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On the Solutions of Two-Scale Difference Equations

ABSTRACT. This paper deals with specific two-scale difference equations which are equivalent to a system of functional equations. Such equations have a continuous solution if the coefficients c_j of the corresponding characteristic polynomial P satisfy condition $|c_j| < 1$ for all j. By means of some functional relations for the solution we show that it is Hölder continuous and we determine the optimal Hölder exponent. Moreover we give a condition which is necessary and sufficient for the differentiability almost everywhere where we apply Borel's normal number theorem. If the coefficients c_j are nonnegative then the solution is a singular function. Special cases are the well-known singular functions of *de Rham* and of *Cantor*.

1 Introduction

A *two-scale difference equation* (*dilation equation*) is a functional equation of the form

$$\varphi\left(\frac{x}{d}\right) = \sum_{j=0}^{p-1} c_j \varphi(x-j) \tag{1.1}$$

with dilation parameter $d > 1$ and complex coefficients c_j where $c_0 c_{p-1} \neq 0$, $p \geq 2$. Such equations especially with $d = 2$ appear in wavelet theory and in subdivision schemes where nontrivial compactly supported Lebesgue-integrable solutions are demanded, cf. [5], [7], [8], [9].

In this paper we consider the two-scale difference equation (1.1) with $d = p$, that means

$$\varphi\left(\frac{x}{p}\right) = \sum_{j=0}^{p-1} c_j \varphi(x-j) \qquad (x \in \mathbb{R}) \tag{1.2}$$

under the condition

$$\sum_{j=0}^{p-1} c_j = 1 \tag{1.3}$$

and we are interested to solutions φ which satisfy the boundary conditions

$$\varphi(x) = 0 \quad \text{for} \quad x < 0, \qquad \varphi(x) = 1 \quad \text{for} \quad x > 1. \tag{1.4}$$

It is easy to see that under these conditions equation (1.2) with (1.3) can be written as system of functional equations. Replacing x in (1.2) by $k + x$ with $k \in \{0, 1, \ldots, p-1\}$ and $x \in [0, 1]$ we get in view of (1.4) the following system of equations

$$\varphi\left(\frac{k+x}{p}\right) = b_k + c_k \varphi(x) \qquad (0 \leq x \leq 1) \tag{1.5}$$

with

$$b_k = \sum_{j=0}^{k-1} c_j \tag{1.6}$$

$k = 0, 1, \ldots, p-1$, cf. [18]. Such systems of equations are intensively investigated by R. Girgensohn, see [14], [15], [16]. If $|c_j| < 1$ for all $j = 0, 1, \ldots, p-1$ then there exists exactly one bounded $\varphi : [0, 1] \mapsto \mathbb{R}$ which satisfies (1.5) with (1.6) and (1.3). This function φ is continuous and given in terms of the p-adic expansion of x by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=0}^{\infty} b_{\xi_n} \prod_{k=1}^{n-1} c_{\xi_k}, \tag{1.7}$$

cf. [14], see also [18, Theorem 2]. In particular, $\varphi(0) = 0$ and $\varphi(1) = 1$ so that φ can be extended by (1.4) to $x \in \mathbb{R}$, and this extended function is a continuous solution of (1.2) and satisfies (1.4). In this sense the two-scale difference equation (1.2) with (1.3) is equivalent to the system of equations (1.5) with (1.6) and (1.3).

The polynomial

$$P(z) = \sum_{j=0}^{p-1} c_j z^j \tag{1.8}$$

with $P(0) \neq 0$ and $P(1) = 1$ is called the *characteristic polynomial* of the equation (1.2). Simple examples are the extended functions of *de Rham* and of *Cantor*.

1. (*De Rham's function*) In case $P(z) = a + (1-a)z$ with $a \in (0, 1)$ equation (1.2) reads

$$\varphi\left(\frac{x}{2}\right) = a\varphi(x) + (1-a)\varphi(x-1) \qquad (x \in \mathbb{R}) \tag{1.9}$$

which in view of (1.4) can be written as system of functional equations

$$\varphi\left(\frac{x}{2}\right) = a\varphi(x), \qquad \varphi\left(\frac{x+1}{2}\right) = a + (1-a)\varphi(x) \tag{1.10}$$

with $0 \leq x \leq 1$ and de Rham's function is the uniquely bounded solution, cf. e.g. [18].

2. (*Cantor's function*) In case $P(z) = (1 + z^2)/2$ equation (1.2) reads

$$\varphi\left(\frac{x}{3}\right) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x-2) \qquad (x \in \mathbb{R}). \tag{1.11}$$

In view of (1.4) this equation can be written as system of equations

$$\varphi\left(\frac{x}{3}\right) = \frac{1}{2}\varphi(x), \qquad \varphi\left(\frac{x+1}{3}\right) = \frac{1}{2}, \qquad \varphi\left(\frac{x+2}{3}\right) = \frac{1}{2} + \frac{1}{2}\varphi(x) \tag{1.12}$$

with $0 \le x \le 1$, and Cantor's function is the unique bounded solution of this system, cf. [21], (see also [20], p. 241).

In case $c_j \ge 0$ for all $j = 0, 1, \ldots, p-1$ one can interpret the coefficients as probabilities $p_j = c_j$ and the solution φ as a distribution function which is a measure-preserving mapping, cf. [4, Section 3]. The figure on p. 37 in [4] shows the graph of φ in case $p = 2$, $p_0 = 0,7$ and $p_1 = 0,3$ (φ is de Rham's function with respect to the parameter $a = 0,7$).

According to (1.4) we are only interested to the solution φ of (1.2) in $[0,1]$. We always assume that $|c_j| < 1$ for all $j = 0, 1, \ldots, p-1$ which guarantees the existence of a continuous solution φ with $\varphi(0) = 0$ and $\varphi(1) = 1$. In the simple case $c_j = \frac{1}{p}$ for all j we have $\varphi(x) = x$ for $x \in [0,1]$. In the following we always exclude this trivial case. We show in this paper that the solution φ of (1.2) with (1.3), (1.4) satisfies some functional relations (Proposition 2.3) and that it has in $[0,1]$ the following properties:

1. If $c_j \ge 0$ for all j then φ is an increasing function (Proposition 2.5).
2. If not $c_j \ge 0$ for all j then in no nonempty subinterval of $[0,1]$ φ has finite variation (Proposition 2.6).
3. If $|c_j| < 1$ for all j then φ is Hölder continuous, i.e.

$$|\varphi(x) - \varphi(y)| \le A|x-y|^\alpha$$

   with the optimal Hölder exponent $\alpha = \min\{-\log_p |c_0|, \ldots, -\log_p |c_{p-1}|\}$ and coefficient A with $1 \le A \le p^{1-\alpha}\frac{p-1}{p^\alpha - 1}$ (Theorem 3.6).
4. If φ is differentiable at the point x_0 then $\varphi'(x_0) = 0$ (Proposition 4.2).
5. If $\min|c_j| \ge \frac{1}{p}$ then φ is nowhere differentiable in [0,1], and if $\min|c_j| < \frac{1}{p}$ then both sets, where φ is differentiable and where φ is not differentiable have positive Hausdorff dimension (Theorem 4.11).
6. If $p\,M_0 < 1$, where $M_0 = |c_0 c_1 \cdots c_{p-1}|^{1/p}$, then φ is differentiable almost everywhere and if $p\,M_0 \ge 1$ then it is almost nowhere differentiable (Theorem 4.12).
7. If $0 \le c_j < 1$ and $\min c_j = 0$ then φ is constant on the components $J_{m,n}$ of an open set $G \subseteq [0,1]$ with Lebesgue measure $|G| = 1$. These intervals can be represented by means of a sequence γ_n (Theorem 5.4, Example 5.6).

Example 1.1 For $0 < a < 1$ the equation

$$\varphi\left(\frac{x}{3}\right) = a\varphi(x) + (1-2a)\varphi(x-1) + a\varphi(x-2) \qquad (x \in \mathbb{R}) \tag{1.13}$$

has a continuous solution φ satisfying (1.4). For $0 \le x \le 1$ we have: φ is increasing for $0 < a < \frac{1}{2}$, φ is Cantor's function for $a = \frac{1}{2}$ and φ does not have finite variation for $\frac{1}{2} < a < 1$. Further, φ is nowhere differentiable for $\frac{2}{3} \le a < 1$. If a_0 is the positive solution of $27a^2(2a-1) = 1$, i.e. $a_0 = 0,5592\ldots$, then φ is differentiable almost everywhere for $0 < a < a_0$ and almost nowhere differentiable for $a_0 \le a < 1$. So it is astonishing that in case $\frac{1}{2} < a < a_0$ the continuous solution φ does not have finite total variation though the derivative vanishes almost everywhere.

Remark 1.2 **1.** Hölder continuity of compactly supported solutions φ of (1.1) are intensive investigated, e.g. for the Hölder exponent there are bounds in terms of the joint spectral radius of two matrices determined of the coefficients c_j, cf. [5, Theorem 4.3], [6], [7].

2. The optimal Hölder exponent $\alpha = \log_3 2$ of Cantor's function is already known from [17].

3. The optimal Hölder exponent $\alpha = \min\{-\log_2 a, -\log_2(1-a)\}$ of de Rham's function was already determined in [2, Section 2]. Remark 2 and Figure 3 in [2] show a comparison with the Hölder exponent $\mu = -\frac{1}{\log 4}\log(2a^2 - 2a + 1)$ obtainable by means of the corresponding joint spectral radius, cf. [6].

2 Functional relations

We start with a replicative relation, cf. [18].

