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B-Nearness on Boolean frames

ABSTRACT. In this paper we define and study BN-proximity, B-nearness, B-farness, B-smallness and B-covering on Boolean frames and investigate relation between them. Also we define and investigate some properties of B-nearness on frames and subframes. Then we define complete nearness space by given B-near frame.

KEY WORDS. BN-Proximity, B-Nearness, B-Covering, B-Smallness, B-Farness, Contigual, Uniform, graded, Complete.

1 Introduction

Nearness on space introduced by Herrlich on 1974. Topics developed in this paper are based on the work of Banaschewski and Dube. We define and study BN-proximity, B-nearness, B-farness, B-smallness and B-covering on Boolean frames and we investigate relation between them.

Also we investigate some properties of B-nearness on frames and subframes. Then we have shown that by given B-Nearness frame we can define a complete nearness space.

2 Background

Definition 1 Let X be a set and let ξ be a subset of P^2X. Consider the following axioms:

$(N1)$ If $\mathcal{A} << \mathcal{B}$ and $\mathcal{B} \in \xi$ then $\mathcal{A} \in \xi$. Where $\mathcal{A} << \mathcal{B}$ iff $\forall A \in \mathcal{A} \exists B \in \mathcal{B}, A \supseteq B$;

$(N2)$ If $\bigcap \mathcal{A} \neq \emptyset$ then $\mathcal{A} \in \xi$;

$(N3)$ $\emptyset \neq \xi \neq P^2X$;

$(N4)$ If $(\mathcal{A} \bigvee \mathcal{B}) \in \xi$ then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$, where $\mathcal{A} \bigvee \mathcal{B} := \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$;

$(N5)$ If $\{cl_\xi A | A \in \mathcal{A}\} \in \xi$ then $\mathcal{A} \in \xi$, where $cl_\xi A := \{x \in X | \{A, \{x\}\} \in \xi\}$.

ξ is called nearness structure on X iff ξ satisfying to above conditions, and the pair (X, ξ) is called a nearness space -shortly a N-space- iff ξ is a nearness structure on X.

If (X, ξ) and (Y, η) are N-spaces then a function $f : X \longrightarrow Y$ is called a nearness preserving map from (X, ξ) to (Y, η) iff $\mathcal{A} \in \xi$ implies $f\mathcal{A} \in \eta$, where $f\mathcal{A} := \{f[A] : A \in \mathcal{A}\}$.

Let (X, ξ) be a N-space, then a subset $\mathcal{A}$ of PX is called a ξ-*cluster* iff $\mathcal{A}$ is maximal element of the set $\xi \setminus \{\emptyset\}$, ordered by inclusion. We call (X, ξ) complete iff every ξ-*cluster* contains an element $\{x\}$ for some $x \in X$.

(X^*, ξ^*) is a completion of (X, ξ) where $\xi^* = \{\Omega \subset PX^* | \cup \{\cap\omega | \omega \in \Omega\} \in \xi\}$ and X^* denotes the set of all ξ-*clusters*.

Definition 2 A *frame* is a complete lattice satisfying the special distribution law,

$(IFD1)$ $\forall\, a \in L, \forall\, S \subseteq L,\, a \wedge \bigvee S = \bigvee \{a \wedge x | x \in S\}$;

and it is called a Boolean frame if it is complementary.

Note, that in this case each element has unique complement, i.e.

$\forall\, a \in L\ \exists!\ a' \in L$ s.t. $a \wedge a' = 0$ and $a \vee a' = 1$

Therefore Boolean frame satisfying in

$(IFD2)$ $\forall\, a \in L, \forall\, S \subseteq L,\, a \vee \bigwedge S = \bigwedge \{a \vee x | x \in S\}$.

Frame homomorphisms between Boolean frames preserve top, bottom (denoted by 1 and 0 respectively) meets, joins and complementary.

3 B-Nearness and BN-proximity on frames

Definition 3 Let L be a Boolean frame and A, B are subsets of L.

$$\begin{aligned} secA &= \{x \in L | \forall a \in A, x \wedge a \neq 0\}\,; \\ stackA &= \{x \in L | \exists a \in A \text{ s.t. } a \leq x\}\,; \\ A \bigvee B &= \{a \vee b | a \in A, b \in B\}\,; \\ A \bigwedge B &= \{a \wedge b | a \in A, b \in B\}\,; \\ A' &= \{a' | a \in A\}\,; \\ st(x, A) &= \bigvee \{a \in A | a \wedge x \neq 0\}\,; \\ st(x, A)^d &= \bigwedge \{a \in A | a \vee x \neq 1\}\,; \end{aligned}$$

The partial order defined by setting $a \leq b$ iff $b = a \vee b$;

$$A << B \text{ iff } \forall a \in A\ \exists b \in B \text{ s.t. } b \leq a;\ (A \text{ corefines } B)$$
$$A \prec B \text{ iff } \forall a \in A\ \exists b \in B \text{ s.t. } a \leq b.\ (A \text{ refines } B)$$

Proposition 1 Let L be a Boolean frame and A, B are subsets of L. Then we have following statements:

(1) $A << B$ iff $A' \prec B'$;

(2) $stack(A \cup B) = stackA \cup stackB$;

(3) $stack\emptyset = \emptyset$;

(4) $stackA = sec^2A$;

(5) $sec^3A = secA$;

Proof:

(1) $A << B$ iff $\forall a \in A$, $\exists b \in B$ s.t. $b \leq a$ so $a' \leq b'$ i.e. $\forall a' \in A'$, $\exists b' \in B'$ s.t. $a' \leq b'$ iff $A' \prec B'$.

(2) and (3) obviously hold.

(4) Let $b \in stackA$ i.e. $\exists a \in A$ s.t. $a \leq b$. Let c be arbitrary member of $secA$ so $\forall a \in A$, $c \wedge a \neq 0$ and since $a \leq b$, it implies $c \wedge b \neq 0$ therefore we have $\forall c \in secA$, $b \wedge c \neq 0$ i.e. $b \in sec^2A$ so $stackA \subseteq sec^2A$.
Now let $b \in sec^2A$, if $b \notin stackA$, then $\forall a \in A$, $a \nleq b$ so $\forall a \in A$, $b' \wedge a \neq 0$ i.e. $b' \in secA$ so $b \notin sec^2A$ which is contradiction so $b \in stackA$ i.e. $sec^2A \subseteq stackA$.
Therefore $stackA = sec^2A$.

(5) Let $c \in sec^3A = stack(secA)$ so $\exists d \in secA$ s.t. $d \leq c$ but $\forall a \in A$, $d \wedge a \neq 0$ therefore $\forall a \in A$, $c \wedge a \neq 0$ i.e. $c \in secA$ so $sec^3A \subseteq secA$. And obviously, $secA \subseteq stack(secA)$ so $sec^3A = secA$.

Definition 4 Let L be a Boolean frame. The relation δ satisfying in the following conditions:

(BP0) $x\delta y$ implies $y\delta x$;

(BP1) $x \leq y$ and $x\delta z$ imply $y\delta z$;

(BP2) $x \wedge y \neq 0$ implies $x\delta y$;

(BP3) $x\delta y$ implies $x \neq 0$;

(BP4) $x\delta(y \vee z)$ implies $x\delta y$ or $x\delta z$;

(BP5) For every $x \in L$ we have $x = \bigwedge \{y \in L | x \leq y, x\overline{\delta}y'\}$.
Relation δ is called *BN-proximity* on L and (L, δ) is *BN-proximal frame.*
If additionally δ satisfies in (BP6),

(BP6) $x\overline{\delta}y$ implies there exist $z \in L$ such that $x\overline{\delta}z$ and $z'\overline{\delta}y$; ($x\overline{\delta}y$ means x and y are not in relation.)
Then δ is called *B-proximity* on L and (L, δ) is *B-proximal frame.*

Let (L, δ_1) and (M, δ_2) be two B-proximal frames.

