Rostock. Math. Kolloq. **66**, 45–67 (2011)

Subject Classification (AMS)
11A63, 39B22, 26A27

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The partial derivatives of de Rham's singular function and power sums of binary digital sums

ABSTRACT. This note is a supplement to the paper [9] on the partial derivatives T_n of de Rham's function $R_a(x)$ with respect to the parameter a at $a = 1/2$. In particular, $T_0(x) = x$ and $T_1(x) = 2T(x)$ where T is Takagi's continuous nowhere differentiable function. We present a new representation of T_n. From this we derive a limit relation at dyadic rational points. Moreover, we show that real linear combinations of T_n with $n \geq 1$ are nowhere differentiable. Thus we are able to prove that the functions which appear e.g. in the well known formula of Coquet for power sums of binary digital sums are nowhere differentiable. Finally, we derive a corresponding formula for power sums of the number of zeros.

KEY WORDS. De Rham's singular function, Takagi's continuous nowhere differentiable function, functional equations, binary digital sums, number of zeros, Stirling numbers.

1 Introduction

For a fixed parameter $a \in (0, 1)$ the system of functional equations

$$\left.\begin{aligned} f\left(\tfrac{x}{2}\right) &= af(x), \\ f\left(\tfrac{x+1}{2}\right) &= a + (1-a)f(x) \end{aligned}\right\} \qquad (x \in [0,1]) \tag{1.1}$$

has a unique bounded solution $f = R_a(x)$ with $R_a(0) = 0$ and $R_a(1) = 1$, cf. [6]. It is $R_{1/2}(x) = x$, but for $a \neq \frac{1}{2}$ de Rham's function $R_a(x)$ is a strictly singular function which is also called Lebesgue singular function, cf. e.g. [1]. In [2] it was shown that for $\ell \in \mathbb{N}$ and $n = 0, 1, \ldots, 2^\ell$ it holds

$$R_a\left(\frac{n}{2^\ell}\right) = a^\ell \sum_{j=0}^{n-1} q^{s(j)} \tag{1.2}$$

where $q = (1-a)/a$ and where $s(j)$ denotes the number of ones in the binary representation of j. As consequence of (1.2) it was shown in [9] that for $q > 0$ it holds

$$\sum_{j=0}^{N-1} q^{s(j)} = N^\alpha G_q(\log_2 N) \tag{1.3}$$

where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is a continuous, 1-periodic function which is connected with de Rham's function by

$$G_q(u) = a^u R_a(2^u) \qquad (u \le 0) \tag{1.4}$$

where $a = \frac{1}{1+q}$. Formula (1.3) was the start point for the proof of explicit formulas for digital sums. For the binomial sum

$$B_k(N) = \sum_{j=0}^{N-1} \binom{s(j)}{k} \tag{1.5}$$

with integer $k \ge 1$ it holds the formula ([9])

$$\frac{1}{N} B_k(N) = \frac{1}{k!} \left(\frac{\log_2 N}{2} \right)^k + \frac{1}{k!} \sum_{\ell=0}^{k-1} (\log_2 N)^\ell F_{k,\ell}(\log_2 N) \tag{1.6}$$

and for the power sum

$$S_k(N) = \sum_{j=0}^{N-1} s(j)^k \tag{1.7}$$

with $k \ge 1$ it holds the formula of Coquet [3], (cf. also [5], [11] and [9])

$$\frac{1}{N} S_k(N) = \left(\frac{\log_2 N}{2} \right)^k + \sum_{\ell=0}^{k-1} (\log_2 N)^\ell G_{k,\ell}(\log_2 N) \tag{1.8}$$

where $F_{k,\ell}(u)$ and $G_{k,\ell}(u)$ are continuous, 1-periodic functions. In this note we show that the functions $F_{k,\ell}(u)$ and $G_{k,\ell}(u)$ are nowhere differentiable. (For $G_{k,\ell}(u)$ this is already known from [5]). In case $k = 1$ both formulas yield the well-known formula of Trollope-Delange ([13], [4]) for the sum of digits

$$\frac{1}{N} \sum_{j=0}^{N-1} s(j) = \frac{1}{2} \log_2 N + F_1(\log_2 N) \tag{1.9}$$

where the 1-periodic function $F_1(u)$ is connected with Takagi's function $T(x)$ by

$$F_1(u) = -\frac{u}{2} - \frac{1}{2^{u+1}} T(2^u) \qquad (u \le 0), \tag{1.10}$$

cf. [8, Theorem 2.1]. In [9] the functions $F_{k,\ell}(u)$ and $G_{k,\ell}(u)$ were expressed by means of the partial derivatives of de Rham's function $R_a(x)$ with respect to the parameter a at $a = \frac{1}{2}$, i.e.

$$T_n(x) = \frac{\partial^n}{\partial a^n} R_a(x) \Big|_{a=1/2} \qquad (x \in [0,1]). \tag{1.11}$$

In particular, $T_0(x) = x$ and $T_1(x) = 2T(x)$ where T is the Takagi function, cf. [9]. We show that for $0 < x \le 1$ we have

$$\frac{1}{x} T_n(x) = (-2)^n (\log_2 x)^n + \sum_{\nu=0}^{n-1} (\log_2 x)^\nu g_{n,\nu}(\log_2 x)$$

where the functions $g_{n,\nu}(u)$ are 1-periodic, continuous and nowhere differentiable. At dyadic points $x = \frac{k}{2^\ell}$ it hold the one-sided limits

$$\lim_{h\to+0} \frac{T_n(x+h) - T_n(x)}{h(\log_2 \frac{1}{h})^n} = 2^n$$

and

$$\lim_{h\to-0} \frac{T_n(x+h) - T_n(x)}{|h|(\log_2 \frac{1}{|h|})^n} = (-1)^{n+1}2^n.$$

Finally, if $s_0(j)$ denotes the number of zeros in the binary expansion of j then

$$\frac{1}{N}\sum_{j=1}^{N-1} s_0(j)^k = \left(\frac{\log_2 N}{2}\right)^k + \frac{(-1)^{k-1}}{N} + \sum_{\ell=0}^{k-1} (\log_2 N)^\ell H_{k,\ell}(\log_2 N) \tag{1.12}$$

where $H_{k,\ell}(u)$ are 1-periodic continuous, nowhere differentiable functions.

In this note we use the Stirling numbers of first and second kind $s^{(1)}_{k,\ell}$, $s^{(2)}_{k,\ell}$ given by

$$k!\binom{x}{k} = \sum_{\ell=0}^{k} s^{(1)}_{k,\ell} x^\ell \tag{1.13}$$

and

$$x^k = \sum_{\ell=0}^{k} s^{(2)}_{k,\ell}\, \ell! \binom{x}{\ell}. \tag{1.14}$$

These numbers are integers. In particular, $s^{(1)}_{k,0} = s^{(2)}_{k,0} = 0$ for $k \geq 1$ and $s^{(1)}_{k,k} = s^{(2)}_{k,k} = 1$ for $k \geq 0$.