Proposition 2.1 *The solution φ of system (1.5), (1.6) satisfies the replicative relation*

$$\sum_{k=0}^{p-1} \varphi\left(\frac{k+x}{p}\right) = \varphi(x) + C \qquad (x \in [0,1]) \tag{2.1}$$

with the constant

$$C = p - 1 - P'(1). \tag{2.2}$$

Proof: Equation (2.1) follows from (1.5) by summation where $x = 0$ yields for the constant in (2.1)

$$C = \sum_{k=1}^{p-1} \varphi\left(\frac{k}{p}\right).$$

From (1.5) and (1.6) we get

$$\varphi\left(\frac{k}{p}\right) = \sum_{j=0}^{k-1} c_j$$

so that

$$\begin{aligned} C &= (p-1)c_0 + (p-2)c_1 + \ldots + c_{p-2} \\ &= (p-1)(c_0 + \ldots + c_{p-1}) - \{c_1 + 2c_2 + \ldots + (p-1)c_{p-1}\} \\ &= (p-1)P(1) - P'(1). \end{aligned}$$

In view of $P(1) = 1$ it follows (2.2). □

In order to derive further functional relations for the solutions of (1.2) we introduce a sequence $C_k(c)$ depending on an arbitrary parameter $c \neq 0$ as follows: For $j \in \{0, 1, \ldots, p-1\}$ we put $C_j(c) = \frac{c_j}{c}$ where c_j are the coefficients of (1.2) and in general by the recursion:

$$C_{kp+j}(c) = C_k(c)C_j(c) \qquad (k \geq 1, j \in \{0, 1, \ldots, p-1\}). \tag{2.3}$$

Obviously, if k has the p-adic representation

$$k = \sum_{\nu=0}^{n} k_\nu p^\nu, \qquad (k_\nu \in \{0, 1, \ldots, p-1\}) \tag{2.4}$$

then we have the explicit representation

$$C_k(c) = \prod_{j=0}^{p-1} \left(\frac{c_j}{c}\right)^{s_j(k)} \tag{2.5}$$

where $s_j(k)$ denotes the total number of occurrences of the digit j in the p-adic expansion (2.4) of k.

Remark 2.2 We use the parameter c in two cases:

1. In case $c = c_0$ we have $C_0(c_0) = 1$ and from (2.3) it is easy to see that the numbers $C_k := C_k(c_0)$ have the generating function

$$G(z) := \prod_{j=0}^{\infty} \frac{1}{c_0} P\left(z^{p^j}\right) = \sum_{k=0}^{\infty} C_k z^k \tag{2.6}$$

which converges for $|z| < 1$. Let us mention that the unit circle is a natural bound of convergence for G, cf. [12].

2. In Section 3 (Hölder continuity) we put $c = \max\{|c_0|, \ldots, |c_{p-1}|\}$ and so we are able to estimate the Hölder coefficient.

In the following we need the function

$$\varphi^*(x) = 1 - \varphi(1-x) \qquad (x \in \mathbb{R}) \tag{2.7}$$

which is the solution of the reversed two-scale difference equation

$$\varphi^*\left(\frac{x}{p}\right) = \sum_{j=0}^{p-1} c_j^* \varphi^*(x-j) \qquad (x \in \mathbb{R}). \tag{2.8}$$

where

$$c_j^* = c_{p-1-j} \tag{2.9}$$

cf. [3].

Proposition 2.3 *The solution φ of system (1.5) satisfies the functional equations*

$$\varphi\left(\frac{k+t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi(t) \quad (0 \le t \le 1) \tag{2.10}$$

where $n \in \mathbb{N}$, $k = 0, 1, \dots, p^n - 1$, $C_k(c)$ from (2.5) with

$$s_0(k) + \ldots + s_{p-1}(k) = n \tag{2.11}$$

and

$$\varphi\left(\frac{k-t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) - c^n C_{k-1}(c)\varphi^*(t) \quad (0 \le t \le 1) \tag{2.12}$$

for $k = 1, 2, \dots, p^n$ with φ^* from (2.7). Moreover

$$\varphi\left(\frac{k}{p^n}\right) = c^n \sum_{j=0}^{k-1} C_j(c). \tag{2.13}$$

Proof: We prove (2.10) by induction on n. For $n = 1$ the equations (2.10) are equivalent to the system (1.5). If (2.10) with (2.11) in $C_k(c)$ holds for a fixed n then for $\frac{j+t}{p}$ instead of t with $j \in \{0, 1, \dots, p-1\}$ and $0 \le t \le 1$ we have

$$\begin{aligned} \varphi\left(\frac{kp+j+t}{p^{n+1}}\right) &= \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi\left(\frac{j+t}{p}\right) \\ &= \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi\left(\frac{j}{p}\right) + c^{n+1}C_k(c)\frac{c_j}{c}\varphi(t). \end{aligned}$$

For $t = 0$ it follows

$$\varphi\left(\frac{kp+j}{p^{n+1}}\right) = \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi\left(\frac{j}{p}\right)$$

and hence we get in view of (2.3)

$$\varphi\left(\frac{kp+j+t}{p^{n+1}}\right) = \varphi\left(\frac{kp+j}{p^{n+1}}\right) + c^{n+1}C_{kp+j}(c)\varphi(t)$$

with $s_0(kp+j)+\ldots+s_{p-1}(kp+j)=n+1$. Thus (2.10) with (2.11) in $C_k(c)$ is proved by induction. Now (2.13) follows from (2.10) for $t=1$ and $\varphi(1)=1$ by summation. Equation (2.10) with $k-1$ instead of k and $1-t$ instead of t yields in view of (2.7)

$$\begin{aligned}\varphi\left(\frac{k-t}{p^n}\right) &= \varphi\left(\frac{k-1}{p^n}\right)+c^nC_{k-1}(c)\varphi(1-t)\\ &= \varphi\left(\frac{k-1}{p^n}\right)+c^nC_{k-1}(c)-c^nC_{k-1}(c)\varphi^*(t).\end{aligned}$$

For $t=0$ it follows

$$\varphi\left(\frac{k}{p^n}\right)=\varphi\left(\frac{k-1}{p^n}\right)+c^nC_{k-1}(c)$$

and hence (2.12). □

Example 2.4 (*De Rham's function*) In case $P(z)=a+(1-a)z$ we have $p=2$ and equation (1.9), i.e. $c_0=a$, $c_1=1-a$. For $c=a$ we have by (2.5) that $C_k=C_k(a)=q^{s_1(k)}$ with $q=\frac{1-a}{a}$ where $s_1(k)$ denotes the number of ones in the dyadic representation of k, and the generating function (2.6) reads

$$G(z)=\prod_{j=0}^{\infty}\left(1+qz^{2^j}\right)=\sum_{k=0}^{\infty}q^{s_1(k)}z^k. \tag{2.14}$$

Formulas (2.10) and (2.13) yield the known relations

$$\varphi\left(\frac{k+t}{2^n}\right)=\varphi\left(\frac{k}{2^n}\right)+a^nq^{s_1(k)}\varphi(t)\quad(0\le t\le 1)$$

and

$$\varphi\left(\frac{k}{2^n}\right)=a^n\sum_{j=0}^{k-1}q^{s_1(j)}$$

for de Rham's function φ, cf. [1].

Proposition 2.5 *In case $c_j\ge 0$ for all $j=0,1,\ldots,p-1$ the solution φ of (1.5) is an increasing function, and in case $c_j>0$ it is strictly increasing.*

Proof: If $c_j\ge 0$ for all j then we have $0\le c_j<1$ since $c_0>0$, $c_{p-1}>0$ and (1.3). Hence the solution φ is continuous. From (2.10) we get for $n\in\mathbb{N}$ and $k=0,1,\ldots,p^n-1$

$$\varphi\left(\frac{k+1}{p^n}\right)\ge\varphi\left(\frac{k}{p^n}\right)$$

so that the continuous function φ is increasing. In case $c_j>0$ for all j equation (2.10) implies

$$\varphi\left(\frac{k+1}{p^n}\right)>\varphi\left(\frac{k}{p^n}\right)$$

so that indeed φ is strictly increasing in $[0,1]$. □

Proposition 2.6 *If not $c_j \geq 0$ for all $j = 0, 1, \ldots, p-1$ then in no nonempty subinterval of $[0, 1]$ the solution φ of (1.2) has finite total variation.*

Proof: If not $c_j \geq 0$ for all j then owing to (1.3) we have $|c_0| + \ldots + |c_{p-1}| > 1$. From (1.5) we get for $k \in \{0, \ldots, p-1\}$

$$\varphi\left(\frac{k+1}{p}\right) - \varphi\left(\frac{k}{p}\right) = c_k$$

and hence

$$\sum_{k=0}^{p-1} \left|\varphi\left(\frac{k+1}{p}\right) - \varphi\left(\frac{k}{p}\right)\right| = \sum_{k=0}^{p-1} |c_k|$$

and by induction on n

$$\sum_{k=0}^{p^n-1} \left|\varphi\left(\frac{k+1}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right)\right| = \left(\sum_{k=0}^{p-1} |c_k|\right)^n .$$

In view of $|c_0| + |c_1| + \ldots + |c_{p-1}| > 1$ it follows that φ does not have finite total variation in $[0, 1]$. From (2.10) we conclude that this is valid also for the intervals $[\frac{k}{p^n}, \frac{k+1}{p^n}]$ with $n \in \mathbb{N}$ and $k = 0, 1, \ldots, p^n - 1$. $\square$

3 Hölder continuity

We assume that $|c_j| < 1$ for all $j = 0, 1, \ldots, p-1$ so that the solution φ of (1.2) is continuous. In order to verify the Hölder continuity of φ we introduce the notation

$$S_k(c) := \sum_{j=0}^{k-1} C_j(c) \tag{3.1}$$

for the sum in (2.13), i.e. we have

$$\varphi\left(\frac{k}{p^n}\right) = c^n S_k(c). \tag{3.2}$$

Lemma 3.1 *The sequence $S_k(c)$ has following properties:*

(i) $S_{pk}(c) = \frac{1}{c} S_k(c) \quad (k \geq 1)$.

(ii) $S_{p^n}(c) = \frac{1}{c^n} \quad (n \geq 0)$.

(iii) $S_{kp^n+\ell}(c) = S_{p^n}(c) S_k(c) + C_k(c) S_\ell(c) \quad (0 \leq k < p, \quad n \geq 1, \quad 0 \leq \ell < p^n)$.

Proof: **(i)** For given $k \geq 1$ we choose n such that $k < p^{n-1}$. From (2.13) and (3.1) we get

$$S_{pk}(c) = \frac{1}{c^n}\varphi\left(\frac{pk}{p^n}\right) = \frac{1}{c^n}\varphi\left(\frac{k}{p^{n-1}}\right) = \frac{1}{c}S_k(c)$$

which implies **(i)**.

(ii) follows from (2.13) and $\varphi(1) = 1$.

