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Ordered and non-ordered non-congruent convex quadrilaterals inscribed in a regular n-gon

ABSTRACT. Using several arguments, some authors showed that the number of non-congruent triangles inscribed in a regular n-gon equals $\{n^2/12\}$, where $\{x\}$ is the nearest integer to x. In this paper, we revisit the same problem, but study the number of ordered and non-ordered non-congruent convex quadrilaterals, for which we give simple closed formulas using Partition Theory. The paper is complemented by a study of two further kinds of quadrilaterals called proper and improper non-congruent convex quadrilaterals, which allows to give a formula that connects the number of triangles and ordered quadrilaterals. This formula can be considered as a new combinatorial interpretation of a certain identity in Partition Theory.

KEY WORDS. Congruent triangles; Congruent quadrilaterals; Ordered quadrilaterals; proper quadrilaterals; Integer partitions.

1 Introduction

In 1938, Anning proposed the following problem [6]: "*From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. How many essentially different possible triangles are there ?*". For any given positive integer $n \geq 3$, let $\Delta(n)$ denote the number of such triangles.

Using a geometric argument, Frame showed that $\Delta(n) = \{n^2/12\}$, where $\{x\}$ is the nearest integer to x. After that, other solutions were proposed by some authors, such as Auluck [2].

In 1978, Reis posed the following natural general problem: *From the vertices of a regular n-gon k are chosen to be the vertices of a k-gon. How many incongruent convex k-gons are there ?*

Let us first specify that two k-gons are called congruent if one k-gon can be moved to the other by rotation or reflection.

For any given positive integers $2 \leq k \leq n$, let $R(n,k)$ denotes the number of such k-gons. In 1979 Gupta [5] gave the solution of Reis's problem, using the Möbius inversion formula.

Theorem 1

$$R(n,k) = \frac{1}{2}\binom{\left\lfloor \frac{n-h_k}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{2k}\sum_{d/\gcd(n,k)} \varphi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1},$$

where $h_k \equiv k \ (mod\ 2)$ *and* $\varphi(n)$ *is the Euler function.*

One can find the first values of $R(n,k)$ in the Online Encyclopedia of Integer Sequences (OEIS) [7] as A004526 for $k=2$, A001399 for $k=3$, A005232 for $k=4$ and A032279 for $k=5$.

The immediate consequence of both Gupta's and Frame's Theorems is the following identity:

$$\left\{\frac{n^2}{12}\right\} = \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor + \frac{1}{6}\binom{n-1}{2} + \frac{\chi(3/n)}{3},$$

where $\chi(3/n) = 1$ if $n \equiv 0\,(mod\ 3)$, 0 otherwise.

In 2004, Shevelev gave a short proof of Theorem 1, using a bijection between the set of convex polygons with the tops in the n-gon splitting points and the set of all (0,1)-configurations with the elements in these points [8].

The aim of this paper is to enumerate two kinds of non-congruent convex quadrilaterals, inscribed in a regular n-gon, the ordered ones which have the sequence of their sides's sizes ordered, denoted by $R_O(n,4)$ and those which are non-ordered denoted by $R_{\overline{O}}(n,4)$, using the Partition Theory. As an example, let us consider Figure 1 showing three quadrilaterals inscribed in a regular 12-gon, the first is not convex, the second is ordered while the third is not. Observe that the second quadrilateral generates 1+1+3+3 as a partition of 8 into four parts, that is why it is called ordered.

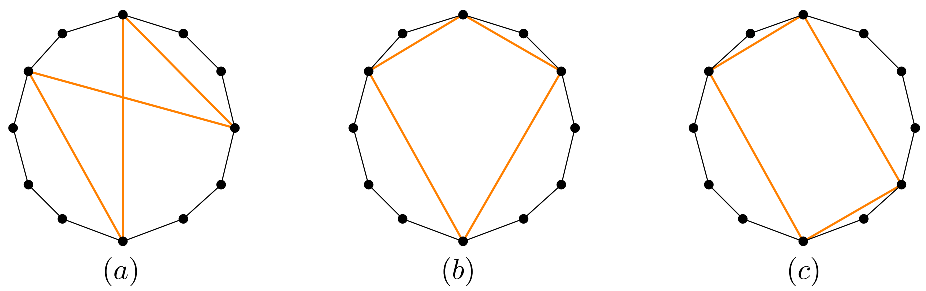

Figure 1

2 Notations and preliminaries

We denote by G_n a regular n-gon and by $\mathbb{N}$ the set of nonnegative integers. The partition of $n \in \mathbb{N}$ into k parts is a tuple $\pi = (\pi_1, \ldots, \pi_k) \in \mathbb{N}^k, k \in \mathbb{N}$, such that

