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René Bartsch

Vietoris hyperspaces as quotients of natural function spaces

ABSTRACT. Hyperspaces form a powerful tool in some branches of mathematics: lots of fractal and other geometric objects can be viewed as fixed points of some functions in suitable hyperspaces - as well as interesting classes of formal languages in theoretical computer sciences, for example (to illustrate the wide scope of this concept). Moreover, there are many connections between hyperspaces and function spaces in topology. Thus results from hyperspaces help to get new results in function spaces and vice versa.

We give here a new description of the Vietoris hyperspace on the family $K(Y)$ of the nonempty compact subsets of a regular topological space Y as quotient of the space $C(\beta D, Y)$, endowed with compact-open topology τ_{co}, where βD is the Stone-Čech-compactification of a discrete space.

1 Preliminary Definitions and Results

For a given set X we denote by $\mathfrak{P}(X)$ the power set of X, by $\mathfrak{P}_0(X)$ the power set without the empty set. By $\mathfrak{F}(X)$ (resp. $\mathfrak{F}_0(X)$) we mean the set of all filters (resp. ultrafilters) on X; if φ is a filter on X, the term $\mathfrak{F}_0(\varphi)$ denotes the set of all ultrafilters on X, which contain φ. For $x \in X$ we denote by $\overset{\bullet}{x} := \{A \subseteq X |\ x \in A\}$ the singleton filter on X, generated by $\{x\}$. With $\mathfrak{S}(X) := \{\overset{\bullet}{x} |\ x \in X\}$ we mean the family of all singleton filters on X.

For families $\mathfrak{A} \subseteq \mathfrak{P}(X)$ and any $M \subseteq X$ we set

$$M^{-\mathfrak{A}} := \{A \in \mathfrak{A} |\ A \cap M \neq \emptyset\}$$

and

$$M^{+\mathfrak{A}} := \{A \in \mathfrak{A} |\ A \cap M = \emptyset\} \ .$$

Then for a topological space (X, τ) on $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ the *lower Vietoris topology* $\tau_{l,\mathfrak{A}}$ is defined by the subbase $\{O^{-\mathfrak{A}} |\ O \in \tau\}$, whereas the *upper Vietoris topology* $\tau_{u,\mathfrak{A}}$ on $\mathfrak{A}$ comes from the

subbase $\{(X\setminus O)^{+\mathfrak{A}} | \, O\in\tau\}$. The *Vietoris topology* on $\mathfrak{A}$ is $\tau_{V,\mathfrak{A}} := \tau_{l,\mathfrak{A}} \vee \tau_{u,\mathfrak{A}}$. In most cases $\mathfrak{A}$ is chosen as the family $Cl(X)$ of the nonempty closed, or $K(X)$ of the nonempty compact subsets of a topological space (X,τ), or as the entire $\mathfrak{P}_0(X)$.
The Vietoris topology on $\mathfrak{A}$ is also generated by the standard *basis* consisting of all sets

$$\langle U_1,...,U_n\rangle_{\mathfrak{A}} := \left\{ A\in\mathfrak{A} \;\middle|\; A\subseteq\bigcup_{i=1}^{n} U_i \;\wedge\; \forall i=1,...,n : A\cap U_i\neq\emptyset \right\}$$

with open $U_1,...,U_n$.
Whenever there is no doubt about $\mathfrak{A}$, we will omit it as sub- and superscript.

We will need some basic facts about the Stone-Čech-compactification of discrete spaces.

A discrete space (D,δ) clearly is T_4 and Hausdorff, so its Stone-Čech-compactification is homeomorphic to its Wallman extension, consisting in this case just of the set $\mathfrak{F}_0(D)$, where the singleton filters are identified with their generating points via $w: D\to\mathfrak{F}_0(D) : w(x) := \overset{\bullet}{x}$, endowed with the topology generated from the base consisting of all sets $\mathfrak{F}_0(M)$, with $M\subseteq D$ (see [4], p. 176 ff).

Proposition 1.1 *Let (D,δ) be a discrete topological space. Then for its Stone-Čech-compactification $(\beta D,\delta^\beta)$ hold*

(a) *For all $M\subseteq D$ in B the closure $\overline{M}$ is clopen.*

(b) *δ^β has a base consisting of clopen sets.*

(c) *All clopen sets C in $(\beta D,\delta^\beta)$ are of the form $C=\overline{C\cap D}$.*

Proof: We use the homeomorphy of $(\beta D,\delta^\beta)$ to the Wallman extension.

