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Local antimaximum principle for the Schrödinger operator in $\mathbb{R}^N$

ABSTRACT. We consider in this paper equations defined in $\mathbb{R}^N$ involving Schrödinger operators with indefinite weight functions and with potentials which tend to infinity at infinity. After recalling the existence of principal eigenvalues and the maximum principle, we study the local antimaximum principle.

KEY WORDS. Schrödinger operator, indefinite weight, antimaximum principle

1 Introduction

We consider in this paper the Schrödinger operator $-\Delta + q$ defined on $\mathbb{R}^N$ associated with the indefinite weight m where q is a potential which satisfies the following hypothesis:

($\mathbf{H_q^1}$) $q \in L^2_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N)$, $p > \frac{N}{2}$, such that $\lim_{|x|\to\infty} q(x) = \infty$ and $q \geq cst > 0$.

and where the weight m satisfies one of the following hypotheses:

($\mathbf{H_m'^1}$) $m \in L^\infty(\mathbb{R}^N)$, m is positive in the open subset $\Omega_m^+ = \{x \in \mathbb{R}^N, m(x) > 0\}$ with non zero measure and m is negative in the open subset $\Omega_m^- = \{x \in \mathbb{R}^N, m(x) < 0\}$ with non zero measure.

($\mathbf{H_m'^2}$) **(i)** $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ $(N \geq 3)$, $meas(\Omega_m^+) > 0$, $meas(\Omega_m^-) > 0$.

(ii) $m = m_1 - m_2$, $m_1 \geq 0$, $m_1 \in L^\infty(\mathbb{R}^N)$, $m_2 \geq 0$, $m_2 \in L^\infty_{loc}(\mathbb{R}^N)$.

Mainly, this paper deals with the local antimaximum principle for the following equation

$$(-\Delta + q)u = \lambda m u + f \text{ in } \mathbb{R}^N, \tag{1.1}$$

where λ is a real parameter and f satisfies the following hypothesis:

($\mathbf{H_f^1}$) $f \in L^2(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$.

As in [2, 3] we introduce the quadratic form

$$(v, w)_q = \int_{\mathbb{R}^N} \nabla v \cdot \nabla w + qvw$$

defined for every pair

$$v, w \in V_q(\mathbb{R}^N) := \{f \in L^2(\mathbb{R}^N,\ (f, f)_q < \infty\}.$$

Notice that $V_q(\mathbb{R}^N)$ is a Hilbert space with the inner product $(v, w)_q$ and the norm

$$\|v\|_q = ((v, v)_q)^{1/2} = \left(\int_{\mathbb{R}^N} [|\nabla v|^2 + qv^2]\right)^{1/2}.$$

The set $D(\mathbb{R}^N)$, which is the set of C^∞ functions with compact supports, is a dense linear subspace of $V_q(\mathbb{R}^N)$. By the Lax-Milgram theorem, the Schrödinger operator $L = -\Delta + q$ in $L^2(\mathbb{R}^N)$ is defined to be the selfadjoint operator in $L^2(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} (Lv)w = (v, w)_q \text{ for all } v, w \in D(\mathbb{R}^N).$$

We denote by $D(L)$ its domain (strong domain) and $V_q(\mathbb{R}^N)$ is its weak domain. In the following a function $u \in V_q(\mathbb{R}^N)$ will be called a solution of (1.1) if it is a weak solution of (1.1) i.e. if $\int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi + qu\phi = \lambda \int_{\mathbb{R}^N} mu\phi + \int_{\mathbb{R}^N} f\phi$ for all $\phi \in D(\mathbb{R}^N)$. We recall that the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.

We add another hypothesis upon the potential q which assures that any element of the weak domain of the operator $L = -\Delta + q$ belongs to the strong domain $D(L)$. It is the following hypothesis.

$(\mathbf{H_q^2})$ There exists a positive constant C such that for all $x \in \mathbb{R}^N$ and all $h \in \mathbb{R}^N$, $h \neq 0$, $|\frac{q(x+h)-q(x)}{|h|}| \leq C\ \sqrt{q(x)}$.

Note that for example, the potential $q(x) = 1 + |x|$ satisfies $(\mathbf{H_q^2})$. And we recall the following proposition in [7], based on the methods of translations due to Nirenberg.

Proposition 1.1 *Assume that the potential q satisfy $(\mathbf{H_q^1})$ and $(\mathbf{H_q^2})$. Let u be a weak solution of $(-\Delta + q)u = f$ in $\mathbb{R}^N$ with $f \in L^2(\mathbb{R}^N)$. Then $u \in H^2(\mathbb{R}^N)$, $qu \in L^2(\mathbb{R}^N)$ and therefore $u \in D(L)$.*

Our assumptions on the weight m guarantee the existence of a unique principal and positive eigenvalue $\lambda_{1,q,m} > 0$ associated with a positive eigenfunction $\phi_{1,q,m} > 0$, and also the existence and uniqueness of a principal negative eigenvalue $\tilde{\lambda}_{1,q,m} < 0$ associated with a positive eigenfunction $\tilde{\phi}_{1,q,m} > 0$ (see [6]). We also recall a variational characterization of these eigenvalues and that will be essential for the proof of the local antimaximum principle. The problem of the existence of principal eigenvalues has been studied for the Laplacian and the p-Laplacian operators associated with a weight, in bounded domains (see for example [14]), in $\mathbb{R}^N$ (see for example [5]), for the Schrödinger operator $-\Delta + q$ associated with a weight m in $\mathbb{R}^N$ (see [6, 7]). We also recall the maximum principle for (1.1): if u is one weak solution of (1.1), if $f \geq 0$ and if $\tilde{\lambda}_{1,q,m} < \lambda < \lambda_{1,q,m}$, then $u \geq 0$. Note that the maximum principle has been extensively studied for equations or systems (see for example [8, 11–13, 21, 23, 25, 26]).

Afterwards we study the local antimaximum principle: we denote by B_R the open ball in $\mathbb{R}^N$ of center 0 and radius R ; if $f \geq 0$, $f \not\equiv 0$, then there exists a constant $\delta = \delta(f, R) > 0$ such that for all $\lambda \in]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution u of (1.1) is negative in $\overline{B}_R$.

