Rostock. Math. Kolloq. **68**, 3 – 12 (2013)

Subject Classification (MSC)
35Q40, 35J50, 46E35

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The limiting Dirac-Sobolev inequality

ABSTRACT. *We prove the critical Dirac-Sobolev inequality for $p \in (1,3)$. It follows that the Dirac Sobolev spaces are equivalent to classical Sobolev spaces if and only if $p \in (1,3)$. We prove the cocompactness of $L^{p^*}(\mathbb{R}^3)$ in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$.* As an application, we prove the existence of minimizers to a class of isoperimetric problems.

KEY WORDS AND PHRASES. cocompact imbeddings, concentration compactness, Dirac operator, minimizers, Sobolev imbeddings, critical exponent

1 Introduction

In [1], Balinsky, Evans and Saito introduced an L^p-seminorm $\|(\alpha \cdot \mathbf{p})u\|_{p,\Omega}$ of a $\mathbb{C}^4$-valued function on an open subset of Ω of $\mathbb{R}^3$ relevant to a massless Dirac operator

$$\alpha \cdot \mathbf{p} = \sum_{j=1}^{3} \alpha_j(-i\partial_j). \tag{1.1}$$

Here $\mathbf{p} = -\mathbf{i}\nabla$, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix} \quad j = 1, 2, 3$$

that use the 2×2 zero matrix 0_2 and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad \sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They proved a family of inequalities for this seminorm, called Dirac-Sobolev inequalities, in order to obtain L^p-estimates of the *zero modes*, i.e. generalized eigenfunctions associated with the eigenvalue 0 of the Dirac operator $(\alpha \cdot \mathbf{p}) + \mathbf{Q}$, where $Q(x)$ is a 4×4 Hermitian matrix-valued potential decaying at infinity.

Let Ω be an open subset of $\mathbb{R}^3$. The first order Dirac-Sobolev space $\mathbf{H}_0^{1,p}(\Omega;\mathbb{C}^4)=\mathbf{H}_0^{1,p}(\Omega)$, $1\le p<\infty$, is the completion of $C_0^\infty(\Omega;\mathbb{C}^4)$ with respect to the norm

$$\|u\|_{D,1,p,\Omega} := \int_\Omega (|u(x)|_p^p + |(\alpha\cdot\mathbf{p})u(x)|_p^p)\mathbf{d}x \tag{1.2}$$

where the p-norm of a vector $a=(a_1,a_2,a_{3,}a_4)^T\in\mathbb{C}^4$ is defined as

$$|a|_p = (\sum_{i=1}^4 |a_i|^p)^{1/p},$$

$u(x)=(u_1(x),u_2(x),u_3(x),u_4(x))^T$, and

$$(\alpha\cdot\mathbf{p})u(x) := \sum_{j=1}^3 \alpha_j p_j u(x) = \sum_{j=1}^3(-i\alpha_j\partial_j u(x)).$$

A completion of $C^\infty(\Omega;\mathbb{C})^4$ with respect to the same norm will be denoted $\mathbf{H}^{1,p}(\Omega)$.

Let β be the fourth Dirac matrix given by

$$\beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix},$$

where 1_2 is the 2×2 identity matrix. It is known that the free massless Dirac operator $\alpha\cdot\mathbf{p}$ as well as the free Dirac operator $\alpha\cdot\mathbf{p}+m\beta$ with positive mass m and the relativistic Schrodinger operator $\sqrt{m^2-\Delta}$ have similar embedding properties in L^2 but not necessarily in L^p for $p\neq 2$. It is also known that for $1<p<\infty$, the usual $W^{1,p}(\Omega)$ Sobolev norm $(\|\psi\|_p^p+\|\nabla\psi\|_p^p)^{1/p}$ is equivalent to the norm $\|\sqrt{1-\Delta}\psi\|_p$, where $\psi:\mathbb{R}^3\mapsto\mathbb{C}$ [14].