2 Partial derivatives

In [9] were introduced the partial derivatives of de Rham's function $R_a(x)$ at $a = \frac{1}{2}$, i.e.

$$T_n(x) = \frac{\partial^n}{\partial a^n} R_a(x)\bigg|_{a=1/2} \qquad (x \in [0,1]). \tag{2.1}$$

Thus $T_0(x) = x$ and $T_1(x) = 2T(x)$ where T is Takagi's function. For $n \geq 1$ the function T_n is continuous and has the symmetry property

$$T_n(1-x) = (-1)^{n+1} T_n(x) \tag{2.2}$$

and for $n \geq 2$ it satisfies the functional equations

$$\left.\begin{array}{rcl} T_n\left(\frac{x}{2}\right) &=& nT_{n-1}(x) + \frac{1}{2}T_n(x) \\ T_n\left(\frac{x+1}{2}\right) &=& -nT_{n-1}(x) + \frac{1}{2}T_n(x) \end{array}\right\} \qquad (x \in [0,1]). \tag{2.3}$$

In [1] were investigated the functions

$$\tilde{T}_n(x) = \frac{1}{n!} T_n(x), \tag{2.4}$$

there with the notation $T_n(x)$. For every $\varepsilon > 0$ there exist constants $C_{n,\varepsilon}$ such that if $0 \le x < x + y \le 1$, then

$$|\tilde{T}_n(x+y) - \tilde{T}_n(x)| \le C_{n,\varepsilon} y^{1-\varepsilon}, \tag{2.5}$$

cf. [1]. By [9, Proposition 4.2] we know that for $n \ge 1$ the derivatives (2.1) of de Rham's function R_a satisfy the functional relations

$$T_n\left(\frac{k+x}{2^\ell}\right) = T_n\left(\frac{k}{2^\ell}\right) + \sum_{\nu=0}^{n} a_\nu T_\nu(x) \tag{2.6}$$

where $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$, $x \in [0,1]$, $T_0(x) = x$ and where a_ν are the constants

$$a_\nu = \binom{n}{\nu} \frac{\partial^{n-\nu}}{\partial a^{n-\nu}} a^{\ell - s(k)}(1-a)^{s(k)} \Big|_{a=1/2} \tag{2.7}$$

which depend on n, k and ℓ. In particular, $a_n = 1/2^\ell$. Moreover, for $k = 0, 1, \ldots, 2^\ell$ it holds

$$T_n\left(\frac{k}{2^\ell}\right) = \frac{n!}{2^{\ell-n}} \sum_{j=0}^{k-1} \sum_{r=0}^{n} (-1)^r \binom{s(j)}{r} \binom{\ell - s(j)}{n-r}. \tag{2.8}$$

Proposition 2.1 *For $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$, $x \in [0,1]$ we have*

$$T_n\left(\frac{k-x}{2^\ell}\right) = T_n\left(\frac{k}{2^\ell}\right) + \sum_{\nu=0}^{n} b_\nu T_\nu(x) \tag{2.9}$$

where b_ν are the constants

$$b_\nu = (-1)^{\nu+1} \binom{n}{\nu} \frac{\partial^{n-\nu}}{\partial a^{n-\nu}} a^{\ell - s(k-1)}(1-a)^{s(k-1)} \Big|_{a=1/2} \tag{2.10}$$

which depend on n, k and ℓ. In particular, $b_n = (-1)^{n+1}/2^\ell$.

Proof: If we denote the coefficients (2.7) more precisely by $a_{\nu,k}$ (for fixed n and ℓ) then from (2.6) with $k-1$ instead of k and $1-x$ instead of x we get

$$\begin{aligned} T_n\left(\frac{k-x}{2^\ell}\right) &= T_n\left(\frac{k-1}{2^\ell}\right) + \sum_{\nu=0}^{n} a_{\nu,k-1} T_\nu(1-x) \\ &= T_n\left(\frac{k-1}{2^\ell}\right) + a_{0,k-1} + \sum_{\nu=0}^{n} (-1)^{\nu+1} a_{\nu,k-1} T_\nu(x) \end{aligned}$$

where we have used (2.2) and $T_0(x) = x$. For $x = 0$ it follows

$$T_n\left(\frac{k}{2^\ell}\right) = T_n\left(\frac{k-1}{2^\ell}\right) + a_{0,k-1}$$

and hence (2.9) with the coefficients b_ν given by (2.10). $\square$

3 Non-differentiability of linear combinations of T_n

The following proposition is a generalization of [1, Theorem 1.5] to linear combinations

$$f_n(x) = \sum_{\nu=1}^{n} c_\nu \tilde{T}_\nu(x) = \sum_{\nu=1}^{n} \frac{c_\nu}{\nu!} T_\nu(x) \qquad (x \in [0,1]) \tag{3.1}$$

with certain constants $c_1, \ldots, c_n$. We will modify a bit the nice proof in [1] where we use largely the same notations.

Proposition 3.1 *If $c_n \neq 0$ then the function $f_n(x)$ from (3.1) is nowhere differentiable.*

Proof: For $x_0 \in [0,1)$ and positive integers k we put $j_k = [2^k x_0]$ such that $0 \leq j_k \leq 2^k - 1$ and

$$\frac{j_k}{2^k} \leq x_0 < \frac{j_k + 1}{2^k}, \qquad k \in \mathbb{N}. \tag{3.2}$$

Observe that $j_{k+1} = 2j_k$ or $j_{k+1} = 2j_k + 1$ where $A = \{k : j_{k+1} = 2j_k\}$ is always infinite and $\mathbb{N} \setminus A = \{k : j_{k+1} = 2j_k + 1\}$ is finite if and only if x_0 is dyadic rational.

For an arbitrary function $f : [0,1] \mapsto \mathbb{R}$ we define

$$\Delta_f(k,j) := \frac{f((j+1) \cdot 2^{-k}) - f(j \cdot 2^{-k})}{2^{-k}} \qquad k \in \mathbb{N}, \quad j = 0, 1, \ldots, 2^k - 1. \tag{3.3}$$

Let be K_n the set of all functions (3.1) with $c_n \neq 0$. We show by induction on n that for no $f \in K_n$ the limit

$$\lim_{k \to \infty} \Delta_f(k, j_k) \tag{3.4}$$

exists. For $n = 1$ this is true since each $f \in K_1$ has the form $f(x) = c_1 \tilde{T}_1(x) = 2c_1 T(x)$ with $c_1 \neq 0$ and the Takagi function $T(x)$ for which the nonexistence of the limit is well known (cf. [12]). Assume for a fixed $n \geq 2$ that for no $f \in K_{n-1}$ the limit (3.4) exists. Now we consider the function $f_n(x)$ from (3.1) with $c_n \neq 0$ which belongs to K_n and assume that there exists a finite number λ such that

$$\lim_{k \to \infty} \Delta_{f_n}(k, j_k) = \lambda. \tag{3.5}$$

It follows

$$\lim_{k \to \infty, k \in A} \Delta_{f_n}(k+1, 2j_k) = \lambda \tag{3.6}$$

and

$$\lim_{k \to \infty, k \notin A} \Delta_{f_n}(k+1, 2j_k + 1) = \lambda \tag{3.7}$$

whenever $\mathbb{N} \setminus A$ is infinite, cf. [1].

