Rostock. Math. Kolloq. **68**, 13 – 43 (2013)

Subject Classification (AMS)
47H04, 47H09, 65JXX

Dieter Schott

Projection kernels of linear operators and convergence considerations

ABSTRACT. In the study of iterative methods used to solve linear operator equations sequences of linear iteration operators (T_k) occur which have a nontrivial projection kernel, that is a linear projector $P \neq O$ satisfying $P = T_k P = P T_k$ for all natural k. The convergence proof for (T_k) or some related operator sequences is simplified if such P is known. It is investigated when projection kernels exist and how they can be determined. Besides, special types of projection kernels are considered.

KEY WORDS. Linear operators, Fejér monotone operators, nonexpansive operators, projectors, orthoprojectors, relaxation of orthoprojectors

1 Introduction

It is remarkable that sequences (T_k) of linear (bounded) operators occuring in iterative methods for linear operator equations or in ergodic theory often have the following property:

(*) There is a projector $P \neq O$ with $T_k P = P T_k = P$ for all $k \in \mathrm{N}$.

Such a projector P is called a (nontrivial) *projection kernel* of (T_k). E.g., if a linear bounded operator T acting on a (real) Banach space X is *asymptotically convergent*, that is, if the power sequence (T^k) is convergent (to a linear bounded operator $T^\infty \neq O$), then (*) is fulfilled for $P = T^\infty$ and $T_k = T^k$. If $T^\infty = O$, then only the trivial projection kernel $P = O$ exists. In both cases the decomposition

$$X = \mathbb{N}(I-T) \oplus \overline{\mathbb{R}(I-T)}, \quad \mathbb{R}(P) = \mathbb{N}(I-T), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I-T)}$$

holds, where I is the identity operator. Reversely, if such a projection kernel P is known, the convergence investigation of (T^k) can be reduced to the invariant subspace $\mathbb{N}(P)$, while $\mathbb{R}(P)$ is the fixed point set of T. More generally the knowledge of a projection kernel P simplifies the convergence proof for (T_k) or for other related sequences. In this section we investigate, when sequences or sets of operators possess projection kernels and how they can be determined. Later we specify also orthogonal, maximal, optimal and attainable projection kernels. The starting point of these investigations is my paper [11]. In the mean time some new aspects, examples and results can be presented.

For motivation we state some results concerning the iterative solution of linear equations with operators acting in BANACH spaces X and Y. Let $\mathcal{L}(X,Y)$ be the algebra of linear bounded operators from X into Y, let

$$Ax = b, \quad A \in \mathcal{L}(X,Y), \quad b \in Y \tag{1.1}$$

be an equation with unknowns $x \in X$ and let (D_k) be a given operator sequence with $D_k \in \mathcal{L}(Y,X)$. Then linear iterative methods

$$x_{k+1} := T_k x_k + D_k b, \quad T_k := I - D_k A, \quad x_0 \in X \text{ arbitrary} \tag{1.2}$$

for the solution of (1.1) can be constructed. The defects $r_k := b - Ax_k$ are obtained by

$$r_{k+1} := S_k r_k, \quad S_k := I - AD_k, \quad r_0 := b - Ax_0 \in Y. \tag{1.3}$$

Explicitly we have the representations

$$x_{k+1} = T_{k,0}x_0 + B_k b, \quad B_k := \sum_{i=0}^{k} T_{k,i+1} D_i, \quad r_{k+1} = S_{k,0} r_0, \tag{1.4}$$

where the product notation $U_{i,j} := U_i \dots U_{j+1} U_j$ for $i \geq j$ is used (see e.g. [1], [12]). If $D_k = D$ is constant for all k, then we get from (1.2) and (1.3) the stationary method

$$x_{k+1} := Tx_k + Db = T^{k+1}x_0 + \sum_{i=0}^{k} T^i Db, \quad r_{k+1} := Sr_k = S^{k+1} r_0 \tag{1.5}$$

with $T := I - DA$ and $S := I - AD$. If (D_k) is cyclic and iteration is considered in cycles, then again stationary methods of type (1.5) arise.

We state now some examples for $X = \mathrm{R}^n$ and $Y = \mathrm{R}^m$ (finite-dimensional case). Then equation (1.1) is a system of m linear equations with n scalar unknowns. Further, the operator A can be identified with a matrix $A \in \mathrm{R}^{m,n}$. The adjoint operator A^* is realized by the transpose A^t of the corresponding matrix A.

Example 1.1 (Stationary iteration) If the method (1.5) is investigated, the convergence of the power sequences (T^k) and (S^k) as well as of the NEUMANN series $(\sum_{i=0}^{k} T^i)$ is of interest.

Example 1.2 (PSH method) see [8] and [6]: p. 53f. We start with matrices E_k which select one or more linearly independent rows of the matrix A in steps k in such a way that each (non-vanishing) row is selected at least once in certain step sections uniformly bounded for all k (as cycles if (E_k) is cyclic). Defining matrices

$$D_k := A^* E_k^* (E_k A A^* E_k^*)^{-1} E_k$$

the corresponding iterative method (1.2) projects in each step k orthogonally onto subspaces of R^n formed by intersection of the hyperplanes corresponding to the rows in $E_k A$. Further, the following can be shown:

a) The sequence (T_k) of orthoprojectors $T_k := I - D_k A$ has the orthogonal projection kernel P with $\mathbb{R}(P) = \mathbb{N}(A)$ and $\mathbb{N}(P) = \mathbb{R}(A^*)$. The product sequence $(T_{k,0})$ converges to this P.

b) The sequence (S_k) of operators $S_k := I - AD_k$ has a projection kernel Q with $\mathbb{N}(Q) = \mathbb{R}(A)$. The product sequence $(S_{k,0})$ converges to this Q.

Example 1.3 (SPA method) see [7] and [6]: p. 38f. We start with matrices F_k which select one or more linearly independent columns of the matrix A in steps k in such a way that each (non-vanishing) column is selected at least once in certain step sections uniformly bounded for all k. Defining matrices

$$D_k := F_k (F_k^* A^* A F_k)^{-1} F_k^* A^*$$

the corresponding iterative method (1.3) projects in each step k orthogonally onto subspaces of R^m spanned by the rows in AF_k. Further, the following can be shown:

a) The sequence (S_k) of orthoprojectors $S_k := I - AD_k$ has the orthogonal projection kernel Q with $\mathbb{R}(Q) = \mathbb{N}(A^*)$ and $\mathbb{N}(Q) = \mathbb{R}(A)$. The product sequence $(S_{k,0})$ converges to this Q.

b) The sequence (T_k) of operators $T_k := I - D_k A$ has a projection kernel P with $\mathbb{R}(P) = \mathbb{N}(A)$. The product sequence $(T_{k,0})$ converges to this P.

The methods described in Example 1.2 and Example 1.3 can be generalized in various ways by conservation of the main results (see [16]).

Example 1.4 (A gradient method for regular systems) see [2]. Let A be a regular quadratic matrix $(m = n)$. We consider row vectors H_k containing the signs of the k-th columns $\vec{a}_k$ of A, i.e. $H_k := (\operatorname{sign} \vec{a}_k)^*$. Now we define matrices

$$D_k := A^* H_k^* (H_k A A^* H_k^*)^{-1} H_k.$$

Then the operators of the iterative methods (1.2) and (1.3) have the following properties:

a) The sequences (T_k) and (S_k) have only the trivial projection kernel O.

b) The product sequences $(T_{k,0})$ and $(S_{k,0})$ converge to O.

Example 1.5 (A general case with operator relations) see [6]: p. 32 and [1]. We consider the general iterative method described in (1.2) and (1.3). If the operator sequence (B_k) occurring in (1.4) converges, say $\lim_{k\to\infty} B_k = B_\infty$, and if moreover

$$D_k A B_\infty = B_\infty A D_k = D_k \quad \text{for all } k,$$

then the following holds:

a) (T_k) has the projection kernel $P = I - B_\infty A$ and $(T_{k,0})$ converges to P.

b) (S_k) has the projection kernel $Q = I - AB_\infty$ and $(S_{k,0})$ converges to Q.

2 Projection kernels of operator sets

Let X be a (real) Banach space. In the following we consider projectors $P \in \mathcal{L}(X)$, that means $P^2 = P$, and sets $\mathcal{T}$ of operators $T \in \mathcal{L}(X)$. We start with a well-known fact.

Proposition 2.1 *A linear projector P is bounded (continuous) and induces the space decomposition*

$$X = \mathbb{R}(P) \oplus \mathbb{N}(P) = \mathbb{N}(I-P) \oplus \mathbb{R}(I-P),$$

where ranges and nullspaces are (closed) linear subspaces of X. Moreover, P is uniquely determined by this decomposition. The operator $I - P$ is a projector, too, with analogue properties.
A projector P is an orthoprojector ($\mathbb{R}(P) \perp \mathbb{N}(P)$) iff P is self-adjoint ($P = P^*$). An orthoprojector P is uniquely determined by its range $\mathbb{R}(P)$ (see e.g. [10], section 5.6).

Now the main concept of the paper is introduced.

Definition 2.1 *The projector P is said to be a*

- left projection kernel *of $\mathcal{T}$ if $P = PT$ for all $T \in \mathcal{T}$ ($P \in \mathbb{K}_l(\mathcal{T})$).*
- right projection kernel *of $\mathcal{T}$ if $P = TP$ for all $T \in \mathcal{T}$ ($P \in \mathbb{K}_r(\mathcal{T})$).*
- projection kernel *of $\mathcal{T}$ if $P = PT = TP$ for all $T \in \mathcal{T}$ ($P \in \mathbb{K}(\mathcal{T})$).*

In brackets the short notations are given. Another expression for $P \in \mathbb{K}(\mathcal{T})$ is that $\mathcal{T}$ has the projection kernel P.

Remark 2.1 If sequences (T_k) are involved, we write simply (T_k) instead of the set notation $\mathcal{T} = \{T_k : k \in \mathrm{N}\}$. If $\mathcal{T} = \{T\}$ contains only one operator T we often write simply T instead.

Trivially, O is a projection kernel of all operator sets $\mathcal{T}$ ($O \in \mathbb{K}(\mathcal{T})$). For completion we define that P is a projection kernel of $\emptyset$, the empty set in $\mathcal{L}(X)$. Although the identity $P = I$ is a projector, it is no projection kernel of $\mathcal{T}$ if $\mathcal{T}$ contains operators $T \neq I$.

By definition P is a projection kernel of $\mathcal{T}$ iff it is both a left and a right projection kernel of $\mathcal{T}$. The following example shows that indeed left or right projection kernels need not to be projection kernels.

Example 2.1 Consider the matrices

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathrm{R}^{3,3}$. Then the following relations are satisfied:

$$P^2 = P = PT = TP = T^*P = PT^*,$$
$$Q^2 = Q = TQ = QT^*, \quad Q \neq QT, \quad Q \neq QT^*.$$

Hence, P is a projection kernel of T and T^*, while Q is neither a projection kernel of T nor of T^*. But Q is a right projection kernel of T and a left projection kernel of T^*.