In [5] the authors explore the relationship of $\mathbf{H}_0^{1,p}(\Omega)$ with the classical Sobolev spaces $W_0^{1,p}(\Omega;\mathbb{C}^4)$ when Ω is a bounded domain. In particular, it is shown that $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ are dense subspaces of $H_0^{1,p}(\Omega)$ and $H^{1,p}(\Omega)$ respectively. The maps

$$\begin{cases} J_{\Omega,0}: & W_0^{1,p}(\Omega)\ni u\mapsto u\in\mathbf{H}_0^{1,p}(\Omega) \\ J_\Omega: & W^{1,p}(\Omega)\ni u\mapsto u\in\mathbf{H^{1,p}(\Omega)} \end{cases}$$

are one to one and continuous for $1\le p<\infty$. They showed that the map $J_{\Omega,0}$ is onto with continuous inverse if $1<p<\infty$ so that the spaces $W_0^{1,p}(\Omega)$ and the space $H_0^{1,p}(\Omega)$ are the same. If $p=1$, the map $J_{\Omega,0}$ is not onto.

In this paper we prove the limiting Dirac-Sobolev inequality on the whole space,

$$\int_{\mathbb{R}^3}|u|^{p^*}\mathrm{d}x \le C_p\left(\int_{\mathbb{R}^3}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x\right)^{\frac{p^*}{p}}, \tag{1.3}$$

where C_p is a positive constant, $p \in (1,3)$, and $p^* = \frac{3p}{3-p}$. It follows that the map J_Ω is onto for $p \in (1,3)$ if Ω is an extension domain. An extension domain is a domain for which every $u \in \mathbf{H^{1,p}(\Omega)}$ there is a $\tilde{u} \in \mathbf{H_0^{1,p}(\Omega')}$ such that $u = \tilde{u}|_\Omega$ where $\Omega \subset \Omega'$. We then prove cocompactness of the embedding $L^{p^*}(\mathbb{R}^3) \subset \dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$, where $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{1.4}$$

See Remark 2.2. We apply this result to show existance of minimizers to isoperimetric problems involving oscillatory nonlinearities with critical growth.

2 A Dirac-Sobolev inequality and the space $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$

In this section we prove inequality (1.3).

Theorem 2.1 *Let $p \in (1,3)$. Then there exists a constant $C_p > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$, (1.3) holds.*

Proof. Let us use the inequality (3.10) of [1], with the choice of parameters $k = p$, $r = 1$ and $\theta = \frac{3p-3}{4p-3}$:

$$\|u\|_{p,B_1}^p \le C\|(\alpha \cdot \mathbf{p})u\|_{p,B_1}^{p\theta} \|u\|_{1,B_1}^{p(1-\theta)}, \ u \in C_0^\infty(B_1;\mathbb{C}^4). \tag{2.1}$$

Using an elementary inequality $s^\theta t^{(1-\theta)} \le C(\lambda s + \lambda^{-\gamma} t)$, $\frac{1}{\gamma} = \frac{1}{\theta} - 1$, that holds for all positive t, s, and λ, and setting $\lambda = \rho^p$, $s = \|u\|_{p,B_1}^p$, and $t = \|u\|_{1,B_1}^p$, one deduces from (2.1), for all positive ρ,

$$\|u\|_{p,B_1}^p \le C\left(\rho^p\|(\alpha \cdot \mathbf{p})u\|_{p,B_1}^p + \rho^{3-3p}\|u\|_{1,B_1}^p\right) \ u \in C_0^\infty(B_1;\mathbb{C}^4). \tag{2.2}$$

By choosing $\rho' = R\rho$ and rescaling the integration domain we will have, for any positive ρ', renamed ρ,

$$\|u\|_{p,B_R}^p \le C\left(\rho^p\|(\alpha \cdot \mathbf{p})u\|_{p,B_R}^p + \rho^{3-3p}\|u\|_{1,B_R}^p\right), \ u \in C_0^\infty(B_R;\mathbb{C}^4), \tag{2.3}$$

for any $R > 0$. We conclude that for any positive ρ,

$$\|u\|_{p,\mathbb{R}^3}^p \le C\left(\rho^p\|(\alpha \cdot \mathbf{p})u\|_{p,\mathbb{R}^3}^p + \rho^{3-3p}\|u\|_{1,\mathbb{R}^3}^p\right), \ u \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4). \tag{2.4}$$