(a) + (b) We have $\overline{w(M)} = \mathfrak{F}_0(M)$: from $\mathfrak{F}_0(M) = \mathfrak{F}_0(D)\setminus\mathfrak{F}_0(D\setminus M)$ we conclude, that $\mathfrak{F}_0(M)$ is closed and of course it contains $w(M)$. So, $\overline{w(M)}\subseteq\mathfrak{F}_0(M)$ follows. If there would be a filter $\varphi\in\mathfrak{F}_0(M)$ which belongs not to $\overline{w(M)}$, then there would exist a base set $\mathfrak{F}_0(S)$, $S\subseteq D$, of δ^β s.t. $\varphi\in\mathfrak{F}_0(S)$ and $\mathfrak{F}_0(S)\cap w(M)=\emptyset$. But this implies $M\cap S=\emptyset$, and thus $\mathfrak{F}_0(S)\cap\mathfrak{F}_0(M)=\emptyset$ - in contradiction to $\varphi\in\mathfrak{F}_0(S)\cap\mathfrak{F}_0(M)$. So, we have indeed $\overline{w(M)}=\mathfrak{F}_0(M)$, which is also open, because it belongs to our defining base of δ^β.

(c) Let $C\subseteq\beta D$ be clopen. Then for all $c\in C$ there exists a basic open set $\mathfrak{F}_0(M_c)$ with $M_c\subseteq D$, s.t. $c\in\mathfrak{F}_0(M_c)\subseteq C$, because C is open. From closedness of C automatically follows compactness, because βD is compact, thus there are finitely many $M_{c_1}, ..., M_{c_n}$ with $C = \bigcup_{i=1}^{n} \mathfrak{F}_0(M_{c_i})$. Now, for such finite union we have generally $\bigcup_{i=1}^{n} \mathfrak{F}_0(M_{c_i}) = \mathfrak{F}_0(\bigcup_{i=1}^{n} M_{c_i})$ and it is clear, that $w(\bigcup_{i=1}^{n} M_{c_i}) = C \cap w(D)$ holds. $\square$

For a topological space (Y, σ) - especially, if it is not T_0 - we define an equivalence relation on Y by

$$x \sim y \;:\Leftrightarrow\; (\forall O \in \sigma : x \in O \leftrightarrow y \in O) \;.$$

Then the quotient space $(Y/\!\sim, \sigma_\sim)$ is obviously T_0; we call it the T-zerofication of (Y, σ). Let $\nu : Y \to Y/\!\sim\, : \nu(y) := [y]_\sim$ be the canonical surjection. Because ν is continuous, the space $(Y/\!\sim, \sigma_\sim)$ is compact, whenever (Y, σ) is.

Proposition 1.2 *Let (X, τ) be a Tychonoff space, (Y, σ) a compact T_3-space and $f : X \to Y$ a continuous function. Then there exists a continuous extension $F : \beta X \to Y$ with $F_{|X} = f$.*

Proof: The T-zerofication $(Y/\!\sim, \sigma_\sim)$ of (Y, σ) is also T_3 (see [3], p. 191), and of course T_0, so it is T_2. Because it is also compact, from the theorem of Stone-Čech we get a continuous extension $G : \beta X \to (Y/\!\sim, \sigma_\sim)$ of $\nu \circ f : X \to Y/\!\sim$, where ν is the canonical surjection from Y to $Y/\!\sim$. Now, let $\alpha : Y/\!\sim\, \to Y$ be a choice function, i.e. $\forall [y]_\sim \in Y_\sim : \alpha([y]_\sim) \in [y]_\sim$.