In various common versions of the antimaximum principle in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, besides the assumption $f \geq 0$, $f \not\equiv 0$ in Ω, it is only assumed that $f \in L^p(\Omega)$ for some $p > N$ (cf [10, Theorem 1 p.222], [25, 26]). The case of the Schrödinger operator on $\mathbb{R}^N$ is more difficult; the hypothesis $f \in L^p(\Omega)$ $(p > N)$ is no longer sufficient (see [2, 3] for the Schrödinger operator with no weight and [7] for the Schrödinger operator with a positive bounded weight m). Therefore the two main difficulties here are the unboundedness of the domain $\mathbb{R}^N$ and the weight m which is not a positive bounded function.

We do not use the ideas expressed in [2, 3, 10, 20, 25, 26] where the antimaximum principle is obtained by a decomposition of the resolvant of the operator near the principal and positive eigenvalue and by projecting on the eigenspace generated by the eigenfunction associated with this eigenvalue. Indeed because of the unboundedness of our domain, we cannot proceed as for example in Hess (see [20]) where the antimaximum principle is studied for the Laplacian operator with an indefinite weight function but in a bounded domain. And furthermore because of our weight which is an indefinite function, we cannot proceed as in [2, 3, 7] where the antimaximum principle is studied for the Schrödinger operator $-\Delta + q$ on $\mathbb{R}^N$ with a positive bounded weight m ($m = 1$ in [2, 3]): more precisely in these former papers $(\int_{\mathbb{R}^N} mu^2)^{1/2}$ must be a norm, equivalent to the usual norm in $L^2(\mathbb{R}^N)$.

Thus for the proof of the local antimaximum principle, we follow a method developed in [15, 24]. This method has been first established for the Laplacian operator in $\mathbb{R}^N$, $N \geq 3$, and $f \in L^\infty(\mathbb{R}^N)$ and for the p-Laplacian operator with a nonpositive weight m at infinity (see [24]), then it has been extended to the p-Laplacian operator in $\mathbb{R}^N$ in [15]. This method is based on a nonexistence result of nonnegative solutions for (1.1) if $\lambda > \lambda_{1,q,m}$ (see Proposition 3.1) and also on estimates given by the regularity C^1 of any solution u of (1.1). We can get this regularity either by using a regularity result of Tolksdorf in [27] and Serrin $L^\infty(B_R)$ estimates for u (see [22]) as in [15, 24] or more classically by the local L^p-regularity theory.

Indeed, first note that any solution u of (1.1) is continuous (see [1, Theorem 0.1 p.3], [23, Theorem 7.1 p.232]). Moreover if $u \in D(L)$ and $(-\Delta+q)u = f \in L^2(\mathbb{R}^N)$ with $f \in L^p_{loc}(\mathbb{R}^N)$ for some p with $2 \leq p < \infty$ then the local L^p-regularity theory yields $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ (see [17, Theorem 9.15 p.241]). In particular, if $p > N$ then $u \in C^1(\mathbb{R}^N)$ by the Sobolev embedding theorem (see [17, Theorem 7.10 p. 155]).

Therefore these results for the Schrödinger operator $-\Delta + q$ associated with an indefinite weight m extend here the results of the antimaximum principle for the Laplacien operator in a bounded domain (see [10, 20, 26]) and for the p-Laplacien operator in $\mathbb{R}^N$ (see [15, 24]). Note that extensions of maximum and antimaximum principles, respectively called ground state positivity and negativity (or also called fundamental positivity and negativity), are given for the Schrödinger operator in $\mathbb{R}^N$ without any weight (see [2, 4]) and with a positive weight (see [7]) but for a potential q which is a perturbation of a radially symmetric potential and for a more restrictive set of functions f. Note in [15, Theorem 4.1] a result where the fundamental negativity in $\mathbb{R}^N$ is not verified for the Laplacien operator associated with an indefinite weight in dimension $N = 1$. Also recall examples given in [2, Example 2.1] and in [3, Example 4.1] where the global antimaximum principle in $\mathbb{R}^N$ is still not verified for the Schrödinger operator with no weight in dimension $N \geq 1$. Finally, we can cite among other papers the works of [16, 18, 19] where the antimaximum principle is studied either for the p-Laplacian operator or an elliptic operator of second order with a bounded weight on a bounded domain.

Our paper is organized as follows: In Section 2 we recall the existence of a principal positive (resp. negative) eigenvalue $\lambda_{1,q,m} > 0$ (resp. $\tilde{\lambda}_{1,q,m} < 0$) associated with a positive eigenfunction $\phi_{1,q,m} > 0$ (resp. $\tilde{\phi}_{1,q,m} > 0$). We also recall the classical maximum principle for (1.1) in the case of an indefinite weight m. In Section 3, we study the local antimaximum principle. Finally in Section 4, we extend the local antimaximum principle to the case of the system (4.1).

2 Existence of principal eigenvalues and maximum principle

First we recall in this section the existence of a unique positive principal eigenvalue $\lambda_{1,q,m}$ and of a unique negative principal eigenvalue $\tilde{\lambda}_{1,q,m}$ (see [6, Theorems 2.1,2.2,3.1]). So we assume in this paper that q satisfies $(\mathbf{H_q^1})$, $(\mathbf{H_q^2})$ and m satisfies $(\mathbf{H_m'^1})$ or $(\mathbf{H_m'^2})$.

Theorem 2.1 *Assume that q satisfies $(\mathbf{H_q^1})$, $(\mathbf{H_q^2})$ and m satisfies $(\mathbf{H_m'^1})$ or $(\mathbf{H_m'^2})$(**i**). Then the operator $-\Delta + q$ associated with the weight m has a unique positive principal eigenvalue $\lambda_{1,q,m}$ associated with a positive eigenfunction $\phi_{1,q,m} \in C^1(\mathbb{R}^N)$ normalized by*

$\int_{\mathbb{R}^N} m\phi_{1,q,m}^2 = 1$, $\lambda_{1,q,m}$ *is simple and* $(\lambda_{1,q,m}, \phi_{1,q,m})$ *satisfy*

$$(-\Delta + q)\phi_{1,q,m} = \lambda_{1,q,m}\ m\ \phi_{1,q,m} \text{ in } \mathbb{R}^N; \quad \lambda_{1,q,m} > 0; \quad \phi_{1,q,m} > 0.$$

$$\lambda_{1,q,m} = \inf\{\frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q(\mathbb{R}^N) \text{ s. t. } \int_{\mathbb{R}^N} m\phi^2 > 0\}, \tag{2.1}$$

and this infimum is achieved for any function $\phi = \alpha\phi_{1,q,m}$ *with* $\alpha \in \mathbb{R}$, $\alpha \neq 0$. *Moreover the operator* $-\Delta + q$ *associated with the weight* m *has a unique negative principal eigenvalue* $\tilde{\lambda}_{1,q,m}$ *associated with a positive eigenfunction* $\tilde{\phi}_{1,q,m}$.