Now we list some simple statements about projection kernels. If a proof is missing it is either obvious or it is a simple consequence of more general statements given later.

Proposition 2.2 *Let $\mathcal{T} \setminus \{I\} \neq \emptyset$. If P is a projection kernel of $\mathcal{T}$, then $I - P$ is not.*

Proof: We assume $P \in \mathbb{K}(\mathcal{T})$. Then the projector $I - P$ satisfies

$$(I - P)T = T(I - P) = T - P \neq I - P$$

for $T \neq I$. But such T are supposed to be in $\mathcal{T}$ by assumption. ■

Proposition 2.3

a) *If P is a projection kernel of $\mathcal{T}_1$, then also of $\mathcal{T}_2 \subseteq \mathcal{T}_1$.*

b) *If P is a projection kernel of both $\mathcal{T}_1$ and $\mathcal{T}_2$, then also of $\mathcal{T}_1 \cup \mathcal{T}_2$.*

c) *Each projector P is a projection kernel of itself ($P \in \mathbb{K}(P)$).*

d) *Each projector P is a projection kernel of I ($P \in \mathbb{K}(I)$).*

e) *If P is a projection kernel of $\mathcal{T}$, then also of $\mathcal{T} \cup \{I\}$.*

f) *If $P \neq O$ is a projection kernel of $\mathcal{T}$, then $O \notin \mathcal{T}$.*

The next statements refer to operations conserving projection kernels.

Lemma 2.1 *If P is a (left, right) projection kernel of both T_1 and T_2, then P is also a (left, right) projection kernel of the products $T_1 \cdot T_2$ and $T_2 \cdot T_1$ as well as of the linear combinations $\lambda_1 T_1 + \lambda_2 T_2$ with $\lambda_1 + \lambda_2 = 1$.*

Proof: We assume that $P \in \mathbb{K}(\{T_1, T_2\})$. By the way, the proofs for $P \in \mathbb{K}_l(\{T_1, T_2\})$ and $P \in \mathbb{K}_r(\{T_1, T_2\})$ are included as parts. Since

$$P = PT_i = T_i P \quad (i = 1, 2)$$

holds, we have for $T := T_1 T_2$:

$$PT = PT_1T_2 = PT_2 = P, \quad TP = T_1T_2P = T_1P = P.$$

Hence T is a projection kernel of P. Analogously this can be shown for $T := T_2T_1$. If $T := \lambda_1 T_1 + \lambda_2 T_2$ and $\lambda_1 + \lambda_2 = 1$, then

$$PT = P(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 PT_1 + \lambda_2 PT_2 = \lambda_1 P + \lambda_2 P = P.$$

Analogously $TP = P$ is proven for this T. ■

Corollary 2.1 *If P is a projection kernel of $\mathcal{T}$, then P is the projection kernel of the generated multiplicative semi-group $[\mathcal{T}](I)$ with identity I and of the affine hull* $\mathrm{aff}(\mathcal{T})$.

The next statement considers the aspect of regular (invertible) transformations in $\mathcal{L}(X)$.

Proposition 2.4 *Let S be regular. If P is a (left, right) projection kernel of $\mathcal{T}$, then P_S is a (left, right) projection kernel of $\mathcal{T}_S$, where $P_S := S^{-1}PS$ and $\mathcal{T}_S := S^{-1}\mathcal{T}S$.*

Proof: Under the given assumptions it is

$$P_S T_S = S^{-1}PS \cdot S^{-1}TS = S^{-1}PTS = S^{-1}PS = P_S,$$
$$T_S P_S = S^{-1}TS \cdot S^{-1}PS = S^{-1}TPS = S^{-1}PS = P_S$$

for all $T_S \in \mathcal{T}_S$. ■

Proposition 2.5 *If P is a (left, right) projection kernel of $\mathcal{T}$, then the dual (adjoint) P^* is a (right, left) projection kernel of $\mathcal{T}^*$.*

Proof: The assertion follows from the equations

$$(P^2)^* = (P^*)^2, \quad (PT)^* = T^*P^*, \quad (TP)^* = P^*T^*. \ ■$$

Now examples for projection kernels of operator sets are given.

Example 2.2 We consider $X = \mathrm{R}^n$ and the sets

$$\mathcal{T}_m = \left\{ \begin{pmatrix} I_{m,m} & O_{m,n-m} \\ O_{n-m,m} & T_{n-m,n-m} \end{pmatrix} \right\}, \quad \mathcal{P}_l = \left\{ \begin{pmatrix} I_{l,l} & O_{l,n-l} \\ O_{n-l,l} & O_{n-l,n-l} \end{pmatrix} \right\}$$

of matrices in $\mathrm{R}^{n,n}$, where l,m, n are natural numbers with $1 \le l \le m \le n$ and m, n fixed. The indices indicate the size of the submatrices. Further, indexed I stands for identity submatrices and indexed O for zero submatrices. The matrices act as linear operators on R^n. The set $\mathcal{T}_m$ is a subring and a subalgebra of $\mathrm{R}^{n,n}$ containing the identity (matrix).
It is easy to check that each operator $P_l \in \mathcal{P}_l$ is a projection kernel of the set $\mathcal{T}_m$. Hence, there are different projection kernels for the same operator set.
Let us fix an operator $T_m = T \in \mathcal{T}_m$. Then each $P_l \in \mathcal{P}_l$ is also a projection kernel of the power sequence (T^k), where obviously $T^k \in \mathcal{T}_m$ for all $k \in \mathrm{N}$.

The example presents matrices in a canonical form. We can produce many other examples applying a regular matrix $S \in \mathrm{R}^{n,n}$ and its inverse, namely

$$\mathcal{P}_l^S = S^{-1}\mathcal{P}_l S, \quad \mathcal{T}_m^S = S^{-1}\mathcal{T}_m S$$

(see Proposition 2.4). Reversely, for an operator set $\mathcal{T}$ and a projection kernel P we can look for regular matrices S transforming the operators into a canonical form.

Example 2.3 Let us consider the matrices

$$P = \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \dots & \vec{e}_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathrm{R}^{n,n},$$

$$T = \begin{pmatrix} \vec{e}_1 & \vec{t}_2 & \dots & \vec{t}_n \end{pmatrix} = \begin{pmatrix} 1 & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & t_{n2} & \dots & t_{nn} \end{pmatrix} \in \mathrm{R}^{n,n} :$$

$$\sum_{i=1}^{n} t_{ij} = 1 \; (1 < j \le n),$$

where $\vec{e}_1$ is the first column of the identity matrix $I \in \mathrm{R}^{n,n}$ and the sums of the columns $\vec{t}_j = (t_{ij})$ are equal to 1. Then P is a projection kernel and also an element of the set $\mathcal{T}$ of all such operators T. By the way, $\mathcal{T}$ is a noncommutative semi-group with respect to matrix multiplication. Further, $P^* = P^t$, the matrix with first row elements 1 and other elements 0, is a projection kernel of $\mathcal{T}^*$ whose operators T^* have the same first row as I and the (other) row sums are always equal to 1.

Example 2.4 Let X be a (real) HILBERT space and $T \in \mathcal{L}(X)$ nonexpansive. Then the orthoprojector P defined by $\mathbb{R}(P) = \mathbb{N}(I - T)$ is a projection kernel of T, its powers T^k ($k \in \mathrm{N}$) and their affine combinations (see Section 6 and Corollary 2.1).

3 Properties of projection kernels

Now we look for simple conditions to determine projection kernels. Obviously the relation $P \in \mathbb{K}(\mathcal{T})$ can be characterized by the behavior of operators $T \in \mathcal{T}$ on $\mathbb{R}(P)$ and $\mathbb{N}(P)$. We introduce the abbreviations

$$\mathbb{N}(I - \mathcal{T}) := \bigcap_{T \in \mathcal{T}} \mathbb{N}(I - T), \quad \overline{\mathbb{R}(I - \mathcal{T})} := \operatorname{span} \bigcup_{T \in \mathcal{T}} \mathbb{R}(I - T). \tag{3.1}$$

Both defined sets are (closed) linear subspaces. The set $\mathbb{N}(I - \mathcal{T})$ is the common fixed point set $\mathbb{F}(\mathcal{T})$ of $\mathcal{T}$. The span operation contains the closure of the corresponding set. In finite-dimensional spaces the closure operation can be omitted.

Lemma 3.1 *The following conditions are equivalent for an operator P and operators in a set $\mathcal{T}$:*

a1) $P = TP$ *for all* $T \in \mathcal{T}$,

b1) $T = P + T(I - P)$ *for all* $T \in \mathcal{T}$,

c1) $T \,|\, \mathbb{R}(P) = I \,|\, \mathbb{R}(P)$ *for all* $T \in \mathcal{T}$,

d1) $\mathbb{R}(P) \subseteq \mathbb{N}(I - \mathcal{T})$.

Proof: The equivalence of a1), b1) and c1) is obvious. Besides, a1) is fulfilled iff the equation $(I - T)P = O$, that means $\mathbb{R}(P) \subseteq \mathbb{N}(I - T)$, holds for all $T \in \mathcal{T}$. Hence also a1) and d1) are equivalent. ■

Lemma 3.2 *The following conditions are equivalent for an operator P and operators in a set $\mathcal{T}$:*

a2) $P = PT$ *for all* $T \in \mathcal{T}$,

b2) $T = P + (I - P)T$ *for all* $T \in \mathcal{T}$,

d2) $\mathbb{N}(P) \supseteq \overline{\mathbb{R}(I - \mathcal{T})}$.

From each of these conditions follows

c2) $T\,\mathbb{N}(P) \subseteq \mathbb{N}(P)$ *for all* $T \in \mathcal{T}$.

Proof: The equivalence of a2) and b2) is obvious. Further a2) is fulfilled iff $P(I-T) = O$, that is $\mathbb{N}(P) \supseteq \mathbb{R}(I-T)$, holds for all $T \in \mathcal{T}$. Since $\mathbb{N}(P)$ is a closed linear subspace, also a2) and d2) are equivalent. Finally, supposing a2), $Px = 0$ supplies $PTx = 0$ for all $x \in X$. But this is c2). ■

Theorem 3.1 *The following conditions are equivalent for a projector P and operators in a set $\mathcal{T}$:*

a) $P = TP = TP$ *for all* $T \in \mathcal{T}$*, that is* $P \in \mathbb{K}(\mathcal{T})$*,*

a') $(I-T)P = P(I-T) = O$ *for all* $T \in \mathcal{T}$*,*

b) $T = I \,|\, \mathbb{R}(P) \oplus T \,|\, \mathbb{N}(P)$ *for all* $T \in \mathcal{T}$*,*

c) $T \,|\, \mathbb{R}(P) = I \,|\, \mathbb{R}(P)$*,* $T\,\mathbb{N}(P) \subseteq \mathbb{N}(P)$ *for all* $T \in \mathcal{T}$*,*

d) $\mathbb{R}(P) \subseteq \mathbb{N}(I-\mathcal{T})$ *and* $\mathbb{N}(P) \supseteq \overline{\mathbb{R}(I-\mathcal{T})}$*.*

Proof: A great part of the assertions is obtained by combination of Lemma 3.1 and Lemma 3.2. Thus a) and d) are equivalent. Further a) and a') are equivalent because a') can be written as $P - TP = P - PT = O$. Since P is a projector $T(I-P) = (I-P)T$ means that $\mathbb{R}(P)$ and $\mathbb{N}(P)$ are invariant linear subspaces of T. Now the equivalence of a), b) and c) is obvious. ■

The conditions b) and c) play an important part for considering convergence of operators. The condition d) is especially useful for determining suitable projection kernels.

