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The existence and C^1-smoothness of local center-unstable manifolds for differential equations with state-dependent delay

ABSTRACT. The purpose of this work is to construct C^1-smooth local center-unstable manifolds at a stationary point for a class of functional differential equations of the form $\dot{x}(t) = f(x_t)$. Here the function f under consideration is defined on an open subset of the space $C^1([-h, 0], \mathbb{R}^n)$, $h > 0$, and satisfies some mild smoothness conditions which are often fulfilled when f represents the right-hand side of a differential equation with state-dependent delay.

KEY WORDS. Center-unstable manifold, functional differential equation, state-dependent delay

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1 Introduction

The interest in delay differential equations (abbreviated by DDE, respectively DDEs) dates back at least to the work [10] of Poisson from the year 1806. Even so, the general theory started to be systematically developed only at the beginning of the second half of the last century. During the 60th and 70th the theory of DDEs became an established field of mathematical research. In that progress, the development of another, more abstract class of differential equations, namely the so-called retarded functional differential equations (abbreviated by RFDE, respectively RFDEs), was essential. The development of the theory of RFDEs has also been started in the second half of the last century. We point out the fundamental work [3] and the newer edition [4] of Hale. Great parts of the theory of RFDEs is now as well understood as that for ordinary differential equations as presented in the monographs [2, 5].

Different DDEs with constant as well as with time- or state-dependent delay can be represented in the more abstract form of an RFDE. Accordingly, after carrying out such a transformation, one may ask whether basic or even far-reaching results for RFDEs may be used to study the original differential equation with delay. It turns out that the solution of this question is essentially dependent on the involved delays of the considered DDE. The reason is that the representation of a DDE in the more abstract form of an RFDE may lead to a loss of smoothness of the right-hand side if the involved delays are not constant. Therefore, the theory of RFDEs is in general not applicable to study DDEs with state-dependent delays and a lot of problems such as linearization and invariant manifolds for differential equations with state-dependent delay at a stationary point stayed open for many years.

In recent times, Walther introduced a modified class of functional differential equations and developed the fundamental theory in the series [13–15] of works under mild smoothness hypothesis. The main idea of Walther's approach is to study an abstract functional differential equation only on a smooth submanifold, the so-called solution manifold, of a function space. He proved that under mild smoothness assumptions the Cauchy problem is well-posed on the solution manifold, and the solutions generate a continuous semiflow with continuously differentiable solution operators. In particular, this framework seems to be often applicable in cases where the corresponding functional differential equation represents a DDE with state-dependent delay. Additionally, in cases of applicability it solves the difficulties concerning the linearization of a semiflow generated by differential equations with state-dependent delays. As long as the problem of linearization had not been solved, heuristical methods based on formal linearization were used for considerations as local stability and instability of stationary points. The work [1] of Cooke and Huang is indicative for such an approach.

In connection with the semiflow from the framework in [13–15] the existence of different types of local invariant manifolds at a stationary point is also well know by now. For instance, in [7] Krisztin considers an abstract class of functional differential equations and proves the existence of local unstable manifolds under a hyperbolicity condition but without knowledge of a semiflow. However, the result in [7] is also applicable in the situation of the semiflow discussed in [13–15]. Additionally, [7] discusses the construction of so-called fast or strong unstable manifolds without the hyperbolicity condition. A proof of the existence of continuously differentiable local stable and local center manifolds at stationary points is contained in the survey paper [6] of Hartung et al. and in the work [8] of Krisztin. The occurrence of continuously differentiable local center-stable manifolds is confirmed by Qesmi and Walther in the recent work [11].

The aim of this work is to prove the existence and C^1-smoothness of local center-unstable manifolds at stationary points for the semiflow from [13–15]. For this purpose, we first follow the approach used in Hartung et al. [6] for the construction of local center manifolds, and apply a modification of the Lyapunov-Perron method contained in Diekmann et al. [2] to establish the existence of Lipschitz continuous local center-unstable manifolds. Hereafter, we employ the techniques from Krisztin [8] to prove C^1-smoothness.

2 The Main Result

Let $h > 0$, $n \in \mathbb{N}$ and $\|\cdot\|_{\mathbb{R}^n}$ a norm in $\mathbb{R}^n$. For abbreviation, let us denote by C the set of all continuous functions from the interval $[-h, 0]$ into $\mathbb{R}^n$, equipped with the norm

$$\|\varphi\|_C := \max_{s\in[-h,0]} \|\varphi(s)\|_{\mathbb{R}^n}$$

of uniform convergence. Analogously, we write C^1 for the Banach space of all continuously differentiable functions $\varphi : [-h, 0] \longrightarrow \mathbb{R}^n$, provided with the norm $\|\varphi\|_{C^1} := \|\varphi\|_C + \|\varphi'\|_C$.

For a given function $x : I \longrightarrow \mathbb{R}^n$ defined on some interval $I \subseteq \mathbb{R}$, and $t \in \mathbb{R}$ with $[t-h, t] \subset I$, the **segment** x_t of x at t is defined by the relation $x_t(\vartheta) := x(t + \vartheta)$, $\vartheta \in [-h, 0]$; that is, by x_t we restrict the function x to $[t - h, t]$ and shift it back to $[-h, 0]$. In particular, if the function x is continuous, then clearly $x_t \in C$.

Let $U \subseteq C^1$ be an open neighborhood of the origin $0 \in C^1$ and a function $f : U \longrightarrow \mathbb{R}^n$ with $f(0) = 0$ be given. Throughout this paper, we consider the functional differential equation

$$\dot{x}(t) = f(x_t) \tag{1}$$

under the following conditions on the right-hand side:

(S 1) f is continuously differentiable, and

(S 2) each derivative $Df(\varphi)$, $\varphi \in U$, extends to a linear map

$$D_e f(\varphi) : C \longrightarrow \mathbb{R}^n,$$

and the induced map

$$U \times C \ni (\varphi, \chi) \longmapsto D_e f(\varphi)\, \chi$$

is continuous.

By a **solution** of the differential equation (1) we understand either a continuously differentiable function $x : [t_0 - h, t_e) \longrightarrow \mathbb{R}^n$ with $t_0 < t_e \leq \infty$ such that $x_t \in U$ for $t_0 \leq t < t_e$ and Eq. (1) holds for $t_0 < t < t_e$, or a continuously differentiable function $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $x_t \in U$ and Eq. (1) holds everywhere in $\mathbb{R}$. Additionally, we will consider solutions on unbounded, right-closed intervals $(-\infty, t_e]$, $-\infty < t_e$, which are defined in an analogous way.

By assumption $x(t) = 0$, $t \in \mathbb{R}$, is a solution of Eq. (1) as $f(0) = 0$. Therefore, the closed subset

$$X_f := \{\varphi \in U \mid \varphi'(0) = f(\varphi)\}$$

of C^1 is not empty. Under the above conditions on f the framework developed in [13–15] implies the following fundamental results. The **solution manifold** X_f is a C^1-submanifold of $U \subseteq C^1$ with codimension n. Each $\varphi \in X_f$ uniquely defines a constant $t_+(\varphi) > 0$ and a (in the forward time direction) non-continuable solution $x^\varphi : [-h, t_+(\varphi)) \longrightarrow \mathbb{R}^n$ of Eq. (1) with initial value $x_0^\varphi = \varphi$. All segments x_t^φ, $0 \leq t < t_+(\varphi)$ and $\varphi \in X_f$, belong to X_f and the equations

$$F(t, \varphi) = x_t^\varphi$$

define a continuous semiflow $F : \Omega \longrightarrow X_f$ on the solution manifold X_f where

$$\Omega = \{(t, \varphi) \in [0, \infty) \times X_f \mid 0 \leq t < t_+(\varphi)\}.$$

For every $t \geq 0$ the solution map at time t, that is, the map

$$F_t : \{\psi \in X_f | \; 0 \leq t < t_+(\psi)\} \ni \varphi \longmapsto F(t, \varphi) \in X_f,$$

is continuously differentiable, and for each $\varphi \in X_f$ the tangent space of X_f at φ is

$$T_\varphi X_f = \left\{\chi \in C^1 \mid \chi'(0) = Df(\varphi)\,\chi\right\}.$$

For all $(t, \varphi) \in \Omega$ and all $\chi \in T_\varphi X_f$ the derivative

$$DF_t\varphi : T_\varphi X_f \longrightarrow T_{F_t(\varphi)} X_f$$

satisfies the equations

$$DF_t(\varphi)\,\chi = v_t^{\varphi,\chi},$$

where $v^{\varphi,\chi} : [-h, t_+(\varphi)) \longrightarrow \mathbb{R}^n$ is the solution of the (linear) initial value problem

$$\begin{cases} \dot{v}(t) = Df(F(t, \varphi))\, v_t \\ v_0 = \chi \end{cases} \tag{2}$$

for $\chi \in T_\varphi X_f$. Here a solution of the Cauchy problem (2) is a continuously differentiable function $v : [-h, t_e(\varphi)) \longrightarrow \mathbb{R}$ such that $v_0 = \chi$, $v_t \in T_{F(t,\varphi)} X_f$ for all $0 \leq t < t_e(\varphi)$ and v satisfies the differential equation for all $0 < t < t_e(\varphi)$.

Obviously, we have $F(t,0) = 0$ for all $t \in \mathbb{R}$; that is, $\varphi_0 := 0 \in X_f$ is a stationary point of the semiflow F. As discussed in Hartung et al. [6] the linearization of F at $\varphi_0 = 0$ is the strongly continuous semigroup $T = \{T(t)\}_{t\geq 0}$ of bounded linear operators $T(t) = D_2F(t,0)$, $t \geq 0$, on the Banach space

$$T_0X_f = \left\{\chi \in C^1 \mid \chi'(0) = Df(0)\,\chi\right\},$$

equipped with the norm $\|\cdot\|_{C^1}$ of C^1. For any $t \geq 0$ the action of $T(t)$ on an element $\chi \in T_0X_f$ is determined by the relation $T(t)\,\chi = v_t^{\chi}$, where $v^{\chi} : [-h,\infty) \longrightarrow \mathbb{R}^n$ is the unique solution of the variational equation

$$\dot{v}(t) = Df(0)\,v_t \tag{3}$$

with initial value $v_0 = \chi$. The infinitesimal generator G of T is given by the linear operator

$$G : \mathcal{D}(G) \ni \chi \longmapsto \chi' \in T_0X_f$$

with domain

$$\mathcal{D}(G) = \left\{\chi \in C^2 \,\middle|\, \chi'(0) = Df(0)\,\chi,\ \ \chi''(0) = Df(0)\,\chi'\right\},$$

where C^2 denotes the set of all twice continuously differentiable functions from $[-h,0]$ into $\mathbb{R}^n$.

Remark 2.1 For the convenience of the reader we repeat that an RFDE on some open subset $V \subset \mathbb{R} \times C$ is an equation of the form

$$\dot{x}(t) = f_e(t,x_t) \tag{4}$$

with a function $f_e : V \longrightarrow \mathbb{R}^n$. A function x is a solution of Eq. (4) on the interval $[t_0-h,t_+)$, if there are $t_0 \in \mathbb{R}$ and $t_+ > t_0$ such that $x : [t_0-h,t_+) \longrightarrow \mathbb{R}^n$ is continuous, $(t,x_t) \in V$ for all $t_0 \leq t < t_+$, and x satisfies Eq. (4) for all $t_0 < t < t_+$. Solutions on unbounded intervals $(-\infty,t_+)$ or $(-\infty,t_+]$ for some $t_+ > -\infty$ are defined in an analogous way.

By assumption (S 2) on f the linear operator $Df(0)$ may be extended to a bounded linear operator $D_ef(0)$ on the larger space C. The operator $L_e := Df_e(0)$ induces the linear autonomous RFDE

$$\dot{v}(t) = L_e\,v_t$$

and the solutions of the associated initial value problem

$$\begin{cases} \dot{v}(t) = L_e\,v_t \\ v_0 = \chi \end{cases} \tag{5}$$

for initial values $\chi \in C$ define a strongly continuous semigroup $T_e = \{T_e(t)\}_{t\geq 0}$ on C as shown, for instance, in Diekmann et al. [2]. The infinitesimal generator of T_e is

$$G_e : \mathcal{D}(G_e) \ni \chi \longmapsto \chi' \in C$$

with the domain

$$\mathcal{D}(G_e) = \Big\{\chi \in C^1 \,\Big|\, \chi'(0) = L_e\,\chi\Big\}$$

which particularly coincides with T_0X_f. We have $T(t)\,\varphi = T_e(t)\,\varphi$ for all $\varphi \in \mathcal{D}(G_e)$ and $t \geq 0$.

For the spectra $\sigma(G_e), \sigma(G) \subset \mathbb{C}$ of the generators G_e, G of both semigroups we have

$$\sigma(G_e) = \sigma(G)$$

by [6]. The spectrum $\sigma(G_e)$ is given by the zeros of a familiar characteristic equation, is discrete and contains only eigenvalues of finite rank, that is, the generalized eigenspaces are finite-dimensional. Setting

$$\sigma_u(G_e) :=\{\lambda \in \sigma(G_e) \mid \mathrm{Re}(\lambda) > 0\},$$
$$\sigma_c(G_e) :=\{\lambda \in \sigma(G_e) \mid \mathrm{Re}(\lambda) = 0\}$$

and

$$\sigma_s(G_e) :=\{\lambda \in \sigma(G_e) \mid \mathrm{Re}(\lambda) < 0\},$$

we obtain the decomposition

$$\sigma(G_e) = \sigma_u(G_e) \cup \sigma_c(G_e) \cup \sigma_s(G_e).$$

As proven in Hale and Verduyn Lunel [5] or in Diekmann et al. [2], for each $\beta \in \mathbb{R}$ the half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > \beta\}$ of $\mathbb{C}$ contains at most a finite number of elements of $\sigma(G_e)$, so that spectral parts $\sigma_u(G_e)$, $\sigma_c(G_e)$ are empty or finite. Hence, the associated realified generalized eigenspaces C_u and C_c, which are called the **unstable** and the **center space** of G_e, respectively, are finite dimensional subspaces of C. In contrast, the **stable space** $C_s \subset C$ of G_e, that is, the realified generalized eigenspace associated to the spectral part $\sigma_s(G_e)$, is infinite-dimensional. The subspaces C_u, C_c and C_s are closed, invariant under $T_e(t)$, $t \geq 0$, and provide a decomposition

$$C = C_u \oplus C_c \oplus C_s \tag{6}$$

of C. The restriction of T_e to the finite dimensional spaces C_u, C_c has a bounded generator so that T_e may be extended to a one-parameter group in each case.

As a consequence of the above decomposition of C we obtain also a decomposition of the smaller Banach space C^1, namely

$$C^1 = C_u \oplus C_c \oplus C_s^1 \tag{7}$$

with the closed subspace $C_s^1 := C_s \cap C^1$ of C^1.

The sets C_u, C_c lie in $\mathcal{D}(G_e) = T_0X_f$ and coincide with the unstable and the center space of G, respectively. The stable space of G is $C_s \cap T_0X_f$. Consequently, we have the decomposition

$$T_0X_f = C_u \oplus C_c \oplus (C_s \cap T_0X_f).$$

All spaces are closed subspaces of T_0X_f and positively invariant under the operators $T(t)$, $t \geq 0$, and T forms a one-parameter group on each of the finite-dimensional subspaces C_u and C_c.