(iii) From (2.10) and (2.13) we get

$$\varphi\left(\frac{k + \frac{\ell}{p^n}}{p}\right) = \varphi\left(\frac{k}{p}\right) + c\,C_k(c)\varphi\left(\frac{\ell}{p^n}\right) = c\,S_k(c) + c^{n+1}C_k(c)S_\ell(c).$$

On the other side we have

$$\varphi\left(\frac{kp^n + \ell}{p^{n+1}}\right) = c^{n+1}S_{kp^n+\ell}(c)$$

and in view of **(ii)** it follows **(iii)**. □

Now we choose the parameter $c = \mathbf{c}$ where

$$\mathbf{c} := \max\{|c_0|, |c_1|, \ldots, |c_{p-1}|\}, \tag{3.3}$$

cf. Remark 2.2. Then $|C_k(\mathbf{c})| \leq 1$ for $k \in \{0, 1, \ldots, p-1\}$ and (2.3) implies

$$|C_k(\mathbf{c})| \leq 1 \qquad (k \in \mathbb{N}_0). \tag{3.4}$$

In view of (1.3) we have $\frac{1}{p} \leq \mathbf{c} < 1$. In case $\mathbf{c} = \frac{1}{p}$ we have $c_j = \frac{1}{p}$ for all $j = 0, 1, \ldots, p-1$ and $\varphi(x) = x$ for $0 \leq x \leq 1$. If we exclude this trivial case then

$$\frac{1}{p} < \mathbf{c} < 1. \tag{3.5}$$

For the parameter $\mathbf{c}$ from (3.3) satisfying (3.5) we put

$$\alpha := -\log_p \mathbf{c}, \tag{3.6}$$

i.e.

$$\mathbf{c}\,p^\alpha = 1 \tag{3.7}$$

and (3.5) implies

$$0 < \alpha < 1. \tag{3.8}$$

Lemma 3.2 *With α from (3.6) and $\mathbf{c}$ from (3.3) the sequence $\frac{1}{k^\alpha}S_k(\mathbf{c})$ is bounded. More precisely, for*

$$K := \sup_k \left\{\left|\frac{1}{k^\alpha}S_k(\mathbf{c})\right|\right\} \tag{3.9}$$

we have the estimate

$$1 \leq K \leq \frac{p-1}{p^\alpha - 1}. \tag{3.10}$$

Proof: According to Lemma 3.1/**(ii)** and (3.7) we have

$$\frac{1}{p^\alpha} S_p(\mathbf{c}) = 1$$

and hence $K \geq 1$. Moreover, by Lemma 3.1/**(i)** and (3.7)

$$\frac{1}{(pk)^\alpha} S_{pk}(\mathbf{c}) = \frac{1}{k^\alpha} S_k(\mathbf{c}) \tag{3.11}$$

so that

$$\sup_k \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| = \limsup_{k\to\infty} \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right|. \tag{3.12}$$

For integer $n \geq 1$ let be

$$K_n := \max\left\{ \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| : p^{n-1} \leq k \leq p^n - 1 \right\}$$

then by (3.11) we have $K_n \leq K_{n+1}$.

Owing to Lemma 3.1/**(iii)** and to (3.11) we have

$$\frac{1}{(kp^n+\ell)^\alpha} S_{kp^n+\ell}(\mathbf{c}) = \left(\frac{kp^n}{kp^n+\ell} \right)^\alpha \frac{1}{k^\alpha} S_k(\mathbf{c}) + \left(\frac{\ell}{kp^n+\ell} \right)^\alpha \frac{C_k(\mathbf{c})}{\ell^\alpha} S_\ell(\mathbf{c}) \tag{3.13}$$

for $k = 1, \ldots, p-1$ and $\ell = 0, 1, \ldots, p^n - 1$.

Hence for $m = kp^n + \ell$ with $k \in \{1, \ldots, p-1\}$ and $\ell \in \{0, 1, \ldots, p^n - 1\}$ we have

$$\frac{1}{m^\alpha} S_m(\mathbf{c}) = (1-\xi)^\alpha \frac{1}{k^\alpha} S_k(\mathbf{c}) + \xi^\alpha C_k(\mathbf{c}) \frac{1}{\ell^\alpha} S_\ell(\mathbf{c}) \tag{3.14}$$

where $\xi = \frac{\ell}{kp^n+\ell}$ with a certain $\ell \in \{0, 1, \ldots, p^n - 1\}$ so that $0 \leq \xi < \frac{1}{k+1}$. By (3.14) and (3.4) we get

$$\begin{aligned} K_{n+1} &\leq (1-\xi)^\alpha \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| + \xi^\alpha |C_k(\mathbf{c})| \left| \frac{1}{\ell^\alpha} S_\ell(\mathbf{c}) \right| \\ &\leq (1-\xi)^\alpha \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| + \xi^\alpha K_n \end{aligned}$$

where $k \in \{1, \ldots, p-1\}$, $\ell \leq p^n$ and in view of $K_n \leq K_{n+1}$ it follows

$$(1-\xi^\alpha) K_n \leq (1-\xi)^\alpha \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right|.$$

Note $1 - \xi^\alpha > 0$ since $0 \leq \xi < \frac{1}{k+1}$ and $\alpha > 0$, cf. (3.8). Consequently,

$$K_n \leq \frac{(1-\xi)^\alpha}{1-\xi^\alpha} \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| \leq M_k \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| \qquad (k \in \{1, \ldots, p-1\}) \tag{3.15}$$

with

$$M_k = \max_{0 \le x \le \frac{1}{k+1}} \frac{(1-x)^\alpha}{1-x^\alpha}.$$

In view of (3.8) the function $f(x) = (1-x)^\alpha/(1-x^\alpha)$ is increasing so that we get $M_k = f(\frac{1}{k+1}) = \frac{k^\alpha}{(k+1)^\alpha - 1}$ and

$$K_n \le \frac{1}{(k+1)^\alpha - 1}|S_k(\mathbf{c})| \qquad (k \in \{1, \ldots, p-1\}).$$

From (3.1) we get in view of $|C_j(\mathbf{c})| \le 1$ that $|S_k(\mathbf{c})| \le k$ so that

$$K_n \le \frac{k}{(k+1)^\alpha - 1} \qquad (k \in \{1, \ldots, p-1\}).$$

The function $g(x) = \frac{x}{(x+1)^\alpha - 1}$ is increasing in $[1, p-1]$ so that $K_n \le g(p-1) = \frac{p-1}{p^\alpha - 1}$ which yields the assertion. □

Remark 3.3 If we carry out the foregoing considerations with the coefficient c_j^* of the reversed equation (2.8) instead of c_j then in view of (2.9) and (3.3) we have $\mathbf{c}^* = \max\{|c_0^*|, \ldots, |c_{p-1}^*|\} = \mathbf{c}$, and hence with the same α from (3.6) we find that the corresponding coefficients $C_j^*(\mathbf{c})$ satisfy $|C_j^*(\mathbf{c})| \le 1$ and that the sums $\frac{1}{k^\alpha} S_k^*(\mathbf{c})$ are bounded where

$$K^* := \sup_k \left| \frac{1}{k^\alpha} S_k^*(\mathbf{c}) \right| \tag{3.16}$$

can be estimates similarly as in (3.10). So

$$K^* \le \frac{p-1}{p^\alpha - 1}. \tag{3.17}$$

Lemma 3.4 *If $|c_j| < 1$ for all $j \in \{0, 1, \ldots, p-1\}$ then for $0 \le t \le 1$, $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p-1\}$ we have*

$$\left| \varphi\left(\frac{k+t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right) \right| \le K \left(\frac{t}{p^n}\right)^\alpha \tag{3.18}$$

and for $k \in \{1, 2, \ldots, p\}$

$$\left| \varphi\left(\frac{k-t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right) \right| \le K^* \left(\frac{t}{p^n}\right)^\alpha. \tag{3.19}$$

Proof: We only prove (3.18). For $t = \frac{k}{p^n}$ with $0 \le k \le p^n$ the representation (2.13) with $c = \mathbf{c}$ implies

$$\frac{\varphi(t)}{t^\alpha} = \frac{\varphi(\frac{k}{p^n})}{(\frac{k}{p^n})^\alpha} = \frac{1}{k^\alpha} \sum_{j=0}^{k-1} C_j(\mathbf{c}) = \frac{1}{k^\alpha} S_k(\mathbf{c})$$

in view of (3.7). By Lemma 3.2 it follows

$$\frac{|\varphi(t)|}{t^\alpha} \le K$$

for these t and hence also for arbitrary $t \in (0,1]$ by continuity. By (2.13) with $c = \mathbf{c}$ we have in view of (3.7)

$$\varphi\left(\frac{k+t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right) = \frac{1}{p^{\alpha n}} C_k(\mathbf{c})\varphi(t)$$

and using (3.4) we get

$$\left|\varphi\left(\frac{k+t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right)\right| \le \left(\frac{t}{p^n}\right)^\alpha \frac{|\varphi(t)|}{t^\alpha} \le \left(\frac{t}{p^n}\right)^\alpha K.$$

In the same way using (2.7) it follows (3.19). □

Proposition 3.5 *If $|c_j| < 1$ for $j = 0, \ldots, p-1$ then for arbitrary $x, y \in [0,1]$ the solution φ satisfies the inequality*

$$|\varphi(x) - \varphi(y)| \le \frac{p^{1-\alpha}(p-1)}{p^\alpha - 1}|x-y|^\alpha$$

with α from (3.6).