Corollary 3.1 *If P is a projection kernel of $\mathcal{T}$, then all operators $T \in \mathcal{T}$ map for all $x \in X$ the affine subspaces $x + \mathbb{N}(P)$ into itself and the affine subspaces $x + \mathbb{R}(P)$ onto the affine subspaces $Tx + \mathbb{R}(P)$.*

Proof: Let be $P \in \mathbb{K}(\mathcal{T})$. First $(I-P)x \in N(P)$ because of $P^2 = P$. Hence

$$x + \mathbb{N}(P) = Px + (I-P)x + \mathbb{N}(P) = Px + \mathbb{N}(P).$$

Having also Theorem 3.1 in mind, we get

$$\begin{aligned} T(x + \mathbb{N}(P)) &= T(Px + \mathbb{N}(P)) = TPx + T\,\mathbb{N}(P) \\ &\subseteq Px + \mathbb{N}(P) = x + \mathbb{N}(P), \\ T(x + \mathbb{R}(P)) &= Tx + T\,\mathbb{R}(P) = Tx + TP\,\mathbb{R}(P) \\ &= Tx + P\,\mathbb{R}(P) = Tx + \mathbb{R}(P). \blacksquare \end{aligned}$$

The corollary shows that the operators T map affine subspaces which are parallel to $\mathbb{R}(P)$ again into such subspaces. Further, all images Tx of x remain in the affine subspace $x + \mathbb{N}(P)$.

Relations between ranges and nullspaces of projectors can be used to define a semi-order between projectors.

Definition 3.1 *We write $P \leq Q$ for two projectors P and Q, if $\mathbb{R}(P) \subseteq \mathbb{R}(Q)$ and $\mathbb{N}(P) \supseteq \mathbb{N}(Q)$. We write $P < Q$ if $P \leq Q$ and $P \neq Q$.*

Proposition 3.1 *If P is a projection kernel of the projector Q, then $P \leq Q$ holds.*

Proof: The assumption $P \in \mathbb{K}(Q)$ implies by Theorem 3.1 the relations $\mathbb{R}(P) \subseteq \mathbb{R}(Q)$ and $\mathbb{N}(P) \supseteq \mathbb{N}(Q)$. By Definition 3.1 this is $P \leq Q$. ■

In Example 2.2 the projectors P_l fulfil the relations $P_l < P_{l+1}$ for $1 \leq l \leq n-1$. The following statement shows how we can construct 'smaller' and 'bigger' projection kernels.

Proposition 3.2 *If P_1 and P_2 are commutable projection kernels of $\mathcal{T}$, then $P = P_1P_2$ and $\tilde{P} = P_1 + P_2 - P_1P_2$ are projection kernels of $\mathcal{T}$ satisfying*

$$\begin{aligned} \mathbb{R}(P) &= \mathbb{R}(P_1) \cap \mathbb{R}(P_2), \quad \mathbb{N}(P) = \operatorname{span}\left(\mathbb{N}(P_1) \cup \mathbb{N}(P_2)\right) \\ \mathbb{R}(\tilde{P}) &= \operatorname{span}\left(\mathbb{R}(P_1) \cup \mathbb{R}(P_2)\right), \quad \mathbb{N}(\tilde{P}) = \mathbb{N}(P_1) \cap \mathbb{N}(P_2). \end{aligned}$$

This means $P \leq P_1, P_2 \leq \tilde{P}$ and $P < \tilde{P}$ for $P_1 \neq P_2$.

Proof: The first part is shown in [11], p. 33. The relations between ranges and nullspaces supply

$$\mathbb{R}(P) \subseteq \mathbb{R}(P_i) \subseteq \mathbb{R}(\tilde{P}), \quad \mathbb{N}(P) \supseteq \mathbb{N}(P_i) \supseteq \mathbb{N}(\tilde{P}) \quad (i = 1, 2).$$

Hence, the relations $P \leq P_1, P_2 \leq \tilde{P}$ follow by Definition 3.1. Finally we suppose $P_1 \neq P_2$. In contrary to the assertion we assume $P = \tilde{P}$. By the above relations we get $\mathbb{R}(P_1) = \mathbb{R}(P_2)$ and $\mathbb{N}(P_1) = \mathbb{N}(P_2)$. Proposition 2.1 shows that $P_1 = P_2$. This is a contradiction. Hence, $P < \tilde{P}$ is true. ■

4 Special kinds of projection kernels

If we investigate the convergence behavior of a operator sequence (T_k), we are interested in projection kernels P with maximal range $\mathbb{R}(P)$, where the operators T_k are the identity (see Theorem 3.1). Further, if the limit of (T_k) is P, then P is in the closure of $\{T_k : k \in \mathrm{N}\}$.

Definition 4.1 *Let P be a (left, right) projection kernel of $\mathcal{T}$. Then P is called*

- nontrivial *iff $P \neq O$,*
- maximal *iff there is no other projection kernel Q of $\mathcal{T}$ with $\mathbb{R}(Q) \supset \mathbb{R}(P)$,*

- orthogonal *iff P is an orthoprojector (in a* HILBERT *space X),*
- attainable *(r.t. operator topology τ), iff P is in the (τ-) closure of $\mathcal{T}$.*

Proposition 4.1 *Generally projection kernels P of an operator set $\mathcal{T}$ are neither uniquely determined nor maximal, orthogonal or attainable.*

This can be seen by the examples. The next example also exposes that the method of projection kernels has limitations.

Example 4.1 For $X = \mathrm{R}^3$ we investigate operators

$$T(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} : \quad c \in \mathrm{R}.$$

The matrices $T(c)$ have the determinant 1 and inverses $T(-c)$. Further, it holds

$$\mathbb{N}(I - T(c)) = \mathbb{N}(I - T(1)) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \ (c \neq 0),$$
$$\mathbb{N}(I - T(0)) = \mathbb{N}(O) = \mathrm{R}^3,$$
$$\mathbb{R}(I - T(c)) = \mathbb{R}(I - T(1)) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \ (c \neq 0),$$
$$\mathbb{R}(I - T(0)) = \mathbb{R}(O) = \{0\}.$$

The set $\mathcal{T}_0$ of all such operators $T(c)$ is a multiplicative commutative group. Now we consider subsets $\mathcal{T}$ only assuming $\mathcal{T} \setminus \{I\} \neq \emptyset$. Hence, $\mathcal{T}$ contains at least one operator $T(c)$ with $c \neq 0$. Then we get

$$\mathbb{N}(I - \mathcal{T}) = \bigcap_{T(c) \in \mathcal{T}} \mathbb{N}(I - T(c)) = \mathbb{N}(I - T(1)),$$
$$\overline{\mathbb{R}(I - \mathcal{T})} = \operatorname{span} \bigcup_{T(c) \in \mathcal{T}} \mathbb{R}(I - T(c)) = \mathbb{R}(I - T(1)).$$

It is easy to check that the set $\mathcal{P} = \mathbb{K}(\mathcal{T})$ of all projection kernels consists of the matrices

$$P(a, b) = \begin{pmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ ab & b & 0 \end{pmatrix} : \quad a, b \in \mathrm{R}.$$

The ranges and nullspaces are

$$\mathbb{R}(P(a,b)) = \text{span}\left\{\begin{pmatrix}0\\1\\b\end{pmatrix}\right\}, \ \mathbb{N}(P(a,b)) = \text{span}\left\{\begin{pmatrix}1\\-a\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

These results show the relations

$$\mathbb{R}(P(a,b)) \subset \mathbb{N}(I-\mathcal{T}), \quad \mathbb{N}(P(a,b)) \supset \mathbb{R}(I-\mathcal{T})$$

such that condition d) in Theorem 3.1 is fulfilled properly, not reaching set equality. For all operators $P(a,b)$ the range is one-dimensional. Hence all these projection kernels are nontrivial and even maximal. Since $\mathcal{T}$ is a set whose closure does not contain operators of $\mathcal{P}$, all projection kernels are not attainable. The power sequence $(T(1)^k) = (T(k))$ is divergent, but $(T_k) = (T(0.5^k))$ tends to I, which is no projection kernel. The constant sequence $(T_k) = (T(1))$ tends to $T(1)$ which is even no projector $(T(1)^2 = T(2) \neq T(1))$. Further, only the projection kernel $P(0,0)$ is orthogonal (self-adjoint).

We turn to the question if always maximal projection kernels exist.

Theorem 4.1 *Each set $\mathcal{T}$ of operators with finite-dimensional subspace $\mathbb{N}(I-\mathcal{T})$ has at least one maximal projection kernel.*

Proof: Because of $O \in \mathbb{K}(\mathcal{T})$ it is $\mathbb{K}(\mathcal{T}) \neq \emptyset$. For $P \in \mathbb{K}(\mathcal{T})$ it holds $\mathbb{R}(P) \subseteq \mathbb{N}(I-\mathcal{T})$ and therefore $\dim \mathbb{R}(P) \leq \dim \mathbb{N}(I-\mathcal{T}) =: n < \infty$. Hence, there is a $\tilde{P} \in \mathbb{K}(\mathcal{T})$ with $n \geq k := \dim \mathbb{R}(\tilde{P}) \geq \dim \mathbb{R}(P)$ for all $P \in \mathbb{K}(\mathcal{T})$. This $\tilde{P}$ is maximal, since the assumption $\mathbb{R}(P) \supset \mathbb{R}(\tilde{P})$ leads to the contradiction $\dim \mathbb{R}(P) > \dim \mathbb{R}(\tilde{P})$. ■

Condition d) of Theorem 3.1 is of special importance for convergence, if equality of the sets is reached in the subset relation. Remember that this was not the case in Example 4.1.