Using the notation $C_{cu} := C_u \oplus C_c$ for the **center-unstable space** of G, we are now able to state our result on the existence of local center-unstable manifolds for the semiflow F at the stationary point $\varphi_0 = 0$.

Theorem 1 (Existence of Local Center-Unstable Manifold) *Suppose in addition to the previous assumptions on f that $\{\lambda \in \sigma(G_e) \mid Re(\lambda) \geq 0\} \neq \emptyset$ or, equivalently, $C_{cu} \neq \{0\}$. Then there are open neighborhoods $C_{cu,0}$ of 0 in C_{cu} and $C_{s,0}^1$ of 0 in C_s^1 with $N_{cu} := C_{cu,0} + C_{s,0}^1 \subseteq U$, and a Lipschitz continuous map $w_{cu} : C_{cu,0} \longrightarrow C_{s,0}^1$ with $w_{cu}(0) = 0$, such that the graph*

$$W_{cu} := \Big\{\varphi + w_{cu}(\varphi) \mid \varphi \in C_{cu,0}\Big\}$$

has the following properties.

(i) *The set W_{cu} belongs to the solution manifold X_f of Eq. (1). Moreover, W_{cu} is a k-dimensional Lipschitz submanifold of X_f where $k := \dim C_{cu}$.*

(ii) *For each solution $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ of Eq. (1) on $(-\infty, 0]$, we have*

$$\Big\{x_t \mid t \leq 0\Big\} \subseteq N_{cu} \quad \Longrightarrow \quad \Big\{x_t \mid t \leq 0\Big\} \subseteq W_{cu}.$$

(iii) *The graph W_{cu} is positively invariant with respect to the semiflow F relative to N_{cu}; that is, if $\varphi \in W_{cu}$ and $t > 0$ then*

$$\Big\{F(s,\varphi) \mid 0 \leq s \leq t\Big\} \subset N_{cu} \quad \Longrightarrow \quad \Big\{F(s,\varphi) \mid 0 \leq s \leq t\Big\} \subset W_{cu}.$$

The submanifold W_{cu} of X_f is called a **local center-unstable manifold** of F at the stationary point $\varphi_0 = 0$. It is C^1-smooth and passes φ_0 tangentially to the center-unstable space C_{cu} as we shall have established by our next theorem.

Theorem 2 **(C^1-Smoothness of Local Center-Unstable Manifold)**
The map

$$w_{cu} : C_{cu,0} \longrightarrow C^1_{s,0}$$

obtained in Theorem 1 is continuously differentiable and $Dw_{cu}(0) = 0$.

In the next three sections we prove the above theorems. Even though the proofs are quite long and at certain points technical, they are nevertheless not difficult to understand. As mentioned in the introduction, we follow the construction of local center manifolds in Hartung et al. [6] and apply the Lyapunov-Perron method to obtain the existence of local center-unstable manifolds as claimed in Theorem 1. The basic idea of this method is to transform the differential equation (1), or more precisely, a smoothed modification of it, into an integral equation such that the corresponding integral operator forms a parameter-dependent contraction in an appropriate Banach space of continuous functions. The fixed points of this contraction define a mapping whose graph forms the desired invariant manifold.

After the described construction, we follow the procedure in Krisztin [8] and show the C^1-dependence of the obtained fixed points on the parameter which leads to the continuous differentiability of the manifolds asserted in Theorem 2.

3 Preliminaries for the Proof of Existence

For the transformation of the considered differential equation into an integral form we will employ a variation-of-constants formula, which is established in Diekmann et al. [2] and involves duality and adjoint semigroups. For the convenience of the reader and to make our exposition self-contained, we repeat some of the relevant material from Diekmann et al. [2] without proofs. Afterwards we discuss some preparatory results.

Duality and Sun-Reflexivity

Recall that for a Banach space X over $\mathbb{R}$ the **dual space** X^* is the set of all continuous linear functionals on X, that is, X^* consists of all continuous linear maps from X into $\mathbb{R}$. We write x^* for elements of X^*, and for $x^* \in X^*$ and $x \in X$ we use the notation $\langle x^*, x\rangle \in \mathbb{R}$ instead of $x^*(x)$. Provided with the norm

$$\|x^*\|_{X^*} := \sup_{\|x\|_X \leq 1} |\langle x^*, x\rangle| \,,$$

where $\|\cdot\|_X$ denotes the norm on X, the dual space X^* becomes also a Banach space over $\mathbb{R}$.

If $A : \mathcal{D}(A) \longrightarrow X$ is a linear operator defined on some dense linear subspace $\mathcal{D}(A)$ in X, then its **adjoint** A^* is defined by

$$\mathcal{D}(A^*) = \Big\{ x^* \in X^* \Big| \; \exists\, y^* \in X^* \text{ with } \langle y^*, x \rangle = \langle x^*, A\, x \rangle \text{ for all } x \in \mathcal{D}(A) \Big\}$$

and then for $x^* \in \mathcal{D}(A^*)$

$$A^* x^* = y^*.$$

If $A : X \longrightarrow X$ is a bounded linear operator, then for each $x^* \in X^*$ the induced map $X \ni x \longmapsto \langle x^*, A\, x \rangle \in \mathbb{K}$ is linear and bounded. Thus, in this case, the relations

$$\langle A^* x^*, x \rangle = \langle x^*, A\, x \rangle$$

for all $x \in X$ and $x^* \in X^*$ uniquely define a bounded linear operator $A^* : X^* \longrightarrow X^*$. In particular, we have $\|A\| = \|A^*\|$.

Consider now the Banach space C and the strongly continuous semigroup $T_e = \{T_e(t)\}_{t \geq 0}$ of bounded linear operators defined by the solutions of the initial value problem (5). For every $t \geq 0$ the adjoint $T_e^*(t)$ of $T_e(t)$ is a linear operator with norm $\|T_e^*(t)\| = \|T_e(t)\|$ on the dual space C^* of C and the family $T_e^* = \{T_e^*(t)\}_{t \geq 0}$ obviously constitutes a semigroup of operators on C^*. We also have $T_e^*(0)\, \varphi^* = \varphi^*$ for all $\varphi^* \in C^*$, but T_e^* is in general not a strongly continuous semigroup. Indeed, if C^* is equipped with the topology given by the norm $\| \cdot \|_{C^*}$, it is not difficult to see that for $\varphi^* \in C^*$ the induced curve

$$[0, \infty) \ni t \longmapsto T^*(t)\, \varphi^* \in C^* \tag{8}$$

is not necessarily continuous. However, the set of all functions $\varphi^\odot \in C^*$ for which the curve (8) is continuous, in other words, $\varphi^\odot \in C^*$ with the property $\|T_e^*(t)\, \varphi^\odot - \varphi^\odot\|_{C^*} \to 0$ as $t \searrow 0$, forms a closed subspace $C^\odot$ of C^*. Furthermore, $T_e^*(t)(C^\odot) \subset C^\odot$ for all $t \geq 0$ so that the family of operators

$$T_e^\odot(t) : C^\odot \ni \varphi^\odot \longmapsto T_e^*(t)\, \varphi^\odot \in C^\odot$$

constitutes a strongly continuous semigroup $T_e^\odot$ on $C^\odot$.

Remark 3.1 It is worth to mention that the family T_e^* of linear operators on C^* is a weak* continuous semigroup, and G_e^* the associated weak* generator. More precisely, if the dual space C^* of C is equipped with the so-called *weak* topology*, that is, the coarsest topology on C^* such that for all $\varphi \in C$ the functions $C^* \ni \varphi^* \longmapsto \langle \varphi^*, \varphi \rangle \in \mathbb{R}$ are continuous, then for each $\varphi^* \in C^*$ the induced curve (8) is continuous. In this way, T_e^* becomes a continuous semigroup and G_e^* its generator.

Similarly, we can repeat the above process with the Banach space $C^{\odot}$ and the strongly continuous semigroup $T_e^{\odot}$. At first, we introduce again the adjoint operators $T_e^{\odot *}(t)$ of $T_e^{\odot}(t)$, $t \geq 0$, on the dual space $C^{\odot *}$ of $C^{\odot}$, and afterwards we restrict the semigroup $T_e^{\odot *} := \{T_e^{\odot *}(t)\}_{t\geq 0}$ to the closed subspace $C^{\odot\odot}$, for which the semigroup is strongly continuous.

The original Banach space C together with the strongly continuous semigroup T_e is $\odot$-reflexive in the sense that there is an isometric linear map $j : C \longrightarrow C^{\odot *}$ with $jC = C^{\odot\odot}$ and $T_e^{\odot *}(t)(j\varphi) = j(T_e(t)\,\varphi)$ for all $\varphi \in C$ and $t \geq 0$. We omit the embedding operator j of C in $C^{\odot *}$ and simply identify the Banach space C with $C^{\odot\odot}$ as usual

The spectrum $\sigma(G_e^{\odot *})$ of the generator $G_e^{\odot *}$ for the semigroup $T_e^{\odot *}$ coincides with $\sigma(G_e)$, and the decomposition (6) of C results in the decomposition

$$C^{\odot *} = C_u \oplus C_c \oplus C_s^{\odot *} \tag{9}$$

of $C^{\odot *}$, where C_u, C_c, and $C_s^{\odot *}$ are closed and invariant under $T_e^{\odot *}$. Furthermore, there are constants $K \geq 1$, $c_s < 0 < c_u$ and $c_c > 0$ with $c_c < \min\{-c_s, c_u\}$ so that the asymptotic behavior of $T^{\odot *}$ on these subspaces is given by

$$\begin{aligned} \|T_e(t)\,\varphi\|_C &\leq K e^{c_u t}\|\varphi\|_C, && t \leq 0,\ \varphi \in C_u, \\ \|T_e(t)\,\varphi\|_C &\leq K e^{c_c |t|}\|\varphi\|_C, && t \in \mathbb{R},\ \varphi \in C_c, \\ \|T_e^{\odot *}(t)\,\varphi^{\odot *}\|_{C^{\odot *}} &\leq K e^{c_s t}\|\varphi^{\odot *}\|_{C^{\odot *}}, && t \geq 0,\ \varphi^{\odot *} \in C_s^{\odot *}. \end{aligned} \tag{10}$$

The decompositions (7), (9) of C^1 and $C^{\odot *}$ induce continuous projections P_u, P_c, P_s and analogously $P_u^{\odot *}$, $P_c^{\odot *}$, $P_s^{\odot *}$ onto subspaces C_u, C_c, C_s^1, and C_u, C_c, $C_s^{\odot *}$, respectively. Also, using the identification of C with $C^{\odot\odot}$ we see at once $C_s^1 = C^1 \cap C_s^{\odot *}$.

The Variation-of-Constants Formula

Next, we proceed with recalling the variation-of-constant formula for solutions of the inhomogeneous linear RFDE

$$\dot{x}(t) = L_e\, x_t + q(t) \tag{11}$$

with given function $q : I \longrightarrow \mathbb{R}^n$ on some interval $I \subset \mathbb{R}$. For this purpose, let $L^\infty([-h,0],\mathbb{R}^n)$ denote the Banach space of all measurable and essentially bounded functions from $[-h,0]$ into $\mathbb{R}^n$, provided with the norm $\|\cdot\|_{L^\infty}$ of essential least upper bound. With the norm

$$\|(\alpha,\varphi)\|_{\mathbb{R}^n\times L^\infty} := \max\{\|\alpha\|_{\mathbb{R}^n}, \|\varphi\|_{L^\infty}\},$$

the product space $\mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$ becomes also a Banach space, which is in particular isometrically isomorphic to the space $C^{\odot *}$. Using the temporary notation $k : C^{\odot *} \longrightarrow \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$ for a norm-preserving isomorphism from $C^{\odot *}$ onto $\mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$, we define elements $r_i^{\odot *} := k^{-1}(e_i, 0) \in C^{\odot *}$, $i = 1, \ldots, n$, where e_i is the i-th canonical basis vector of $\mathbb{R}^n$. Clearly, the family $\{r_1^{\odot *}, \ldots, r_n^{\odot *}\}$ constitutes a basis of the linear subspace $Y^{\odot *} := k^{-1}(\mathbb{R}^n \times \{0\})$ of $C^{\odot *}$, and the requirement $l(e_i) = r_i^{\odot *}$ for $i = 1, \ldots, n$ uniquely determines a linear bijective mapping $l : \mathbb{R}^n \longrightarrow Y^{\odot *}$ with $\|l\| = \|l^{-1}\| = 1$.

For reals $a \leq b \leq c$ and a (norm) continuous function $w : [a, b] \longrightarrow C^{\odot *}$ the **weak* integral**

$$\int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau \in C^{\odot *} \tag{12}$$

is defined by

$$\langle \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau, \varphi^{\odot} \rangle := \int_a^b \langle T_e^{\odot *}(c - \tau)\, w(\tau), \varphi^{\odot} \rangle\, d\tau$$

for $\varphi^{\odot} \in C^{\odot}$. Furthermore, set

$$\int_b^a T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau := - \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau$$

as usual. It turns out that, under the above condition on w, this weak* integral belongs to C (more precisely, to $C^{\odot\odot} = j(C)$). Additionally, one obtains the formulas

$$T_e^{\odot *}(t) \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau = \int_a^b T_e^{\odot *}(c + t - \tau)\, w(\tau)\, d\tau \tag{13}$$

for all $t \geq 0$,

$$P_\lambda^{\odot *} \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau = \int_a^b T_e^{\odot *}(c - \tau)\, P_\lambda^{\odot *}\, w(\tau)\, d\tau \tag{14}$$

with $\lambda \in \{s, c, u\}$, and finally the inequality

$$\left\| \int_a^b T_e^{\odot *}(c - \tau)\, w(\tau)\, d\tau \right\|_{C^{\odot *}} \leq \int_a^b \left\| T_e^{\odot *}(c - \tau)\, w(\tau) \right\|_{C^{\odot *}} d\tau. \tag{15}$$

If $q : I \longrightarrow \mathbb{R}^n$ is a continuous function defined on some interval $I \subseteq \mathbb{R}$ and if the function $x : I + [-h, 0] \longrightarrow \mathbb{R}^n$ is a solution of the inhomogeneous RFDE (11), then the curve $u : I \ni t \longmapsto x_t \in C$ satisfies the abstract integral equation

$$u(t) = T_e(t - s)\, u(s) + \int_s^t T_e^{\odot *}(t - \tau)\, Q(\tau)\, d\tau \tag{16}$$

for all $s, t \in I$ with $s \leq t$, where $Q : [s, t] \ni \tau \longmapsto l(q(\tau)) \in Y^{\odot *}$. On the other hand, if $Q : I \longrightarrow Y^{\odot *}$ is continuous, and if $u : I \longrightarrow C$ is a solution of Eq. (16) then there is a continuous function $x : I + [-h, 0] \longrightarrow \mathbb{R}^n$ with $x_t = u(t)$, $t \in I$, solving the differential equation (11) for the inhomogeneity $q : I \ni \tau \longmapsto l^{-1}(Q(\tau)) \in \mathbb{R}^n$. In this sense we have a one-to-one correspondence between solutions for Eq.s (11) and (16).

Preliminary Results on Inhomogeneous Linear Equations

As the last step to prepare the construction of local center-unstable manifolds for Eq. (1), we establish the existence and some properties of special solutions of the integral equation (16). In doing so, we will need certain Banach spaces which are introduced below.