Proof: For given $x, y \in [0,1]$ with $h = y - x > 0$ we assume that

$$\frac{1}{p^n} \le h < \frac{1}{p^{n-1}}.$$

Let be $k = [p^n x]$ and $t_\mu = \frac{k+\mu}{p^n}$ $(\mu = 0, 1, \ldots)$. Then we have

$$t_0 \le x < t_1 < \ldots < t_m < y \le t_{m+1}$$

where $1 \le m \le p-1$ since $t_1 = \frac{k+1}{p^n} \le x + \frac{1}{p^n} \le x + h = y$ and $t_{p+1} = \frac{k+p+1}{p^n} > x + \frac{1}{p^{n-1}} > x + h = y$. We use

$$|\varphi(y) - \varphi(x)| \le |\varphi(t_1) - \varphi(x)| + |\varphi(y) - \varphi(t_m)| + \sum_{\mu=2}^{m-1} |\varphi(t_{\mu+1}) - \varphi(t_\mu)|.$$

We denote $a_1 = t_1 - x$, $a_k = t_k - t_{k-1}$ for $k = 2, \ldots, m-1$, and $a_m = y - t_m$ then $a_1 + \ldots + a_m = y - x$ and by Lemma 3.4

$$|\varphi(y) - \varphi(x)| \le K^* a_1^\alpha + K \sum_{\mu=2}^{m} a_\mu^\alpha \le K_{\max}\left(a_1^\alpha + \ldots + a_m^\alpha\right)$$

with $K_{\max} := \max\{K, K^*\}$. According to (3.8) the function $t \mapsto t^\alpha$ is concave and applying Jensen's inequality

$$\frac{a_1^\alpha + \ldots + a_m^\alpha}{m} \le \left(\frac{a_1 + \ldots + a_m}{m}\right)^\alpha$$

we find in view of $m \le p$ and (3.8)

$$|\varphi(y) - \varphi(x)| \le K_{\max} m^{1-\alpha}(y-x)^\alpha \le K_{\max} p^{1-\alpha}(y-x)^\alpha.$$

Finally, from (3.9) and (3.17) we get

$$K_{\max} p^{1-\alpha} \le p^{1-\alpha}\frac{p-1}{p^\alpha - 1}$$

and the proposition is proved. □

Now we know that φ is Hölder continuous with exponent α from (3.6). Next we show that α is the optimal Hölder exponent and we determine also the optimal Hölder coefficient.

Theorem 3.6 *If $|c_j| < 1$ for $j = 0, \ldots, p-1$ then the solution φ of the equation (1.2) is Hölder continuous with the optimal Hölder exponent α from (3.6), i.e.*

$$\alpha = \min\left\{-\log_p |c_0|, \ldots, -\log_p |c_{p-1}|\right\}$$

where $0 < \alpha < 1$, cf. (3.8), and the optimal Hölder coefficient

$$A := \sup_{k,\ell} \frac{1}{k^\alpha}\left|\sum_{j=0}^{k-1} C_{\ell+j}(\mathbf{c})\right| \tag{3.20}$$

which satisfies

$$1 \le A \le \frac{p^{1-\alpha}(p-1)}{p^\alpha - 1}, \tag{3.21}$$

i.e. we have

$$|\varphi(x) - \varphi(y)| \le A\,|x-y|^\alpha \tag{3.22}$$

for arbitrary $x, y \in [0,1]$.

Proof: **1.** First we show (3.22) with α from (3.6) and A from (3.20). For $y = \frac{\ell}{p^n}$ and $x = y + \frac{k}{p^n}$ with $0 \le \ell < k+\ell \le p^n$ the representation (2.13) with $c = \mathbf{c}$ implies

$$\frac{\varphi(x)-\varphi(y)}{(x-y)^\alpha} = \frac{\varphi(\frac{k+\ell}{p^n}) - \varphi(\frac{\ell}{p^n})}{(\frac{k}{p^n})^\alpha} = \frac{1}{k^\alpha}\sum_{j=\ell}^{k+\ell-1} C_j(\mathbf{c})$$

in view of (3.7). Hence, we get (3.22) for p-adic rational $x, y \in [0,1]$ where A is finite by Proposition 3.5. Continuity of φ implies that (3.22) is valid for all x and y in $[0,1]$.

2. We show that α is the optimal Hölder exponent. Assume that φ is Hölder continuous with an exponent $\beta > \alpha$, i.e. for all $x, y \in [0,1]$ we have

$$|\varphi(x) - \varphi(y)| \le B|x-y|^\beta \tag{3.23}$$

with a certain constant B.

From (1.5) we get for $k = 0, 1, \ldots, p-1$ by induction on n that

$$\varphi\left(\frac{k(p^n-1)+t(p-1)}{p^n(p-1)}\right) = b_k \sum_{\nu=0}^{n-1} c_k^\nu + c_k^n \varphi(t) \qquad (0 \le t \le 1).$$

Putting $t = 0$ and $t = 1$ we get in view of $\varphi(0) = 0$ and $\varphi(1) = 1$ that

$$\varphi\left(\frac{k(p^n-1)+p-1}{p^n(p-1)}\right) - \varphi\left(\frac{k(p^n-1)}{p^n(p-1)}\right) = c_k^n. \tag{3.24}$$

Now we choose $k \in \{0, 1, \ldots, p-1\}$ such that $|c_k| = \mathbf{c}$, cf. (3.3). In (3.24) we put $y = \frac{k(p^n-1)}{p^n(p-1)}$, $x = y + \frac{1}{p^n}$ and obtain in view of $x - y = \frac{1}{p^n}$, $|c_k| = \mathbf{c}$ and (3.6) that

$$|\varphi(x) - \varphi(y)| = (x-y)^\alpha.$$

According to (3.23) we get

$$\left(\frac{1}{p^n}\right)^\alpha \le B \left(\frac{1}{p^n}\right)^\beta,$$

i.e. $p^{n(\beta-\alpha)} \le B$, which yields a contradiction for large n. Hence, α is the optimal Hölder exponent and it follows that A from (3.20) is the optimal Hölder coefficient. The estimate $A \ge 1$ follows from (3.22) with $x = 0$, $y = 1$ in view of $\varphi(0) = 0$, $\varphi(1) = 1$. The above estimate of A follows from Proposition 3.5. $\square$

Remark 3.7 Note that in limit case $\alpha = 1$ we get $A = 1$ in accordance with $\varphi(x) = x$ for $0 \le x \le 1$.

A detail discussion of the Hölder continuity of de Rham's function and of solutions of certain two-scale difference equations you can find in [2, Section 2 and Section 5.2]. In [11, Proposition 10.1] it was shown the Hölder continuity of Cantor's function with optimal exponent $\alpha = \frac{\log 2}{\log 3}$ and coefficient $A = 1$.

4 Differentiability

As before we exclude the case $c_j = \frac{1}{p}$ for all $j \in \{0, 1, \ldots, p-1\}$ where $\varphi(x) = x$ for $0 \le x \le 1$. First we give a general statement on the differentiability.

4.1 General statements

We start with the following simple lemma, cf. [15].

Lemma 4.1 *Let $f : [0,1] \mapsto \mathbb{R}$ have a finite right-hand derivative $f'_+(x_0)$ at the point $x_0 \in [0,1)$. If (u_n) and (v_n) are sequences in $[0,1]$ such that $x_0 < u_n < v_n$, $v_n \to x_0$ and $u_n - x_0 \le L(v_n - u_n)$ with a constant L then*

$$\frac{f(v_n) - f(u_n)}{v_n - u_n} \to f'_+(x_0) \qquad (n \to \infty).$$

Proposition 4.2 *If the solution φ of (1.2) is differentiable at x_0 then $\varphi'(x_0) = 0$.*

Proof: Assume, at $x_0 \in [0,1)$ there exists the finite derivative $\varphi'(x_0) \neq 0$. For $n \in \mathbb{N}$ and $k = 0, 1, \dots, p^n - 1$ we put $x_{k,n} = \frac{k}{p^n}$ and $N_{a,b} = \{k \in \mathbb{N} : a \le k \le b\}$. If $x_{k',n} \le x_0 < x_{k'+1,n}$ then for each $k \in N_{k'+1,k'+2p-1}$ we put $u_{k,n} = x_{k,n}$ and $v_{k,n} = x_{k+1,n}$ so that $x_0 < u_{k,n} < v_{k,n}$ and $u_{k,n} - x_0 \le p(v_{k,n} - u_{k,n})$. Applying (2.10) with $t = 1$ we get

$$\frac{\varphi(v_{k,n}) - \varphi(u_{k,n})}{v_{k,n} - u_{k,n}} = p^n c_0^n C_k$$

just as

$$\frac{\varphi(v_{k+1,n}) - \varphi(u_{k+1,n})}{v_{k+1,n} - u_{k+1,n}} = p^n c_0^n C_{k+1}.$$

In view of $\varphi'_+(x_0) \neq 0$ it follows by Lemma 4.1 that for $k \in N_{k'+1,k'+p}$ we have

$$\frac{C_{k+1}}{C_k} \to 1 \qquad (n \to \infty).$$

The set $N_{k'+1,k'+2p-1}$ contains a section of the form $N_{d,d+p-2}$ with $d = pk_0 < k' + p$. For $k \in N_{d,d+p-2}$, i.e. $k = pk_0 + j$ with $j = 0, 1, \dots, p-2$ we have by (2.3) with $c = c_0$ that $C_k = C_{pk_0+j} = \frac{c_j}{c_0} C_{k_0}$ and it follows

$$\frac{c_{j+1}}{c_j} \to 1 \qquad (n \to \infty),$$

i.e. $c_{j+1} = c_j$. So by (1.3) it follows $c_j = \frac{1}{p}$ for $j = 0, \dots, p-1$. □

Proposition 4.3 *The set E of points $x \in [0,1]$ where φ is differentiable has the Lebesgue measure 0 or 1.*

Proof: The set E is Lebesgue measurable with the measure $|E|$. We show that E is homogeneous, that means for each nonempty interval $[a,b]$ in $[0,1]$ we have $|E \cap [a,b]| = (b-a)|E|$. Equation (2.10) with $c = c_0$, $C_k = C_k(c_0)$ implies

$$\frac{1}{p^n} \varphi'\left(\frac{k+t}{p^n}\right) = c_0^n C_k \varphi'(t) \quad (t \in E).$$

Put $E_{k,n} := E \cap [\frac{k}{p^n}, \frac{k+1}{p^n}]$ we have $|E_{k,n}| = |E_{k',n}|$ $(0 \le k, k' < p^n)$ and hence

$$E = \bigcup_{k=0}^{p^n-1} E_{k,n}$$

implies $|E_{k,n}| = \frac{1}{p^n}|E|$. It follows $|E \cap [a,b]| = (b-a)|E|$ for each interval $[a,b] \subset [0,1]$ and hence $|E| = 0$ or $|E| = 1$ by a theorem of Lebesgue. □

4.2 Special difference quotients

Now, for given $x \in [0,1]$ we investigate the special difference quotients

$$\Delta_n(x) := \frac{\varphi(\frac{k+1}{p^n}) - \varphi(\frac{k}{p^n})}{1/p^n} \tag{4.1}$$

with $k = [p^n x]$, i.e.