Now we recall the maximum principle for (1.1) (see ([6, Theorem 3.2])).

Theorem 2.2 *Assume that* q *satisfies* $(\mathbf{H_q^1})$, $(\mathbf{H_q^2})$ *and* m *satisfies* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})(\mathbf{i})$. *Assume that* $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ *and* u *is a solution of* (1.1). *If* $\tilde{\lambda}_{1,q,m} < \lambda < \lambda_{1,q,m}$, *then* $u \geq 0$.

We conclude this section by adding the following proposition. We follow here [15, Proposition 2.1].

Proposition 2.1 *Assume that* q *satisfies* $(\mathbf{H_q^1})$, $(\mathbf{H_q^2})$ *and* m *satisfies* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})(\mathbf{i})$-$(\mathbf{ii})$. *Then any minimizing sequence* (u_k) *of* $\lambda_{1,q,m}$ *admits a subsequence which converges weakly in* $V_q(\mathbb{R}^N)$ *to some* u *which realizes the infimum* (2.1); *and so there exists* $\alpha \in \mathbb{R}$, $\alpha \neq 0$ *such that* $u = \alpha\phi_{1,q,m}$.

Proof: First note that

$$\lambda_{1,q,m} = \inf\{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2],\ \phi \in V_q(\mathbb{R}^N) \text{ s. t. } \int_{\mathbb{R}^N} m\phi^2 = 1\}. \tag{2.2}$$

Let now (u_k) be a minimizing sequence. Then (u_k) is a bounded sequence in $V_q(\mathbb{R}^N)$ and there exists $u \in V_q(\mathbb{R}^N)$ such that for a subsequence (u_k) converges weakly to u in $V_q(\mathbb{R}^N)$ (and strongly in $L^2(\mathbb{R}^N)$, and for a subsequence, still denoted by (u_k), $u_k \to u$ a.e. in $\mathbb{R}^N$).

If m satisfies $(\mathbf{H_m'^1})$, since the weight m is bounded, by the Lebesgue dominated convergence theorem we get that $1 = \int_{\mathbb{R}^N} mu_k^2 \to \int_{\mathbb{R}^N} mu^2$ as $k \to \infty$. Moreover since (u_k) converges weakly to u in $V_q(\mathbb{R}^N)$, we have $\|u\|_q \leq \liminf \|u_k\|_q = \lambda_{1,q,m}$. Therefore u realizes the infimum (2.2) and so u is on the form $u = \alpha\phi_{1,q,m}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

If now m satisfies $(\mathbf{H_m'^2})$, note that $\int_{\mathbb{R}^N} m_1u_k^2 \to \int_{\mathbb{R}^N} m_1u^2$ as $k \to \infty$. Recall that $1 = \int_{\mathbb{R}^N} mu_k^2 = \int_{\mathbb{R}^N} m_1u_k^2 - \int_{\mathbb{R}^N} m_2u_k^2$. Thus

$$\int_{\mathbb{R}^N} m_2u^2 \leq \liminf \int_{\mathbb{R}^N} m_2u_k^2 = \int_{\mathbb{R}^N} m_1u^2 - 1.$$

Therefore $\int_{\mathbb{R}^N} mu^2 \geq 1$ and there exists $\beta \in]0,1]$ such that $\int_{\mathbb{R}^N} m(\beta u)^2 = 1$. Moreover since (u_k) converges weakly to u in $V_q(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} [|\nabla u|^2 + qu^2] \leq \liminf \int_{\mathbb{R}^N} [|\nabla u_k|^2 + qu_k^2] = \lambda_{1,q,m}, \tag{2.3}$$

and from the variational characterization (2.2) of $\lambda_{1,q,m}$ we also have

$$\lambda_{1,q,m} \leq \int_{\mathbb{R}^N} [|\nabla(\beta u)|^2 + q(\beta u)^2] = \beta^2 \int_{\mathbb{R}^N} [|\nabla u|^2 + qu^2]. \tag{2.4}$$

From (2.3) and (2.4) we get $\beta^2 \geq 1$ and therefore $\beta = 1$. So here again u realizes the infimum (2.2) and therefore u is on the form $u = \alpha\phi_{1,q,m}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$. $\square$

3 The local antimaximum principle

In this section we consider the equation (1.1) where m satisfies $(\mathbf{H_m'^1})$ or $(\mathbf{H_m'^2})$, q satisfies $(\mathbf{H_q^1})$, $(\mathbf{H_q^2})$ and f satisfies $(\mathbf{H_f^1})$. Let u be a weak solution of (1.1). Recall that $u \in C^1(\mathbb{R}^N)$. First we recall the Picone identity.

Lemma 3.1 *Let Ω be a domain in $\mathbb{R}^N$. For $\psi, u \in C^1(\Omega)$ with $\psi \geq 0$ and $u > 0$ in Ω, we have $|\nabla\psi|^2 - \nabla u \cdot \nabla(\frac{\psi^2}{u}) \geq 0$ in Ω.*

Proposition 3.1 *If $f \geq 0$, $f \not\equiv 0$, then (1.1) has no solution if $\lambda = \lambda_{1,q,m}$ and has no nonnegative solution if $\lambda > \lambda_{1,q,m}$.*

Proof: First assume that $\lambda = \lambda_{1,q,m}$ and there exists a solution u for (1.1). Multiplying (1.1) by $\phi_{1,q,m}$ as a test function, we obtain that $\int_{\mathbb{R}^N} f\phi_{1,q,m} = 0$ and so we get a contradiction since $f \geq 0$, $f \not\equiv 0$, $\phi_{1,q,m} > 0$.

Assume now that $\lambda > \lambda_{1,q,m}$ and there exists a nonnegative solution u for (1.1). Let $R > 0$ and c_R a positive constant sufficiently large such that $c_R + \lambda m - q \geq 0$ in B_R. Note that $-\Delta u + c_R u = (c_R + \lambda m - q)u + f \geq 0$ in B_R. Applying the strong maximum principle in B_R (see [17, Theorem 8.19 p.198]) (or as in [15, 24] the Vázquez maximum principle given in [28, Theorems 1,5]) we obtain that $u > 0$ in B_R for any R sufficiently large and so $u > 0$ in $\mathbb{R}^N$.