Put $\Delta_\nu(k,j) = \Delta_{\tilde{T}_\nu}(k,j)$ then $\Delta_0(k,j) = 1$ since $\tilde{T}_0(x) = x$ and by (3.1) we have

$$\Delta_{f_n}(k,j) = \sum_{\nu=1}^{n} c_\nu \Delta_\nu(k,j).$$

In view of

$$\Delta_\nu(k+1,2j) - \Delta_\nu(k+1,2j+1) = 4\Delta_{\nu-1}(k,j), \qquad (\nu \geq 1) \tag{3.8}$$

cf. [1], we find

$$\begin{aligned}\Delta_{f_n}(k+1,2j_k) - \Delta_{f_n}(k+1,2j_k+1) &= \sum_{\nu=1}^{n} 4c_\nu \Delta_{\nu-1}(k,j_k) \\ &= 4c_1\Delta_0(k,j_k) + \sum_{\mu=1}^{n-1} 4c_{\mu+1}\Delta_\mu(k,j_k)\end{aligned}$$

and hence

$$\Delta_{f_n}(k+1,2j_k) - \Delta_{f_n}(k+1,2j_k+1) = 4c_1 + \Delta_f(k,j_k) \tag{3.9}$$

where f is the function

$$f(x) = 4c_2\tilde{T}_1(x) + \cdots + 4c_n\tilde{T}_{n-1}(x). \tag{3.10}$$

Obviously,

$$\Delta_{f_n}(k+1,2j_k) + \Delta_{f_n}(k+1,2j_k+1) = 2\Delta_{f_n}(k,j_k). \tag{3.11}$$

Now we consider two cases:

1. If x_0 is not dyadic rational, i.e. $\mathbb{N} \setminus A$ is infinite, then (3.5), (3.6) and (3.7) imply

$$\lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k) = \lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k+1) = \lambda.$$

2. If x_0 is dyadic rational, i.e. $\mathbb{N} \setminus A$ is finite, then there exists k_0 such that $j_{k+1} = 2j_k$ for $k > k_0$ and (3.6) can be written as

$$\lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k) = \lambda. \tag{3.12}$$

Now, (3.11), (3.5) and (3.12) imply

$$\lim_{k\to\infty} \Delta_{f_n}(k+1,2j_k+1) = \lambda.$$

So in both cases from (3.9) we get $\lim_{k\to\infty} \Delta_f(k,j_k) = -4c_1$ for f from (3.10) which belongs to K_{n-1} since $c_n \neq 0$. This is a contradiction to the induction hypothesis. Thus $f_n(x)$ with $c_n \neq 0$ is not differentiable at $x_0 \in [0,1)$ which is valid also at $x_0 = 1$ in view of (2.2). □

Remark 3.2 The proof makes use of the recursion (3.8) which in [1] was derived by a system of infinitely many difference equations for the functions $\tilde{T}_n(x)$, cf. [1, Corollary 2.5].

Theorem 3.3 *If $g_\nu(x)$ $(\nu = 1, \ldots, n)$ are differentiable functions for $x \in [0,1]$ then the function*

$$f(x) = \sum_{\nu=1}^{n} g_\nu(x) T_\nu(x) \qquad (x \in [0,1])$$

is differentiable at a point x_0 if and only if $g_\nu(x_0) = 0$ for $\nu = 1, \ldots, n$.

Proof: For $x_0 \in [0,1]$ we consider $h \neq 0$ such that also $x_0 + h \in [0,1]$. Obviously,

$$\frac{f(x_0+h) - f(x_0)}{h} = \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = \sum_{\nu=1}^{n} \frac{g_\nu(x_0+h) - g_\nu(x_0)}{h} T_\nu(x_0+h), \qquad \Sigma_2 = \sum_{\nu=1}^{n} g_\nu(x_0) \frac{T_\nu(x_0+h) - T_\nu(x_0)}{h}.$$

Note that Σ_1 converges as $h \to 0$ since $g_\nu(x)$ is differentiable and $T_\nu(x)$ is continuous and that Σ_2 is convergent by Proposition 3.1 if and only if $g_\nu(x_0) = 0$ for all $\nu = 1, \ldots, n$. $\square$

4 Relations to periodic functions

In [9] were introduced the continuous, 1-periodic functions $F_k(u)$ given for $u \leq 0$ by

$$F_k(u) = \frac{\partial^k}{\partial q^k} a^u R_a(2^u)\Big|_{q=1} \qquad (u \leq 0). \tag{4.1}$$

In particular, $F_0(u) = 1$ and $F_1(u)$ is the function from (1.10) which appears in the formula (1.9) of Trollope-Delange. For $k \geq 1$ the 1-periodic functions $F_k(u)$ have the representations

$$F_k(u) = \frac{1}{2^{u+k}} \sum_{\ell=0}^{k} \frac{P_{k,\ell}(u)}{2^\ell} T_\ell(2^u) \qquad (u \leq 0) \tag{4.2}$$

with the binomial polynomials

$$P_{k,\ell}(u) = (-1)^k \frac{k!}{\ell!} \binom{u+k-1}{k-\ell} \qquad (0 \leq \ell \leq k) \tag{4.3}$$

of degree $k - \ell$ and the partial derivatives T_ℓ from (2.1). In particular,

$$P_{k,0}(u) = (-1)^k u(u+1)\cdots(u+k-1), \qquad P_{k,k}(u) = (-1)^k, \tag{4.4}$$

cf. [9, Proposition 5.1]. From (2.4), (2.5) and (4.2) it follows

Proposition 4.1 *For $h > 0$ and $\varepsilon > 0$ we have*

$$|F_k(u+h) - F_k(u)| \leq A_{k,\varepsilon} h^{1-\varepsilon}$$

with a constant $A_{k,\varepsilon}$.

A consequence of Theorem 3.3 and (4.2) is the following

Proposition 4.2 *If the functions $h_k(u)$ are differentiable then*

$$F(u) = \sum_{k=1}^{n} h_k(u) F_k(u)$$

is differentiable at u_0 if and only if $h_k(u_0) = 0$ for all $k \in \{1, 2, \ldots, n\}$.

If we put $P_{k,\ell}(u) = 0$ for $\ell > k$ then for $n \in \mathbb{N}$ equation (4.2) can also be written in the matrix form

$$(1, 2F_1(u), \ldots, 2^n F_n(u))^\top = \mathbf{A}_n \left(\frac{1}{2^u}, \frac{1}{2^{u+1}} T_1(u), \ldots, \frac{1}{2^{u+n}} T_n(u) \right)^\top \tag{4.5}$$

with the lower triangular matrix $\mathbf{A}_n = (P_{k,\ell}(u))$, $0 \le k, \ell \le n$.

Lemma 4.3 *For arbitrary integer $n \ge 1$ the matrix $\mathbf{A}_n$ is invertible and for the inverse matrix it holds $\mathbf{A}_n^{-1} = \mathbf{A}_n$.*

Proof: We have to show that $\mathbf{B}_n = (b_{k,\ell}) = \mathbf{A}_n^2$ is the unit matrix, i.e. $b_{k,\ell} = \delta_{k,\ell}$. We have

$$b_{k,\ell} = \sum_{j=0}^{n} P_{k,j}(u) P_{j,\ell}(u) = \sum_{j=\ell}^{k} P_{k,j}(u) P_{j,\ell}(u)$$

and hence $b_{k,\ell} = 0$ for $0 \le k \le \ell - 1$. In view of $P_{\ell,\ell}(u) = (-1)^\ell$ we get $b_{\ell,\ell} = 1$. Now let be $k \ge \ell + 1$. Note that

$$P_{k,\ell}(u) = (-1)^k \binom{k}{\ell} (u + k - 1)(u + k - 2) \cdots (u + \ell)$$

so that

$$P_{k,j}(u) P_{j,\ell}(u) = (-1)^{k+j} \binom{k}{j} \binom{j}{\ell} (u + k - 1)(u + k - 2) \cdots (u - \ell)$$

and therefore

$$b_{k,\ell} = (-1)^k (u - k - 1)(u - k - 2) \cdots (u - \ell) \sum_{j=\ell}^{k} (-1)^j \binom{k}{j} \binom{j}{\ell}.$$

Now

$$\binom{k}{j} \binom{j}{\ell} = \binom{k}{\ell} \binom{k-\ell}{j-\ell}$$

and

$$\sum_{j=\ell}^{k} (-1)^j \binom{k-\ell}{j-\ell} = (-1)^\ell (1-1)^{k-\ell} = 0.$$

Hence $b_{k,\ell} = 0$ for $k \ge \ell + 1$. $\square$

As consequence we get from (4.5)

Proposition 4.4 *The partial derivatives (2.1) of de Rham's function $R_a(x)$ have the representations*

$$\frac{1}{2^{u+k}}T_k(2^u) = \sum_{\ell=0}^{k} P_{k,\ell}(u)2^\ell F_\ell(u) \qquad (u \le 0) \tag{4.6}$$

with the polynomials (4.3) and the 1-periodic functions (4.1).

