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Two-Scale Difference Equations and Power Sums related to Digital Sequences

ABSTRACT. This paper uncovers a connection between two-scale difference equations and the representation of sums of sequences which satisfy a certain multiplicative recurrence formula. For certain digital power sums related with such a sequence we derive a formula which in case of usual power sums yields the known representation of power sums by means of Bernoulli polynomials.

KEY WORDS. Two-scale difference equations, digital sums, Bernoulli polynomials, Appell polynomials, generating functions

1 Introduction

Let $p > 1$ be an integer and C_n the sequence which is given by the p initial values $C_0 = 1$, $C_1, \ldots, C_{p-1}$ such that

$$C := C_0 + \cdots + C_{p-1} > 0 \tag{1.1}$$

and which satisfies the recurrence formula

$$C_{kp+r} = C_k C_r \qquad (k \in \mathbb{N},\ r = 0, \ldots, p-1). \tag{1.2}$$

In this paper we derive a formula for the sum

$$S_m(N) = \sum_{n=0}^{N-1} C_n n^m \tag{1.3}$$

where $m \in \mathbb{N}_0$. In the simple case $C_n = 1$ for all n we have the usual power sum which can be expressed by means of the Bernoulli polynomials $B_m(t)$ in the form

$$\sum_{n=0}^{N-1} n^m = \tilde{B}_m(N) \tag{1.4}$$

where

$$\tilde{B}_m(t) = \frac{1}{m+1}\{B_{m+1}(t) - B_{m+1}\}. \tag{1.5}$$

Digital sums were investigated by many authors, cf. e.g. [4], [13], [3], [12], [5], [6], [10].

Under the condition

$$|C_r| < C \quad \text{for} \quad r = 0, 1, \ldots, p-1 \tag{1.6}$$

we show that for the digital sum (1.3) it holds

$$\sum_{n=0}^{N-1} C_n n^m = N^\alpha \sum_{\mu=0}^{m} N^\mu F_{m,\mu}\left(\log_p N\right) \tag{1.7}$$

with $\alpha = \log_p C$ and 1-periodic continuous functions $F_{m,\mu}$ which can be expressed by means of the solutions of certain two-scale difference equations (Theorem 4.1).

In order to derive formula (1.7) we quote some facts on the two-scale difference equation

$$\lambda\varphi\left(\frac{t}{p}\right) = \sum_{r=0}^{p-1} c_r \varphi(t-r) \qquad (t \in \mathbb{R}) \tag{1.8}$$

with $\lambda \neq 0$ and complex coefficients c_r where $c_0 \neq 0$ and

$$\sum_{r=0}^{p-1} c_r = 1, \tag{1.9}$$

cf. [11] where equation (1.8) with $\lambda = 1$ was studied in detail. In [7] and [8] it was investigated a system of simple functional equations which is equivalent to equation (1.8) with $\lambda = 1$, cf. [11, p. 60]. It is known that under the condition $|c_r| < 1$ equation (1.8) with $\lambda = 1$ has a continuous solution φ_0 satisfying

$$\varphi_0(t) = 0 \quad \text{for} \quad t < 0, \qquad \varphi_0(t) = 1 \quad \text{for} \quad t > 1 \tag{1.10}$$

and that φ_0 is even Hölder continuous cf. [11, Theorem 3.6]. The solution $\varphi = \varphi_0$ has the Laplace transform

$$\mathcal{L}\{\varphi_0\} = \frac{1}{z}\Phi(z) \tag{1.11}$$

where

$$\Phi(z) = \prod_{j=1}^{\infty} P\left(e^{-z/p^j}\right) \tag{1.12}$$

with the polynomial

$$P(w) = \sum_{r=0}^{p-1} c_r w^r, \tag{1.13}$$

cf. [1], [2].

The iterated integrals φ_n $(n \in \mathbb{N})$ of φ_0, defined recursively by

$$\varphi_n(t) = \int_0^t \varphi_{n-1}(\tau)d\tau$$

are solutions of (1.8) with $\lambda = p^n$. For $t > 1$ the solution φ_n is a polynomial

$$\varphi_n(t) = p_n(t) \qquad (t > 1) \tag{1.14}$$

of degree n with the main term $\frac{1}{n!}t^n$. We remark that the polynomials p_n have the property $p_n'(t) = p_{n-1}(t)$, i.e. they are Appell polynomials, cf. [1], [2]. The generating function reads

$$e^{tz}\Phi(z) = \sum_{n=0}^{\infty} p_n(t) z^n \qquad (t \in \mathbb{R}) \tag{1.15}$$

with Φ from (1.12). The coefficients of the power series

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.16}$$

can be calculated recursively by $a_0 = \Phi(0) = 1$ and

$$a_n = \frac{1}{p^n - 1} \sum_{k=1}^{n} (-1)^k \frac{a_{n-k}}{k!} \sum_{r=1}^{p-1} r^k c_r \quad (n \in \mathbb{N}) \tag{1.17}$$

cf. [2, Proposition 2.6] where $p = 2$, and the polynomials p_n in (1.15) have the representation

$$p_n(t) = \sum_{k=0}^{n} \frac{a_{n-k}}{k!} t^k. \tag{1.18}$$

We also need the power series

$$\frac{1}{\Phi(z)} = \sum_{n=0}^{\infty} b_n z^n \tag{1.19}$$

where the coefficients b_n are determined by $b_0 = 1$ and the equations

$$a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n = 0 \qquad (n > 1). \tag{1.20}$$