Let now $(\psi_k)_k$ be a convergent sequence to $\phi_{1,q,m}$ in $V_q(\mathbb{R}^N)$, $\psi_k \geq 0$, $\psi_k \in D(\mathbb{R}^N)$. Applying the Picone identity, we get

$$\int_{\mathbb{R}^N} \left(|\nabla\psi_k|^2 - \nabla u \cdot \nabla(\frac{\psi_k^2}{u}) \right) = \|\psi_k\|_q^2 - \lambda \int_{\mathbb{R}^N} m\psi_k^2 - \int_{\mathbb{R}^N} f\frac{\psi_k^2}{u} \geq 0.$$

Since $(\psi_k)_k$ is a convergent sequence to $\phi_{1,q,m}$ in $V_q(\mathbb{R}^N)$, we have

$$\|\psi_k\|_q^2 \to \|\phi_{1,q,m}\|_q^2 = \lambda_{1,q,m} \int_{\mathbb{R}^N} m\phi_{1,q,m}^2 \text{ as } k \to \infty.$$

If m satisfies $(\mathbf{H_m'^1})$, note that $\psi_k \to \phi_{1,q,m}$ in $L^2(\mathbb{R}^N)$ as $k \to \infty$ and (at least for a subsequence still denoted by (ψ_k)) there exists $h \in L^2(\mathbb{R}^N)$ such that $\psi_k \to \phi_{1,q,m}$ a.e. in $\mathbb{R}^N$ and $|\psi_k| \le h$ a.e. in $\mathbb{R}^N$ for all k. So, since $m \in L^\infty(\mathbb{R}^N)$, there exists a positive constant C such that $|m\psi_k^2 - m\phi_{1,q,m}^2| \le C(h^2 + \phi_{1,q,m}^2)$ a.e. in $\mathbb{R}^N$. Applying the Lebesgue dominated convergence Theorem, we deduce that

$$\int_{\mathbb{R}^N} m\psi_k^2 \to \int_{\mathbb{R}^N} m\phi_{1,q,m}^2 \text{ as } k \to \infty. \tag{3.1}$$

By the same way, if m satisfies $(\mathbf{H_m'^2})$, recall that $N \ge 3$ and $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ with a continuous embedding, so note that $\psi_k \to \phi_{1,q,m}$ in $L^{2^*}(\mathbb{R}^N)$ as $k \to \infty$ with $2^* = \frac{2N}{N-2}$ and (at least for a subsequence still denoted by (ψ_k)) there exists $h \in L^{2^*}(\mathbb{R}^N)$ such that $\psi_k \to \phi_{1,q,m}$ a.e. in $\mathbb{R}^N$ and $|\psi_k| \le h$ a.e. in $\mathbb{R}^N$ for all k. So $|m\psi_k^2 - m\phi_{1,q,m}^2| \le |m|(h^2 + \phi_{1,q,m}^2)$ a.e. in $\mathbb{R}^N$. Applying the Lebesgue dominated convergence Theorem, we still get (3.1).

Finally, note that $\frac{f\psi_k^2}{u} \ge 0$ and $\frac{f\psi_k^2}{u} \in L^1(\mathbb{R}^N)$ since ψ_k has a compact support, $f \in L^\infty_{loc}(\mathbb{R}^N)$, $u \in L^\infty_{loc}(\mathbb{R}^N)$. By the Fatou lemma we get that

$$\int_{\mathbb{R}^N} \frac{f\phi_{1,q,m}^2}{u} \le \liminf \int_{\mathbb{R}^N} \frac{f\psi_k^2}{u}.$$

So by the Lebesgue dominated convergence Theorem and Fatou Lemma, we obtain

$$(\lambda_{1,q,m} - \lambda) \int_{\mathbb{R}^N} m\phi_{1,q,m}^2 - \int_{\mathbb{R}^N} f\frac{\phi_{1,q,m}^2}{u} \ge 0.$$

And we get a contradiction since the first term of this estimate is negative and the second term is negative too. $\square$

We give now the local antimaximum principle.

Theorem 3.1 *Let $f \ge 0$, $f \not\equiv 0$. Then for any $R > 0$ there exists a positive constant $\delta = \delta(f, R) > 0$ such that for any $\lambda \in]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution u of (1.1) is negative in $\overline{B}_R$.*

Proof: We follow [15, 24]. Assume by contradiction that for some $R > 0$ there exist $\lambda_k > \lambda_{1,q,m}$, $\lambda_k \searrow \lambda_{1,q,m}$, a solution u_k of

$$(-\Delta + q)u_k = \lambda_k m u_k + f \text{ in } \mathbb{R}^N, \tag{3.2}$$

and $x_k \in \overline{B}_R$ such that $u_k(x_k) \ge 0$.

First we show that $\lim_{k\to\infty}\|u_k\|_q=\infty$. On the contrary, assume that $(\|u_k\|_q)_k$ is a bounded sequence. Therefore, from the compact embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$, for a subsequence, there exists $u\in V_q(\mathbb{R}^N)$ such that $(u_k)_k$ converges to u, weakly in $V_q(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. So for all $\phi\in D(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(\nabla u_k\cdot\nabla\phi+qu_k\phi)\to\int_{\mathbb{R}^N}(\nabla u\cdot\nabla\phi+qu\phi)\text{ as }k\to\infty$$

and by the Lebesgue dominated convergence Theorem

$$\int_{\mathbb{R}^N}mu_k\phi\to\int_{\mathbb{R}^N}mu\phi\text{ as }k\to\infty.$$

Therefore as $k\to\infty$, we get from (3.2) that $(-\Delta+q)u=\lambda_{1,q,m}mu+f$ in $\mathbb{R}^N$ which contradicts Proposition 3.1. So $\lim_{k\to\infty}\|u_k\|_q=\infty$.

