Rostock. Math. Kolloq. **65**, 15–28 (2010)

Subject Classification (AMS)
**35J08, 35J57, 35Q30**

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# Green's matrix of the Stokes system in a convex polyhedron

ABSTRACT. The paper deals with the Dirichlet problem for the stationary Stokes system in a convex three-dimensional polyhedron. The author proves Hölder estimates for the elements of Green's matrix and their derivatives.

KEY WORDS. Stokes system, Green's matrix

## 1 Introduction

The present paper is concerned with the Green matrix $G(x,\xi) = \big(G_{i,j}(x,\xi)\big)_{i,j=1}^4$ of the boundary value problem

$$-\Delta u + \nabla p = f, \quad -\nabla\cdot u = g \;\text{ in } \Omega, \tag{1}$$

$$u = 0 \;\text{ on } \partial\Omega, \tag{2}$$

where $\Omega$ is a convex polyhedron in $\mathbb{R}^3$. It is well-known that the elements of the Green matrix satisfy the estimate

$$\big|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi)\big| \le c\,|x-\xi|^{-1-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|} \tag{3}$$

for arbitrary multi-indices $\alpha$ and $\beta$ if the boundary of the domain $\Omega$ is smooth (of class $C^\infty$). For nonsmooth domains this result fails. If the domain $\Omega$ is of polyhedral type, then the derivatives of the elements of the Green matrix can be estimated by a function which depends not only on $|x-\xi|$ but also on the distances of $x$ and $\xi$ from the vertices and edges of the domain. Such estimates are given in papers of Maz'ya and Plamenevskiĭ [5], Maz'ya and Rossmann [6], Rossmann [8] (see also the monograph by Maz'ya and Rossmann [7]). Using these estimates, it was shown in [8] and [7, Section 11.5] that (3) is satisfied for $|\alpha| \le 1-\delta_{i,4}$ and $|\beta| \le 1-\delta_{j,4}$ if $\Omega$ is a convex polyhedron. The goal of the present paper is to prove that the functions $G_{i,j}(x,\xi)$ and their derivatives satisfy even a Hölder estimate

$$\frac{\left|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi) - \partial_x^\alpha \partial_\xi^\beta G_{i,j}(y,\xi)\right|}{|x-y|^\sigma} \\ \le c\left(|x-\xi|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|} + |y-\xi|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|}\right) \tag{4}$$

for $|\alpha| \le 1-\delta_{i,4}$ and $|\beta| \le 1-\delta_{j,4}$. Here $\sigma$ is a sufficiently small positive number depending on the domain $\Omega$. For $i \ne 4$, the estimate (4) was proved in [8] (see also [7, Section 11.5]). However, the proof given in [8] does not work in the case $i = 4$. We modify here the proof of the paper [8] and obtain the estimate (4) for $i = 4$. As a consequence of (4), also the estimate

$$\frac{\left|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi) - \partial_x^\alpha \partial_\eta^\beta G_{i,j}(x,\eta)\right|}{|\xi-\eta|^\sigma} \\ \le c\left(|x-\xi|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|} + |x-\eta|^{-1-\sigma-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|}\right) \tag{5}$$

holds for $|\alpha| \le 1-\delta_{i,4}$, $|\beta| \le 1-\delta_{j,4}$.

Analogous results were obtained for the Green function of the Laplace equation and some other second order equations and systems including the Lamé system. In papers by Grüter, Widman [2] and Fromm [1] it was shown that the Green function $\mathcal{G}(x,\xi)$ of the Laplace equation satisfies the estimate

$$\left|\partial_x^\alpha \partial_\xi^\beta \mathcal{G}(x,\xi)\right| \le c\,|x-\xi|^{-1-|\alpha|-|\beta|}$$

for $|\alpha|, |\beta| \le 1$ if $\Omega$ is an arbitrary (not necessarily polyhedral) convex domain. For a wider class of differential equations, we refer also to the paper by Kozlov [4]. Hölder estimates for the derivatives $\partial_x^\alpha \partial_\xi^\beta \mathcal{G}(x,\xi)$ of orders $|\alpha| \le 1$ and $|\beta| \le 1$ were proved in [2] for domains with $C^{1,\sigma}$ boundary and by Guzman, Leykekhman, Rossmann and Schatz [3] for convex domains of polyhedral type. In [7, Subsection 5.1.5], one can find these estimates for a class of second order differential equations and systems in convex polyhedral domains. Note that the Hölder estimates do not hold for general convex domains (see the counter-example in [2]).