The corresponding Appell polynomials

$$q_n(t) = \sum_{k=0}^{n} \frac{b_{n-k}}{k!} t^k \tag{1.21}$$

have the generating function

$$\frac{e^{tz}}{\Phi(z)} = \sum_{n=0}^{\infty} q_n(t) z^n. \tag{1.22}$$

This paper is organized as follows: At first we show that the solution $\varphi = \varphi_n$ of the two-scale difference equation (1.8) with $\lambda = p^n$ has for $k \le p^\ell$ the representation

$$\varphi_n\left(\frac{k}{p^\ell}\right) = \frac{c_0^\ell}{p^{n\ell}} \sum_{j=1}^{k} C_{k-j} p_n(j) \tag{1.23}$$

where p_n are the polynomials (1.18), (Theorem 2.1). This formula is the start point for the representation (1.7) of digital power sums. In Section 3 we prove (1.7) in the case $m = 0$, i.e.

$$S_0(N) = \sum_{n=0}^{N-1} C_n = N^\alpha F_0(\log_p N) \tag{1.24}$$

(Theorem 3.2), and give some properties of the 1-periodic continuous function F_0 under the condition (1.6), for instance that F_0 is Hölder continuous and that F_0 is differentiable almost everywhere if $p|C_0C_1\cdots C_{p-1}|^{1/p} < C$, (Proposition 3.5). By means of a Toeplitz theorem we prove the convergence of the arithmetical mean

$$\frac{1}{p^n}\sum_{N=1}^{p^n}\frac{1}{N^\alpha}S_0(N) \tag{1.25}$$

as $n \to \infty$ (Proposition 3.7). In Section 4 we prove the main result of this paper, namely the representation (1.7), (Theorem 4.1). In the simple case $C_n = 1$ for all n formula (1.7) turns over into the known representation (1.4) for the usual power sums, cf. Remark 4.2. For the specific power sums (1.3) where N is a power of p we have the representation

$$S_m(p^k) = (-1)^m m! p^{\alpha k}\sum_{\mu=0}^{m} p^{\mu k} a_\mu b_{m-\mu} \tag{1.26}$$

with $\alpha = \log_p C$ and the coefficients a_n from (1.16) and b_n from (1.19), (Proposition 5.2), and we prove for positive integers k, ℓ

$$S_m(p^{k+\ell}) = \sum_{\mu=0}^{m}\binom{m}{\mu} p^{k\mu} S_\mu(p^\ell) S_{m-\mu}(p^k) \tag{1.27}$$

(Proposition 5.6).

2 Functional relations

For given coefficients $c_0, c_1, \ldots, c_{p-1}$ of the two-scale difference equation (1.8) we define a sequence C_n by $C_n = \frac{c_n}{c_0}$ for $n = 0, 1, \ldots, p-1$ and for $n \geq p$ by the recursion

$$C_{kp+r} = C_k C_r \qquad (k \geq 1, r \in \{0, 1, \ldots, p-1\}). \tag{2.1}$$

If n has the p-adic representation

$$n = \sum n_i p^i, \qquad (n_i \in \{0, 1, \ldots, p-1\}) \tag{2.2}$$

then we have

$$C_n = \prod_{r=1}^{p-1} C_r^{s_r(n)} \tag{2.3}$$

where $s_r(n)$ denotes the total number of occurrences of the digit r in the representation (2.2) of n, cf. [11, p. 63].

The numbers C_n have the generating function

$$G(z) = \prod_{j=0}^{\infty} \frac{1}{c_0} P\left(z^{p^j}\right) = \sum_{n=0}^{\infty} C_n z^n \tag{2.4}$$

which converges for $|z| < 1$, cf. [11, Remark 2.2.1.].

In the following we want to generalize Proposition 2.3 from [11] for φ_n.

Theorem 2.1 *For $\ell \in \mathbb{N}$ and non-negative integers $k < p^\ell$ the solution $\varphi = \varphi_n$ of (1.8) with $\lambda = p^n$ satisfies the equations*

$$\varphi_n\left(\frac{k+t}{p^\ell}\right) = \frac{c_0^\ell}{p^{n\ell}} \sum_{j=0}^{k} C_j \varphi_n(t+k-j) \qquad (0 \le t \le 1). \tag{2.5}$$

Moreover, for $k \le p^\ell$ we have

$$\varphi_n\left(\frac{k}{p^\ell}\right) = \frac{c_0^\ell}{p^{n\ell}} \sum_{j=1}^{k} C_{k-j} p_n(j) \tag{2.6}$$

where p_n are the polynomials (1.18).

Proof: In (1.8) with $\lambda = p^n$ we replace t by $k+t$ with $0 \le k \le p-1$ and get in view of $C_r = \frac{c_r}{c_0}$ for $0 \le r \le p-1$

$$\begin{aligned}
\varphi_n\left(\frac{k+t}{p}\right) &= \frac{1}{p^n} \sum_{r=0}^{p-1} c_r \varphi_n(k+t-r) \\
&= \frac{c_0}{p^n} \sum_{r=0}^{p-1} C_r \varphi_n(k+t-r) \\
&= \frac{c_0}{p^n} \sum_{j=0}^{k} C_j \varphi_n(k+t-j)
\end{aligned}$$

since $\varphi_n(t) = 0$ for $t \le 0$. So (2.5) is true for $\ell = 1$. Assume that (2.5) is valid for a fixed ℓ. Replace t by $\frac{s+t}{p}$ with $0 \le s \le p-1$ we get

$$\begin{aligned}
\varphi_n\left(\frac{kp+s+t}{p^{\ell+1}}\right) &= \frac{c_0^\ell}{p^{n\ell}} \sum_{j=0}^{k} C_j \varphi_n\left(\frac{p(k-j)+s+t}{p}\right) \\
&= \frac{c_0^\ell}{p^{n\ell+n}} \sum_{j=0}^{k} \sum_{r=0}^{p-1} C_j c_r \varphi_n(pk+s-pj-r+t) \\
&= \frac{c_0^{\ell+1}}{p^{n(\ell+1)}} \sum_{j=0}^{k} \sum_{r=0}^{p-1} C_{jp+r} \varphi_n(t+kp+s-pj-r).
\end{aligned}$$

