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# The infimal convolution can be used to easily prove the classical Hahn-Banach theorem

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ABSTRACT. By using a particular case of the infimal convolution, we provide an instructive proof for the dominated Hahn-Banach extension theorem.

Former proofs have only used this convolution rather implicitly.

KEY WORDS. Infimal convolution, Hahn-Banach theorem

## 1 Hahn-Banach extensions and the infimal convolution

**Notation 1.1** Suppose that $X$ *is a vector space over* $\mathbb{R}$ and $p$ *is a positively homogeneous, subadditive function of* $X$ *to* $\mathbb{R}$.

Moreover, assume that $V$ *is a subspace of* $X$ and $\varphi$ *is a linear function of* $V$ *to* $\mathbb{R}$ such that $\varphi$ is dominated by $p$ on $V$ in the sense that $\varphi(v) \le p(v)$ *for all* $v \in V$.

Under the above assumptions, the subsequent dominated extension theorem was first proved by Banach in [1, pp. 227–29] with reference to his former paper in 1929. At some later pages, he also mentions the pioneering works of Riesz in 1907, Helly in 1912, and Hahn in 1927. See the reliable historical notes of Saccoman [10].

The term *Hahn-Banach theorem* has been coined to the following theorem of Banach, or an important consequence of it proved earlier by Hahn, after a paper of Bohnenblust and Sobczcyk in 1938 who proved a complex form Hahn's theorem independently of the works of Murray in 1936 and Sukhomlinov in 1938. See the excellent surveys of Buskes [5] and Narici and Beckenstein [9].

**Theorem 1.2** *There exists a linear function* $f$ *of* $X$ *to* $\mathbb{R}$ *that extends* $\varphi$ *and is dominated by* $p$ *on* $X$.

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This theorem is usually proved with the help of an elementary, but tricky computation and some important, non-direct consequence of the Axiom of Choice such as the well-ordering principle or transfinite induction, and Zorn's lemma or Hausdorff's maximal principle. See Bridges [3, pp. 261–262] for a nice instructive treatment.

In the present note, we shall show that the computational part of the proof can be put into a proper perspective with the help of the $p * \varphi$ particular case of the infimal convolution. The latter notion was already intensively studied by Moreau [8] and Strömberg [11] with several applications. See also [13], [6] and [14] for some further results.

However, up till now, it has only been implicitly used in the proofs of the Hahn-Banach theorems. Unfortunately, the second author in [12] considered the intersection convolution to be a more convenient tool for proving linear extension theorems than the infimal one. Though, he observed that the former one is only a particular case of an obvious extension of the latter one.

In the sequel, in addition to Notation 1.1, we shall only need the following two fundamental definitions.

**Definition 1.3** If $U$ is a linear subspace of $X$ containing $V$, then a linear function $\psi$ of $U$ to $\mathbb{R}$, that extends $\varphi$ and is dominated by $p$ on $U$, will be called a *Hahn-Banach extension* of $\varphi$ to $U$.

**Remark 1.4** By using this definition, the assertion of Theorem 1.2 can be briefly expressed by saying that there exists a Hahn-Banach extension $f$ of $\varphi$ to $X$.

**Definition 1.5** The function $q = p * \varphi$, defined by

$$q(x) = \inf_{v \in V} \big( p(x-v) + \varphi(v) \big)$$

for all $x \in X$, will be called the *infimal convolution* of $p$ and $\varphi$.

**Remark 1.6** The above definition can be put a more instructive form by observing that

$$q(x) = \inf \big\{ p(u) + \varphi(v) : \quad u \in X, \quad v \in V, \quad x = u + v \big\}$$

for all $x \in X$. Note that the latter form can be applied to more general situations.

The close relationship that exists between the Hahn-Banach extensions and the infimal convolution can already be nicely clarified by the following

**Theorem 1.7** *If $\psi$ is a Hahn-Banach extension of $\varphi$ to $U$, then for any $u \in U$ we have*

$$-q(-u) \leq \psi(u) \leq q(u).$$

**Proof:** For any $v \in V$, we have

$$\psi(u) = \psi(\, u - v + v\,) = \psi(u - v) + \psi(v) \leq p(u - v) + \varphi(v)\,.$$

Hence, we can already infer that

$$\psi(u) \leq \inf_{v \in V} \big( p(u - v) + \varphi(v) \big) = q(u)\,.$$

Now, by writing $-u$ in place of $u$, we can see that

$$-\psi(u) = \psi(-u\,) \leq q\,(-u\,)\,, \quad \text{and thus} \quad -\,q\,(-u\,) \leq \psi(u)$$

also holds.