Then

$$F : \beta X \to Y : F(x) := \begin{cases} f(x) & ; \quad x \in X \\ \alpha \circ G(x) & ; \quad x \in \beta X \setminus X \end{cases}$$

is continuous by [3], prop. 4.1.4(4), and is obviously an extension of f. $\square$

Proposition 1.3 *Let (Y, σ) be a topological T_3-space, $K \subseteq Y$ compact and $O \subseteq Y$ open with $K \subseteq O$. Then an open set U exists with $K \subseteq U \subseteq \overline{U} \subseteq O$. Especially, $\overline{K} \subseteq O$ holds.*

Proof: $K \subseteq O$ just means $K \cap (Y \setminus O) = \emptyset$ and $(Y \setminus O)$ is closed. Thus, by T_3, for every element $k \in K$ there are $U_k, V_k \in \sigma$ s.t. $k \in U_k, Y \setminus O \subseteq V_k$ and $U_k \cap V_k = \emptyset$. The U_k's cover K, so by compactness a finite subcover $U_{k_1}, ..., U_{k_n}$ exists. Let $U := \bigcup_{i=1}^{n} U_{k_i}$ and $V := \bigcap_{i=1}^{n} V_{K_i}$, so U, V are open, $U \cap V = \emptyset$, $K \subseteq U$ and $Y \setminus O \subseteq V$ hold, i.e.

$$K \subseteq U \subseteq Y \setminus V \subseteq O \;.$$

Now, $Y \setminus V$ is closed, so we get

$$\overline{K} \subseteq \overline{U} \subseteq \overline{Y \setminus V} = Y \setminus V \subseteq O \;.$$

$\square$

2 Vietoris Hyperstructure as final w.r.t. Function Spaces

Remember a wide class of function space structures, defined for Y^X or $C(X,Y)$: the so called set–open topologies, examined in [1], [5]. According to [5], we use the following convention: Let X and Y be sets and $A \subseteq X$, $B \subseteq Y$; then let be $(A,B) := \{f \in Y^X |\ f(A) \subseteq B\}$. Now let X be a set, (Y,σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Then the topology $\tau_{\mathfrak{A}}$ on Y^X (resp. $C(X,Y)$), which is defined by the open subbase $\{(A,W)|\ A \in \mathfrak{A}, W \in \sigma\}$ is called the *set–open topology, generated by* $\mathfrak{A}$, or shortly the $\mathfrak{A}$*–open topology.*

We know

Lemma 2.1 [cf. [2], lemma 3.4] Let (X,τ), (Y,σ) be topological spaces, let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and $\mathcal{H} \subseteq Y^X$ be endowed with $\tau_{\mathfrak{A}}$. Then the map

$$\mu_X : \mathcal{H} \to \mathfrak{P}_0(Y)^{\mathfrak{A}} :\ f \to \mu_X(f) : \forall A \in \mathfrak{A} : \mu_X(f)(A) := f(A),$$

is open, continuous and bijective onto its image, for $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology σ_V, and $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ with the generated pointwise topology.

Now, the pointwise topology on $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ is just the product topology on $\prod_{A \in \mathfrak{A}} \mathfrak{P}_0(Y)_A$ (with all $\mathfrak{P}_0(Y)_A$ being copies of $\mathfrak{P}_0(Y)$). By chosing $\mathcal{H} := C(X,Y)$, $\mathfrak{A} := K(X)$ and consequently replacing $\mathfrak{P}_0(Y)$ by $K(Y)$, we have the following situation:

$$\begin{tikzcd}
C(X,Y) \arrow[r, "\mu_X"] & K(Y)^{K(X)} \arrow[r, phantom, "\cong"] & \prod_{A \in K(X)} K(Y)_A \arrow[dl, "\pi_A"] \\
& (K(Y), \sigma_V) &
\end{tikzcd}$$

Of course, by π_A we mean the canonical projection from the product to the factor $K(Y)_A = K(Y)$.

From lemma 2.1 we get the continuity of μ_X, if $C(X,Y)$ is equipped with compact-open topology, thus in this case all compositions $\pi_A \circ \mu_X$ are continuous, too.

Moreover, μ_X is even a homeomorphism onto its image and the product structure is initial w.r.t. the projections. So, the question arises, whether or not the Vietoris topology σ_V on $K(Y)$ is final w.r.t. all $\pi_A \circ \mu_X$.