Now set $v_k=\frac{u_k}{\|u_k\|_q}$. Then v_k satisfies

$$(-\Delta+q)v_k=\lambda_k mv_k+\frac{f}{\|u_k\|_q}\text{ in }\mathbb{R}^N. \tag{3.3}$$

Since $(v_k)_k$ is a bounded sequence in $V_q(\mathbb{R}^N)$, as before, for a subsequence, there exists $v\in V_q(\mathbb{R}^N)$ such that $(v_k)_k$ converges to v, weakly in $V_q(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. And v satisfies

$$(-\Delta+q)v=\lambda_{1,q,m}mv\text{ in }\mathbb{R}^N.$$

Since $\lambda_{1,q,m}$ is a simple eigenvalue then there exists $\beta\in\mathbb{R}$ such that

$$v=\beta\phi_{1,q,m}.$$

First note that if we multiply (3.3) by $\phi_{1,q,m}$ as a test function and if we integrate over $\mathbb{R}^N$, we get

$$\lambda_{1,q,m}\int_{\mathbb{R}^N}m\phi_{1,q,m}v_k=\lambda_k\int_{\mathbb{R}^N}m\phi_{1,q,m}v_k+\int_{\mathbb{R}^N}\frac{f}{\|u_k\|_q}\phi_{1,q,m}.$$

Therefore since $\int_{\mathbb{R}^N}\frac{f}{\|u_k\|_q}\phi_{1,q,m}>0$ and $\lambda_{1,q,m}<\lambda_k$ we have $\int_{\mathbb{R}^N}m\phi_{1,q,m}v_k<0$. So if the weight m satisfies $(\mathbf{H'^1_m})$, passing to the limit we get that $\int_{\mathbb{R}^N}m\phi_{1,q,m}v\le 0$ and $\beta\le 0$. Therefore we will consider three cases for β (and the case $\beta>0$ only when the weight m satisfies $(\mathbf{H'^2_m})$).

If $\beta=0$ then $v=0$. Note that $\|v_k\|_q^2=\lambda_k\int_{\mathbb{R}^N}mv_k^2+\int_{\mathbb{R}^N}\frac{fv_k}{\|u_k\|_q}$ and $\|v_k\|_q^2\le\lambda_k\int_{\mathbb{R}^N}pv_k^2+\int_{\mathbb{R}^N}\frac{fv_k}{\|u_k\|_q}$ with $p:=m$ if m satisfies $(\mathbf{H'^1_m})$ and $p:=m_1$ if m satisfies $(\mathbf{H'^2_m})$. By the Lebesgue dominated convergence theorem, we get that $\int_{\mathbb{R}^N}pv_k^2\to 0$ and $\int_{\mathbb{R}^N}\frac{fv_k}{\|u_k\|_q}\to 0$ as $k\to\infty$. So we have $\|v_k\|_q\to 0$ as $k\to\infty$, which is impossible since $\|v_k\|_q=1$. Therefore $\beta\neq 0$.

If now $\beta < 0$ then $v < 0$ in $\mathbb{R}^N$. But $(v_k)_k$ converges to v in $C^1_{loc}(\mathbb{R}^N)$ and uniformly on all ball B_R. So v_k is negative in $\overline{B}_R$ for k sufficiently large, which contradicts the existence of the sequence x_k.

So we consider the last case $\beta > 0$ (and in fact only when the weight m satisfies $(\mathbf{H'^2_m})$). We will show that $v_k \geq 0$ i.e. $v_k^- \equiv 0$ in $\mathbb{R}^N$ for k sufficiently large. On the contrary, assume that $v_k^- \not\equiv 0$. Multiplying (3.3) by v_k^- and integrating over $\mathbb{R}^N$, we get that

$$0 < \|v_k^-\|_q^2 = \lambda_k \int_{\mathbb{R}^N} m(v_k^-)^2 - \int_{\mathbb{R}^N} \frac{f v_k^-}{\|u_k\|_q} \leq \lambda_k \int_{\mathbb{R}^N} m(v_k^-)^2.$$

So $r_k := \int_{\mathbb{R}^N} m(v_k^-)^2 > 0$. Moreover by the variational characterization of $\lambda_{1,q,m}$ we have

$$\frac{\|v_k^-\|_q^2}{\int_{\mathbb{R}^N} m(v_k^-)^2} \to \lambda_{1,q,m} \text{ as } k \to \infty \text{ i.e. } \lim_{k\to\infty} \|w_k\|_q^2 = \lambda_{1,q,m} \text{ with } w_k = \frac{1}{r_k^{1/2}} v_k^-.$$

So (w_k) is a minimizing sequence for $\lambda_{1,q,m}$ in (2.2) and from the simplicity of the eigenvalue $\lambda_{1,q,m}$, using Proposition 2.1, for a subsequence, we deduce that $(w_k)_k$ converges to $\alpha\phi_{1,q,m}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, weakly in $V_q(\mathbb{R}^N)$ (and strongly in $L^2(\mathbb{R}^N)$). But $(v_k)_k$ converges to $\beta\phi_{1,q,m} > 0$ in $C^1_{loc}(\mathbb{R}^N)$. So v_k is positive on the unit ball B_1 for k sufficiently large and so $v_k^- \equiv 0$, $w_k \equiv 0$ on B_1. Thus $\alpha = 0$. Therefore we get a contradiction.

So $v_k \geq 0$ in $\mathbb{R}^N$ for k sufficiently large and v_k satisfies (3.3) with $\lambda_k > \lambda_{1,q,m}$. This contradicts Proposition 3.1 and this concludes the proof of the local antimaximum principle theorem. □

Note that we obtain the same kind of local antimaximum principle for $\lambda < \tilde{\lambda}_{1,q,m}$ since $\tilde{\lambda}_{1,q,m} = -\lambda_{1,q,-m}$ and the equation $(-\Delta + q)u = \lambda m u + f$ is equivalent to $(-\Delta + q)u = (-\lambda)(-m)u + f$. To conclude this section, as in [15], we give a result for the semi-global antimaximum principle.

Proposition 3.2 *Assume that there exists $R_0 \in \mathbb{R}$, $R_0 > 0$ such that $m \leq 0$ in $B^C_{R_0} := \mathbb{R}^N \setminus B_{R_0}$. Assume also that $f \geq 0$, $f \not\equiv 0$, and there exists a constant $C \geq 0$ such that $f \leq -Cm\phi_{1,q,m}$ in $B^C_{R_0}$. Let $\delta = \delta(f, R_0)$ be given by Theorem 3.1. Then for any $\lambda \in]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution u of (1.1) satisfies $u \leq \frac{C}{\lambda - \lambda_{1,q,m}}\phi_{1,q,m}$ in $\mathbb{R}^N$.*

Proof: Let $C' = \frac{C}{\lambda - \lambda_{1,q,m}}$ (with $\lambda \in]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$) and $v = u - C'\phi_{1,q,m}$. We want to prove that $v^+ = 0$ in $\mathbb{R}^N$. First note that $v^+ = 0$ in B_{R_0} by Theorem 3.1. Moreover we have

$$(-\Delta + q)v = (\lambda - \lambda_{1,q,m})mu + \lambda_{1,q,m} m v + f \text{ in } \mathbb{R}^N. \tag{3.4}$$