A frame homomorphism $f : L \to M$ is called frame B-proximal homomorphism iff

$$x\overline{\delta_1}y \text{ implies } f(x)\overline{\delta_2}f(y)$$

We denote the corresponding category by **BProxFrm**.

Definition 5 Let L be a Boolean frame and ξ be a subset of PL:

(BN1) If $A << B$ and $B \in \xi$ then $A \in \xi$;

(BN2) If $\bigwedge A \neq 0$ then $A \in \xi$;

(BN3) $\emptyset \neq \xi \neq PL$;

(BN4) If $(A \bigvee B) \in \xi$ then $A \in \xi$ or $B \in \xi$;

Then ξ is called *preB-nearness* on L, and the pair (L, ξ) is called *preB-nearness frame* iff it satisfies in (BN1)-(BN3).
ξ is called *semiB-nearness* on L, and the pair (L, ξ) is called *semiB-nearness frame* iff it satisfies in (BN1)-(BN4).
ξ is called *B-nearness* on L, and the pair (L, ξ) is called *B-nearness frame* iff it additionally satisfies:

(BN5) For each $x \in L$, $x = \bigwedge \{y \in L | x \leq st(y, A)^d, \text{ for some } A \notin \xi\}$.

Let (L, ξ) and (M, η) be two B-nearness frames.
A frame homomorphism $f : L \to M$ is called B-nearness homomorphism iff
$A \in \eta \Rightarrow f^{-1}(A) \in \xi$.

The corresponding category denoted by **BNFrm**.

Proposition 2 Let L be a Boolean frame, $y \in L$ and $A \subseteq L$ where $\bigwedge A = 0$ then $st(y, A)^d \leq y$.

Proof: We know for every $y \in L$

$$y = y \vee 0 = y \vee (\bigwedge \{z | z \in A\}) = \bigwedge \{y \vee z | z \in A\}.$$

And since L is Boolean if for all $z \in A$, $y \vee z = 1$ then $\forall z \in A$, $y' \leq z$ so $y' \leq \bigwedge A$ i.e. $y' = 0$ so $y = 1$ and then $y = \wedge\emptyset$ so we have

$$\begin{aligned} y &= \bigwedge \{y \vee z | y \vee z \neq 1, z \in A\} \\ &= y \vee (\bigwedge \{z \in A | z \vee y \neq 1\}) \\ &= y \vee st(y, A)^d \;\Rightarrow\; st(y, A)^d \leq y \end{aligned}$$

Proposition 3 Let L be a Boolean frame and ξ is a preB-nearness on L then $x \leq st(y, A)^d$ for some $A \notin \xi$ iff $x \leq y$ and $\{y', x\} \notin \xi$.

Proof: Let $x \leq st(y, A)^d$ for some $A \notin \xi$. Since $A \notin \xi$ by (BN2), $\bigwedge A = 0$ and by Proposition 2, $st(y, A)^d \leq y$ so $x \leq \bigwedge \{z | z \vee y \neq 1, z \in A\} \leq y$. Now for every $z \in A$ either $z \vee y \neq 1$ or $z \vee y = 1$. If $z \vee y \neq 1$ then $x \leq z$ and if $z \vee y = 1$ then $y' \leq z$ i.e. $A << \{y', x\}$ and since $A \notin \xi$ by (BN1), $\{y', x\} \notin \xi$.

Conversely, let $x \leq y$ and $\{y', x\} \notin \xi$. For $A = \{y', x\}$, if $y = 1$ then $st(y, A)^d = 1$ and if $y \neq 1$ then $st(y, A)^d = x$. In any case $x \leq st(y, A)^d$ for some $A \notin \xi$.

Remark 1 Let (L, ξ) be a semiB-nearness frame then ξ is B-nearness frame iff it satisfies in the following condition:

(BN5$'$) For each $x \in L$, $x = \bigwedge \{y \in L | x \leq y, \textit{ and } \{y', x\} \notin \xi\}$.

Theorem 1 Let L be a Boolean frame. Set of all preB-nearness on L ordered by set inclusion is a lattice when its bottom is $\{A \subset L | \bigwedge A \neq 0\}$ and its top is $\{A \subset L | 0 \notin A\}$.

Remark 2 If (X, ξ) is a B-nearness frame then the relation δ on L defined by

$$x \delta y \text{ iff } \{x, y\} \in \xi$$

is a BN-proximity.

Proof: We show that δ satisfying in the (BP0)-(BP5).

To (BP0): Let $x\delta y$ then $\{x, y\} \in \xi$ and equivalently $\{y, x\} \in \xi$ i.e. $y \delta x$.

To (BP1): Let $x \leq y$ and $x \delta z$ i.e. $\{x, z\} \in \xi$ since $\{y, z\} << \{x, z\}$ by (BN1) we have $\{y, z\} \in \xi$ i.e. $y \delta z$.

To (BP2): Let $x \wedge y \neq 0$ so by (BN2), $\{x, y\} \in \xi$ i.e. $x \delta y$.

To (BP3): Let $x \delta y$ i.e. $\{x, y\} \in \xi$. If $x = 0$ then for every $A \subset L$, $A << \{x, y\}$ and by (BN1), $A \in \xi$ i.e. $\xi = L$ which is contradiction to (BN3). So $x \neq 0$.

To (BP4): Let $x\delta(y \vee z)$ i.e. $\{x, (y \vee z)\} \in \xi$ but we have
$\{x, y\} \vee \{x, z\} = \{x, x \vee z, y \vee x, y \vee z\} << \{x, (y \vee z)\}$ so by (BN1), $\{x, y\} \vee \{x, z\} \in \xi$ and by (BN4), $\{x, y\} \in \xi$ or $\{x, z\} \in \xi$ i.e. $x\delta y$ or $x \delta z$.

To (BP5): By (BN5′) we have for every $x \in L$, $x = \bigwedge\{y \in L | x \leq y,\ and\ \{y', x\} \notin \xi\}$ i.e. $x = \bigwedge\{y \in L | x \leq y,\ \text{and}\ x\bar{\delta}y'\}$.

Therefore δ is an *BN-proximity* on L.

4 B-Farness, B-Smallness and B-Covering on frames

Definition 6 Let L be a Boolean frame let $\bar{\xi}$ be a subset of PL satisfying the following conditions:

(BF1) If $A << B$ and $A \in \bar{\xi}$ then $B \in \bar{\xi}$;

(BF2) If $A \in \bar{\xi}$ then $\bigwedge A = 0$;

(BF3) $\emptyset \neq \bar{\xi} \neq PL$;

(BF4) If $A \in \bar{\xi}$ and $B \in \bar{\xi}$ then $(A \bigvee B) \in \bar{\xi}$;

Then $\bar{\xi}$ is called *B-farness* on L, and the pair $(L, \bar{\xi})$ is called a *B-farness frame* iff it additionally satisfies:

(BF5) For each $x \in L$, $x = \bigwedge\{y \in L | x \leq st(y, A)^d,\ for\ some\ A \in \bar{\xi}\}$.