Lemma 2.2 *Let (X,τ), (Y,σ) be topological spaces and let σ_V be the Vietoris topology on $K(Y)$. Then for every $\mathfrak{O} \in \sigma_V$ and every $A \in K(X)$ the set $(\pi_A \circ \mu_X)^{-1}(\mathfrak{O}) \subseteq C(X,Y)$ is open w.r.t. the compact-open topology.*

Proof: Let $A \in K(X)$ be given and let $F \in Cl(Y)$ be a closed subset of Y. Then $(\pi_A \circ \mu_X)^{-1}(F^+) = \{f \in C(X,Y) |\ f(A) \subseteq Y \setminus F\} = (A, Y \setminus F) \in \tau_{co}$. Let now $O \in \sigma$ be given, then

$(\pi_A \circ \mu_X)^{-1}(O^-) = \{f \in C(X,Y) |\ f(A) \cap O \neq \emptyset\} = \bigcup_{a \in A}(\{a\}, O) \in \tau_{co}$.

So, because the F^+ and O^- form a subbase of σ_V, for $\mathfrak{O} \in \sigma_V$ the preimage $(\pi_A \circ \mu_X)^{-1}(\mathfrak{O})$ is an element of τ_{co}. $\square$

Corollary 2.3 *Let (Y, σ) be a topological space. For every topological space let $C(X,Y)$ be equipped with compact-open topology.*

Then the Vietoris topology σ_V on $K(Y)$ is contained in the final topology w.r.t. all $\pi_A \circ \mu_{(X,\tau)}$, $(X, \tau) \in \mathcal{B}$, $A \in K(X, \tau)$, for every class $\mathcal{B}$ of topological spaces.

Theorem 2.4 *Let (Y, σ) be a T_3-space and let $(K(Y), \sigma_V)$ be its Vietoris Hyperspace of compact subsets. Let furthermore δ be the discrete topology on $Y \times Y$ and denote by (Z, ζ) the Stone-Čech-compactification of $(Y \times Y, \delta)$.*

Then σ_V is the final topology on $K(Y)$ w.r.t. $\pi_Z \circ \mu_Z : C(Z,Y) \to K(Y)$, where $C(Z,Y)$ is endowed with compact-open topology τ_{co}.

Proof: From Lemma 2.2 we know that σ_V is contained in the final topology w.r.t. $\pi_Z \circ \mu_Z$, so we only have to show, that every open set of the final topology also belongs to σ_V. Let $\mathfrak{O}$ be an open set of the final topology, i.e. $(\pi_Z \circ \mu_Z)^{-1}(\mathfrak{O}) \in \tau_{co}$, and let $A \in \mathfrak{O}$.

We want to show, that there exist finitely many open sets $U_1, ..., U_m \in \sigma$ s.t.
$A \in \langle U_1, ..., U_m \rangle_{K(Y)} \subseteq \mathfrak{O}$.

At first, chose any surjection s from Y onto $A \subseteq Y$. Then extend it to a surjection $f_A : Y \times Y \to A$ by $f_A(y_1, y_2) := s(y_1)$, just meaning, that f_A maps such pairs with equal first component to the same image.

Now, endowing $Y \times Y$ with discrete topology, we get f_A being continuous. So, if (Z, ζ) denotes the Stone-Čech-compactification of the discrete $Y \times Y$, there exists a continuous extension $F_A : Z \to A$ of f_A, by proposition 1.2.

Because F_A is an extension of f_A, we have

$$\forall (a,b), (c,d) \in Y \times Y : a = c \implies F_A(a,b) = F_A(c,d) \ . \qquad (1)$$

Because $\mathfrak{O}$ is open in the final topology, there are finitely many compact subsets $K_1, ..., K_n \in K(Z)$ and open subsets $O_1, ..., O_n \in \sigma$ s.t. $F_A \in \bigcap_{i=1}^n (K_i, O_i) \subseteq (\pi_X \circ \mu_X)^{-1}(\mathfrak{O})$.

We will improve the sets K_i and O_i a little in an appropriate manner.