So (2.5) is proved by induction. Formula (2.6) follows by summation in view of $\varphi_n(0) = 0$ and (1.14) for the polynomials $p_n(t)$ from (1.18). □

Remark 2.2 Formula (2.6) yields in case $n = 0$ the known representations

$$\varphi_0\left(\frac{k+t}{p^\ell}\right) = \varphi_0\left(\frac{k}{p^\ell}\right) + c_0^\ell C_k \varphi_0(t) \quad (0 \le t \le 1) \tag{2.7}$$

and

$$\varphi_0\left(\frac{k}{p^\ell}\right) = c_0^\ell \sum_{j=0}^{k-1} C_j \tag{2.8}$$

for the solution $\varphi = \varphi_0$ of equation (1.8) with $\lambda = 1$, cf. [11].

From (2.5) and (2.6) we get in view of (1.14) the following result.

Corollary 2.3 *For $\ell \in \mathbb{N}_0$ and non-negative integers $k < p^\ell$ the solution $\varphi = \varphi_n$ of (1.8) with $\lambda = p^n$ satisfies*

$$\varphi_n\left(\frac{k+t}{p^\ell}\right) = \frac{c_0^\ell C_k}{p^{n\ell}} \varphi_n(t) + \frac{c_0^\ell}{p^{n\ell}} p_{nk}(t) \qquad (0 \le t \le 1) \tag{2.9}$$

with the polynomials

$$p_{nk}(t) = \sum_{j=1}^{k} C_{k-j} p_n(j+t) \tag{2.10}$$

and $p_n(t)$ from (1.18).

We remark that (2.9) with (2.10) is already known for the iterated integrals of de Rham's function, cf. [2, (3.16) and Theorem 3.1].

3 Digital sums

Let C_n be an arbitrary sequence with the properties $C_0 = 1$, (1.1) and (1.2). In order to obtain a formula for the sum (1.3) with $m = 0$, i.e.

$$S_0(N) = \sum_{n=0}^{N-1} C_n \tag{3.1}$$

we consider the two-scale difference equation

$$\varphi\left(\frac{t}{p}\right) = \frac{1}{C} \sum_{r=0}^{p-1} C_r \varphi(t-r) \tag{3.2}$$

with C from (1.1). In the following we assume that (1.6) is satisfies so that equation (3.2) has a continuous solution $\varphi = \varphi_0$ satisfying (1.10) since the quotients $c_r = \frac{C_r}{C}$ satisfy (1.9) and $|c_r| < 1$. For $0 \le t \le 1$ we have in view of $C_0 = 1$ and (1.10)

$$\varphi_0\left(\frac{t}{p}\right) = \frac{1}{C}\varphi_0(t) \qquad (0 \le t \le 1).$$

We put

$$\alpha := \log_p C \tag{3.3}$$

so that $p^\alpha = C$ and

$$\frac{\varphi_0(\frac{t}{p})}{(\frac{t}{p})^\alpha} = \frac{\varphi_0(t)}{t^\alpha} \qquad (0 < t \le 1). \tag{3.4}$$

Hence, the function

$$f_0(t) := \frac{\varphi_0(t)}{t^\alpha} \qquad (0 < t \le 1) \tag{3.5}$$

has the property: $f_0(\frac{t}{p}) = f_0(t)$ so that it can be extended for all $t > 0$ by

$$f_0(pt) = f_0(t) \tag{3.6}$$

where $f_0(t)$ is continuous for $t > 0$.

Proposition 3.1 *If (1.6) is satisfies then for $N \in \mathbb{N}$ the sum $S_0(N)$ from (3.1) can be represented as*

$$S_0(N) = N^\alpha f_0(N) \tag{3.7}$$

with α from (3.3) and the continuous function f_0 from (3.5) and (3.6).

Proof: Because of (1.6) equation (3.2) has a continuous solution φ_0 satisfying (1.10). For $N \le p^\ell$ we have by (2.8) the formula

$$S_0(N) = C^\ell \varphi_0\left(\frac{N}{p^\ell}\right). \tag{3.8}$$

For arbitrary N we choose ℓ so large that $p^\ell > N$. In view of (3.8), (3.3) and (3.5) we have

$$S_0(N) = C^\ell \varphi_0\left(\frac{N}{p^\ell}\right) = N^\alpha \left(\frac{p^\ell}{N}\right)^\alpha \varphi_0\left(\frac{N}{p^\ell}\right) = N^\alpha f_0\left(\frac{N}{p^\ell}\right).$$

Owing to (3.6) it follows (3.7). □

According to (3.6) the function

$$F_0(u) := f_0(p^u) \qquad (u \in \mathbb{R}) \tag{3.9}$$

has the period 1 and in virtue of (3.5) we have by Proposition 3.1 :

Theorem 3.2 *If (1.6) is satisfies then for $N \in \mathbb{N}$ the sum $S_0(N)$ from (3.1) can be represented as*

$$S_0(N) = N^\alpha F_0(\log_p N) \tag{3.10}$$

with α from (3.3) and an -1 periodic continuous function F_0 which is given by

$$F_0(u) = \frac{\varphi_0(p^u)}{p^{\alpha u}} = C^{-u}\varphi_0(p^u) \qquad (u \leq 0) \tag{3.11}$$

where φ_0 is the solution of (3.2) satisfying (1.10).

Remark 3.3 Note that from (3.10) and (3.11) for $N = p^k$ we get in view of $F_0(k) = F_0(0) = 1$ that

$$S_0(p^k) = \sum_{n=0}^{p^k-1} C_n = p^{k\alpha} = C^k \tag{3.12}$$

with C from (1.1).