Remark 4.5 According to $P_{1,0}(u) = -u$, $P_{1,1}(u) = -1$, $F_0(u) = 1$ and $F_1(u)$ in (1.9) we get

$$\frac{1}{2^{u+1}}T_1(2^u) = -u - 2F_1(u) \qquad (u \le 0).$$

Putting $x = 2^u$ and using the fact that $T_1(x) = 2T(x)$ where $T(x)$ is the Takagi function, we find

$$\frac{1}{x}T(x) = -\log_2 x - 2F_1(\log_2 x) \qquad (0 < x \le 1), \tag{4.7}$$

cf. [8, Formula (2.5)].

By means of (4.6) we can give a new representation of T_n using the explicit representation of the polynomials $P_{k,\ell}(u)$ of degree $k-\ell$

$$P_{k,\ell}(u) = \sum_{j=0}^{k-\ell} c_{k,\ell,j}u^j. \tag{4.8}$$

In view of (4.3) and the Stirling numbers of first kind $s^{(1)}_{k,\ell}$ given by (1.13) it is easy to compute the coefficients

$$c_{k,\ell,j} = (-1)^k \binom{k}{\ell} \sum_{r=0}^{k-\ell-j} s^{(1)}_{k-\ell,j+r} \binom{j+r}{r} (k-1)^r. \tag{4.9}$$

In particular, the coefficient of $u^{k-\ell}$ reads

$$c_{k,\ell,k-\ell} = (-1)^k \binom{k}{\ell} \tag{4.10}$$

which can be seen directly from (4.3).

Theorem 4.6 *For $n \ge 1$ the derivatives (2.1) of de Rham's function R_a have the representations*

$$\frac{1}{x}T_n(x) = (-2)^n(\log_2 x)^n + \sum_{\nu=0}^{n-1}(\log_2 x)^\nu g_{n,\nu}(\log_2 x) \qquad (0 < x \le 1) \tag{4.11}$$

where $g_{n,\nu}(u)$ are 1-periodic functions given by

$$g_{n,\nu}(u) = 2^n \sum_{\ell=0}^{n-\nu} c_{n,\ell,\nu}2^\ell F_\ell(u) \tag{4.12}$$

with the coefficients from (4.9). They are continuous and nowhere differentiable.

Proof: For $u \le 0$ we have by (4.6) and (4.8)

$$\begin{aligned}\frac{1}{2^{u+k}} T_k(2^u) &= \sum_{\ell=0}^{k} \sum_{j=0}^{k-\ell} c_{k,\ell,j} u^j 2^\ell F_\ell(u) \\ &= \sum_{j=0}^{k} \sum_{\ell=0}^{k-j} c_{k,\ell,j} u^j 2^\ell F_\ell(u).\end{aligned}$$

For $k = n$ we get

$$\begin{aligned}\frac{1}{2^{u+n}} T_n(2^u) &= \sum_{\nu=0}^{n} u^\nu \sum_{\ell=0}^{n-\nu} c_{n,\ell,\nu} 2^\ell F_\ell(u) \\ &= (-1)^n u^n + \sum_{\nu=0}^{n-1} u^\nu \sum_{\ell=0}^{n-\nu} c_{n,\ell,\nu} 2^\ell F_\ell(u)\end{aligned}$$

where we have used that $c_{n,0,n} = (-1)^n$ and $F_0(u) = 1$. With $u = \log_2 x$ it follows (4.11) with (4.12). Obviously, the function $g_{n,\nu}(u)$ is 1-periodic and continuous. By (4.12) we have

$$g_{n,\nu}(u) = 2^{2n-\nu} c_{n,n-\nu,\nu} F_{n-\nu}(u) + 2^n \sum_{\ell=0}^{n-\nu-1} c_{n,\ell,\nu} 2^\ell F_\ell(u)$$

where according to (4.10) it is $c_{n,n-\nu,\nu} = (-1)^n \binom{n}{\nu} \neq 0$. Therefore, by Proposition 4.2 the function $g_{n,\nu}(u)$ is nowhere differentiable. $\square$

5 Limit relations

For the Takagi function T it is known that at each dyadic point $x = \frac{k}{2^\ell}$ it holds

$$\lim_{h \to 0} \frac{T(x+h) - T(x)}{h \log_2 \frac{1}{h}} = 1, \tag{5.1}$$

cf. [7, Proposition 3.2]. We remember $T_1(x) = 2T(x)$ so that the following result is a generalization of (5.1).

Proposition 5.1 *For $n \ge 1$ the derivatives (2.1) of de Rham's function R_a satisfy at each dyadic rational point $x = \frac{k}{2^\ell}$ the limit relations*

$$\lim_{h \to +0} \frac{T_n(x+h) - T_n(x)}{h (\log_2 \frac{1}{h})^n} = 2^n \tag{5.2}$$

and

$$\lim_{h \to -0} \frac{T_n(x+h) - T_n(x)}{|h| (\log_2 \frac{1}{|h|})^n} = (-1)^{n+1} 2^n. \tag{5.3}$$

Proof: For $x = 0$ equation (5.2) is a consequence of Theorem 4.6. Let $x = \frac{k}{2^\ell}$ and $0 < h < 1/2^\ell$. According to (2.6) we have

$$T_n(x+h) - T_n(x) = \sum_{\nu=0}^{n} a_\nu T_\nu(2^\ell h)$$

where $a_n = 1/2^\ell$ so that

$$\frac{T_n(x+h) - T_n(x)}{h(\log_2 \frac{1}{h})^n} = \frac{T_n(2^\ell h)}{2^\ell h(\log_2 \frac{1}{h})^n} + \sum_{\nu=0}^{n-1} a_\nu \frac{T_\nu(2^\ell h)}{h(\log_2 \frac{1}{h})^n}.$$

In view of $(\log_2 \frac{1}{h})^\nu \sim (\log_2 \frac{1}{2^\ell h})^\nu$ as $h \to 0$ it follows (5.2) by Proposition 4.6.

According to (2.9) we have

$$T_n(x-h) - T_n(x) = \sum_{\nu=0}^{n} b_\nu T_\nu(2^\ell h)$$

where $b_n = (-1)^{n+1}/2^\ell$ and hence

$$\frac{T_n(x-h) - T_n(x)}{h(\log_2 \frac{1}{h})^n} = (-1)^{n+1} \frac{T_n(2^\ell h)}{2^\ell h(\log_2 \frac{1}{h})^n} + \sum_{\nu=0}^{n-1} b_\nu \frac{T_\nu(2^\ell h)}{h(\log_2 \frac{1}{h})^n}$$

which implies (5.3). □

Remark 5.2 Relations (5.2) and (5.3) imply that at dyadic rational points $x = \frac{k}{2^\ell}$ there exists the improper derivative

$$\lim_{h \to 0} \frac{T_n(x+h) - T_n(x)}{h} = +\infty,$$

whenever $n \geq 2$ is even, whereas for odd n it holds

$$\lim_{h \to 0} \frac{T_n(x+h) - T_n(x)}{|h|} = +\infty,$$

i.e. T_n with odd n has at x a local minimum. Note that in case $n = 3$ there are further points x where T_3 has a local minimum, cf. Theorem 6.24 in [1].