Definition 4.2

- *A (right) projection kernel P of $\mathcal{T}$ is said to be* right optimal *if $\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T})$.*
- *A (left) projection kernel P of $\mathcal{T}$ is said to be* left optimal *if $\mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})}$.*
- *A projection kernel P of $\mathcal{T}$ is said to be* optimal *if $\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T})$ as well as $\mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})}$.*

Theorem 4.2 *If P_s is a (left, right) optimal projection kernel of $\mathcal{T}$, then P_s is maximal.*

Proof: a) Let us assume that $P_s \in \mathbb{K}(\mathcal{T})$ is right optimal. Then $\mathbb{R}(P_s) = \mathbb{N}(I-\mathcal{T})$. Since $\mathbb{R}(P) \subseteq \mathbb{N}(I-\mathcal{T})$ for all $P \in \mathbb{K}(\mathcal{T})$ by Theorem 3.1, this P_s is maximal.

b) Let us assume that $P_s \in \mathbb{K}(\mathcal{T})$ is left optimal. Then $\mathbb{N}(P_s) = \overline{\mathbb{R}(I-\mathcal{T})}$. Supposing that P_s is not maximal there is a $\tilde{P} \in \mathbb{K}(\mathcal{T})$ with $\mathbb{R}(\tilde{P}) \supset \mathbb{R}(P_s)$. Considering again Theorem 3.1 it is also $\mathbb{N}(\tilde{P}) \supseteq \mathbb{N}(P_s)$. Since both $\tilde{P}$ and P_s are projectors we get

$$\mathbb{R}(\tilde{P}) \oplus \mathbb{N}(\tilde{P}) \supset X = \mathbb{R}(P_s) \oplus \mathbb{N}(P_s).$$

This is a contradiction. ■

Theorem 4.3 *The operator set $\mathcal{T}$ has an optimal projection kernel iff*

$$X = \mathbb{N}(I-\mathcal{T}) \oplus \overline{\mathbb{R}(I-\mathcal{T})}.$$

In this case the projector P with $\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T})$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})}$ is the optimal and also the unique maximal projection kernel of $\mathcal{T}$.

Proof: a) Let P be an optimal projection kernel of $\mathcal{T}$. Then $\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T})$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})}$ by definition. Hence

$$X = \mathbb{R}(P) \oplus \mathbb{N}(P) = \mathbb{N}(I-\mathcal{T}) \oplus \overline{\mathbb{R}(I-\mathcal{T})}.$$

b) Let be $X = \mathbb{N}(I-\mathcal{T}) \oplus \mathbb{R}(I-\mathcal{T})$. Then we consider the projector P with $\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T})$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})}$. Consequently, P is an optimal projection kernel of $\mathcal{T}$ by definition. By Theorem 4.2 this P is also maximal. Assuming another optimal or maximal projection kernel $\tilde{P} \neq P$ we would get $\mathbb{R}(\tilde{P}) = \mathbb{R}(P)$ and $\mathbb{N}(\tilde{P}) = \mathbb{N}(P)$. For projectors this means $\tilde{P} = P$ by Proposition 2.1 in contradiction with the assumption. ■

Remark 4.1 The optimal projection kernel P of $\mathcal{T}$ is shortly denoted by $P = \mathbb{K}_o(\mathcal{T})$. If $\mathcal{T}$ has more than one maximal projection kernel, then $\mathcal{T}$ has no optimal projection kernel ($\mathbb{K}_o(\mathcal{T}) = \emptyset$). This shows that the set $\mathcal{T}$ in Example 4.1 has no optimal projection kernel. If P is the optimal projection kernel of $\mathcal{T}$, where $\mathcal{T} \setminus \{I\} \neq \emptyset$, then $I-P$ is the projector with $\mathbb{R}(I-P) = \overline{\mathbb{R}(I-\mathcal{T})}$ and $\mathbb{N}(I-P) = \mathbb{N}(I-\mathcal{T})$. This projector is no projection kernel of $\mathcal{T}$ (see Proposition 2.2).

5 Optimal projection kernels

Now we consider optimal projection kernels of an operator set in more detail.

Lemma 5.1 *If P is the optimal projection kernel of $\mathcal{T}_1$ and a projection kernel of $\mathcal{T}_2$, then P is the optimal projection kernel of $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2$.*

Proof: By Theorem 3.1 we have

$$\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T}_1) \subseteq \mathbb{N}(I-\mathcal{T}_2), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T}_1)} \supseteq \overline{\mathbb{R}(I-\mathcal{T}_2)}.$$

It follows

$$\mathbb{N}(I-\mathcal{T}) = \mathbb{N}(I-\mathcal{T}_1) \cap \mathbb{N}(I-\mathcal{T}_2) = \mathbb{N}(I-\mathcal{T}_1) = \mathbb{R}(P),$$
$$\overline{\mathbb{R}(I-\mathcal{T})} = \overline{\operatorname{span}(\mathbb{R}(I-\mathcal{T}_1)} \cup \overline{\mathbb{R}(I-\mathcal{T}_2))} = \overline{\mathbb{R}(I-\mathcal{T}_1)} = \mathbb{N}(P).$$

This is the assertion. ■

Corollary 5.1 *If P is a projection kernel of $\mathcal{T}$ and $P \in \mathcal{T}$, then P is the optimal projection kernel of $\mathcal{T}$.*

Proof: By Proposition 2.1 it holds

$$X = \mathbb{R}(P) \oplus \mathbb{N}(P) = \mathbb{N}(I-P) \oplus \mathbb{R}(I-P).$$

Hence P is the optimal projection kernel of itself. Since P is a projection kernel of $\mathcal{T}$, then P is the optimal projection kernel of $\mathcal{T} \cup \{P\} = \mathcal{T}$ by Lemma 5.1. ■

Lemma 5.2 *If P is the optimal projection kernel of $\mathcal{T}$, then P is the optimal projection kernel of the generated semi-group $[\mathcal{T}](I)$ with identity I and of the affine hull $\operatorname{aff}(\mathcal{T})$.*

Proof: Let $P = \mathbb{K}_o(\mathcal{T})$. Consequently $P \in \mathbb{K}(\mathcal{T})$. By Corollary 2.1 we have $P \in \mathbb{K}([\mathcal{T}](I))$ and $P \in \mathbb{K}(\operatorname{aff}(\mathcal{T}))$. Since $\mathcal{T}_1 := \mathcal{T}$ is a subset of both $\mathcal{T}_2 := [\mathcal{T}](I)$ and $\mathcal{T}_3 := \operatorname{aff}(\mathcal{T})$ the assertion follows now by Lemma 5.1. ■

Corollary 5.2 *An operator T as well as the corresponding sets $\mathcal{T} := \{T^k : k \in \mathrm{N}\}$ and $\mathcal{S} := \operatorname{aff}(\mathcal{T})$ have an optimal projection kernel iff*

$$X = \mathbb{N}(I-T) \oplus \overline{\mathbb{R}(I-T)}.$$

In this case the projector P with $\mathbb{R}(P) = \mathbb{N}(I-T)$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I-T)}$ is the optimal projection kernel of T, $\mathcal{T}$ and $\mathcal{S}$.

Proof: The operators $S \in \mathcal{S}$ have the representations

$$S = S_k(T) = \sum_{i=0}^{k} \alpha_i T^i, \quad \sum_{i=0}^{k} \alpha_i = 1.$$

Considering the coefficient relation of the α_i, each polynomial $P_k(\lambda) := 1 - S_k(\lambda)$ has the zero 1. Hence, in each operator $I - S$ a factor $I - T$ can be separated. This implies

$$\mathbb{N}(I-T) \subseteq \mathbb{N}(I-S), \quad \mathbb{R}(I-T) \supseteq \mathbb{R}(I-S)$$

for all $S \in \mathcal{S}$. Observing $\{T\} \subseteq \mathcal{T} \subseteq \mathcal{S}$ we obtain

$$\mathbb{N}(I-T) = \mathbb{N}(I-\mathcal{T}) = \mathbb{N}(I-\mathcal{S}), \quad \overline{\mathbb{R}(I-T)} = \mathbb{R}(I-\mathcal{T}) = \mathbb{R}(I-\mathcal{S}).$$

Now Theorem 4.3 shows the assertions. ■

Remark 5.1 In Corollary 5.2 the space decomposition

$$X = \mathbb{N}(I-T) \oplus \overline{\mathbb{R}(I-T)}$$

occurs. Operators $I-T$ with this property are called *decomposition regular* (short: d-regular) in [14]. This paper contains more material about such operators. The given decomposition of X is also necessary for the convergence of (T^k) (see Section 9).

Example 5.1 Now we come back to Example 2.2 discussing matrices

$$T_m = \begin{pmatrix} I_{m,m} & O_{m,n-m} \\ O_{n-m,m} & T_{n-m,n-m} \end{pmatrix}, \quad P_l = \begin{pmatrix} I_{l,l} & O_{l,n-l} \\ O_{n-l,l} & O_{n-l,n-l} \end{pmatrix},$$
$$1 \le l \le m \le n$$

in $\mathrm{R}^{n,n}$, where the corresponding operators P_l are stated to be projection kernels of the corresponding operators T_m. Let us choose $m < n$. Further let $\mathcal{T}$ be a set of matrices T_m, where at least one $T_m = \hat{T}_m$ has rank n. Then, using the coordinate unit vectors $\vec{e}_i$ $(i = 1, 2, \ldots, n)$, in other words the columns of the identity $I_{n,n}$, and the linear subspaces

$$\mathrm{V}_{i,j} := \mathrm{span}\,\{\vec{e}_i, \ldots, \vec{e}_j\}, \quad 1 \le i \le j \le n,$$

we get for T_m the relations

$$\mathbb{R}(I-T_m) \subseteq \mathbb{R}(I-\hat{T}_m) = \mathrm{V}_{m+1,n}, \quad \mathbb{N}(I-T_m) \supseteq \mathbb{N}(I-\hat{T}_m) = \mathrm{V}_{1,m}$$

and finally for the set $\mathcal{T}$ the result

$$\overline{\mathbb{R}(I-\mathcal{T})} = \mathrm{V}_{m+1,n}, \quad \mathbb{N}(I-\mathcal{T}) = \mathrm{V}_{1,m}.$$

Further, it is

$$\mathbb{R}(P_l) = \mathrm{V}_{1,l}, \quad \mathbb{N}(P_l) = \mathrm{V}_{l+1,n}, \quad l \le m.$$

Hence, we have a chain of orthogonal projection kernels P_l, where the maximal one, namely P_m, is the optimal one. Moreover, P_m is attainable iff $O_{n-m,n-m}$ is in the closure of the set of submatrices $T_{n-m,n-m}$ belonging to the matrices $T_m \in \mathcal{T}$. For instance, this is the case if $\mathcal{T}$ consists of all possible T_m, because $O_{n-m,n-m}$ is a submatrix of $P_m \in \mathcal{T}$.