Remark 3.4 In the case $C_r = 1$ for all $r = 0, 1, \dots, p-1$ we have $C = p$ and $\alpha = 1$. Equation (3.2) has the trivial solution $\varphi_0(t) = t$ for $0 \leq t \leq 1$, $f_0(t) = 1$ for $t > 0$, $F_0(u) = 1$ for all $u \in \mathbb{R}$ and we get $S_0(N) = N$ for the sum (3.1).

In the following we exclude the trivial case $C_n = 1$ for all n.

Proposition 3.5 *If (1.6) is satisfies then the 1-periodic continuous function $F_0(u)$ from (3.11) has the following properties:*

1. F_0 is Hölder continuous.
2. *If $pM_0 < C$ where $M_0 = |C_0 C_1 \cdots C_{p-1}|^{1/p}$ then F_0 is differentiable almost everywhere and if $pM_0 \geq C$ then it is almost nowhere differentiable.*
3. *F_0 has finite total variation on $[0, 1]$ if and only if $C_r \geq 0$ for $r = 0, 1, \dots, p-1$. In this case we have*

$$\bigvee_0^1 (F_0) \leq 2C - 2. \tag{3.13}$$

Proof: It is known that in case $|c_r| < 1$ the solution $\varphi = \varphi_0$ of (1.8) with $\lambda = 1$ is Hölder continuous, cf. [11, Theorem 3.6]. This implies in view of $c_r = \frac{C_r}{C}$ with C from (1.1), (3.5) and (3.9) the first property of F_0. Analogously, the second property is a consequence of [11, Theorem 4.12].

In order to prove the third property first we consider the case $C_r \geq 0$ where the solution $\varphi = \varphi_0$ of (3.2) is increasing, cf. [11, Proposition 5.1]. We show that for f_0 from (3.5) it holds

$$\bigvee_{1/p}^1 (f_0) \leq 2C - 2. \tag{3.14}$$

Let $\frac{1}{p} = t_0 < t_1 < \ldots < t_n = 1$ be some decomposition of $[\frac{1}{p}, 1]$. Because of the identity

$$2(aA - bB) = (a + b)(A - B) + (A + B)(a - b) \tag{3.15}$$

it holds

$$2|aA - bB| \leq |a + b||A - B| + |A + B||a - b|.$$

Using this inequality with $a = \frac{1}{t_i^\alpha}$, $b = \frac{1}{t_{i+1}^\alpha}$, $A = \varphi_0(t_i)$ and $B = \varphi_0(t_{i+1})$ we have in view of $\max|\varphi_0(t)| = \varphi_0(1) = 1$ and (3.5)

$$\begin{aligned} 2|f_0(t_i) - f_0(t_{i+1})| &\leq \left|\frac{1}{t_i^\alpha} + \frac{1}{t_{i+1}^\alpha}\right| |\varphi_0(t_i) - \varphi_0(t_{i+1})| + 2\left|\frac{1}{t_i^\alpha} - \frac{1}{t_{i+1}^\alpha}\right| \\ &\leq 2\max\{p^\alpha, 1\}|\varphi_0(t_i) - \varphi_0(t_{i+1})| + 2\left|\frac{1}{t_i^\alpha} - \frac{1}{t_{i+1}^\alpha}\right|. \end{aligned}$$

Since $p^\alpha = C > 1$ and $\varphi_0(.)$ is increasing we get by summation

$$\bigvee_{1/p}^{1}(f_0) \leq C\left(\varphi_0(1) - \varphi_0\left(\frac{1}{p}\right)\right) + |p^\alpha - 1| = C\left(1 - \frac{1}{C}\right) + (C - 1)$$

where we have used $\varphi_0(1) = 1$, $\varphi_0(\frac{1}{p}) = \frac{1}{p^\alpha} = \frac{1}{C}$, cf. (3.4) with $t = 1$, and (3.3). So we have proved (3.14) which implies (3.13) in virtue of (3.9).

Now we consider the case that $C_r \geq 0$ is not true for all $r = 0, 1, \ldots, p - 1$. Then by [11, Proposition 2.6] the solution $\varphi = \varphi_0$ of (3.2) does not have finite total variation on $[0, 1]$. According to (2.7) this is true also for the subinterval $[\frac{k}{p}, \frac{k+1}{p}]$ if $C_k \neq 0$. This implies

$$\bigvee_{1/p}^{1}(\varphi_0) = \infty \tag{3.16}$$

since in view of (1.6) it is impossible that $C_r = 0$ for all $r = 1, 2, \ldots, p - 1$.

From (3.15) we get

$$2|aA - bB| \geq |a + b||A - B| - |A + B||a - b|$$

and with the same notations as before

$$2|f_0(t_i) - f_0(t_{i+1})| \geq 2\min\{p^\alpha, 1\}|\varphi_0(t_i) - \varphi_0(t_{i+1})| - 2M\left|\frac{1}{t_i^\alpha} - \frac{1}{t_{i+1}^\alpha}\right|$$

where $M = \max\{|\varphi_0(t)|\}$ for $\frac{1}{p} \leq t \leq 1$. In view of $p^\alpha > 1$ it follows

$$\sum_{i=0}^{n-1} |f_0(t_i) - f_0(t_{i+1})| \geq \sum_{i=0}^{n-1} |\varphi_0(t_i) - \varphi_0(t_{i+1})| - M(p^\alpha - 1)$$

which implies

$$\bigvee_{1/p}^{1}(f_0) = \infty$$

according to (3.16). Finally, (3.9) yields that F_0 does not have finite total variation on [0,1]. □

Remark 3.6 Note that according to (2.7) the solution φ_0 is constant on $[\frac{k}{p}, \frac{k+1}{p}]$ if $C_k = 0$ for some $k \leq p-1$. We remark that the suppositions of Proposition 2.6 in [11] are to add by $c_j \neq 0$ for all $j = 0, 1, \ldots, p-1$.