Start point for the proof of (5.1) in [7] was the fact that for $0 < x \leq \frac{1}{2}$ the Takagi function T satisfies the estimate

$$x \log_2 \frac{1}{x} \leq T(x) \leq x \log_2 \frac{1}{x} + cx \tag{5.4}$$

with a constant $c < \frac{2}{3}$, cf. [7, Lemma 3.1]. By [10, Lemma 2.1] the estimate (5.4) is valid for $0 < x \leq 1$.

Proposition 5.3 *The Takagi function T satisfies for $0 < x \leq 1$ the estimate (5.4) with the optimal constant $c = 2 - \log_2 3 = 0,415\ldots$ where on the right-hand side we have equality if and only if $x = \frac{1}{3} \cdot 2^{1-\ell}$ $(\ell = 0, 1, 2, \ldots)$.*

Proof: For the Takagi function T we know that

$$\frac{1}{x}T(x) = -\log_2 x - 2F_1(\log_2 x) \qquad (0 < x \leq 1)$$

where $F_1(u)$ is the the fractal function in (1.9), cf. (4.7). The assertion follows by Proposition 2.2 and Proposition 2.5 in [8] in view of $c = -2\min F_1(.) = -2(\frac{\log 3}{\log 4} - 1) = 2 - \log_2 3$. $\square$

Proposition 5.4 *For $n \geq 1$ the 1-periodic functions $F_n(u)$ given by (4.2) for $u \leq 0$ satisfy at each point u with $2^u = \frac{k}{2^\ell}$ the limit relations*

$$\lim_{h\to+0} \frac{F_n(u+h) - F_n(u)}{h(\log_2 \frac{1}{h})^n} = \frac{(-1)^n}{2^n}\ln 2 \tag{5.5}$$

and

$$\lim_{h\to-0} \frac{F_n(u+h) - F_n(u)}{|h|(\log_2 \frac{1}{|h|})^n} = \frac{-1}{2^n}\ln 2. \tag{5.6}$$

Proof: For $2^u = \frac{k}{2^\ell} < 1$ and $h > 0$ such that $2^{u+h} \leq 1$ we have

$$\frac{1}{2^{u+h}}T_n(2^{u+h}) - \frac{1}{2^u}T_n(2^u) = \frac{1}{2^u}\left\{T_n(2^{u+h}) - T_n(2^u)\right\} + \frac{1}{2^u}\left(\frac{1}{2^h} - 1\right)T_n(2^{u+h})$$

and by (5.2) the asymptotic relation

$$\frac{1}{2^{u+h}}T_n(2^{u+h}) - \frac{1}{2^u}T_n(2^u) \sim 2^n(2^h - 1)\left(\log_2 \frac{1}{2^{u+h} - 2^u}\right)^n \qquad (h \to +0).$$

In view of $(2^h - 1)/h \to \ln 2$ as $h \to 0$ as well as

$$\log_2 \frac{1}{2^{u+h} - 2^u} = -u + \log_2 \frac{1}{2^h - 1}$$

and

$$\log_2 \frac{1}{2^h - 1} = \log_2 \frac{h}{2^h - 1} + \log_2 \frac{1}{h} \sim \log_2 \frac{1}{h} \qquad (h \to +0)$$

we get

$$\frac{1}{2^{u+h}}T_n(2^{u+h}) - \frac{1}{2^u}T_n(2^u) \sim 2^n h \ln 2\left(\log_2 \frac{1}{h}\right)^n \qquad (h \to +0).$$

By (4.2) we have

$$F_n(u) = \frac{1}{2^{u+n}}\frac{(-1)^n}{2^n}T_n(2^u) + \frac{1}{2^{u+n}}\sum_{\ell=0}^{n-1}\frac{P_{n,\ell}(u)}{2^\ell}T_\ell(2^u) \qquad (u \leq 0)$$

and it follows

$$\frac{F_n(u+h) - F_n(u)}{h(\log_2 \frac{1}{h})^n} \sim \frac{(-1)^n}{2^n}\ln 2\frac{T_n(2^{u+h}) - T_n(2^u)}{h(\log_2 \frac{1}{h})^n} \qquad (h \to +0).$$

Hence (5.2) implies (5.5) at u with $2^u = \frac{k}{2^\ell} < 1$ which is true for arbitrary u with $2^u = \frac{k}{2^\ell}$ since $F_k(u)$ is an 1-periodic function. $\square$

6 Binomial and Power sums

In [9] it was shown that for integer $k \geq 1$ it holds

$$\frac{\partial^k}{\partial q^k} N^\alpha = \frac{N^\alpha}{(1+q)^k} \sum_{\ell=1}^{k} (\log_2 N)^\ell a_{k,\ell} \tag{6.1}$$

with certain coefficients $a_{k,\ell}$ which satisfy a recurrence relation. However, we have overlooked that $a_{k,\ell}$ is the Stirling number $s_{k,\ell}^{(1)}$ of first kind, given by (1.13). By a hint of L. Berg this can be seen as follows: We have $N^\alpha = (1+q)^\beta$ with $\beta = \log_2 N$ and hence

$$\frac{\partial^k}{\partial q^k} N^\alpha = \beta(\beta-1)\cdots(\beta-k+1)(1+q)^{\beta-k}.$$

In view of (1.13) it follows (6.1) with

$$a_{k,\ell} = s_{k,\ell}^{(1)}. \tag{6.2}$$

Theorem 6.1 *For the binary binomial sum (1.5) with integer $k \geq 1$ we have the explicit formula*

$$\frac{1}{N} B_k(N) = \frac{1}{k!} \left(\frac{\log_2 N}{2} \right)^k + \frac{1}{k!} \sum_{\ell=0}^{k-1} (\log_2 N)^\ell F_{k,\ell}(\log_2 N) \tag{6.3}$$

where

$$F_{k,\ell}(u) = \frac{1}{2^\ell} \binom{k}{\ell} F_{k-\ell}(u) + \sum_{j=0}^{k-\ell-1} \binom{k}{j} \frac{s_{k-j,\ell}^{(1)}}{2^{k-j}} F_j(u) \tag{6.4}$$

with the Stirling numbers of first kind $s_{k,\ell}^{(1)}$ and the 1-periodic functions $F_k(u)$ from (4.1). In particular, $F_{k,0}(u) = F_k(u)$ and $F_{k,k}(u) = 1/2^k$. For $\ell < k$ the functions $F_{k,\ell}(u)$ are continuous, nowhere differentiable and of period 1.

Proof: In view of (6.2) and $s_{\ell,\ell}^{(1)} = 1$ the representation (6.3) with (6.4) is already proved in [9, Theorem 5.3] where $F_{k,\ell}(u)$ $(\ell < k)$ is continuous and of period 1. By Proposition 4.2 the function $F_{k,\ell}(u)$ is nowhere differentiable since the coefficient of $F_{k-\ell}(u)$ is different from zero. □

Remarks 6.2 **1.** By Proposition 5.4 it holds that if 2^u is dyadic rational then for $\ell < k$ the functions $F_{k,\ell}$ from (6.4) satisfy the limit relations

$$\lim_{h \to +0} \frac{F_{k,\ell}(u+h) - F_{k,\ell}(u)}{h(\log_2 \frac{1}{h})^{k-\ell}} = \frac{(-1)^{k-\ell}}{2^k} \binom{k}{\ell} \ln 2 \tag{6.5}$$

and

$$\lim_{h \to -0} \frac{F_{k,\ell}(u+h) - F_{k,\ell}(u)}{|h|(\log_2 \frac{1}{|h|})^{k-\ell}} = \frac{-1}{2^k} \binom{k}{\ell} \ln 2. \tag{6.6}$$

2. In case $k = 1$ formula (6.3) yields the formula (1.9) of Trollope-Delange and in case $k = 2$ we get

$$\frac{1}{N}B_2(N) = \frac{1}{2}\left(\frac{\log_2 N}{2}\right)^2 + \frac{\log_2 N}{2}\left\{-\frac{1}{4} + F_1(\log_2 N)\right\} + \frac{1}{2}F_2(\log_2 N).$$

(In the corresponding formula in [9, p. 70$_2$] the term $\frac{1}{2}F_1(L)$ is to cancel and in the previous formula the term $\binom{m}{2}F_1(u)$ is to replace by $\binom{m-1}{2}F_1(u)$).