Let X be a Banach space with norm $\|\cdot\|_X$. For every $\eta \geq 0$ we define the linear space

$$C_\eta((-\infty,0],X) = \left\{ g \in C((-\infty,0],X) \,\middle|\, \sup_{s\in(-\infty,0]} e^{\eta s}\,\|g(s)\|_X < \infty \right\}$$

where $C((-\infty,0],X)$ denotes the Banach space of all continuous functions from the interval $(-\infty,0]$ into X. Providing $C_\eta((-\infty,0],X)$ with the weighted supremum norm given by

$$\|g\|_{C_\eta} = \sup_{s\in(-\infty,0]} e^{\eta s}\|g(t)\|_X,$$

we obtain a one-parameter family of Banach spaces with the scaling property

$$C_{\eta_1}((-\infty,0],X) \subseteq C_{\eta_2}((-\infty,0],X)$$

for all $\eta_1 \leq \eta_2$ and

$$\|g\|_{C_{\eta_1}} \geq \|g\|_{C_{\eta_2}}$$

for all $g \in C_{\eta_1}((-\infty,0],X)$. To simplify notation, we use the abbreviations Y_η, C^0_η, and C^1_η, for the spaces $C_\eta((-\infty,0],Y^{\odot *})$, $C_\eta((-\infty,0],C)$, and $C_\eta((-\infty,0],C^1)$, respectively, which are mainly regarded in the sequel.

From now on, let us denote by $P^{\odot *}_{cu}$ the projection of $C^{\odot *}$ along $C^{\odot *}_s$ onto the center-unstable space C_{cu}, that is, $P^{\odot *}_{cu} := P^{\odot *}_u + P^{\odot *}_c$. For a given function $Q : (-\infty,0] \longrightarrow Y^{\odot *}$ we formally introduce a mapping $\mathcal{K}^{cu}\,Q$ from $(-\infty,0]$ into $C^{\odot *}$ by

$$(\mathcal{K}^{cu}\,Q)(t) = \int_0^t T^{\odot *}_e(t-\tau)\,P^{\odot *}_{cu}\,Q(\tau)\,d\tau + \int_{-\infty}^t T^{\odot *}_e(t-\tau)\,P^{\odot *}_s\,Q(\tau)\,d\tau \tag{17}$$

for $t \leq 0$. Note that the right-hand side of Eq. (17) may not be well-defined for arbitrary Q. However, in our next result we show that for maps $Q \in Y_\eta$ with $\eta \in \mathbb{R}$ such that $c_c < \eta < \min\{-c_s, c_u\}$ the integrals in (17) do not only exist, but the functions $\mathcal{K}^{cu}\,Q$ form also solutions for the abstract integral equation (16).

Proposition 3.2 *Let $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$ be given. Then Eq. (17) induces a bounded linear map*

$$\tilde{\mathcal{K}} : Y_\eta \ni Q \longmapsto \mathcal{K}^{cu}\,Q \in C^0_\eta.$$

In addition, for every $Q \in Y_\eta$ the function $u = \tilde{\mathcal{K}}\,Q$ is a solution of the integral equation

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T^{\odot *}_e(t-\tau)\,Q(\tau)\,d\tau \tag{18}$$

for $-\infty < s \leq t \leq 0$, and the only one in C^0_η satisfying $P^{\odot *}_{cu}\,u(0) = 0$.

Proof: The proof falls naturally into three parts. In the first one, we show that, under the stated assumption on $\eta \in \mathbb{R}$, the formal expression (17) forms indeed a well-defined mapping $\mathcal{K}^{cu} Q$ from $(-\infty, 0]$ into C for all $Q \in Y_\eta$. Afterwards we prove that $\tilde{\mathcal{K}}$ is a bounded linear operator and finally we conclude the part of the proposition concerning the abstract integral equation. From now on to the end of the proof, we fix $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$.

1. In order to see $(\mathcal{K}^{cu} Q)(t) \in C$ for all $Q \in Y_\eta$ and $t \leq 0$, recall that for given $Q \in Y_\eta$ and $t \leq 0$ both

$$\int_0^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau = -\int_t^0 T_e^{\odot *}(-\tau)\, T_e^{\odot *}(t)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau$$

and

$$I(s) := \int_s^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau$$

with $s \leq t$ belong to C. Hence, it remains to prove the convergence of $I(s)$ in C as $s \to -\infty$. To show this, we assume $\{s_k\}_{k\in\mathbb{N}} \subset (-\infty, t]$ with $s_k \to -\infty$ as $k \to \infty$. Then, by inequality (15) and the estimate (10) for the action of $T_e^{\odot *}$ on the center space,

$$\begin{aligned}
\|I(s_{k_2}) - I(s_{k_1})\|_{C^{\odot *}} &= \left\| \int_{s_{k_2}}^{s_{k_1}} T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \right\|_{C^{\odot *}} \\
&\leq \int_{s_{k_2}}^{s_{k_1}} \left\| T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau) \right\|_{C^{\odot *}} d\tau \\
&\leq K \left\| P_s^{\odot *} \right\| \int_{s_{k_2}}^{s_{k_1}} e^{c_s(t-\tau)}\, \|Q(\tau)\|_{C^{\odot *}}\, d\tau \\
&\leq e^{c_s t} K \left\| P_s^{\odot *} \right\| \int_{s_{k_2}}^{s_{k_1}} e^{-(c_s+\eta)\tau} e^{\eta\tau} \|Q(\tau)\|_{C^{\odot *}} d\tau \\
&\leq e^{c_s t} K \|P_s^{\odot *}\| \|Q\|_{Y_\eta} \int_{s_{k_2}}^{s_{k_1}} e^{-(c_s+\eta)\tau}\, d\tau \\
&\leq \frac{-e^{c_s t}}{c_s+\eta} K \|P_s^{\odot *}\| \|Q\|_{Y_\eta} \left[e^{-(c_s+\eta)s_{k_1}} - e^{-(c_s+\eta)s_{k_2}} \right] \\
&\leq \frac{-e^{c_s t}}{c_s+\eta} K \|P_s^{\odot *}\| \|Q\|_{Y_\eta} e^{-(c_s+\eta)s_{k_1}}
\end{aligned}$$

for all $k_1, k_2 \in \mathbb{N}$ with $s_{k_1} \geq s_{k_2}$. Thus, $\{I(s_k)\}_{k\in\mathbb{N}}$ constitutes a Cauchy sequence in C. In particular, $I := \lim_{k\to\infty} I(s_k)$ exists. Furthermore, in the same manner we see that for any another given sequence $\{\tilde{s}_k\}_{k\in\mathbb{N}} \subset (-\infty, t]$ of reals with $\tilde{s}_k \to -\infty$, we also have $\|I(\tilde{s}_k) - I\|_{C^{\odot *}} \to 0$ as $k \to \infty$. This implies the desired conclusion $I = \lim_{s\to-\infty} I(s)$. Hence, $(\mathcal{K}^{cu} Q)(t) \in C$ for all $Q \in Y_\eta$ and $t \leq 0$.

2. The technical results in Diekmann et al. [2, Chapter III.2] on the continuous dependence of the weak* star integral on parameters and estimates (10) enable to show that the induced

curve $(-\infty,0] \ni t \longmapsto (\mathcal{K}^{cu}\,Q)(t) \in C$ is continuous for every $Q \in Y_\eta$. Consequently, Eq. (17) defines by $Q \longmapsto \mathcal{K}^{cu}\,Q$ a mapping from Y_η into $C((-\infty,0],C)$. This map is also linear. In addition, we claim $\mathcal{K}^{cu}\,Q \in C^0_\eta$ for all $Q \in Y_\eta$. To this end, consider the apparent inequality

$$\begin{aligned} e^{\eta t}\|(\mathcal{K}^{cu}\,Q)(t)\|_{C^{\odot *}} \leq e^{\eta t} & \left\|\int_0^t T_e^{\odot *}(t-\tau)\,P_c^{\odot *}\,Q(\tau)\,d\tau\right\|_{C^{\odot *}} \\ & + e^{\eta t}\left\|\int_0^t T_e^{\odot *}(t-\tau)\,P_u^{\odot *}\,Q(\tau)\,d\tau\right\|_{C^{\odot *}} \\ & + e^{\eta t}\left\|\int_{-\infty}^t T_e^{\odot *}(t-\tau)\,P_s^{\odot *}\,Q(\tau)\,d\tau\right\|_{C^{\odot *}} \end{aligned}$$

for fixed $Q \in Y_\eta$ and $t \leq 0$. Using the inequalities (15) and (10) as in the part above, we estimate the first term on the right-hand side by

$$\begin{aligned} e^{\eta t}\left\|\int_0^t T_e^{\odot *}(t-\tau)\,P_c^{\odot *}\,Q(\tau)\,d\tau\right\|_{C^{\odot *}} & \leq -e^{\eta t}\int_0^t \left\|T_e^{\odot *}(t-\tau)\,P_c^{\odot *}\,Q(\tau)\right\|_{C^{\odot *}}\,d\tau \\ & \leq -Ke^{\eta t}\int_0^t e^{c_c|t-\tau|}\|P_c^{\odot *}\,Q(\tau)\|_{C^{\odot *}}d\tau \\ & = -K\int_0^t e^{(c_c-\eta)(\tau-t)}e^{\eta\tau}\,\left\|P_c^{\odot *}\,Q(\tau)\right\|_{C^{\odot *}}\,d\tau \\ & \leq -K\|P_c^{\odot *}\|\int_0^t e^{(c_c-\eta)(\tau-t)}e^{\eta\tau}\|Q(\tau)\|_{C^{\odot *}}d\tau \\ & \leq K\|P_c^{\odot *}\|\|Q\|_{Y_\eta}\int_t^0 e^{(c_c-\eta)(\tau-t)}\,d\tau \\ & \leq K\|P_c^{\odot *}\|\|Q\|_{Y_\eta}\frac{1}{\eta-c_c}\,. \end{aligned}$$

In the same manner we can see that

$$e^{\eta t}\left\|\int_0^t T_e^{\odot *}(t-\tau)\,P_u^{\odot *}\,Q(\tau)\,d\tau\right\|_{Y^{\odot *}} \leq K\|P_u^{\odot *}\|\|Q\|_{Y_\eta}\frac{1}{c_u+\eta}$$

and

$$e^{\eta t}\left\|\int_{-\infty}^t T_e^{\odot *}(t-\tau)\,P_s^{\odot *}\,Q(\tau)\,d\tau\right\|_{Y^{\odot *}} \leq K\|P_s^{\odot *}\|\|Q\|_{Y_\eta}\frac{1}{-c_s-\eta}.$$

Summarizing, we get

$$e^{\eta t}\|(\mathcal{K}^{cu}\,Q)(t)\|_{Y^{\odot *}} \leq K\|Q\|_{Y_\eta}\left(\frac{\|P_c^{\odot *}\|}{\eta-c_c}+\frac{\|P_u^{\odot *}\|}{c_u+\eta}-\frac{\|P_s^{\odot *}\|}{c_s+\eta}\right), \tag{19}$$

and thus $\mathcal{K}^{cu}\,Q \in C^0_\eta$. It follows that $Q \longmapsto \mathcal{K}^{cu}\,Q$ forms a linear mapping $\tilde{\mathcal{K}}$ from Y_η into C^0_η, which in particular is bounded as claimed.

3. Given any $Q \in Y_\eta$ define $\delta(t,s) := (\mathcal{K}^{cu}\,Q)(t) - T_e(t-s)\big((\mathcal{K}^{cu}\,Q)(s)\big)$ for all reals $-\infty < s \leq t \leq 0$. Then, by the linearity and formula (13), we get

$$\begin{aligned}
\delta(t,s) &= \int_0^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&\quad - T_e(t-s)\left(\int_0^s T_e^{\odot *}(s-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_{-\infty}^s T_e^{\odot *}(s-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \right) \\
&= \int_0^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&\quad - \int_0^s T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau - \int_{-\infty}^s T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&= \int_s^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau + \int_s^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&= \int_s^t T_e^{\odot *}(t-\tau)\, Q(\tau)\, d\tau,
\end{aligned}$$

which yields that $u := \mathcal{K}^{cu}\,Q$ satisfies Eq. (18) for all $-\infty < s \leq t \leq 0$. Moreover, in view of Eq. (14) for the relation of the weak* integrals and projections on the decomposition of $C^{\odot *}$, for $t = 0$ we have

$$\begin{aligned}
u(0) &= (\mathcal{K}^{cu}\,Q)(0) \\
&= \int_{-\infty}^0 T_e^{\odot *}(-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \\
&= P_s^{\odot *}\left(\int_{-\infty}^0 T_e^{\odot *}(-\tau)\; Q(\tau)\, d\tau \right)
\end{aligned}$$

implying $P_{cu}^{\odot *}\, u(0) = 0$.

So the assertion of the proposition follows if we are able to prove that u is the only solution of Eq. (18) in C_η^0 with vanishing C_{cu} component at $t = 0$. For this purpose, suppose $v \in C_\eta^0$ is also a solution of (18) for $-\infty < s \leq t \leq 0$ with $P_{cu}^{\odot *} v(0) = 0$. Then the difference $w = u - v$ belongs to C_η^0, has a vanishing C_{cu} component at $t = 0$, and satisfies the equation

$$w(t) = T_e(t-s)\, w(s) \tag{20}$$

for all $-\infty < s \leq t \leq 0$. Furthermore, w can be extended by

$$t \longmapsto \begin{cases} w(t), & \text{for } t \leq 0, \\ T_e(t)\, w(0), & \text{for } t \geq 0 \end{cases}$$

to a solution $\tilde{w} : \mathbb{R} \longrightarrow C$ of Eq. (20) for all $-\infty < s \le t < \infty$. Since

$$\begin{aligned}\sup_{t\ge 0} e^{-\eta t}\|w(t)\|_C &= \sup_{t\ge 0} e^{-\eta t}\|T_e(t)\,w(0)\|_C \\ &\le K \sup_{t\ge 0} e^{-\eta t} e^{c_s t}\|w(0)\|_C \\ &= K\|w(0)\|_C\end{aligned}$$

due to $(c_s - \eta) < 0$ we get

$$\begin{aligned}\sup_{t\in\mathbb{R}} e^{-\eta|t|}\|\tilde{w}(t)\|_C &\le \sup_{t\le 0} e^{\eta t}\|\tilde{w}(t)\|_C + \sup_{t\ge 0} e^{-\eta t}\|\tilde{w}(t)\|_C \\ &= \|w\|_{C^0_\eta} + K\|w(0)\|_C < \infty.\end{aligned}$$

Now from Diekmann et al. [2, Lemma 2.4 in Section IX.2] it follows $w(0) \in C_u$ and $\tilde{w}(0) \in C_c$. As $w(0) = \tilde{w}(0)$ and $C_u \cap C_c = \{0\}$, we conclude $\tilde{w}(0) = w(0) = 0$, and so by Eq. (20),

$$0 = T_e(s)w(0) = T_e(s)\,T_e(-s)\,w(s) = T_e(0)\,w(s) = u(s) - v(s)$$

for all $-\infty < s \le 0$. This completes the proof. $\square$

Next, we prove a smoothing property of the integral equation (21). This property will be useful in combination with our preceding result.

