IAN SCHINDLER, CYRIL TINTAREV

The limiting Dirac-Sobolev inequality

ABSTRACT. We prove the critical Dirac-Sobolev inequality for \( p \in (1, 3) \). It follows that the Dirac Sobolev spaces are equivalent to classical Sobolev spaces if and only if \( p \in (1, 3) \). We prove the cocompactness of \( L^p(\mathbb{R}^3) \) in \( \dot{H}^{1,p}(\mathbb{R}^3) \). As an application, we prove the existence of minimizers to a class of isoperimetric problems.

KEY WORDS AND PHRASES. cocompact imbeddings, concentration compactness, Dirac operator, minimizers, Sobolev imbeddings, critical exponent

1 Introduction

In [1], Balinsky, Evans and Saito introduced an \( L^p \)-seminorm \( \Vert (\alpha \cdot p)u \Vert_{p,\Omega} \) of a \( \mathbb{C}^4 \)-valued function on an open subset of \( \Omega \) of \( \mathbb{R}^3 \) relevant to a massless Dirac operator

\[
\alpha \cdot p = \sum_{j=1}^{3} \alpha_j (-i\partial_j).
\]

(1.1)

Here \( p = -i\nabla \), and \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is the triple of \( 4 \times 4 \) Dirac matrices

\[
\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix} \quad j = 1, 2, 3
\]

that use the \( 2 \times 2 \) zero matrix \( 0_2 \) and the triple of \( 2 \times 2 \) Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

They proved a family of inequalities for this seminorm, called Dirac-Sobolev inequalities, in order to obtain \( L^p \)-estimates of the zero modes, i.e. generalized eigenfunctions associated with the eigenvalue 0 of the Dirac operator \( (\alpha \cdot p) + Q \), where \( Q(x) \) is a \( 4 \times 4 \) Hermitian matrix-valued potential decaying at infinity.
Let $\Omega$ be an open subset of $\mathbb{R}^3$. The first order Dirac-Sobolev space $H^{1,p}_0(\Omega; \mathbb{C}^4) = H^{1,p}_0(\Omega)$, $1 \leq p < \infty$, is the completion of $C^\infty(\Omega; \mathbb{C}^4)$ with respect to the norm

$$\|u\|_{D,1,p,\Omega} := \int_\Omega \left(|u(x)|^p_p + |(\alpha \cdot p) u(x)|^p_p\right) dx,$$

where the $p$-norm of a vector $a = (a_1, a_2, a_3, a_4)^T \in \mathbb{C}^4$ is defined as

$$|a|^p_p = \left(\sum_{i=1}^4 |a_i|^p\right)^{1/p},$$

and $u(x) = (u_1(x), u_2(x), u_3(x), u_4(x))^T$, and

$$(\alpha \cdot p) u(x) := \sum_{j=1}^3 \alpha_j p_j u(x) = \sum_{j=1}^3 (-i\alpha_j \partial_j u(x)).$$

A completion of $C^\infty(\Omega; \mathbb{C})^4$ with respect to the same norm will be denoted $H^{1,p}(\Omega)$.

Let $\beta$ be the fourth Dirac matrix given by

$$\beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix},$$

where $1_2$ is the $2 \times 2$ identity matrix. It is known that the free massless Dirac operator $\alpha \cdot p$ as well as the free Dirac operator $\alpha \cdot p + m\beta$ with positive mass $m$ and the relativistic Schrödinger operator $\sqrt{m^2 - \Delta}$ have similar embedding properties in $L^2$ but not necessarily in $L^p$ for $p \neq 2$. It is also known that for $1 < p < \infty$, the usual $W^{1,p}(\Omega)$ Sobolev norm $(||\psi||^p_p + ||\nabla \psi||^p_p)^{1/p}$ is equivalent to the norm $||\sqrt{1-\Delta}\psi||_p$, where $\psi : \mathbb{R}^3 \mapsto \mathbb{C}$ [14].