## 2 The Green matrix for the Stokes system

Let $\Omega$ be a bounded polyhedron in $\mathbb{R}^3$, the boundary $\partial\Omega$ of which consists of the plane faces $\Gamma_j$, $j = 1,\dots,N$, the edges $M_k$, $k = 1,\dots,l$, and the vertices $x^{(1)},\dots,x^{(d)}$. Throughout this paper, we assume that $\Omega$ is convex. As is known, the boundary value problem (1), (2) is solvable in $W^{1,2}(\Omega)^3 \times L_2(\Omega)$ for arbitrary $f \in W^{-1,2}(\Omega)^3$ and $g \in L_2(\Omega)$ satisfying the condition

$$\int_\Omega g(x)\,dx = 0.$$

The solution $(u,p)$ is unique up to vectors $(0,c)$, where $c$ is a constant. Let $\phi$ be an infinitely differentiable function in $\Omega$ which vanishes in a neighborhood of the edges such that

$$\int_\Omega \phi(x)\, dx = 1.$$

The matrix

$$G(x,\xi) = \big(G_{i,j}(x,\xi)\big)_{i,j=1}^4$$

is called *Green's matrix* for the problem (1), (2) if the vector functions $\vec{G}_j = (G_{1,j}, G_{2,j}, G_{3,j})^t$ and the function $G_{4,j}$ are solutions of the problem

$$\begin{aligned}
&-\Delta_x \vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)\,(\delta_{1,j},\delta_{2,j},\delta_{3,j})^t \quad \text{ for } x,\xi\in\Omega,\\
&-\nabla_x\cdot\vec{G}_j(x,\xi) = \big(\delta(x-\xi)-\phi(x)\big)\,\delta_{4,j} \quad \text{ for } x,\xi\in\Omega,\\
&\vec{G}_j(x,\xi) = 0 \;\text{ for } x\in\partial\Omega,\; \xi\in\Omega
\end{aligned}$$

and $G_{4,j}$ satisfies the condition

$$\int_\Omega G_{4,j}(x,\xi)\,\phi(x)\,dx = 0 \text{ for } \xi\in\Omega,\; j=1,2,3,4.$$

As was shown in [5] (see also [6, Theorem 4.5]), there exists a uniquely determined Green matrix $G(x,\xi)$ such that the vector functions $x\to\zeta(x,\xi)\,\big(\vec{G}_j(x,\xi), G_{4,j}(x,\xi)\big)$ belong to the space $\overset{\circ}{W}{}^{1,2}(\Omega)^3\times L_2(\Omega)$ for each $\xi\in\Omega$ and for every infinitely differentiable function $\zeta(\cdot,\xi)$ equal to zero in a neighborhood of the point $x=\xi$. Note that

$$G_{i,j}(x,\xi) = G_{j,i}(\xi,x) \quad \text{ for } x,\xi\in\Omega,\; i,j=1,2,3,4. \tag{6}$$

**Remark 1** It is also possible (and perhaps even more natural) to define the columns $(\vec{G}_j, G_{4,j})$ of the Green matrix as the unique solutions of the problem

$$\begin{aligned}
&-\Delta_x \vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)\,(\delta_{1,j},\delta_{2,j},\delta_{3,j})^t \quad \text{ for } x,\xi\in\Omega,\\
&-\nabla_x\cdot\vec{G}_j(x,\xi) = \big(\delta(x-\xi)-(\text{mes}(\Omega))^{-1}\big)\,\delta_{4,j} \quad \text{ for } x,\xi\in\Omega,\\
&\vec{G}_j(x,\xi) = 0 \;\text{ for } x\in\partial\Omega,\; \xi\in\Omega, \quad \int_\Omega G_{4,j}(x,\xi)\,dx = 0 \text{ for } \xi\in\Omega,
\end{aligned}$$