Proposition 3.7 *If (1.6) is satisfies then for the sums $S_0(N)$ from (3.1) we have*

$$\frac{1}{p^n}\sum_{N=1}^{p^n}\frac{1}{N^\alpha}S_0(N) \to c \qquad (n\to\infty) \tag{3.17}$$

where

$$c = \int_{1/p}^{1} f_0(t)dt \tag{3.18}$$

with f_0 from (3.5) and (3.6).

Proof: The sum in (3.17) can be written as

$$\frac{1}{p^n}\sum_{N=1}^{p^n}\frac{1}{N^\alpha}S_0(N) = \sum_{m=0}^{n} t_{n,m}A_m \tag{3.19}$$

with

$$t_{n,0} := \frac{1}{p^n}, \qquad t_{n,m} := \frac{p^m - p^{m-1}}{p^n} \qquad (1 \leq m \leq n)$$

and

$$A_0 := 1, \qquad A_m := \frac{1}{p^m - p^{m-1}}\sum_{N=p^{m-1}+1}^{p^m}\frac{1}{N^\alpha}S_0(N) \qquad (1\leq m\leq n).$$

For the numbers $t_{n,m}$ we have $t_{n,m} > 0$, $t_{n,0}+t_{n,1}+\cdots+t_{n,n} = 1$ and $t_{n,m} \to 0$ as $n\to\infty$ for fixed m, so that by a known Toeplitz theorem the sum (3.19) converges to c from (3.18) if

$$A_m \to \int_{1/p}^{1} f_0(t)dt \qquad (m\to\infty). \tag{3.20}$$

According to (3.7) with the continuous function f_0 from (3.5) and (3.6) we have for $m \geq 1$

$$\begin{aligned} A_m &= \frac{1}{p^m - p^{m-1}}\sum_{N=p^{m-1}+1}^{p^m} f_0(N) \\ &= \frac{1}{p^m - p^{m-1}}\sum_{N=p^{m-1}+1}^{p^m} f_0\left(\frac{N}{p^m}\right) \end{aligned}$$

where we have used (3.6). With the substitution $k = N - p^{m-1}$ we get

$$A_m = \frac{1}{p^m - p^{m-1}} \sum_{k=1}^{p^m - p^{m-1}} f_0\left(\frac{1}{p} + \frac{k}{p^m}\right)$$

and in view of the continuity of f_0 it follows (3.20). □

Example 3.8 (*Digital exponential sums*) We consider the sequence $C_n = q^{s(n)}$ with $q > 0$, where $s(n)$ denotes the number of ones in the binary representation of n. This sequence satisfies relation (1.2) with $p = 2$, $C_0 = 1$ and $C_1 = q$. The corresponding two-scale difference equation (3.2) reads

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-a)\varphi(t-1) \qquad (t \in \mathbb{R}) \tag{3.21}$$

with $a = \frac{1}{1+q}$ and the solution $\varphi = \varphi_0$ satisfying (1.10) which clearly depend on the parameter a. (cf. de Rham's function [10]). By Theorem 3.2 we have for the sum

$$S_0(N) = \sum_{n=0}^{N-1} q^{s(n)} \tag{3.22}$$

the exact formula

$$S_0(N) = N^\alpha F_0(\log_2 N)$$

where $\alpha = \log_2(1+q)$ and where $F_0(u)$ is a continuous, 1-periodic function which is connected with de Rham's function φ_0, i.e. the solution of (3.21), by

$$F_0(u) = a^u \varphi_0(2^u) \qquad (u \le 0),$$

cf. also [10, Theorem 2.1]. Let us mention that in case $q = 2$ the sum (3.22) is equal to the number of odd binomial coefficients in the first N rows of Pascal's triangle and that the sum (3.22) was already investigated by many authors, cf. e.g. [12], [6], [10].

Example 3.9 (*Cantor's function*) We consider the sequence C_n where $C_n = 0$ if the triadic representation of n contains the digit 1, elsewhere $C_n = 0$. This sequence satisfies relation (1.2) with $p = 3$, $C_0 = 1$, $C_1 = 0$ and $C_2 = 1$. Note that for the generating function (2.4) we have

$$G(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{k=0}^{\infty} z^{\gamma_k} = 1 + z^2 + z^6 + z^8 + z^{18} + z^{20} + z^{24} + z^{26} + \cdots$$

with strictly increasing exponents $\gamma_0 = 0, \gamma_1 = 2, \gamma_2 = 6, \gamma_3 = 8$ and so on, where it holds with $\varepsilon_\mu \in \{0, 1\}$:

$$n = \sum_{\mu=0}^{m} \varepsilon_\mu 2^\mu \qquad \Longrightarrow \qquad \gamma_n = 2 \sum_{\mu=0}^{m} \varepsilon_\mu 3^\mu, \tag{3.23}$$

cf. [11, Formula (5.9)]. For the sum (3.1) it follows

$$S_0(N) = \sum_{n=0}^{N-1} C_n = k+1 \qquad \text{for} \quad \gamma_k + 1 \le N < \gamma_{k+1}. \tag{3.24}$$

By means of Theorem 3.2 this sum can also represented by means of Cantor's function. Cantor's function is the solution φ_0 of (3.2) restricted to [0,1] with $p=3$, $C_0=1$, $C_1=0$, $C_2=1$ and $C=2$, i.e. $\varphi=\varphi_0$ is solution of

$$\varphi\left(\frac{t}{3}\right) = \frac{1}{2}\varphi(t) + \frac{1}{2}\varphi(t-2) \qquad (t \in \mathbb{R})$$

satisfying (1.10), cf. [9, Section 5], [11, Example 5.6]. By Theorem 3.2 the sum (3.24) can be expressed as follows:

$$S_0(N) = N^\alpha F_0(\log_3 N) \tag{3.25}$$

where $\alpha = \log_3 2$ and where F_0 is a continuous periodic function with period 1 which is given by

$$F_0(u) = \frac{1}{2^u}\varphi_0(2^u) \qquad (u \le 0) \tag{3.26}$$

with Cantor's function φ_0.

