \ge \ \mathcal{A} := \alpha \frac{\lambda_1 - d - \mu_1^+}{\mu_1^+ - \mu_1^-} + \beta \frac{b}{\mu_1^+ - \mu_1^-},$$

and from (3.12), we obtain

$$\|h^\perp\|_{L^q} \le \mathcal{B} := C_3 \left(\frac{\lambda_2 - d - \mu_1^-}{\lambda_2 - \lambda_1} \|f^\perp\|_{L^q} + \frac{b}{\lambda_2 - \lambda_1} \|g^\perp\|_{L^q} \right).$$

From the computation above we can choose $\delta_2 = \frac{K\mathcal{A}}{\mathcal{B}}$ which does not depend on μ, and the result follows. □

Proof of Theorem 3, $a < d$:
We deduce this theorem from Theorem 2 by change of variables. Set $\hat{a} = d$, $\hat{d} = a$, $\hat{u} = v$, $\hat{v} = -u$ and $\hat{f} = g$, $\hat{g} = -f$. $\hat{f} \ge 0$, $\hat{g} \ge 0$, imply $\hat{u} < 0$, $\hat{v} > 0$. We get Theorem 3. □

3.4.3 Proof of Theorem 4

Since $a < d$, we have $t^* = \frac{d-a+\sqrt{D}}{-2c} > 0$. With now the change of variable $w = -u + t^* v$, as in [4] (see also [11]) , we can write the system as

$$-\Delta u = \gamma_1 u + (b/t^*) w + f \ in\, \Omega, \tag{3.13}$$

$$-\Delta v = \gamma_1 v - cw + g \ in\, \Omega \tag{3.14}$$

$$-\Delta w = \gamma_2 w + (t^* g - f) \ in\, \Omega, \tag{3.15}$$
$$u = v = w = 0 \ on\ \partial\Omega.$$

Now $\mu < \mu_1^-$, and it follows from $(L2)$ in Lemma 3.1 that $\gamma_1 < \gamma_2 < \lambda_1$. From (3.15) it follows from the Maximum Principle that $w > 0$. Then in (3.14) $-cw + g > 0$, and again by the Maximum Principle $v > 0$. Finally, since $(b/t^*)w + f > 0$ in (3.13), again by the Maximum Principle $u > 0$. □

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Authors:

J. Fleckinger
Institut de Mathématique -
CEREMATH-UT1
Université de Toulouse,
31042 Toulouse Cedex,
France

e-mail: jfleckinger@gmail.com

J. Hernández
Departamento de Matemáticas
Universidad Autónoma,
28049 Madrid,
Spain

e-mail: jesus.hernandez@uam.es

F. de Thélin
Institut de Mathématique
Université de Toulouse,
31062 Toulouse Cedex,
France

e-mail: francoisdethelin@yahoo.fr