Proposition 3.3 *Suppose that $Q \in Y_\eta$ for some $\eta \ge 0$. If $u \in C^0_\eta$ satisfies the abstract integral equation*

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T_e^{\odot *}(t-\tau)\,Q(\tau)\,d\tau \tag{21}$$

for all $-\infty < s \le t \le 0$, then $u \in C^1_\eta$ and

$$\|u\|_{C^1_\eta} \le \big(1 + e^{\eta h}\|L_e\|\big)\|u\|_{C^0_\eta} + e^{\eta h}\|Q\|_{Y_\eta}.$$

Proof: Consider the mapping $q : (-\infty, 0] \longrightarrow \mathbb{R}^n$ defined by $q(t) = l^{-1}(Q(t))$, $-\infty < t \le 0$. Of course, $q \in C((-\infty,0],\mathbb{R}^n)$. Moreover, since

$$\begin{aligned}\sup_{t\in(-\infty,0]} e^{\eta t}\|q(t)\|_{\mathbb{R}^n} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|l^{-1}(Q(t))\|_{\mathbb{R}^n} \\ &= \sup_{t\in(-\infty,0]} e^{\eta t}\|Q(t)\|_{Y^{\odot *}} \\ &= \|Q\|_{Y_\eta}\end{aligned}$$

we see at once $q \in C_\eta((-\infty,0],\mathbb{R}^n)$ with $\|q\|_{C_\eta} = \|Q\|_{Y_\eta}$.

By assumption, u satisfies Eq. (21) such that, taking into account our discussion about the one-to-one correspondence between solutions for (11) and (16), the function $x : (\infty, 0] \longrightarrow \mathbb{R}^n$ given by $x(t) = u(t)(0)$ is a solution of the differential equation

$$\dot{x}(t) = L_e\,x_t + q(t)$$

for all $-\infty < t \le 0$. Accordingly, x is everywhere continuously differentiable, x_t belongs to C^1 for all $-\infty < t \le 0$, and the map $(-\infty, 0] \ni t \longmapsto u(t) = x_t \in C^1$ is continuous. Furthermore, by the differential equation for x and the estimate for q, we have

$$\begin{aligned}\|\dot{x}(t)\|_{\mathbb{R}^n} &\le \|L_e\|\|x_t\|_C + \|q(t)\|_{\mathbb{R}^n}\\ &\le \|L_e\|\|u(t)\|_C + e^{-\eta t}\|q\|_{C_\eta}\\ &\le e^{-\eta t}(\|L_e\|\|u\|_{C^0_\eta} + \|Q\|_{Y_\eta})\end{aligned}$$

and therefore

$$\begin{aligned}\sup_{t\in(-\infty,0]} e^{\eta t}\|\dot{x}_t\|_C &= \sup_{t\in(-\infty,0]}\left(e^{\eta t}\sup_{\vartheta\in[-h,0]}\|\dot{x}(t+\vartheta)\|_{\mathbb{R}^n}\right)\\ &\le (\|L_e\|\|u\|_{C^0_\eta} + \|Q\|_{Y_\eta})\sup_{t\in(-\infty,0]}\left(e^{\eta t}\sup_{\vartheta\in[-h,0]} e^{-\eta(t+\vartheta)}\right)\\ &\le e^{\eta h}(\|L_e\|\|u\|_{C^0_\eta} + \|Q\|_{Y_\eta}),\end{aligned}$$

for all $-\infty < t \le 0$. From this, it follows that $u \in C^1_\eta$ and

$$\begin{aligned}\|u\|_{C^1_\eta} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|u(t)\|_{C^1}\\ &= \sup_{t\in(-\infty,0]} e^{\eta t}\|x_t\|_{C^1}\\ &= \sup_{t\in(-\infty,0]} e^{\eta t}(\|x_t\|_C + \|\dot{x}_t\|_C)\\ &\le \|u\|_{C^0_\eta} + e^{\eta h}(\|L_e\|\|u\|_{C^0_\eta} + \|Q\|_{Y_\eta})\end{aligned}$$

as claimed. □

As an easy consequence of the last two results we conclude that the formal definition (17) generates a bounded linear mapping from the Banach space Y_η into C^1_η for $c_c < \eta < \min\{-c_s, c_u\}$.

Corollary 3.4 *For each $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$, relation (17) defines a bounded linear mapping*

$$\mathcal{K}_\eta : Y_\eta \ni Q \longmapsto \mathcal{K}^{cu}\, Q \in C^1_\eta$$

with

$$\|\mathcal{K}_\eta\| \le K(1 + e^{\eta h}\|L_e\|)\left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta}\right) + e^{\eta h}.$$

Moreover, for all $Q \in Y_\eta$ the function $u = \mathcal{K}_\eta Q$ is a solution of

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T_e^{\odot *}(t-\tau)\,Q(\tau)\,d\tau$$

for $-\infty < s \le t \le 0$, and the only one in C^1_η with $P_{cu}^{\odot *}\,u(0) = 0$.

Proof: Apply Propositions 3.2 and 3.3, taking into account the estimate (19) for the bound of the linear map $\tilde{\mathcal{K}}$. □

Remark 3.5 Observe that the bounds of the linear maps $\mathcal{K}_\eta$ in the above corollary are given by a continuous function in η. This will be a crucial point in the proof of Theorem 2.

4 The Construction of Local Center-Unstable Manifolds

This section is devoted to the actual proof of Theorem 1 about the existence of local center-unstable manifolds for Eq. (1). Throughout the proof, we consider the differential equation (1) in the equivalent form

$$\dot{x}(t) = L\,x_t + r(x_t) \tag{22}$$

with the linear part

$$L := Df(0)$$

and the nonlinearity

$$r : U \ni \varphi \longmapsto f(\varphi) - L\,\varphi \in \mathbb{R}^n. \tag{23}$$

Obviously, r also satisfies the same smoothness conditions (S 1) and (S 2) as f and we have $r(0) = 0$ and $Dr(0) = 0$.

The proof is organized as follows. In the first part, we modify the nonlinearity r outside a small neighborhood of the origin and assign the resulting differential equation to an abstract integral equation by the variation-of-constants formula. Then, using the changes on the nonlinearity in combination with the auxiliary conclusions of the last section, we show that the associated integral operator forms a parameter-dependent contraction in C^1_η for an appropriate $\eta > 0$. In the final step, we prove that the graph of this contraction is an invariant manifold for the modified differential equation and that a part of this graph also satisfies the assertions of Theorem 1.

Smoothing Modification of the Nonlinearity

As the Banach space C_{cu} is finite-dimensional, there exists a norm $\|\cdot\|_{cu}$ on C_{cu} being infinitely often continuously differentiable on $C_{cu}\setminus\{0\}$. Introducing the projection operator $P_{cu} := P_c + P_u$ of C^1 along C^1_s onto the center-unstable space C_{cu} and defining

$$\|\varphi\|_1 = \max\left\{\|P_{cu}\,\varphi\|_{cu}, \|P_s\,\varphi\|_{C^1}\right\} \tag{24}$$

for $\varphi \in C^1$, we get a second norm on C^1, which is equivalent to $\|\cdot\|_{C^1}$.

Let $\varrho : [0,\infty) \longrightarrow \mathbb{R}$ be a C^∞-smooth function with $\varrho(t) = 1$ for $0 \le t \le 1$, $0 < \varrho(t) < 1$ for $1 < t < 2$, and $\varrho(t) = 0$ for all $t \ge 2$. Further, let the map $\hat{r} : C^1 \longrightarrow \mathbb{R}^n$ be given by

$$\hat{r}(\varphi) = \begin{cases} r(\varphi), & \text{for } \varphi \in U, \\ 0, & \text{for } \varphi \notin U. \end{cases}$$

Using these two functions, we introduce for all $\delta > 0$ the smoothing modification

$$r_\delta : C^1 \ni \varphi \longmapsto \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\varphi) \in \mathbb{R}^n$$

of the nonlinearity r, where we write φ_{cu}, φ_s for the components $P_{cu}\,\varphi$, $P_s\,\varphi$ of φ, respectively. For every $\gamma > 0$ let $B_\gamma(0) = \{\varphi \in C^1 \,|\, \|\varphi\|_1 < \gamma\}$ denote the open ball in C^1 of radius γ with respect to the $\|\cdot\|_1$-norm and centered at the origin. Since $U \subset C^1$ is open and r continuously differentiable due to property (S 1), we find a sufficiently small $\delta_0 > 0$ with $B_{2\delta_0}(0) \subset U$, so that the restriction $r|_{B_{2\delta_0}(0)}$ of r to $B_{2\delta_0}(0)$ together with the associated derivative $Dr|_{B_{2\delta_0}(0)}$ are both bounded. Subsequently, for small reals $\delta > 0$, the modifications of r in a neighborhood of the origin are also bounded and continuously differentiable with bounded derivatives. More precisely, the following result holds.

Corollary 4.1 *For all reals $0 < \delta < \delta_0$ the restriction of the map r_δ to the strip*

$$S := \left\{\psi \in C^1 \middle|\; \|\psi_s\|_1 < \delta\right\}$$

in C^1 is a bounded, C^1-smooth function with bounded derivative. Moreover,

$$r_\delta(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot r(\varphi)$$

for all $\varphi \in S$.

Proof: Given any positive constant $0 < \delta < \delta_0$ suppose that $\varphi \in S$. Then, by definition of r_δ in combination with the inequality $\|\varphi_s\|_{C^1} \le \|\varphi_s\|_1$ we get

$$r_\delta(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot r(\varphi).$$

Consequently, we have $r_\delta(\varphi) = r(\varphi)$ for all $\varphi \in S$ with $\|\varphi\|_1 \le \delta$, and $r_\delta(\varphi) = 0$ for all $\varphi \in S$ with $\|\varphi\|_1 \ge 2\delta$. Since r, ϱ are C^1-smooth and the norm $\|\cdot\|_1$ continuously differentiable on $C_{cu} \setminus \{0\}$ by assumption, the restriction of r_δ to the strip S is clearly also continuously differentiable. Moreover, using the above expressions for r_δ on S together with the boundedness of r and Dr on $B_{2\delta_0}(0) \subset U$, we conclude that both r_δ and Dr_δ are bounded on S as claimed. $\square$

For sufficiently small $\delta > 0$, the functions r_δ are even globally bounded and Lipschitz continuous with constants continuously depending on δ, as proved in [9].

Proposition 4.2 [*Proposition* II.2 *in Krisztin et al.* [9]] *Under the above assumptions there exists* $\delta_1 \in (0, \delta_0)$ *and a monotone increasing* $\lambda : [0, \delta_1] \longrightarrow [0, 1]$ *with* $\lambda(0) = 0$ *and* $\lambda(\delta) \searrow 0$ *as* $\delta \searrow 0$ *such that*

$$\|r_\delta(\varphi)\|_{\mathbb{R}^n} \leq \delta \cdot \lambda(\delta)$$

and

$$\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} \leq \lambda(\delta) \cdot \|\varphi - \psi\|_{C^1}$$

for all $0 < \delta \leq \delta_1$ *and* $\varphi, \psi \in C^1$.

Using the modification r_δ of the nonlinearity r, we introduce for each $0 < \delta \leq \delta_1$ the retarded functional differential equation

$$\dot{x}(t) = L\, x_t + r_\delta(x_t), \qquad -\infty < t \leq 0, \tag{25}$$

and the associated abstract integral equations

$$u(t) = T_e(t - s)\, u(s) + \int_s^t T_e^{\odot *}(t - \tau)\, l(r_\delta(u(\tau)))\, d\tau, \qquad -\infty < s \leq t \leq 0. \tag{26}$$

We have now a one-to-one correspondence in the following sense: If $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ is a continuously differentiable solution of RFDE (25), then $u : (-\infty, 0] \longmapsto x_t \in C^1$ is a solution of Eq. (26). On the other hand, for a continuous mapping $u : (-\infty, 0] \longrightarrow C^1$ satisfying integral equation (26), the function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ defined by $x(t) = u(t)(0)$, $-\infty < t \leq 0$, forms a continuously differentiable solution of(25).

Center-Unstable Manifolds of the Smoothed Equation

Until the end of this section fix $\eta \in \mathbb{R}$ satisfying the estimate

$$c_c < \eta < \min\{-c_s, c_u\}. \tag{27}$$

Then we find a constant $0 < \delta < \delta_1$ with

$$\|\mathcal{K}_\eta\|\, \lambda(\delta) < \frac{1}{2} \tag{28}$$

where the mappings $\mathcal{K}_\eta$ and λ are defined in Corollary 3.4 and Proposition 4.2, respectively. Below, we construct a parameter-dependent contraction on the Banach space C^1_η, such that the fixed points will form solutions for the abstract integral equation (26). For this purpose, we assign to Eq. (26) an integral operator. We begin with the nonlinear part.

Corollary 4.3 *Let R denote the map, which assigns to $u \in C((-\infty,0],C^1)$ the mapping $(-\infty,0] \ni s \longmapsto l(r_\delta(u(s))) \in Y^{\odot *}$ in $C((-\infty,0],Y^{\odot *})$. Then R maps C^1_η into Y_η, and the induced mapping $R_{\delta\eta} : C^1_\eta \ni u \longmapsto R(u) \in Y_\eta$ satisfies*

$$\|R_{\delta\eta}(u)\|_{Y_\eta} \leq \delta\,\lambda(\delta) \tag{29}$$

and

$$\|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta} \leq \lambda(\delta)\|u - v\|_{C^1_\eta} \tag{30}$$

for all $u, v \in C^1$.

Proof: First, note that R indeed assigns a continuous function from $(-\infty,0]$ into $Y^{\odot *}$ to a function $u \in C((-\infty,0],C^1)$, as the mappings l and r_δ are continuous. Given $u \in C^1_\eta$, Proposition 4.2 implies

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\|R(u)(t)\|_{Y^{\odot *}} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|l(r_\delta(u(t)))\|_{Y^{\odot *}} \\
&= \sup_{t\in(-\infty,0]} e^{\eta t}\|r_\delta(u(t))\|_{\mathbb{R}^n} \\
&\leq \sup_{t\in(-\infty,0]} e^{\eta t}\delta\lambda(\delta) \\
&= \delta\lambda(\delta).
\end{aligned}$$

This shows $R(C^1_\eta) \subset Y_\eta$ and in particular the boundedness of $R_{\delta\eta}$ by $\delta\lambda(\delta)$ as claimed. Using the Lipschitz continuity of r_δ from Proposition 4.2, we also see that $R_{\delta\eta}$ is Lipschitz continuous with Lipschitz constant $\lambda(\delta)$, and the corollary follows. $\square$

Remark 4.4 The mapping $R : C((-\infty,0],C^1) \longrightarrow C((-\infty,0],Y^{\odot *})$ in the last result is called the **substitution** or the **Nemitsky operator** of the map $C^1 \ni \varphi \longmapsto l(r_\delta(\varphi)) \in Y^{\odot *}$ on $(-\infty,0]$.

Next, we consider the linear part of the integral equation (26) and prove that it constitutes a bounded linear operator from the center-unstable space into C^1_η.