Now, as an immediate consequence of this theorem, we can also state

**Corollary 1.8** *If $\psi$ is as in Theorem 1.7 and $q$ is odd on $U$, then $q$ is an extension of $\psi$.*

**Proof:** In this case, for any $u \in U$, we have

$$q\,(-u\,) = -q(u)\,, \qquad \text{and hence} \qquad -\,q\,(-u\,) = q(u)\,.$$

Therefore, by Theorem 1.7, $\psi(u) = q(u)$ is also true.

Thus, in particular, we can also state

**Corollary 1.9** *If $U$ is a subspace of $X$ such that $V \subset U$ and $q$ is odd on $U$, then there exists at most one Hahn-Banach extension $\psi$ of $\varphi$ to $U$.*

## 2 Further inequalities for the function $q$

**Theorem 2.1** *For any $x \in X$, we have*

$$-p\,(-x) \leq q(x) \leq p\,(x)\,.$$

**Proof:** For any $v \in V$, we have

$$\begin{aligned} 0 = \varphi(0) = \varphi(-v\,) + \varphi(v) &\leq p(-v\,) + \varphi(v) \\ &= p\,(-x + x - v\,) + \varphi(v) \leq p(-x\,) + p\,(\,x - v\,) + \varphi(v)\,, \end{aligned}$$

and thus

$$-p\,(-x\,) \leq p\,(\,x - v\,) + \varphi(v)\,.$$

Hence, we can already infer that

$$-p(-x) \le \inf_{v \in V} \big( p(x-v) + \varphi(v) \big) = q(x) .$$

Moreover, we can at once see that

$$q(x) = \inf_{v \in V} \big( p(x-v) + \varphi(v) \big) \le p(x-0) + \varphi(0) = p(x)$$

also holds.

Now, as an immediate consequence of this theorem, we can also state

**Corollary 2.2** *$q$ is a real-valued function of $X$ such that $q(0)=0$.*

From Theorem 2.1, by writing $-x$ in place $x$, we can also immediately get

**Corollary 2.3** *For any $x \in X$, we have*

$$-p(-x) \le -q(-x) \le p(x) .$$

In addition to the above results and Theorem 1.7, it is also worth proving the following

**Theorem 2.4** *For any $x \in X$, we have*

$$-q(-x) \le q(x) .$$

**Proof:** For any $v, s \in V$ we have

$$\begin{aligned}
-p(x-v) - \varphi(v) &= -p(x-v) - \varphi(v) - \varphi(s) + \varphi(s) \\
&= -p(x-v) + \varphi(-v-s) + \varphi(s) \le -p(x-v) + p(-v-s) + \varphi(s) \\
&= -p(x-v) + p(x-v-x-s)) + \varphi(s) \le -p(x-v) + p(x-v) + p(-x-s) + \varphi(s) \\
&= p(-x-s) + \varphi(s) .
\end{aligned}$$

Hence, we can infer that

$$-p(x-v) - \varphi(v) \le \inf_{s \in V} \big( p(-x-s) + \varphi(s) \big) = q(-x) .$$

Now, we can already see that

$$-q(-x) \le p(x-v) + \varphi(v) ,$$

and thus

$$-q(-x) \le \inf_{v \in V} \big( p(x-v) + \varphi(v) \big) = q(x) .$$

also holds.

This theorem makes the less obvious part of the proof of Theorem 2.1 superfluous. Moreover, it immediately yields the following

**Corollary 2.5** *$q$ is a superodd function of $X$ in the sense that $-q(x) \leq q(-x)$ for all $x \in X$.*

**Remark 2.6** Later, we shall see that the function $q$ is not, in general, odd. Therefore, in contrast to Corollary 1.8, it cannot usually be a Hahn–Banach extension of $\varphi$ to $X$.