Multiplying (3.4) by v^+ and integrating over $\mathbb{R}^N$, since $v^+ = 0$ in B_{R_0}, we get

$$0 \leq \int_{B^C_{R_0}} [|\nabla v^+|^2 + q|v^+|^2] = \lambda_{1,q,m} \int_{B^C_{R_0}} m|v^+|^2 + \int_{B^C_{R_0}} [(\lambda - \lambda_{1,q,m})mu + f]v^+.$$

Since $f \le -Cm\phi_{1,q,m}$ in $B_{R_0}^C$ and $m \le 0$ in $B_{R_0}^C$, we obtain

$$0 \le \int_{B_{R_0}^C} [(\lambda - \lambda_{1,q,m})mu + f]v^+ \le (\lambda - \lambda_{1,q,m}) \int_{B_{R_0}^C} m|v^+|^2 \le 0.$$

Therefore $\int_{B_{R_0}^C} [|\nabla v^+|^2 + q|v^+|^2] = 0$ and $v^+ = 0$ in $B_{R_0}^C$. □

Theorem 3.2 *Assume that there exists $R_0 \in \mathbb{R}$, $R_0 > 0$ such that $m \le 0$ in $B_{R_0}^C := \mathbb{R}^N \setminus B_{R_0}$. Assume also that $f \ge 0$, $f \not\equiv 0$, f with compact support. Then there exists $\delta := \delta(f)$ a positive constant such that for any $\lambda \in]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution u of (1.1) is negative in $\mathbb{R}^N$.*

Proof: Let $R_1 \ge R_0$ such that *suppf* $\subset B_{R_1}$ and let $\delta := \delta(f, R_1)$ given by Theorem 3.1. Let $\lambda \in]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$ and u a solution of (1.1). From Proposition 3.2 with $C = 0$ we get that $u \le 0$ in $\mathbb{R}^N$. Moreover from Theorem 3.1 we have $u < 0$ in $\overline{B}_{R_1}$.

Let now $x \in \overline{B}_{R_1}^C := \mathbb{R}^N \setminus \overline{B}_{R_1}$ and $r > 0$ such that $B(x,r) \cap B_{R_1} \ne \emptyset$ and $B(x,r) \cap suppf = \emptyset$. Let c_r be a positive constant such that $c_r + \lambda m - q > 0$ in $B(x,r)$ the open ball of center x and radius r. Since $(-\Delta)(-u) + c_r(-u) = (c_r + \lambda m - q)(-u) \ge 0$ in $B(x,r)$, by the strong maximum principle, we get that $-u \equiv 0$ or $-u > 0$ in $B(x,r)$. Since $u < 0$ in $\overline{B}_{R_1}$ we deduce that $-u > 0$ in $B(x,r)$. So $u(x) < 0$ and this concludes the proof of the semi-global antimaximum principle. □

4 Study of a linear elliptic system

In this section, we study the antimaxium principle for the following system

$$(-\Delta + q_i)u_i = \lambda \left(m_i u_i + \sum_{j=1; j \ne i}^{n} m_{ij} u_j \right) + f_i \text{ in } \mathbb{R}^N, \; i = 1, \cdots, n, \tag{4.1}$$

where each of the potentials q_i satisfy $(\mathbf{H_q^1})$-$(\mathbf{H_q^2})$, each of the weights m_i satisfy the hypothesis $(\mathbf{H_m'^1})$ and each of the functions f_i satisfy the hypothesis $(\mathbf{H_f^1})$. We denote by M the $n \times n$-matrix given by $M = (m_{ij})$ with $m_{ii} = m_i$. We will consider the following hypotheses:

$(\mathbf{H_M^1})$ For all $i \ne j$, $m_{ij} \in L^\infty(\mathbb{R}^N)$ and $m_{ij} > 0$.

$(\mathbf{H_M^2})$ M is a symmetric matrix.

$(\mathbf{H_M^3})$ $\Omega := \cap_{i=1}^n \Omega_i^+$ is an open subset of $\mathbb{R}^N$ with non zero measure and with $\Omega_i^+ := \{x \in \mathbb{R}^N, m_i(x) > 0\}$.

We also consider the following system:

$$(-\Delta + q_i)u_i = \lambda \left(m_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij} u_j \right) \text{ in } \mathbb{R}^N, \; i = 1, \cdots, n. \tag{4.2}$$

We recall from [7] the existence of a positive and simple eigenvalue associated with a positive eigenfunction for (4.2).

Theorem 4.1 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$-$(\mathbf{H_q^2})$ *and each of the weights m_i satisfy* $(\mathbf{H_m'^1})$. *Assume also that* $(\mathbf{H_M^1})$-$(\mathbf{H_M^3})$ *are satisfied. Then there exists a unique principal eigenvalue $\Lambda_{1,M} > 0$ associated with a positive eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M}) \in V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ for the system (4.2) (and $\phi_{i,M} > 0$ for all i). The eigenvalue $\Lambda_{1,M}$ is simple and verifies*

$$\Lambda_{1,M} = \inf \left\{ \frac{\sum_{i=1}^n \|u_i\|_{q_i}^2}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j}, \; u = (u_1, \cdots, u_n) \in V \right.$$

$$\left. \textit{such that } \sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j > 0 \right\}. \tag{4.3}$$

We recall from [7] the maximum principle for (4.1).

Theorem 4.2 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$-$(\mathbf{H_q^2})$ *and each of the weights m_i satisfy* $(\mathbf{H_m'^1})$. *Assume also that* $(\mathbf{H_M^1})$-$(\mathbf{H_M^3})$ *are satisfied. Assume that $f_i \in L^2(\mathbb{R}^N)$ for all i. If $0 \leq \lambda < \Lambda_{1,M}$, then the system (4.1) satisfies the maximum principle: if $f = (f_1, \cdots, f_n) \geq 0$, then $u_i \geq 0$ for all i with $u = (u_1, \cdots, u_n)$ solution of (4.1).*

Now we study the local antimaximum principle for (4.1). As for one equation, note that any solution $u = (u_1, \cdots, u_n)$ of (4.1) satisfies $u \in (C^1(\mathbb{R}^N))^n$. We now extend Proposition 3.1 to the system (4.1).