In [5] the authors explore the relationship of $H^{1,p}_0(\Omega)$ with the classical Sobolev spaces $W^{1,p}_0(\Omega; \mathbb{C}^4)$ when $\Omega$ is a bounded domain. In particular, it is shown that $W^{1,p}_0(\Omega)$ and $W^{1,p}(\Omega)$ are dense subspaces of $H^{1,p}_0(\Omega)$ and $H^{1,p}(\Omega)$ respectively. The maps

$$\begin{cases} J_{\Omega,0} : W^{1,p}_0(\Omega) \ni u \mapsto u \in H^{1,p}_0(\Omega) \\ J_{\Omega} : W^{1,p}(\Omega) \ni u \mapsto u \in H^{1,p}(\Omega) \end{cases}$$

are one to one and continuous for $1 \leq p < \infty$. They showed that the map $J_{\Omega,0}$ is onto with continuous inverse if $1 < p < \infty$ so that the spaces $W^{1,p}_0(\Omega)$ and the space $H^{1,p}_0(\Omega)$ are the same. If $p = 1$, the map $J_{\Omega,0}$ is not onto.

In this paper we prove the limiting Dirac-Sobolev inequality on the whole space,

$$\int_{\mathbb{R}^3} |u|^p dx \leq C_p \left( \int_{\mathbb{R}^3} |(\alpha \cdot p) u|^p dx \right)^{p/2},$$

(1.3)
where $C_p$ is a positive constant, $p \in (1, 3)$, and $p^* = \frac{3p}{3-p}$. It follows that the map $J_\Omega$ is onto for $p \in (1, 3)$ if $\Omega$ is an extension domain. An extension domain is a domain for which every $u \in \mathcal{H}^{1,p}(\Omega)$ there is a $\tilde{u} \in \mathcal{H}^0_0(\Omega')$ such that $u = \tilde{u}|_\Omega$ where $\Omega \subset \Omega'$. We then prove cocompactness of the embedding $L^p(\mathbb{R}^3) \subset \mathcal{H}^{1,p}(\mathbb{R}^3)$, where $\mathcal{H}^{1,p}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$ with respect to the norm

$$
\|u\| = \left(\int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p dx\right)^{\frac{1}{p}}.
$$

(1.4)

See Remark 2.2. We apply this result to show existence of minimizers to isoperimetric problems involving oscillatory nonlinearities with critical growth.

## 2 A Dirac-Sobolev inequality and the space $\mathcal{H}^{1,p}(\mathbb{R}^3)$

In this section we prove inequality (1.3).

**Theorem 2.1** Let $p \in (1, 3)$. Then there exists a constant $C_p > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$, (1.3) holds.

**Proof.** Let us use the inequality (3.10) of [1], with the choice of parameters $k = p$, $r = 1$ and $\theta = \frac{3p-3}{4p-3}$.

$$
\|u\|^p_{p,B_1} \leq C\|((\alpha \cdot \mathbf{p})u\|^p_{p,B_1} \|u\|^p_{1,B_1}, \quad u \in C_0^\infty(B_1; \mathbb{C}^4).\quad (2.1)
$$

Using an elementary inequality $s^{\theta t^{(1-\theta)}} \leq C(\lambda s + \lambda^{-\gamma}t)$, $\frac{1}{\gamma} = 1 - \frac{3}{4p} - 1$, that holds for all positive $t$, $s$, and $\lambda$, and setting $\lambda = \rho^p$, $s = \|u\|^p_{p,B_1}$, and $t = \|u\|^p_{1,B_1}$, one deduces from (2.1), for all positive $\rho$,

$$
\|u\|^p_{p,B_1} \leq C(\rho^p\|((\alpha \cdot \mathbf{p})u\|^p_{p,B_1} + \rho^{3-3p}\|u\|^p_{1,B_1}), \quad u \in C_0^\infty(B_1; \mathbb{C}^4).\quad (2.2)
$$

By choosing $\rho' = R^p \rho$ and rescaling the integration domain we will have, for any positive $\rho'$, renamed $\rho$,

$$
\|u\|^p_{p,B_R} \leq C(\rho'\|((\alpha \cdot \mathbf{p})u\|^p_{p,B_R} + \rho'^{3-3p}\|u\|^p_{1,B_R}), \quad u \in C_0^\infty(B_R; \mathbb{C}^4),\quad (2.3)
$$

for any $R > 0$. We conclude that for any positive $\rho$,

$$
\|u\|^p_{p,\mathbb{R}^3} \leq C(\rho^p\|((\alpha \cdot \mathbf{p})u\|^p_{p,\mathbb{R}^3} + \rho^{3-3p}\|u\|^p_{1,\mathbb{R}^3}), \quad u \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^4).\quad (2.4)
$$