Definition 7 Let L be a Boolean frame let γ be a subset of PL satisfying the following conditions:

(BS1) If $A << B$ and $A \in \gamma$ then $B \in \gamma$;

(BS2) For $0 \in L$, $\{0\} \in \gamma$;

(BS3) $\emptyset \neq \gamma \neq PL$;

(BS4) If $A \cup B \in \gamma$ then $A \in \gamma$ or $A \in \gamma$;

Then γ is called *B-smallness* on L, and the pair (L, γ) is called a *B-smallness frame* iff it additionally satisfies:

(BS5) For each $x \in L$, $x = \bigwedge\{y \in L | x \leq st(y, A)^d,\ \text{for some}\ secA \notin \gamma\}$.

Definition 8 [2] Let L be a Boolean frame let μ be a subset of PL satisfying the following conditions:

(BC1) If $A \prec B$ and $A \in \mu$ then $B \in \mu$;

(BC2) If $A \in \mu$ then $\bigvee A = 1$;

(BC3) $\emptyset \neq \mu \neq PL$;

(BC4) If $A \in \mu$ and $B \in \mu$ then$(A \bigwedge B) \in \mu$;

Then μ is called *B-covering* on L, and the pair (L, μ) is called a *B-covering frame* iff it additionally satisfies:

(BC5) For each $x \in L$, $x = \bigvee \{y \in L | st(y, A) \leq x, \text{ for some } A \in \mu\}$.

Let (L, μ_1) and (M, μ_2) be two B-covering frames, a frame homomorphism $f : L \to M$ is called B-covering homomorphism iff $A \in \mu_1 \Rightarrow \{f(a) : a \in A\} =: f(A) \in \mu_2$.

We denote the corresponding category with **BCFrm**.

Proposition 4 Let L be a Boolean frame and ξ be a B-nearness on L. Then

(i) $\overline{\xi} = \{A \subseteq L | A \notin \xi\}$ is the B-farness structure on L induced by ξ;

(ii) $\mu = \left\{A \subseteq L | A' \in \overline{\xi}\right\}$ is the B-covering structure on L induced by ξ;

(iii) $\gamma = \{A \subset L | \forall B \in \mu, B \cap stackA \neq \emptyset\}$ is the B-smallness structure on L induced by ξ.

Proof:

(i) We prove that $\overline{\xi}$ is a B-farness on L.

To (BF1): Let $A << B$ and $A \in \overline{\xi}$ then $A \notin \xi$ so by (BN1), $B \notin \xi$ i.e. $B \in \overline{\xi}$.

To (BF2): If $A \in \overline{\xi}$ i.e. $A \notin \xi$ so $\wedge A = 0$.

To (BF3): Since $\xi \neq \emptyset$, $\overline{\xi} \neq PL$. And since $\xi \neq PL$, $\overline{\xi} \neq \emptyset$.

To (BF4): Let $A \in \overline{\xi}$ and $B \in \overline{\xi}$ i.e. $A \notin \xi$ and $B \notin \xi$ so by (BN4), $A \vee B \notin \xi$ i.e. $A \vee B \in \overline{\xi}$.

To (BF5): Let $x \in L$, we know $x = \bigwedge \left\{y \in L | x \leq st(y, A)^d, \text{ for some } A \notin \xi\right\}$ i.e. $x = \bigwedge \left\{y \in L | x \leq st(y, A)^d, \text{ for some } A \in \overline{\xi}\right\}$.

So $\overline{\xi}$ is a B-farness on L which induced by ξ.

(ii) We prove that μ is a B-covering on L.

To (BC1): Let $A \prec B$ and $A \in \mu$ i.e. $A' << B'$ and $A' \in \overline{\xi}$ so by (BF1), $B' \in \overline{\xi}$ i.e. $B \in \mu$.

To (BC2): Let $A \in \mu$ i.e. $A' \in \overline{\xi}$ so by (BF2), $\wedge A' = 0$ so $\vee A = 1$.

To (BC3): Since $\overline{\xi} \neq \emptyset$, $\mu \neq \emptyset$ and since $\overline{\xi} \neq PL$, $\mu \neq PL$.

To (BC4): Let $A \in \mu$ and $B \in \mu$ i.e. $A' \in \overline{\xi}$ and $B' \in \overline{\xi}$ so by (BF4), $A' \vee B' \in \overline{\xi}$ i.e. $(A \wedge B)' \in \overline{\xi}$ so $A \wedge B \in \mu$.

To (BC5): Let $x \in L$, we know $x' = \bigwedge \left\{y' \in L | x' \leq st(y', A')^d, \text{ for some } A' \in \overline{\xi}\right\}$ so $x = \bigvee \left\{y \in L | x' \leq st(y', A')^d, \text{ for some } A' \in \overline{\xi}\right\}$.

But $x' \leq st(y', A')^d$ means $x' \leq \bigwedge \{a' \in A' | a' \vee y' \neq 1\}$ i.e.

$\bigvee \{a \in A | a \wedge y \neq 0\} \leq x$ i.e. $st(y, A) \leq x$ so

$x = \bigvee \left\{y \in L | st(y, A) \leq x, \text{ for some } A' \in \overline{\xi}\right\}$ i.e.

$x = \bigvee \{y \in L | st(y, A) \leq x,$ for some $A \in \mu\}$.

So μ is a B-covering on L which induced by ξ.

(iii) We prove that γ is a B-smallness on L.

To (BS1): If $A << B$ and $A \in \gamma$ then for every $D \in \mu$, $D \cap stackA \neq \emptyset$ i.e. for arbitrary $D \in \mu$, there exists $d \in D$ s.t. $a \leq d$ for some $a \in A$. Also we know $A << B$ so there exists $b \in B$ s.t. $b \leq a$ therefore $b \leq a \leq d$ i.e. $d \in stackB$ so $D \cap stackB \neq \emptyset$ i.e. $B \in \gamma$.

To (BS2): We know $stack\,\{0\} = \{x \in L | 0 \leq x\} = L$ and obviously for every$B \in \mu$, $B \cap L \neq \emptyset$. So $\{0\} \in \gamma$.

To (BS3): Since $\{0\} \in \gamma$ so $\gamma \neq \emptyset$.

We know $stack\emptyset = \emptyset$ and for every $B \in \mu$, $B \cap \emptyset = \emptyset$ so $\emptyset \notin \gamma$ i.e. $\gamma \neq PL$.

To (BS4): Let $A \cup B \in \gamma$ so for every $D \in \mu$, $D \cap stack(A \cup B) \neq \emptyset$ i.e. $D \cap (stackA \cup stackB) \neq \emptyset$ so $(D \cap stackA) \cup (D \cap stackB) \neq \emptyset$ i.e. either $D \cap stackA \neq \emptyset$ or $D \cap stackB \neq \emptyset$.

$$\text{Let for} \quad D_1 \in \mu, \quad D_1 \cap stackA = \emptyset \quad \text{and} \quad D_1 \cap stackB \neq \emptyset \tag{1}$$

$$\text{Let for} \quad D_2 \in \mu, \quad D_2 \cap stackA \neq \emptyset \quad \text{and} \quad D_2 \cap stackB = \emptyset \tag{2}$$

Since $D_1 \in \mu$ and $D_2 \in \mu$, by (BC4), $D_1 \wedge D_2 \in \mu$.

So either $(D_1 \wedge D_2) \cap stackA \neq \emptyset$ or $(D_1 \wedge D_2) \cap stackB \neq \emptyset$.