It is remarkable that the intervals $J_{m,n}$, where Cantor's function φ_0 is constant, have the form

$$J_{m,n} = \left(\frac{\gamma_{m-1}+1}{3^n}, \frac{\gamma_m}{3^n}\right) \qquad (n = 1,2,3,\ldots, \quad m = 1,2,\ldots,2^n)$$

with $\varphi_0(t) = \frac{m}{2^n}$ for $t \in J_{m,n}$, cf. [11, Formula (5.11)]. Let us mention that in [6, Section 5] it was considered a sequence $h(n)$, defined by

$$h\left(\sum_i 2^{e_i}\right) = \sum_i 3^{e_i}$$

with strictly increasing exponents e_i, and in virtue of (3.23) we see that $h(n) = \frac{1}{2}\gamma_n$. In [6] it was mentioned that $h(1) < h(2) < \cdots < h(n)$ is the "minimal" sequence of n positive integers not containing an arithmetic progression. By means of the Mellin transformation it was shown [6, Theorem 5.1]:

$$H(N) := \sum_{n<N} h(n) = N^{\rho+1}F(\log_2 N) - \frac{1}{4}N$$

where $\rho = \log_2 3$ and where $F(u)$ is an 1-periodic function which has the Fourier series

$$F(u) = \frac{1}{3\log 2}\sum_{k\in\mathbb{Z}} \zeta(\rho+\chi_k)\frac{e^{2\pi iku}}{(\rho+\chi_k)(\rho+\chi_k+1)}$$

with $\chi_k = 2\pi ik/\log 2$ and Riemann's zeta function $\zeta(.)$.

4 Power sums related to digital sequences

Now we investigate the sum

$$S_m(N) = \sum_{n=0}^{N-1} C_n n^m \tag{4.1}$$

with $m \in \mathbb{N}_0$, where C_n is an arbitrary sequence with $C_0 = 1$ and (1.1) satisfying (1.2). For this we consider the two-scale difference equations (1.8) with $\lambda = p^n$ $(n \in \mathbb{N}_0)$ and $c_r = \frac{C_r}{C}$ with C from (1.1). By Theorem 2.1 we have for the solutions $\varphi(t) = \varphi_n(t)$ that

$$\varphi_n\left(\frac{t}{p}\right) = \frac{1}{Cp^n}\varphi_n(t) \qquad (0 \le t \le 1)$$

since $\varphi_n(t) = 0$ for $t < 0$. Choosing α_n so that $p^{\alpha_n} = Cp^n$ i.e.

$$\alpha_n = n + \log_p C \tag{4.2}$$

then

$$\frac{\varphi_n(\frac{t}{p})}{(\frac{t}{p})^{\alpha_n}} = \frac{\varphi_n(t)}{t^{\alpha_n}} \qquad (0 < t \le 1).$$

Hence, the functions

$$f_n(t) := \frac{\varphi_n(t)}{t^{\alpha_n}} \qquad (0 < t \le 1) \tag{4.3}$$

have the property $f_n(\frac{t}{p}) = f_n(t)$ so that they can be extended for all $t > 0$ by

$$f_n(pt) = f_n(t) \qquad (t > 0). \tag{4.4}$$

Theorem 4.1 *If (1.6) is satisfies then for $N \in \mathbb{N}$ the sum $S_m(N)$ from (4.1) can be represented as*

$$S_m(N) = N^\alpha \sum_{\mu=0}^{m} N^\mu F_{m,\mu}(\log_p N) \tag{4.5}$$

where $\alpha = \log_p C$ and where $F_{m,\mu}(u)$ are 1-periodic continuous functions which have the representations

$$F_{m,\mu}(u) = (-1)^m m!\, b_{m-\mu} \sum_{\nu=0}^{\mu} \frac{(-1)^\nu}{\nu!} f_{\mu-\nu}(p^u) \tag{4.6}$$

with the coefficients b_n from (1.19) and $f_n(.)$ from (4.3) and (4.4).

Proof: For given $N \in \mathbb{N}$ we choose ℓ such that $p^\ell \ge N$. From (2.6) with $n = m$ and $k = N$ we get

$$\varphi_m\left(\frac{N}{p^\ell}\right) = \frac{c_0^\ell}{p^{m\ell}} \sum_{j=1}^{N} C_{N-j} p_m(j)$$

where φ_m is the continuous solution of (1.8) with $\lambda = p^m$ satisfying (1.10). With $j = N - n$ it follows in view of $c_0 = \frac{1}{C}$ and $p^m C = p^{\alpha_m}$

$$\sum_{n=0}^{N-1} C_n p_m(N-n) = \frac{p^{m\ell}}{c_0^\ell}\varphi_m\left(\frac{N}{p^\ell}\right) = N^{\alpha_m}\left(\frac{p^\ell}{N}\right)^{\alpha_m}\varphi_m\left(\frac{N}{p^\ell}\right).$$