Let us apply (2.4) to functions $\chi_j(|u|)$, where $\chi_j(t) = 2^{-j}\chi(2^j t)$, $j \in \mathbb{Z}$ and $\chi \in C_0^\infty((\frac{1}{2},4),[0,3])$, such that $\chi(t) = t$ whenever $t \in [1,2]$ and $|\chi'| \le 2$. Then we obtain, with the values

$\rho = \rho_j$ to be determined,

$$\begin{aligned}\int_{|u|\in[2^j,2^{j+1}]}|u|^p\mathrm{d}x &\le \int \chi_j(u)^p\mathrm{d}x\\ &\le C\left(\rho_j^p\int_{|u|\in[2^{j-1},2^{j+2}]}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x+\right.\\ &\qquad \left.\rho_j^{3-3p}\left(\int_{|u|\in[2^{j-1},2^{j+2}]}|u|\mathrm{d}x\right)^p\right).\end{aligned}$$

Taking into account the upper and lower bounds of $|u|$ on the respective sets of integration, we have

$$\begin{aligned}2^{(p-p^*)j}\int_{|u|\in[2^j,2^{j+1}]}|u|^{p^*}\mathrm{d}x &\le C\rho_j^p\int_{|u|\in[2^{j-1},2^{j+2}]}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x+\\ &\qquad C2^{p(1-p^*)(j-1)}\rho_j^{3-3p}\left(\int_{|u|\in[2^{j-1},2^{j+2}]}|u|^{p^*}\mathrm{d}x\right)^p.\end{aligned}$$

If we substitute $\rho_j = 2^{-\frac{p^3(1-p)j+pp^*-p}{3-3p}}\rho$, take the sum over $j\in\mathbb{Z}$, and note that each of the intervals $[2^{j-1},2^{j+2}]$, $j\in\mathbb{Z}$, overlaps with the others not more than four times, we get

$$\int|u|^{p^*}\mathrm{d}x \le C\left(\rho^p\int_{\mathbb{R}^3}|(\alpha\cdot\mathbf{p})u|^p\mathrm{d}x+\rho^{3-3p}\left(\int_{\mathbb{R}^3}|u|^{p^*}\mathrm{d}x\right)^p\right)$$

Setting $\rho = \left(\frac{1}{2C}\right)^{\frac{1}{3-3p}}\left(\int u^{p^*}\right)^{\frac{1}{3}}$ and collecting similar terms we arrive at (1.3). □

Inequality (1.3) defines a continuous imbedding of $L^{p^*}(\mathbb{R}^3;\mathbb{C}^4)$ into $\dot H_D^{1,p}(\mathbb{R}^3)$.

Remark 2.2 Note that (1.4) does indeed define a norm on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$, since $(\alpha\cdot\mathbf{p})u=0$ implies $|\nabla u|^2=0$ which yields $u=$ const. Since $u=0$ outside of a compact set, the value of this constant is zero. We have therefore a Banach space $\dot H_D^{1,p}(\mathbb{R}^3)$ into which $L^{p^*}(\mathbb{R}^3)$ is continuously imbedded. It should be noted, however, that the space $\dot H_D^{1,p}(\mathbb{R}^3)$ is equivalent to the usual gradient-norm space $\mathcal{D}^{1,p}(\mathbb{R}^3;\mathbb{C}^4)$ *if and only if* $p\in(1,3)$. If $p>1$, consider the gradient norm and the Dirac-gradient norm (1.4) on $C_0^\infty(B_R;\mathbb{C}^4)$, which are equivalent Sobolev norms in $W_0^{1,p}(B_R;\mathbb{C}^4)$ and $\mathbf{H}_0^{1,p}(B_R)$ respectively. Since these norms are scale-invariant, they are equivalent (by Theorem 1.3 (ii) of [5]) on the balls B_R with bounds independent of R and thus, these norms are equivalent on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$, and, consequently $\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)=\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$. As a further consequence, the map J_Ω defined in the introduction is onto if $p\in(1,3)$ and Ω is an extension domain. From this observation, we obtain