Example 5.2 It is interesting to discuss Example 2.3 in more detail. There is stated that the multiplicative semi-group $\mathcal{T}$ of all matrices

$$T = \begin{pmatrix} \vec{e}_1 & \vec{t}_2 & \ldots & \vec{t}_n \end{pmatrix} \in \mathrm{R}^{n,n} : \quad \sum_{i=1}^{n} t_{ij} = 1 \quad (1 < j \le n)$$

has the projection kernel

$$P = \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \ldots & \vec{e}_1 \end{pmatrix} \in \mathcal{T}.$$

It can be shown that

$$\mathbb{R}(P) = \text{span}\{\vec{e}_1\}, \quad \mathbb{N}(P) = \text{span}\{\vec{e}_2 - \vec{e}_1,\ \vec{e}_3 - \vec{e}_1,\ \ldots,\ \vec{e}_n - \vec{e}_1\}.$$

The nullspace of P contains all vectors with coordinate sums 0. Further, each vector $\vec{x} \in \mathbb{N}(P)$ has the basis representation

$$\vec{x} = x_2\,(\vec{e}_2 - \vec{e}_1) + x_3\,(\vec{e}_3 - \vec{e}_1) + \ldots + x_n\,(\vec{e}_n - \vec{e}_1).$$

The matrices $S := I - T$ have the form

$$\begin{pmatrix} \vec{0} & \vec{s}_2 & \ldots & \vec{s}_n \end{pmatrix} \in \mathrm{R}^{n,n} : \quad \sum_{i=1}^{n} s_{ij} = 0 \quad (1 < j \le n).$$

Since P is a projection kernel of all operators T, we have by Theorem 3.1

$$\mathbb{R}(I - T) = \mathbb{R}(S) \subseteq \mathbb{N}(P), \quad \mathbb{N}(I - T) = \mathbb{N}(S) \supseteq \mathbb{R}(P).$$

Indeed, these relations are also a consequence of the above results. Now we consider a subset $\mathcal{T}_s$ of $\mathcal{T}$. If

$$\dim \mathbb{R}(I - \mathcal{T}_s) = n - 1,$$

then it holds

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}_s), \quad \mathbb{N}(P) = \mathbb{R}(I - \mathcal{T}_s).$$

Hence, P is the optimal projection kernel of $\mathcal{T}_s$. Especially this is the case if there is a matrix $\hat{T} \in \mathcal{T}_s$ with rank $(I - \hat{T}) = n - 1$. Such a matrix is

$$\hat{T} = \begin{pmatrix} \vec{e}_1 & 2\vec{e}_2 - \vec{e}_1 & 2\vec{e}_3 - \vec{e}_1 & \ldots & 2\vec{e}_n - \vec{e}_1 \end{pmatrix} \in \mathcal{T}$$

with full rank n since

$$I - \hat{T} = \begin{pmatrix} \vec{0} & \vec{e}_1 - \vec{e}_2 & \vec{e}_1 - \vec{e}_3 & \ldots & \vec{e}_1 - \vec{e}_n \end{pmatrix}$$

has indeed rank $n - 1$. This means also that P is the optimal projection kernel of $\mathcal{T}$. Additionally P is then attainable, because $P \in \mathcal{T}$. The operator $\hat{T}$ has interesting properties, for instance

$$(I - \hat{T})^2 = -(I - \hat{T}) = I - P.$$

Hence $\hat{T} - I$ is a projector. But observe that $I - \hat{T}$ and $I - P$ are not in $\mathcal{T}$ and are no projection kernels of $\mathcal{T}$. A simple consideration shows

$$\hat{T}^n = P - 2^n(I - \hat{T}) \in \mathcal{T}$$

for all integers n. Hence, the sequence $(\hat{T}^n)$ is divergent (for natural n) while the sequence $(\hat{T}^{-n})$ of the inverses converges to P. Since

$$I - \hat{T}^{-1} = -\frac{1}{2}\,(I - \hat{T})$$

has also rank $n - 1$, the sequence $(\hat{T}^{-n})$ has the optimal and attainable projection kernel P.

Example 5.3 Let us investigate a more general approach to the set $\mathcal{T}$ of all matrices

$$T = \begin{pmatrix} \vec{e}_1 & \vec{t}_2 & \dots & \vec{t}_n \end{pmatrix} \in \mathrm{R}^{n,n} : \quad \sum_{i=1}^{n} t_{ij} = 1 \quad (1 < j \leq n)$$

just investigated in Example 5.2. If subsets $\mathcal{T}_s$ of $\mathcal{T}$ are considered, then possibly P is not the optimal projection kernel. On the other hand, if $Q \in \mathcal{T}$ is any projector, then we can find subsets with Q as an optimal projection kernel. Let us look at the special projectors

$$P_k = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_k & \vec{e}_1 & \dots & \vec{e}_1 \end{pmatrix} \in \mathcal{T} \quad (1 \leq k < n).$$

For $k = 1$ we have $P_k = P$ (see Example 5.2). The case $k = n$ supplies $P_n = I$ which is not of interest. For arbitrary k we get

$$\begin{aligned} \mathbb{R}(P_k) &= \text{span}\,\{\vec{e}_1,\ \vec{e}_2,\ \dots,\ \vec{e}_k\}, \\ \mathbb{N}(P_k) &= \text{span}\,\{\vec{e}_1 - \vec{e}_{k+1},\ \vec{e}_1 - \vec{e}_{k+2},\ \dots,\ \vec{e}_1 - \vec{e}_n\}. \end{aligned}$$

The sets $\mathcal{T}_k$ of matrices

$$\begin{gathered} T_k = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_k & \vec{t}_{k+1} & \dots & \vec{t}_n \end{pmatrix}, \\ \vec{e}_i - \vec{t}_i \in \mathbb{N}(P_k) \quad (1 \leq k < n,\ k+1 \leq i \leq n) \end{gathered}$$

are again semi-groups of operators containing P_k. Now $\mathcal{T}_k$ has the projection kernel P_k because of

$$I - T_k = \begin{pmatrix} \vec{0} & \vec{0} & \dots & \vec{0} & \vec{e}_{k+1} - \vec{t}_{k+1} & \dots & \vec{e}_n - \vec{t}_n \end{pmatrix}$$

and

$$\mathbb{R}(P_k) \subseteq \mathbb{N}(I - T_k), \quad \mathbb{N}(P_k) \supseteq \mathbb{R}(I - T_k).$$

There are special matrices

$$\hat{T}_k = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_k & 2\vec{e}_{k+1} - \vec{e}_1 & \dots & 2\vec{e}_n - \vec{e}_1 \end{pmatrix}$$

with

$$I - \hat{T}_k = \begin{pmatrix} \vec{0} & \vec{0} & \dots & \vec{0} & \vec{e}_1 - \vec{e}_{k+1} & \dots & \vec{e}_1 - \vec{e}_n \end{pmatrix}$$

and

$$\mathbb{R}(P_k) = \mathbb{N}(I - \hat{T}_k), \quad \mathbb{N}(P_k) = \mathbb{R}(I - \hat{T}_k).$$

Hence, P_k is the optimal projection kernel of $\mathcal{T}_k$. Besides, the relations

$$\mathcal{T} \supseteq \mathcal{T}_k \supset \mathcal{T}_{k+1}, \quad \mathbb{R}(P) \subseteq \mathbb{R}(P_k) \subset \mathbb{R}(P_{k+1}), \quad \mathbb{N}(P) \supseteq \mathbb{N}(P_k) \supset \mathbb{N}(P_{k+1})$$

are fulfilled.

Let X be a (real) HILBERT space. We turn to optimal projection kernels which are orthogonal.

Proposition 5.1 *Let $\mathcal{T}$ possess the optimal projection kernel P. Then the following conditions are equivalent:*

a) *P is orthogonal ($P = P^*$),* b) $\mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T}^*)}$, c) $\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}^*)$.

Proof: Using Theorem 4.3 the assumption $P \in \mathbb{K}_o(\mathcal{T})$ implies

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})}.$$

Then it follows

$$\begin{aligned} \mathbb{N}(P^*) &= \mathbb{R}(P)^\perp = \mathbb{N}(I - \mathcal{T})^\perp = \overline{\mathbb{R}(I - \mathcal{T}^*)}, \\ \mathbb{R}(P^*) &= \mathbb{N}(P)^\perp = \overline{\mathbb{R}(I - \mathcal{T})}^\perp = \mathbb{N}(I - \mathcal{T}^*). \end{aligned}$$

Because of the equivalences

$$P = P^* \quad \Leftrightarrow \quad \mathbb{N}(P) = \mathbb{N}(P^*) \quad \Leftrightarrow \quad \mathbb{R}(P) = \mathbb{R}(P^*)$$

the assertion is true. ∎

Self-adjoint operators $T = T^*$ trivially satisfy $\mathbb{N}(I - T) = \mathbb{N}(I - T^*)$. But all operators with this property have an outstanding property.

Theorem 5.1 *If the operators T in $\mathcal{T}$ have the property $\mathbb{N}(I - T) = \mathbb{N}(I - T^*)$, then $\mathcal{T}$ has an orthogonal optimal projection kernel, namely the orthoprojector P with*

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})} = \overline{\mathbb{R}(I - \mathcal{T}^*)}.$$

Proof: It is known that the linear subspaces

$$\mathbb{N} := \mathbb{N}(I - \mathcal{T}), \quad \mathbb{R} := \overline{\mathbb{R}(I - \mathcal{T}^*)}$$

are orthogonal complements (see e.g. [13]). Hence an orthogonal projector P is defined by $R(P) = \mathbb{N}$ and $\mathbb{N}(P) = \mathbb{R}$. The property $\mathbb{N}(I - T) = \mathbb{N}(I - T^*)$ implies

$$\overline{\mathbb{R}(I - T)} = \mathbb{N}(I - T^*)^\perp = \mathbb{N}(I - T)^\perp = \overline{\mathbb{R}(I - T^*)}.$$

This means also

$$\overline{\mathbb{R}(I - \mathcal{T}^*)} = \overline{\mathbb{R}(I - \mathcal{T})}.$$

Now Theorem 4.3 shows that P is the optimal projection kernel of $\mathcal{T}$. ∎

Remark 5.2 The property $\mathbb{N}(I-T)=\mathbb{N}(I-T^*)$ is equivalent to the orthogonal decomposition

$$X=\mathbb{N}(I-T)\oplus\overline{\mathbb{R}(I-T)},\quad \overline{\mathbb{R}(I-T)}=\mathbb{N}(I-T)^\perp,$$

considering the orthogonality relation $\mathbb{N}(I-T^*)^\perp=\overline{\mathbb{R}(I-T)}$ (see e.g. [13]). Not only self-adjoint, but also *normal operators* T, defined by the commutation relation $TT^*=T^*T$, have the property $\mathbb{N}(I-T)=\mathbb{N}(I-T^*)$ (see e.g. [21], p. 331 – 332). Moreover all products $T=P_kP_{k-1}\dots P_1$ of orthoprojectors P_i $(i=1,2,\dots,k)$ fulfil this condition. Here it is

$$\begin{aligned}\mathbb{N}(I-T)=\mathbb{N}(I-P_kP_{k-1}\dots P_1)&=\bigcap_{i=1}^{k}\mathbb{R}(P_i)=\bigcap_{i=1}^{k}\mathbb{R}(P_i^*)\\ &=\mathbb{N}(I-P_1^*\dots P_{k-1}^*P_k^*)=\mathbb{N}(I-T^*).\end{aligned}$$

Moreover, T is nonexpansive. In [17], p. 183f. a more general result is proven, namely if so-called relaxations T_i of orthoprojectors P_i replace P_i. In section 6 we will see that arbitrary nonexpansive operators T satisfy $\mathbb{N}(I-T)=\mathbb{N}(I-T^*)$ (see Remark 6.1 and the text after it).

