Rostock. Math. Kolloq. **68**, 65 – 69 (2013)

Subject Classification (AMS)
46A20, 46B50

Harry Poppe

An abstract Alaoglu theorem

1 Introduction

We start with the paper [6] and repeat the definition of the dual space X^d of the space X w. r. t. another space Y. By Corollary 3.3 of this paper was given a (strong) generalization of the Alaoglu theorem for normed spaces. The Alaoglu theorem concerns the compactness of subsets $H \subseteq X^d$, where for H the pointwise topology τ_p is considered. Of course for the classical Alaoglu theorem we have:

$$(X, \|\cdot\|) \text{ is a Banachspace}, \quad Y = \mathbb{K}, \quad X^d = X', \quad H = B(X') \subseteq X',$$

the norm-closed ball, and the topology τ_p here is nothing else than the weak-star topology.

What is the aim of this paper? We prove a general τ_p-compactness theorem for special subsets $H \subseteq X^d$ (Theorem 2.3). This theorem includes both the generalized Alaoglu theorem of [6], Corollary 3.3 and the Alaoglu theorem for locally convex topological vector spaces (see for instance [5]).

2 The τ_p-compactness theorem

Definition 2.1 *Let (X, τ), (Y, σ) be topological spaces. Moreover let X, Y be spaces with finitely many algebraic operations such that X, Y belong to the same class of such spaces. We assume that we can assign to each algebraic operation in X an algebraic operation in Y (in a naturel manner).*

$X^d = \{h : X \to Y |$ *h is continuous and h is a homomorphism with respect to each pair of corresponding algebraic operations in X and in Y respectively*$\}$*; X^d is called the (first) dual space of X with respect to Y or the Y-dual of X.*

Remark 2.2 1. A theory of this general concept of duality is developed in [6], [2], [3] (definitions of X^d and of the second Y-dual space X^{dd} together with the corresponding toolbox).

Hereby in [2] the definition of X^d and that of the other notions (definitions, propositions and theorems occuring in [2]) is presented very precisely using the language of universal algebra.

2. Clearly, a closed subset of a compact topological space is compact too. Hence, we not only look for the τ_p-compactness of some sets $H \subseteq X^d$ but also for the τ_p-closedness of H.

3. We still provide a notion of relative compactness we need:

   if X is a topological space, $A \subseteq X$ is called relatively compact in X iff holds: each open cover of X has a finite subcover which covers A, or equivalently:

   for each ultrafilter π on X:

$$A \in \pi \Longrightarrow \exists\, x \in X,\ \pi \longrightarrow x.$$

   For the definition and especially the properties of this notion see [8], [1]. Sometimes one defines: A is relatively compact iff the closure $\overline{A}$ is compact, a notion somewhat stronger than the first one.

   For regular spaces the two notions coincide.

Theorem 2.3 *Let (X, τ), (Y, σ) be topological spaces with algebraic structure according to definition 2.1 and let us consider X^d. We assume that (Y, σ) is Hausdorff such that all algebraic operations in Y are continuous with respect to σ. Moreover there is a bornology $\underline{B}$ in Y.*

We assume that there exists a family $(K_x)_{x\in X}$ such that $\forall\, x \in X$, $K_x \in \underline{B}$ and $K_x \neq \emptyset$; for the product $\Pi\{K_x|x \in X\}$ we consider the Tychonoff-topology; thus $\Pi\{K_x|x \in X\}$ is a subspace of $(Y^X; \tau_p)$.

$$H = \{h \in X^d | \forall\, x \in X : \ h(x) \in K_x\};$$

identifying

$$h \equiv (h(x))_{x\in X} \quad \text{we get} \quad H \subseteq \Pi\{K_x|x \in X\}.$$

Finally, we assume

1. *For $(Y; \underline{B})$ holds: $\forall\, B \in \underline{B}$: B is relatively compact in Y.*
2. *Either*

   (a) *H is τ_p-closed in Y^X, or*

(b) *$\forall\, x \in X$: K_x is closed and H is closed in $\Pi\{K_x|x \in X\}$.*

Then H is compact and Hausdorff in (Y^X, τ_p) and hence in (X^d, τ_p) too.

