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Compactness in function spaces with splitting topologies

1 Introduction

Let (X, τ), (Y, σ) be topological spaces, and let be $\emptyset \neq \mathfrak{A} \subseteq \mathfrak{P}(X)$. We consider the set-open topology $\tau_{\mathfrak{A}}$ for Y^X or for $C(X, Y)$, generated by the family $\mathfrak{A}$, and we assume that $\tau_p \subseteq \tau_{\mathfrak{A}}$ holds, where τ_p denotes the pointwise topology. For $H \subseteq C(X, Y)$ we want to characterize the $\tau_{\mathfrak{A}}$-compactness of H. We will need the condition that H is evenly continuous on each $A \in \mathfrak{A}$. Hence we consider both sets $C(X, Y)$ and $C(A, Y)$ and of course we can link these spaces by the map $q_A : q_A(f) := f_{|A}$, the restriction of f to the subspace $(A, \tau_{|A})$ of (X, τ). So we have $q_A : C(X, Y) \to C(A, Y)$.

Using these maps, we give a new and interesting proof of a "final" kind of the Ascoli theorem, as former derived by use of hyperspaces in [1].

Most notions used here are standard and explanations can be found in standard books on general topology such as [3], [4], [5]. Concerning some more special notions we refer to [2], more explanations can be found in [6] and [1], too.

2 The continuity of the map q_A

Now let be $B \subseteq X$ with $\emptyset \neq B \neq X$; let $\mathfrak{A} \subseteq \mathfrak{P}(X)$, $\mathfrak{B} \subseteq \mathfrak{P}(B)$, $\mathfrak{A} \neq \emptyset$ and $\mathfrak{B} \neq \emptyset$. Then we can consider the set-open topologies $\tau_{\mathfrak{A}}$ on Y^X and $\tau_{\mathfrak{B}}$ on Y^B respectively, and for fixed B we have our map $q_B : Y^X \to Y^B : q_B(f) := f_{|B}$. Here at first the question arises, when is $q_B : (Y^X, \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ continuous? (Remark: If $q_B : (Y^X, \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ is continuous, then $q_B : (C(X, Y), \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ is continuous, and we know that $q_B(C(X, Y)) \subseteq C(B, Y)$ so we find $q_B : (C(X, Y), \tau_{\mathfrak{A}}) \to (C(B, Y), \tau_{\mathfrak{B}})$ being continuous.)

Proposition 2.1 *If $\mathfrak{B} \subseteq \mathfrak{A}$ holds, then $q_B : (Y^X, \tau_{\mathfrak{A}}) \to (Y^B, \tau_{\mathfrak{B}})$ is continuous.*

Proof: For the generating subbase-elements of our topologies we use the symbols $(Z,V)_B := \{g \in Y^B | \; g(Z) \subseteq V\}$ and $(Z,V)_X := \{f \in Y^X | \; f(Z) \subseteq V\}$ with elements Z of $\mathfrak{B}$ or $\mathfrak{A}$, respectively, and open subsets V of Y.

To prove continuity of q_B, it is enough to show that the preimage of every subbase element of $\tau_{\mathfrak{B}}$ belongs to $\tau_{\mathfrak{A}}$, so let $Z \in \mathfrak{B} \subseteq \mathfrak{P}(B)$ and $V \in \sigma$ be given. Then we have $q_B^{-1}((Z,V)_B) = \{f \in Y^X | \; f_{|B} \in (Z,V)_B\} = \{f \in Y^X | \; f_{|B}(Z) \subseteq V\} = \{f \in Y^X | \; f(Z) \subseteq V\} = (Z,V)_X \in \tau_{\mathfrak{A}}$. ■

Some options to define suitable families $\mathfrak{A}, \mathfrak{B}$:

1. Let $\mathcal{E}$ be a property, which is defined for subsets of topological spaces (such as compactness, relative compactness or closedness, for example; but even such "non-topological" defined things as finiteness may be considered). The family of all subsets of a topological space (X,τ) having property $\mathcal{E}$ w.r.t. τ is denoted by $\mathcal{E}(X,\tau)$.[1] Then we can define $\mathfrak{A} := \mathcal{E}(X,\tau)$ and $\mathfrak{B} := \mathcal{E}(B,\tau_{|B})$.
2. We start with a family $\mathfrak{A} \subseteq \mathfrak{P}(X)$ and define $\forall B \in \mathfrak{A} : \mathfrak{A}_B := \{A \in \mathfrak{A} \mid A \subseteq B\}$.

3 Basic lemmas

We provide a few lemmas, which are very useful for our considerations.