Finally, the operators $T\in\mathcal{T}$ itself can be orthoprojectors.

Corollary 5.3 *If the operators $T\in\mathcal{T}$ are orthoprojectors ($T^2=T=T^*$), then $\mathcal{T}$ has an orthogonal optimal projection kernel, namely the orthoprojector P with*

$$\mathbb{R}(P)=\bigcap_{T\in\mathcal{T}}\mathbb{R}(T),\quad \mathbb{N}(P)=\operatorname{span}\bigcup_{T\in\mathcal{T}}\mathbb{N}(T).$$

Proof: If the operators T are orthoprojectors, then we get

$$\mathbb{R}(T)=\mathbb{N}(I-T)=\mathbb{N}(I-T^*),\quad \mathbb{N}(T)=\mathbb{R}(I-T)=\mathbb{R}(I-T^*),$$

where both $\mathbb{R}(T)$ and $\mathbb{N}(T)$ are closed. This means also

$$\mathbb{N}(I-\mathcal{T})=\bigcap_{T\in\mathcal{T}}\mathbb{R}(T),\quad \overline{\mathbb{R}(I-\mathcal{T})}=\operatorname{span}\bigcup_{T\in\mathcal{T}}\mathbb{N}(T).$$

Now the assertion follows if we take Theorem 5.1 into account. ■

The assumptions and hence the assertions of Corollary 5.3 are fulfilled in Example 1.2 for the set $\mathcal{T}$ of sequence members T_k and in Example 1.3 for the set $\mathcal{T}$ of sequence members S_k. It turns out that these assertions also hold for a bigger class than that of orthoprojectors. We follow this topic in Section 6.

Proposition 5.2 *Let $\mathcal{T}$ be a set of orthoprojectors and P be a further orthoprojector. Then the following conditions are equivalent:*

a) *P is a projection kernel of $\mathcal{T}$.*

b) *P is a left projection kernel of $\mathcal{T}$.*

c) *P is a right projection kernel of $\mathcal{T}$.*

Proof: Obviously the conditions b) and c) follow from condition a). Now we want to show the reversions. Starting with the relations

$$P^2 = P, \quad P = PT, \quad P = TP \quad \text{for all } T \in \mathcal{T}$$

the transition to the adjoint operators supplies

$$(P^*)^2 = P^*, \quad P^* = T^*P^*, \quad P^* = P^*T^* \quad \text{for all } T \in \mathcal{T}.$$

Observing the assumptions $P^* = P$ and $T^* = T$ for all $T \in \mathcal{T}$ we get correspondingly

$$P^2 = P, \quad P = TP, \quad P = PT \quad \text{for all } T \in \mathcal{T}.$$

Hence, a left (right) projection kernel of $\mathcal{T}$ is also a right (left) projection kernel of $\mathcal{T}$ and consequently also a projection kernel of $\mathcal{T}$. ■

6 Nonexpansive operators and projection kernels

Let X be a (real) HILBERT space. Nonexpansive operators play an important part in the fixed point theory. Here we study the linear case.

Definition 6.1 *A linear operator T is called*

a) nonexpansive*, if $\|Tx\| \le \|x\|$ for all x.*

b) isometric*, if $\|Tx\| = \|x\|$ for all x.*

c) contractive*, if $\|Tx\| \le k\,\|x\|$ for all x and a number $k < 1$.*

d) FEJÉR monotone*, if $\|Tx\| < \|x\|$ for all $x \notin \mathbb{N}(I-T)$.*

e) strongly FEJÉR monotone*, if $\|Tx\| \le k\,\|x\|$ for all $x \in \mathbb{N}(I-T)^\perp$ and a number $k < 1$.*

These concepts are also defined for nonlinear operators (see e.g. [4], [24], [22]). In my papers [17] and [18] linear (strongly) FEJÉR monotone operators are called (strong) *relaxations*. The concepts are renamed to get a better coordination between linear and nonlinear theory.

Remark 6.1 Nonexpansive operators T are characterized by $\|T\| \le 1$. They induce via $I - T$ the orthoprojector $P = P(T)$, where

$$\begin{aligned} X = \mathbb{R}(P) \oplus \mathbb{N}(P), \quad \mathbb{R}(P) \perp \mathbb{N}(P), \\ \mathbb{R}(P) = \mathbb{N}(I-T) = \overline{R(I-T)}^{\perp} = \mathbb{N}(I-T^*), \\ \mathbb{N}(P) = \overline{R(I-T)} = \mathbb{N}(I-T)^{\perp} = \overline{R(I-T^*)} \end{aligned}$$

is the corresponding decomposition of X (see [17]: p. 182). By Corollary 5.2 and the property $\mathbb{R}(P) \perp \mathbb{N}(P)$ the projector P is the orthogonal optimal projection kernel of T. Moreover, $\mathbb{N}(P) = \mathbb{N}(I-T)^{\perp}$ is an invariant linear subspace under T (for another proof see [17]: p. 180). Further

$$\|T - P\| = \|T(I-P)\| = \|T \,|\, \mathbb{R}(I-P)\| = \|T \,|\, \mathbb{N}(P)\| =: \nu \le 1,$$

where the number ν measures the deviation of T from P. We call $P = P(T)$ also the *eigenprojection* of T.

Theorem 6.1 *If $\mathcal{T}$ consists of nonexpansive operators T, then $\mathcal{T}$ has an orthogonal optimal projection kernel, namely the orthoprojector P with*

$$\mathbb{R}(P) = \mathbb{N}(I-\mathcal{T}), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I-\mathcal{T})} = \overline{\mathbb{R}(I-\mathcal{T}^*)}.$$

Proof: By Remark 6.1 nonexpansive operators T satisfy $\mathbb{N}(I-T) = \mathbb{N}(I-T^*)$. Hence, the assertion follows immediately from Theorem 5.1. ■

The set of all nonexpansive operators is a multiplicative semi-group with identity. This set can be divided again into semi-groups of nonexpansive operators with the same eigenprojection P.

Isometric operators T satisfy $\|T\| = 1$. Contractive operators T are norm reducing for $x \neq 0$ and fulfil $\|T\| \le k < 1$.

Fejér monotone operators T are nonexpansive, but not isometric, since they are norm reducing outside their fixed point sets $\mathbb{N}(I-T)$. Strongly Fejér monotone operators T are contractive on the invariant subspace $\mathbb{N}(I-T)^{\perp}$ (see Definition 6.1). Here it is

$$\nu = \nu(T) := \|T \,|\, \mathbb{N}(I-T)^{\perp}\| \le k < 1.$$

For $\mathbb{N}(I-T) = \{0\}$ strongly Fejér monotone operators and contractive operators coincide. In [17] and [18] a (strongly) Fejér monotone T with eigenprojection P is said to be a (strong) relaxation of its carrier P. In [18] you can find an example of T which is Fejér monotone, but not strong.

Since Fejér monotone operators play an important part in a certain class of iterative solution methods (see [16]), we mention some further facts about them.

Example 6.1 Let $P \neq I$ be an orthoprojector. Then the operators

$$T = (1-\lambda)I + \lambda P, \quad |1-\lambda| < 1$$

are self-adjoint strongly FEJÉR monotone operators with the same eigenprojection P (*scalar relaxations*). Here it is $\nu(T) = |1-\lambda|$.

If T is in one of the operator classes of Definition 6.1, then the same is true for T^*. We state this for one class.

Proposition 6.1 ([17]: p. 182, [18]: p. 33, p. 37) *T is (strongly) FEJÉR monotone iff T^* is (strongly) FEJÉR monotone. Thereby both have the same eigenprojection.*

Proposition 6.2 ([17]: p. 184, [18]: p. 40) *If T is (strongly) FEJÉR monotone, then T^k, TT^* and T^*T are (strongly) FEJÉR monotone with the same eigenprojection as T.*

Theorem 6.2 ([17]: p. 183) *Let $\mathcal{T}$ be a set of FEJÉR monotone operators T with eigenprojections $P = P(T)$. Then each projection kernel of $\mathcal{T}$ is a projection kernel of $\mathcal{P} := \{P = P(T) : T \in \mathcal{T}\}$ and vice versa.*

This statement also holds if $\mathcal{T}$ is a set of nonexpansive operators. The proof is the same as in the paper [17].

7 Attainable projection kernels

For simplicity we use in $\mathcal{L}(X)$ the strong operator topology τ_s. Then $\mathcal{L}(X)$ becomes a local convex topological vector space (see e.g. [23]: p. 110). But, corresponding results can also be obtained for the uniform and the weak operator topology.
First we investigate the relation between attainable and optimal projection kernels. In this section we use generalized sequences $(T_\alpha)_{\alpha \in J}$ of operators, where α is an element of an index set J. Shortly we write (T_α). Such sequences are also called MOORE-SMITH sequences. Further, we use the notation $\mathcal{T}_J := \{T_\alpha : \alpha \in J\}$ for the set of sequence members.

Lemma 7.1 *If $\mathcal{T} \subseteq \mathcal{L}(X)$ and $S \in \overline{\mathcal{T}}$, then*

$$\mathbb{N}(S) \subseteq \mathbb{R}(I-S) \subseteq \overline{\mathbb{R}(I-\mathcal{T})}, \quad \mathbb{R}(S) \supseteq \mathbb{N}(I-S) \supseteq \mathbb{N}(I-\mathcal{T}).$$

Proof: Since S belongs to $\overline{\mathcal{T}}$, there is a generalized sequence (T_α) in $\mathcal{T}$ with limit S (see e.g. [24]: p. 205). Further, we have for arbitrary $x \in X$ and all $\alpha \in J$

$$(I - T_\alpha)x \in \mathbb{R} := \overline{\mathbb{R}(I-\mathcal{T})}.$$

Hence, it holds also

$$(I - S)\,x = \lim_{\alpha}(I - T_\alpha)\,x \in \mathbb{R}$$

because $\mathbb{R}$ is closed. Moreover, we get

$$(I - S)\,x = \lim_{\alpha}(I - T_\alpha)\,x = 0 \quad \text{for all } x \in \mathbb{N} := \mathbb{N}(I - \mathcal{T}).$$

This implies $\mathbb{R}(I - S) \subseteq \mathbb{R}$ and $\mathbb{N}(I - S) \supseteq \mathbb{N}$. The relations $\mathbb{N}(S) \subseteq \mathbb{R}(I - S)$ and $\mathbb{R}(S) \supseteq \mathbb{N}(I - S)$ are obvious. ■

Theorem 7.1 *Each attainable (left, right) projection kernel P of $\mathcal{T}$ is a (left, right) optimal (left, right) projection kernel of $\mathcal{T}$.*

Proof: According to Lemma 3.2 and Lemma 3.1, respectively, we have

$$\mathbb{N}(P) \supseteq \overline{\mathbb{R}(I - \mathcal{T})}, \quad \mathbb{R}(P) \subseteq \mathbb{N}(I - \mathcal{T})$$

for a left (right) projection kernel P of $\mathcal{T}$, respectively. Lemma 7.1 supplies for $S = P$ the relations

$$\mathbb{N}(P) \subseteq \overline{\mathbb{R}(I - \mathcal{T})}, \quad \mathbb{R}(P) \supseteq \mathbb{N}(I - \mathcal{T}),$$

respectively. This shows

$$\mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T})}, \quad \mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}),$$

respectively. Hence, the assertions hold by Definition 4.2. ■

Remark 7.1 Theorem 7.1 shows that an operator set $\mathcal{T}$ has at most one attainable projection kernel P, because there is at least one optimal projection kernel.