Moreover, we shall also see that $q$ is not, in general, even. Therefore, it cannot usually be a seminorm even if $p$ is so. However, due to the linearity of $\varphi$, it will turn out to have some better additivity and homogeneity properties than $p$.

## 3 Additivity and homogeneity properties of $q$

**Theorem 3.1** *For any $x \in X$ and $v \in V$, we have*

$$q(x+v) = q(x) + \varphi(v).$$

**Proof:** For any $s \in V$, we have

$$q(x) = \inf_{t \in V} \big( p(x-t) + \varphi(t) \big) \leq p\big(x-(s-v)\big) + \varphi(s-v),$$

and thus

$$q(x) + \varphi(v) \leq p\big(x-(s-v)\big) + \varphi(s-v) + \varphi(v) = p(x+v-s) + \varphi(s).$$

Hence, we can already infer that

$$q(x) + \varphi(v) \leq \inf_{s \in V} \big( p(x+v-s)) + \varphi(s) \big) = q(x+v).$$

Now, we can easily see that

$$q(x+v) = q(x+v) + \varphi(0) = q(x+v) + \varphi(-v) + \varphi(v) \leq q(x) + \varphi(v)$$

also holds.

From this theorem, by using Corollary 2.2, we can immediately derive

**Corollary 3.2** *$q$ is an extension of $\varphi$.*

**Proof:** Namely, by Theorem 3.1 and Corollary 2.2, we have

$$q(v) = q(0+v) = q(0) + \varphi(v) = 0 + \varphi(v) = \varphi(v)$$

for all $v \in V$.

Now, as an immediate consequence of Theorem 3.1 and Corollary 3.2, we can also state

**Corollary 3.3** *$q$ is an $X\times V$–additive function of $X$ in the sense that*

$$q\,(x+v\,) = q(x) + q(v)$$

*for all $x \in X$ and $v \in V$.*

Concerning the function $q$, we can also easily prove the following

**Theorem 3.4** *$q$ is a subadditive function of $X$.*

**Proof:** If $x, y \in X$, then by Definition 1.3 and Corollary 2.2, for any $\varepsilon > 0$ there exist $s, t \in V$ such that

$$p\,(x-s) + \varphi(s) < q(x) + \varepsilon \qquad \text{and} \qquad p\,(y-t) + \varphi(t) < q(y) + \varepsilon\,.$$

Now, we can already see that

$$\begin{aligned} q\,(x+y\,) &= \inf_{v\in V}\big(p\,(x+y-v\,) + \varphi(v)\big) \\ &\le p\big(x+y-(\,s+t\,)\big) + \varphi\,(\,s+t\,) \le p(\,x-s\,) + p\,(\,y-t\,) + \varphi(s) + \varphi(t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le q(x) + q(y) + 2\,\varepsilon\,. \end{aligned}$$

Hence, by letting $\varepsilon$ tend to $0$, we can infer that

$$q\,(x+y\,) \le q(x) + q(y)\,.$$

**Remark 3.5** This theorem makes the proof of Theorem 2.4 superfluous. Namely, by Theorem 3.4 and [4, Theorem 4.3], the function $q$ is superodd.

Moreover, by the above theorems, we can also at once state that $q$ is $\mathbb{N}$–subhomogeneous and $\{0\}\cup\mathbb{N}^{-1}$–superhomogeneous.

However, the latter facts are of no particular importance for us now since we can also prove the following

**Theorem 3.6** *$q$ is a positively homogeneous function of $X$.*

**Proof:** For any $x \in X$, $v \in V$ and $\lambda \in \mathbb{R}$, with $\lambda > 0$, we have

$$q(x) = \inf_{s\in V}\big(p\,(x-s\,) + \varphi(s)\big) \le p\big(x-\lambda^{-1}v\,\big) + \varphi\big(\lambda^{-1}v\,\big)\,,$$

and thus

$$\lambda\, q(x) \le \lambda\, p\big(x-\lambda^{-1}v\,\big) + \lambda\,\varphi\big(\lambda^{-1}v\,\big) = p\,(\,\lambda\, x - v\,) + \varphi(v)\,.$$

Hence, we can already infer that

$$\lambda\, q(x) \leq \inf_{v \in V} \big( p(\lambda\, x - v) + \varphi(v) \big) = q\,(\,\lambda\, x\,)\,.$$

Now, we can easily see that

$$q\,(\,\lambda\, x\,) = \lambda\, \lambda^{-1} q\,(\,\lambda\, x\,) \leq \lambda\, q\,(\,\lambda^{-1} \lambda\, x\,) = \lambda\, q(x)$$

also holds.