Lemma 3.1 *Let (X,τ) a topological space, (Y,σ) a Hausdorff topological space. Let ζ be a topology (*$\lim$ *a convergence structure) on $C(X,Y)$ with $\tau_p \leq \zeta$ ($\tau_p \leq \lim$) and let $\mathcal{H} \subseteq C(X,Y)$ be compact w.r.t. ζ (resp. $\lim$). The $\mathcal{H}$ is τ_p-closed in Y^X.*

Proof: Because of $\tau_p \leq \zeta$ ($\tau_p \leq \lim$) the compactness of $\mathcal{H}$ w.r.t. τ_p follows from assumtion. So, $\mathcal{H}$ is τ_p-closed in Y^X as a compact subset of the Hausdorff-space (Y^X, τ_p). ■

Lemma 3.2 *Let $(X,\tau), (Y,\sigma)$ topological spaces; let $\emptyset \neq B \subseteq X$ and $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{P}(B)$ be given with the properties:*

(1) *$\forall Z \subseteq B : Z$ is $\tau_{|B}$-closed $\implies Z \in \mathfrak{B}$ and*

(2) *$\forall f \in C(B,Y) : f(B)$ is a T_3-subspace of Y.*

Then the set-open topology $\tau_{\mathfrak{B}}$ is conjoining for $C(B,Y)$.

[1] Although any dependence of our property $\mathcal{E}$ on τ is not required, it remains still allowed, so, we respect τ as a parameter. Somewhat more precise: such an "property" $\mathcal{E}$ is just a map from the class of all topological spaces to the class of all sets fulfilling the condition, that the image $\mathcal{E}(X,\tau)$ of every topological space (X,τ) is a subset of $\mathfrak{P}(X)$.

Proof: We will show, that the evaluation map

$$\omega : B \times C(B,Y) \to Y : \omega(x,f) := f(x)$$

is continuous w.r.t. $\tau \times \tau_{\mathfrak{B}}, \sigma$. For arbitrary $x \in B$ and $f \in C(B,Y)$ let $V \in \sigma$ be given with $\omega(x,f) \in V$. Because $f(B)$ is T_3 by assumtion and $V \cap f(B)$ is open in $f(B)$, there exist a closed subset Z of $f(B)$ and an open subset W of $f(B)$ such that

$$f(x) \in W \subseteq Z \subseteq f(B) \cap V \ .$$

since $f : B \to (Y,\sigma)$ ist continuous, it is continuous, too, viewed as a map from B onto $f(B)$ w.r.t. $\sigma_{|f(B)}$. Thus $f^{-1}(Z)$ is closed and $f^{-1}(W)$ is open in B, and of course, $x \in f^{-1}(W)$ holds. So, by assumption (1), we have $f^{-1}(Z) \in \mathfrak{B}$ and consequently $(f^{-1}(Z), V) \in \tau_{\mathfrak{B}}$. Now, $f(f^{-1}(Z)) \subseteq Z \subseteq V$ implies $f \in (f^{-1}(Z), V)$, so $(f^{-1}(Z), V)$ is an open $\tau_{\mathfrak{B}}$-neighborhood of f in $C(B,Y)$ and obviously, $f^{-1}(W)$ is an open neighborhood of x in B. Now we have $\omega(f^{-1}(W) \times (f^{-1}(Z), V)) \subseteq V$, thus ω is continuous. ∎

Lemma 3.3 *Let $(X,\tau), (Y,\sigma)$ be topological spaces; let $\emptyset \neq \mathfrak{A} \subseteq \mathfrak{P}(X)$ be given and for every $B \in \mathfrak{A}$ let $\mathfrak{A}_B$ be a subset of $\mathfrak{P}(B)$ such that $B \in \mathfrak{A}_B$. Now we consider a filter $\mathcal{F}$ on Y^X and a function $f \in Y^X$. Assume*

$$\forall B \in \mathfrak{A} : \ q_B(\mathcal{F}) \xrightarrow{\tau_{\mathfrak{A}_B}} f_{|B} \ .$$

Then we have $\mathcal{F} \xrightarrow{\tau_{\mathfrak{A}}} f$ in Y^X.

Proof: The sets $(B,V)_X$ with $B \in \mathfrak{A}$ and $V \in \sigma$ form a subbase of $\tau_{\mathfrak{A}}$, so we have to show, that $\mathcal{F}$ contains all such neighborhoods of f.

To do this, let $B \in \mathfrak{A}$, $V \in \sigma$ with $f \in (B,V)_X$ be given; we have $f(B) \subseteq V$ and hence $f_{|B}(B) \subseteq V$; by this way $f_{|B} = q_B(f) \in (B,V)_B = \{h \in Y^B | \ h(B) \subseteq V\}$; since $B \in \mathfrak{A}_B$, $(B,V)_B$ is an open subbase-element of $\tau_{\mathfrak{A}_B}$ in Y^B. Since $q_B(\mathcal{F}) \longrightarrow f_{|B}$ w.r.t. $\tau_{\mathfrak{A}_B}$, there exists $A \in \mathcal{F}$ such that $q_B(A) \subseteq (B,V)_B$ and so follows $A \subseteq (B,V)_X$ implying $(B,V)_X \in \mathcal{F}$. ∎

4 $\tau_{\mathfrak{A}}$-compactness

Now, we want to formulate and prove the compactness criterion.

