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Improved nearness research II

ABSTRACT. When applying in consequence the new created concept "Bounded Topology" [8] hence "classical structures" like nearness structures [5], convergence structures [8] and syntopogenous structures [8] will be analyzed in connexion with neighbourhood structures [11] or supertopologies [4], respectively. In this context "nearness" is presented as special paranearness, "convergence" as special b-convergence and being "syntopogenous" as special case of b-syntopogenous, leading us accordingly to a <u>general</u> theory of his <u>own</u>! Now, in this paper we will study certain superclan spaces, whichever are in one-to-one correspondence to <u>strict</u> topological extensions. Here, we should mention that the presented concept is <u>not</u> of <u>utmost</u> generality, but then the reader is referred to [9].

KEY WORDS AND PHRASES. LEADER proximity; supertopological space; LODATO space; supernear space; superclan space; Bounded Topology

1 Basic concepts

As usual $\underline{P}X$ denotes the power set of a set X, and we use $\mathcal{B}^X \subset \underline{P}X$ to denote <u>a collection</u> of <u>bounded</u> subsets of X, also known as $\underline{B}$-sets, e.g. $\mathcal{B}^X$ has the following properties:

(b$_1$) $\varnothing \in \mathcal{B}^X$;

(b$_2$) $B_2 \subset B_1 \in \mathcal{B}^X$ imply $B_2 \in \mathcal{B}^X$;

(b$_3$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$.

Then, for $\underline{B}$-sets $\mathcal{B}^X, \mathcal{B}^Y$ a function $f : X \longrightarrow Y$ is called <u>bounded</u> iff f satisfies (b), e.g.

(b) $\{f[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^Y$.

Definition 1.1 *For a set X, we call a tripel $(X, \mathcal{B}^X, N)$ consisting of X, $\underline{B}$-set $\mathcal{B}^X$ and a near-operator $N : \mathcal{B}^X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ a* <u>*supernearness space*</u> *(shortly* <u>*supernear space*</u>*) iff the following axioms are satisfied, e.g.*

(sn$_1$) $B \in \mathcal{B}^X$ *and* $\rho_2 << \rho_1 \in N(B)$ *imply* $\rho_2 \in N(B)$, *where* $\rho_2 << \rho_1$ *iff* $\forall F_2 \in \rho_2 \exists F_1 \in \rho_1 \; F_2 \supset F_1$;

(sn$_2$) $B \in \mathcal{B}^X$ *implies* $\mathcal{B}^X \notin N(B) \neq \varnothing$;

(sn$_3$) $\rho \in N(\varnothing)$ *implies* $\rho = \varnothing$;

(sn$_4$) $x \in X$ *implies* $\{\{x\}\} \in N(\{x\})$;

(sn$_5$) $B_1 \subset B_2 \in \mathcal{B}^X$ *imply* $N(B_1) \subset N(B_2)$;

(sn$_6$) $B \in \mathcal{B}^X$ *and* $\rho_1 \vee \rho_2 \in N(B)$ *imply* $\rho_1 \in N(B)$ *or* $\rho_2 \in N(B)$, *where* $\rho_1 \vee \rho_2 := \{F_1 \cup F_2 : F_1 \in \rho_1, F_2 \in \rho_2\}$;

(sn$_7$) $B \in \mathcal{B}^X, \rho \subset \underline{P}X$ *and* $\{cl_N(F) : F \in \rho\} \in N(B)$ *imply* $\rho \in N(B)$, *where* $cl_N(F) := \{x \in X : \{F\} \in N(\{x\})\}$.

If $\rho \in N(B)$ for some $B \in \mathcal{B}^X$, then we call ρ a <u>*B-near collection*</u> *in N. For supernear spaces $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ a bounded function $f : X \longrightarrow Y$ is called* <u>*sn-map*</u> *iff it satisfies* (sn), *e.g.*

(sn) $B \in \mathcal{B}^X$ *and* $\rho \in N(B)$ *imply* $\{f[F] : F \in \rho\} =: f\rho \in M(f[B])$.

We denote by SN the corresponding category.

Example 1.2 (i) For a nearness space (X, ξ) let $\mathcal{B}^X$ be $\underline{B}$-set. Then we consider the tripel $(X, \mathcal{B}^X, N_\xi)$, where

$$N_\xi(\varnothing) := \{\varnothing\} \text{ and}$$
$$N_\xi(\varnothing) := \{\rho \subset \underline{P}X : \{B\} \cup \rho \in \xi\}, \text{ otherwise.}$$

(ii) For a topological space (X, t) given by closure operator t let $\mathcal{B}^X$ be $\underline{B}$-set. Then we consider the tripel $(X, \mathcal{B}^X, N_t)$, where $N_t(\varnothing) := \{\varnothing\}$ and $N_t(B) := \{\rho \subset \underline{P}X : \exists x \in B x \in \bigcap\{t(F) : F \in \rho\}\}$, otherwise.

(iii) For a LODATO space $(X, \mathcal{B}^X, \delta)$ with $\delta \subset \mathcal{B}^X \times \underline{P}X$ we consider the tripel $(X, \mathcal{B}^X, N_\delta)$, where $N_\delta(\varnothing) := \{\varnothing\}$ and $N_\delta(B) := \{\rho \subset \underline{P}X : \rho \subset \delta(B)$ and $\{B\} \cup \rho \subset \cap\{\delta(F) : F \in \rho \cap \mathcal{B}^X\}\}$, otherwise, with $\delta(B) := \{A \subset X : B\delta A\}$. Hereby, following conditions must be satisfied:

(bp$_0$) $B \in \mathcal{B}^X$ implies $cl_\delta(B) \in \mathcal{B}^X$, where $cl_\delta(B) := \{x \in X : \{x\}\delta B\}$;

(bp$_1$) $\varnothing\overline{\delta}A$ and $B\overline{\delta}\varnothing$ (e.g. $\varnothing$ is not in relation to A, and analogously this is also holding for B;

(bp$_2$) $B\delta(A_1 \cup A_2)$ iff $B\delta A_1$ or $B\delta A_2$;

(bp$_3$) $x \in X$ implies $\{x\}\delta\{x\}$;

(bp$_4$) $B_1 \subset B_2 \in \mathcal{B}^X$ and $B_1\delta A$ imply $B_2\delta A$;

(bp$_5$) $B \in \mathcal{B}^X$ and $B\delta A$ with $A \subset cl_\delta(C)$ imply $B\delta C$;

(bp$_6$) $B_1 \cup B_2 \in \mathcal{B}^X$ and $(B_1 \cup B_2)\delta A$ imply $B_1\delta A$ or $B_2\delta A$;

(bp$_7$) $A, B \subset X, cl_\delta(B) \in \mathcal{B}^X$ and $cl_\delta(B)\delta A$ imply $B\delta A$;

(bp$_8$) $B_1, B_2 \in \mathcal{B}^X$ and $B_1\delta B_2$ imply $B_2\delta B_1$.