Corollary 2.3 *Let $p\in(1,3)$, $\Omega\subset\mathbb{R}^3$ be and extension domain, and $q\in[p,p^*]$ then*

$$\|u\|_{q,\Omega}\le C_{p,q}\left(\int_\Omega(|(\alpha\cdot\mathbf{p})u|^p+|u|^p)\mathrm{d}x\right)^{1/p}. \tag{2.5}$$

Remark 2.4 If $p = 1$, by Proposition 4.4 of [5] $\dot{H}^{1,1}(\mathbb{R}^3;\mathbb{C}^4)$ is strictly smaller than $\dot{\mathbf{H}}^{1,1}(\mathbb{R}^3)$.

3 Cocompactness of Dirac-Sobolev imbeddings

We recall the following definitions:

Definition 3.1 *Let u_k be a sequence in a Banach space E and D be a set of linear isometries acting on E. We say that u_k converges $D-$weakly to u, which we denote*

$$u_k \overset{D}{\rightharpoonup} u,$$

if for all ϕ in E',

$$\lim_{k\to\infty} \sup_{g\in D}(g\phi, u_k - u) = 0.$$

Remark 3.2 It follows immediately from 3.1 that if a bounded sequence u_k is not $D-$weakly convergent to 0, then there exists a sequence $g_k \in D$ and a $w \neq 0 \in E$ such that $g_k^* u_k \rightharpoonup w$.

Definition 3.3 *Let B be a Banach space continuously embedded in E. We say that B is* cocompact *in E with respect to D if $u_k \overset{D}{\rightharpoonup} u$ in E implies $u_k \to u$ in B.*

Let $\delta_{\mathbb{R}}$ be the group of dilations,

$$h_s u(x) = p^{\frac{3-p}{p}s} u(p^s x),$$

let D_G be the group of translations,

$$g_y u = u(\cdot - y),\ y \in \mathbb{R}^3,$$

and let

$$D := \delta_{\mathbb{R}} \times D_G.$$

We will denote by $D_{\mathbb{Z}}$ the subgroup, $s \in \mathbb{Z}$, $y \in \mathbb{Z}^3$. Note that both $\|u\|_{p^*}$and $\|u\|_{\dot{\mathbf{H}}}$are invariant under D and $D_{\mathbb{Z}}$. Furthermore, cocompactness with respect to D is equivalent to cocompactness with respect to $D_{\mathbb{Z}}$ (Lemma 5.3, [15]).

Theorem 3.4 *Let $p \in (1, 3)$. Then $L^{p^*}(\mathbb{R}^3)$is cocompactly embedded in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$ with respect to D.*

Proof. Assume u_k is D-weakly convergent to zero in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$. Since $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ is dense in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$ and the latter is continuously imbedded into $L^{p^*}(\mathbb{R}^3)$, we may assume without loss of generality that $u_k \in C_0^\infty(\mathbb{R}^3)$. Let $\chi \in C_0^\infty((\frac{1}{p}, p^2); [0, p^2 - 1])$, be such that $\chi(t) = t$ for $t \in [1, p]$ and $|\chi'| \leq \frac{p}{p-1}$. By the Dirac-Sobolev inequality (2.5), for every $y \in \mathbb{Z}^3$,

$\left(\int_{(0,1)^3+y}\chi(|u_k|)^{p^*}\mathrm{d}x\right)^{p/p^*} \leq C\int_{(0,1)^3+y}(|(\alpha\cdot\mathbf{p})u_k|^p+\chi(u_k)^p)\mathrm{d}x.$

Since $\chi(t)^{p^*}\leq Ct^p$, this gives

$$\begin{aligned}
&\int_{(0,1)^3+y}\chi(|u_k|)^{p^*}\mathrm{d}x\\
\leq\ & C\left(\int_{(0,1)^3+y}(|(\alpha\cdot\mathbf{p})u_k|^p+\chi(u_k)^p)\mathrm{d}x\right)\left(\int_{(0,1)^3+y}\chi(|u_k|)^{p^*}\mathrm{d}x\right)^{1-p/p^*}\\
\leq\ & C\left(\int_{(0,1)^3+y}(|(\alpha\cdot\mathbf{p})u_k|^p+\chi(u_k)^p)\mathrm{d}x\right)\left(\int_{(0,1)^3+y}u_k^p\mathrm{d}x\right)^{1-p/p^*}.
\end{aligned}$$