Next, we consider the formula (1.8) of Coquet for the sum of digital power sums.

Theorem 6.3 *For the power sum (1.7) it holds the formula of Coquet*

$$\frac{1}{N}S_k(N) = \left(\frac{\log_2 N}{2}\right)^k + \sum_{\ell=0}^{k-1}(\log_2 N)^\ell G_{k,\ell}(\log_2 N) \tag{6.7}$$

where

$$G_{k,\ell}(u) = \sum_{j=0}^{k-\ell}\sum_{n=\ell+j}^{k}\binom{n}{j}\frac{s^{(1)}_{n-j,\ell}}{2^{n-j}}s^{(2)}_{k,n}F_j(u) \tag{6.8}$$

with the Stirling numbers of the first and second kind given by (1.13), (1.14) and the 1-periodic functions $F_j(u)$ from (4.1). So $G_{k,k}(u) = 1/2^k$ and for $\ell < k$ they are continuous, nowhere differentiable 1-periodic functions which can be written as

$$G_{k,\ell}(u) = \frac{1}{2^\ell}\binom{k}{\ell}F_{k-\ell}(u) + \sum_{j=0}^{k-\ell-1}a_jF_j(u) \tag{6.9}$$

with certain constants a_j which depend on k and ℓ.

Proof: In view of (6.2) the representation (6.7) with (6.8) is already proved in [9, Theorem 6.1] where $G_{k,\ell}(u)$ is continuous and of period 1. Obviously, the function $G_{k,\ell}(u)$ has the form

$$G_{k,\ell}(u) = \sum_{j=0}^{k-\ell}a_jF_j(u)$$

where the constants a_j depend on k and ℓ. From (6.8) we get for the main coefficient $a_{k-\ell}$ the term

$$a_{k-\ell} = \binom{k}{\ell}\frac{s^{(1)}_{\ell,\ell}}{2^\ell}s^{(2)}_{k,k} = \frac{1}{2^\ell}\binom{k}{\ell}$$

which yields representation (6.9). By Proposition 4.2 the function $G_{k,\ell}(u)$ $(\ell < k)$ is nowhere differentiable since $a_{k-\ell} \neq 0$. □

Remarks 6.4 **1.** In view of (6.9) the statements for $F_{k,\ell}$ in Remarks 6.2/1. are valid also for the functions $G_{k,\ell}$.

2. In case $k = 1$ formula (6.7) yields the formula of Trollope-Delange (1.9) and in case $k = 2$ we get the formula of Coquet [3]

$$\frac{1}{N} S_2(N) = \left(\frac{\log_2 N}{2}\right)^2 + \log_2 N \left\{\frac{1}{4} + F_1(\log_2 N)\right\} + G(\log_2 N)$$

where $G(u) = F_1(u) + F_2(u)$.

Proposition 6.5

For every integer $k \geq 1$ we have

$$\frac{\partial^k}{\partial t^k} N^\alpha G_q(\log_2 N)\Big|_{t=0} = N\left(\frac{\log_2 N}{2}\right)^k + N\sum_{\ell=0}^{k-1} (\log_2 N)^\ell G_{k,\ell}(\log_2 N)$$

where $q = e^t$ and $\alpha = \log_2(1+e^t)$.

Proof: With $q = e^t$ we get from (1.3)

$$\sum_{j=0}^{N-1} e^{ts(j)} = N^\alpha G_q(\log_2 N) \tag{6.10}$$

where $\alpha = \log_2(1+e^t)$ and where the 1-periodic function G_q is connected with de Rham's function by (1.4) with $a = \frac{1}{1+q}$. It follows

$$\sum_{j=0}^{N-1} s(j)^k = \frac{\partial^k}{\partial t^k} N^\alpha G_q(\log_2 N)\Big|_{t=0}$$

and by (6.7) the assertion. □

7 The number of zeros

If $2^n \leq j < 2^{n+1}$ then the number of zeros is $s_0(j) = n + 1 - s(j)$ where $s(j)$ denotes the number of ones.

Lemma 7.1

For $q > 0$ and $2^n \leq N < 2^{n+1}$ we have

$$\sum_{j=1}^{N-1} \left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{q^{n+1}} N^\alpha G_q(\log_2 N) - q + \left(q - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)^n \tag{7.1}$$

where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is a continuous, 1-periodic function given by (1.4).

Proof: By formula (1.2) we get for $2^n \le N < 2^{n+1}$

$$\begin{aligned} R_a\left(\frac{N}{2^{n+1}}\right) - R_a\left(\frac{1}{2}\right) &= a^{n+1}\sum_{j=2^n}^{N-1} q^{s(j)} \\ &= a^{n+1}q^{n+1}\sum_{j=2^n}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)}. \end{aligned}$$

Moreover, (1.2) yields $R_a(1/2^r) = a^r$. If $2^{r-1} \le j < 2^r$ the number of zeros is $s_0(j) = r - s(j)$ and by (1.2) we get

$$\begin{aligned} R_a\left(\frac{2^r}{2^n}\right) - R_a\left(\frac{2^{r-1}}{2^n}\right) &= a^n\sum_{j=2^{r-1}}^{2^r-1} q^{s(j)} \\ &= a^n q^r\sum_{j=2^{r-1}}^{2^r-1}\left(\frac{1}{q}\right)^{s_0(j)} \end{aligned}$$

and hence

$$\sum_{j=2^{r-1}}^{2^r-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{a^\ell q^r}\left(a^{n-r} - a^{n-r+1}\right) = \frac{1-a}{(aq)^r} = \frac{1}{(aq)^{r-1}}.$$

In view of $aq = 1 - a$ and

$$\sum_{r=1}^{n}\frac{1}{(aq)^{r-1}} = \frac{1-\frac{1}{(aq)^n}}{1-\frac{1}{aq}} = -q\left(1 - \frac{1}{(aq)^n}\right) = -q + \frac{1}{a^n q^{n-1}}$$

we get

$$\sum_{j=1}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{(1-a)^{n+1}}R_a\left(\frac{N}{2^{n+1}}\right) - \frac{1}{a^n q^{n+1}} - q + \frac{1}{a^n q^{n-1}}$$

i.e.

$$\sum_{j=1}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{q^{n+1}a^{n+1}}R_a\left(\frac{N}{2^{n+1}}\right) - q + \frac{q^2+1}{a^n q^{n-1}}.$$