Now, as a useful consequence of Theorem 3.6 and Corollary 2.5, we can also prove the following

**Corollary 3.7** *$q$ is an $\mathbb{R}$–superhomogeneous function of $X$ in the sense that*

$$\lambda\, q\,(x) \leq q\,(\,\lambda\, x\,)$$

*for all $\lambda \in \mathbb{R}$ and $x \in X$.*

**Proof:** By Corollary 2.5 and Theorem 3.6, for any $x \in X$ and $\lambda \in \mathbb{R}$, with $\lambda < 0$, we also have

$$\lambda\, q(\,x) = (-\lambda\,)\big(-q(x)\big) \leq (-\lambda\,)\, q(-x\,) = q\,\big(\,(-\lambda\,)(-x\,)\big) = q\,(\,\lambda\, x\,)\,.$$

From this corollary, by writing $-\lambda$ and $-x$ in place of $\lambda$ and $x$, respectively, we can immediately infer

**Corollary 3.8** *For any $\lambda \in \mathbb{R}$ and $x \in X$, we have*

$$-q\,(\,\lambda\, x\,) \leq \lambda\, q\,(-x\,)\,.$$

## 4 An instructive proof of the Hahn-Banach theorem

We first state the following basic theorem whose proof may be left to the reader.

**Theorem 4.1** *If $a \in X$ such that $a \notin V$ and*

$$U = \mathbb{R}\, a + V = \big\{\, \lambda\, a + v\ :\quad \lambda \in \mathbb{R}\,, \quad v \in V\,\big\}\,,$$

*then*

(1) *$U$ is the smallest linear subspace of $X$ such that $a \in U$ and $V \subset U$;*

(2) *for each $u \in U$ there exists a unique pair $(\,\lambda_u,\, v_u) \in \mathbb{R} \times V$ such that $u = \lambda_u a + v_u$;*

(3) *the mappings $u \mapsto \lambda_u$ and $u \mapsto v_u$, where $u \in U$, are linear functions of $U$ to $\mathbb{R}$ and $V$, respectively, such that $\lambda_v = 0$ and $v_v = v$ for all $v \in V$.*

Now, by using this theorem and our former results on the infimal convolution, we can quite easily prove the following simple, but important particular case of a slight improvement of Theorem 1.2.

**Theorem 4.2** *If $a \in X$ such that $a \notin V$, then there exists a linear function $\psi$ of the subspace $U = \mathbb{R}\,a + V$ to $\mathbb{R}$ that extends $\varphi$ and satisfies*

$$-q(-u) \le \psi(u) \le q(u)$$

*for all $u \in U$.*

**Proof:** Note that if $\psi$ is as above, then under the notation of Theorem 4.1, for any $u \in U$, we have

$$\psi(u) = \psi(\lambda_u a + v_u) = \lambda_u \psi(a) + \psi(v_u) = \lambda_u \psi(a) + \varphi(v_u).$$

Moreover, we also have

$$-q(-a) \le \psi(a) \le q(a).$$

Therefore, to prove the theorem, we may naturally define a function $\psi$ of $U$ to $\mathbb{R}$ such that

$$\psi(u) = \lambda_u q(a) + \varphi(v_u)$$

for all $u \in U$. Now, by using Theorem 4.1, we can easily see that $\psi$ is a linear extension of $\varphi$.

Therefore, by Theorem 1.7, we need only show that $\psi$ is dominated by $p$ on $U$. For this, note that by Corollary 3.7 and Theorems 3.1 and 2.1 we have

$$\psi(u) = \lambda_u q(a) + \varphi(v_u) \le q(\lambda_u a) + \varphi(v_u) = q(\lambda_u a + v_u) = q(u) \le p(u)$$

for all $u \in U$.

**Remark 4.3** Note that, in the above proof, instead of $q(a)$ we may take any number $b \in \mathbb{R}$ with

$$-q(-a) \le b \le q(a).$$

Therefore, the required extension $\psi$ of $\varphi$ is unique if and only if the function $q$ is odd at the point $a$.