Proposition 4.1 *Let $(X,\tau), (Y,\sigma)$ be topological spaces, let $\mathcal{H} \subseteq C(X,Y)$ and let $\emptyset \neq \mathfrak{A} \subseteq \mathfrak{P}(X)$ be given. Moreover, for every $B \in \mathfrak{A}$ let $\mathfrak{B}_B$ be a nonempty subset of $\mathfrak{P}(B)$. Assume $\tau_p \leq \tau_{\mathfrak{A}}$.*

1. *If $\mathcal{H}$ is $\tau_{\mathfrak{A}}$-compact and if*

   (i) *(Y,σ) is Hausdorff,*

   (ii) $\forall B \in \mathfrak{A} : \mathfrak{B}_B \subseteq \mathfrak{A}$,

   (iii) $\forall B \in \mathfrak{A}, Z \subseteq B :$ *Z $\tau_{|B}$-closed* $\implies Z \in \mathfrak{B}_B$,

   (iv) $\forall B \in \mathfrak{A}, f \in C(B,Y) :$ *$f(B)$ is a T_3-subspace of Y*

   hold, then we have:

   (a) $\forall x \in X : \mathcal{H}(x)$ *is relatively compact in Y.*

   (b) *$\mathcal{H}$ is evenly continuous on each $B \in \mathfrak{A}$.*

   (c) *$\mathcal{H}$ is τ_p-closed in Y^X.*

2. *Let (a), (b), (c) be true and let hold*

   (ii) $\forall B \in \mathfrak{A} : \mathfrak{B}_B \subseteq \mathfrak{A}$,

   (v) $\forall B \in \mathfrak{A} : B \in \mathfrak{B}_B$,

   (vi) $\forall B \in \mathfrak{A} :$ *the set-open topology $\tau_{\mathfrak{B}_B}$ is splitting in $C(B,Y)$.*

   Then $\mathcal{H}$ is $\tau_{\mathfrak{A}}$-compact in $C(X,Y)$.

Proof: (1) By lemma 3.1 we get (c); moreover by the proof of lemma 3.1 we know that $\mathcal{H}$ is τ_p-compact, too, and hence $\mathcal{H}$ is τ_p-relatively compact in Y^X, but then we obtain (a) by the Tychonoff-theorem for relatively compact sets (see [2], [1]). Now by condition (ii) and by proposition 2.1 we get: $\forall B \in \mathfrak{A} : q_B(\mathcal{H})$ is $\tau_{\mathfrak{B}_B}$-compact in $C(B,Y)$. (iii) and (iv) yield that $\tau_{\mathfrak{B}_B}$ is conjoining and hence $\mathcal{H}$ is evenly continuous on B since Y is Hausdorff (see theorem 32 in [2]). Thus we got (b).

(2) By (a), $\mathcal{H}$ is τ_p-relatively compact in Y^X and hence τ_p-compact by (c). Let $\mathcal{F}$ be an ultrafilter on $C(X,Y)$ such that $\mathcal{H} \in \mathcal{F}$; by the τ_p-compactness of $\mathcal{H}$ there exists $f \in \mathcal{H}$ with $\mathcal{F} \xrightarrow{\tau_p} f$; now, for all $B \in \mathfrak{A}$ the map $q_B : (C(X,Y),\tau_p) \to (C(B,Y),\tau_p)$ is continuous, implying that $q_B(\mathcal{F}) \xrightarrow{\tau_p} q_B(f) = f_{|B}$ in $C(B,Y)$ yielding by (b) that $q_B(\mathcal{F}) \xrightarrow{c} q_B(f)$ in $C(B,Y)$ holds. By (vi) we get $q_B(\mathcal{F}) \xrightarrow{\tau_{\mathfrak{B}_B}} q_B(f)$, thus $\mathcal{F} \xrightarrow{\tau_{\mathfrak{A}}} f$, by lemma 3.3 - showing that $\mathcal{H}$ is $\tau_{\mathfrak{A}}$-compact. ■

Assume $\mathfrak{A} := \{A \subseteq X |\ A \text{ compact}\}$ and for all $B \in \mathfrak{A}$ let $\mathfrak{B}_B := \{Z \subseteq B |\ Z \text{ compact}\}$. Then for the families $\mathfrak{A}$, $\mathfrak{B}_B$ the assumptions (ii) ... (vi) are obviously valid. So, we get:

Corollary 4.2 *Let $(X,\tau),(Y,\sigma)$ be topological spaces, (Y,σ) Hausdorff. Let $\mathcal{H} \subseteq C(X,Y)$ be given and consider the compact-open topology τ_{co} on $C(X,Y)$. Then are equivalent:*

(1) *$\mathcal{H}$ is τ_{co}-compact.*

(2) (a) *$\forall x \in X : \mathcal{H}(x)$ is relatively compact in Y,*

(b) *$\mathcal{H}$ is evenly continuous on every compact set $K \subseteq X$,*

(c) *$\mathcal{H}$ is in Y^X τ_p-closed.*

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