(iv) For a preLEADER space $(X, \mathcal{B}^X, \delta)$ with $\delta \subset \mathcal{B}^X \times \underline{P}X$ only satisfies (bp$_1$) to (bp$_5$) we consider the tripel $(X, \mathcal{B}^X, N^\delta)$, where $N^\delta(B) := \{\rho \subset \underline{P}X : \rho \subset \delta(B)\}$ for each $B \in \mathcal{B}^X$.

Definition 1.3 *For preLEADER spaces $(X, \mathcal{B}^X, \delta), (Y, \mathcal{B}^Y, \gamma)$ a bounded function $f : X \longrightarrow Y$ is called p-map iff f satisfied* (p)*, e.g.*

(p) *$B \in \mathcal{B}^X, A \subset X$ and $B\delta A$ imply $f[B]\gamma f[A]$. By LOSP respectively pLESP we denote the corresponding categories.*

Definition 1.4 *TEXT denotes the category, whose objects are triples $E := (e, \mathcal{B}^X, Y)$ - called topological extensions - where $X := (X, cl_X), Y := (Y, cl_Y)$ are topological spaces (given by closure operators) with $\underline{B}$-set $\mathcal{B}^X$, and $e : X \longrightarrow Y$ is a function satisfying the following conditions:*

(tx$_1$) *$A \in \underline{P}X$ implies $cl_X(A) = e^{-1}[cl_Y(e[A])]$, where e^{-1} denotes the inverse image under e;*

(tx$_2$) *$cl_Y(e[X]) = Y$, which means the image of X under e is dense in Y. Morphisms in TEXT have the form $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$, where $f : X \longrightarrow X', g : Y \longrightarrow Y'$ are continuous maps such that f is bounded, and the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "e'"'] & Y' \end{tikzcd} .$$

If $(f,g):(e,\mathcal{B}^X,Y)\longrightarrow(e',\mathcal{B}^{X'},Y')$ *and* $(f',g'):(e',\mathcal{B}^{X'},Y')\longrightarrow(e'',\mathcal{B}^{X''},Y'')$*, are TEXT-morphisms, then they can be* composed *according to the rule:*

$$(f',g')\circ(f,g):=(f'\circ f,g'\circ g):(e,\mathcal{B}^X,Y)\longrightarrow(e'',\mathcal{B}^{X''},Y''),$$

where "o" denotes the composition of maps.

Remark 1.5 Observe, that axiom (tx_1) in this definition is automatically satisfied if $e: X\longrightarrow Y$ is a topological embedding. Moreover, we only admit an ordinary $\underline{B}$-set $\mathcal{B}^X$ on X which need not be necessary coincide with the power $\underline{P}X$. In addition we mention that such an extension is called strict iff it satisfies (tx_3), e.g.

(tx_3) $\{cl_Y(e[A]):A\subset X\}$ forms a base for the closed subsets of Y [1].

By STREXT we denote the corresponding full subcategory of TEXT.

(v) For a topological extension $E:=(e,\mathcal{B}^X,Y)$ we consider the tripel $(X,\mathcal{B}^X,N_e)$, where

$N_e(\varnothing):=\{\varnothing\}$ and

$N_e(B):=\{\rho\subset\underline{P}X: y\in\cap\{cl_Y(e[F]):F\in\rho\}$ for some $y\in e[B]\}$, otherwise.

2 Some important isomorphisms

With respect to above examples, first let us focus our attention to some special classes of supernear spaces.

Definition 2.1 *A supernear space* $(X,\mathcal{B}^X,N)$ *is called* saturated *iff* $\mathcal{B}^X$ *is, e.g.*

(s) $X\in\mathcal{B}^X$.

Remark 2.2 Note, that in above case $\mathcal{B}^X$ coincide with the power $\underline{P}X$. (Also compare with examples (i) or (ii), respectively). Moreover, we claim that the full subcategory SN^S of SN, whose objects are the saturated supernear spaces is bireflective in SN. Concretely, for a supernear space $(X,\mathcal{B}^X,N)$ we put: $N^S(B):=N(B)$ for each $B\in\mathcal{B}^X$ and $N^S(B):=\{\rho\subset\underline{P}X:\exists x\in X\exists B^*\in\mathcal{B}^X(x\in B\supset B^*$ and $\rho\in N(\{x\})\cup N(B^*))\}$ for each $B\in\underline{P}X\setminus\mathcal{B}^X$, hence $(X,\underline{P}X,N^S)$ is saturated supernear space and $1_X:(X,\mathcal{B}^X,N)\longrightarrow(X,\underline{P}X,N^S)$ to be the bireflection in demand!

Definition 2.3 *A supernear space* $(X,\mathcal{B}^X,N)$ *is called*

(i) paranearness space *(paranear space) iff it is symmetric, hence N additionally satisfies* (sy)*, e.g.*

(sy) $B \in \mathcal{B}^X \setminus \{\varnothing\}$ *and* $\rho \in N(B)$ *imply* $\{B\} \cup \rho \in \cap\{N(A) : A \in (\rho \cap \mathcal{B}^X) \cup \{B\}\}$;

(ii) *pointed iff* N *satisfies (pt), e.g.*

(pt) $B \in \mathcal{B}^X \setminus \{\varnothing\}$ *implies* $N(B) = \cup\{N(\{x\}) : x \in B\}$. *By PN respectively PT-SN we denote the corresponding full subcategory of SN.*

Theorem 2.4 *The category NEAR of nearness spaces and nearness preserving maps is isomorphic to the full subcategory* PN^S *of PN, whose objects are the saturated paranear spaces.*

Proof: According to example (i). Conversely, we consider for a saturated paranear space $(Y, \mathcal{B}^Y, M)$:

$$\mu_M := \{\mathcal{A} \subset \underline{P}X : \mathcal{A} \in \cap\{M(A) : A \in \mathcal{A}\}\}.$$

□

Theorem 2.5 *The category TOP of topological spaces and continuous maps is isomorphic to the full subcategory PT-*SN^S *of PT-SN, whose objects are the saturated pointed supernear spaces.*

Proof: According to example (ii) and by respecting (sn_7) in definition 1.1. □

Definition 2.6 *Let be given a supernear space* $(X, \mathcal{B}^X, N)$. *For* $B \in \mathcal{B}^X \mathcal{C} \in GRL(X)$ *is called B-clan in N iff it satisfies*

(cla_1) $B \in \mathcal{C} \in N(B)$;

(cla_2) $A \in \mathcal{C}$ *and* $A \subset cl_N(F)$ *imply* $F \in \mathcal{C}$, *where* $GRL(X) := \{\gamma \subset \underline{P}X : \gamma$ *is grill* $\}$, *and* $\gamma \subset \underline{P}X$ *is called grill (Choquet [3]) iff*

(gri_1) $\varnothing \notin \gamma$;

(gri_2) $G_1 \cup G_2 \in \gamma$ *iff* $G_1 \in \gamma$ *or* $G_2 \in \gamma$.