(a) For each K_i and every $k \in K_i$ there is an open neighbourhood U_k of k, s.t. $F_A(U_k) \subseteq O_i$, because F_A is continuous. Now, ζ has a base consisting of clopen sets B of the form $B = \overline{B \cap (Y \times Y)}$. So, there exist always such a clopen $B_k \subseteq U_k$ with $k \in B_k$ and $F_A(B_k) \subseteq O_i$. The family of all $B_k, k \in K_i$ is an open cover of K_i and consequently there is a finite subcover $\{B_{k_1}, ..., B_{k_l}\}$, by compactness of K_i. Now let

$$K'_i := \bigcup_{j=1}^{l} B_{k_j}$$

and observe, that K'_i as a finite union of clopen sets is clopen again, hence it is compact and of the form $K'_i = \overline{K'_i \cap (Y \times Y)}$. Furthermore we have $K_i \subseteq K'_i$ and consequently

$$F_A \in (K'_i, O_i) \subseteq (K_i, O_i) \ .$$

(b) We want to have our K's saturated in the sense, that whenever $(a, b) \in K \cap (Y \times Y)$ holds, then $\{a\} \times Y \subseteq K$ also holds. So, let us define

$$D_i := \bigcup_{\substack{a \in Y, \exists b \in Y: \\ (a,b) \in K'_i \cap (Y \times Y)}} \{a\} \times Y$$

and then $K''_i := \overline{D_i}$. From the continuity of F_A follows

$$F_A(K''_i) = F_A(\overline{D_i}) \subseteq \overline{F_A(D_i)} \tag{2}$$

and from (1) we get

$$F_A(D_i) = F_A\left(K'_i \cap (Y \times Y)\right) \ . \tag{3}$$

Of course, $F_A\left(K'_i \cap (Y \times Y)\right) \subseteq F_A(K'_i)$ and $F_A(K'_i)$ is compact and fulfills $F_A(K'_i) \subseteq O_i$, so by proposition 1.3 we get from (Y, σ) being T_3

$$F_A(K''_i) \subseteq \overline{F_A(D_i)} \subseteq \overline{F_A(K'_i)} \subseteq O_i \ . \tag{4}$$

Note, that all K''_i are compact and clopen again, by construction as a closure of a subset of $Y \times Y$ in the Stone-Čech-compactification (Z, ζ) of the discrete $Y \times Y$. Clearly, $K''_i \supseteq K'_i$ holds, yielding $(K''_i, O_i) \subseteq (K'_i, O_i)$, thus

$$F_A \in \bigcap_{i=1}^{n} (K''_i, O_i) \subseteq \bigcap_{i=1}^{n} (K'_i, O_i) \ . \tag{5}$$

(c) To cover Z (resp. A) with our compact sets, we add $K_0'' := Z$ (resp. $O_0 := Y$) and find of course

$$F_A \in \bigcap_{i=1}^{n}(K_i'', O_i) \ = \bigcap_{i=0}^{n}(K_i'', O_i)\ .$$

For each $z \in Z$ define

$$I(z) := \{i \in \{0, ..., n\} \mid z \in K_i''\}$$

and then

$$C(z) := \bigcap_{i \in I(z)} K_i'' \setminus \left(\bigcup_{j \in \{0,...,n\} \setminus I(z)} K_j'' \right) \tag{6}$$

as well as

$$V(z) := \bigcap_{i \in I(z)} O_i\ . \tag{7}$$

Obviously for every $z \in Z$ we have

$$F_A(C(z)) \subseteq F_A \left(\bigcap_{i \in I(z)} K_i'' \right) \subseteq \bigcap_{i \in I(z)} O_i \ = V(z)$$

implying $F_A \in (C(z), V(z))$.

The family of all $C(z)$ covers Z, because every $z \in Z$ is contained at least in it's own $C(z)$. Observe, that different $C(z_1)$ and $C(z_2)$ are disjoint: if $y \in C(z_1) \cap C(z_2)$ exists, then $I(z_1) = I(y) = I(z_2)$ follows, implying $C(z_1) = C(z_2)$ by (6).

Obviously, there are only finitely many different sets $C(z), V(z)$, because they are uniquely determined by $I(z)$, which is a subset of $\{0, ..., n\}$ and this set has just finitely many subsets. So, for simplicity, let us denote them by $C_1, ..., C_m$ and $V_1, ..., V_m$, respectively.

It is clear, that the C_j's are clopen (thus compact) and saturated in the sense of paragraph (b), by construction (6) from just clopen saturated K_i'''s.