$$\frac{k}{p^n} \le x < \frac{k+1}{p^n}. \tag{4.2}$$

Applying (2.10) with $c = c_0$ and $t = 1$ we get in view of $\varphi(1) = 1$

$$\Delta_n(x) = p^n c_0^n C_k. \tag{4.3}$$

In order to get a suitable representation for C_k we need a mean value M. For $\lambda_j \in [0,1]$ with $\lambda_0 + \cdots + \lambda_{p-1} = 1$ we introduce the mean value $M = M(\lambda_0, \ldots, \lambda_{p-1})$ by

$$M := \prod_{j=0}^{p-1} |c_j|^{\lambda_j}. \tag{4.4}$$

Lemma 4.4 *Let $x \in [0,1]$ and $n \in \mathbb{N}$. Then for $C_k = C_k(c_0)$ from (2.5) with $k = [p^n x]$ we have*

$$C_k = \frac{1}{c_0^n} \prod_{j=0}^{p-1} c_j^{s_j(k)} \tag{4.5}$$

with

$$s_0(k) + s_1(k) + \ldots + c_{p-1}(k) = n \tag{4.6}$$

where $s_j(k)$ is the number of the digit j in the p-adic representation of k. Further

$$|C_k|^{1/n} = \frac{1}{|c_0|}\, e_n(x)\, M(\lambda_0, \ldots, \lambda_{p-1}) \tag{4.7}$$

where

$$e_n(x) := \prod_{j=0}^{p-1} |c_j|^{\varepsilon_j(n)} \tag{4.8}$$

with $\varepsilon_j(n) = \frac{1}{n} s_j(k) - \lambda_j$.

Proof: If $x = 0, \xi_1 \xi_2 \ldots$ is the p-adic expansion of x then $k = k(n) = [p^n x]$ has the form $k = \xi_n + \xi_{n-1} p + \ldots + \xi_1 p^{n-1}$. So $s_0(k) + \ldots + s_{p-1}(k) = n$ and by (2.5) we get

$$C_k = \prod_{j=0}^{p-1} \left(\frac{c_j}{c_0}\right)^{s_j(k)} = \frac{1}{c_0^n} \prod_{j=0}^{p-1} c_j^{s_j(k)},$$

i.e. (4.5) is proved. Formula (4.7) with (4.8) is a simple consequence of (4.5). □

Lemma 4.5 *Let $x \in [0,1]$ and $n \in \mathbb{N}$. Then for $\Delta_n(x)$ from (4.1) we have*

$$|\Delta_n(x)| = \prod_{j=0}^{p-1} a_j^{s_j(k)}$$

with $a_j = p\,|c_j|$ so that $a_0 a_1 \dots a_{p-1} = 1$.

Proof: Formulas (4.3) and (4.5) imply

$$\Delta_n(x) = p^n \prod_{j=0}^{p-1} c_j^{s_j(k)}.$$

In view of (4.6) it follows

$$\Delta_n(x) = \prod_{j=0}^{p-1} (p\,c_j)^{s_j(k)}$$

which proved the assertion. □

Next we consider special sets of real numbers, cf. [13, Chapter 10]. Let $x = 0, \xi_1 \xi_2 \dots$ be the representation of a number $x \in (0,1)$ to the base p and $d_j(x|_n)$ the total number of occurrence of the digit $j \in \{0, 1, \dots, p-1\}$ in the first n places $0, \xi_1 \dots \xi_{n-1}$. That means

$$d_j(x|_n) = s_j(k) \tag{4.9}$$

where $k = [p^n x]$. For $\lambda_j \in [0,1]$ with $\lambda_0 + \ldots + \lambda_{p-1} = 1$ let $F = F(\lambda_0, \dots, \lambda_{p-1})$ be the set

$$F := \left\{ x \in [0,1] : \lim_{n \to \infty} \frac{1}{n} d_j(x|_n) = \lambda_j \qquad \forall \quad j = 0, 1, \dots, p-1 \right\}. \tag{4.10}$$

It is known that F has the Hausdorff dimension

$$dim_H F = -\frac{1}{log\, p} \sum_{j=0}^{p-1} \lambda_j \log \lambda_j \tag{4.11}$$

with the convention $0 \log 0 = 0$, cf. [13]. Further, the numbers $x \in F(p^{-1}, \dots, p^{-1})$ are called *normal numbers* with respect to the base p and Borel's normal number theorem says that $F(p^{-1}, \dots, p^{-1})$ is a set of Lebesgue measure 1.

The following proposition is the basis for the investigation of φ concerning the differentiability.

Proposition 4.6 *For $x \in F(\lambda_0, \dots, \lambda_{p-1})$ we have*

$$\lim_{n \to \infty} |\Delta_n(x)|^{1/n} = p\, M(\lambda_0, \dots, \lambda_{p-1}) \tag{4.12}$$

with M from (4.4).

Proof: By Lemma 4.4 and (4.3) we get

$$|\Delta_n(x)|^{1/n} = p\, e_n(x)\, M(\lambda_0, \dots, \lambda_{p-1}).$$

Using (4.9) for $x \in F(\lambda_0, \dots, \lambda_{p-1})$ we have $\frac{1}{n}s_j(k) = \frac{1}{n}d_j(x|_n) \to \lambda_j$ as $n \to \infty$. Hence $e_n(x) \to 1$ as $n \to \infty$ and it follows the assertion. □

4.3 The case $p\,M < 1$

We need further lemmata.

Lemma 4.7 *Let be $x \in F(\lambda_0, \dots, \lambda_{p-1})$ with the p-adic expansion $x = 0, \xi_1\xi_2 \dots$ where $\xi_{n-j} = p-1$ for $j = 1, 2, \dots, r_n$. If $\lambda_{p-1} < 1$ then $\frac{r_n}{n} \to 0$ as $n \to \infty$.*

Proof: For $x \in F$ we have $\frac{1}{n-r_n}d_{p-1}(x|_{n-r_n}) \to \lambda_{p-1}$ and $\frac{1}{n}d_{p-1}(x|_n) \to \lambda_{p-1}$ as $n \to \infty$. By supposition we have $d_{p-1}(x|_n) = d_{p-1}(x|_{n-r_n}) + r_n$ and hence

$$\frac{1}{n}d_{p-1}(x|_n) = \frac{n-r_n}{n}\frac{1}{n-r_n}d_{p-1}(x|_{n-r_n}) + \frac{r_n}{n}.$$

Certainly $0 \le \frac{r_n}{n} \le 1$, i.e. the sequence $\frac{r_n}{n}$ is bounded. If s is the limit of a convergent subsequence then in view of (4.10) it follows $\lambda_{p-1} = (1-s)\lambda_{p-1} + s$, i.e $(1-\lambda_{p-1})s = 0$ and hence $s = 0$ since $\lambda_{p-1} < 1$. So $\frac{r_n}{n} \to 0$ as $n \to \infty$. □

Lemma 4.8 *Let be $x \in F(\lambda_0, \dots, \lambda_{p-1})$ with $\lambda_{p-1} < 1$ and $k = k(n) = [p^n x]$ then for $\mu = 0, 1, \dots, p$ we have*

$$\lim_{n\to\infty} \left|\frac{C_{k(n)+\mu}}{C_{k(n)}}\right|^{1/n} = 1$$

with $C_k = C_k(c_0)$ from (2.5).

Proof: Let be $x = 0, \xi_1\xi_2 \dots$ the p-dic expansion of x then $k = k(n) = [p^n x]$ has the form $k = \xi_n + \xi_{n-1}p + \dots + \xi_1 p^{n-1}$. For $\mu \ge 0$ with $\xi_{n-1} \le \xi_{n-1} + \mu < p$ we have $k + \mu = (\xi_n + \mu) + \xi_{n-1}p + \dots + \xi_1 p^{n-1}$ and by (2.5) it holds

$$C_{k+\mu} = C_k \frac{C_{\xi_n+\mu}}{C_{\xi_n}}.$$

Now we consider $\mu \in \{1, \dots, p\}$ with $p \le \xi_{n-1} + \mu \le 2p-1$. If $\xi_{n-1} < p-1$ then we have $k + \mu = (\xi_n + \mu - p) + (\xi_{n-1}+1)p + \xi_{n-2}p^2 + \dots + \xi_1 p^{n-1}$ and by (2.5) it holds

$$C_{k+\mu} = C_k \frac{C_{\xi_n+\mu-p}}{C_{\xi_n}} \frac{C_{\xi_{n-1}+1}}{C_{\xi_{n-1}}}.$$

If $\xi_{n-j} = p-1$ for $j = 1, \ldots, n_r$ and $\xi_{n-n_r-1} < p-1$ then k has the representation

$$k = \xi_n + (p-1)p + \ldots + (p-1)p^{r_n} + \xi_{n-n_r-1}p^{r_n+1} + \ldots + \xi_1 p^{n-1}$$

and we have

$$k + \mu = (\xi_n + \mu - p) + (\xi_{n-r_n-1} + 1)p^{r_n+1} + \ldots + \xi_1 p^{n-1}.$$

According to (2.5) we get

$$C_{k+\mu} = C_k \frac{C_{\xi_n+\mu-p}}{C_{\xi_n}} \left(\frac{C_{p-1}}{C_0}\right)^{r_n} \frac{C_{\xi_{n-r_n-1}+1}}{C_{\xi_{n-r_n}}}.$$

Put $C = \max\{|C_i|/|C_j|\}$ $(i, j = 0, 1, \ldots, p-1)$ then we get

$$\frac{1}{C^{r_n+2}} \leq \left|\frac{C_{k+\mu}}{C_k}\right| \leq C^{r_n+2}$$

and in view of Lemma 4.7 it follows the assertion. □

Proposition 4.9 *If $p\,M(\lambda_0, \ldots, \lambda_{p-1}) < 1$ where $\lambda_{p-1} < 1$ then the solution φ of (1.2) is differentiable at each point $x \in F(\lambda_0, \ldots, \lambda_{p-1})$ with $\varphi'(x) = 0$.*