Then $(X, \mathcal{B}^X, N)$ *is called superclan space iff* N *satisfies* (cla), *e.g.*

(cla) $B \in \mathcal{B}^X \setminus \{\varnothing\}$ *and* $\rho \in N(B)$ *imply the existence of B-clan* $\mathcal{C} \in GRL(X) \rho \subset \mathcal{C}$.

Moreover, if $(X, \mathcal{B}^X, N) \in \underline{P}N$ *satisfies* (cla), *we analogously call it paraclan space!*

Remark 2.7 Here, we note that each pointed supernear space is always a superclan space by making use of the fact that for each $B \in \mathcal{B}^X$ with $x \in B$ $\{T \subset X : x \in cl_N(T)\} =: x_N$ is B-clan in N, and x_N is maximal in $N(\{x\}) \setminus \{\varnothing\}$, ordered by inclusion!

Theorem 2.8 *The category BUN of bunch-determined nearness spaces and related maps* [2] *is isomorphic to the full subcategory CLA-PN^S of PN^S, whose objects are the saturated paraclan spaces.*

Proof: Compare with theorem 2.4. $\square$

Definition 2.9 *A paranear space $(X, \mathcal{B}^X, N)$ is called* <u>*round*</u> *iff it satisfies* (r), *e.g.*

(r) $B \in \mathcal{B}^X$ *implies* $cl_N(B) \in \mathcal{B}^X$.

Theorem 2.10 *The full subcategory R-PN of PN, whose objects are the round paranear spaces is bireflective in PN.*

Proof: For a paranear space $(X, \mathcal{B}^X, N)$ we set:

$\mathcal{B}^X_N := \{\mathcal{D} \subset X : \exists B \in \mathcal{B}^X cl_N(B) \supset \mathcal{D}\}$ and

$N_r(\varnothing) := \{\varnothing\}$ respectively

$N_r(\mathcal{D}) := \{\rho \subset \underline{P}X : \exists B \in \mathcal{B}^X \{D\} \cup \rho \in N(B)\}$, otherwise.

Then the tripel $(X, \mathcal{B}^X, N_r)$ is a round paranear space and $1_X : (X, \mathcal{B}^X, N) \longrightarrow (X, \mathcal{B}^X, N_r)$ to be the bireflection in demand! $\square$

Corollary 2.11 *If $(X, \mathcal{B}^X, N)$ is paraclan space then $(X, \mathcal{B}^X_N, N_r)$ as well.*

Definition 2.12 *A round paranear space $(X, \mathcal{B}^X, N)$ is called* <u>*LOproximal*</u> *iff it satisfies* (LOp), *e.g.*

(LOp) $B \in \mathcal{B}^X \backslash \{\varnothing\}, \rho \in p_N(B)$ *and* $\{B\} \cup \rho \subset \cap\{p_N(F) : F \in \rho \cap \mathcal{B}^X\}$ *imply* $\rho \in N(B)$, *where* $B p_N A$ *iff* $\{A\} \in N(B)$.

Theorem 2.13 *The category LOSP is isomorphic to the full subcategory LO-PN of R-PN, whose objects are the LOproximal paranear spaces.*

Proof: According to example (iii). Conversely, we consider the near-relation "p_N" as defined in 2.12. Moreover we note that for a paranear space $(X, \mathcal{B}^X, N)$ the near-operator N is <u>dense</u>, e.g. by satisfying $(d) B \subset X$ and $cl_N(B) \in \mathcal{B}^X$ imply $N(cl_N(B)) = N(B)$, and moreover it is <u>connected</u>, e.g. by satisfying

(cnc) $B_1 \cup B_2 \in \mathcal{B}^X$ implies $N(B_1 \cup B_2) = N(B_1) \cup N(B_2)$.

$\square$

Remark 2.14 Now, we mention that in the "saturated case" LOproximal paranear spaces and LODATO proximity spaces [10] essentially are the same!

Proposition 2.15 *Let (Y, t) be a symmetric topological space given by closure operator t and $\mathcal{B}^X$ $\underline{B}$-set with $X \subset Y$. We set $\mathcal{B}^X_t := \{D \subset X : \exists B \in \mathcal{B}^X t(B) \supset D\}$ and $D\delta_t A$ iff $t(D) \cap t(A) \neq \emptyset$. Then $(X, \mathcal{B}^X_t, \delta_t)$ is LODATO space.*

Remark 2.16 Now, surely it seems to be of interest to characterize those LODATO spaces whichever are induced by a topologival space Y as above so that bounded and arbitrary sets are near iff their closures meet in Y. But this problem already has been solved under more general conditions in [9].

Remark 2.17 Returning to nearness spaces we already know that in general subspaces of topological nearness spaces need not to be topological again, hence Bentley [2] has called them subtopological. But now here, we will give an extended description of this definition in term of supernear spaces as follows:

Definition 2.18 *A supernear space $(X, \mathcal{B}^X, N)$ is called supergrill space if N satisfies* (gri), *e.g.*

(gri) $B \in \mathcal{B}^X$ *and* $\rho \in N(B)$ *imply the existence of* $\gamma \in GRL(X) \cap N(B)$ *with* $\rho \subset \gamma$.

Remark 2.19 We point out that this definition generalize that of 2.6. Moreover, if $(X, \mathcal{B}^X, N) \in$ PN satisfies (gri), we analogously call it a paragrill space. By G-SN respectively G-PN we denote the corresponding full subcategory of SN respectively PN.

Proposition 2.20 *For a nearness space (X, ξ) the following statements are equivalent:*

(i) (X, ξ) *is subtopological;*

(ii) $(X, \underline{P}X, N_\xi)$ *is paragrill space.*

Remark 2.21 According to example (iv) we also note that $(X, \mathcal{B}^X, N^\delta)$ is a supergrill space.

Definition 2.22 *A supergrill space $(X, \mathcal{B}^X, N)$ then is called conic iff N satisfies* (c), *e.g.*

(c) $B \in \mathcal{B}^X$ *implies* $\{F \subset X : \exists \rho \in N(B) F \in \rho\} =: \cup N(B) \in N(B)$.