If $(D_1 \wedge D_2) \cap stackA \neq \emptyset$, there exists $d_1 \wedge d_2$ where $d_1 \in D_1$ and $d_2 \in D_2$ s.t. $a \leq (d_1 \wedge d_2)$ for some $a \in A$ therefore $a \leq d_1$ and $a \leq d_2$, i.e. $D_1 \cap stackA \neq \emptyset$ and $D_2 \cap stackA \neq \emptyset$ which is contradiction to (1). Similarly if $(D_1 \wedge D_2) \cap stackB \neq \emptyset$, we have $D_1 \cap stackB \neq \emptyset$ and $D_2 \cap stackB \neq \emptyset$ which is contradiction to (2).

So either for all $D \in \mu$, $D \cap stackA \neq \emptyset$ or for all $D \in \mu$, $D \cap stackB \neq \emptyset$ i.e. either $A \in \gamma$ or $B \in \gamma$.

To (BS5): Let $x \in L$ so $x = \bigwedge \left\{y \in L | x \leq st(y, A)^d,\right.$ for some $\left.A \in \overline{\xi}\right\}$ i.e.

$x = \bigwedge \left\{y \in L | x \leq st(y, A)^d,\right.$ for some $\left.A' \in \mu\right\}$.

If $A' \in \mu$ then $secA \notin \gamma$ since $A' \cap stack(secA) = A' \cap secA = \emptyset$.

And if $secA \notin \gamma$ then there is $B \in \mu$ s.t. $B \cap stack(secA) = \emptyset$, i.e. $B \cap secA = \emptyset$, i.e. for every $b \in B$ there is $a \in A$ s.t. $b \wedge a = 0$ so $b \leq a'$ therefore we have $B \prec A'$ and by (BC1) it implies $A' \in \mu$.

So equivalently we have $x = \bigwedge \left\{y \in L | x \leq st(y, A)^d,\right.$ for some $\left.secA \notin \gamma\right\}$.

Therefore γ is a B-smallness on L which induced by ξ.

Proposition 5 Let L be a Boolean frame and ξ, $\overline{\xi}$, γ and μ be respectively B-nearness, B-farness, B-smallness and B-covering structures induced by each other on L. Then following relations are hold

(1) $A \in \xi$ iff $A \notin \overline{\xi}$;

(2) $A \in \overline{\xi}$ iff $A \notin \xi$;

(3) $A \in \overline{\xi}$ iff $A' \in \mu$;

(4) $A \in \mu$ iff $A' \in \overline{\xi}$;

(5) $A \in \mu$ iff $\forall\ B \in \xi\ A \cap secB \neq \emptyset$;

(6) $A \in \xi$ iff $\forall\ B \in \mu$, $B \cap secA \neq \emptyset$;

(7) $A \in \gamma$ iff $\forall\ B \in \mu$, $B \cap stackA \neq \emptyset$;

(8) $A \in \mu$ iff $\forall B \in \gamma$, $A \cap stackB \neq \emptyset$;

(9) $A \in \xi$ iff $secA \in \gamma$;

(10) $A \in \gamma$ iff $secA \in \xi$;

(11) $A \in \gamma$ iff $\forall B \in \overline{\xi}\ \exists a \in A\ \exists b \in B$, $a \wedge b = 0$;

(12) $A \in \overline{\xi}$ iff $\forall B \in \gamma$, $\exists a \in A\ \exists b \in B$, $a \wedge b = 0$.

Proof:

By Proposition 4, (1) - (4) is clear.

(5) Let $A \in \mu$ then $A' \in \overline{\xi}$ and let $B \in \xi$ we prove that $A \cap secB \neq \emptyset$ i.e. $\exists\ a \in A$ s.t. $a \in secB$ i.e. $\exists\ a \in A$ s.t. $\forall\ b \in B$, $a \wedge b \neq 0$. If not so $\forall a \in A\ \exists\ b \in B$ s.t. $a \wedge b = 0$ i.e. $b \leq a'$ therefore $\forall a' \in A'\ \exists\ b \in B$ s.t. $b \leq a'$ i.e. $A' << B$ and since $A' \in \overline{\xi}$ by (BF1) $B \in \overline{\xi}$ which is a contradiction.

Conversely, let $\forall\ B \in \xi\ A \cap secB \neq \emptyset$. If $A \notin \mu$ then $A' \in \xi$. But we have $A \cap secA' = \emptyset$ since for every $a \in A$, $a \wedge a' = 0$ and $a' \in A'$ so $a \notin secA'$ which is contradiction so $A \in \mu$.

(6) Similar to (5). And by Proposition 4, (7) is clear.

(9) Let $A \notin \xi$ i.e. $A' \in \mu$ then we have $A' \cap secA = \emptyset$ i.e. $A' \cap stack(secA) = \emptyset$ so $secA \notin \gamma$.

Conversely, let $secA \notin \gamma$ i.e. there exists $B \in \mu$ s.t. $B \cap stack(secA) = \emptyset$ so $B \cap secA = \emptyset$ i.e. $\forall b \in B$, $\exists a \in A$ s.t. $a \wedge b = 0$ so $b \leq a'$ therefore $B \prec A'$ and by (BC1) it implies $A' \in \mu$ so $A \notin \xi$.

(10) Let $A \in \gamma$ i.e. $\forall B \in \mu$, $B \cap stackA \neq \emptyset$.

We consider $B = \{d' \in L | \forall a \in A, d \wedge a \neq 0\}$. For every $d' \in B$, there is not any $a \in A$ s.t. $a \leq d'$ i.e. for every $d' \in B$, $d' \notin stackA$ so $\{d' \in L | \forall a \in A, d \wedge a \neq 0\} \notin \mu$ therefore $\{d \in L | \forall a \in A, d \wedge a \neq 0\} \in \xi$ i.e. $secA \in \xi$.

Conversely, $secA \in \xi$ by (9), $sec^2 A \in \gamma$ i.e. $stackA \in \gamma$ and since $stackA << A$ by (BS1) it implies $A \in \gamma$.

(8) Let $A \in \mu$ and $B \in \gamma$ then by (7) we have $A \cap stackB \neq \emptyset$.

Conversely, let $\forall B \in \gamma$, $A \cap stackB \neq \emptyset$, if $A \notin \mu$ then $A' \in \xi$ and by (9) $secA' \in \gamma$ and by assumption $A \cap stack(secA') \neq \emptyset$ i.e. $A \cap secA' \neq \emptyset$ which is contradiction so $A \in \mu$.

(11) By (7) is clear. And (12) by (8) is clear.

Proposition 6 If (L, ξ) is a B-nearness frame and $\overline{\xi}$ and μ are respectively corresponding B-farness and B-covering then the following conditions are equivalent:

(C) If every finite corefinement of A belongs to ξ then A belongs to ξ;

(C') If $A \in \overline{\xi}$ then there exists a finite corefinement B of A with $B \in \overline{\xi}$;

(C'') If $A \in \mu$ then there exists a finite refinements B of A with $B \in \mu$.

Proof: $(C) \Leftrightarrow (C')$: Obviously.

$(C') \Rightarrow (C'')$: Let $A \in \mu$ then $A' \in \overline{\xi}$ so by (C') there exists a finite $B' \in \overline{\xi}$ s.t. $B' << A'$ i.e. $B \prec A$ and since B' is finite B-farness so B is finite B-covering. So there exists a finite refinements B of A with $B \in \mu$.