Proof: Choose $\varepsilon > 0$ so that

$$q := (1+\varepsilon)^3 \, p\, M(\lambda_0, \ldots, \lambda_{p-1}) < 1.$$

For fixed $x \in F(\lambda_0, \ldots, \lambda_{p-1})$ it holds $e_n(x) \to 1$ as $n \to \infty$, cf. (4.8), (4.10) and (4.9). Hence there is a number n_0 such that for $n \geq n_0$ we have

$$e_n(x) < 1 + \varepsilon, \tag{4.13}$$

$$K^{1/n} < 1 + \varepsilon \tag{4.14}$$

with $K = \max|\varphi(t)|$ for $0 \leq t \leq 1$ and by Lemma 4.8

$$|C_{k+\mu}|^{1/n} < (1+\varepsilon)|C_k|^{1/n} \tag{4.15}$$

where $k = [p^n x]$. Now, let $y = x + h < 1$ with $h > 0$ (the case $h < 0$ is analogous) and

$$\frac{1}{p^n} \leq h < \frac{1}{p^{n-1}}$$

with $n \geq n_0$. Note that $h \to 0$ is equivalent to $n \to \infty$. Put $t_\mu = \frac{k+\mu}{p^n}$ $(\mu = 0, 1, \ldots)$ then we have $t_0 < x < t_1 < \ldots < t_m < x + h \leq t_{m+1}$ where $1 \leq m \leq p-1$ since $t_1 = \frac{k+1}{p^n} \leq x + \frac{1}{p^n} \leq x + h$ and $t_{p+1} = \frac{k+p+1}{p^n} > x + \frac{1}{p^{n-1}} > x + h$. We use

$$|\varphi(x+h) - \varphi(x)| \leq |\varphi(x+h) - \varphi(t_m)| + |\varphi(t_1) - \varphi(x)| + \sum_{\mu=1}^{m-1} |\varphi(t_{\mu+1}) - \varphi(t_\mu)|.$$

Put $x = \frac{k+1-t}{p^n}$ with suitable $0 \le t < 1$ then by (2.10) (with $a = c_0$, $C_k = C_k(c_0)$) we have

$$\varphi(x) - \varphi(t_1) = c_0^n C_{k+1} \varphi(1-t)$$

and hence

$$\frac{|\varphi(t_1) - \varphi(x)|}{h} = \frac{1}{h} |c_0|^n |C_{k+1}| |\varphi(1-t)| \le p^n |c_0|^n |C_{k+1}| K$$

where $K = \max |\varphi|$. Applying Lemma 4.4 we get

$$p|c_0||C_{k+1}|^{1/n} K^{1/n} = p|c_0||C_k|^{1/n} \left| \frac{C_{k+1}}{C_k} \right|^{1/n} K^{1/n} = pMe_n(x) \left| \frac{C_{k+1}}{C_k} \right|^{1/n} K^{1/n}$$

and using (4.13), (4.14) and (4.15) it follows

$$p|c_0||C_{k+1}|^{1/n} K^{1/n} < (1+\varepsilon)^3 pM = q$$

so that

$$\frac{|\varphi(x) - \varphi(t_1)|}{h} < q^n. \tag{4.16}$$

Since $t_m < x + h \le t_{m+1}$ we have $x + h = \frac{k+m+\tau}{p^n}$ with suitable $0 < \tau \le 1$ and by (2.10)

$$\varphi(x+h) - \varphi(t_m) = c_0^n C_{k+m} \varphi(\tau).$$

Therefore

$$\frac{|\varphi(x+h) - \varphi(t_m)|}{h} \le \frac{1}{h} |c_0|^n |C_{k+m}| K \le p^n |c_0|^n |C_{k+m}| K < q^n \tag{4.17}$$

where we have again used (4.13), (4.14) and (4.15).

Moreover, by (2.10) it holds

$$\varphi(t_{\mu+1}) - \varphi(t_\mu) = c_0 C_{k+\mu}$$

and hence again

$$\frac{|\varphi(t_{\mu+1}) - \varphi(t_\mu)|}{h} = \frac{1}{h} |c_0|^n |C_{k+\mu}| \le p^n |c_0|^n |C_{k+\mu}| < q^n. \tag{4.18}$$

Form (4.16), (4.17) and (4.18) it follows in view of $m \le p - 1$

$$\frac{|\varphi(x+h) - \varphi(x)|}{h} < (p+1) q^n.$$

This implies $\varphi'_+(x) = 0$. In the same way $\varphi'_-(x) = 0$. □

4.4 The case $p\,M = 1$

We investigate $\Delta_n(x)$ from (4.1) under the condition

$$p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} = 1. \tag{4.19}$$

The following proof due to A. Meister (personal communication).

Lemma 4.10 *Assume that it holds (4.19) and that $a_j = p|c_j|$ for $j = 0, 1, \ldots, p-1$. If not $a_0 = a_1 = \ldots = a_{p-1} = 1$ then the set of x with the property $\Delta_n(x) \to 0$ as $n \to \infty$ has the measure zero.*

Proof: Let $x = 0, \xi_1 \xi_2 \ldots$ where the digits ξ_j are independent and identically distributed on the discrete set $\{0, 1, \ldots, p-1\}$. Since

$$d_j(x|_n) = \sum_{k=1}^{n} \chi_j(\xi_k)$$

we have by Lemma 4.5 and (4.9)

$$\log|\Delta_n(x)| = \sum_{k=1}^{n} \sum_{j=0}^{p-1} \chi_j(\xi_k) \log a_j = \sum_{k=1}^{n} \log a_{\xi_k}$$

where $\log a_{\xi_k}$ are independent and identically distributed. Moreover,

$$E(\log a_{\xi_k}) = \sum_{j=0}^{p-1} \frac{1}{p} \log a_j = \frac{1}{p} \log\left(\prod_{j=0}^{p-1} a_j\right) = 0$$

since by (4.19) we have $a_0 a_1 \cdots a_{p-1} = 1$, and it is

$$\sigma^2 = E(\log^2 a_{\xi_k}) = \sum_{j=0}^{p-1} \frac{1}{p} (\log^2 a_j) > 0$$

since not all a_j are equal to 1. The law of iterated logarithm says

$$\limsup_{n\to\infty} \frac{\log|\Delta_n(x)|}{\sqrt{2\sigma^2 n \log\log n}} = +1 \qquad (a.s.)$$

and

$$\liminf_{n\to\infty} \frac{\log|\Delta_n(x)|}{\sqrt{2\sigma^2 n \log\log n}} = -1 \qquad (a.s.).$$

This implies the assertion. $\square$

4.5 On the differentiability of the solution

After the foregoing preparations we are able to give the main results concerning differentiability of the solution φ of (1.2). As already mentioned in the Introduction we again exclude the trivial case $c_j = \frac{1}{p}$ for all $j = 0, 1, \ldots, p-1$.

Theorem 4.11 *The solution φ of (1.2) has the property:*

1. *If* $\min |c_j| \geq \frac{1}{p}$ *then φ is nowhere differentiable in* $[0,1]$.

2. *If* $\min |c_j| < \frac{1}{p}$ *then both sets, where φ is differentiable and where φ does not have a finite derivative, have positive Hausdorff dimension.*

Proof: 1. If $|c_j| \geq \frac{1}{p}$ for all $j = 0, 1, \ldots, p-1$ then $a_j = p\,|c_j| \geq 1$ and for each $x \in [0,1]$ we have by Lemma 4.5 that $|\Delta_n(x)| \geq 1$ for all $n \in \mathbb{N}$. So φ is not differentiable at x according to Proposition 4.2.

2. If $\min |c_j| < \frac{1}{p}$ then in view of (1.3) there are indices k and ℓ such that $|c_k| < \frac{1}{p}$ and $|c_\ell| > \frac{1}{p}$. For the mean value (4.4) we have $M(\lambda_0, \ldots, \lambda_{p-1}) = |c_k| < \frac{1}{p}$ if $\lambda_k = 1$ and $\lambda_j = 0$ for $j \neq k$. Hence, there exist such $\lambda'_j > 0$ (with λ'_k nearly by 1 and $\lambda'_{p-1} < 1$) that $p\,M(\lambda'_0, \ldots, \lambda'_{p-1}) < 1$. By Proposition 4.9 we have $\varphi'(x) = 0$ for $x \in F(\lambda'_0, \ldots, \lambda'_{p-1})$ and by (4.11) this set F has positive Hausdorff dimension. Moreover, $M(\lambda_0, \ldots, \lambda_{p-1}) = |c_\ell| > \frac{1}{p}$ if $\lambda_\ell = 1$ and $\lambda_j = 0$ for $j \neq \ell$, so that there are $\lambda''_j > 0$ such that $p\,M(\lambda''_0, \ldots, \lambda''_{p-1}) > 1$. For $x \in F(\lambda''_0, \ldots, \lambda''_{p-1})$ it fails $\Delta_n(x) \to 0$ by Proposition 4.6 so that φ is not differentiable at x according to Proposition 4.2, and by (4.11) also this set F has positive Hausdorff dimension. □

Theorem 4.12 *The solution φ of (1.2) has in $[0,1]$ the property:*

1. *If* $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} < 1$ *then* $\varphi'(x) = 0$ *almost everywhere.*

2. *If* $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} \geq 1$ *then φ is almost nowhere differentiable.*

Proof: We consider $x \in F(p^{-1}, \ldots, p^{-1})$ and remember that this set has the Lebesgue measure 1 by Borel's normal number theorem.

1. If $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} < 1$ then by Proposition 4.9 we have $\varphi'(x) = 0$ for each $x \in F(p^{-1}, \ldots, p^{-1})$.

2.1. If $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} = 1$ then by Proposition 4.10 the set of all $x \in F(p^{-1}, \ldots, p^{-1})$ with $\limsup |\Delta_n(x)| > 0$ has the measure 1. For all these x the derivative does not exist according to Proposition 4.2.

2.2. If $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} > 1$ then for each $x \in F(p^{-1}, \ldots, p^{-1})$ we have according to Proposition 4.6 that $|\Delta_n(x)| \to \infty$ as $n \to \infty$ and hence the derivative does not exist owing to Proposition 4.2. □

Remark 4.13 **1.** Note that Proposition 4.3 is a consequence of Theorem 4.12.