In virtue of (4.2) and (4.4) it follows

$$\sum_{n=0}^{N-1} C_n p_m(N-n) = N^{\alpha+m} f_m(N)\ . \tag{4.7}$$

Next we write $p_m(N-n)$ as polynomial with respect to n. By Taylor's formula

$$p_m(N-n) = \sum_{\mu=0}^{m} p_m^{(\mu)}(N)\frac{(-n)^\mu}{\mu!} = \sum_{\mu=0}^{m}\frac{(-1)^\mu}{\mu!}p_{m-\mu}(N)n^\nu$$

where we have used that $p_m(t)$ are Appell polynomials. It follows

$$\sum_{n=0}^{N-1} C_n p_m(N-n) = \sum_{\mu=0}^{m}\frac{(-1)^\mu}{\mu!}p_{m-\mu}(N)\sum_{n=0}^{N-1} C_n n^\mu$$

and comparison with (4.7) yields in view of (4.1) that

$$N^{\alpha+m} f_m(N) = \sum_{\mu=0}^{m}\frac{(-1)^\mu}{\mu!}p_{m-\mu}(N)S_\mu(N).$$

Multiplication by z^m and summation over m we get in view of the Cauchy product and (1.15)

$$\begin{aligned}\sum_{m=0}^{\infty} N^{\alpha+m} f_m(N) z^m &= \sum_{n=0}^{\infty} p_n(N) z^n \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}S_m(N) z^m \\ &= e^{Nz}\Phi(z)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}S_m(N) z^m.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}S_m(N) z^m &= \frac{e^{-Nz}}{\Phi(z)}\sum_{m=0}^{\infty} N^{\alpha+m} f_m(N) z^m \\ &= \sum_{n=0}^{\infty} q_n(-N) z^n \sum_{m=0}^{\infty} N^{\alpha+m} f_m(N) z^m\end{aligned}$$

where we have used (1.22) with $t = -N$. Comparison of coefficients implies in view of the Cauchy product

$$\frac{(-1)^m}{m!}S_m(N) = \sum_{n=0}^{m} q_{m-n}(-N) N^{\alpha+n} f_n(N).$$

Moreover, for the Appell polynomials $q_n(t)$ we have by (1.21) the representation

$$q_{m-n}(-N) = \sum_{k=0}^{m-n} \frac{b_{m-n-k}}{k!}(-1)^k N^k$$

so that with the substitution $\mu = n + k$ we get

$$\begin{aligned}
\frac{(-1)^m}{m!} S_m(N) &= N^\alpha \sum_{n=0}^{m} \sum_{k=0}^{m-n} \frac{b_{m-n-k}}{k!}(-1)^k N^{n+k} f_n(N) \\
&= N^\alpha \sum_{\mu=0}^{m} \sum_{n=0}^{\mu} (-1)^{\mu-n} \frac{b_{m-\mu}}{(\mu-n)!} N^\mu f_n(N) \\
&= N^\alpha \sum_{\mu=0}^{m} b_{m-\mu} N^\mu \sum_{n=0}^{\mu} (-1)^{\mu-n} \frac{1}{(\mu-n)!} f_n(N)
\end{aligned}$$

and it follows (4.5) with (4.6). □

Remark 4.2 In the simple case $C_n = 1$ for all $n \in \mathbb{N}_0$ the sum (4.1) is the usual power sum. In this case equation (1.8) with $\lambda = 1$ has the solution $\varphi_0(t) = t$ for $0 \leq t \leq 1$ so that the iterated integrals are $\varphi_n(t) = \frac{1}{(n+1)!} t^{n+1}$ in $[0, 1]$. From (4.2) we get $\alpha_n = n + 1$ so that $f_n(t) = \frac{1}{(n+1)!}$ for all $t > 0$. Hence, the functions $F_{m,\mu}$ from (4.6) are constant and it easy to see that (4.5) yields the known representation (1.4) with the Bernoulli polynomials.

In the following we again exclude the trivial case $C_n = 1$ for all n.

Proposition 4.3 *If (1.6) is satisfies then the 1-periodic continuous functions $F_{m,\mu}(u)$ from (4.6) have the following properties:*

1. *Each of the functions $F_{m,\mu}$ is Hölder continuous.*
2. *If $pM_0 < C$ where $M_0 = |C_0 C_1 \cdots C_{p-1}|^{1/p}$ then each $F_{m,\mu}$ is differentiable almost everywhere and if $pM_0 \geq C$ then each $F_{m,\mu}$ is almost nowhere differentiable.*
3. *Each of the functions $F_{m,\mu}$ has finite total variation on $[0, 1]$ if and only if $C_r \geq 0$ for all $r = 0, 1, \ldots, p - 1$.*

Proof: Owing to (4.6) and (4.3) we see in view of the fact that φ_n are the iterated integrals of φ_0, that the analytic properties as differentiability of $F_{m,\mu}$ are determined by the function f_0. So the assertions are consequences of Proposition 3.5. □

5 Specific power sums

We consider the sum (4.1) for $N = p^k$, i.e.

$$S_m(p^k) = \sum_{n=0}^{p^k-1} C_n n^m. \tag{5.1}$$

In order to get a simple formula for this sum we need the following lemma.

Lemma 5.1 *For the* 1*-periodic function* $F_{m,\mu}(.)$ *from* (4.6) *we have*

$$F_{m,\mu}(0) = (-1)^m m!\, b_{m-\mu} a_\mu \tag{5.2}$$

with the coefficients a_n *from* (1.16) *and* b_n *from* (1.19).