Theorem 2.23 *The category pLESP is isomorphic to the full subcategory CG-SN of G-SN, whose objects are the conic supergrill spaces.*

Proof: According to example (iv) in connexion with the definition of "p_N" in 2.12. $\square$

Definition 2.24 *A preLEADER space* $(X, \mathcal{B}^X, \delta)$ *then is called* <u>*LEADERspace*</u> *iff* δ *in addition satisfies* (bp$_6$) *in* (iii).

Remark 2.25 We point out that in the "saturated" case LEADER spaces and LEADER proximity spaces [6] essentially are the same. Moreover, each supertopological space [4] $(X, \mathcal{B}^X, \Theta)$, where $\Theta : \mathcal{B}^X \longrightarrow \mathrm{FIL}(X) := \{\mathcal{F} \subset \underline{P}X : \mathcal{F} \text{ is filter}\}$ satisfies the following conditions, e.g.

(stop$_1$) $\Theta(\varnothing) = \underline{P}X$;

(stop$_2$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply $U \supset B$;

(stop$_3$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply there exists a set $V \in \Theta(B)$ such that always $U \in \Theta(B')\forall B' \in \mathcal{B}^X B' \subset V$ is leading us to the preLEADER space $(X, \mathcal{B}^X, \delta_\Theta)$ by setting $B\delta_\Theta A$ iff $A \in \sec\Theta(B)$. If in addition $(X, \mathcal{B}^X, \Theta) \in$ ASTOP [11], then $(X, \mathcal{B}^X, \delta_\Theta)$ is LEADER space, too. The above assignment now is "bi-functoriell", hence STOP can be considered as a subcategory of CG-SN. In the second case we note that the corresponding supergrill operator N^{δ_Θ} is in addition linked, hence it satisfies (l), e.g.

(l) $B_1 \cup B_2 \in \mathcal{B}^X$ and $\rho \in N^{\delta_\Theta}(B_1 \cup B_2)$ imply $\{F\} \in N^{\delta_\Theta}(B_1) \cup N^{\delta_\Theta}(B_2)$ for each $F \in \rho$.

Definition 2.26 *A conic supergrill space* $(X, \mathcal{B}^X, N)$ *then is called* <u>*LEproximal*</u> *iff* N *is linked. By LE-SN we denote the full subcategory of SN.*

Theorem 2.27 *The category LE-SN is isomorphic to the full subcategory LESP of pLESP, whose objects are the LEADER spaces.*

Remark 2.28 According to 2.25 we also note that ASTOP now can be considered as subcategory of LE-SN.

Proposition 2.29 *Let* (Y, t) *be a topological space given by closure operator* t *and* $\mathcal{B}^X$ $\underline{B}$*-set with* $X \subset Y$. *We set* $B\delta^t A$ *iff* $B \cap t(A) \neq \varnothing$ *for each* $B \in \mathcal{B}^X$ *and* $A \subset X$. *Then* $(X, \mathcal{B}^X, \delta^t)$ *is LEADER space*

Proof: straightforward. □

Remark 2.30 According to 2.16 now it seems to be of interest to characterize those LEADER spaces, whichever are included by a topological space Y as above so that a bounded set B is near to an arbitrary one iff B intersects its closure in Y. But we will solve this problem under more <u>general</u> conditions in a <u>forthcoming</u> paper!

Remark 2.31 Returning to conic supergrill spaces we point out that for such a space $(X, \mathcal{B}^X, N)$ and for each $B \in \mathcal{B}^X \backslash \{\varnothing\} \cup N(B)$ is a B-clan in N. hence, we claim that conic supergrill spaces even are superclan spaces!

Theorem 2.32 *The category CG-SN is bicoreflective in G-SN.*

Proof: For a supergrill space $(X, \mathcal{B}^X, N)$ we set for each $B \in \mathcal{B}^X$:

$$N_C(B) := \{\rho \subset \underline{P}X : \{cl_N(F) : F \in \rho\} \subset \cup N(B)\}.$$

Then $(X, \mathcal{B}^X, N_c)$ is a conic supergrill space and $1_X : (X, \mathcal{B}^X, N_c) \longrightarrow (X, \mathcal{B}^X, N)$ to be the bicoreflection in demand. First, we only show that N_C satisfies (sn$_7$): Let be $\{cl_{N_c}(A) : A \in \mathcal{A}\} \in N_c(B)$ for $B \in \mathcal{B}^X$, we have to verify $cl_N(A) \in \cup N(B)$ for each $A \in \mathcal{A}$.

$A \in \mathcal{A}$ implies $cl_N(cl_{N_c}(A)) \in \cup N(B)$ by hypothesis. We claim now that the statement $cl_{N_c}(A) \subset cl_N(A)$ is valid. $x \in cl_{N_c}(A)$ implies $\{A\} \in N_c(\{x\})$, hence $cl_N(A) \in \cup N_c(\{x\})$. We can find $\rho \in N(\{x\})$ such that $cl_N(A) \in \rho$. Consequently $\{cl_N(A)\} \in N(\{x\})$ follows, which shows $\{A\} \in N(\{x\})$, hence $x \in cl_N(A)$ results.

Altogether we get $cl_N(A) \supset cl_N(cl_N(A)) \supset cl_N(cl_{N_c}(A))$ implying $cl_N(A) \in \cup N(B)$, since $\cup N(B) \in \text{GRL}(X)$. Secondly, we prove $\cup N_c(B) \in \text{GRL}(X)$ for each $B \in \mathcal{B}^X$. Let be given $B \in \mathcal{B}^X$, evidently $\varnothing \notin \cup N_c(B)$. Now, if $F_1 \in \cup N_c(B)$ and $F_1 \subset F_2 \subset X$, then there exists $\rho_1 \in N_c(B) F_1 \in \rho_1$. Consequently $\{cl_N(A) : A \in \rho_1\} \subset \cup N(B)$ follows by definition. We put $\rho_2 := \{F_2\}$, hence $\rho_2 \in N_C(B)$, because $\{cl_N(F) : F \in \rho_2\} = \{cl_N(F_2)\}$ and $cl_N(F_2) \supset cl_N(F_1) \in \cup N(B)$ implies $cl_N(F_2) \in \cup N(B)$. But $F_2 \in \{F_2\} = \rho_2$ immediately leading us to $F_2 \in \cup N_c(B)$. At last let be $F_1 \cup F_2 \in \cup N_c(B)$, hence there exists $\rho \in N_c(B) F_1 \cup F_2 \in \rho$ By definition $\{cl_N(F) : F \in \rho\} \subset \cup N(B)$ is valid showing that $cl_N(F_1) \cup cl_N(F_2) \supset cl_N(F_1 \cup F_2) \in \cup N(B)$. Consequently, $cl_N(F_1) \in \cup N(B)$ or $cl_N(F_2) \in \cup N(B)$ results, since $\cup N(B) \in \text{GRL}(X)$. If $cl_N(F_1) \in \cup N(B)$ then we put $\rho_1 := \{F_1\}$,hence $F_1 \in \cup N_c(B)$ results.