Corollary 4.5 *For each $\varphi \in C_{cu}$, the curve $(-\infty,0] \ni t \longmapsto T_e(t)\,\varphi \in C^1$ belongs to C^1_η, and $S_\eta : C^1 \supset C_{cu} \longrightarrow C^1_\eta$ defined by $(S_\eta\,\varphi)(t) = T_e(t)\varphi$ for $\varphi \in C_{cu}$ and $t \leq 0$ is a bounded linear operator with*

$$\|S_\eta\| \leq K\big(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|\big). \tag{31}$$

Proof: To start with, recall that T_e defines a group on $C_{cu} \subset C^1$ and coincides with T. Thus, for all $\varphi \in C_{cu}$, the curve $(-\infty,0] \ni t \longmapsto T_e(t)\,\varphi \in C_{cu}$ takes values in C^1 and is in fact a continuous map from $(-\infty,0]$ into C^1. Furthermore, we have

$$\|T_e(t)\,\varphi\|_{C^1} = \|T_e(t)\,\varphi\|_C + \left\|\tfrac{d}{dt}T_e(t)\,\varphi\right\|_C$$

and

$$\tfrac{d}{dt}(T_e(t)\,\varphi) = T_e(t)\,G_e\,\varphi = T_e(t)\,\varphi'$$

for $\varphi \in C_{cu}$. Hence, by the exponential trichotomy under our assumption (27), it follows

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\,\varphi\|_{C^1} &= \sup_{t\in(-\infty,0]} e^{\eta t}\Big(\|T_e(t)\,\varphi\|_C + \|T_e(t)\,\varphi'\|_C\Big) \\
&\le \sup_{t\in(-\infty,0]} e^{\eta t}\,\Big(\|T_e(t)\,P_c^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi\|_C \\
&\qquad + \|T_e(t)\,P_c^{\odot *}\,\varphi'\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi'\|_C\Big) \\
&\le \sup_{t\in(-\infty,0]} e^{\eta t}\Big(\|T_e(t)\,P_c^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_c^{\odot *}\,\varphi'\|_C\Big) \\
&\quad + \sup_{t\in(-\infty,0]} e^{\eta t}\Big(\|T_e(t)\,P_u^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi'\|_C\Big) \\
&\le K \sup_{t\in(-\infty,0]} e^{-(c_c-\eta)t}\Big(\|P_c^{\odot *}\,\varphi\|_C + \|P_c^{\odot *}\,\varphi'\|_C\Big) \\
&\qquad + K \sup_{t\in(-\infty,0]} e^{(\eta+c_u)t}\Big(\|P_u^{\odot *}\,\varphi\|_C + \|P_u^{\odot *}\,\varphi'\|_C\Big) \\
&\le K\|P_c^{\odot *}\|\big(|\varphi\|_C + \|\varphi'|_C\big) + \\
&\qquad K\|P_u^{\odot *}\|\big(\|\varphi\|_C + \|\varphi'\|_C\big) \\
&= K\big(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|\big)\,\|\varphi\|_{C^1}.
\end{aligned}$$

Accordingly, $S_\eta\varphi \in C^1_\eta$ for $\varphi \in C_{cu}$, and thus S_η is well-defined. In addition, the mapping S_η is obviously linear by definition, and

$$\|S_\eta\varphi\|_{C^1_\eta} \le K(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|)$$

for $\|\varphi\|_{C^1} \le 1$. Therefore, inequality (31) holds and this completes the proof. $\square$

Using Corollaries 3.4, 4.3, and 4.5 to guarantee the well-definedness, we introduce the mapping $\mathcal{G}_\eta$ from the product space $C^1_\eta \times C_{cu}$ into C^1_η given by

$$\mathcal{G}_\eta(u,\varphi) := S_\eta\,\varphi + \mathcal{K}_\eta \circ R_{\delta\eta}(u). \tag{32}$$

In the next proposition we prove that each function $\varphi \in C_{cu}$ uniquely determines a solution of $u = \mathcal{G}_\eta(u,\varphi)$ in C^1_η.

Proposition 4.6 *For each $\varphi \in C_{cu}$, the mapping $\mathcal{G}_\eta(\,\cdot\,,\varphi) : C^1_\eta \longrightarrow C^1_\eta$ has exactly one fixed point $u = u(\varphi)$. Moreover, the associated solution operator*

$$\tilde{u}_\eta : C_{cu} \ni \varphi \longmapsto u(\varphi) \in C^1_\eta \tag{33}$$

of $u = \mathcal{G}_\eta(u,\varphi)$ is (globally) Lipschitz continuous.

Proof: We begin with the claim that, for given $\varphi \in C_{cu}$, $\mathcal{G}_\eta(\,\cdot\,,\varphi)$ maps sufficiently large closed balls centered at the origin into themselves. Indeed, for fixed $\varphi \in C_{cu}$ we find a positive real $\gamma > 0$ with $2\|S_\eta\|\,\|\varphi\|_{C^1} \leq \gamma$ so that both estimates (28) and (30) together imply

$$\begin{aligned}\|\mathcal{G}_\eta(u,\varphi)\|_{C^1_\eta} &= \|S_\eta\,\varphi + \mathcal{K}_\eta \circ R_{\delta\eta}(u)\|_{C^1_\eta}\\ &\leq \|S_\eta\,\varphi\|_{C^1_\eta} + \|\mathcal{K}_\eta \circ R_{\delta\eta}(u)\|_{C^1_\eta}\\ &\leq \|S_\eta\|\|\varphi\|_{C^1} + \lambda(\delta)\|\mathcal{K}_\eta\|\,\|u\|_{C^1_\eta}\\ &\leq \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma\end{aligned}$$

for all $u \in C^1_\eta$ with $\|u\|_{C^1_\eta} \leq \gamma$. Hence, $\mathcal{G}_\eta(\,\cdot\,,\varphi)$ maps $\left\{u \in C^1_\eta |\ \|u\|_{C^1_\eta} \leq \gamma\right\}$ into itself. The mapping $\mathcal{G}_\eta(\,\cdot\,,\varphi)$, $\varphi \in C_{cu}$, is also a contraction since, by application of (28) and (30),

$$\begin{aligned}\|\mathcal{G}_\eta(u,\varphi) - \mathcal{G}_\eta(v,\varphi)\|_{C^1_\eta} &= \|\mathcal{K}_\eta \circ R_{\delta\eta}(u) - \mathcal{K}_\eta \circ R_{\delta\eta}(v)\|_{C^1_\eta}\\ &\leq \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta}\\ &\leq \lambda(\delta)\|\mathcal{K}_\eta\|\,\|u - v\|_{C^1_\eta}\\ &\leq \frac{1}{2}\|u - v\|_{C^1_\eta}\end{aligned}$$

for all $u, v \in C^1_\eta$. Consequently, using the Banach contraction principle, we find a unique $u(\varphi) \in C^1_\eta$ satisfying $u = \mathcal{G}_\eta(u,\varphi)$.

To see the global Lipschitz continuity of $\tilde{u}_\eta : C_{cu} \ni \varphi \longmapsto u(\varphi) \in C^1_\eta$, assume $\varphi, \psi \in C_{cu}$. Using the two inequalities (28) and (30) once more, we see

$$\begin{aligned}\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta} &= \|\mathcal{G}_\eta(\tilde{u}_\eta(\varphi),\varphi) - \mathcal{G}_\eta(\tilde{u}_\eta(\psi),\psi)\|_{C^1_\eta}\\ &= \|S_\eta(\varphi - \psi) + \mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\varphi)) - \mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\psi))\|_{C^1_\eta}\\ &\leq \|S_\eta\|\,\|\varphi - \psi\|_{C^1} + \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(\tilde{u}_\eta(\varphi)) - R_{\delta\eta}(\tilde{u}_\eta(\psi))\|_{Y_\eta}\\ &\leq \|S_\eta\|\,\|\varphi - \psi\|_{C^1} + \lambda(\delta)\,\|\mathcal{K}_\eta\|\,\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}\\ &\leq \|S_\eta\|\,\|\varphi - \psi\|_{C^1} + \frac{1}{2}\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}.\end{aligned}$$

Therefore

$$\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta} \leq 2\|S_\eta\|\,\|\varphi - \psi\|_{C^1},$$

which completes the proof. □

For all $\varphi \in C_{cu}$, the associated fixed point $\tilde{u}(\varphi)$ of the last proposition forms a solution of Eq. (26) in C^1_η with the property that its component in the center-unstable space at $t = 0$ is just given by φ, as shown in the following.

Corollary 4.7 *For all $\varphi \in C_{cu}$ the mapping $\tilde{u}_\eta(\varphi)$ is a solution of the abstract integral equation (26) with $P_{cu}(\tilde{u}_\eta(\varphi)(0)) = \varphi$.*

Proof: The proof is straightforward. Given $\varphi \in C_{cu}$ define $z = \tilde{u}_\eta(\varphi) - S_\eta\,\varphi$. By Corollary 3.4, we have

$$z(t) = T_e(t-s)\,z(s) + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau, \qquad -\infty < s \leq t \leq 0,$$

and $P_{cu}\,z(0) = P_{cu}^{\odot *}\,z(0) = 0$. From this we conclude

$$\begin{aligned}
\tilde{u}_\eta(\varphi)(t) - T_e(t)\,\varphi &= \tilde{u}_\eta(\varphi)(t) - (S_\eta\,\varphi)(t) \\
&= z(t) \\
&= T_e(t-s)\,z(s) + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau \\
&= T_e(t-s)\,\tilde{u}_\eta(\varphi)(s) - T_e(t-s)(S_\eta\varphi)(s) \\
&\qquad + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau \\
&= T_e(t-s)\,\tilde{u}_\eta(\varphi)(s) - T_e(t)\,\varphi + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau
\end{aligned}$$

for all $-\infty < s \leq t \leq 0$ and

$$\begin{aligned}
P_{cu}(\tilde{u}_\eta(\varphi)(0)) - \varphi &= P_{cu}(\tilde{u}_\eta(\varphi)(0)) - P_{cu}\,\varphi \\
&= P_{cu}(\tilde{u}_\eta(\varphi)(0)) - P_{cu}((S_\eta\varphi)(0)) \\
&= P_{cu}\,z(0) = 0
\end{aligned}$$

Adding $T_e(t)\,\varphi$ and φ, respectively, yields the assertion. □

By the discussed one-to-one correspondence of solutions for the differential equation (25) and the associated abstract integral equation (26), the above corollary shows that for all $\varphi \in C_{cu}$ there exists a continuously differentiable function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ satisfying $x_t = \tilde{u}(\varphi)(t)$ for $-\infty < t \leq 0$ and solving Eq. (26) on $(-\infty, 0]$. The set W^η consisting of all segments of these solutions at time $t = 0$, that is, the set

$$W^\eta := \left\{ \tilde{u}_\eta(\varphi)(0) \mid \varphi \in C_{cu} \right\},$$

is called the **global center-unstable manifold** of RFDE (25) at the stationary point $0 \in C^1$. Note that W^η can also be represented as the graph of the operator

$$w^\eta : C_{cu} \ni \varphi \longmapsto P_s(\tilde{u}_\eta(\varphi)(0)) \in C_s^1.$$

Indeed, applying Corollary 4.7, we see at once

$$W^\eta = \{\varphi + w^\eta(\varphi) \mid \varphi \in C_{cu}\}.$$

We close this subsection with the conclusion that the values of every solution $v \in C_\eta^1$ of the abstract integral equation (26) belong to the global center-unstable manifold W^η.

Proposition 4.8 *Suppose that $v \in C^1_\eta$ is a solution of Eq. (26). Then*

$$v(t) \in W^\eta$$

for all $t \leq 0$.

Proof: Assuming $v \in C^1_\eta$ satisfies the abstract integral equation

$$u(t) = T_e(t-s)\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(u(\tau)))\, d\tau$$

for $-\infty < s \leq t \leq 0$, we begin with the claim that $v(0) \in W^\eta$. In order to see this, let $z : (-\infty, 0] \longrightarrow C^1$ be defined by $z(t) = v(t) - T_e(t)\, P_{cu}\, v(0)$. As

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\|z(t)\|_{C^1} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|v(t) - T_e(t)\, P_{cu}\, v(0)\|_{C^1} \\
&\leq \sup_{t\in(-\infty,0]} e^{\eta t}\|v(t)\|_{C^1} \\
&\qquad + \sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\, P_{cu}\, v(0)\|_{C^1} \\
&\leq \|v\|_{C^1_\eta} + \sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\, P_c\, v(0)\|_{C^1} \\
&\qquad + \sup_{t\in(-\infty,0]} e^{\eta t}\|T_e(t)\, P_u\, v(0)\|_{C^1} \\
&\leq \|v\|_{C^1_\eta} + K \sup_{t\in(-\infty,0]} e^{-(c_c-\eta)t}\|P_c\, v(0)\|_{C^1} \\
&\qquad + K \sup_{t\in(-\infty,0]} e^{(c_u+\eta)t}\|P_u\, v(0)\|_{C^1} \\
&\leq \|v\|_{C^1_\eta} + K\|P_c\|\, \|v(0)\|_{C^1} + K\|P_u\|\|v(0)\|_{C^1} \\
&\leq \big(1 + K\|P_c\| + K\|P_u\|\big)\|v\|_{C^1_\eta} < \infty,
\end{aligned}$$

we have $z \in C^1_\eta$. Moreover, for all $s \leq t \leq 0$, we have

$$\begin{aligned}
z(t) &= v(t) - T_e(t)\, P_{cu}\, v(0) \\
&= T_e(t-s)\, v(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(v(\tau)))\, d\tau - T_e(t)\, P_{cu}\, v(0) \\
&= T_e(t-s)\, v(s) - T_e(t-s)\, T_e(s)\, P_{cu}\, v(0) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(v(\tau)))\, d\tau \\
&= T_e(t-s)\, z(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(v(\tau)))\, d\tau.
\end{aligned}$$

Since furthermore $R_{\delta\eta}(v) \in Y_\eta$ by Corollary 4.3 and $P_{cu}^{\odot *}\, z(0) = P_{cu}\, z(0) = 0$, we obtain $z = \mathcal{K} \circ R_{\delta\eta}(v)$ due to Corollary 3.4. Hence, by definition

$$v(t) = z(t) + T_e(t)\, P_{cu}\, v(0) = (\mathcal{K}_\eta \circ R_{\delta\eta}(v))(t) + T_e(t)\, P_{cu}\, v(0)$$

for all $t \le 0$, or equivalently,

$$v = \mathcal{K}_\eta \circ R_{\delta\eta}(v) + S_\eta(P_{cu}\, v(0)) = \mathcal{G}(v, P_{cu}\, v(0)).$$

This implies $v(0) = \mathcal{G}(v, P_{cu}\, v(0))(0) = \tilde{u}_\eta(P_{cu}\, v(0))(0) \in W^\eta$ as claimed.

The proof of $v(t) \in W^\eta$ as $t < 0$ may now be reduced to the above claim as follows. For given $t_0 < 0$ consider the translation

$$\hat{v} : (-\infty, 0] \ni s \longmapsto v(t_0 + s) \in C^1.$$

Obviously, we have $\hat{v} \in C^1_\eta$ and $\hat{v}$ is a solution of Eq. (26). Therefore $v(-t_0) = \hat{v}(0) \in W^\eta$ by the above claim. This completes the proof. $\square$

Remark 4.9 Note that by application of the above result we easily deduce the identity

$$\tilde{u}_\eta(\varphi)(t) = \tilde{u}_\eta(P_{cu}\, \tilde{u}_\eta(\varphi)(t))(0)$$

for all $\varphi \in C_{cu}$ and $t \le 0$.

Proof of Theorem 1

In this final part of the present section we complete the proof of Theorem 1 on the existence of Lipschitz continuous local center-unstable manifolds. We conclude that in a neighborhood of the origin, the global center-unstable manifold W^η of Eq. (25) has the properties asserted in Theorem 1.