Proposition 4.1 *If $f_i \geq 0$, $f_i \not\equiv 0$ for all i, then (4.1) has no solution if $\lambda = \Lambda_{1,M}$ and has no nonnegative solution if $\lambda > \Lambda_{1,M}$.*

Proof: First assume that $\lambda = \Lambda_{1,M}$ and there exists a solution $u = (u_1, \cdots, u_n)$ for (4.1). Multiplying each equation of (4.1) by $\phi_{i,M}$ as a test function, integrating over $\mathbb{R}^N$ and adding all these equations, since M is a symmetric matrix, we obtain that $\sum_{i=1}^n \int_{\mathbb{R}^N} f_i \phi_{i,M} = 0$ and so we get a contradiction since $f_i \geq 0$, $f_i \not\equiv 0$, $\phi_{i,M} > 0$.

Assume now that $\lambda > \Lambda_{1,M}$ and there exists a nonnegative solution $u = (u_1, \cdots, u_n)$ for (4.1) i.e. $u_i \geq 0$ for all i. Let $R > 0$ and c_R a positive constant sufficiently large such that $c_R + \lambda m_i - q_i \geq 0$ in B_R for any i. Note that for any i

$$-\Delta u_i + c_R u_i = (c_R + \lambda m_i - q_i)u_i + \lambda \sum_{j=1;j\neq i}^{n} m_{ij} u_j + f_i \geq 0 \text{ in } B_R.$$

Applying the strong maximum principle in B_R, since $\lambda > 0$, $m_{ij} > 0$, $u_j \geq 0$, $f_i \geq 0$, $f_i \not\equiv 0$, we obtain that $u_i > 0$ in B_R for any R sufficiently large and so $u_i > 0$ in $\mathbb{R}^N$.

Let now for each $i = 1, \cdots, n$ $(\psi_{ik})_k$ be a convergent sequence to $\phi_{i,M}$ in $V_{q_i}(\mathbb{R}^N)$, $\psi_{ik} \geq 0$, $\psi_{ik} \in D(\mathbb{R}^N)$. Applying the Picone identity, we get

$$\int_{\mathbb{R}^N} \left(|\nabla \psi_{ik}|^2 - \nabla u_i \cdot \nabla(\frac{\psi_{ik}^2}{u_i}) \right) =$$
$$\|\psi_{ik}\|_{q_i}^2 - \lambda \int_{\mathbb{R}^N} m_i \psi_{ik}^2 - \lambda \sum_{j=1;j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij} u_j \frac{\psi_{ik}^2}{u_i} - \int_{\mathbb{R}^N} f_i \frac{\psi_{ik}^2}{u_i} \geq 0.$$

Therefore

$$\sum_{i=1}^{n} \int_{\mathbb{R}^N} \left(|\nabla \psi_{ik}|^2 - \nabla u_i \cdot \nabla(\frac{\psi_{ik}^2}{u_i}) \right) = \sum_{i=1}^{n} \|\psi_{ik}\|_{q_i}^2 - \lambda \sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i \psi_{ik}^2$$
$$-\lambda \sum_{i,j=1;j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij} u_j \frac{\psi_{ik}^2}{u_i} - \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i \frac{\psi_{ik}^2}{u_i} \geq 0.$$

Since $(\psi_{ik})_k$ is a convergent sequence to $\phi_{i,M}$ in $V_{q_i}(\mathbb{R}^N)$, we have

$$\|\psi_{ik}\|_{q_i}^2 \to \|\phi_{i,M}\|_{q_i}^2 = \Lambda_{1,M}(\int_{\mathbb{R}^N} m_i \phi_{i,M}^2 + \sum_{j=1;i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} \phi_{i,M} \phi_{j,M}) \text{ as } k \to \infty.$$

Passing to the limit by the Lebesgue dominated convergence Theorem and Fatou Lemma, we get

$$(\Lambda_{1,M} - \lambda) \sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i \phi_{i,M}^2 - \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i \frac{\phi_{i,M}^2}{u_i}$$
$$+\Lambda_{1,M} \sum_{i,j=1;i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} \phi_{i,M} \phi_{j,M} - \lambda \sum_{i,j=1;j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij} u_j \frac{\phi_{iM}^2}{u_i} \geq 0.$$

Since $\lambda > \Lambda_{1,M}$ and $\sum_{i,j=1;i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} \phi_{i,M} \phi_{j,M} > 0$ thus

$$(\Lambda_{1,M} - \lambda) \sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i \phi_{i,M}^2 - \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i \frac{\phi_{i,M}^2}{u_i} - \lambda \sum_{i,j=1;i<j}^{n} \int_{\mathbb{R}^N} \frac{m_{ij}}{u_i u_j} (u_j \phi_{i,M} - u_i \phi_{j,M})^2 \geq 0.$$

And we get a contradiction since all the two first terms of this estimate are negative and the third term is nonpositive. □

We give now the local antimaximum principle.

Theorem 4.3 *Let $f = (f_1, \cdots, f_n)$, $f_i \geq 0$, $f_i \not\equiv 0$ for all i. Then for any $R > 0$ there exists a positive constant $\delta = \delta(f, R) > 0$ such that for any $\lambda \in]\Lambda_{1,M}, \Lambda_{1,M} + \delta[$, any solution $u = (u_1, \cdots, u_n)$ of (4.1) is negative in $\overline{B}_R$ i.e. $u_i < 0$ in $\overline{B}_R$ for all i.*

Proof: Assume by contradiction that for some $R>0$ there exist $\lambda_k > \Lambda_{1,M}$, $\lambda_k \searrow \Lambda_{1,M}$, a solution $u_k = (u_{1k}, \cdots, u_{nk})$ of

$$(-\Delta + q_i)u_{ik} = \lambda_k(m_i u_{ik} + \sum_{j=1;j\neq i}^{n} m_{ij}u_{jk}) + f_i \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n \tag{4.4}$$

and $i_k \in \{1, \cdots, n\}$, $x_{i_k,k} \in \overline{B}_R$ such that $u_{i_k k}(x_{i_k,k}) \geq 0$.

First we show that $\lim_{k\to\infty} \|u_{ik}\|_{q_i} = \infty$ for at least one i. On the contrary, assume that $(\|u_{ik}\|_{q_i})_k$ is a bounded sequence for all i. Therefore, from the compact embedding of $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$, for a subsequence, there exists $u_i \in V_{q_i}(\mathbb{R}^N)$ such that $(u_{ik})_k$ converges to u_i, weakly in $V_{q_i}(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. Passing to the limit in (4.4) as in Theorem 3.1 we get

$$(-\Delta + q_i)u_i = \Lambda_{1,M}(m_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij}u_j) + f_i \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n$$

which contradicts Proposition 4.1. So $\lim_{k\to\infty} \|u_{ik}\|_{q_i} = \infty$ for at least one i.