Analogously, this also holds in the second case. Evidently, $1_X : (X, \mathcal{B}^X, N_c) \longrightarrow (X, \mathcal{B}^X, N)$ is sn-map. Now, let be given $(Y, \mathcal{B}^Y, M) \in \text{CG-SN}$ and sn-map $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N)$, we have to prove $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N_c)$ is sn-map. For $B \in \mathcal{B}^Y$ and $\rho \in M(B)$ we must show $f\rho \in N_c(f[B])$, which means $\{cl_N(A) : A \in f\rho\} \subset \cup N(f[B])$. $A \in f\rho$ implies $A = f[F]$ for some $F \in \rho$. By supposition $f\rho \in N(f[B])$ follows, and $cl_N(A) = cl_N(f[F]) \supset f[cl_M(F)] \supset f[F] \in f\rho \in \cup N(f[B])$ is valid. Consequently, $cl_N(A) \in \cup N(f[B])$ results! $\square$

Remark 2.33 As mentioned in 2.7 we already know, that pointed supernear spaces are superclan spaces as well. Moreover, in the next, we will show that PT-SN can be "nicely embedded" in SN as follows:

Theorem 2.34 *PT-SN is bicoreflective subcategory of SN.*

Proof: For a supernear space $(X, \mathcal{B}^X, N)$ we set:

$N_P(\varnothing) := \{\varnothing\}$ and

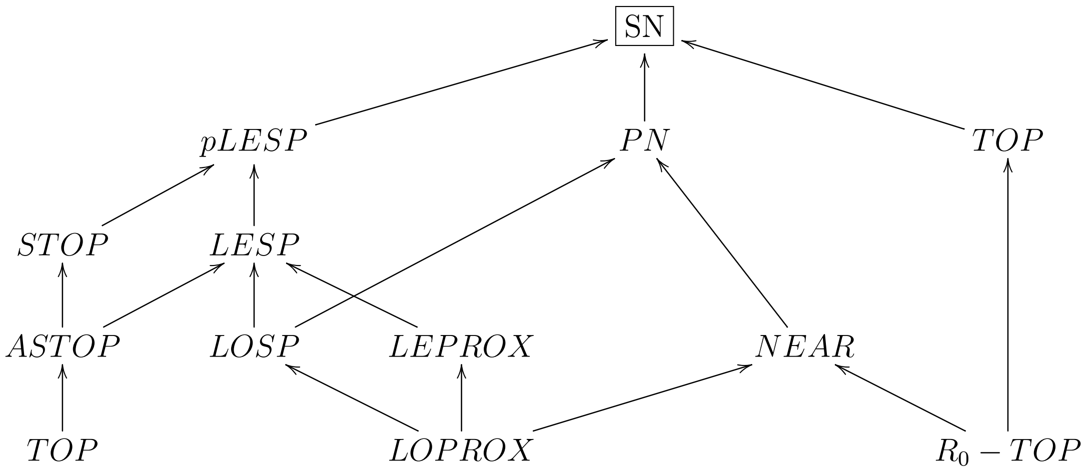
$N_P(B) := [\mathcal{A} \subset \underline{P}X : \exists x \in B \exists \gamma \in N(\{x\}) \cap \mathrm{GRL}(X) \{cl_N(A) : A \in \mathcal{A}\} \subset \gamma\}$, otherwise.

Then $(X, \mathcal{B}^X, N_P)$ is pointed supernear space and $1_X : (X, \mathcal{B}^X, N_P) \longrightarrow (X, \mathcal{B}^X, N)$ to be the bicoreflection in demand. First, we will show that N_P satisfies (sn_7). Let be $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\{cl_{N_P}(A) : A \in \mathcal{A}\} \in N_P(B)$, then we can choose $x \in B$ and $\gamma \in N(\{x\}) \cap \mathrm{GRL}(X)$ such that $\{cl_N(F) : F \in \{cl_{N_P}(A) \in \mathcal{A}\}\} \subset \gamma$. In showing $\mathcal{A} \in N_P(B)$ we have to verify $cl_N(A) \in \gamma$ for each $A \in \mathcal{A}$: $A \in \mathcal{A}$ implies $cl_N(cl_{N_P}(A)) \in \gamma$ by hypothesis. Now, we claim that $cl_{N_P}(A) \subset cl_N(A)$, because $x \in cl_{N_P}(A)$ implies $\{A\} \in N_P(\{x\})$, hence there exists $\gamma' \in N(\{x\}) \cap \mathrm{GRL}(X) \{cl_N(A)\} \subset \gamma'$. Then $\{cl_N(A)\} \in N(\{x\})$ is valid, and consequently $\{A\} \in N(\{x\})$ follows which shows $x \in cl_N(A)$. Altogether we have $cl_N(A) \supset cl_N(cl_N(A)) \supset cl_N(cl_{N_P}(A)) \in \gamma$, hence $cl_N(A) \in \gamma$ results! Evidently, N_P fulfills the axioms (sn_1) to (sn_5).

to (sn_6): $\mathcal{A}_1 \vee \mathcal{A}_2 \in N_P(B)$ for $B \in \mathcal{B}^X \backslash \{\varnothing\}$ implies the existence of $x \in B$ and $\gamma \in N(\{x\}) \cap \mathrm{GRL}(X)$ so that $\{cl_N(A) : A \in \mathcal{A}_1 \vee \mathcal{A}_2\} \subset \gamma$. If supposing $\mathcal{A}_1, \mathcal{A}_2 \notin N_P(B)$ we get $\{cl_N(A_1) : A_1 \in \mathcal{A}_1\} \not\subset \gamma$ and $\{cl_N(A_2) : A_2 \in \mathcal{A}_2\} \not\subset \gamma$, hence there exist $A_1 \in \mathcal{A}_1 cl_N(A_1) \notin \gamma$ and $A_2 \in \mathcal{A}_2 cl_N(A_2) \notin \gamma$ implying $A_1 \cup A_2 \in \mathcal{A}$ and $cl_N(A_1) \cup cl_N(A_2) \notin \gamma$. Consequently $cl_N(A_1 \cup A_2) \notin \gamma$ follows, since $\gamma \in \mathrm{GRL}(X)$. On the other hand $cl_N(A_1 \cup A_2) \in \gamma$ by hypothesis is leading us to a contradiction! By definition N_P is pointed and $1_X : (X, \mathcal{B}^X, N_P) \longrightarrow (X, \mathcal{B}^X, N))$ sn-map. Now, let be given a pointed supernear space $(Y, \mathcal{B}^Y, M)$ and sn-map $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N)$, we will show that $f : (Y, \mathcal{B}^Y, M) \longrightarrow (X, \mathcal{B}^X, N_P)$ is sn-map as well. Without restriction let be $B \in \mathcal{B}^Y \setminus \{\varnothing\}$ and $\mathcal{A} \in M(B)$, hence by hypothesis there exists $y \in B$ such that $\mathcal{A} \in M(\{y\})$. Since f is sn-map $f\mathcal{A} \in N(\{f(y)\})$ follows with $f(y) \in f[B]$. But $f(y)_N \in N(\{f(y)\}) \cap \mathrm{GRL}(X)$, according to 2.7. Now, for $F \in f\mathcal{A}$ we will show that $cl_N(F) \in f(y)_N$. $F \in f\mathcal{A}$ implies $F = f[A]$ for some $A \in \mathcal{A}$. We claim $\{f[A]\} \in N(\{f(y)\})$. By hypothesis $f\mathcal{A} \in N(\{f(y)\})$, hence $\{f[A]\} << f\mathcal{A}$, which shows $\{f[A]\} \in N(\{f(y)\})$, and at last $f\mathcal{A} \in N_P(f[B])$ results.