Hence

$$\sum_{j=1}^{N-1}\left(\frac{1}{q}\right)^{s_0(j)} = \frac{1}{q^{n+1}}N^\alpha G_q(\log_2 N) - q + \frac{q^2+1}{a^n q^{n-1}}$$

with $a = \frac{1}{1+q}$ which yields the representation (7.1). $\square$

With $q = e^t$ we get from (7.1)

$$\sum_{j=1}^{N-1} e^{-ts_0(j)} = e^{-t(n+1)} N^\alpha G_q(\log_2 N) - e^t + (e^t - e^{-t})(1 + e^{-t})^n \tag{7.2}$$

where $\alpha = \log_2(1 + e^t)$ and $n = [\log_2 N]$ since $2^n \le N \le 2^{n+1} - 1$ and it follows for every integer $k \ge 1$

$$(-1)^k \sum_{j=1}^{N-1} s_0(j)^k = A_k(N) + B_k(N) - 1 \tag{7.3}$$

where

$$A_k(N) = \frac{\partial^k}{\partial t^k} \left[e^{-t(n+1)} N^\alpha G_q(\log_2 N) \right] \Big|_{t=0} \tag{7.4}$$

and

$$B_k(N) = \frac{\partial^k}{\partial t^k} \left[(e^t - e^{-t})(1 + e^{-t})^n \right] \Big|_{t=0}. \tag{7.5}$$

Lemma 7.2 *For (7.4) we have the representations*

$$A_k(N) = (-1)^k N \left(\frac{\log_2 N}{2} \right)^k + N \sum_{\ell=0}^{k-1} (\log_2 N)^\ell A_{k,\ell}(\log_2 N) \tag{7.6}$$

where $A_{k,\ell}(u)$ are 1-periodic function given for $0 \le u < 1$ by

$$A_{k,\ell}(u) = \sum_{i=0}^{\ell} (-1)^i \sum_{m=i}^{k} \binom{k}{m} \binom{m}{i} (u-1)^{m-i} G_{k-m,\ell-i}(u) \tag{7.7}$$

with the functions $G_{k,\ell}(u)$ from (6.8).

Proof: We put $L = \log_2 N$. Observe that

$$\frac{\partial^k}{\partial t^k} \left[e^{-t(n+1)} N^\alpha G_q(L) \right] = \sum_{m=0}^{k} \binom{k}{m} (-n-1)^m e^{-t(n+1)} \frac{\partial^{k-m}}{\partial t^{k-m}} \left[N^\alpha G_q(L) \right].$$

It follows by (7.4) and Proposition 6.5 with $n = [\log_2 N]$

$$A_k(N) = \sum_{m=0}^{k} \binom{k}{m} (-n-1)^m N \sum_{j=0}^{k-m} L^j G_{k-m,j}(L)$$

with the 1-periodic functions $G_{k-m,j}(u)$ from (6.8). For $2^n \le N \le 2^{n+1} - 1$ we write $N = 2^{n+u_N}$ with $0 \le u_N < 1$. In view of $L = \log_2 N = n + u_N$ we have $G_{k-m,j}(L) = G_{k-m,j}(u_N)$ and

$$A_k(N) = N \sum_{m=0}^{k} \binom{k}{m} (u_N - 1 - L)^m \sum_{j=0}^{k-m} L^j G_{k-m,j}(u_N).$$

We want to sort the right-hand side by powers of $L = \log_2 N$. From

$$A_k(N) = N \sum_{m=0}^{k} \binom{k}{m} \sum_{i=0}^{m} \binom{m}{i} (u_N - 1)^{m-i} (-L)^i \sum_{j=0}^{k-m} L^j G_{k-m,j}(u_N)$$

we get

$$A_k(N) = N \sum_{\ell=0}^{k} L^\ell A_{k,\ell}(u_N)$$

with

$$A_{k,\ell}(u) = \sum_{i+j=\ell} (-1)^i \sum_{m=i}^{k} \binom{k}{m} \binom{m}{i} (u-1)^{m-i} G_{k-m,j}(u)$$

which can be written as (7.7). In particular,

$$A_{k,k}(u) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} G_{k-i,k-i}(u) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{2^{k-i}} = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} (-2)^i = \frac{(-1)^k}{2^k}$$

where we have used (6.9) and $F_0(u) = 1$. If we continue the functions $A_{k,\ell}(u)$ to 1-periodic functions on $\mathbb{R}$ then we also get $A_{k,\ell}(u_N) = A_{k,\ell}(L)$ since $u_N = L - n$, and it follows (7.6). □

Remark 7.3 In particular, for $0 \leq u < 1$ we get by (7.7) in case $k = 1$

$$A_{1,0}(u) = u - 1 + F_1(u)$$

and in case $k = 2$

$$A_{2,0}(u) = u^2 - 2u + 2 + (1 - 2u)F_1(u) + F_2(u),$$
$$A_{2,1}(u) = \frac{1}{4} - (u - 1) - F_1(u)$$

where we have used (6.8) with the 1-periodic functions $F_j(u)$ from (4.1).

Now, for integer $k \geq 1$ we compute (7.5). Applying Leibniz formula it is easy to see that

$$B_k(N) = 2^n \sum_{i=0}^{k-1} b_{k,i} n^i \tag{7.8}$$

with certain coefficients $b_{k,i}$. The first sums read

$$B_1(N) = 2 \cdot 2^n, \quad B_2(N) = -2n \cdot 2^n, \qquad B_3(N) = (n^2 + 2n + 2)2^n. \tag{7.9}$$

Lemma 7.4 *For (7.5) we have the representations*

$$B_k(N) = N\sum_{\ell=0}^{k-1}(\log_2 N)^\ell B_{k,\ell}(\log_2 N) \tag{7.10}$$

where $B_{k,\ell}(u)$ are 1-periodic functions given for $0 \le u < 1$ by

$$B_{k,\ell}(u) = \frac{1}{2^u}\sum_{i=\ell}^{k-1} b_{k,i}\binom{i}{\ell}(-u)^{i-\ell} \tag{7.11}$$

with the numbers $b_{k,i}$ from (7.8).

Proof: Starting with (7.8) we prove (7.10) with (7.11). As before we write $N = 2^{n+u_N}$ with $0 \le u_N < 1$ so that $L = \log_2 N = n + u_N$, $2^n = 2^{L-u_N} = N/2^{u_N}$ and

$$n^i = (L - u_N)^i = \sum_{\ell=0}^{i}\binom{i}{\ell}L^\ell(-u_N)^{i-\ell}.$$

From (7.8) we get

$$B_k(N) = N\sum_{\ell=0}^{k-1}(\log_2 N)^\ell B_{k,\ell}(u_N)$$

with $B_{k,\ell}(u)$ from (7.11) for $0 \le u < 1$. If we $B_{k,\ell}$ continue to 1-periodic functions on $\mathbb{R}$ then we have $B_{k,\ell}(\log_2 N) = B_{k,\ell}(u_N)$ since $N = 2^{n+u_N}$. So we get (7.10) with (7.11). □

Remark 7.5 In particular, for $0 \ge u < 1$ we get by (7.11), (7.8) and (7.9) in case $k = 1$

$$B_{1,0}(u) = 2 \cdot \frac{1}{2^u}$$

and in case $k = 2$

$$B_{2,0}(u) = \frac{u}{2^{u-1}}, \quad B_{2,1}(u) = -\frac{1}{2^{u-1}}.$$

Lemma 7.6 *For $\ell < k$ the 1-periodic function $A_{k,\ell}(u)$ given for $0 \le u < 1$ by (7.7) is nowhere differentiable.*

Proof: We apply Proposition 4.2. According to (7.7) and (6.9) the function $A_{k,\ell}(u)$ has the form

$$A_{k,\ell}(u) = \sum_{j=0}^{k-\ell} h_j(u)F_j(u) \qquad (0 \le u < 1)$$

where

$$h_{k-\ell}(u) = \sum_{i=0}^{\ell}(-1)^i\binom{k}{i}\frac{1}{2^{k-i}}\binom{k-i}{\ell-i}.$$