For $G \in \bigcap_{j=1}^{m}(C_j, V_j) = \bigcap_{z \in Z}(C(z), V(z))$ we find

$$\begin{array}{rcl} \forall i \in \{0, ..., n\} & : & \forall z \in K_i'' \quad : \quad i \in I(z) \\ & & \quad \Longrightarrow \quad\quad : \quad G(z) \in V(z) \subseteq O_i \\ \Longrightarrow & : & G(K_i'') \subseteq O_i\ . \end{array}$$

Consequently, we have

$$F_A \in \bigcap_{j=1}^{m}(C_j, V_j) \subseteq \bigcap_{i=0}^{n}(K_i'', O_i) \tag{8}$$

(d) At last, let us chose for every $j = 1, ..., m$ an open set $U_j \in \sigma$ s.t. $F_A(C_j) \subseteq U_j \subseteq \overline{U_j} \subseteq V_j$ holds, as provided by proposition 1.3. Of course, we have then automatically $F_A \in (C_j, U_j) \subseteq (C_j, V_j)$.
So, because the C_j's cover Z, the $F_A(C_j)$'s cover A, and so the U_j's do.

With these U_j, $j = 1, ..., m$, we show $A \in \langle U_1, ..., U_j \rangle_{K(Y)} \subseteq \mathfrak{O}$.

$A \in \langle U_1, ..., U_m \rangle_{K(Y)}$ is clear, because the U_j's cover A, as seen in paragraph (d), and $\emptyset \neq F_A(C_j) \subseteq A \cap U_j$ for all $j = 1, ..., m$.

Let

$$B \in \langle U_1, ..., U_j \rangle_{K(Y)} \tag{9}$$

be given.

Because every C_j is nonempty clopen and saturated in the sense of paragraph (b), $C_j \cap (Y \times Y)$ has the cardinality of Y. So, there exists a surjection $t_j : C_j \cap (Y \times Y) \to U_j \cap B$ (the range is not empty by (9)).

Now, define

$$t : (Y \times Y) \to B : \ t(x, y) := t_j(x, y) \text{ for } (x, y) \in C_j$$

This t is well defined, because the C_j's are pairwise disjoint and cover Z by paragraph (c), and it is a surjection onto B, because the U_j's cover B by (9) and the t_j are surjections onto $U_j \cap B$. Our t is continuous w.r.t. the discrete topology on $Y \times Y$, so it extends to a continuous $T : Z \to B$.

By construction we have for each $j \in \{1, ..., m\}$

$$T\left(C_j \cap (Y \times Y)\right) \subseteq U_j \ , \tag{10}$$

implying $T(C_j) = T\left(\overline{C_j \cap (Y \times Y)}\right) \subseteq \overline{T\left(C_j \cap (Y \times Y)\right)} \subseteq \overline{U_j}$ by continuity, thus $T(C_j) \subseteq V_j$ by choice of U_j in paragraph (d).

We find $T \in \bigcap_{j=1}^{m} (C_j, V_j) \subseteq (\pi_Z \circ \mu_Z)^{-1}(\mathfrak{O})$, yielding $B = \pi_Z \circ \mu_Z(T) \in \mathfrak{O}$. This works for every $B \in \langle U_1, ..., U_m \rangle_{K(Y)}$, thus we have indeed $\langle U_1, ..., U_m \rangle_{K(Y)} \subseteq \mathfrak{O}$. Consequently, $\mathfrak{O}$ is a union of Vietoris-open subsets of $K(Y)$, just meaning $\mathfrak{O} \in \sigma_V$. □

Remark 2.5 Of course, $Y \times Y$ with discrete topology is homeomorphic to Y with discrete topology for infinite Y. So, we used $Y \times Y$ here just for convenience concerning the description of the „saturated“ subsets within the proof. Moreover, even for finite Y this proof works fine, but wouldn't do so with Y instead of $Y \times Y$.

Corollary 2.6 *Let (Y, σ) be a T_3-space. For every topological space let $C(X, Y)$ be equipped with compact-open topology. Let $\mathcal{B}$ be a class of topological spaces, that contains the Stone-Čech-compactification of a discrete space with cardinality at least card(Y).*
Then the Vietoris topology σ_V on $K(Y)$ is the final topology w.r.t. all $\pi_A \circ \mu_{(X,\tau)}$, $(X, \tau) \in \mathcal{B}$, $A \in K(X, \tau)$.