$\square$

Remark 2.35 The following diagram illustrates the relationship between important former mentioned categories:



3 Topological extensions and related superclan spaces

Taking into account example (v), we will now consider the problem for finding a one-to-one corresponding between certain topological extensions and their related supernear spaces. It turns out that there exists an interesting one between pointed supernear spaces and some strict topological extensions.

Lemma 3.1 *For a topological extension* $(e, \mathcal{B}^X, Y), (X, \mathcal{B}^X, N_e)$ *is a pointed supernear space such that* $cl_{N_e} = cl_X$.

Proof: First, we will show the equality of the closure operators. So, let $A \in \underline{P}X$ and $x \in cl_X(A)$. Then by (tx_1) $e(x) \in cl_Y(e[A])$ hence $\{A\} \in N_e(\{x\})$, and $x \in cl_{N_e}(A)$ follows. Conversely, let $x \in cl_{N_e}(A)$, then $\{A\} \in N_e(\{x\})$. Consequently there exists $y \in e[\{x\}] = \{e(x)\}$ with $y \in cl_Y(e[A])$. Hence $y = e(x)$, and as a consequence of (tx_1) we get $x \in e^{-1}[cl_Y(e[A])] \subset cl_X(A)$, which was to be proven. Secondly, it is easy to check the axioms (sn_1) to (sn_6).

to (sn_7): Let be $\{cl_{N_e}(F) : F \in \rho\} \in N_e(B)$ for $\rho \subset \underline{P}X, B \in \mathcal{B}^X$ and without restriction $B \neq \varnothing$, then there exists $y \in e[B]$ with $y \in \cap\{cl_Y(e[A]) : A \in \{cl_{N_e}(F) : F \in \rho\}\}$. For $F \in \rho$ we get $y \in cl_Y(e[cl_{N_e}[F]]) = cl_Y(e[cl_X(F)])$ according to the first approved equality. Consequently, $y \in cl_Y(cl_Y(e[F])) \subset cl_Y(e[F])$ results, which shows $\rho \in N_e(B)$, according to (tx_1). By definition N_e is automatically pointed.

$\square$

Theorem 3.2 *Let F : TEXT $\longrightarrow$ PT-SN be defined by:*

(a) *For a TEXT-object $(e, \mathcal{B}^X, Y)$ we put $F(e, \mathcal{B}^X, Y) := (X, \mathcal{B}^X, N_e)$;*

(b) *for a TEXT-morphism $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ we put $F(f, g) := f$. Then F : TEXT $\longrightarrow$ PT-SN is a functor.*

Proof: With respect to 3.1 we already know that $F(e, \mathcal{B}^X, Y)$ is an object of PT-SN. Let $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ be a TEXT-morphism such that $F(e, \mathcal{B}^X, Y) = (X, \mathcal{B}^X, N_e)$ and $F(e', \mathcal{B}^{X'}, Y') = (X', \mathcal{B}^{X'}, N_{e'})$. It has to be shown that $f : (X, \mathcal{B}^X, N_e) \longrightarrow (X', \mathcal{B}^{X'}, N_{e'})$ preserves B-near collections for each $B \in \mathcal{B}^X$. Without loss of generality, let be $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\rho \in N_e(B)$, hence there exists $y \in e[B]$ such that $y \in \cap\{cl_Y(e[F]) : F \in \rho\}$. Our goal is to verify that $f\rho \in N_{e'}(f[B])$. By hypothesis we have $g(y) \in g[e[B]] = e'[f[B]]$. On the other hand let $D \in f\rho$. We have to verify that $g(y) \in cl_{Y'}(e'[D])$. As $D = f[F]$ for some $F \in \rho, y \in cl_Y(e[F])$. Consequently, $g(y) \in g(cl_Y(e[F])) \subset cl_{Y'}(g(e[F]]) = cl_{Y'}(e'(f[F])) = cl_{Y'}(e'[D])$, which results in $f\rho \in N_{e'}(f[B])$ according to the definitions in 1.4. Then the remainder is clear. □

4 Pointed supernear spaces and strict topological extensions

In the previous paragraph we have found a functor from TEXT to PT-SN. Now, we are going to introduce a related one from PT-SN to STREXT.

Lemma 4.1 *Let $(X, \mathcal{B}^X, N)$ be a supernear space. We put $X^C := \{\mathcal{C} \subset \underline{P}X : \mathcal{C}$ is B-clan in N for some $B \in \mathcal{B}^X\}$, and for each $A^C \subset X^C$ we set: $cl_{X^C}(A^C) := \{\mathcal{C} \in X^C : \triangle A^C \subset \mathcal{C}\}$, where $\triangle A^C := \{F \subset X : \forall \mathcal{C} \in A^C F \in \mathcal{C}\}$, so that by convention $\triangle A^C = \underline{P}X$ if $A^C = \varnothing$. Then cl_{X^C} is a topological closure operator on X^C.*