Summing the above inequalities over all $y\in\mathbb{Z}^3$, and noting that by (1.3) $\|u_k\|_{p^*}\leq C$, therefore $|\left\{u_k\geq\frac{1}{p}\right\}|\leq C$ from which we can conclude $\int_{\mathbb{R}^3}\chi(u_k)^p\leq C$, we obtain

$$\int_{\mathbb{R}^3}\chi(|u_k|)^{p^*}\leq C\sup_{y\in\mathbb{Z}^3}\left(\int_{(0,1)^3+y}|u_k|^p\right)^{1-p/p^*}. \tag{3.1}$$

Let $y_k\in\mathbb{Z}^3$ be such that

$$\sup_{y\in\mathbb{Z}^3}\left(\int_{(0,1)^3+y}|u_k|^p\right)^{1-p/p^*}\leq 2\left(\int_{(0,1)^3+y_k}|u_k|^p\right)^{1-p/p^*}.$$

Since u_k converges to zero D-weakly, $u_k(\cdot-y_k)\rightharpoonup 0$ in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$, and thus it follows from Theorem 1.3 (ii) in [5] and the fact that $(0,1)^3$ is an extension domain that $u_k(\cdot-y)\to 0$ in $L^p((0,1)^3;\mathbb{C}^4)$. Therefore,

$$\int_{(0,1)^3+y_k}|u_k|^p=\int_{(0,1)^3}|u_k(\cdot-y_k)|^p\to 0.$$

Substituting into (3.1), we obtain

$$\int_{\mathbb{R}^3}\chi(|u_k|)^{p^*}\mathrm{d}x\to 0.$$

Let

$$\chi_j(t)=p^j\chi(p^{-j}t)),\ \ j\in\mathbb{Z}.$$

Since for any sequence $j\in\mathbb{Z}$, $h_{j_k}u_k$ converges to zero D-weakly, we have also, with arbitrary $j_k\in\mathbb{Z}$,

$$\int_{\mathbb{R}^3}\chi_{j_k}(|u_k|)^{p^*}\mathrm{d}x\to 0. \tag{3.2}$$

For $j\in\mathbb{Z}$, we have

$$\left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*} \mathrm{d}x\right)^{p/p^*} \leq C \int_{\{p^{j-1} \leq |u_k| \leq p^{j+2}\}} |(\alpha \cdot \mathbf{p}) u_k|^p \mathrm{d}x,$$

which can be rewritten as

$$\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*} \mathrm{d}x \leq C \int_{\{p^{j-1} \leq |u_k| \leq p^{j+2}\}} |(\alpha \cdot \mathbf{p}) u_k|^p \mathrm{d}x \left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*} \mathrm{d}x\right)^{1-\frac{p}{p^*}}. \tag{3.3}$$

Adding the inequalities (3.3) over $j \in \mathbb{Z}$ and taking into account that the sets $\{x \in \mathbb{R}^3 : 2^{j-1} \leq |u_k| \leq 2^{j+2}\}$ cover $\mathbb{R}^3$ with uniformly finite multiplicity, we obtain

$$\int_{\mathbb{R}^3} |u_k|^{p^*} \mathrm{d}x \leq C \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p}) u_k|^p \mathrm{d}x \sup_{j \in \mathbb{Z}} \left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*}\right)^{1-p/p^*}. \tag{3.4}$$

Let j_k be such that

$$\sup_{j \in \mathbb{Z}} \left(\int_{\mathbb{R}^3} \chi_j(|u_k|)^{p^*}\right)^{1-p/p^*} \leq 2 \left(\int_{\mathbb{R}^3} \chi_{j_k}(|u_k|)^{p^*}\right)^{1-p/p^*}.$$