$$n = \pi_1 + \cdots + \pi_k, \ 1 \le \pi_1 \le \cdots \le \pi_k,$$

where the nonnegative integers π_i are called parts. We denote the number of partitions of n into k parts by $p(n,k)$, the number of partitions of n into parts less than or equal to k by $P(n,k)$ and by $q(n,k)$ we denote the number of partitions of n into k distinct parts. We sometimes write a partition of n into k parts $\pi = (\pi_1^{f_1}, \ldots, \pi_s^{f_s})$, where $\sum_{i=1}^{s} f_i = k$, the value of f_i is termed as frequency of the part π_i. For $m \in \mathbb{N}, m \le k$, we denote the number of partitions of n into k parts $\pi = (\pi_1^{f_1}, \ldots, \pi_s^{f_s})$ for which $1 \le f_i \le m$ and $f_j = m$ for at least one $j \in \{1, \ldots, s\}$ by $c_m(n,k)$. For example $c_2(12,4) = 10$, since such partitions are exactly 1128, 1137, 1146, 1155, 1227, 1335, 1344, 2235, 2244, 2334. Let $\delta(n) \equiv n \ (mod\ 2)$, so that $\delta(n) = 1$ or 0, $\lfloor x \rfloor$ the integer part of x and finally $\{x\}$ the nearest integer to x.

3 Main results

In this section, we give the explicit formulas of $R_O\,(n,4)$ and $R_{\overline{O}}\,(n,4)$.

Theorem 2 *For $n \ge 4$,*

$$R_O\,(n,4) = \left\{ \frac{n^3}{144} + \frac{n^2}{48} - \frac{n\delta(n)}{16} \right\}.$$

Proof: First of all, notice that

$$R_O\,(n,4) = p(n,4). \tag{1}$$

Indeed, each ordered convex quadrilateral $ABCD$ inscribed in G_n can be viewed as a quadruple of integers (x,y,z,t), abbreviated for convenience as a word $xyzt$, such that:

$$\begin{cases} n - 4 = x + y + z + t; \\ 0 \le x \le y \le z \le t, \end{cases} \tag{2}$$

where x, y, z and t represent the number of vertices between A and B, B and C, C and D and finally between D and A, respectively. It should be noted, that the number of solutions of System (2) equals $p(n,4)$, by setting $x' = x+1$, $y' = y+1$, $z' = z+1$ and $t' = t+1$.

Now, let $g(z)$ be the known generating function of $p(n,4)$ [3]:

$$g\,(z) = \frac{z^4}{(1-z)\,(1-z^2)\,(1-z^3)\,(1-z^4)}.$$

From expanding $g(z)$ into partial fractions, we obtain

$$g(z)=\frac{1}{32\,(1+z)^2}-\frac{13}{288\,(1-z)^2}-\frac{1}{24\,(1-z)^3}+\frac{1}{24\,(1-z)^4}+\frac{1-z^2}{8\,(1-z^4)}-\frac{1-z}{9\,(1-z^3)}.$$

Via straightforward calculations, it can be proved that

$$g\left(z\right)=\sum_{n\geq 0}\left(\frac{(-1)^n\,(n+1)}{32}-\frac{13\,(n+1)}{288}-\frac{(n+1)\,(n+2)}{48}+\frac{\left(1+\frac{11}{6}n+n^2+\frac{1}{6}n^3\right)}{24}+\epsilon\left(n\right)\right)z^n,$$

where $\epsilon\left(n\right)\in\left\{-\frac{17}{72},-\frac{1}{8},-\frac{1}{9},-\frac{1}{72},0,\frac{1}{72},\frac{1}{9},\frac{1}{8},\frac{17}{72}\right\}$.

Thus, we have

$$g\left(z\right)=\sum_{n\geq 0}\left(\frac{n^3}{144}+\frac{n^2}{48}+\frac{\left((-1)^n-1\right)n}{32}+\beta\left(n\right)\right)z^n,$$

where

$\beta\left(n\right)\in\left\{-\frac{5}{16},-\frac{1}{4},-\frac{29}{144},-\frac{3}{16},-\frac{5}{36},-\frac{1}{8},-\frac{13}{144},-\frac{11}{144},-\frac{1}{16},-\frac{1}{36},-\frac{1}{72},0,\frac{5}{144},\frac{7}{144},\frac{1}{9},\frac{23}{144},\frac{2}{9},\frac{7}{72}\right\}$.