Proof: First, we note that for any $\mathcal{C} \in X^C, \mathcal{C} \notin cl_{X^C}(\varnothing)$, because $\varnothing \notin \mathcal{C}$ according to 2.6 and (sn$_2$) respectively. Now, let $A_1^C \subset A_2^C$. Then $\triangle A_2^C \subset \triangle A_1^C$ which yields $cl_{X^C}(A_1^C) \subset cl_{X^C}(A_2^C)$. Further, let A_1^C and A_2^C be subsets of X^C. Let $\mathcal{C}$ be an elements of X^C and suppose $\mathcal{C} \notin cl_{X^C}(A_1^C) \cup cl_{X^C}(A_2^C)$. Then we have $\triangle A_1^C \not\subset \mathcal{C}$ and $\triangle A_2^C \not\subset \mathcal{C}$. Choose $F_1 \in \triangle A_1^C$ with $F_1 \notin \mathcal{C}$ and $F_2 \in \triangle A_2^C$ with $F_2 \notin \mathcal{C}$, hence $F_1 \cup F_2 \notin \mathcal{C}$, according to 2.6. On the other hand, we have $F_1 \cup F_2 \in \triangle(A_1^C \cup A_2^C)$, and consequently $\mathcal{C} \notin cl_{X^C}(A_1^C \cup A_2^C)$ results. Now, let $\mathcal{C}$ be the element of $cl_{X^C}(cl_{X^C}(A^C))$ and suppose $\mathcal{C} \notin cl_{X^C}(A^C)$. Choose $F \in \triangle A^C$ $F \notin \mathcal{C}$. By hypothesis we have $\triangle cl_{X^C}(A^C) \subset \mathcal{C}$, hence $F \notin \triangle cl_{X^C}(A^C)$. Choose $\mathcal{D} \in cl_{X^C}(A^C)$ $F \notin \mathcal{D}$. Then $\triangle A^C \subset \mathcal{D}$, hence $F \in \mathcal{D}$, which leads us to a contradiction! □

Theorem 4.2 *For supernear spaces $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ let $f : X \longrightarrow Y$ be a sn-map. Define a function $f^C : X^C \longrightarrow Y^C$ by setting for each $\mathcal{C} \in X^C$: $f^C(\mathcal{C}) := \{D \subset Y : f^{-1}[cl_M(D)] \in \mathcal{C}\}$. Then the following statements are valid:*

(i) $f^C : (X^C, cl_{X^C}) \longrightarrow (Y^C, cl_{Y^C})$ *is a continuous map;*

(ii) *the equality* $f^C \circ e_X = e_Y \circ f$ *holds, where* $e_X : X \longrightarrow X^C$ *denotes that function which assigns the* $\{x\}$*-clan* x_N *to each* $x \in X$.

Proof: First, let $\mathcal{C} \in X^C$, we must show that $f^C(\mathcal{C}) \in Y^C$. $f^C(\mathcal{C}) \in \mathrm{GRL}(Y)$, since $\mathcal{C} \in \mathrm{GRL}(X)$ and f^{-1} respectively cl_M are compatible with finite union. By hypothesis $\mathcal{C} \in N(B)$ for some $B \in \mathcal{B}^X$, hence $f\mathcal{C} \in N(f[B])$, because f is sn-map. Now, we will show that $\{cl_M(D) : D \in f^C(\mathcal{C})\} << f\mathcal{C}$. $cl_M(D)$ for some $D \in f^C(\mathcal{C})$ implies $f^{-1}[cl_M(D)] \in \mathcal{C}$, hence $cl_M(D) \supset f[f^{-1}[cl_M(D)]] \in f\mathcal{C}$. According to (sn$_7$), $f^C(\mathcal{C}) \in M(f[B])$ follows. $f[B] \in f^C(\mathcal{C})$, since $f^{-1}[cl_M(f[B])] \supset f^{-1}[f[cl_N(B)]] \supset B \in \mathcal{C}$ by hypothesis.

At last, let be $D \in f^C(\mathcal{C})$ and $D \subset cl_M(F)$, we have to verify $F \in f^C(\mathcal{C})$. By supposition $f^{-1}[cl_M(D)] \in \mathcal{C}$. $f^{-1}[cl_M(D)] \subset cl_N(f^{-1}[cl_M(F)])$, because $x \in f^{-1}[cl_M(D)]$ implies $f(x) \in cl_M(D)$; but $cl_M(D) \subset cl_M(cl_M(F)) \subset cl_M(F)$, hence $f(x) \in cl_M(F)$. Consequently, $x \in f^{-1}[cl_M(F)] \subset cl_N(f^{-1}[cl_M(F)])$ results. Since $\mathcal{C}$ satisfies (cla$_2$), $f^{-1}[cl_M(F)] \in \mathcal{C}$ is valid, which shows $F \in f^C(\mathcal{C})$.

to (i): Let $A^C \subset X^C, \mathcal{C} \in cl_{X^C}(A^C)$ and suppose $f^C(\mathcal{C}) \notin cl_{Y^C}(f^C[A^C])$. Then $\triangle f^C[A^C] \not\subset f^C(\mathcal{C})$, hence $D \notin f^C(\mathcal{C})$ for some $D \in \triangle f^C[A^C]$, which means $f^{-1}[cl_M(D)] \notin \mathcal{C}$.

But $\triangle A^C \subset \mathcal{C}$ implies $f^{-1}[cl_M(D)] \notin \mathcal{D}$ for some $\mathcal{D} \in A^C$. Therefore $D \notin f^C(\mathcal{D})$, which leads us to a contradiction, because $D \in \triangle f^C[A^C]$.

to (ii): Let x be an element of X. We will prove that the equality $f^C(e_X(x)) = e_Y(f(x))$ is valid. To this end let $T \in e_Y(f(x))$, hence $f(x) \in cl_M(T)$, and consequently $x \in f^{-1}[cl_M(T)]$ follows, which shows $f^{-1}[cl_M(T)] \in x_N = e_X(x)$. Thus, $T \in f^C(e_X(x))$ which proves the inclusion $e_Y(f(x)) \subset f^C(e_X(x))$.

Consequently, since $e_Y(f(x))$ is maximal in $M(\{f(x)\}) \setminus \{\varnothing\}$ (see 2.7 and note also that $\{cl_M(D) : D \in f^C(e_X(x))\} << fx_N \in M(\{f(x)\})$, since by hypothesis f is sn-map) we obtain the desired equality.

□

Theorem 4.3 *Let* $G : SN \longrightarrow STREXT$ *be defined as follows:*

(a) *For any supernear space* $(X, \mathcal{B}^X, N)$ *we put* $G(X, \mathcal{B}^X, N) := (e_X, \mathcal{B}^X, X^C)$ *with* $X := (X, cl_N)$ *and* $X^C := (X^C, cl_{X^C})$;

(b) *for any sn-map* $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$ *we put:* $G(f) := (f, f^C)$.