Now, as a slight improvement of Theorem 1.2, we can also prove the following

**Theorem 4.4** *There exists a linear function $f$ of $X$ to $\mathbb{R}$ that extends $\varphi$ and satisfies*

$$-q(-x) \le f(x) \le q(x)$$

*for all $x \in X$.*

**Proof:** Denote by $\Psi$ the family of all Hahn–Banach extensions $\psi$ of $\varphi$. Then, it is clear $\Psi$ is a nonvoid partially ordered set with the ordinary set inclusion.

Moreover, if $\Phi$ is a nonvoid totally ordered subset of $\Psi$, then it can be easily seen that $\phi = \cup \Phi$ is an upper bound of $\Phi$ in $\Psi$. Thus, by Zorn's lemma, there exists a maximal element $f$ of $\Psi$.

Now, by Theorem 1.7, it remains only to show that the domain $D_f$ of $f$ is $X$. For this, note that if for some $a \in X$ we have $a \notin D_f$, then by Theorem 4.2 and 2.1 there exists a Hahn-Banach extension $\psi$ of $f$ to the subspace $U = \mathbb{R}\, a + D_f$. However, this contradicts the maximality of $f$.

**Remark 4.5** Note that if $f$ is as in the above theorem, then by Theorem 2.1 $f$ is, in particular, a Hahn-Banach extension of $\varphi$ to $X$.

Now, as a useful consequence of our former results, we can briefly prove the following

**Theorem 4.6** *The following assertions are equivalent:*

(1) $q$ *is odd* $X$*;*

(2) $q$ *is a Hahn-Banach extension of* $\varphi$ *to* $X$*;*

(3) *there exists a unique Hahn-Banach extension* $f$ *of* $\varphi$ *to* $X$*;*

(4) *there exists at most one Hahn-Banach extension* $f$ *of* $\varphi$ *to* $X$*.*

**Proof:** By Corollary 1.9, it is clear that (1) implies (4). Moreover, from Theorems 4.4 and 2.1, we can see that there exists a Hahn-Banach extension $f$ of $\varphi$ to $X$. Therefore, (4) implies (3). Moreover, if (1) holds, then by Corollary 1.8 we necessarily have $f = q$. Therefore, (1) also implies (2).

Now, since the implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (4) trivially hold, we need only show that (4) also implies (1). For this, note that if (1) does not hold, then there exists $a \in X$ such that

$$q(-a) \neq -q(a).$$

Hence, by Corollary 3.2 and Theorem 2.4, we can infer that

$$a \notin V \qquad \text{and} \qquad -q(-a) < q(a).$$

Now, by Remark 4.3 and Theorem 2.1, we can construct two Hahn-Banach extensions $\psi_1$ and $\psi_2$ of $\varphi$ to $U = \mathbb{R}\, a + V$ such that $\psi_1(a) \neq \psi_2(a)$. Moreover, by Theorems 4.4 and 2.1, we can state that there exist some Hahn-Banach extensions $f_1$ and $f_2$ of $\psi_1$ and $\psi_2$ to $X$, respectively. Thus, (4) does not also hold. This proves the required implication.

**Remark 4.7** Sections 7 and 11 of [9] and [5] respectively, show that the question of the uniqueness of the Hahn–Banach extension has also been intensively studied by several authors. Moreover, some further uniqueness results can also be found on the MathSciNet. However, the above simple convolutional characterization seems to be new.

## 5 A simple illustrating example to Theorems 4.2 and 2.4

**Example 5.1** Take

$$a = (-1\,,\,1) \qquad \text{and} \qquad V = \mathbb{R}\,(1\,,\,1)\,.$$

Moreover, define

$$\varphi(\,s\,,\,s\,) = s \qquad \text{and} \qquad p\,(\,s\,,\,t\,) = \max\big\{\,|\,s\,|\,,\;|\,t\,|\,\big\}$$

for all $s\,,\,t \in \mathbb{R}\,.$

Then, it is clear that $V$ is a linear subspace of $\mathbb{R}^2$ such that $a \notin V$. Moreover, for any $(\,s\,,\,t\,) \in \mathbb{R}^2$, by taking