Our proof starts with the following series of definitions depending on the constant $\delta > 0$ from condition (28):

$$\begin{aligned} C_{cu,0} &:= \Big\{\varphi \in C_{cu} \mid \|\varphi\|_1 < \delta\Big\}, \\ C^1_{s,0} &:= \Big\{\varphi \in C^1_s \mid \|\varphi\|_1 < \delta\Big\}, \\ N_{cu} &:= C_{cu,0} + C^1_{s,0}, \\ w_{cu} &:= w^\eta\big|_{C_{cu,0}}, \end{aligned}$$

and

$$W_{cu} := \Big\{\varphi + w_{cu}(\varphi) \mid \varphi \in C_{cu,0}\Big\}.$$

Given an open neighborhood V of 0 in X_f, note that one may choose $\delta > 0$ with $W_{cu} \subset V$. Applying Corollary 3.4 and estimate (29) of Corollary 4.3, we obtain for all $\varphi \in C_{cu,0}$

$$
\begin{aligned}
\|w_{cu}(\varphi)\|_1 &= \|w^\eta(\varphi)\|_1 \\
&= \|P_s(\tilde{u}_\eta(\varphi)(0))\|_{C^1} \\
&= \|\tilde{u}_\eta(\varphi)(0) - P_{cu}(\tilde{u}_\eta(\varphi)(0))\|_{C^1} \\
&= \|\mathcal{G}_\eta(\tilde{u}_\eta(\varphi),\varphi)(0) - P_{cu}(\mathcal{G}_\eta(\tilde{u}_\eta(\varphi),\varphi)(0))\|_{C^1} \\
&= \|(S_\eta\,\varphi)(0) + (\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0) - P_{cu}((S_\eta\,\varphi)(0)) \\
&\qquad - P_{cu}((\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0))\|_{C^1} \\
&= \|(\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0)\|_{C^1} \\
&\leq \|\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi))\|_{C^1_\eta} \\
&\leq \|\mathcal{K}_\eta\|\|R_{\delta\eta}(\tilde{u}(\varphi))\|_{Y_\eta} \\
&\leq \|\mathcal{K}_\eta\|\delta\lambda(\delta),
\end{aligned}
\tag{34}
$$

and thus, $w_{cu}(C_{cu,0}) \subset C^1_{s,0}$ by assumption (28). The mapping w_{cu} is also Lipschitz continuous, because for all $\varphi, \psi \in C_{cu,0}$ we have

$$
\begin{aligned}
\|w_{cu}(\varphi) - w_{cu}(\psi)\|_{C^1} &= \|w^\eta(\varphi) - w^\eta(\psi)\|_{C^1} \\
&= \|P_s(\tilde{u}_\eta(\varphi)(0)) - P_s(\tilde{u}_\eta(\psi)(0))\|_{C^1} \\
&\leq \|P_s\|\|\tilde{u}_\eta(\varphi)(0) - \tilde{u}_\eta(\psi)(0)\|_{C^1} \\
&\leq \|P_s\|\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}
\end{aligned}
$$

and the operator $\tilde{u}_\eta$ is (globally) Lipschitz continuous due to Proposition 4.6. Moreover, since $\mathcal{G}_\eta(0,0) = 0$ by definition, we have $\tilde{u}_\eta(0) = 0$ and hence $w_{cu}(0) = 0$. Consequently, Theorem 1 follows if we verify properties (i) - (iii) for W_{cu}, which is done below.

Proof of Assertion (ii): Assuming that $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ is a solution of the differential equation (1) with $x_t \in N_{cu}$, $t \leq 0$, we have to show $x_t \in W_{cu}$ for all $t \leq 0$. To this end, notice that by definition $\|P_{cu}\,x_t\|_1 < \delta$ and $\|P_s\,x_t\|_1 < \delta$ so that Corollary 4.1 yields $r(x_t) = r_\delta(x_t)$ for all $t \leq 0$. Therefore x satisfies the smoothed differential equation (25) as well. Setting $u(t) = x_t$, $t \leq 0$, we consequently obtain a solution of the smoothed abstract integral equation (26). In particular, as u is bounded on $(-\infty, 0]$, we conclude that $u \in C^1_\eta$, and hence $u(t) \in W^\eta$, $t \leq 0$, by Proposition 4.8. This implies $x_t \in W_{cu}$ for all $t \leq 0$, which is the desired conclusion. □

Proof of Assertion (iii): Assume that for a function $\varphi \in W_{cu}$ and $t_N > 0$ we have $\{F(t,\varphi) \mid 0 \leq s \leq t_N\} \subset N_{cu}$. To deduce $\{F(t,\varphi) \mid 0 \leq s \leq t_N\} \subset W_{cu}$ from this, consider the function

$$
v(t) = \begin{cases} \tilde{u}_\eta(P_{cu}\,\varphi)(t_N + t), & \text{for } t \leq -t_N, \\ F(t_N + t, \varphi), & \text{for } -t_N \leq t \leq 0, \end{cases}
$$

where $\tilde{u}_\eta(P_{cu}\,\varphi) \in C^1_\eta$ is the solution of Eq. (26) with $\tilde{u}_\eta(P_{cu}\,\varphi)(0) = \varphi$ from Corollary 4.7. As v takes values in C^1, it is continuous at the questionable point $t = -t_N$ in view of the limits

$$\lim_{t \nearrow -t_N} v(t) = \lim_{t \nearrow -t_N} \tilde{u}_\eta(P_{cu}\,\varphi)(t_N + t) = \tilde{u}_\eta(P_{cu}\,\varphi)(0) = \varphi$$

and

$$\lim_{t \searrow -t_N} v(t) = \lim_{t \searrow -t_N} F(t_N + t, \varphi) = F(0, \varphi) = \varphi.$$

In addition, v is bounded in the $\|\cdot\|_{C^1_\eta}$-norm due to

$$\sup_{t \in (-\infty, 0]} e^{\eta t} \|v(t)\|_{C^1} \leq \max\left\{ \|\tilde{u}_\eta(P_{cu}\,\varphi)\|_{C^1_\eta},\ \max_{t \in [0, t_N]} \|F(t, \varphi)\|_{C^1} \right\} < \infty,$$

we have $v \in C^1_\eta$. Moreover, we claim that v is also a solution of Eq. (26). Indeed, suppose $s, t \in (-\infty, 0]$ with $s \leq t$. Then the cases $s \leq t \leq -t_N < 0$ and $-t_N \leq s \leq t \leq 0$ are obvious, whereas in the situation $s \leq -t_N \leq t \leq 0$, we get

$$\begin{aligned}
v(t) - T_e(t-s)\,v(s) &= v(t) - T_e(t+t_N)\,T_e(-t_N - s)\,v(s) \\
&= T_e(t+t_N)\,v(-t_N) + \int_{-t_N}^{t} T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&\qquad - T_e(t+t_N)\,T_e(-t_N - s)\,v(s) \\
&= T_e(t+t_N)\Big(v(-t_N) - T_e(-t_N - s)\,v(s)\Big) \\
&\qquad + \int_{-t_N}^{t} T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&= T_e(t+t_N) \int_{s}^{-t_N} T_e^{\odot *}(-t_N - \tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&\qquad + \int_{-t_N}^{t} T_e^{\odot *}(-t_N - \tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&= \int_{s}^{-t_N} T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau + \int_{-t_N}^{t} T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&= \int_{s}^{t} T_e^{\odot *}(t-\tau)\,l(r_\delta(v(\tau)))\,d\tau\,.
\end{aligned}$$

Thus, v is a solution of Eq. (26) in C^1_η as claimed.

Now Proposition 4.8 shows $v(t) \in W^\eta$ for all $t \leq 0$. Consequently, for constants $0 \leq t \leq t_N$ we have

$$F(t, \varphi) = v(t - t_N) \in N_{cu} \cap W^\eta,$$

and hence $F(t, \varphi) \in W_{cu}$, which proves our assertion. □

Proof of Assertion (i): It remains to prove that W_{cu} is contained in the solution manifold X_f of Eq. (1), and that W_{cu} forms a Lipschitz submanifold of dimension $\dim C_{cu}$. For the first part, let $\varphi \in W_{cu}$ be given. Then from Corollary 4.7 it follows that the equations $x_t = \tilde{u}_\eta(P_{cu}\,\varphi)(t)$, $t \le 0$, define a continuously differentiable function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ satisfying the smoothed differential equation (26) on $(-\infty, 0]$ and $x_0 = \varphi$. In particular, $\dot{\varphi}(0) = L\,\varphi + r_\delta(\varphi)$. As $\varphi \in W_{cu} \subset N_{cu}$ and in addition $r_\delta = r$ on N_{cu} due to Corollary 4.1 we conclude

$$\dot{\varphi}(0) = L\,\varphi + r(\varphi) = f(\varphi) \in X_f.$$

This proves $W_{cu} \subset X_f$.

To see the second part of the assertion, we consider an n-dimensional complementary space E of $Y = T_0 X_f$ in the Banach space C^1. We claim that there is no loss of generality in assuming $E \subset C^1_s$. In fact, let $\{e_1, \dots, e_n\}$ denote a basis of E. Then by the decomposition $C^1 = C_{cu} \oplus C^1_s$ according to Eq. (7) we get for each $i = 1, \dots, n$

$$e_i = u_i + s_i$$

with uniquely determined $u_i \in C_{cu}$ and $s_i \in C^1_s$. As the center-unstable space C_{cu} is contained in Y, we conclude that $s_i \notin Y$ for all $i = 1, \dots, n$.

Define vectors $\hat{e}_i = e_i - u_i$ for $i = 1, \dots, n$ and suppose we have

$$\sum_{i=1}^{n} \lambda_i\,\hat{e}_i = 0$$

with reals λ_i, $i = 1, \dots, n$. Using the definition of $\hat{e}_i$, we obtain

$$E \ni \sum_{i=1}^{n} \lambda_i\,e_i = \sum_{i=1}^{n} \lambda_i\,u_i \in C_{cu}.$$

Since $C_{cu} \cap E = \{0\}$ it follows $\lambda_i = 0$ for all $i \in \{1, \dots, n\}$. Thus, the elements $\hat{e}_i$, $i = 1, \dots, n$, generate an n-dimensional subspace $\hat{E}$ of C^1, which is complementary to Y in C^1. In particular, $\hat{E} \subset C^1_s$.

In view of the above, we suppose now that indeed $E \subset C^1_s$, which leads to

$$C^1_s = E \oplus (C^1_s \cap Y),$$
$$Y = C_{cu} \oplus (C^1_s \cap Y),$$

and

$$C^1 = E \oplus (C^1_s \cap Y) \oplus C_{cu} = E \oplus Y.$$

Let $P_Y : C^1 \longrightarrow C^1$ denote the projection operator of the Banach space C^1 onto Y along E. Then we find an open neighborhood V of 0 in X_f such that the restriction of P_Y to

V forms a manifold chart of X_f with a C^1-smooth inverse mapping from $Y_0 = P_Y(V)$ onto V. Additionally, we may assume that $\delta > 0$ is sufficient small such that $W_{cu} \subset V$ and $P_Y W_{cu} \subset Y_0$. Consequently, we shall have established the assertion if we prove that $P_Y W_{cu}$ is an $\dim C_{cu}$-dimensional Lipschitz submanifold of the Banach space Y. But this is clear, since

$$P_Y W_{cu} = \left\{ P_Y(\varphi + w_{cu}(\varphi)) \,|\, \varphi \in C_{cu,0} \right\} = \left\{ \varphi + P_Y\, w_{cu}(\varphi) \mid \varphi \in C_{cu,0} \right\}$$

and $w_{cu}(\varphi) \in C_s^1$ for all $\varphi \in C_{cu,0}$. Therefore, for every $\varphi \in C_{cu,0}$ we obviously have $P_Y w_{cu}(\varphi) \in C_s^1 \cap Y$, so that $P_Y W_{cu}$ is the graph of the map

$$\left\{ \varphi \in C_{cu} \,|\, \|\varphi\|_1 < \delta \right\} \ni \chi \longmapsto P_Y\, w_{cu}(\chi) \in C_s^1 \cap Y.$$

In particular, the above map is Lipschitz continuous. This finishes the proof of the assertion (i) and so of Theorem 1 as a whole. □

5 The C^1-Smoothness of Local Center-Unstable Manifolds

Having proved the existence of local center-unstable manifolds in the last section, below we establish Theorem 2, asserting the C^1-smoothness of these manifolds. For this purpose, we follow very closely the procedure in the proof of smoothness of local center manifolds in Krisztin [8] and show that the technique also works in our situation.

Auxiliary Results

The main idea of the proof for Theorem 2 is to employ the following abstract lemma stating under which conditions the fixed points of a parameter-dependent contraction form a C^1-smooth mapping of the involved parameter.

Lemma 5.1 (Lemma II.8 in Krisztin et al. [9]) *Let X, Λ denote two Banach spaces over $\mathbb{R}$, let $\mathcal{P} \subset \Lambda$ be open, and let a map $\xi : X \times \mathcal{P} \longrightarrow X$ and a real $\kappa \in [0, 1)$ be given satisfying*

$$\|\xi(x, p) - \xi(\tilde{x}, p)\|_X \leq \kappa \|x - \tilde{x}\|_X$$

for all $x, \tilde{x} \in X$ and all $p \in \mathcal{P}$. Consider a convex subset $\mathcal{M}$ of X and a map $\Phi : \mathcal{P} \longrightarrow \mathcal{M}$ with the property that for every $p \in \mathcal{P}$, the element $\Phi(p)$ is the unique fixed point of the induced map $\xi(\cdot, p) : X \longrightarrow X$. Furthermore, suppose that the following hypotheses hold.

(i) *The restriction $\xi_0 = \xi\big|_{\mathcal{M}\times\mathcal{P}}$ of the mapping ξ has a partial derivative*

$$D_2\xi_0 : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(\Lambda, X),$$

and $D_2\xi_0$ is continuous.

(ii) *There exist a Banach space X_1 over $\mathbb{R}$ and a continuous injective map $j : X \longrightarrow X_1$ such that the composed map $k = j \circ \xi_0$ is continuously differentiable with respect to $\mathcal{M}$ in the sense that there is a continuous map*

$$B : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X, X_1)$$

such that for every $(x, p) \in \mathcal{M} \times \mathcal{P}$ and every $\varepsilon^* > 0$ one finds a real $\delta^* > 0$ guaranteeing

$$\left\| k(\tilde{x}, p) - k(x, p) - B(x, p)\big(\tilde{x} - x\big) \right\|_{X_1} \leq \varepsilon^* \left\| \tilde{x} - x \right\|_X$$

for all $\tilde{x} \in \mathcal{M}$ with $\|\tilde{x} - x\|_X \leq \tilde{\delta}$.

(iii) *There exist maps*

$$\xi^{(1)} : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X, X)$$

and

$$\xi_1^{(1)} : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X_1, X_1)$$

such that

$$B(x, p)\, \tilde{x} = \big(j \circ \xi^{(1)}(x, p)\big)(\tilde{x}) = \big(\xi_1^{(1)}(x, p) \circ j\big)(\tilde{x})$$

for all $(x, p, \tilde{x}) \in \mathcal{M} \times \mathcal{P} \times X$ and

$$\left\| \xi^{(1)}(x, p) \right\| \leq \kappa$$

as well as

$$\left\| \xi_1^{(1)}(x, p) \right\| \leq \kappa$$

on $\mathcal{M} \times \mathcal{P}$.

(iv) *The map*

$$\mathcal{M} \times \mathcal{P} \ni (x, p) \longmapsto j \circ \xi^{(1)}(x, p) \in \mathcal{L}(X, X_1)$$

is continuous.