Proof: From (4.6) with $u = 0$ we get

$$F_{m,\mu}(0) = (-1)^m m!\, b_{m-\mu} d_\mu$$

with

$$d_\mu = \sum_{\nu=0}^{\mu} \frac{(-1)^\nu}{\nu!} f_{\mu-\nu}(1). \tag{5.3}$$

Multiplication by t^μ and summation over μ yields in view of the Cauchy product

$$\begin{aligned} \sum_{\mu=0}^{\infty} d_\mu z^\mu &= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} z^\nu \sum_{\mu=0}^{\infty} f_\mu(1) z^\mu \\ &= e^{-z} \sum_{\mu=0}^{\infty} f_\mu(1) z^\mu. \end{aligned}$$

Further, by (4.3) we have $f_n(1) = \varphi_n(1)$ and by (1.14) also $\varphi_n(1) = p_n(1)$. Hence, in view of (1.15) with $t = 1$ we get

$$\sum_{\mu=0}^{\infty} f_\mu(1) z^\mu = \sum_{n=0}^{\infty} p_n(1) z^n = e^z \Phi(z).$$

It follows

$$\sum_{\mu=0}^{\infty} d_\mu z^\mu = \Phi(z)$$

so that $d_\mu = a_\mu$ according to (1.16). □

Theorem 4.1 and Lemma 5.1 imply

Proposition 5.2 *The sum* (4.1) *for* $N = p^k$ *with* $k \in \mathbb{N}$ *reads*

$$S_m(p^k) = (-1)^m m! p^{\alpha k} \sum_{\mu=0}^{m} p^{\mu k} a_\mu b_{m-\mu} \tag{5.4}$$

where $\alpha = \log_p C$ *with* C *from* (1.1), a_n *from* (1.16) *and* b_n *from* (1.19).

Remark 5.3 Formula (5.4) for $m = 0$ yields

$$S_0(p^k) = \sum_{n=0}^{p^k-1} C_n = p^{\alpha k} = C^k$$

in accordance with (3.12).

If we introduce the polynomials

$$P_m(t) := \sum_{\mu=0}^{m} t^\mu a_\mu b_{m-\mu} \tag{5.5}$$

then in virtue of (5.4) we have

$$S_m(p^k) = (-1)^m m! p^{\alpha k} P_m(p^k). \tag{5.6}$$

Lemma 5.4 *The polynomials $P_m(t)$ have the generating function*

$$\sum_{m=0}^{\infty} P_m(t) z^m = \frac{\Phi(tz)}{\Phi(z)}. \tag{5.7}$$

with Φ from (1.16), cf. also (1.12).

Proof: By multiplication of the power series (1.16) with tz in place of z and (1.19) we get by means of the Cauchy product

$$\begin{aligned}\Phi(tz)\frac{1}{\Phi(z)} &= \sum_{n=0}^{\infty} a_n (tz)^n \sum_{n=0}^{\infty} b_n z^n \\ &= \sum_{m=0}^{\infty} \left(\sum_{\mu=0}^{m} t^\mu a_\mu b_{m-\mu} \right) z^m\end{aligned}$$

and in view of (5.5) it follows (5.7). $\square$

Proposition 5.5 *The polynomials P_m from (5.5) satisfy the relation*

$$P_m(st) = \sum_{\mu=0}^{m} s^\mu P_\mu(t) P_{m-\mu}(s). \tag{5.8}$$

Proof: By repeated application of (5.7) we get

$$\begin{aligned}\sum_{m=0}^{\infty} P_m(st) z^m &= \frac{\Phi(stz)}{\Phi(z)} = \frac{\Phi(stz)}{\Phi(sz)} \cdot \frac{\Phi(sz)}{\Phi(z)} \\ &= \sum_{m=0}^{\infty} P_m(t)(sz)^m \sum_{m=0}^{\infty} P_m(s) z^m \\ &= \sum_{m=0}^{\infty} \left(\sum_{\mu=0}^{m} s^\mu P_\mu(t) P_{m-\mu}(s) \right) z^m\end{aligned}$$

where we have used the Cauchy product. Comparison of coefficients yields (5.8). $\square$

Proposition 5.6 *For positive integers k, ℓ the sums (5.1) satisfy the relation*

$$S_m(p^{k+\ell}) = \sum_{\mu=0}^{m} \binom{m}{\mu} p^{k\mu} S_\mu(p^\ell) S_{m-\mu}(p^k). \tag{5.9}$$

Proof: Applying (5.6) and (5.8) we get

$$\begin{aligned}
\frac{(-1)^m}{m!} S_m(p^{k+\ell}) &= p^{\alpha(k+\ell)} P_m(p^{k+\ell}) \\
&= p^{\alpha(k+\ell)} \sum_{\mu=0}^{m} p^{k\mu} P_\mu(p^\ell) P_{m-\mu}(p^k) \\
&= \sum_{\mu=0}^{m} p^{\mu} \frac{(-1)^\mu}{\mu!} S_\mu(p^\ell) \frac{(-1)^{m-\mu}}{(m-\mu)!} S_{m-\mu}(p^k)
\end{aligned}$$

which implies (5.9). □

Remark 5.7 Let us mention that in the simple case $C_n = 1$ for all n the polynomials (5.5) can be represented as

$$P_m(t) = \frac{(-1)^m}{m!} \cdot \frac{1}{t} \tilde{B}_m(t) \tag{5.10}$$

with the polynomials $\tilde{B}_m(t)$ from (1.5) which as is known have the generating function

$$\frac{e^{tz} - 1}{e^z - 1} = \sum_{m=0}^{\infty} \frac{\tilde{B}_m(t)}{m!} z^m \qquad (|z| < 2\pi). \tag{5.11}$$

In order to see (5.10) we note that in case $C_n = 1$ for all n the polynomial (1.13) has the form

$$P(w) = \frac{1}{p}(1 + w + \cdots + w^{p-1}) = \frac{1 - w^p}{p(1-w)}$$

so that for Φ from (1.12) we obtain

$$\Phi(z) = \frac{1 - e^{-z}}{z}.$$

Therefore

$$\frac{\Phi(-tz)}{\Phi(-z)} = \frac{e^{tz} - 1}{t(e^z - 1)}$$

and in virtue of (5.11) and Lemma 5.4 it follows (5.10).

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