$$\lambda\,(\,s\,,\,t\,) = 2^{-1}(\,t - s\,) \qquad \text{and} \qquad v\,(\,s\,,\,t\,) = 2^{-1}(\,s + t\,)\,(\,1\,,\,1\,)\,,$$

we can easily check that

$$(\,s\,,\,t\,) = \lambda\,(\,s\,,\,t\,)\,a + v\,(\,s\,,\,t\,)\,.$$

Therefore, $\mathbb{R}^2 = \mathbb{R}\,a + V$. Moreover, we can also at once state that $\varphi$ is a linear function of $V$ to $\mathbb{R}$ and $p$ is a norm on $\mathbb{R}^2$ such that

$$|\varphi\,(\,s\,,\,s\,)\,| = |\,s\,| = p\,(\,s\,,\,s\,)$$

for all $s \in \mathbb{R}$. Thus, in particular, $\varphi$ is dominated by $p$ on $V$.

Therefore, by Theorem 4.2, there exists a linear function $\psi$ of $\mathbb{R}^2$ to $\mathbb{R}$ that extends $\varphi$ and satisfies

$$-q\,(\,-s\,,\,-t\,) \leq \psi\,(\,s\,,\,t\,) \leq q\,(\,s\,,\,t\,)$$

for all $s\,,\,t \in \mathbb{R}$, with $q = p * \varphi$. Moreover, by the proof Theorem 4.2, we can take

$$\psi\,(\,s\,,\,t\,) = \lambda\,(\,s\,,\,t\,)\,q(a) + \varphi\,\big(\,v\,(\,s\,,\,t\,)\big) = 2^{-1}(\,t - s\,)\,q\,(a) + 2^{-1}(\,s + t\,)$$

for all $s\,,\,t \in \mathbb{R}\,.$

Now, by drawing pictures of the functions involved, we can also easily see that

$$\begin{aligned} q(a) &= \inf_{v \in V}\big(\,p(\,a - v\,) + \varphi(v)\big) = \inf_{s \in \mathbb{R}}\big(\,p((-1\,,\,1\,) - (\,s\,,\,s\,)) + \varphi\,(\,s\,,\,s\,)\big) \\ &= \inf_{s \in \mathbb{R}}\,\big(\max\big\{\,|\,1 + s\,|\,,\;|\,1 - s\,|\,\big\} + s\,\big) = \inf_{s \in \mathbb{R}}\,\big(\,1 + \,|\,s\,|\, + s\,\big) = 1\,. \end{aligned}$$

Therefore,

$$\psi(s,t) = 2^{-1}(t-s) + 2^{-1}(s+t) = t$$

for all $s, t \in \mathbb{R}$.

**Remark 5.2** Quite similarly, we can also see that $q(-a) = 1$. Therefore,

$$-q(-a) = -1 < 1 = q(a).$$

Thus, the superodd function $q$ fails to be odd at the point $a$.

In this respect, it is also worth noticing that, by Corollary 3.2, $q$ is an extension of $\varphi$. Thus, it is also not even.

**Remark 5.3** Now, by using our former observations, we can also state that if $\psi$ is a Hahn-Banach extension of $\varphi$ to $\mathbb{R}^2$, then there exists $b \in [-1, 1]$ such that

$$\begin{aligned} \psi(s,t) &= \lambda(s,t)\,b + \varphi\big(v(s,t)\big) \\ &= 2^{-1}(t-s)\,b + 2^{-1}(s+t) = 2^{-1}(1-b)\,s + 2^{-1}(1+b)\,t \end{aligned}$$

for all $s, t \in \mathbb{R}$. Hence, by taking

$$c = 2^{-1}(1-b),$$

we can already infer that $0 \le c \le 1$ such that

$$\psi(s,t) = cs + (1-c)t$$

for all $s, t \in \mathbb{R}$.

Conversely, we can also note that if $\psi$ is of the above form for some $c \in [0, 1]$, then $\psi$ is a linear extension of $\varphi$ to $\mathbb{R}^2$ such that

$$\begin{aligned} \psi(s,t) \le |\psi(s,t)| &= |cs + (1-c)t| \le c|s| + (1-c)|t| \\ &\le cp(s,t) + (1-c)p(s,t) = p(s,t) \end{aligned}$$

for all $s, t \in \mathbb{R}$. Thus, we have obtained all the Hahn-Banach extensions of $\varphi$ to $\mathbb{R}^2$.