Then the map $j \circ \Phi : \mathcal{P} \longrightarrow X_1$ is continuously differentiable and its derivative satisfies

$$D(j \circ \Phi)(p) = \xi_1^{(1)}(\Phi(p), p) \circ D(j \circ \Phi)(p) + j \circ D_2\xi_0(\Phi(p), p)$$

for all $p \in \mathcal{P}$.

To verify the hypotheses of the last lemma in our situation, we will need another auxiliary result on some smoothness properties of Nemitsky operators between scaled Banach spaces. This result is a negligible modification of Lemma II.6 in Krisztin et al. [9] and Lemma 3.1 in Krisztin [8].

Lemma 5.2 *Given any two Banach spaces E, F over $\mathbb{R}$, consider for a real $\eta \geq 0$ the scaled Banach spaces $E_\eta := C_\eta((-\infty,0],E)$ and $F_\eta := C_\eta((-\infty,0],F)$. Further, let $q: U \longrightarrow F$ be a continuous and bounded map defined on some subset $U \subset E$ and let $\mathfrak{M}((-\infty,0],U)$, $\mathfrak{M}((-\infty,0],F)$ denote the sets of all mappings from the interval $(-\infty,0]$ into U, F, respectively. Then for the induced substitution operator*

$$\tilde{q} : \mathfrak{M}((-\infty,0],U) \longrightarrow \mathfrak{M}((-\infty,0],F)$$

defined by

$$\tilde{q}(u)(t) = q(u(t))$$

for all $u \in \mathfrak{M}((-\infty,0],U)$ and $t \leq 0$ the following holds.

(i) *If $\eta, \tilde{\eta} \geq 0$, then $\tilde{q}(\mathfrak{M}((-\infty,0],U) \cap E_\eta) \subset F_{\tilde{\eta}}$.*

(ii) *If U is open, if q is continuously differentiable with a bounded derivative Dq and $0 \leq \eta \leq \tilde{\eta}$, then, for all $u \in C((-\infty,0],U)$, the linear map*

$$A(u) : \mathfrak{M}((-\infty,0],E) \longrightarrow \mathfrak{M}((-\infty,0],F),$$

given by

$$A(u)(v)(t) := Dq(u(t))v(t)$$

for $v \in \mathfrak{M}((-\infty,0],E)$ and $t \leq 0$, satisfies

$$A(u)(E_\eta) \subset F_{\tilde{\eta}}$$

and

$$\sup_{\|v\|_{E_\eta} \leq 1} \|A(u)(v)\|_{F_{\tilde{\eta}}} \leq \sup_{x \in U} \|Dq(x)\|,$$

the induced linear maps

$$A_{\eta\tilde{\eta}}(u) : E_\eta \longrightarrow F_{\tilde{\eta}}$$

are continuous and in case $\eta < \tilde{\eta}$, the map

$$A_{\eta\tilde{\eta}} : \big(C((-\infty,0],U) \cap E_\eta\big) \ni u \longmapsto A_{\eta\tilde{\eta}}(u) \in \mathcal{L}(E_\eta, F_{\tilde{\eta}})$$

is continuous as well.

(iii) *If additionally to the hypothesis stated above there holds $\eta < \tilde{\eta}$ and the set U is convex, then for every $\tilde{\varepsilon} > 0$ and $u \in C((-\infty,0],U) \cap E_\eta$ there exists $\tilde{\delta} > 0$ such that for every $v \in C((-\infty,0],U) \cap E_\eta$ with $\|v-u\|_{E_\eta} < \tilde{\delta}$ we have*

$$\|\tilde{q}(v) - \tilde{q}(u) - A_{\eta\tilde{\eta}}(u)(v-u)\|_{F_{\tilde{\eta}}} \leq \tilde{\varepsilon}\,\|v-u\|_{E_\eta}.$$

Proof: We adopt the proof of Lemma 3.1 in Krisztin [8] which falls naturally into three steps.

1. *The proof of (i).* Assuming $u \in (\mathfrak{M}((-\infty,0],U) \cap E_\eta)$, we see at once that the continuity of u and q implies the one of

$$(-\infty,0] \ni t \longmapsto \tilde{q}(u)(t) = q(u(t)) \in F.$$

Moreover, the boundedness of q leads to

$$\sup_{t\in(-\infty,0]} e^{\tilde{\eta}t}\|q(u(t))\|_F \leq \sup_{t\in(-\infty,0]} e^{\tilde{\eta}t} \sup_{t\in(-\infty,0]} \|q(u(t))\|_F \leq \sup_{x\in U} \|q(x)\|_F < \infty,$$

and thus $\|\tilde{q}(u)\|_{F_{\tilde{\eta}}} < \infty$. Consequently, we have $\tilde{q}(u) \in F_{\tilde{\eta}}$, which is the desired conclusion.

2. *The proof of (ii).* We begin with the observation that for all elements $u \in C((-\infty,0],U)$ the map $A(u)$ is well-defined, linear and that under the stated assumption the image $A(u)v \in \mathfrak{M}((-\infty,0],F)$ of an element $v \in E_\eta$, that is, the map

$$[0,\infty) \ni t \longmapsto Dq(u(t))\, v(t) \in F,$$

is continuous. As in this situation we also have

$$\begin{aligned} e^{\tilde{\eta}t}\, \|Dq(u(t))\, v(t)\|_F &\leq e^{(\tilde{\eta}-\eta)t} e^{\eta t}\, \|v(t)\|_E \sup_{x\in U} \|Dq(x)\| \\ &\leq \sup_{t\in(-\infty,0]} e^{\eta t}\|v(t)\|_E \sup_{x\in U} \|Dq(x)\| \\ &\leq \|v\|_{E_\eta} \sup_{x\in U} \|Dq(x)\| < \infty \end{aligned}$$

due to the boundedness of Dq on U, we conclude $A(u)(E_\eta) \subset F_{\tilde{\eta}}$ and additionally

$$\sup_{\|v\|_{E_\eta}\leq 1} \|A(u)v\|_{F_{\tilde{\eta}}} \leq \sup_{x\in U} \|Dq(x)\|.$$

In particular, this shows the continuity of the maps $A_{\eta\tilde{\eta}} : E_\eta \longmapsto F_{\tilde{\eta}}$.

The only point remaining of assertion (ii) concerns the continuity of the map

$$A_{\eta\tilde{\eta}} : C((-\infty,0],U) \cap E_\eta \ni u \longmapsto A_{\eta\tilde{\eta}}(u) \in \mathcal{L}(E_\eta, F_{\tilde{\eta}})$$

in case $\eta < \tilde{\eta}$. To see this, choose $u \in C((-\infty,0],U) \cap E_\eta$ and let $\tilde{\varepsilon} > 0$ be given. As $\eta < \tilde{\eta}$ and Dq is bounded on U, there clearly is a real $t_0 < 0$ satisfying

$$2e^{(\tilde{\eta}-\eta)t} \sup_{x\in U} \|Dq(x)\| < \tilde{\varepsilon}$$

for all $t \leq t_0$. Furthermore, in view of the continuity of u and Dq we find a constant $\tilde{\delta} > 0$ such that

$$B_t(u) = \left\{ y \in E \,\middle|\, \|y - u(t)\|_E < \tilde{\delta} e^{-\eta t_0} \right\} \subset U$$

as $t_0 \leq t \leq 0$ and such that additionally

$$\|Dq(y) - Dq(u(t))\| < \tilde{\varepsilon}$$

holds for all $y \in B_t$. Consequently, if $\tilde{u} \in C((-\infty,0],U) \cap E_\eta$ with $\|\tilde{u} - u\|_{E_\eta} < \tilde{\delta}$, and if $v \in E_\eta$ with $\|v\|_{E_\eta} \leq 1$, then the above estimates yield

$$e^{\tilde{\eta}t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big)v(t) \right\|_F \leq \tilde{\varepsilon}$$

for all $t \leq 0$. Indeed, in case $t \leq t_0$ we see

$$\begin{aligned} e^{\tilde{\eta}t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big)v(t) \right\|_F &\leq 2e^{(\tilde{\eta}-\eta)t} e^{\eta t} \|v(t)\|_E \sup_{x \in U} \|Dq(x)\| \\ &\leq 2e^{(\tilde{\eta}-\eta)t} \|v\|_{E_\eta} \sup_{x \in U} \|Dq(x)\| \\ &< \tilde{\varepsilon}, \end{aligned}$$

whereas, for $t_0 < t \leq 0$, we first conclude

$$\|\tilde{u}(t) - u(t)\|_E < \tilde{\delta} e^{-\eta t} < \tilde{\delta} e^{-\eta t_0}$$

and hence

$$\begin{aligned} e^{\tilde{\eta}t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big)v(t) \right\|_F &\leq e^{(\tilde{\eta}-\eta)t} e^{\eta t} \|v(t)\|_E \|Dq(\tilde{u}(t)) - Dq(u(t))\| \\ &\leq \|v\|_{E_\eta} \|Dq(\tilde{u}(t)) - Dq(u(t))\| \\ &< \tilde{\varepsilon}. \end{aligned}$$

This shows

$$\|A_{\eta\tilde{\eta}}(\tilde{u}) - A_{\eta\tilde{\eta}}(u)\| \leq \tilde{\varepsilon},$$

and the continuity of $A_{\eta\tilde{\eta}}$ is proved.

3. *The proof of (iii).* Note that from the additional assumption on the convexity of the open set U in E it is easy to check that the set $C((-\infty,0],U) \cap E_\eta$ is convex as well. Hence, for all $u, v \in C((-\infty,0],U) \cap E_\eta$ and all $t \leq 0$ we have

$$\begin{aligned} & e^{\tilde{\eta}t} \left\| q(v(t)) - q(u(t)) - Dq(u(t))\big(v(t) - u(t)\big) \right\|_F \\ &\quad = e^{\tilde{\eta}t} \left\| \int_0^1 \Big(Dq\big(sv(t) + (1-s)u(t)\big) - Dq\big(u(t)\big) \Big) \big(v(t) - u(t)\big)\, ds \right\|_F \\ &\quad \leq e^{(\tilde{\eta}-\eta)t} e^{\eta t} \|v(t) - u(t)\|_E \\ &\qquad\qquad \cdot \max_{s \in [0,1]} \left\| Dq\big(sv(t) + (1-s)u(t)\big) - Dq\big(u(t)\big) \right\| \\ &\quad \leq e^{(\tilde{\eta}-\eta)t} \|v - u\|_{E_\eta} \\ &\qquad\qquad \cdot \max_{s \in [0,1]} \left\| Dq\big(s\, v(t) + (1-s)\, u(t)\big) - Dq\big(u(t)\big) \right\|. \end{aligned} \tag{35}$$

Fix $u \in C((-\infty,0],E) \cap E_\eta$ and $\tilde\varepsilon > 0$. Then, using $\eta < \tilde\eta$, we find constants $t_0 < 0$ and $\tilde\delta \geq 0$ as in the last part. Let now an arbitrary $v \in C((-\infty,0],U) \cap E_\eta$ with $\|v-u\|_{E_\eta} < \tilde\delta$ be given. Then, in the situation $t \leq t_0$, the estimate (35) and the choice of the real t_0 yield

$$\begin{aligned} e^{\tilde\eta t} & \big\| q(v(t)) - q(u(t)) - Dq(u(t))\big(v(t)-u(t)\big)\big\|_F \\ & \leq e^{(\tilde\eta-\eta)t}\|v-u\|_{E_\eta} \\ & \qquad \cdot \max_{s\in[0,1]} \big\|Dq\big(sv(t)+(1-s)u(t)\big) - Dq\big(u(t)\big)\big\| \\ & \leq 2e^{(\tilde\eta-\eta)t} \max_{x\in U}\|Dq(x)\|\|v-u\|_{E_\eta} \\ & < \tilde\varepsilon\|v-u\|_{E_\eta} \end{aligned}$$

On the other hand, if $t_0 < t \leq 0$, then we have

$$\|v(t)-u(t)\|_E \leq \tilde\delta e^{-\eta t} < \tilde\delta e^{-\eta t_0}.$$

This implies $sv(t)+(1-s)u(t) \in B_t(u)$ for all $0 \leq s \leq 1$ and hence, by inequality (35), we get again

$$\begin{aligned} e^{\tilde\eta t} & \big\| q(v(t)) - q(u(t)) - Dq(u(t))\big(v(t)-u(t)\big)\big\|_F \\ & \leq e^{(\tilde\eta-\eta)t}\|v-u\|_{E_\eta} \\ & \qquad \cdot \max_{s\in[0,1]} \big\|Dq\big(sv(t)+(1-s)u(t)\big) - Dq\big(u(t)\big)\big\| \\ & < \tilde\varepsilon e^{(\tilde\eta-\eta)t}\|v-u\|_{E_\eta} \\ & < \tilde\varepsilon\|v-u\|_{E_\eta}. \end{aligned}$$

Combining these yields

$$\big\| \tilde q(v) - \tilde q(u) - A_{\eta\tilde\eta}(u)\big(v-u\big)\big\|_{F_{\tilde\eta}} \leq \tilde\varepsilon \,\|v-u\|_{E_\eta}\,,$$

and the proof is complete. □

Proof of Theorem 2

After the preparatory results above, we return to the local center-unstable manifolds from the last section and prove Theorem 2.

We start our proof with the observation that an important, but probably inconspicuous point of our construction of the invariant manifolds in the foregoing section was the choice of a constant $\eta > 0$ satisfying condition (27), that is,

$$c_c < \eta < \min\{-c_s, c_u\},$$

and hereafter the choice of a second constant $0 < \delta < \delta_1$ satisfying condition (28), that is,

$$\|\mathcal{K}_\eta\|\lambda(\delta) < \frac{1}{2}.$$

Now, recall from Corollary 3.4 that $\mathcal{K}_\eta$ is a bounded linear map from the Banach space Y_η into C^1_η. Moreover, the bound of $\mathcal{K}_\eta$ satisfies the inequality

$$\|\mathcal{K}_\eta\| < c(\eta) \tag{36}$$

with the continuous map $c : (c_c, \min\{-c_s, c_u\}) \longrightarrow [0, \infty)$ given by

$$c(\eta) = K\Big(1 + e^{\eta h}\|L_e\|\Big)\left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta}\right) + e^{\eta h}.$$

Hence, fixing a constant $\eta_1 > 0$ with $c_c < \eta_1 < \min\{-c_u, c_s\}$ and additionally a constant $0 < \delta < \delta_1$ with

$$c(\eta_1)\lambda(\delta) < \frac{1}{2},$$

we clearly find a real $c_c < \eta_0 < \eta_1$ such that the estimate

$$c(\eta)\lambda(\delta) < \frac{1}{2} \tag{37}$$

is fulfilled for all $\eta_0 \le \eta \le \eta_1$. As an immediate consequence, we see that for any $\eta_0 \le \eta \le \eta_1$ the pair (η, δ) satisfies both conditions (27), (28), and thus the construction in the last section works for any such choice of constants.