Now we turn to the question under which conditions the limits T_∞ of generalized sequences (T_α) are projection kernels of these sequences.

Proposition 7.1 *If the limit T_∞ of (T_α) exists, then the following equivalences hold:*

a) $T_\alpha T_\infty = T_\infty$ *for all* $\alpha \in J \Leftrightarrow T_\infty \in \mathbb{K}_+(\mathcal{T}_J) \;\Leftrightarrow\; \mathbb{R}(T_\infty) = \mathbb{N}(I - \mathcal{T}_J)$,

b) $T_\infty T_\alpha = T_\infty$ *for all* $\alpha \in J \Leftrightarrow T_\infty \in \mathbb{K}_-(\mathcal{T}_J) \;\Leftrightarrow\; \mathbb{N}(T_\infty) = \overline{\mathbb{R}(I - \mathcal{T}_J)}$.

Proof: a) We start with the first part and conclude cyclically. The relation $T_\alpha T_\infty = T_\infty$ implies $T_\infty^2 = T_\infty$ by limit transition. Hence T_∞ is a right projection kernel of $\mathcal{T}_J$, that is $T_\infty \in \mathbb{K}_+(\mathcal{T}_J)$. Since T_∞ is attainable by assumption, Theorem 7.1 shows that T_∞ is right optimal, that is $\mathbb{R}(T_\infty) = \mathbb{N}(I - \mathcal{T}_J)$ by Definition 4.2. This relation implies again $T_\alpha T_\infty = T_\infty$ for all α by Lemma 3.1. Hence, the cycle is closed.
b) This part can be obtained analogously. ■

Theorem 7.2 *If the limit T_∞ of (T_α) exists, then the following statements are equivalent:*

1) $T_\alpha T_\infty = T_\infty T_\alpha = T_\infty$ *for all* $\alpha \in J$,
2) $T_\infty \in \mathbb{K}(\mathcal{T}_J)$,
3) $\mathbb{R}(T_\infty) = \mathbb{N}(I - \mathcal{T}_J)$, $\mathbb{N}(T_\infty) = \overline{\mathbb{R}(I - \mathcal{T}_J)}$.

Proof: Combining a) and b) in Proposition 7.1 we arrive at the assertion. ■

Theorem 7.2 shows: if the limit T_∞ of (T_α) is a projection kernel, then it is an optimal one.

Example 7.1 (Semi-group of operators) Let $\mathcal{T}$ be a semi-group of operators $T \in \mathcal{L}(X)$ with identity I. Further, let exist a generalized sequence (T_α) of operators $T_\alpha \in \mathcal{L}(X)$ with the following properties:

a) (T_α) is (uniformly) bounded,
b) $T_\alpha x \in \overline{\text{co}}(\{Tx :\ T \in \mathcal{T}\})$ for all α and for all $x \in X$,
c) $\lim_\alpha T_\alpha x$ exists for all $x \in X$,
d) $\lim_\alpha (I - T)T_\alpha x = \lim_\alpha T_\alpha (I - T)x = 0$ for all $x \in X$ and for all $T \in \mathcal{T}$.

Then the limit operator P defined by $Px := \lim_\alpha T_\alpha x$ is a projection kernel of $\text{co}(\mathcal{T})$. This result can be derived from [5], p. 220 − 222. In our context, P is moreover the attainable and optimal projection kernel of $\text{co}(\mathcal{T})$.

Theorem 7.3 *Let P be a projection kernel of $\mathcal{T}_J = \{T_\alpha : \alpha \in J\}$. Then the following conditions are equivalent:*

a) (T_α) *converges to* P.
b) (T_α) *converges on* $\mathbb{N}(P)$ *to the null operator* O.

In both cases P is the optimal projection kernel of $\mathcal{T}_J$, that means

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}_J), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T}_J)}.$$

Proof: It is supposed that $P \in \mathbb{K}(\mathcal{T}_J)$. According to Theorem 3.1 the operators T_α have the direct sum representation

$$T_\alpha = I \,|\, \mathbb{R}(P) \oplus T_\alpha \,|\, \mathbb{N}(P) = P \,|\, \mathbb{R}(P) \oplus T_\alpha \,|\, \mathbb{N}(P).$$

Hence, a) and b) are equivalent. Under the condition a) P is attainable. Consequently P is by Theorem 7.1 the optimal projection kernel of $\mathcal{T}_J$, where the given range and nullspace follow by Definition 4.2. ■

8 Projection kernels of operator products

Considering iterative methods, beside operators T_k also product operators

$$T_{k,0} := T_k \dots T_1 T_0$$

occur. Hence, especially the limit behavior of $(T_{k,0})$ is of interest. Finally, we introduce the set notations

$$\mathcal{T}_N := \{T_k : \; k \in \mathrm{N}\}, \quad \mathcal{T}_{N,0} := \{T_{k,0} : \; k \in \mathrm{N}\}$$

for the corresponding sequences (T_k) and $(T_{k,0})$.

Theorem 8.1 *If the product sequence $(T_{k,0})$ converges to a (left, right) projection kernel P of (T_k), then P is a (left, right) optimal (left, right) projection kernel of (T_k) and $(T_{k,0})$.*

Proof: Let P be a (left, right) projection kernel of (T_k) with $\lim_{k\to\infty} T_{k,0} = P$. Let us consider the identities

$$I - T_{k,0} = \sum_{i=0}^{k} T_k \dots T_{i+1}(I - T_i) = \sum_{i=0}^{k} (I - T_i)\, T_{i-1} \dots T_0.$$

Since $x \in \mathbb{N}(I - \mathcal{T}_N)$ implies $(I - T_k)x = 0$ for all k, it implies also $(I - T_{k,0})x = 0$ for all k. Hence, by limit transition it is $(I - P)x = 0$. This shows

$$\mathbb{R}(P) = \mathbb{N}(I - P) \supseteq \mathbb{N}(I - \mathcal{T}_{N,0}) \supseteq \mathbb{N}(I - \mathcal{T}_N).$$

On the other hand the identities verify $\mathbb{R}(I - T_{k,0}) \subseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}$ for all k and by limit transition also $\mathbb{R}(I - P) \subseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}$. In more detail, we have even

$$\mathbb{N}(P) = \mathbb{R}(I - P) \subseteq \overline{\mathbb{R}(I - \mathcal{T}_{N,0})} \subseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}.$$

But, by Lemma 3.1 and Lemma 3.2 it holds

$$\mathbb{R}(P) \subseteq \mathbb{N}(I - \mathcal{T}_N), \quad \mathbb{N}(P) \supseteq \overline{\mathbb{R}(I - \mathcal{T}_N)}$$

for a left and right projection kernel of $\mathcal{T}_N$, respectively.
Consequently, the assertion is true. ■

Proposition 8.1 *If the limit $T_{\infty,0}$ of $(T_{k,0})$ exists, then it follows*

a) $T_k T_{\infty,0} = T_{\infty,0}$ *for all* $k \Leftrightarrow T_{\infty,0} \in \mathbb{K}_+(\mathcal{T}_N) \;\Leftrightarrow\; \mathbb{R}(T_{\infty,0}) = \mathbb{N}(I - \mathcal{T}_N)$,

b) $T_{\infty,0} T_k = T_{\infty,0}$ *for all* $k \Leftrightarrow T_{\infty,0} \in \mathbb{K}_-(\mathcal{T}_N) \;\Leftrightarrow\; \mathbb{N}(T_{\infty,0}) = \overline{\mathbb{R}(I - \mathcal{T}_N)}$.

Proof: a) The relation $T_kT_{\infty,0} = T_{\infty,0}$ for all k implies $T_{k,0}T_{\infty,0} = T_{\infty,0}$ for all k. By limit transition we get $T^2_{\infty,0} = T_{\infty,0}$. This means $T_{\infty,0} \in \mathbb{K}_+(\mathcal{T}_N)$ considering the first relation. Since $T_{\infty,0}$ is attainable, Theorem 7.1 shows that $T_{\infty,0}$ is right optimal. By Definition 4.2 the relation $\mathbb{R}(T_{\infty,0}) = \mathbb{N}(I - \mathcal{T}_N)$ holds in this case.
Assertion b) is shown analogously. ■

Theorem 8.2 *If the limit $T_{\infty,0}$ of $(T_{k,0})$ exists, then the following conditions are equivalent:*

1) $T_kT_{\infty,0} = T_{\infty,0}T_k = T_{\infty,0}$ *for all* k,
2) $T_{\infty,0} \in \mathbb{K}(\mathcal{T}_N)$,
3) $\mathbb{R}(T_{\infty,0}) = \mathbb{N}(I - \mathcal{T}_N), \quad \mathbb{N}(T_{\infty,0}) = \overline{\mathbb{R}(I - \mathcal{T}_N)}$.

Proof: The assertion follows by combination of a) and b) in Proposition 8.1. ■

Theorem 8.3 *Let P be a projection kernel of $\mathcal{T}_N$. Then the following conditions are equivalent:*

a) $(T_{k,0})$ *converges to* P.
b) $(T_{k,0})$ *converges on* $\mathbb{N}(P)$ *to the null operator* O.

In both cases P is the optimal projection kernel of $\mathcal{T}_N$, that means

$$\mathbb{R}(P) = \mathbb{N}(I - \mathcal{T}_N), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - \mathcal{T}_N)}.$$

Proof: By assumption it is $P \in \mathbb{K}(\mathcal{T}_N)$. Then $P \in \mathbb{K}(\mathcal{T}_{N,0})$ follows using Lemma 2.1 or Corollary 2.1. Applying Theorem 7.3 with $(T_{k,0})$ instead of (T_α), the statements a) and b) are shown to be equivalent. If a) or b) are supposed, Theorem 8.1 supplies that the projection kernel P is optimal. By definition P has the stated range and nullspace. ■

Remark 8.1 If $(T_{k,0})$ converges to O, then there is only the trivial projection kernel $P = O$. Further, it is

$$\mathbb{N}(I - \mathcal{T}_N) = \mathbb{R}(O) = \{0\}, \quad \overline{\mathbb{R}(I - \mathcal{T}_N)} = \mathbb{N}(O) = X.$$

Reversely, if these space conditions hold for (T_k), then $(T_{k,0})$ converges to O.