Using the previous estimate and (3.2) we see that the right hand side of (3.4) converges to zero. Thus $u_k \to 0$ in L^{p^*}. □

4 Existence of minimizers

We consider the class of functions $F \in C_{\mathrm{loc}}(\mathbb{R})$satisfying

$$F(p^{\frac{3-p}{p}j} s) = p^{3j} F(s), \; s \in \mathbb{R}, j \in \mathbb{Z}. \tag{4.1}$$

This class is characterized by continuous functions on the intervals $[1, p^{\frac{3-p}{p}}]$ and $[-p^{\frac{3-p}{p}}, -1]$ satisfying $F(p^{\frac{3-p}{p}}) = p^3 F(1)$ and $F(-p^{\frac{3-p}{p}}) = p^3 F(-1)$, extended to $(0, \infty)$ and $(-\infty, 0)$ by (4.1). It is immediate that there exists positive constants C_1 and C_2 such that

$$C_1 |s|^{p^*} \leq |F(s)| \leq C_2 |s|^{p^*}. \tag{4.2}$$

It also follows from (4.1) that for $h_j \in \delta_{\mathbb{Z}}$,

$$\int_{\mathbb{R}^3} F(h_j u) \mathrm{d}x = \int_{\mathbb{R}^3} F(u) \mathrm{d}x, \; \text{for j} \in \mathbb{Z}, \; \text{u} \in \mathrm{L}^{\mathrm{p}^*}(\mathbb{R}^3).$$

The functional

$$G(u) = \int_{\mathbb{R}^3} F(u) \mathrm{d}x$$

is continuous on $L^{p^*}(\mathbb{R}^3)$ and thus on $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$.

Theorem 4.1 *There exists a minimizer to the following isoperimetric problem.*

$$\inf_{G(u)=1} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \mathrm{d}x \tag{4.3}$$

in $\dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$.

Proof. Let u_k be a minimizing sequence. By (4.2) and (2.5), u_k is bounded. By Theorem 3.4 and (4.2), u_k cannot converge $D-$weakly to 0. By Theorem 2 in [13] (see also [12]), (4.2), and using the facts: $\|gw\|^p_{\dot{\mathbf{H}}} = \|w\|^p_{\dot{\mathbf{H}}}$ and $G(gw) = G(w)$, we may write (in our notation) $\|u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)}\|_{L^{p^*}} \to 0$ with $g_k^{(n)} \in D_{\mathbb{Z}}$, $w^{(n)} \in \dot{\mathbf{H}}^{1,p}(\mathbb{R}^3)$,

$$\|u_k\|^p_{\dot{\mathbf{H}}} \geq \sum_{n\in\mathbb{N}} \|w^{(n)}\|^p_H, \text{ and} \tag{4.4}$$

$$1 = G(u_k) = \sum_{n\in\mathbb{N}} G(w^{(n)}) + o(1). \tag{4.5}$$

Since $G(u_k) = 1$, (4.5) implies that at least one $w^{(n)} \neq 0$. We will denote this $w^{(n)}$ by w. From the proof of Theorem 2 in [13] it is immediate that

$$\|u_k\|^p_H = \|w\|^p_H + \|u_k - w\|^p_H + o(1). \tag{4.6}$$

From (4.5) we deduce that

$$G(u_k) = G(w) + G(u_k - w) + o(1). \tag{4.7}$$

Assume $G(w) = \lambda$. We imbed problem (4.3) in the continuous family of problems

$$\alpha(t) := \inf_{G(u)=t} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \mathrm{d}x.$$

From the change of variables $u(t^{1/3}\cdot)$, we see that $\alpha(t) = \inf_{G(u)=1} t^{(1-p/3)} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \mathrm{d}x = t^{(1-p/3)}\alpha(1)$, so $\alpha(t)$ is a strictly concave function. From (4.6), we deduce that $\alpha(1) = \alpha(\lambda) + \alpha(1-\lambda)$. Since $\alpha(t)$ is strictly concave, this is only possible if $\lambda = 1$. Therefore $G(w) = 1$ and w solves problem (4.3). □

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received: January 28, 2014

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