Let us apply (2.4) to functions $\chi_j(|u|)$, where $\chi_j(t) = 2^{-j}\chi(2^jt)$, $j \in \mathbb{Z}$ and $\chi \in C_0^\infty(\frac{1}{2}, 4)$, $[0, 3])$, such that $\chi(t) = t$ whenever $t \in [1, 2]$ and $|\chi'| \leq 2$. Then we obtain, with the values
\( \rho = \rho_j \) to be determined,

\[
\int_{|u| \in [2^{j-1}, 2^j]} |u|^p \, dx \leq \int \chi_j(u)^p \, dx \\
\leq C \left( \rho_j^p \int_{|u| \in [2^{j-1}, 2^j]} |(\alpha \cdot p)u|^p \, dx + \rho_j^{3-3p} \left( \int_{|u| \in [2^{j-1}, 2^j]} |u| \, dx \right)^p \right).
\]

Taking into account the upper and lower bounds of \(|u|\) on the respective sets of integration, we have

\[
2^{(p-p^*)j} \int_{|u| \in [2^{j-1}, 2^j]} |u|^{p^*} \, dx \leq C \rho_j^p \int_{|u| \in [2^{j-1}, 2^j]} |(\alpha \cdot p)u|^p \, dx + C 2^{p(1-p^*)j} \rho_j^{3-3p} \left( \int_{|u| \in [2^{j-1}, 2^j]} |u|^{p^*} \, dx \right)^p.
\]

If we substitute \( \rho_j = 2 \frac{\rho_j^{3-3p} \left( \int_\Omega |u|^{p^*} \, dx \right) \rho_j^{p^*} \rho_j^{3-3p}}{\rho_j^{3-3p}} \), take the sum over \( j \in \mathbb{Z} \), and note that each of the intervals \([2^{j-1}, 2^j], j \in \mathbb{Z}\), overlaps with the others not more than four times, we get

\[
\int |u|^{p^*} \, dx \leq C \left( \rho_j^p \int_\mathbb{R}^3 |(\alpha \cdot p)u|^p \, dx + \rho_j^{3-3p} \left( \int_{\mathbb{R}^3} |u|^{p^*} \, dx \right)^p \right)
\]

Setting \( \rho = \left( \frac{1}{2^{j+3}} \right)^{\frac{1}{3-3p}} \left( \int u^{p^*} \right)^{\frac{1}{4}} \) and collecting similar terms we arrive at (1.3).

Inequality (1.3) defines a continuous imbedding of \( L^{p^*}(\mathbb{R}^3; \mathbb{C}^4) \) into \( \dot{H}_D^{1,p}(\mathbb{R}^3) \).

**Remark 2.2** Note that (1.4) does indeed define a norm on \( C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \), since \((\alpha \cdot p)u = 0\) implies \(|\nabla u|^2 = 0\) which yields \( u = \text{const} \). Since \( u = 0 \) outside of a compact set, the value of this constant is zero. We have therefore a Banach space \( \dot{H}_D^{1,p}(\mathbb{R}^3) \) into which \( L^{p^*}(\mathbb{R}^3) \) is continuously imbedded. It should be noted, however, that the space \( \dot{H}_D^{1,p}(\mathbb{R}^3) \) is equivalent to the usual gradient-norm space \( \mathcal{D}^{1,p}(\mathbb{R}^3; \mathbb{C}^4) \) if and only if \( p \in (1, 3) \). If \( p > 1 \), consider the gradient norm and the Dirac-gradient norm (1.4) on \( C_0^\infty(B_R; \mathbb{C}^4) \), which are equivalent Sobolev norms in \( W_0^{1,p}(B_R; \mathbb{C}^4) \) and \( H_0^{1,p}(B_R) \) respectively. Since these norms are scale-invariant, they are equivalent (by Theorem 1.3 (ii) of [3]) on the balls \( B_R \) with bounds independent of \( R \) and thus, these norms are equivalent on \( C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \), and, consequently \( \mathcal{D}(\mathbb{R}^3; \mathbb{C}^4) = \dot{H}_D^{1,p}(\mathbb{R}^3) \). As a further consequence, the map \( J_\Omega \) defined in the introduction is onto if \( p \in (1, 3) \) and \( \Omega \) is an extension domain. From this observation, we obtain

**Corollary 2.3** Let \( p \in (1, 3), \Omega \subset \mathbb{R}^3 \) be and extension domain, and \( q \in [p, p^*] \) then

\[
\|u\|_{q, \Omega} \leq C_{p,q} \left( \int_\Omega \left( |(\alpha \cdot p)u|^p + |u|^p \right) \, dx \right)^{1/p}.
\] (2.5)
Remark 2.4 If \( p = 1 \), by Proposition 4.4 of [5] \( \dot{H}^{1,1}(\mathbb{R}^3; \mathbb{C}^4) \) is strictly smaller than \( \dot{H}^{1,1}(\mathbb{R}^3) \).