$j=1,2,3,4$. However, then the derivatives of the vector $\vec{G}_4(\cdot,\xi)$ with respect to the variable $x$ cannot be continuous on the edges for any $\xi\in\Omega$. The reason is that from the boundary condition $\vec{G}_4(\cdot,\xi) = 0$ on $\partial\Omega$ it follows that $\nabla_x\cdot\vec{G}_4(\cdot,\xi) = 0$ on the edges $M_k$. This contradicts the equation $\nabla_x\cdot\vec{G}_4(\cdot,\xi)|_{M_k} = (\text{mes}(\Omega))^{-1}$ which follows from the Stokes system. In particular, then the functions $G_{1,4}, G_{2,4}, G_{3,4}$ cannot satisfy the Hölder estimate (4) for $|\alpha|=1,\ \beta=0$.

## 3 Point estimates for the elements of Green's matrix

For every $\nu = 1,\dots,d$, let $I_\nu$ denote the set of all indices $k$ such that the vertex $x^{(\nu)}$ is an endpoint of the edge $M_k$. Furthermore, let $\mathcal{U}_\nu$ and $\mathcal{V}_\nu$ be convex neighborhoods of the vertex $x^{(\nu)}$. We assume that

$$\mathcal{U}_\nu \subset \mathcal{V}_\nu \quad \text{and} \quad \bigcup_{\nu=1}^{d} \mathcal{U}_\nu \supset \overline{\Omega}.$$

Moreover, we suppose that there exists a positive number $\varepsilon_0$ such that

$$\operatorname{dist}(\mathcal{U}_\nu, \Omega\backslash\mathcal{V}_\nu) > \varepsilon_0 \quad \text{and} \quad \operatorname{dist}\Big(\mathcal{V}_\nu\,, \bigcup_{k\notin I_\nu} M_k\Big) > \varepsilon_0$$

for $\nu = 1,\dots,d$. In the sequel, $\Lambda_\nu$ and $\mu'$ are certain real numbers which depend on the domain $\Omega$,

$$1 < \Lambda_\nu \le 2, \quad 1 < \mu' \le 2.$$

More precisely, we define $\mu' = \min(2, \pi\theta_1^{-1},\dots,\pi\theta_l^{-1})$, where $\theta_k$ denotes the inner angle at the edge $M_k$. For every vertex $x^{(\nu)}$, we denote by $\lambda_\nu$ the greatest real number such that the strip $1 < \operatorname{Re}\lambda < \lambda_\nu$ is free of eigenvalues of the operator pencil $\mathfrak{A}_\nu(\lambda)$ introduced in [8, Section 3] (see also [7, Subsection 11.1.2]). Then $\Lambda_\nu = \min(\lambda_\nu, 2)$.

The distance of the point $x$ from the vertex $x^{(\nu)}$ is denoted by $\rho_\nu(x)$, the distance from the edge $M_k$ by $r_k(x)$. Furthermore, let

$$r(x) = \min\big(r_1(x),\dots,r_l(x)\big).$$

We will use in this paper the following estimates of Green's matrix which are proved in [7, 8]. First we consider the case, where $x$ and $\xi$ lie in the neighborhood $\mathcal{V}_\nu$ of the same vertex $x^{(\nu)}$.

**Lemma 1** 1) *Let $x,\xi \in \Omega\cap\mathcal{V}_\nu$ and $\rho_\nu(\xi) < \rho_\nu(x)/2$. Then*

$$\big|\partial_x^\alpha \partial_\xi^\beta G_{4,j}(x,\xi)\big| \le c\,\rho_\nu(x)^{-2-\Lambda_\nu-|\alpha|+\varepsilon}\,\rho_\nu(\xi)^{\Lambda_\nu-|\beta|-\varepsilon}\Big(\frac{r(x)}{\rho_\nu(x)}\Big)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}\Big(\frac{r(\xi)}{\rho_\nu(\xi)}\Big)^{\mu'-|\beta|-\varepsilon}$$

*for $j \ne 4$ and*

$$\big|\partial_x^\alpha G_{4,4}(x,\xi)\big| \le c\,\rho_\nu(x)^{-3-|\alpha|}\Big(\frac{r(x)}{\rho_\nu(x)}\Big)^{\min(0,\mu'-1-|\alpha|-\varepsilon)},$$