In view of

$$\binom{k}{i}\binom{k-i}{\ell-i} = \binom{k}{\ell}\binom{\ell}{i}$$

we get

$$h_{k-\ell}(u) = \frac{1}{2^k}\binom{k}{\ell}\sum_{i=0}^{\ell}(-2)^i\binom{\ell}{i} = (-1)^\ell\frac{1}{2^k}\binom{k}{\ell}$$

such that $h_{k-\ell}(u) \neq 0$ for $0 \leq u < 1$. By Proposition 4.2 the function $A_{k,\ell}(u)$ is nowhere differentiable. □

Theorem 7.7 *If $s_0(j)$ denotes the number of zeros in the binary expansion of the integer j then for integer $k \geq 1$ we have*

$$\frac{1}{N}\sum_{j=1}^{N-1} s_0(j)^k = \left(\frac{\log_2 N}{2}\right)^k + \frac{(-1)^{k-1}}{N} + \sum_{\ell=0}^{k-1}(\log_2 N)^\ell H_{k,\ell}(\log_2 N) \tag{7.12}$$

where

$$H_{k,\ell}(u) = (-1)^k A_{k,\ell}(u) + (-1)^k B_{k,\ell}(u) \tag{7.13}$$

with the functions $A_{k,\ell}$ from (7.6) and $B_{k,\ell}$ from (7.10). They are 1-periodic functions which are continuous and nowhere differentiable.

Proof: The representation (7.12) follows from (7.3) in view of (7.6), (7.10) and (7.13) where $H_{k,k}(u) = 1/2^k$ since $B_{k,k}(u) = 0$. For $\ell < k$ the functions $A_{k,\ell}(u)$ are nowhere differentiable (Lemma 7.6) and $B_{k,\ell}(u)$ from (7.11) are differentiable in [0,1) so that $H_{k,\ell}(u)$ are nowhere differentiable. By Lemma 7.2 we know that the 1-periodic functions $H_{k,\ell}(u)$ are continuous in $[0, 1)$ and that $H_{k,\ell}(1-0)$ there exist. It remains to show that $H_{k,\ell}(1-0) = H_{k,\ell}(1)$. For that we show that for integer n it holds

$$S(n) = \sum_{\ell=0}^{k} n^\ell\{H_{k,\ell}(1) - H_{k,\ell}(1-0)\} = o(1) \qquad (n \to \infty)$$

which is possible only if $H_{k,\ell}(1) - H_{k,\ell}(1-0) = 0$ for $\ell = k, k-1, \ldots, 0$. We write $S(n) = \Sigma_1(n) + \Sigma_2(n)$ where

$$\Sigma_1(n) = \sum_{\ell=0}^{k} n^\ell\{H_{k,\ell}(1) - H_{k,\ell}(1 + \log_2(1 - 2^{-n})\},$$

$$\Sigma_2(n) = \sum_{\ell=0}^{k} n^\ell\{H_{k,\ell}(1 + \log_2(1 - 2^{-n}) - H_{k,\ell}(1-0)\}$$

and investigate both sums separately.

1. Using (7.12) we get for $s_0(N-1)^k$ the representation

$$\sum_{\ell=0}^{k} \left\{ N(\log_2 N)^\ell H_{k,\ell}(\log_2 N) - (N-1)(\log_2(N-1))^\ell H_{k,\ell}(\log_2(N-1)) \right\}.$$

As $N \to \infty$ we get the asymptotic equation

$$\frac{1}{N} s_0(N-1)^k = \sum_{\ell=0}^{k} (\log_2 N)^\ell \left\{ H_{k,\ell}(\log_2 N) - H_{k,\ell}(\log_2(N-1)) \right\} + o(1)$$

since in view of

$$(\log_2(N-1))^\ell = (\log_2 N + \log_2(1-1/N))^\ell = (\log_2 N)^\ell + \frac{(\log_2 N)^{\ell-1}}{N} O(1)$$

and $(\log_2 N)^{\ell-1}/N \to 0$ we have

$$(\log_2(N-1))^\ell H_{k,\ell}(\log_2(N-1)) = (\log_2 N)^\ell H_{k,\ell}(\log_2(N-1)) + o(1).$$

We choose $N = 2^n$ with integer n. Note that $s_0(2^n - 1) = 0$ so that

$$0 = \sum_{\ell=0}^{k} n^\ell \{ H_{k,\ell}(n) - H_{k,\ell}(\log_2(2^n - 1)) \} + o(1) \qquad (n \to \infty),$$

and in view of $\log_2(2^n - 1) = n + \log_2(1 - 2^{-n})$ and $H_{k,\ell}(u+1) = H_{k,\ell}(u)$ we get $\Sigma_1(n) = o(1)$ as $n \to \infty$.

2. Now, we consider the sum $\Sigma_2(n)$. In view of (7.13), (7.6), (7.7), (6.8) and the fact that $B_{k,\ell}(u)$ are continuous differentiable in $[0,1)$ (Lemma 7.2) we conclude that each function $H_{k,\ell}$ can be written as

$$H_{k,\ell}(u) = \sum_{j=0}^{k-\ell} f_j(u) F_j(u) \qquad (0 \le u < 1)$$

with certain continuous differentiable functions $f_j(u)$ which depend on k and ℓ. By Proposition 4.1 the functions $F_j(u)$ are Hölder continuous with Hölder exponents $1-\varepsilon$ where $\varepsilon > 0$. It follows that for $0 \le u < 1$ the function $H_{k,\ell}(u)$ is Hölder continuous which is true for $0 \le u \le 1$ if we choose $H_{k,\ell}(1-0)$ for $u = 1$. So we get

$$|H_{k,\ell}(1-0) - H_{k,\ell}(1 + \log_2(1 - 2^{-n}))| \le C_\varepsilon |\log_2(1 - 2^{-n})|^{1-\varepsilon}$$

with $\varepsilon > 0$ and in view of $|\log_2(1 - 2^{-n})| \sim 2^{-n}$ and $n^\ell / 2^{n(1-\varepsilon)} = o(1)$ as $n \to \infty$ we get $\Sigma_2(n) = o(n)$.

Consequently, $S(n) = o(n)$ as $n \to \infty$ and the functions $H_{k,\ell}(u)$ are continuous. $\square$

Remark 7.8 In view of Remarks 7.3 and 7.5 we get in case $k = 1$ the known representation

$$\frac{1}{N}\sum_{j=1}^{N-1} s_0(j) = \frac{1}{2}\log_2 N + \frac{1}{N} + H_{1,0}(\log_2 N)$$

with the 1-periodic function $H_{1,0}(u)$, given for $0 \le u < 1$ by

$$H_{1,0}(u) = \frac{1-u}{2} - 2^{1-u} + \frac{1}{2^u}T(2^{u-1})$$

cf. [8, Theorem 3.2], and in case $k = 2$

$$\frac{1}{N}\sum_{j=1}^{N-1}(s_0(j))^2 = \left(\frac{1}{2}\log_2 N\right)^2 - \frac{1}{N} + H_{2,0}(\log_2 N) + \log_2 N H_{2,1}(\log_2 N)$$

with the 1-periodic functions $H_{2,0}(u)$, $H_{2,1}(u)$, given for $0 \le u < 1$ by

$$H_{2,0}(u) = u^2 - 2u + 2 + (1-2u)F_1(u) + F_2(u) + \frac{u}{2^{u-1}}$$

and

$$H_{2,1}(u) = \frac{1}{4} - (u-1) - \frac{1}{2^{u-1}} - F_1(u).$$

Acknowledgement. The author wishes to thank L. Berg for his hint to the Stirling numbers.

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received: August 31, 2011

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