Theorem 8.4 *If the operators T_k are* Fejér *monotone with eigenprojections P_k ($k = 1, \ldots, m$), then the product $T_{m,1} = T_m \ldots T_2T_1$ is* Fejér *monotone with eigenprojection P defined by*

$$\mathbb{R}(P) = \bigcap_{k=1}^{m} \mathbb{R}(P_k), \quad \mathbb{N}(P) = \operatorname{span} \bigcup_{k=1}^{m} \mathbb{N}(P_k).$$

Thereby P is the orthogonal optimal projection kernel of both (P_k) and (T_k).

Proof: The first part is shown in [17], p. 183. The last part follows by Theorem 5.1, if the relations

$$\mathbb{N}(I - T_k) = \mathbb{N}(I - T_k^*) = \mathbb{R}(P_k) = \mathbb{N}(I - P_k) = \mathbb{N}(I - P_k^*), \quad \mathbb{R}(I - P_k) = \mathbb{N}(P_k)$$

are observed (see also Corollary 5.3). ■

9 Power sequences and related series

Now we turn to the special case $T_k = T$ for all k. Then $T_{k,0} = T^{k+1}$. Thus we arrive at power sequences (T^k) and their convergence properties.

Lemma 9.1 *The following statements are equivalent:*

a) (T^k) *converges strongly (to an operator* $T^\infty \in \mathcal{L}(X)$*).*

b) (T^k) *converges strongly to the projector* P *given by* $\mathbb{R}(P) = \mathbb{N}(I - T)$ *and* $\mathbb{N}(P) = \overline{\mathbb{R}(I - T)}$.

c) *There is a projector* P *with* $\mathbb{R}(P) = \mathbb{N}(I - T)$ *such that* $\tilde{T} := T|\mathbb{N}(P) \in \mathcal{L}(\mathbb{N}(P))$ *and* $(\tilde{T}^k)$ *converges strongly to* $O \in \mathbb{N}(P)$.

Proof: The equivalence of b) and c) is a consequence of Theorem 8.3. It remains to show that the limit operator of (T^k) is a projector in $\mathcal{L}(X)$ with given range and nullspace. This is done e.g. in [15], pp. 6 – 8. ■

Remark 9.1 Several authors have proven in different ways and in different spaces that the limit of a convergent power sequence (T^k) is a projector P (see e.g. [1]: p. 367, [3]: p. 567, [9], [19]: p. 179, [20]: p. 351). This projector P in the above lemma is the optimal projection kernel of T. Hence, the existence of an optimal projection kernel for T is necessary for the convergence of (T^k). In other words, $I - T$ has to be decomposition regular (see [14]):

$$X = \mathbb{N}(I - T) \oplus \overline{\mathbb{R}(I - T)}.$$

Proposition 9.1 ([18], p. 35 – 36) *Let* T *be a strongly* FEJÉR *monotone operator. Then the sequence* (T^k) *converges (uniformly, r.t. the operator norm) to the eigenprojection* $P(T)$ *of* T.

Lemma 9.1 shows that powers sequences (T^k) of strongly FEJÉR monotone operators converge uniformly to O on $\mathbb{N}(P) = \mathbb{R}(I - T)$, where the range of $I - T$ is closed in this case. If $\mathbb{R}(I - T) = X$, then (T^k) converges uniformly to O. The latter statement fits to the following well-known facts.

Proposition 9.2 *These conditions are equivalent:*

a) *The sequence (T^k) converges uniformly to O.*

b) *There is a natural n such that $\|T^n\| < 1$.*

c) *The* NEUMANN *series $\left(\sum_{i=0}^{k} T^i\right)$ converges uniformly.*

Theorem 9.1 *The following two statements are equivalent:*

a) *The sequence (T^k) converges.*

b) *The sequence (T^{nk}) converges for a fixed n and all n-th roots of unity which are different from 1 are no eigenvalues of T.*

Under one of these conditions a) or b) both sequences converge to their optimal projection kernel P with $\mathbb{R}(P) = \mathbb{N}(I - T)$ and $\mathbb{N}(P) = \overline{\mathbb{R}(I - T)}$.

Proof: The equivalence of a) and b) is given in [15]: p. 11 and in [3]: p. 568 for the special case $n = 2$. The consequence is shown by Lemma 9.1. ■

Corollary 9.1 *If the natural power of an operator T is a projector P, say $T^n = P$, and all n-th roots of unity which are different from 1 are no eigenvalues of T, then it holds also $T^k = P$ for all members of (T^k) with $k \geq n$. Thereby P is the optimal projection kernel of (T^k).*

Proof: By assumption we have $T^n = P$. Considering $T^{nk} = P$ for all k and Theorem 9.1 both sequences (T^k) and (T^{nk}) converge to the common optimal projection kernel P such that also $PT = TP = P$ holds. Hence,

$$T^{n+1} = T \cdot T^n = T \cdot P = P.$$

By induction we get $T^k = P$ for all $k \geq n$. ■

Example 9.1 Let us choose

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where P is a projector. Then it is $T^3 = P$. The eigenvalues 0, 1, i and $-$i of T are no third roots of unity except for 1. Thus Corollary 9.1 can be applied for $n = 3$. This implies $T^k = P$ for $k \geq 3$. Direct computation also confirms the result. Further P is the optimal projection kernel of (T^k), i.e.

$$\mathbb{R}(P) = \mathbb{N}(I - T), \quad \mathbb{N}(P) = \overline{\mathbb{R}(I - T)} = \mathbb{R}(I - T).$$

This can also be shown directly. Observe that the given matrices fits Example 5.2. The matrix

$$I - T = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

has rank 3. This again proves the optimality of P in the referred context.

Example 9.2 We consider an operator

$$T := I + B, \quad B \in \mathrm{R}^{n,n}, \quad B \neq O, \quad B^2 = O.$$

Then we get

$$T^k = I + kB.$$

Hence, the limit T^∞ of (T^k) does not exist. Therefore $I - T$ and B are not decomposition regular (see Remark 9.1). A projector $P \in \mathrm{R}^{n,n}$ is a projection kernel of (T^k) iff $PB = BP = O$. But such a projector cannot be optimal (see again Remark 9.1). Example 4.1 shows that the conditions for B given above can be fulfilled. We choose

$$B := c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $T(c) := I + B$, where $B^2 = O$ and $BP(a,b) = P(a,b)B = O$ holds for the projection kernels $P(a,b)$ of operators $T(c)$.

The following example shows that the optimal projection kernel of T are sometimes obtained by limits of more general sequences which can converge if (T^k) diverges.

Example 9.3 (Means of operator powers) Let X be a reflexive BANACH space and $T \in \mathcal{L}(X)$ an operator with a (uniformly) bounded sequence (T^k) of powers. Then the sequence of CESÀRO means

$$T_k := \frac{1}{k+1} \sum_{i=0}^{k} T^i$$

converges strongly to the optimal projection kernel P of (T^k) (see also [23]: p. 214).

Similar results can be obtained also by other means of operator powers.

References

[1] **Berg, L. :** *General iterative methods for the evaluation of linear equations.* Numer. Funct. Anal. and Optimiz. 1,4 (1979), 365 – 381

[2] **Berg, L. :** *Ein Gradientenverfahren zur Lösung linearer Gleichungssysteme.* Beitr. Num. Math. 10 (1981), 7 – 13

[3] **Browder, F. E.**, and **Petryshyn, W. V. :** *The solution by iteration of linear functional equations in Banach spaces.* Bull. Amer. Math. Soc. 72 (1966) 566 – 570

[4] **Deimling, K. :** *Nonlinear Functional Analysis.* Springer-Verlag, Berlin u. a. 1985

[5] **Eberlein, W. F. :** *Abstract ergodic theorems and weak almost periodic functions.* Trans. Amer. Math. Soc. 67 (1949) 217 – 240

[6] **Maeß, G. :** *Iterative Lösung linearer Gleichungssysteme.* Nova Acta Leopoldina, Neue Folge, Band 52, Nummer 238, Halle (Saale) 1979

[7] **Maeß, G.**, and **Peters, W. :** *Lösung inkonsistenter linearer Gleichungssysteme und Bestimmung einer Pseudoinversen für rechteckige Matrizen durch Spaltenapproximation.* ZAMM, 58 (1978) 233 – 237

[8] **Peters, W. :** *Lösung linearer Gleichungssysteme durch Projektion auf Schnitträume von Hyperebenen und Berechnung einer verallgemeinerten Inversen.* Beitr. Num. Math. 5 (1976) 129 – 146

[9] **Peters, W.**, and **Schott, D. :** *Über Nullräume, Wertebereiche und Relationen von Operatoren, die bei stationären Iterationsverfahren zur Lösung linearer Gleichungen auftreten.* Rostock. Math. Kolloq. 17, 71 – 85 (1981)

[10] **Rynne, B. P.**, and **Youngson M. A. :** *Linear Functional Analysis.* Springer, London 2008

[11] **Schott, D. :** *Projektionskerne einer Operatorfolge.* Beitr. Anal. 18, 31 – 41 (1981)

[12] **Schott, D. :** *Zur Konvergenz von Operatorfolgen im Zusammenhang mit Iterationsverfahren.* Math. Nachr. 104, 253 – 270 (1981)

[13] **Schott, D. :** *Annulatoren und Dualitätssätze.* Rostock. Math. Kolloq. 18, 51 – 67 (1981)

[14] **Schott, D. :** *Decomposition properties of linear continuous operators in Banach spaces.* Demonstr. Math. 16, 151 – 163 (1983)

[15] **Schott, D. :** *About asymptotic convergence of linear continuous operators in Banach spaces.* Demonstr. Math. 16, 635 – 650 (1983)

[16] **Schott, D. :** *Konvergenzsätze für Verallgemeinerungen von PSH- und SPA-Verfahren.* Math. Nachr. 118, 89 – 103 (1984)

[17] **Schott, D. :** *Relaxations of orthogonal projectors.* Serdica 16, 177 – 187 (1990)

[18] **Schott, D. :** *Strong relaxations of orthogonal projectors.* Rostock. Math. Kolloq. 42, 31 – 46 (1990)

[19] **Schulz, J. :** *Über die Konvergenz der Neumannschen Reihe in linearen topologischen Räumen.* Beitr. Anal. 12, 177 – 183 (1978)

[20] **Tanabe, K. :** *Characterization of linear stationary iterative processes for solving a singular system of linear equations.* Numer. Math. 22, 349 – 359 (1974)

[21] **Taylor, A. E. :** *Introduction to Functional Analysis.* Wiley, New York 1958

[22] **Vasin, V. V.**, and **Eremin, I. I. :** *Operators and Iterative Processes of Fejér Type – Theory and Applications.* De Gruyter, Berlin et. al. 2009

[23] **Yosida, K. :** *Functional analysis.* Springer, Berlin 1965

[24] **Zeidler, E. :** *Nonlinear Functional Analysis and Its Applications, Part I: Fixed Point Theorems.* Teubner 1976

received: March 3, 2014

Author:

Dieter Schott
Hochschule Wismar
Fakultät für Ingenieurwissenschaften
Bereich Elektrotechnik und Informatik,
Philipp-Müller-Str. 14
D-23966 Wismar

e-mail: dieter.schott@hs-wismar.de