This characterization of the Vietoris hyperspace of the nonempty *compact* subsets of a regular space as a quotient (or more generally as a final object of a given class of spaces under certain mappings) includes an easy possibilty to characterize the Vietoris hyperspace of the nonempty *closed* subsets for Hausdorff T_4-spaces.

Lemma 2.7 *Let (Y, σ) be a Hausdorff T_4-space. Then its Vietoris hyperspace on the nonempty closed subsets $(Cl(Y), \sigma_V)$ is homeomorphic to a subspace of the Vietoris hyperspace $(K(\beta Y), \sigma^\beta)$ of compact subsets of the Stone-Čech-compactification of (Y, σ).*

Proof:

(1) The map
$$\alpha : Cl(Y) \to K(\beta Y) : \alpha(A) := \overline{A}^{\beta Y}$$
is injective: Let $A_1 \neq A_2 \in Cl(Y)$ be given, say w.l.o.g. $\exists a \in A_1 \setminus A_2$. Because Y is Tychonoff, we get a continuous $f : Y \to [0, 1]$ such that $f(a) = 0$ and $f(A_2) = \{1\}$, and then by the theorem of Stone-Čech a continuous extension $F : \beta Y \to [0, 1]$ with $F(a) = f(a) = 0$ and $F(A_2) = f(A_2) = \{1\}$, thus by continuity $F(\overline{A_2}^{\beta Y}) \subseteq \overline{F(A_2)} = \overline{\{1\}} = \{1\}$, implying $a \notin \overline{A_2}^{\beta Y}$ and consequently $\overline{A_1}^{\beta Y} \neq \overline{A_2}^{\beta Y}$.

(2) α is continuous w.r.t. to $\sigma_V, (\sigma^\beta)_V$:

- Let $O \in \sigma^\beta$ and
$$\begin{aligned} A_0 \in \alpha^{-1}\left(O^{-K(\beta Y)}\right) &= \{A \in Cl(Y) | \; \overline{A}^{K(\beta Y)} \cap O \neq \emptyset\} \\ &= \{A \in Cl(Y) | \; A \cap O \neq \emptyset\} \end{aligned}$$
be given.
Because Y is a dense subspace of βY, we get $\emptyset \neq O \cap Y \in \sigma$ and $A_0 \in (O \cap Y)^{-Cl(Y)} \subseteq \alpha^{-1}\left(O^{-K(\beta Y)}\right)$. Thus $\alpha^{-1}\left(O^{-K(\beta Y)}\right)$ is open in σ_V.
- Let $O \in \sigma^\beta$ and
$$A_0 \in \alpha^{-1}\left((\beta Y \setminus O)^{+K(\beta Y)}\right) = \{A \in Cl(Y) | \; \overline{A}^{K(\beta Y)} \subseteq O\}$$
be given.

Now, βY is T_3 and $\overline{A_0}^{K(\beta Y)}$ is compact, so by proposition 1.3 we get an $U_0 \in \sigma^\beta$ with $\overline{A_0}^{K(\beta Y)} \subseteq U_0 \subseteq \overline{U_0}^{K(\beta Y)} \subseteq O$. So, we have $A_0 \subseteq U_0 \cap Y \in \sigma$ and furthermore $\forall A \in (Y \setminus U_0)^{+_{Cl(Y)}} : \overline{A}^{K(\beta Y)} \subseteq \overline{U_0}^{K(\beta Y)} \subseteq O$, yielding $A_0 \in (Y \setminus U_0)^{+_{Cl(Y)}} \subseteq \alpha^{-1}\left((\beta Y \setminus O)^{+_{K(\beta Y)}}\right)$. Consequently, $\alpha^{-1}\left((\beta Y \setminus O)^{+_{K(\beta Y)}}\right)$ is open in σ_V.

Note, that we didn't use T_4 so far.

(3) α is an open map onto its image.
Let $U_1, ..., U_n \in \sigma$ be given.