### 3 Cocompactness of Dirac-Sobolev imbeddings

We recall the following definitions:

**Definition 3.1** Let \( u_k \) be a sequence in a Banach space \( E \) and \( D \) be a set of linear isometries acting on \( E \). We say that \( u_k \) converges \( D \)-weakly to \( u \), which we denote \( u_k \xrightarrow{D} u \), if for all \( \phi \) in \( E' \),

\[
\lim_{k \to \infty} \sup_{g \in D} (g\phi, u_k - u) = 0.
\]

**Remark 3.2** It follows immediately from 3.1 that if a bounded sequence \( u_k \) is not \( D \)-weakly convergent to 0, then there exists a sequence \( g_k \in D \) and a \( w \neq 0 \in E \) such that \( g_k u_k \rightharpoonup w \).

**Definition 3.3** Let \( B \) be a Banach space continuously embedded in \( E \). We say that \( B \) is cocompact in \( E \) with respect to \( D \) if \( u_k \xrightarrow{D} u \) in \( E \) implies \( u_k \to u \) in \( B \).

Let \( \delta_{\mathbb{R}} \) be the group of dilations,

\[
h_s u(x) = p^{\frac{3-p}{p}} s^{p} u(p^s x),
\]

let \( D_G \) be the group of translations,

\[
g_y u = u(\cdot - y), \quad y \in \mathbb{R}^3,
\]

and let

\[
D := \delta_{\mathbb{R}} \times D_G.
\]

We will denote by \( D_Z \) the subgroup, \( s \in \mathbb{Z}, y \in \mathbb{Z}^3 \). Note that both \( \|u\|_{p'} \) and \( \|u\|_{\dot{H}^s} \) are invariant under \( D \) and \( D_Z \). Furthermore, cocompactness with respect to \( D \) is equivalent to cocompactness with respect to \( D_Z \) (Lemma 5.3, [15]).

**Theorem 3.4** Let \( p \in (1, 3) \). Then \( L^p(\mathbb{R}^3) \) is cocompactly embedded in \( \dot{H}^{1,p}(\mathbb{R}^3) \) with respect to \( D \).

**Proof.** Assume \( u_k \) is \( D \)-weakly convergent to zero in \( \dot{H}^{1,p}(\mathbb{R}^3) \). Since \( C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \) is dense in \( \dot{H}^{1,p}(\mathbb{R}^3) \) and the latter is continuously imbedded into \( L^p(\mathbb{R}^3) \), we may assume without loss of generality that \( u_k \in C_0^\infty(\mathbb{R}^3) \). Let \( \chi \in C_0^\infty((\frac{1}{p}, p^2); [0, p^2 - 1]) \), be such that \( \chi(t) = t \) for \( t \in [1, p] \) and \( |\chi'| \leq \frac{p}{p-1} \). By the Dirac-Sobolev inequality (2.5), for every \( y \in \mathbb{Z}^3 \),
\[ \left( \int_{(0,1)^3+y} \chi(|u_k|)^p \right)^{p/p^*} \leq C \int_{(0,1)^3+y} |(\alpha \cdot p) u_k|^p + \chi(u_k)^p \, dx. \]

Since \( \chi(t)^{p^*} \leq C\|t\|_{p^*} \), this gives
\[
\int_{(0,1)^3+y} \chi(|u_k|)^{p^*} \, dx \\
\leq C \left( \int_{(0,1)^3+y} |(\alpha \cdot p) u_k|^p + \chi(u_k)^p \, dx \right) \left( \int_{(0,1)^3+y} \chi(|u_k|)^{p^*} \, dx \right)^{1-p/p^*} \\
\leq C \left( \int_{(0,1)^3+y} |(\alpha \cdot p) u_k|^p + \chi(u_k)^p \, dx \right) \left( \int_{(0,1)^3+y} u_k^p \, dx \right)^{1-p/p^*}.
\]