Now set $v_{jk} = \frac{u_{jk}}{\sum_{i=1}^n \|u_{ik}\|_{q_i}}$ for all j. Then v_{ik} satisfies

$$(-\Delta + q_i)v_{ik} = \lambda_k(m_i v_{ik} + \sum_{j=1;j\neq i}^{n} m_{ij}v_{jk}) + \frac{f_i}{\sum_{i=1}^n \|u_{ik}\|_{q_i}} \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n. \tag{4.5}$$

Since $(v_{ik})_k$ is a bounded sequence in $V_{q_i}(\mathbb{R}^N)$, as before, for a subsequence, there exists $v_i \in V_{q_i}(\mathbb{R}^N)$ such that $(v_{ik})_k$ converges to v_i, weakly in $V_{q_i}(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. And $v = (v_1, \cdots, v_n)$ satisfies

$$(-\Delta + q_i)v_i = \Lambda_{1,M}(m_i v_i + \sum_{j=1;j\neq i}^{n} m_{ij}v_j) \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n.$$

Since $\Lambda_{1,M}$ is a simple eigenvalue then there exists $\beta \in \mathbb{R}$ such that

$$v = \beta\Phi_{1,M} \text{ i.e. for all } i = 1, \cdots, n,\ v_i = \beta\phi_{i,M}.$$

We will consider three cases for β.

If $\beta = 0$ then $v = 0$. Note that for any $i = 1, \cdots, n$,

$$\|v_{ik}\|^2_{q_i} = \lambda_k \int_{\mathbb{R}^N} m_i v_{ik}^2 + \lambda_k \sum_{j=1;j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij}v_{jk}v_{ik} + \int_{\mathbb{R}^N} \frac{f_i v_{ik}}{\sum_{i=1}^n \|u_{ik}\|_{q_i}}.$$

By the Lebesgue dominated convergence Theorem, we get that $\|v_{ik}\|_{q_i} \to 0$ as $k \to \infty$ for all i, which is impossible since $\sum_{i=1}^n \|v_{ik}\|_{q_i} = 1$. Therefore $\beta \neq 0$.

If now $\beta < 0$ then $v < 0$ in $\mathbb{R}^N$. But for any i, $(v_{ik})_k$ converges to v_i in $C^1_{loc}(\mathbb{R}^N)$. So v_{ik} is negative in $\overline{B}_R$ for k sufficiently large, which contradicts the existence of the sequence $x_{i_k,k}$.

So we consider the last case $\beta > 0$. We will show that for any i, $v_{ik} \geq 0$ i.e. $v_{ik}^- \equiv 0$ in $\mathbb{R}^N$ for k sufficiently large. On the contrary, assume that there exists i_0 such that $v_{i_0k}^- \not\equiv 0$. Denote by

$$D(u) := \sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} u_i u_j \text{ for } u = (u_1, \cdots, u_n) \in V.$$

Multiplying (4.5) by v_{ik}^- and integrating over $\mathbb{R}^N$, we get that

$$0 < \sum_{i=1}^{n} \|v_{ik}^-\|_{q_i}^2 = \lambda_k \Big(\sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i (v_{ik}^-)^2 + \sum_{i,j=1;i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} v_{jk}^- v_{ik}^-\Big)$$

$$-\lambda_k \sum_{i,j=1;i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} v_{jk}^+ v_{ik}^- - \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{f_i v_{ik}^-}{\sum_{i=1}^n \|u_{ik}\|_{q_i}}.$$

So $0 < \sum_{i=1}^n \|v_{ik}^-\|_{q_i}^2 \leq \lambda_k D(v_k^-)$ with $v_k^- = (v_{1k}^-, \cdots, v_{nk}^-)$ and therefore $D(v_k^-) > 0$ and $\Lambda_{1,M} = \lim_{k\to\infty} \frac{\|v_k^-\|_V^2}{D(v_k^-)}$ with $\|v_k^-\|_V^2 = \sum_{i=1}^n \|v_{ik}^-\|_{q_i}^2$.

Let $w_k = \frac{1}{D(v_k^-)^{1/2}} v_k^-$. So (w_k) is a minimizing sequence for (4.3) and from the simplicity of the eigenvalue $\Lambda_{1,M}$, for a subsequence, we deduce that $(w_k)_k$ converges to $\alpha\Phi_{1,M}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, weakly in $V(\mathbb{R}^N)$ (and strongly in $(L^2(\mathbb{R}^N))^n$). Indeed first note that (w_k) is a bounded sequence in V so there exists $w \in V$ such that (w_k) converges to w weakly in V and strongly in $(L^2(\mathbb{R}^N))^n$. Note also that $D(w_k) = 1$ and so $\frac{\|w_k\|_V^2}{D(w_k)} \to \Lambda_{1,M}$ as $k \to \infty$. Moreover $D(w_k) \to D(w)$ as $k \to \infty$ and $\|w\|_V \leq \liminf \|w_k\|_V = \sqrt{\Lambda_{1,M}}$ since (w_k) converges weakly to w in V. So using the variational characterization (4.3) of $\Lambda_{1,M}$ we get that $\frac{\|w\|_V^2}{D(w)} = \Lambda_{1,M}$. Thus w realizes the infimum of $\Lambda_{1,M}$ and from the simplicity of the eigenvalue $\Lambda_{1,M}$ we deduce the existence of a real $\alpha \neq 0$ such that $w = \alpha\Phi_{1,M}$.

But $(v_k)_k$ converges to $\beta\Phi_{1,M} > 0$ in $(C^1_{loc}(\mathbb{R}^N))^n$. So for all i, v_{ik} is positive on the unit ball B_1 for k sufficiently large and so $v_{ik}^- \equiv 0$, $w_{ik} \equiv 0$ on B_1. Thus $\alpha = 0$. Therefore we get a contradiction.

So $v_{ik} \geq 0$ in $\mathbb{R}^N$ for all $i = 1, \cdots, n$ and for k sufficiently large and v_{ik} satisfies (4.5) with $\lambda_k > \Lambda_{1,M}$. This contradicts Proposition 4.1 and this concludes the proof of the local antimaximum principle theorem. □

These results can be extended to the system (4.1) with weights m_i satisfying $(\mathbf{H'^2_m})$ (see [9] for the existence of a principal, positive and simple eigenvalue $\Lambda_{1,M}$).

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