Below, we show the assertion of Theorem 2 for the map w^{η_1}. Hereby, remember that w^{η_1} may be also written as the composition

$$w^{\eta_1} = P_s \circ \mathrm{ev}_0 \circ \tilde{u}_{\eta_1}$$

with the projection operator P_s of C^1 along the center-unstable space C_{cu} onto C^1_s, the evaluation map

$$\mathrm{ev}_0 : C^1_{\eta_1} \ni u \longmapsto u(0) \in C^1$$

and the fixed point operator $\tilde{u}_{\eta_1} : C_{cu} \longrightarrow C^1_{\eta_1}$ defined by (33). Since P_s and ev_0 are both bounded linear maps, for a conclusion on the C^1-smoothness of w^{η_1} we are obviously reduced to proving the continuous differentiability of $\tilde{u}_{\eta_1}$ on C_{cu}. By application of Lemmata 5.1, 5.2, we show that $\tilde{u}_{\eta_1}$ is indeed continuously differentiable on C_{cu} in the following.

Consider the open neighborhood

$$O_\delta := \big\{\psi \in C^1 \mid \|P_s\psi\|_1 < \delta\big\}$$

of the origin in C^1. The set O_δ is clearly convex, and from Corollary 4.1 and Proposition 4.2 we see that the restriction of the function r_δ to O_δ is bounded, C^1-smooth and has a bounded derivative with

$$\sup_{\varphi \in O_\delta} \|Dr_\delta(\varphi)\| \le \lambda(\delta).$$

Additionally, we claim

$$\big\{\tilde{u}_\eta(\varphi)(t) \,\big|\, \varphi \in C_{cu}, t \le 0\big\} \subset O_\delta$$

for all $\eta_0 \le \eta \le \eta_1$. Indeed, combining the inequalities (29), (36) and (37) yields

$$\begin{aligned}
\|w^\eta(\varphi)\|_1 &= \|P_s\, \tilde{u}_\eta(\varphi)(0)\|_{C^1} \\
&= \big\|\big(\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\varphi))\big)(0)\big\|_{C^1} \\
&\le \big\|\big(\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\varphi))\big)\big\|_{C^1_\eta} \\
&\le \|\mathcal{K}_\eta\|\, \|R_{\delta\eta}(\tilde{u}_\eta(\varphi))\|_{Y_\eta} \\
&\le c(\eta)\, \delta\, \lambda(\delta) \\
&< \delta
\end{aligned}$$

as $\varphi \in C_{cu}$ and $\eta_0 \le \eta \le \eta_1$. Thus, in view of Remark 4.9 we obtain

$$\|P_s\, \tilde{u}_\eta(\varphi)(t)\|_1 = \|P_s\, \tilde{u}_\eta(P_{cu}\, \tilde{u}_\eta(\varphi)(t))(0)\|_1 = \|w^\eta(P_{cu}\, \tilde{u}_\eta(\varphi)(t))\|_1 < \delta$$

for all $(\varphi, \eta, t) \in C_{cu} \times [\eta_0, \eta_1] \times (-\infty, 0]$, as claimed. Now, setting $E := C^1$, $F := Y^{\odot *}$, $O := O_\delta$, $q := l \circ r_\delta$, $\eta := \eta_0$, $\tilde{\eta} := \eta_1$ and applying Lemma 5.2, we conclude that the linear maps

$$A(u) : \mathfrak{M}((-\infty, 0], C^1) \longrightarrow \mathfrak{M}((-\infty, 0], Y^{\odot *})$$

define a continuous map $A_{\eta_0\eta_1}$ from the convex set

$$\mathcal{M} := \Big\{u \in C^1_{\eta_0} \mid u(t) \in O_\delta \text{ for all } t \in (-\infty, 0]\Big\}$$

into the Banach space $\mathcal{L}(C^1_{\eta_0}, Y_{\eta_1})$. In addition, we see that $A_{\eta_0\eta_1}$ has the property that for every point $u \in \mathcal{M}$ and every real $\tilde{\varepsilon} > 0$ there is a constant $\tilde{\delta}(\tilde{\varepsilon}) > 0$ such that for all $v \in \mathcal{M}$ with $\|v - u\|_{C^1_{\eta_0}} \le \tilde{\delta}$ we have $R_{\delta\eta_1}(u), R_{\delta\eta_1}(v) \in Y_{\eta_1}$ and

$$\big\|R_{\delta\eta_1}(u) - R_{\delta\eta_1}(v) - A_{\eta_0\eta_1}(u)\big(v - u\big)\big\|_{Y_{\eta_1}} \le \tilde{\varepsilon}\|v - u\|_{C^1_{\eta_0}}. \tag{38}$$

Next, we are going to employ Lemma 5.1. To this end, we regard the inclusion map

$$j_{\eta_0\eta_1} : C^1_{\eta_0} \ni u \longmapsto u \in C^1_{\eta_1}.$$

As $\eta_0 < \eta_1$, this map obviously is well-defined and is trivially linear and bounded. Moreover, for all $\varphi \in C_{cu}$, $j_{\eta_0\eta_1}$ maps the fixed point $\tilde{u}_{\eta_0}(\varphi)$ of $\mathcal{G}_{\eta_0}(\,\cdot\,, \varphi)$ defined in Proposition 4.6 onto the fixed point $\tilde{u}_{\eta_1}(\varphi)$ of $\mathcal{G}_{\eta_1}(\,\cdot\,, \varphi)$. Indeed, since for a given $\varphi \in C_{cu}$ we have

$$\begin{aligned}
\mathcal{G}_{\eta_1}(j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)), \varphi) &= S_{\eta_1}\, \varphi + \mathcal{K}_{\eta_1} \circ R_{\delta\eta_1}(j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi))) \\
&= T_e(\,\cdot\,)\, \varphi + \mathcal{K}^{cu}\, R(\tilde{u}_{\eta_0}(\varphi)) \\
&= j_{\eta_0\eta_1}\big(S_{\eta_0}\, \varphi + \mathcal{K}_{\eta_0} \circ R_{\delta\eta_0}(\tilde{u}_{\eta_0}(\varphi))\big) \\
&= j_{\eta_0\eta_1}\big(\mathcal{G}_{\eta_0}(\tilde{u}_{\eta_0}(\varphi), \varphi)\big) \\
&= j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)),
\end{aligned}$$

$j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi))$ is a fixed point of $\mathcal{G}_{\eta_1}(\,\cdot\,,\varphi) : C^1_{\eta_1} \longrightarrow C^1_{\eta_1}$ and from the uniqueness of the fixed point there actually follows

$$j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)) = \tilde{u}_{\eta_1}(\varphi).$$

Set $X := C^1_{\eta_0}$, $X_1 := C^1_{\eta_1}$, $\Lambda := \mathcal{P} = C_{cu}$, $\xi := \mathcal{G}_{\eta_0}$, $j := j_{\eta_0\eta_1}$ and $\kappa := 1/2$. Then we see at once that $\tilde{u}_{\eta_0}(P) \subset \mathcal{M}$, and this implies that the unique fixed point of $\xi(\,\cdot\,,\varphi) : X \longrightarrow X$ is given by the value $\Phi(\varphi)$ of the map

$$\Phi : \mathcal{P} \ni \varphi \longmapsto \tilde{u}_{\eta_0}(\varphi) \in \mathcal{M}.$$

Additionally, for each $\varphi \in C_{cu}$ the map $\xi(\,\cdot\,,\varphi) = \mathcal{G}_{\eta_0}(\,\cdot\,,\varphi)$ is Lipschitz continuous with Lipschitz constant κ due to the proof of Proposition 4.6. Thus, for an application of Lemma 5.1 with the above choice of spaces, maps and reals it remains to confirm conditions (i) - (iv). This point is done below in detail.

Verification of hypothesis (i): Observe that for the restriction ξ_0 of the map ξ to $\mathcal{M} \times \mathcal{P}$ we have

$$\xi_0(u,\varphi) = \mathcal{G}_{\eta_0}(u,\varphi) = S_{\eta_0}\,\varphi + \mathcal{K}_{\eta_0} \circ R_{\delta\eta_0}(u).$$

Consequently, ξ_0 is partially differentiable with respect to the second variable, and for every $(u,\varphi) \in \mathcal{M} \times \mathcal{P}$ its derivative $D_2\xi_0(u,\varphi) \in \mathcal{L}(\Lambda, X)$ is given by

$$D_2\xi_0(u,\varphi)\psi = S_{\eta_0}\psi$$

for all $\psi \in C_{cu}$. Obviously, $D_2\xi_0 : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(\Lambda, X)$ is a constant map and thus in particular continuous. This shows hypothesis (i) of Lemma 5.1.

Verification of hypothesis (ii): The mapping $k = j \circ \xi_0$ reads

$$k(u,\varphi) = S_{\eta_1}\,\varphi + \mathcal{K}_{\eta_1} \circ R_{\delta\eta_1}(j(u)),$$

and the map

$$B : \mathcal{M} \times \mathcal{P} \ni (u,\varphi) \longmapsto \mathcal{K}_{\eta_1} \circ (A_{\eta_0\eta_1}(u)) \in \mathcal{L}(X, X_1)$$

is of course continuous as $\mathcal{K}_{\eta_1}$, $A_{\eta_0\eta_1}$ are so. Consider next an arbitrary point $(u,\varphi) \in \mathcal{M} \times \mathcal{P}$ and $\varepsilon^* > 0$. Choosing

$$\delta^* = \tilde{\delta}\left(\frac{\varepsilon^*}{1 + \|\mathcal{K}_{\eta_1}\|}\right)$$

with the constant $\tilde{\delta}$ from estimate (38), we find that for all points $v \in \mathcal{M}$ with $\|v - u\|_{C^1_{\eta_0}} < \delta^*$

we have

$$\begin{aligned}
&\big\|k(v,\varphi) - k(u,\varphi) - B(u,\varphi)\big(v-u\big)\big\|_{X^1} \\
&\qquad = \big\|\mathcal{K}_{\eta_1}(R(v)) - \mathcal{K}_{\eta_1}(R(u)) - \mathcal{K}_{\eta_1}\big(A_{\eta_0\eta_1}(u)(v-u)\big)\big\|_{C^1_{\eta_1}} \\
&\qquad \leq \|\mathcal{K}_{\eta_1}\| \, \big\|R(v) - R(u) - A_{\eta_0\eta_1}(u)\big(v-u\big)\big\|_{Y_{\eta_1}} \\
&\qquad \leq \|\mathcal{K}_{\eta_1}\| \, \frac{\varepsilon^*}{1+\|\mathcal{K}_{\eta_1}\|} \|v-u\|_{C^1_{\eta_0}} \\
&\qquad \leq \varepsilon^* \|v-u\|_{C^1_{\eta_0}}.
\end{aligned}$$

Thus, condition (ii) is satisfied.

Verification of hypothesis (iii): Next we note that for every $u \in \mathcal{M}$ and all $v \in X$ we have

$$\begin{aligned}
A(u)(v)(t) &= Dq(u(t))v(t) \\
&= D(l \circ r_\delta)(u(t))v(t) \\
&= Dl(r_\delta(u(t))) \circ Dr_\delta(u(t))v(t) \\
&= l \circ Dr_\delta(u(t))v(t)
\end{aligned}$$

for $t \leq 0$. Since $\sup_{\varphi \in O_\delta} \|Dr_\delta(\varphi)\| \leq \lambda(\delta)$ and $\|\mathcal{K}_{\eta_0}\| \leq c(\eta_0)$, and $\|l\| = 1$, it is obvious that for every $u \in \mathcal{M}$, the induced map

$$\mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u)) \in \mathcal{L}(X, X)$$

satisfies

$$\|\mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u))\| \leq c(\eta_0)\lambda(\delta).$$

In the same manner we see that for all $u \in \mathcal{M}$

$$\mathcal{K}_{\eta_1} \circ (A_{\eta_1\eta_1}(u)) \in \mathcal{L}(X_1, X_1)$$

with

$$\|\mathcal{K}_{\eta_0} \circ (A_{\eta_1\eta_1}(u))\| \leq c(\eta_1)\lambda(\delta).$$

Define

$$\xi^{(1)} : \mathcal{M} \times \mathcal{P} \ni (u,\varphi) \longmapsto \mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u)) \in \mathcal{L}(X, X)$$

and

$$\xi_1^{(1)} : \mathcal{M} \times \mathcal{P} \ni (u,\varphi) \longmapsto \mathcal{K}_{\eta_1} \circ (A_{\eta_1\eta_1}(u)) \in \mathcal{L}(X_1, X_1).$$

Then, for all $(u,\varphi,v)\in\mathcal{M}\times\mathcal{P}\times X$, we get

$$\begin{aligned} B(u,\varphi)v &= \big(\mathcal{K}_{\eta_1}\circ(A_{\eta_0\eta_1}(u))\big)(v)\\ &= \mathcal{K}^{cu}(A(u)v)\\ &= j\big(\xi^{(1)}(u,\varphi)v\big)\\ &= \xi_1^{(1)}(u,\varphi)\big(j(v)\big). \end{aligned}$$

Moreover, in view of the choice of η_0,η_1 and δ due to Eq. (37) we have

$$\big\|\xi^{(1)}(u,\varphi)\big\|_X\leq\kappa$$

and

$$\Big\|\xi_1^{(1)}(u,\varphi)\Big\|_{X_1}\leq\kappa$$

for all $(u,\varphi)\in\mathcal{M}\times\mathcal{P}$. This shows that hypothesis (iii) is valid too.

Verification of hypothesis (iv): Finally, we find that the map

$$\mathcal{M}\times\mathcal{P}\ni(x,p)\longmapsto j\circ\xi^{(1)}(x,p)\in\mathcal{L}(X,X_1)$$

satisfies

$$j\big(\xi^{(1)}(u,\varphi)v\big)=\big(j\circ\mathcal{K}_{\eta_0}\circ(A_{\eta_0\eta_0}(u))\big)(v)=\mathcal{K}^{cu}(A(u)v)=B(u,\varphi)v$$

for all $(u,\varphi,v)\in\mathcal{M}\times\mathcal{P}\times X$. As B is continuous, the continuity of the map

$$\mathcal{M}\times\mathcal{P}\ni(x,p)\longrightarrow j\circ\xi^{(1)}(x,p)\in\mathcal{L}(X,X_1)$$

follows, and this is precisely condition (iv) of Lemma 5.1.

As by the above all assumptions of Lemma 5.1 are fulfilled, we conclude that the map

$$\tilde{u}_{\eta_1}=j\circ\Phi:C_{cu}\longrightarrow C^1_{\eta_1}$$

is in fact continuously differentiable. So, if we prove that additionally we have $Dw_{cu}(0)=0$, the assertion of Theorem 2 follows. But this is easily seen in consideration of the formula

$$D\tilde{u}_{\eta_1}(\varphi)=\xi_1^{(1)}(\tilde{u}_{\eta_0}(\varphi),\varphi)\circ D\tilde{u}_{\eta_1}(\varphi)+j\circ D_2\xi_0(\tilde{u}_{\eta_0}(\varphi),\varphi)$$

for the derivative of $\tilde{u}_{\eta_1}$ at $\varphi\in C_{cu}$. Indeed, by $Dr_\delta(0)=0$, we first obtain $A(0)=0$ and $\xi_1^{(1)}(0,0)=0$. Thus, in consideration of $\tilde{u}_{\eta_0}(0)=0$ we get

$$D\tilde{u}_{\eta_1}(0)\psi=j\circ D_2\xi_0(0,0)\psi=S_{\eta_1}\psi$$

for all $\psi\in C_{cu}$. This implies

$$Dw^{\eta_1}(0)\psi=\big(P_s\circ\mathrm{ev}_0\circ D\tilde{u}_{\eta_1}(0)\big)(\psi)=P_s\psi=0$$

on C_{cu}. Consequently, we get

$$Dw^{\eta_1}(0)=0$$

and this completes the proof of Theorem 2.

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