Summing the above inequalities over all \( y \in \mathbb{Z}^3 \), and noting that by (1.3) \( \|u_k\|_{p^*} \leq C \), therefore \( \left\{ \{ u_k \geq \frac{1}{p} \} \right\} \leq C \) from which we can conclude \( \int_{\mathbb{R}^3} \chi(u_k)^p \leq C \), we obtain
\[
\int_{\mathbb{R}^3} \chi(|u_k|)^{p^*} \leq C \sup_{y \in \mathbb{Z}^3} \left( \int_{(0,1)^3+y} u_k^p \right)^{1-p/p^*}.
\]

Let \( y_k \in \mathbb{Z}^3 \) be such that
\[
\sup_{y \in \mathbb{Z}^3} \left( \int_{(0,1)^3+y} u_k^p \right)^{1-p/p^*} \leq 2 \left( \int_{(0,1)^3+y_k} u_k^p \right)^{1-p/p^*}.
\]

Since \( u_k \) converges to zero \( D \)-weakly, \( u_k(\cdot - y_k) \to 0 \) in \( H^{1,p}(\mathbb{R}^3) \), and thus it follows from Theorem 1.3 (ii) in \([5]\) and the fact that \((0,1)^3\) is an extension domain that \( u_k(\cdot - y) \to 0 \) in \( L^p((0,1)^3; \mathbb{C}^4) \). Therefore,
\[
\int_{(0,1)^3+y_k} u_k^p = \int_{(0,1)^3} |u_k(\cdot - y_k)|^p \to 0.
\]

Substituting into (3.1), we obtain
\[
\int_{\mathbb{R}^3} \chi(|u_k|)^{p^*} \, dx \to 0.
\]

Let
\[
\chi_j(t) = t^j \chi(p^{-j}t), \quad j \in \mathbb{Z}.
\]

Since for any sequence \( j \in \mathbb{Z} \), \( h_j u_k \) converges to zero \( D \)-weakly, we have also, with arbitrary \( j_k \in \mathbb{Z} \),
\[
\int_{\mathbb{R}^3} \chi_{j_k}(|u_k|)^{p^*} \, dx \to 0.
\]

For \( j \in \mathbb{Z} \), we have
Interpolation of cocompact imbeddings

\[
\left( \int_{\mathbb{R}^3} \chi_j(|u_k|)^p \, dx \right)^{p/p^*} \leq C \int_{\{p^{j-1} \leq |u_k| \leq p^{j+2}\}} |(\alpha \cdot p)u_k|^p \, dx,
\]

which can be rewritten as

\[
\int_{\mathbb{R}^3} \chi_j(|u_k|)^p \, dx \leq C \int_{\{p^{j-1} \leq |u_k| \leq p^{j+2}\}} |(\alpha \cdot p)u_k|^p \, dx \left( \int_{\mathbb{R}^3} \chi_j(|u_k|)^p \, dx \right)^{1-\frac{p}{p^*}}. \tag{3.3}
\]

Adding the inequalities (3.3) over \( j \in \mathbb{Z} \) and taking into account that the sets \( \{ x \in \mathbb{R}^3 : 2^{j-1} \leq |u_k| \leq 2^{j+2} \} \) cover \( \mathbb{R}^3 \) with uniformly finite multiplicity, we obtain

\[
\int_{\mathbb{R}^3} |u_k|^p \, dx \leq C \int_{\mathbb{R}^3} |(\alpha \cdot p)u_k|^p \, dx \sup_{j \in \mathbb{Z}} \left( \int_{\mathbb{R}^3} \chi_j(|u_k|)^p \, dx \right)^{1-\frac{p}{p^*}}. \tag{3.4}
\]

Let \( j_k \) be such that

\[
\sup_{j \in \mathbb{Z}} \left( \int_{\mathbb{R}^3} \chi_j(|u_k|)^p \, dx \right)^{1-\frac{p}{p^*}} \leq 2 \left( \int_{\mathbb{R}^3} \chi_{j_k}(|u_k|)^p \, dx \right)^{1-\frac{p}{p^*}}.
\]

Using the previous estimate and (3.2) we see that the right hand side of (3.4) converges to zero. Thus \( u_k \to 0 \) in \( L^{p^*} \).