*where $\varepsilon$ is an arbitrarily small positive number.*

2) *Let $x,\xi \in \Omega\cap\mathcal{V}_\nu$ and $\rho_\nu(\xi) > 2\rho_\nu(x)$. Then*

$$\begin{aligned}\big|\partial_x^\alpha \partial_\xi^\beta G_{4,j}(x,\xi)\big| \;&\le\; c\,\rho_\nu(\xi)^{-2-|\alpha|-|\beta|}\Big(\frac{\rho_\nu(x)}{\rho_\nu(\xi)}\Big)^{\min(0,\Lambda_\nu-1-|\alpha|-\varepsilon)}\\ &\quad\times\Big(\frac{r(x)}{\rho_\nu(x)}\Big)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}\Big(\frac{r(\xi)}{\rho_\nu(\xi)}\Big)^{\mu'-|\beta|-\varepsilon}\end{aligned}$$

*for* $j \neq 4$ *and*

$$\left|\partial_x^\alpha G_{4,4}(x,\xi)\right| \le c\,\rho_\nu(\xi)^{-3-|\alpha|} \left(\frac{\rho_\nu(x)}{\rho_\nu(\xi)}\right)^{\min(0,\Lambda_\nu-1-|\alpha|-\varepsilon)} \left(\frac{r(x)}{\rho_\nu(x)}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}$$

**Lemma 2** *Let* $x,\xi \in \Omega\cap\mathcal{V}_\nu$ *and* $\rho_\nu(x)/2 < \rho_\nu(\xi) < 2\rho_\nu(x)$. *If* $|x-\xi| > \min(r(x),r(\xi))$, *then*

$$\left|\partial_x^\alpha\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,|x-\xi|^{-2-\delta_{j,4}-|\alpha|-|\beta|} \left(\frac{r(x)}{|x-\xi|}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}$$

*for* $|\beta| \le 1-\delta_{j,4}$. *If* $|x-\xi| < \min(r(x),r(\xi))$, *then*

$$\left|\partial_x^\alpha\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,|x-\xi|^{-2-\delta_{j,4}-|\alpha|-|\beta|}$$

*for all multi-indices* $\alpha$ *and* $\beta$.

In the next lemma, we consider the case, where $x$ and $\xi$ lie in neighborhoods of different vertices. Then by [8, Theorem 4.3], the following estimates hold.

**Lemma 3** *Suppose that* $\mu \neq \nu$, $x \in \Omega\cap\mathcal{U}_\mu$, $\xi\in\Omega\cap\mathcal{U}_\nu$, $\xi \notin \mathcal{V}_\mu$. *Then*

$$\left|\partial_x^\alpha\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\mu(x)^{\min(0,\Lambda_\mu-1-|\alpha|-\varepsilon)}\,\rho_\nu(\xi)^{\Lambda_\nu-|\beta|-\varepsilon} \left(\frac{r(x)}{\rho_\mu(x)}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}$$

*for* $j\neq 4$, $|\beta|\le 1$ *and*

$$\left|\partial_x^\alpha G_{4,4}(x,\xi)\right| \le c\,\rho_\mu(x)^{\min(0,\Lambda_\mu-1-|\alpha|-\varepsilon)} \left(\frac{r(x)}{\rho_\mu(x)}\right)^{\min(0,\mu'-1-|\alpha|-\varepsilon)}.$$

We also need some sharper estimates for the derivatives $\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)$, where $\rho=\rho(x)=|x-x^{(\nu)}|$. If we apply Lemmas 1–3, we obtain upper bounds for these derivatives, where the factors

$$\frac{r(x)}{\rho_\nu(x)} \quad\text{and}\quad \frac{r(x)}{|x-\xi|}$$

appear with the negative exponent $\mu'-2-\varepsilon$. Since the derivative $\partial_\rho$ is tangent on the faces $\Gamma_j$ adjacent to the vertex $x^{(\nu)}$, this exponent can be replaced by zero (cf. [6, Remark 4.2] and [7, Remark 10.4.6]). In particular, the following assertions hold.