$(C') \Leftarrow (C'')$: Let $A \in \overline{\xi}$ so $A' \in \mu$ and by (C'') there exists a finite $B' \in \mu$ s.t. $B' \prec A'$ so $B << A$ and since B' is finite B-covering so B is finite B-farness i.e. there exists finite corefinement of A belongs to $\overline{\xi}$.

Definition 9 [2] A B-nearness frame is called *contigual* iff it satisfies the condition (C).

Theorem 2 Let (L, ξ) be a B-nearness frame then $\xi_c = \{A \subset L | \forall B << A, (B \text{ finite } \Rightarrow B \in \xi)\}$ is the smallest contigual B-nearness structure on L contains ξ that we call it contigual B-nearness structure on L generated by ξ. In addition $\xi_f = (\xi_c)_f$ where $\xi_f = \{A \in \xi | A \text{ finite}\}$.

Proof: First we show ξ_c is a B-nearness on L.

To (BN1): Let $A_1 << A_2$ and $A_2 \in \xi_c$. We have $\forall B << A_1$, $B << A_2$ so if B is finite it implies that $B \in \xi$ which means $A_1 \in \xi_c$.

To (BN2): Let $\bigwedge A \neq 0$ and $B << A$ so $\bigwedge B \neq 0$ therefore $B \in \xi$ so $A \in \xi_c$.

To (BN3): By (BN2), $\emptyset \in \xi_c$ so $\xi_c \neq \emptyset$. And $\{0\} \notin \xi_c$ so $\xi_c \neq PL$.

To (BN4): Let $A\bigvee B\in\xi_c$ if $A\notin\xi_c$ and $B\notin\xi_c$ then $\exists\ C<<A$ and C is finite but $C\notin\xi$ and $\exists\ D<<B$ and D is finite but $D\notin\xi$ so $C\bigvee D\notin\xi$ and it is finite. But since $C<<A$ and $D<<B$ we have $\forall\ c_i\in C\ \exists\ a_i\in A$ s.t. $a_i\leq c_i$ and $\forall\ d_j\in D\ \exists\ b_j\in B$ s.t. $b_j\leq d_j$ therefore $\forall c_i\vee d_j\in C\bigvee D\ \exists\ a_i\vee b_j\in A\bigvee B$ s.t. $a_i\vee b_j\leq c_i\vee d_j$ i.e. $C\bigvee D<<A\bigvee B$ and since $C\bigvee D\notin\xi$ and it is finite it is contradiction to $A\bigvee B\in\xi_c$.

To (BN5′): Let $x\in L$, then $T=\{y\in L|x\leq y,\{x,y'\}\notin\xi\}$ and
$S=\{y\in L|x\leq y,\{x,y'\}\notin\xi_c\}$. Let $y\in T$ so $x\leq y$ and $\{x,y'\}\notin\xi$ if $y\notin S$ so $\{x,y'\}\in\xi_c$ by definition of ξ_c, $\{x,y'\}\in\xi$ that is contradiction. So we have $T\subseteq S$ therefore $\bigwedge S\leq\bigwedge T$ and we know $\bigwedge T=x$ also by definition of S, x is its lower bound so $x=\bigwedge S$.

Therefore ξ_c is a B-nearness on L.

Now let every finite corefinement of A belongs to ξ_c if $A\notin\xi_c$ then $\exists B<<A$ where B is finite and $B\notin\xi$ so $B\notin\xi_c$ that is contradiction so ξ_c is contigual.

Now we show that ξ_c is the smallest contigual B-nearness contains ξ.

Let $A\in\xi$ so by (BN1) for every $B<<A$, we have $B\in\xi$ therefore $A\in\xi_c$. i.e. $\xi\subset\xi_c$. Suppose η be an arbitrary contigual B-nearness contains ξ and $A\in\xi_c$, so $\forall B<<A$, if B is finite then $B\in\xi$ therefore $B\in\eta$ and since η is contigual, $A\in\eta$ i.e. $\xi_c\subset\eta$ so ξ_c is the smallest contigual B-nearness contains ξ.

And obviously, $\xi_f=(\xi_c)_f$.

Proposition 7 If (L,ξ) is a B-nearness frame and $\overline{\xi}$, μ and γ are respectively corresponding B-farness, B-covering and B-smallness then the following conditions are equivalent:

- (U) If $A\in\overline{\xi}$ then there exists $B\in\overline{\xi}$ such that $\left\{st(b,B)^d|b\in B\right\}<<A$;
- (U') If $A\in\mu$ then there exists $B\in\mu$ such that $\{st(b,B)|b\in B\}\prec A$;
- (U'') If $A\notin\gamma$ then $\exists B\subset L$ s.t. $secB\notin\gamma$ and $\left\{st(b,B)^d|b\in B\right\}<<secA$.

Proof:

($(U)\Rightarrow(U')$) Let $A\in\mu$ i.e. $A'\in\overline{\xi}$ then by (U), there exists $B'\in\overline{\xi}$ s.t. $\left\{st(b',B')^d|b'\in B'\right\}<<A'$ i.e. for every $b'\in B'$ there exists $a'\in A'$ s.t. $a'\leq st(b',B')^d$ i.e. $a'\leq\bigwedge\{c'\in B'|c'\vee b'\neq 1\}$ so $\bigvee\{c\in B|c\wedge b\neq 0\}\leq a$. Therefore for every $b\in B$ there exists $a\in A$ s.t. $st(b,B)\leq a$ i.e. $\{st(b,B)|b\in B\}\prec A$.

($(U)\Leftarrow(U')$) Let $A\in\overline{\xi}$ i.e. $A'\in\mu$ then by (U') there exists $B'\in\mu$ such that $\{st(b',B')|b'\in B'\}\prec A'$ i.e. for every $b'\in B'$ there exists $a'\in A'$ s.t. $st(b',B')\leq a'$ i.e. $\bigvee\{c'\in B'|c'\wedge b'\neq 0\}\leq a'$ so $a\leq\bigwedge\{c\in B|c\vee b\neq 1\}$. Therefore for every $b\in B$, there exists $a\in A$ s.t. $a\leq st(b,B)^d$ i.e. $\left\{st(b,B)^d|b\in B\right\}<<A$.

($(U) \Rightarrow (U'')$) If $A \notin \gamma$ then $secA \in \overline{\xi}$ so by (U) there exists $B \in \overline{\xi}$ such that $\left\{st(b, B)^d | b \in B\right\} << secA$ and $B \in \overline{\xi}$ implies $secB \notin \gamma$.

($(U) \Leftarrow (U'')$) If $A \in \overline{\xi}$ i.e. $secA \notin \gamma$ then by (U'') $\exists B \subset L$ s.t. $secB \notin \gamma$ and $\left\{st(b, B)^d | b \in B\right\} << sec^2 A = stackA$ and we know $stackA << A$ so $\left\{st(b, B)^d | b \in B\right\} << A$ also $secB \notin \gamma$ implies $B \in \overline{\xi}$.

Definition 10 [2] B-nearness frame (L, ξ) is called *uniform* iff it satisfies to condition (U).

We denote by **UBCFrm** the category of uniform B-covering frames and B-covering homomorphisms.

Also we denote by **UBNFrm** the category of uniform B-nearness frames and B-nearness homomorphisms.

5 Relation between B-Nearness on frames and subframes

Let L be a Boolean frame and $a \in L$ then $\downarrow a = \{x \in L | x \leq a\}$ is Boolean frame with $\wedge$ and $\vee$ defined as in L. The top of $\downarrow a$ is a and the bottom of $\downarrow a$ is the bottom of L.