Proof: $\forall\, x \in X$, $K_x \subseteq Y$ is relatively compact in $Y \implies \Pi\{K_x|x \in X\}$ is relatively compact in $(Y^X; \tau_p)$ by the Tychonoff theorem. Y Hausdorff $\implies \forall\, x \in X : K_x$ is Hausdorff $\implies \Pi\{K_x|x \in X\}$ is Hausdorff. By 2. (a) H is τ_p-closed in Y^X and since $H \subseteq \Pi\{K_x|x \in X\}$, H is relatively compact in $(Y^X; \tau_p)$ too and hence H is compact and Hausdorff in (Y^X, τ_p).

We have $H \subseteq X^d \subseteq Y^X$ and hence H is Hausdorff and compact in $(X^d; \tau_p)$, too.

By 2. (b) each K_x is relatively compact and closed yielding that K_x is compact and thus $\Pi\{K_x|x \in X\}$ is a Hausdorff and compact topological space again by Tychonoff. Then H being closed in $\Pi\{K_x|x \in X\}$, is compact in $\Pi\{K_x|x \in X\}$ and hence H is Hausdorff and compact in (Y^X, τ_p) and in (X^d, τ_p) respectively.

Corollary 2.4 *Let X, Y be Hausdorff locally convex topological vector spaces (shortly: l. c. s.); $\underline{B} = \{B \subseteq Y | B$ is bounded$\}$; clearly $\underline{B}$ is a bornology. Only the vector space operations are the algebraic operations in X and Y and these operations are continuous w. r. t. the topologies of X and Y. Now let Y be a Montel space, which means a Hausdorff barreled l. c. s. such that $\forall\, B \subseteq Y : B$ bounded $\implies B$ is relatively compact.*

With these spaces X, Y, their properties and with the other assumptions from Theorem 2.3 the assertions of Theorem 2.3 hold here.

3 Application

We will deduce from Theorem 2.3 or Corollary 2.4 respectively the classical Alaoglu theorem for l. c. s. (sometimes called Alaoglu-Bourbaki theorem) and the generalized τ_p-theorem in Corollary 3.3 of [6].

A. Let X be a Hausdorff l. c. s. and $Y = \mathbb{K}$; $\mathbb{R}$ and $\mathbb{C}$ are Montel spaces (as normed Euclidian spaces) and thus the vector space operations in $\mathbb{K}$ are continuous. $X^d = \{h : X \to \mathbb{K} \,|\, h$ linear and continuous$\} = X'$. Now let U be a neighborhood of $o \in X$ and $U^o = \{h \in X^d | \forall\, x \in U : |h(x)| \leq 1\}$ the polar of U; let $H = U^o$; $\forall\, x \in U$,

$$K_x = \{y \in \mathbb{K} \,|\, |y| \leq 1\},$$

where $|\cdot|$ is the $\mathbb{K}$-norm; $\forall\, z \in X \backslash U$, $\{z\}$ is bounded in X and hence there exists $\lambda_z > 0 :\ z \in \lambda_z U$.

Hence $\forall\, z \in X\backslash U$, $\{\lambda > 0 \,|\, z \in \lambda U\} \neq \emptyset$; by the axiom of choice there exists a vector $(\lambda_z)_{z\in X\backslash U} : \forall\, z \in X\backslash U : z \in \lambda_z U$.

$$\forall\, x \in U : \;\; \lambda_x := 1; \; \forall\, x \in X : \;\; K_x = \{y \in \mathbb{K} \,|\, |y| \leq \lambda_x\};$$

each K_x is bounded and closed in $\mathbb{K}$ and hence compact implying that K_x is relatively compact too, of course each K_x is Hausdorff.

We show:

1. $H = U^o \subseteq \Pi\{K_x \,|\, x \in X\}$.
2. U^o is closed in $(\mathbb{K}^X; \tau_p)$.

Then Corollary 2.4 shows:

U^o is τ_p-compact and Hausdorff in $\mathbb{K}^X$ and in X^d too, thus showing the assertion of the Alaoglu-Bourbaki theorem.