Then $G : SN \longrightarrow STREXT$ *is a functor.*

Proof: With respect to (sn$_7$) cl_N is topological, and by 4.1 this also holds for cl_{X^C}. Therefore we get topological spaces with $\underline{B}$-set $\mathcal{B}^X$, and $e_X : X \longrightarrow X^C$ is a map according to 4.2. Now, we have to verify that $(e_X, \mathcal{B}^X, X^C)$ satisfies the axioms (tx$_1$) to (tx$_3$).

to (tx$_1$): Let A be a subset of X and suppose $x \in cl_N(A)$. Since $\triangle e_X[A] = \{T \subset X : A \subset cl_N(T)\}$ we get $e_X(x) \in cl_{X^C}(e_X[A])$, hence $x \in e_X^{-1}[cl_{X^C}[e_X[A]]]$ follows. Conversely, let x be an element of $e_X^{-1}[cl_{X^C}(e_X[A])]$, then by definition we have $e_X(x) \in cl_{X^C}(e_X[A])$, and consequently the statement $\triangle e_X[A] \subset e_X(x)$ results. In applying the above mentioned equation we get $A \in e_X(x)$, which means $x \in cl_N(A)$.

to (tx$_2$): Let $\mathcal{C} \in X^C$ and suppose $\mathcal{C} \notin cl_{X^C}(e_X[X])$. By definition we get $\triangle e_X[X] \not\subset \mathcal{C}$, so that there exists a set $F \in \triangle e_X[X] F \notin \mathcal{C}$.

Consequently, the inclusion $X \subset cl_N(F)$ holds. By hypothesis $\mathcal{C}$ is B-clan for some $B \in \mathcal{B}^X$, hence $B \in \mathcal{C}$ according to (cla$_1$), and $B \subset X \subset cl_N(F)$ follows, which imply $F \in \mathcal{C}$ according to (cla$_2$). But this is a contradiction, hence $\mathcal{C} \in cl_{X^C}(e_X[X])$ holds.

to (tx$_3$): Let $\mathcal{C} \in X^C$ and let A^C be closed in X^C with $\mathcal{C} \notin A^C$. Then $\mathcal{C} \notin cl_{X^C}(A^C)$ and so $\triangle A^C \not\subset \mathcal{C}$. There exists $F \in \triangle A^C$ such that $F \notin \mathcal{C}$. Now, for each $\mathcal{D} \in A^C$ we have $F \in \mathcal{D}$, which implies $\triangle e_X[F] \subset \mathcal{D}$, and so at last $\mathcal{D} \in cl_{X^C}(e_X[F])$ results. On the other hand since $F \notin \mathcal{C}$ we have $\triangle e_X[F] \not\subset \mathcal{C}$, and so $\mathcal{C} \notin cl_{X^C}(e_X[F])$.

□

Now it is interesting to see, how the composite functor $F \circ G$ works on the category PT-SN.

Theorem 4.4 *Let G : PT-SN $\longrightarrow$ TEXT and F : TEXT $\longrightarrow$ PT-SN be the functors given in theorem 3.2 and 4.3. For each object $(X, \mathcal{B}^X, N)$ of PT-SN let $t_(X, \mathcal{B}^X, N)$ denote the identity map $t_(X, \mathcal{B}^X, N) := id_X : F(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$. Then $t : \mathcal{F} \circ G \longrightarrow 1_{PT\text{-}SN}$ is natural equivalence from $F \circ G$ to the identity functor $1_{PT\text{-}SN}$, i.e. $id_X : F(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$ is in <u>both</u> directions a sn-map for each object $(X, \mathcal{B}^X, N)$, and the following diagram commutes for each sn-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$:*

$$\begin{array}{ccc} F(G(X, \mathcal{B}^X, N)) & \xrightarrow{id_X} & (X, \mathcal{B}^X, N) \\ {\scriptstyle F(G(f))}\downarrow & & \downarrow{\scriptstyle f} \\ F(G(Y, \mathcal{B}^Y, M))) & \xrightarrow[id_Y]{} & (Y, \mathcal{B}^Y, M). \end{array}$$

Proof: The commutativity of the diagram is obvious, because $F(G(f)) = f$.

It remains to prove that in each case $F(G(X, \mathcal{B}^X, N)) \xrightarrow{id_X} (X, \mathcal{B}^X, N) \xrightarrow{id_X} F(G(X, \mathcal{B}^X, N))$ is sn-map for any object $(X, \mathcal{B}^X, N) \in$ PT-SN. To fix the notation, let N_1 be such that

$F(G(X,\mathcal{B}^X,N)) = F(e_X,\mathcal{B}^X,X^C) = (X,\mathcal{B}^X,N_1)$. First we show that for each $B \in \mathcal{B}^X \backslash \{\varnothing\}, \rho \in N_1(B)$ implies $\rho \in N(B)$. To this end assume that $\rho \in N_1(B)$, then there exists $\mathcal{C} \in e_X[B]$ such that $\mathcal{C} \in \cap\{cl_{X^C}(e_X[F]) : F \in \rho\}$. We have $\mathcal{C} = e_X(x)$ for some $x \in B$, hence $\mathcal{C} \in N(B)$ according to 2.7 and 4.2, respectively. $\rho \subset \mathcal{C}$, because $F \in \rho$ implies $\mathcal{C} \in cl_{X^C}(e_X[F])$, and in consequence $\triangle e_X[F] \subset \mathcal{C}$ results. Since $F \in \triangle e_X[F]$ we get $F \in \mathcal{C}$, which shows $\rho \in N(B)$, according to (sn_1). Conversely, let be $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\rho \in N(B)$, we have to show that $\rho \in N_1(B)$.

In assuming the above we get $\rho \in N(\{x\})$ for some $x \in B$, since $(X,\mathcal{B}^X,N)$ is pointed. But $x_N = e_X(x) \in e_X[B]$. We have to show that for each $F \in \rho$ the statement $x_N \in cl_{X^C}(e_X[F])$ is valid. So let be $F \in \rho$ and $T \in \triangle e_X[F]$. By hypothesis $F \subset cl_N(T)$ results with $F \in x_N$, hence $x \in cl_N(F)$, and consequently we get $T \in x_N$, which concludes the proof.

Now, in making this part of searching more transparent, we give a short characterization of the subject as follows: □

Comment 1 Let be given an arbitrary supernear space $(X,\mathcal{B}^X,N)$. Then his property of being pointed can be described in such a way that there exists a topological space Y in which it is densely "embedded", so that non-empty B-near collections are characterized by the fact, that its closure meet in Y by the image of an element of B. Hence, we can resume, that pointed supernear spaces can be strictly extended in such a manner!

Corollary 4.5 *If $(X,\mathcal{B}^X,N)$ is separated, which means N satisfies* (sep), *e.g.*

(sep) *$x,z \in X$ and $\{\{z\}\} \in N(\{x\})$ imply $x = z$, then $e_X : X \longrightarrow X^C$ is injective! Conversely, for a T_1-extension $(e,\mathcal{B}^X,Y)$, where e is a topological embedding, and Y is a T_1-space, then $(X,\mathcal{B}^X,N_e)$ is separated!*

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