Let (L, ξ) be a B-nearness frame and $\overline{\xi}$ its corresponding farness. Let $a \in L$. For each $A \subseteq L$

$$a \wedge A = \{a \wedge x | x \in A\}$$

and

$$a \wedge \xi = \{A \in P(\downarrow a) | A \in \xi\} \quad \text{and} \quad a \wedge \overline{\xi} = \left\{A \in P(\downarrow a) | A \in \overline{\xi}\right\}$$

Theorem 3 If (L, ξ) is a B-nearness frame and $a \in L$ then $a \wedge \xi$ is a nearness on $\downarrow a$ and $a \wedge \overline{\xi}$ is corresponding B-farness on $\downarrow a$.

Proof: (BN1) to (BN4) are obvious. We prove only (BN5).

To (BN5): Let $x \in \downarrow a$ and

$$\begin{aligned} S &= \left\{y \in L | x \leq st(y, A)^d \text{ for some } A \notin \xi\right\} \text{ and} \\ T &= \left\{w \in \downarrow a | x \leq st(w, B)^d \text{ for some } B \notin a \wedge \xi\right\} \end{aligned}$$

Let $z \in S$ by (BN2) and proposition 2, $x \leq z$ and since $x \leq a$ so $x \leq a \wedge z$. Choose $A \notin \xi$ s.t. $x \leq st(z, A)^d$, since $A \ll a \wedge A$ so by (BN1), $a \wedge A \notin \xi$ and by definition, $a \wedge A \notin a \wedge \xi$

$$\begin{aligned} st((a \wedge z), (a \wedge A))^d &= \bigwedge \{a \wedge h | h \in A, (a \wedge z) \vee (a \wedge h) \neq a\} \\ &= \bigwedge \{a \wedge h | h \in A, a \wedge (z \vee h) \neq a\} \\ &= a \wedge (\bigwedge \{h \in A | a \nleq z \vee h\} \end{aligned}$$

Obviously $\{h \in A | a \nleq z \vee h\} \subset \{h \in A | z \vee h \neq 1\}$ so $\bigwedge \{h \in A | z \vee h \neq 1\} \leq \bigwedge \{h \in A | a \nleq z \vee h\}$ and we have $x \leq st(z, A)^d = \bigwedge \{h \in A | z \vee h \neq 1\}$ so $x \leq \bigwedge \{h \in A | a \nleq z \vee h\}$ and since $x \leq a$ therefore $x \leq a \wedge (\bigwedge \{h \in A | a \nleq z \vee h\}$ i.e. $x \leq st((a \wedge z), (a \wedge A))^d$ so $a \wedge z \in T$ therefore $a \wedge S \subseteq T$ so $\bigwedge T \leq \bigwedge (a \wedge S) \leq \bigwedge S$ since ξ is B-nearness so $x = \bigwedge S$ so $\bigwedge T \leq x$ and for any $w \in T$ by (BN2) and proposition 2, we have $x \leq w$ i.e. x is a lower bound for T so $\bigwedge T = x$.

So $a \wedge \xi$ is a B-nearness on $\downarrow a$ generated by ξ.

Now we show that $a \wedge \overline{\xi}$ is its corresponding B-farness on $\downarrow a$

$$\overline{a \wedge \xi} = \{A \subset P(\downarrow a) | A \notin a \wedge \xi\} \text{ i.e. } \overline{a \wedge \xi} = \{A \subset P(\downarrow a) | A \notin \xi\} \text{ i.e.}$$
$$\overline{a \wedge \xi} = \{A \subset P(\downarrow a) | A \in \overline{\xi}\} = a \wedge \overline{\xi}.$$

Proposition 8 Let $a \leq b$ in L and $\overline{\xi}_b$ and $\overline{\xi}_a$ are respectively B-farnesses on $\downarrow b$ and $\downarrow a$ generated by an unknown B-farness on L then $\overline{\xi}_b$ and $\overline{\xi}_a$ satisfy on the following relations :

(i) If $D \in \overline{\xi}_b$ then $D << C$ for some $C \in \overline{\xi}_a$.

(ii) If $C \in \overline{\xi}_a$ and $C << D$ for some $D \in P(\downarrow b)$ then $D \in \overline{\xi}_b$

Proof: Obviously.

Definition 11 B-nearness frame (L, ξ) is called *graded* iff $\{A, secA\} \subseteq \xi$ implies $\xi(A) \in \xi$, where $\xi(A) = \{x \in L | (\{x\} \cup A) \in \xi\}$.

Proposition 9 Let (L, ξ) be a B-nearness frame and $a \in L$ then following result holds

(i) If (L, ξ) is graded then $a \wedge \xi$ is also a graded B-nearness on $\downarrow a$.

(ii) If (L, ξ) is contigual then $a \wedge \xi$ is also a contigual B-nearness on $\downarrow a$.

Proof: (i) Let $\{A, sec_{\downarrow a} A\} \subset a \wedge \xi$ since $A \in P(\downarrow a)$, $sec_{\downarrow a} A = a \wedge sec_L A$ and $sec_L A << a \wedge sec_L A$ so $\{A, sec_L A\} \subset \xi$ then $\xi(A) \in \xi$ but $(a \wedge \xi)(A) \subseteq \xi(A)$ so $(a \wedge \xi)(A) \in \xi$ and $(a \wedge \xi)(A) \subset P(\downarrow a)$ so $(a \wedge \xi)(A) \in a \wedge \xi$.

(ii) Let $A \subseteq \downarrow a$ and every finite corefinemen of A in $\downarrow a$ belongs to $a \wedge \xi$. If there exists B finite subset of L s.t. $B << A$ and $B \notin \xi$, since $B << a \wedge B$ then by (BN1), $a \wedge B \notin \xi$ but since $A \in \downarrow a$ and $B << A$ so $\forall\, a \wedge b \in a \wedge B\ \exists\, x \in A$ s.t. $x \leq a \wedge b$ i.e. $a \wedge B << A$ then by assumption $a \wedge B \in a \wedge \xi$ and so $a \wedge B \in \xi$ that is contradiction so every finite corefinemen of A in L belongs to ξ, therefore $A \in \xi$ and so $A \in a \wedge \xi$.

Let L be a Boolean frame and $a \in L$ then $\uparrow a = \{x \in L | a \leq x\}$ is Boolean frame with $\wedge$ and $\vee$ defined as in L. The top of $\uparrow a$ is the top of L and the bottom of $\uparrow a$ is a.

Let (L, ξ) be a nearness frame and $\overline{\xi}$ its corresponding B-farness. Let $a \in L$.

For each $A \subseteq L$

$$a \vee A = \{a \vee x | x \in A\}$$

and

$$a \vee \overline{\xi} = \left\{B \in P(\uparrow a) | a \vee A << B, \text{ for some } A \in \overline{\xi}\right\}$$

and $a \vee \xi$ is its corresponding B-nearness.

Theorem 4 If $\overline{\xi}$ is a B-farness on L and $a \in L$ then $a \vee \overline{\xi}$ is a B-farness on $\uparrow a$.