1. $\forall\, h \in U^o : \forall\, x \in U$, $|h(x)| \leq 1 = \lambda_x \Longrightarrow h(x) \in K_x$; $x \in X\backslash U \Longrightarrow x \in \lambda_x U \Longrightarrow x = \lambda_x u$, $u \in U \Longrightarrow h(x) = \lambda_x h(u) \Longrightarrow |h(x)| = \lambda_x |h(u)| \leq \lambda_x \Longrightarrow h(x) \in K_x$. Hence $U^o \subseteq \Pi\{K_x | x \in X\}$.
2. We know that the polar set U^o is equicontinuous and hence U^o is evenly continuous; let (h_i) be a net from U^o, $h_i \xrightarrow{\tau_p} h \in \mathbb{K}^X$; $\forall\, i, h_i$ is linear $\Longrightarrow h$ is linear by proposition 3.1 of [6]. Since U^o is evenly continuous and $h_i \xrightarrow{\tau_p} h$ we get $h_i \xrightarrow{c} h$ (continuous convergence) by Theorem 31 of [4] (see also [7]).

   Now $\mathbb{K}$ is a regular topological space and thus h is continuous by Theorem 30 of [4].

   Hence $h \in X^d$; $\forall\, x \in U$, $h_i(x) \longrightarrow h(x)$, $\forall\, i$, $|h_i(x)| \leq 1 \Longrightarrow |h(x)| \leq 1$, meaning that $h \in U^o$ and thus U^o is closed in $(\mathbb{K}^X, \tau_p)$.

B. Now let our l. c. s. X, Y be Banach spaces.

   But that means that $Y = (Y, \|\cdot\|)$ is a finite-dimensional normed space because Y is a Montel space.

In [6] there was defined:

$$\forall\, c \in \mathbb{R}, \; c > 0 : H_c = \{h \in X^d \,|\, \|h\| \leq c\},$$

where $\|\cdot\|$ is the operator norm,

$$\forall\, c > 0, \; \forall\, x \in X : K_{x,c} = \{y \in Y |\, \|y\| \leq c\|x\|\}.$$

Then $H_c \subseteq \Pi\{K_{x,c}|\, x \in X\}$ and in Theorem 3.2 of [6] was shown that H_c is closed w. r. t. the Tychonoff-topology in $\Pi\{K_{x,c}|\, x \in X\}$. For a fixed $c > o$, $K_{x,c}$ is bounded and closed in Y and hence compact too, because Y is finite-dimensional.

Theorem 2.3 then shows that H_c is τ_p-compact in Y^X and in X^d respectively. But this is the assertion of Corollary 3.3 of [6].

References

[1] **Bartsch, R. :** *Allgemeine Topologie.* Oldenburg Wissenschaftsverlag GmbH (2007)

[2] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space I.* in Guiseppe Di Maio and Somashekhar Naimpally (Ed.), Theory and Applications of Proximity, Nearness and Uniformity, Quaderni di Matematica , 22, Palermo (2009), 275 – 297

[3] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space II.* International Journal of Pure and Applied Mathematics, 84, No 5 (2013), 651 – 667

[4] **Bartsch, R., Denker, P.**, and **Poppe, H. :** *Ascoli-Arzela-Theory based on continuous convergence in an (almost) non-Hausdorff setting.* in Categorical Topology, Dordrecht (1996)

[5] **Meise, R.**, and **Vogt, D. :** *Introduction to Functional Analysis.* Clarendon Press · Oxford (1997)

[6] **Poppe, H. :** *A closededness theorem for normed spaces.* Demonstratio Matematica 41, No 1 (2008), 123 – 127

[7] **Poppe, H. :** *Compactness in General Function Spaces.* Deutscher Verlag der Wissenschaften Berlin (1974)

[8] **Poppe, H. :** *On locally defined topological notions.* Questions and Answers in General Topology 13 (1995), 39 – 53

received: March 25, 2014

Author:

Harry Poppe
Universität Rostock
Institut für Mathematik
18051 Rostock
Germany

e-mail: harry.poppe@uni-rostock.de