2. Assume that φ is an increasing solution of (1.2) but not $\varphi(x) = x$ for all $x \in [0,1]$. Then by Proposition 2.5 together with Proposition 2.6 we have $c_j \geq 0$ for all $j = 0, 1, \ldots, p-1$ but not $c_j = \frac{1}{p}$ for all j and in view of (1.3)

$$(c_0 c_1 \cdots c_{p-1})^{1/p} < \frac{c_0 + c_1 + \ldots + c_{p-1}}{p} = \frac{1}{p}$$

so that $\varphi'(x) = 0$ almost everywhere by Theorem 4.12. So for an increasing solution φ of (1.2) we have besides of $\varphi(x) = x$ for $x \in [0,1]$ that $\varphi'(x) = 0$ almost everywhere.

5 Singular solutions

A nonconstant $\varphi : [0,1] \mapsto [0,1]$ is called (strictly) singular, if it is continuous and (strictly) increasing with $\varphi'(x) = 0$ almost everywhere. We remember that in case $c_j = \frac{1}{p}$ for $j \in \{0, 1, \ldots, p-1\}$ the solution φ of (1.2) reads $\varphi(x) = x$ for $0 \leq x \leq 1$ and that we exclude this trivial case. As already mentioned in Remark 3.3.1 we use the parameter $c = c_0$ and write short C_k for $C_k(c_0)$.

From Proposition 2.5 and Proposition 4.2 we get

Proposition 5.1 *If $c_j \geq 0$ for all $j = 0, 1, \ldots, p-1$ then the solution φ of (1.2) is a singular function and if $c_j > 0$ for all j then it is strictly singular.*

Lemma 5.2 *If φ is a solution of equation (1.2) satisfying (1.4) then φ cannot vanish in a neighborhood of $x = 0$.*

Proof: Assume that $\varphi(x) = 0$ for $x < \varepsilon_0$ where $\varepsilon_0 > 0$. In view of

$$\varphi\left(\frac{x}{p}\right) = c_0 \varphi(x) \qquad (0 \leq x \leq 1)$$

and $c_0 \neq 0$ implies $\varphi(x) = 0$ for $x < p\,\varepsilon_0$. In view of $p > 1$ it follows $\varepsilon_0 = 0$ since $\varphi(x) = 1$ for $x > 1$. □

Proposition 5.3 *Let be $0 \leq c_j < 1$ with $\min c_j = 0$. Then the solution φ of (1.2) is constant on the components (a_i, b_i) of an open set G with Lebesgue measure $|G| = 1$. The endpoints a_i and b_i are of the form $\frac{k}{p^n}$ where we have:*

- $\frac{k}{p^n} = a_i \iff C_{k-1} \neq 0, \quad C_k = 0,$
- $\frac{k}{p^n} = b_i \iff C_{k-1} = 0, \quad C_k \neq 0,$
- $\frac{k}{p^n} \in G \iff C_{k-1} = 0, \quad C_k = 0.$

Proof: Assume that $c_{k_0} = 0$ where $1 \leq k_0 \leq p-2$. From (1.5) it follows that φ is constant on the interval

$$I_{k_0} = \left(\frac{k_0}{p}, \frac{k_0+1}{p}\right).$$

By repeated application of (1.5) we see that φ is constant on the intervals

$$I_{k_1,k_0} = \left(\frac{k_1}{p} + \frac{k_0}{p^2}, \frac{k_1}{p} + \frac{k_0+1}{p^2}\right)$$

where $k_1 \neq k_0$, $0 \leq k_1 \leq p-1$, and in general

$$I_{k_{n-1},\ldots,k_0} = \left(\frac{k_{n-1}}{p} + \ldots + \frac{k_1}{p^{n-1}} + \frac{k_0}{p^n}, \frac{k_{n-1}}{p} + \ldots + \frac{k_1}{p^{n-1}} + \frac{k_0+1}{p^n}\right),$$

where $k_\nu \neq k_0$ and $0 \leq k_\nu \leq p-1$ for $\nu > 0$. Obviously, $I_{k_{n-1},\ldots,k_0}$ has the Lebesgue measure $|I_{k_{n-1},\ldots,k_0}| = \frac{1}{p^n}$. These intervals are pairwise different and hence the union G_0 has the Lebesgue measure

$$|G_0| = \sum_{n=1}^{\infty} \frac{(p-1)^{n-1}}{p^n} = \frac{1}{p(1-\frac{p-1}{p})} = 1.$$

The left endpoint of $I_{k_{n-1},\ldots,k_0}$ has the form $\frac{k}{p^n}$ with

$$k = k_{n-1}p^{n-1} + k_{n-2}p^{n-2} + \ldots + k_1 p + k_0$$

so that $c_{k_0} = 0$ implies $C_k = 0$, cf. (2.3). It follows from (2.10) that

$$\varphi\left(\frac{k+t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) \qquad (0 \leq t \leq 1),$$

i.e. φ is constant on $I_{k_{n-1},\ldots,k_0}$. If G is an open set such that φ is constant on each component (a_i, b_i) of G then $G_0 \subseteq G \subseteq [0,1]$ and hence $|G| = 1$ too.

Now let (a_i, b_i) be a maximal interval where φ is constant. Choose n so large that $b_i - a_i > \frac{2}{p^n}$ then there is an integer k such that

$$\frac{k-1}{p^n} < a_i \leq \frac{k}{p^n} \tag{5.1}$$

and $\frac{k+1}{p^n} < b_i$, i.e. φ is constant on $[\frac{k}{p^n}, \frac{k+1}{p^n}]$.

We show that $a_i = \frac{k}{p^n}$ and that $C_{k-1} \neq 0$, $C_k = 0$. By Lemma 5.2 $\varphi(x)$ cannot vanish in a neighborhood of $x = 0$ which is true also for $\varphi^*(x) = 1 - \varphi(1-x)$ since $c_{p-1} > 0$. Therefore in view of (5.1) equation (2.12) implies that φ is not constant in a neighborhood of $\frac{k}{p^n}$ which implies $a_i = \frac{k}{p^n}$ and $C_{k-1} \neq 0$. Moreover, equation (2.10) for $t = 1$ yields

$$\varphi\left(\frac{k+1}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) + c_0^n C_k$$

and hence $c_0^n C_k = 0$ must be. It follows $C_k = 0$ since $c_0 > 0$.

Conversely, let be $C_k = 0$ and $C_{k-1} \neq 0$. Then equation (2.10) implies

$$\varphi\left(\frac{k+t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) \qquad (0 \le t \le 1),$$

i.e. φ is constant on $[\frac{k}{p^n}, \frac{k+1}{p^n}]$. Moreover, equation (2.12) implies that φ is not constant in a neighborhood of $\frac{k}{p^n}$ so that it is a left endpoint a_i of an interval of constancy. In the same manner the another assertions can be proved. □

In case $0 \le c_j < 1$ and $\min c_j = 0$ equation (1.2) can be written in the form

$$\varphi\left(\frac{x}{p}\right) = \sum_{n=0}^{q-1} c_{\gamma_n}\varphi(x-\gamma_n) \qquad (x \in \mathbb{R}) \tag{5.2}$$

where q is an integer with $1 \le q \le p-1$ and where γ_n are nonnegative integers with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{q-1} = p-1$. The characteristic polynomial of equation (5.2) reads $P(z) = c_0 + c_{\gamma_1} z^{\gamma_1} + \cdots + c_{p-1} z^{p-1}$ and (2.6) has the form

$$G(z) = \prod_{j=0}^{\infty} \frac{1}{c_0} P(z^{p^j}) = \sum_{n=0}^{\infty} C_{\gamma_n} z^{\gamma_n} \tag{5.3}$$

with strictly increasing integers γ_n where it holds with $\varepsilon_\mu \in \{0, 1, \ldots, q-1\}$:

$$n = \sum_{\mu=0}^{m-1} \varepsilon_\mu q^\mu \qquad \Longrightarrow \qquad \gamma_n = \sum_{\mu=0}^{m-1} \gamma_{\varepsilon_\mu} p^\mu. \tag{5.4}$$

In particular, if $n = \sum_{\mu=0}^{m-1}(q-1)q^\mu = q^m - 1$ then $\gamma_n = \sum_{\mu=0}^{m-1}(p-1)p^\mu = p^m - 1$ and

$$\gamma_{qn+r} = p\gamma_n + \gamma_r \qquad (r \in \{0, 1, \ldots, q-1\}). \tag{5.5}$$

Theorem 5.4 *The open intervals $J_{m,n} \subseteq [0,1]$ where the solution φ of (5.2) is constant have the form*

$$J_{m,n} = \left(\frac{\gamma_{m-1}+1}{p^n}, \frac{\gamma_m}{p^n}\right) \qquad (n = 1, 2, \ldots, \quad m = 1, 2, \ldots, q^n - 1) \tag{5.6}$$

provided that $\gamma_{m-1} + 1 < \gamma_m$.

Proof: We apply Proposition 5.3. If (a_i, b_i) is a maximal interval of constancy then by Proposition 5.3 and the definition of γ_n we have $a_i = \frac{\gamma_k+1}{p^n}$ and $b_i = \frac{\gamma_m}{p^n}$ with suitable k, m. Since the sequence γ_n is strictly increasing it follows $k = m-1$, i.e. $(a_i, b_i) = J_{m,n}$ from (5.11) with the given indices there. □

Remark 5.5 **1.** Observe that $J_{qm,n+1} = J_{m,n}$ since in view of (5.5) we have for the left endpoint

$$\gamma_{qm-1} + 1 = \gamma_{q(m-1)+q-1} + 1 = p\gamma_{m-1} + \gamma_{q-1} + 1 = p(\gamma_{m-1} + 1)$$

where we have used $\gamma_{q-1} = p - 1$, and for the right endpoint $\gamma_{qm} = p\gamma_m$. So we can see again that the nonempty intervals $J_{m,n}$ coincide or they are disjoint.

2. Note that

$$\sum_{m=1}^{q^n-1} |J_{m,n}| = \sum_{m=1}^{q^n-1} \frac{\gamma_m - \gamma_{m-1} - 1}{p^n} = \frac{\gamma_{q^n-1} - \gamma_0 - q^n}{p^n} = \frac{p^n - 1 - q^n}{p^n} \to 1$$

as $n \to \infty$.