Proof:

To (BF1): Let $C << D$ and $C \in a \vee \overline{\xi}$ so $\exists A \in \overline{\xi}$ s.t. $a \vee A << C$ and since $C << D$ so $a \vee A << D$ i.e. $D \in a \vee \overline{\xi}$

To (BF2): Let $C \in a \vee \overline{\xi}$. If $\bigwedge C \neq a$ then $\exists\ z > a$ s.t. $\bigwedge C = z$ but since $C \in a \vee \overline{\xi}$ so $\exists A \in \overline{\xi}$ s.t. $a \vee A << C$ i.e. $\forall a \vee x \in a \vee A\ \exists c \in C$ s.t. $c \leq a \vee x$ so $\forall a \vee x \in a \vee A$, $z \leq a \vee x$ i.e. $z \leq \bigwedge(a \vee A) = a \vee (\bigwedge A)$ but $\bigwedge A = 0$ therefore $z \leq a$ that is contradiction to assumption. so $\bigwedge C = a$.

To (BF3): For every $A \in \overline{\xi}$, $a \vee A << \{a\}$ i.e. $\{a\} \in a \vee \overline{\xi}$ so $a \vee \overline{\xi} \neq \emptyset$. Now let $b > a$ by (BF2) we know $\{b\} \notin a \vee \overline{\xi}$ so $a \vee \overline{\xi} \neq P(\uparrow a)$.

To (BF4): Let $C \in a \vee \overline{\xi}$ and $D \in a \vee \overline{\xi}$ so $\exists\ A \in \overline{\xi}$ s.t. $a \vee A << C$ and $\exists\ B \in \overline{\xi}$ s.t. $a \vee B << D$ so $A \bigvee B \in \overline{\xi}$. Now let $a \vee (x \vee y) \in a \vee (A \bigvee B)$ where $x \in A$ and $y \in B$ i.e. $(a \vee x) \vee (a \vee y) \in a \vee (A \bigvee B)$ since $a \vee x \in a \vee A$ and $a \vee A << C$, $\exists\ c \in C$ s.t. $c \leq a \vee x$ similarly $\exists\ d \in D$ s.t. $d \leq a \vee y$ therefore $c \vee d \leq (a \vee x) \vee (a \vee y) = a \vee (x \vee y)$ i.e. $a \vee (A \bigvee B) << C \bigvee D$. and since $C \bigvee D \in P(\uparrow a)$ then $C \bigvee D \in a \vee \overline{\xi}$.

To (BF5): Let $x \in \uparrow a$ and

$$S = \left\{y \in L | x \leq st(y, A)^d \text{ for some } A \in \overline{\xi}\right\} \text{ and}$$
$$T = \left\{w \in \uparrow a | x \leq st(w, B)^d \text{ for some } B \in a \vee \overline{\xi}\right\}$$

Let $z \in S$ then by (BF2) and proposition 2, $x \leq z$ and $a \leq x$ so $z \in \uparrow a$. Now we choose $A \in \overline{\xi}$ s.t. $x \leq st(z, A)^d$. Obviously, $a \vee A \in a \vee \overline{\xi}$.

$$\begin{aligned} st(z, (a \vee A))^d &= \bigwedge \{a \vee h | h \in A \text{ and } z \vee (a \vee h) \neq 1\} \\ &= a \vee (\bigwedge \{h \in A | z \vee h \neq 1\}) \\ &\geq \bigwedge \{h \in A | z \vee h \neq 1\} \\ &= st(z, A)^d \geq x \end{aligned}$$

So $z \in T$ i.e. $S \subseteq T$ therefore $\bigwedge T \leq \bigwedge S$ and since $\bigwedge S = x$ and similar to proposition 2, x is a lower bound for every $w \in T$ so we have $\bigwedge T = x$.

so $a \vee \overline{\xi}$ is a B-farness on $\uparrow a$ generated by $\overline{\xi}$.

Proposition 10 Let (L, ξ) be a B-nearness frame and $a \in L$ then following result holds

(i) If (L, ξ) is uniform then $a \vee \xi$ is also a uniform B-nearness on $\uparrow a$;

(ii) If (L, ξ) is contigual then $a \vee \xi$ is also a contigual B-nearness on $\uparrow a$.

Proof: (i) Let $C \in a \vee \overline{\xi}$ then there exists $A \in \overline{\xi}$ such that $a \vee A << C$ and since $\overline{\xi}$ is uniform so $\exists B \in \overline{\xi}$ s.t. $\left\{st(b, B)^d | b \in B\right\} << A$ i.e. $\forall b \in B$, $\exists x \in A$ s.t. $x \leq \bigwedge \{b_i \in B | b_i \vee b \neq 1\}$ therefore we have $\forall a \vee b \in a \vee B (\in a \vee \overline{\xi})$, $\exists a \vee x \in a \vee A$ s.t. $a \vee x \leq \bigwedge \{a \vee b_i \in a \vee B | b_i \vee b \neq 1\}$.

But $\{a \vee b_i \in a \vee B | b_i \vee b \vee a \neq 1\} \subseteq \{a \vee b_i \in a \vee B | b_i \vee b \neq 1\}$ so
$\bigwedge \{a \vee b_i \in a \vee B | b_i \vee b \neq 1\} \leq \bigwedge \{a \vee b_i \in a \vee B | (a \vee b_i) \vee (a \vee b) \neq 1\}$
then we have $a \vee x \leq \bigwedge \{a \vee b_i \in a \vee B | (a \vee b_i) \vee (a \vee b) \neq 1\}$ i.e.
$\left\{st(a \vee b, a \vee B)^d | a \vee b \in a \vee B\right\} << a \vee A$ and since $a \vee A << C$ so
$\left\{st(a \vee b, a \vee B)^d | a \vee b \in a \vee B\right\} << C$ i.e. $a \vee \xi$ is uniform.

(ii) Let $C \in a \vee \overline{\xi}$ then there exists $A \in \overline{\xi}$ s.t. $a \vee A << C$. But $\overline{\xi}$ is contigual so there exists finite $B \in \overline{\xi}$ s.t. $B << A$ so $a \vee B$ is finite and it belongs to $a \vee \overline{\xi}$ and also $a \vee B << a \vee A$ so $a \vee B << C$ i.e. $a \vee \xi$ is contigual.

6 B-Nearness frame and Complete Near Space

Definition 12 Let (L, ξ) be a B-nearness frame. A nonempty subset A of L is called ξ-*cluster* iff A is a maximal element of the set ξ ordered by set inclusion.

Theorem 5 Let (L, ξ) be a B-nearness frame, X^* be set of all ξ-clusters and $\xi^* = \{\Omega \subset PX^* | \bigcup\{\bigcap \omega | \omega \in \Omega\} \in \xi\}$ then (X^*, ξ^*) is complete nearness space induced by B-nearness frame (L, ξ).

Proof: We prove that (X^*, ξ^*) satisfies in all conditions of nearness space.

To $(N1)$ i.e. Let $\Omega_1 << \Omega_2$ and $\Omega_2 \in \xi^*$ then $\Omega_1 \in \xi^*$.

If $\Omega_1 \notin \xi^*$ then $\bigcup\{\bigcap \omega | \omega \in \Omega_1\} \notin \xi$ therefore $\forall A_i \in X^*$, $\bigcup\{\bigcap \omega | \omega \in \Omega_1\} \nsubseteq A_i$ i.e. $\forall A_i \in X^*$, $\exists x_i \in \bigcup\{\bigcap \omega | \omega \in \Omega_1\}$ s.t. $x_i \notin A_i$ but for some $\omega \in \Omega_1$, $x_i \in \bigcap \omega$ say $x_i \in \bigcap \omega_i$ when $\omega_i \in \Omega_1$.