Let $A \in \langle U_1, ..., U_n\rangle_{Cl(Y)}$.
We have $A \subseteq \bigcup_{i=1}^n U_i \;\Rightarrow\; A \cap \left(\bigcap_{i=1}^n (Y \setminus U_i)\right) = \emptyset$. So, A and $\bigcap_{i=1}^n (Y \setminus U_i)$ are disjoint closed subsets of Y, which can be separated by a continuous function from Y to $[0,1]$, according to T_4. This function extends to a continuous function from βY to $[0,1]$ by the Stone-Čech theorem, yielding $\emptyset = \overline{A}^{\beta Y} \cap \overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}$, so we have $\overline{A}^{\beta Y} \subseteq \beta Y \setminus \left(\overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}\right)$.

Furthermore, $A \cap U_i \neq \emptyset$ implies $A \not\subseteq Y \setminus U_i$, and this yields by the same argument as in (1), that $\overline{A}^{\beta Y} \not\subseteq \overline{Y \setminus U_i}^{\beta Y}$, thus $\overline{A}^{\beta Y} \cap \left(\beta Y \setminus \left(\overline{Y \setminus U_i}^{\beta Y}\right)\right) \neq \emptyset$.

So, let $V_0 := \beta Y \setminus \left(\overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}\right)$ and for $i = 1, ..., n$ we define $V_i := \beta Y \setminus (\overline{Y \setminus U_i}^{\beta Y}) \in \sigma^\beta$.

Note, that $\bigcup_{i=1}^n \left(\beta Y \setminus \left(\overline{Y \setminus U_i}^{\beta Y}\right)\right) \subseteq \beta Y \setminus \left(\overline{\bigcap_{i=1}^n (Y \setminus U_i)}^{\beta Y}\right)$ holds, i.e.

$$\bigcup_{i=1}^n V_i \subseteq V_0 \; . \tag{11}$$

So we get $\alpha(A) \in \langle V_0, V_1, ..., V_n\rangle_{K(\beta Y)}$ from the above.
This for all $A \in \langle U_1, ..., U_n\rangle_{Cl(Y)}$ yields

$$\alpha(\langle U_1, ..., U_n\rangle_{Cl(Y)}) \subseteq \langle V_0, V_1, ..., V_n\rangle_{K(\beta Y)} \; . \tag{12}$$

If otherwise $A \in Cl(Y)$ is given with $\alpha(A) = \overline{A}^{\beta Y} \in \langle V_0, V_1, ..., V_n\rangle_{K(\beta Y)}$, then for $i = 1, ..., n$ we get from $\emptyset \neq \overline{A}^{\beta Y} \cap V_i = \overline{A}^{\beta Y} \cap \left(\beta Y \setminus (\overline{Y \setminus U_i}^{\beta Y})\right)$, that $\overline{A}^{\beta Y} \not\subseteq \overline{Y \setminus U_i}^{\beta Y}$ and consequently $A \not\subseteq Y \setminus U_i$, thus $A \cap U_i \neq \emptyset$.

Moreover, from $\overline{A}^{\beta Y} \subseteq (\bigcup_{k=0}^{n} V_k) \; = \; V_0$ we get

$$\begin{aligned} A \; &\subseteq \; Y \cap \overline{A}^{\beta Y} \subseteq Y \cap \left(\beta Y \setminus \overline{\bigcap_{i=1}^{n} Y \setminus U_i}^{\beta Y} \right) \\ &\subseteq \; Y \setminus \overline{\bigcap_{i=1}^{n} Y \setminus U_i}^{\beta Y} \subseteq Y \setminus \bigcap_{i=1}^{n} Y \setminus U_i \\ &\subseteq \; \bigcup_{i=1}^{n} (Y \setminus (Y \setminus U_i)) \; = \; \bigcup_{i=1}^{n} U_i \end{aligned}$$

This yields

$$\alpha^{-1}(\langle V_0, V_1, ..., V_n \rangle_{K(\beta Y)}) \subseteq \langle U_1, ..., U_n \rangle_{Cl(Y)} \; . \tag{13}$$

So, from (12) and (13) we get

$$\alpha(\langle U_1, ..., U_n \rangle_{Cl(Y)}) = \alpha(Cl(Y)) \cap \langle V_0, V_1, ..., V_n \rangle_{K(\beta Y)} \; .$$

□

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Author:

René Bartsch
Technische Universität Darmstadt
Fachbereich Mathematik
Schlossgartenstr. 7
D-64289 Darmstadt (Germany)

e-mail: rbartsch@mathematik.tu-darmstadt.de
(or math@marvinius.net)