Example 5.6 (*Cantor's function.*) We know that Cantor's function φ is the to [0,1] restricted solution of (1.2) with $c_0 = \frac{1}{2}$, $c_1 = 0$, $c_1 = \frac{1}{2}$, i.e.

$$\varphi\left(\frac{x}{3}\right) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x-2) \qquad (x \in \mathbb{R})$$

satisfying (1.4). Here $P(z) = (1 + z^2)/2$ and the generating function (2.6) reads

$$G(z) = \prod_{j=0}^{\infty}\left(1 + z^{2\cdot 3^j}\right) = \sum_{k=0}^{\infty} C_k z^k \tag{5.7}$$

where $C_k = 0$ if the triadic representation of k contains the digit 1, elsewhere $C_k = 1$. Hence, G can be written as

$$G(z) = \sum_{n=0}^{\infty} z^{\gamma_n} = 1 + z^2 + z^6 + z^8 + z^{18} + z^{20} + z^{24} + z^{26} + \dots \tag{5.8}$$

with strictly increasing exponents $\gamma_0 = 0$, $\gamma_1 = 2$, $\gamma_2 = 6$, $\gamma_3 = 8$ and so on. It holds with $\varepsilon_\mu \in \{0, 1\}$:

$$n = \sum_{\mu=0}^{m-1} \varepsilon_\mu 2^\mu \qquad \Longrightarrow \qquad \gamma_n = 2\sum_{\mu=0}^{m-1} \varepsilon_\mu 3^\mu \tag{5.9}$$

and it is easy to see that

$$\gamma_{k-1} + \gamma_{2^n-k} = 3^n - 1 \qquad (n = 1, 2, \dots, \quad k = 1, 2, \dots, 2^n). \tag{5.10}$$

The open intervals $J_{m,n}$ where Cantors function φ is constant have the form

$$J_{m,n} = \left(\frac{\gamma_{m-1} + 1}{3^n}, \frac{\gamma_m}{3^n}\right) \qquad (n = 1, 2, \dots, \quad m = 1, 2, \dots, 2^n - 1) \tag{5.11}$$

with $\varphi(x) = \frac{m}{2^n}$ for $x \in J_{m,n}$.

6 Subadditivity

In this section we investigate the subadditivity of the solution φ of (1.2), i.e.

$$\varphi(x+y) \le \varphi(x) + \varphi(y) \tag{6.1}$$

for all $x, y \in \mathbb{R}$. For this purpose again we consider the sequence $S_k(\mathbf{c})$ from (3.1) with $c = \mathbf{c}$ from (3.3).

Lemma 6.1 *Assume that $c_j \ge 0$ for all $j = 0, 1, \ldots, p-1$ and that for $0 \le k, \ell < p$ we have*

$$S_k(\mathbf{c}) + S_\ell(\mathbf{c}) \ge \begin{cases} S_{k+\ell}(\mathbf{c}) & \text{if} \quad k+\ell < p \\ S_{k+\ell-p}(\mathbf{c}) + S_p(\mathbf{c}) & \text{if} \quad k+\ell \ge p \end{cases} \tag{6.2}$$

then for all nonnegative integers k, ℓ it holds

$$S_{k+\ell}(\mathbf{c}) \le S_k(\mathbf{c}) + S_\ell(\mathbf{c}). \tag{6.3}$$

Proof: First note that by (2.5) we have $C_j(\mathbf{c}) \ge 0$ for all $j \in \mathbb{N}$. We shall prove the inequality (6.3) for nonnegative integers $k, \ell < p^n$ by induction with respect to n where as abbreviation we write S_k in place of $S_k(\mathbf{c})$. For $n = 0$ the inequality is true by (6.2). Assume that (6.3) is true for $0 \le k, \ell < p^n$. For integers $0 \le k, \ell < p^{n+1}$ we write $k = pk' + i$ and $\ell = p\ell' + j$ with $0 \le k', \ell' < p^n$ and $i, j \in \{0, 1, \ldots, p-1\}$. We consider two cases:

1. Let be $i + j < p$. Then in view of Lemma 3.1/(iii) we have

$$\begin{aligned}
S_{p(k'+\ell')+i+j} &= S_p S_{k'+\ell'} + C_{k'+\ell'}(\mathbf{c}) S_{i+j} \\
&\le S_p(S_{k'} + S_{\ell'}) + C_{k'+\ell'}(\mathbf{c})(S_i + S_j) \\
&\le S_p S_{k'} + S_p S_{\ell'} + C_{k'}(\mathbf{c}) S_i + C_{\ell'}(\mathbf{c}) S_j \\
&= S_{pk'+i} + S_{p\ell'+j}
\end{aligned}$$

where we have used that (3.4) and that $C_{k'+\ell'}(\mathbf{c}) \le \min\{C_{k'}(\mathbf{c}), C_{\ell'}(\mathbf{c})\}$ according to (2.5). So $S_{k+\ell} \le S_k + S_\ell$.

2. In case $i + j \ge p$ we have $0 \le i + j - p < p - 1$. Applying Lemma 3.1/(iii) and assumption of induction we get

$$S_{k'+\ell'+1} = S_{k'+\ell'} + C_{k'+\ell'}(\mathbf{c}) \le S_{k'} + S_{\ell'} + C_{k'+\ell'}(\mathbf{c})$$

and

$$\begin{aligned}
S_{p(k'+\ell'+1)+i+j-p} &= S_p S_{k'+\ell'+1} + C_{k'+\ell'+1}(\mathbf{c}) S_{i+j-p} \\
&\le S_p\{S_{k'} + S_{\ell'} + C_{k'+\ell'}(\mathbf{c})\} + C_{k'+\ell'+1}(\mathbf{c}) S_{i+j-p} \\
&\le S_p S_{k'} + S_p S_{\ell'} + C_{k'+\ell'}(\mathbf{c})(S_p + S_{i+j-p}) \\
&\le S_p S_{k'} + S_p S_{\ell'} + C_{k'+\ell'}(\mathbf{c})(S_i + S_j)
\end{aligned}$$

where we have used (6.2) and $C_{k'+\ell'+1}(\mathbf{c}) \le C_{k'+\ell'}(\mathbf{c})$ according to (2.5) and (3.4). Hence we have $S_{k+\ell} \le S_k + S_\ell$ again. $\square$

Theorem 6.2 *If (6.2) is satisfies then the solution φ of (1.2) is subadditive, i.e.*

$$\varphi(x+y) \le \varphi(x) + \varphi(y) \qquad (x, y \in \mathbb{R}). \tag{6.4}$$

Proof: For $x = \frac{k}{p^n}$, $y = \frac{\ell}{p^n}$ in [0,1] with $x + y \le 1$ the assertion follows from (3.2) in view of (6.3), and for arbitrary $x, y \in [0,1]$ with $x + y \le 1$ by continuity of φ. Now from (1.4) it is easy to see that the inequality is true for all $x, y \in \mathbb{R}$. $\square$

Example 6.3 (*De Rham's function*) We know that de Rham's function φ is the to [0,1] restricted solution φ of (1.2) with $c_0 = a$, $c_1 = 1 - a$, $a \in (0,1)$, i.e.

$$\varphi\left(\frac{x}{2}\right) = a\varphi(x) + (1-a)\varphi(x-1) \qquad (x \in \mathbb{R})$$

satisfying (1.4), cf. Example 2.4. For $0 < a < \frac{1}{2}$ de Rham's function φ fails to be subadditive since $2\varphi(\frac{1}{2}) = 2a < 1 = \varphi(1)$. In case $\frac{1}{2} \le a < 1$ we have $\mathbf{c} = \max\{a, 1-a\} = a$ and $C_k = C_k(a) = q^{s(k)}$ with $q = \frac{1-a}{a}$, where $s(k)$ denotes the number of ones in the dyadic representation of k, i.e. $C_0 = 1$, $C_1 = q$, $C_2 = q$, $C_3 = q^2$, $C_4 = q$, $C_5 = q^2$ and for $S_k = S_k(a)$ we have $S_1 = 1$, $S_2 = 1 + q$, $S_3 = 1 + 2q$, $S_4 = 1 + 2q + q^2$, $S_5 = 1 + 3q + q^2$. So inequality (6.2) is satisfies if $0 < q \le 1$, i.e. $\frac{1}{2} \le a < 1$ and for these a we have (6.3), cf. [2, Lemma 2.2]. Hence, for $\frac{1}{2} \le a < 1$ the extended de Rham function is subadditive owing to Theorem 6.2.

Finally we consider once more two-scale difference equation (1.13).

Example 6.4 (*Equation* (1.13)) For $0 < a < 1$ let φ be the continuous solution of

$$\varphi\left(\frac{x}{3}\right) = a\varphi(x) + (1-2a)\varphi(x-1) + a\varphi(x-2) \qquad (x \in \mathbb{R})$$

satisfying (1.4). For $0 < a \le \frac{1}{2}$ the coefficients are nonnegative. In case $0 < a < \frac{1}{3}$ the solution φ fails to be subadditive since $\varphi(\frac{2}{3}) = 1 - a > 2a = 2\varphi(\frac{1}{3})$. In case $\frac{1}{3} \le a \le \frac{1}{2}$ we have $\mathbf{c} = \max\{a, 1-2a, a\} = a$ and $C_k = C_k(a) = \varrho^{s_1(k)}$ with $\varrho = \frac{1-2a}{a}$, where $s_1(k)$ denotes the number of ones in the triadic representation of k. So $C_0 = 1$, $C_1 = \varrho$, $C_2 = 1$, $C_3 = \varrho$, $C_4 = \varrho^2$, $C_5 = \varrho$, $C_6 = 1$ and for $S_k = S_k(a)$ we have $S_1 = 1$, $S_2 = 1 + \varrho$, $S_3 = 2 + \varrho$, $S_4 = 2 + 2\varrho$, $S_5 = 2 + 2\varrho + \varrho^2$. Inequality (6.2) is satisfies if $\varrho \ge 0$ (from $S_2 \le S_1 + S_1$) and if $\varrho \le 1$ (from $S_2 + S_2 \ge S_1 + S_3$). So for $\frac{1}{3} \le a \le \frac{1}{2}$ it holds (6.3), and the solution φ of (1.13) is subadditive according to Theorem 6.2. In particular, Cantor's function ($a = \frac{1}{2}$) is subadditive, cf. also [22, Section 3.2.4], [10].

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