**Remark 5.4** Now, if $s, t \in \mathbb{R}$ such that $\lambda(s,t) \ge 0$, i.e, $s \le t$, then by using Theorems 3.1 and 3.6 we can also easily see that

$$\begin{aligned} q(s,t) &= q\big(\lambda(s,t)\,a + v(s,t)\big) \\ &= \lambda(s,t)\,q(a) + \varphi\big(v(s,t)\big) = 2^{-1}(t-s) + 2^{-1}(s+t) = t. \end{aligned}$$

Hence, because of the symmetry of $s$ and $t$ in the formula

$$q(s,t)=\inf_{r\in\mathbb{R}}\big(p((s,t)-(r,r))+\varphi(r,r)\big)=\inf_{r\in\mathbb{R}}\big(\max\{|s-r|,|t-r|\}+r\big),$$

we can already infer that

$$q(s,t)=\max\{s,t\}$$

for all $s,t\in\mathbb{R}$.

Thus, in particular

$$q(s,s)=\varphi(s,s)\qquad\text{and}\qquad q(s,-s)=|\varphi(s)|$$

and

$$q\big(|s|,|t|\big)=p(s,t)$$

for all $s,t\in\mathbb{R}$.

The value $q(s,t)$ can also be computed directly by observing that

$$\begin{aligned}\max\{|s-r|,|t-r|\}+r&=\big|r-2^{-1}(s+t)\big|+2^{-1}|s-t|+r\\&=\big|r-2^{-1}(s+t)\big|+r-2^{-1}(s+t)+2^{-1}(s+t)+2^{-1}|s-t|=\\&=\big|r-2^{-1}(s+t)\big|+r-2^{-1}(s+t)+\max\{s,t\}\end{aligned}$$

for all $r,s,t\in\mathbb{R}$.

## References

[1] **Banach, S. :** *Théorie des Opérations Linéaires.* Druk M. Garasińki, Warszawa 1932 (A Polish edition had appeared one year earlier)

[2] **Beg, I. :** *Fuzzy multivalued functions.* Bull. Allahabad Math. Soc. **21**, 41–104 (2006)

[3] **Bridges, D. S. :** *Foundation of Real and Abstract Analysis.* Springer, New York 1998

[4] **Burai, P.**, and **Száz, Á. :** *Relationships between homogeneity, subadditivity and convexity properties.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **16**, 77–87 (2005)

[5] **Buskes, G. :** *The Hahn-Banach Theorem surveyed.* Dissertationes Math. **327**, 1–49 (1993)

[6] **Figula, Á.**, and **Száz, Á. :** *Graphical relationships between the infimum and intersection convolutions.* Math. Pannon. **21**, 23–35 (2010)

[7] **Glavosits, T.**, and **Száz, Á. :** *A Hahn–Banach type generalization of the Hyers–Ulam theorem.* Tech. Rep., Inst. Math., Univ. Debrecen **3**, 1 – 4 (2009)

[8] **Moreau, J. J. :** *Inf-convolution, sous-additivité, convexité des fonctions numériques.* J. Math. Pures Appl. **49**, 109 – 154 (1970)

[9] **Narici, L.**, and **Beckenstein, E. :** *The Hahn-Banach theorem : life and times.* Topology Appl. **77**, 193 – 211 (1997)

[10] **Saccoman, J. J. :** *Extension theorems by Helly and Riesz revisited.* Riv. Mat. Univ. Parma **16**, 223 – 230 (1990)

[11] **Strömberg, T. :** *The operation of infimal convolution.* Dissertationes Math. **352**, 1 – 58 (1996)

[12] **Száz, Á. :** *The intersection convolution of relations and the Hahn–Banach type theorems.* Ann. Polon. Math. **69**, 235 – 249 (1998)

[13] **Száz, Á. :** *The infimal convolution can be used to derive extension theorems from the sandwich ones.* Acta Sci. Math. (Szeged), to appear

[14] **Száz, Á. :** *A reduction theorem for a generalized infimal convolution.* Tech. Rep., Inst. Math., Univ. Debrecen **11**, 1 – 4 (2009)

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