Since $p\left(n,4\right)$ is an integer and $\left|\beta\left(n\right)\right|<1/2$, we get

$$p\left(n,4\right)=\left\{\frac{n^3}{144}+\frac{n^2}{48}+\frac{\left((-1)^n-1\right)n}{32}\right\}. \tag{3}$$

Hence, the result follows. □

Remark 3 Andrews and Eriksson said that the method used in the proof above dates back to Cayley and MacMahon [1, p. 58]. Using the same method [1, p. 60], they proved the following formula for $P(n,4)$:

$$P\left(n,4\right)=\left\{\frac{(n+1)\left(n^2+23n+85\right)}{144}-\frac{(n+4)\left\lfloor\frac{n+1}{2}\right\rfloor}{8}\right\}.$$

Because $p(n,k)=P(n-k,k)$ (see for example [4]), it follows:

$$p\left(n,4\right)=\left\{\frac{n^3}{144}+\frac{n^2}{12}-\frac{n}{8}-\frac{n\left\lfloor\frac{n-1}{2}\right\rfloor}{8}\right\}. \tag{4}$$

Note that the formula (3) seems a little bit simpler than (4).

To give an explicit formula for $R_{\overline{O}}\left(n,4\right)$, we need the following lemma.

Lemma 4 *For $n\geq 4$,*

$$c_2\left(n,4\right)=p\left(n,4\right)-q\left(n,4\right)-\left\lfloor\frac{n-1}{3}\right\rfloor.$$

Proof: By definition of $c_m(n,k)$ in Section 2, it easily follows that

$$c_2(n,4) = p(n,4) - (q(n,4) + c_3(n,4) + \chi(4/n)),$$

where $\chi(4/n) = 1$ if $n \equiv 0 \ (mod\ 4)$, 0 otherwise.

Furthermore, $c_3(n,4)$ can be considered as the number of integer solutions of the equation

$$3x + y = n, \text{ with } 1 \leq y \neq x \geq 1.$$

Since $x \neq y$, the solution $x = y = n/4$, when 4 divides n, must be removed. Then, by taking $y = 1$, one can get $c_3(n,4) = \left\lfloor \frac{n-1}{3} \right\rfloor - \chi(4|n)$. This completes the proof. □

Now we can derive the following theorem.

Theorem 5 *For $n \geq 4$,*

$$R_{\overline{O}}(n,4) = \left\{ \frac{n^3}{144} + \frac{n^2}{48} - \frac{n\delta(n)}{16} \right\} + \left\{ \frac{(n-6)^3}{144} + \frac{(n-6)^2}{48} - \frac{(n-6)\delta(n)}{16} \right\} - \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Proof: First of all, notice that $q(n,k) = p(n - k(k-1)/2, k)$ [1]. Then from Theorem 2 we get

$$q(n,4) = p(n-6,4) = \left\{ \frac{(n-6)^3}{144} + \frac{(n-6)^2}{48} - \frac{(n-6)\delta(n)}{16} \right\}.$$

Therefore, it is enough to prove that

$$R_{\overline{O}}(n,4) = p(n,4) + q(n,4) - \left\lfloor \frac{n-1}{3} \right\rfloor. \tag{5}$$

In fact, each non-ordered convex quadrilateral may be obtained by permuting exactly two parts of some partition of n into four parts, which is associated from System (2) to a unique ordered convex quadrilateral. For example, in Figure 1 above, the ordered convex quadrilateral (b) assimilated to the solution 1133 of 8 or to the partition 2244 of 12, generates the non-ordered convex quadrilateral (c) via the permutation 1313. Obviously, not every partition of n can generate a non-ordered convex quadrilateral, those having three equal parts or four equal parts cannot. Also, each partition of n into four distinct parts $xyzt$ generates two non-ordered convex quadrilaterals, each one corresponds to one of the two following permutations $xytz$ and $xzyt$. On the other hand, each partition of n into two equal parts, like $xxyz$, with y and z both of them $\neq x$, generates only one non-ordered convex quadrilateral, corresponding to the unique permutation $xyxz$. Thus,

$$R_{\overline{O}}(n,4) = 2q(n,4) + c_2(n,4). \tag{6}$$

Hence, from Lemma 4 the assertion follows. □

Remark 6 By substituting $k = 4$ in Theorem 1, we get

$$R(n,4) = \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor}{2} + \frac{1}{8}\binom{n-1}{3} + \frac{n(1-\delta(n))}{16} + \alpha,$$

where

$$\alpha = \begin{cases} \frac{1}{8} & if\ n \equiv 0\ (mod\ 4), \\ -\frac{1}{8} & if\ n \equiv 2\ (mod\ 4), \\ 0 & otherwise. \end{cases}$$