**4 Existence of minimizers**

We consider the class of functions \( F \in C_{\text{loc}}(\mathbb{R}) \) satisfying

\[
F(p^{\frac{3}{2^j}}s) = p^{3j}F(s), \quad s \in \mathbb{R}, j \in \mathbb{Z}. \tag{4.1}
\]

This class is characterized by continuous functions on the intervals \([1, p^{\frac{3}{2^j}}] \) and \([-p^{\frac{3}{2^j}}, -1]\) satisfying \( F(p^{\frac{3}{2^j}}) = p^3F(1) \) and \( F(-p^{\frac{3}{2^j}}) = p^3F(-1) \), extended to \((0, \infty)\) and \((-\infty, 0)\) by (4.1). It is immediate that there exists positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1|s|^{p^*} \leq |F(s)| \leq C_2|s|^{p^*}. \tag{4.2}
\]

It also follows from (4.1) that for \( h_j \in \delta_{\mathbb{Z}} \),

\[
\int_{\mathbb{R}^3} F(h_j u) \, dx = \int_{\mathbb{R}^3} F(u) \, dx, \quad \text{for } j \in \mathbb{Z}, \quad u \in L^{p^*}(\mathbb{R}^3).
\]

The functional

\[
G(u) = \int_{\mathbb{R}^3} F(u) \, dx
\]

is continuous on \( L^{p^*}(\mathbb{R}^3) \) and thus on \( \dot{H}^{1,p}(\mathbb{R}^3) \).
Theorem 4.1. There exists a minimizer to the following isoperimetric problem.

$$\inf_{G(u)=1} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \, dx$$ \hspace{1cm} (4.3)

in $\tilde{H}^{1,p}(\mathbb{R}^3)$.

Proof. Let $u_k$ be a minimizing sequence. By (4.2) and (2.5), $u_k$ is bounded. By Theorem 3.4 and (4.2), $u_k$ cannot converge $D-$weakly to 0. By Theorem 2 in [13] (see also [12]), (4.2), and using the facts: $\|gw\|^p_{\tilde{H}} = \|w\|^p_{\tilde{H}}$, and $G(gw) = G(w)$, we may write (in our notation)

$$\|u_k\|^p_{\tilde{H}} \geq \sum_{n \in \mathbb{N}} \|w^{(n)}\|^p_{\tilde{H}}, \text{ and}$$

$$1 = G(u_k) = \sum_{n \in \mathbb{N}} G(w^{(n)}) + o(1).$$ \hspace{1cm} (4.5)

Since $G(u_k) = 1$, (4.5) implies that at least one $w^{(n)} \neq 0$. We will denote this $w^{(n)}$ by $w$. From the proof of Theorem 2 in [13] it is immediate that

$$\|u_k\|^p_{\tilde{H}} = \|w\|^p_{\tilde{H}} + \|u_k - w\|^p_{\tilde{H}} + o(1).$$ \hspace{1cm} (4.6)

From (4.5) we deduce that

$$G(u_k) = G(w) + G(u_k - w) + o(1).$$ \hspace{1cm} (4.7)

Assume $G(w) = \lambda$. We imbed problem (4.3) in the continuous family of problems

$$\alpha(t) := \inf_{G(u)=t} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \, dx.$$ 

From the change of variables $u(t^{1/3})$, we see that $\alpha(t) = \inf_{G(u)=1} t^{(1-p/3)} \int_{\mathbb{R}^3} |(\alpha \cdot \mathbf{p})u|^p \, dx = t^{(1-p/3)} \alpha(1)$, so $\alpha(t)$ is a strictly concave function. From (4.6), we deduce that $\alpha(1) = \alpha(\lambda) + \alpha(1 - \lambda)$. Since $\alpha(t)$ is strictly concave, this is only possible if $\lambda = 1$. Therefore $G(w) = 1$ and $w$ solves problem (4.3). \hfill \Box

References


Interpolation of cocompact imbeddings


I. Schindler, C. Tintarev

received: January 28, 2014

Authors:

Ian Schindler
MIP-Ceremath UMR 5219,
University of Toulouse 1,
21 allee de Brienne,
31000 Toulouse,
France

e-mail: ian.schindler@univ-tse1.fr

Cyril Tintarev
Department of Mathematics,
Uppsala University,
P.O.Box 480,
75 106 Uppsala,
Sweden

e-mail: tintarev@math.uu.se