Since $\Omega_1 << \Omega_2$, $\forall \omega_1 \in \Omega_1$, $\exists \omega_2 \in \Omega_2$ s.t. $\omega_2 \subseteq \omega_1$ we have $\bigcap \omega_1 \subseteq \bigcap \omega_2$. And since $\forall A_i \in X^*$, $\exists x_i \in \bigcap \omega_i$ where $x_i \notin A_i$ so $\exists \omega'_i \in \Omega_2$ s.t. $\omega'_i \subseteq \omega_i$ so $\bigcap \omega_i \subseteq \bigcap \omega'_i$ therefore $x_i \in \bigcap \omega'_i$ and so $x_i \in \bigcup\{\bigcap \omega | \omega \in \Omega_2\}$ where $x_i \notin A_i$ i.e. $\forall A_i \in X^*$, $\bigcup\{\bigcap \omega | \omega \in \Omega_2\} \nsubseteq A_i$ so $\bigcup\{\bigcap \omega | \omega \in \Omega_2\} \notin \xi$ i.e. $\Omega_2 \notin \xi^*$ that is contradiction so $\Omega_1 \in \xi^*$.

To $(N2)$ i.e. If $\bigcap \Omega \neq \emptyset$ then $\Omega \in \xi^*$.

Let $\bigcap \Omega \neq \emptyset$ so $\exists A \in X^*$ s.t. $\forall \omega \in \Omega$, $A \in \omega$ therefore
$\forall \omega \in \Omega$, $\bigcap \omega \subseteq A$ so $\bigcup\{\bigcap \omega | \omega \in \Omega\} \subseteq A$ and since $A \in \xi$ so $\bigcup\{\bigcap \omega | \omega \in \Omega\} \in \xi$ i.e. $\Omega \in \xi^*$.

To $(N3)$ i.e. $\emptyset \neq \xi^* \neq P^2 X^*$.

Since $\xi \neq \emptyset$ so $\exists A \in X^*$ and by definition of X^* obviously $\{\{A\}\} \in \xi^*$ so $\xi^* \neq \emptyset$. And if $\{\emptyset\} \in \xi^*$ then $\bigcup\{\bigcap \emptyset\} (= L) \in \xi$ that is contradiction to $\xi \neq PL$ so $\xi^* \neq P^2 X^*$.

To $(N4)$ i.e. If $\Omega_1 \bigvee \Omega_2 \in \xi^*$ then $\Omega_1 \in \xi^*$ or $\Omega_2 \in \xi^*$,
where $\Omega_1 \bigvee \Omega_2 = \{\omega_1 \cup \omega_2 | \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$.

Let $\Omega_1 \bigvee \Omega_2 \in \xi^*$. If $\Omega_1 \notin \xi^*$ and $\Omega_2 \notin \xi^*$ then
$\Omega_1 \notin \xi^* \Rightarrow \bigcup\{\bigcap \omega_i | \omega_i \in \Omega_1\} \notin \xi$, we call $U_1 = \bigcup\{\bigcap \omega_i | \omega_i \in \Omega_1\}$ and
$\Omega_2 \notin \xi^* \Rightarrow \bigcup\{\bigcap \omega_j | \omega_j \in \Omega_2\} \notin \xi$, we call $U_2 = \bigcup\{\bigcap \omega_j | \omega_j \in \Omega_2\}$.

Since $U_1 \notin \xi$ and $U_2 \notin \xi$ then $U_1 \bigvee U_2 \notin \xi$ i.e.

$$\{x \vee y | x \in U_1, y \in U_2\} \notin \xi \tag{I}$$

Let $x \in U_1$ then $\exists \omega_1 \in \Omega_1$ s.t. $x \in \bigcap \omega_1$ therefore $\forall A_i \in \omega_1$, $x \in A_i$ so $\forall A_i \in \omega_1$, $A_i \bigcup \{x \vee z\} << A_i$ where $z \in L$ so $A_i \bigcup \{x \vee z\} \in \xi$ but since each A_i is ξ-cluster so $\forall A_i \in \omega_1$, $x \vee z \in A_i$ for any $z \in L$.

Similarly Let $y \in U_2$ then $\exists \omega_2 \in \Omega_2$ s.t. $y \in \bigcap \omega_2$ therefore $\forall B_j \in \omega_2$, $y \in B_j$ so $\forall B_j \in \omega_2$, $y \vee z \in B_j$ for any $z \in L$. So we have
$\forall A_i \in \omega_1$ and $\forall B_j \in \omega_2$, $x \vee y \in A_i$ and $x \vee y \in B_j$ i.e. $x \vee y \in \bigcap(\omega_1 \cup \omega_2)$ so $x \vee y \in \bigcup\{\bigcap(\omega_i \cup \omega_j) | \omega_i \in \Omega_1, \omega_j \in \Omega_2\}$ therefore
$\{x \vee y | x \in U_1, y \in U_2\} \subseteq \bigcup\{\bigcap(\omega_i \cup \omega_j) | \omega_i \in \Omega_1, \omega_j \in \Omega_2\}$
and by (I) we have $\bigcup\{\bigcap(\omega_i \cup \omega_j) | \omega_i \in \Omega_1, \omega_j \in \Omega_2\} \notin \xi$ i.e. $\Omega_1 \bigvee \Omega_2 \notin \xi^*$ that is contradiction so $\Omega_1 \in \xi^*$ or $\Omega_2 \in \xi^*$.

To $(N5)$ i.e. If $\{cl^*_\xi \omega | \omega \in \Omega\} \in \xi^*$ then $\Omega \in \xi^*$ where $cl^*_\xi \omega = \{A \in X^* | \{\omega, \{A\}\} \in \xi^*\}$.

Let $\omega \subseteq X^*$ so $cl^*_\xi \omega = \{A \in X^* | ((\bigcap \omega) \cup A) \in \xi\}$ but since A is ξ-cluster so it is maximal in ξ then $cl^*_\xi \omega = \{A \in X^* | \bigcap \omega \subseteq A\}$ therefore $\forall \omega \subseteq X^*$, $\bigcap \omega \subseteq \bigcap cl^*_\xi \omega$.
Now let $\{cl^*_\xi \omega | \omega \in \Omega\} \in \xi^*$ so $\bigcup\{\bigcap(cl^*_\xi \omega) | \omega \in \Omega\} \in \xi$ and we know
$\bigcap \omega \subseteq \bigcap cl^*_\xi \omega$ so $\bigcup\{\bigcap \omega | \omega \in \Omega\} \subseteq \bigcup\{\bigcap(cl^*_\xi \omega) | \omega \in \Omega\}$ so
$\bigcup\{\bigcap \omega | \omega \in \Omega\} \in \xi$ i.e. $\Omega \in \xi^*$.
Therefore (X^*, ξ^*) is a nearness space.

Now we have to show that (X^*, ξ^*) is complete.

Let $\Omega \in \xi^*$ be a ξ^*-*cluster* by definition $\bigcup\{\bigcap \omega | \omega \in \Omega\} \in \xi$ so there exists ξ-*cluster*, A, s.t. $\bigcup\{\bigcap \omega | \omega \in \Omega\} \subset A$. We consider $\Omega' = \Omega \cup \{\{A\}\}$ obviously $\bigcup\{\bigcap \omega' | \omega' \in \Omega'\} \subset A$ therefore $\Omega' \in \xi^*$, since Ω is ξ^*-*cluster* so $\{A\} \in \Omega$ therefore (X^*, ξ^*) is a complete nearness space.

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