Knowing furthermore that

$$R(n,4) = R_O(n,4) + R_{\overline{O}}(n,4),$$

the following identity follows according to Theorem 1 and Theorem 5:

$$\begin{aligned} \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor}{2} + \frac{1}{8}\binom{n-1}{3} + \frac{n(1-\delta(n))}{16} + \alpha &= 2\left\{\frac{n^3}{144} + \frac{n^2}{48} - \frac{n\delta(n)}{16}\right\} + \\ &+ \left\{\frac{(n-6)^3}{144} + \frac{(n-6)^2}{48} - \frac{(n-6)\delta(n)}{16}\right\} - \\ &- \left\lfloor \frac{n-1}{3} \right\rfloor. \end{aligned}$$

4 Connecting formula between $\Delta(n)$ and $R_O(n,4)$

There are two further kinds of quadrilaterals inscribed in G_n, the proper ones, those which do not use the sides of G_n and the improper ones, those using them. In Figure 2 below, two quadrilaterals inscribed in G_{12} are shown, the first one is proper while the second is not.

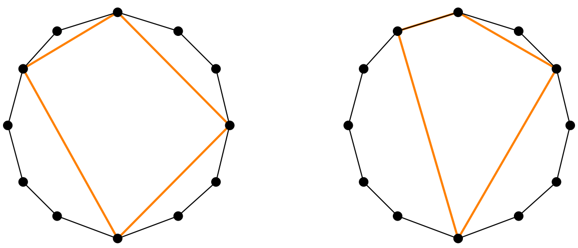

Figure 2

Let denote the number of these two kinds of quadrilaterals by $R_O^P(n,4)$ and $R_O^{\overline{P}}(n,4)$, respectively. The goal of this section is to prove the following theorem.

Theorem 7 *For $n \geq 4$,*

$$\Delta\,(n) = R_O(n+1,4) - R_O(n-3,4).$$

Proof: Note first that an improper ordered quadrilateral is formed by at least one side of G_n, hence the concatenation of the vertices of one of such sides gives a triangle inscribed in G_{n-1}, as shown in Figure 3.

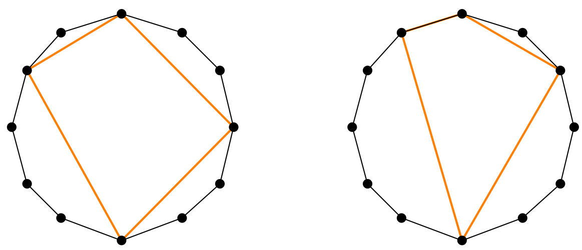

Figure 3

Thus we have

$$R_O^{\overline{P}}(n,4) = \Delta\,(n-1)\,.$$

On the other hand, it is obvious to see that

$$R_O^{P}(n,4) = p(n-4,4).$$

Then from (1), we get

$$R_O^{P}(n,4) = R_O(n-4,4).$$

Since

$$R_O(n,4) = R_O^{P}(n,4) + R_O^{\overline{P}}(n,4),$$

we obtain

$$R_O(n,4) = R_O(n-4,4) + \Delta\,(n-1)\,.$$

So, the theorem has been proved by substituting n by $n+1$. □

Remark 8 The well-known recurrence relation [4, p. 373],

$$p(n,k) = p(n+1,k+1) - p(n-k,k+1), \tag{7}$$

implies by setting $k=3$,

$$p(n,3) = p(n+1,4) - p(n-3,4). \tag{8}$$

Thus, as we can see, the formula of Theorem 7 can be considered as a combinatorial interpretation of identity (8).

For $k \leq n$, we have the following generalization, using the same arguments to prove Theorem 7.

Theorem 9 *For $n \geq k$,*

$$R_O(n,k) = R_O(n+1,k+1) - R_O(n-k,k+1).$$

The formula of Theorem 9 can be considered as a combinatorial interpretation of the recurrence formula (7).

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