**Lemma 4** 1) *Suppose that* $x,\xi\in\Omega\cap\mathcal{V}_\nu$ *and* $|\beta|\le 1-\delta_{j,4}$. *Then*

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\nu(x)^{-3-\Lambda_\nu+\varepsilon}\,\rho_\nu(\xi)^{\Lambda_\nu-|\beta|-\varepsilon} \quad \textit{for } \rho_\nu(\xi)<\rho_\nu(x)/2,\ j\neq 4,$$

$$\left|\partial_\rho G_{4,4}(x,\xi)\right| \le c\,\rho_\nu(x)^{-4} \quad \textit{for } \rho_\nu(\xi)<\rho_\nu(x)/2,$$

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\nu(x)^{\Lambda_\nu-2-\varepsilon}\,\rho_\nu(\xi)^{-1-\Lambda_\nu-\delta_{j,4}-|\beta|+\varepsilon} \quad \textit{for } \rho_\nu(\xi)>2\rho_\nu(x),$$

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,|x-\xi|^{-3-\delta_{j,4}-|\beta|} \quad \textit{for } \rho_\nu(x)/2<\rho_\nu(\xi)<2\rho_\nu(x).$$

2) *If* $\mu\neq\nu$, $x\in\Omega\cap\mathcal{U}_\mu$, $\xi\in\Omega\cap\mathcal{U}_\nu$, $\xi\notin\mathcal{V}_\mu$, *then*

$$\left|\partial_\rho\partial_\xi^\beta G_{4,j}(x,\xi)\right| \le c\,\rho_\mu(x)^{\Lambda_\mu-2-\varepsilon}$$

*for* $|\beta|\le 1-\delta_{j,4}$.

## 4 Hölder estimates

Our goal is to prove that

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,\big(|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} + |y-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}\big) \tag{7}$$

for sufficiently small $\sigma > 0$, where $c$ is a constant independent of $x$ and $\xi$.

**Lemma 5** *Let $m$ be an arbitrary positive number, and let $0 < \sigma < 1$. Then the estimate (7) is satisfied for $|\beta| \le 1 - \delta_{j,4}$, $x, y, \xi \in \Omega$, $|x-\xi| < m|x-y|$.*

**Proof:** By (3),

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)\big|}{|x-y|^\sigma} \le c\,\frac{|x-\xi|^{-2-\delta_{j,4}-|\beta|}}{|x-y|^\sigma} \le c\,m^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

for $|x-\xi| < m|x-y|$. Analogously,

$$\frac{\big|\partial_\xi^\beta G_{i,j}(x,\xi)\big|}{|x-y|^\sigma} \le c\,(m+1)^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

since $|y-\xi| < (m+1)|x-y|$. This proves the lemma. □

The last lemma allows us to restrict ourselves to the case $|x-y| < \delta|x-\xi|$, where $\delta$ is an arbitrary fixed positive number. We assume in the sequel that $\sigma$ is a positive number satisfying the inequalities

$$\sigma < \mu' - 1 \quad \text{and} \quad \sigma < \Lambda_\nu - 1 \ \text{ for } \nu = 1, \dots, d \tag{8}$$

and show that

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \tag{9}$$

for $x \ne y$, $|x-y| < \delta|x-\xi|$, $\beta| \le 1-\delta_{j,4}$. Here $\delta$ is a sufficiently small positive number. We may assume without loss of generality that $x$ and $y$ lie in the neighborhood $\mathcal{U}_1$ of the vertex $x^{(1)}$ and that $x^{(1)}$ coincides with the origin. In the subsequent three lemmas, we assume moreover that there exists an index $k \in I_1 = \{j : x^{(1)} \in \overline{M}_j\}$ such that

$$r_k(x) = \min_{j\in I_1} r_j(x) \quad \text{and} \quad r_k(y) = \min_{j\in I_1} r_j(y), \tag{10}$$

**Lemma 6** *Suppose that $\xi \in \Omega$ and that $x, y$ are points in $\Omega \cap \mathcal{U}_1$ satisfying the conditions (10) and $|x-y| < \delta|x-\xi|$, where $\delta$ is a sufficiently small positive number. Furthermore, we assume that $\sigma$ satisfies the inequalities (8) and that there exists a real number $t \in (0,1)$ such that $y - x^* = t(x - x^*)$, where $x^*$ denotes the nearest point to $x$ on the edge $M_k$. Then the estimate (9) is satisfied for $|\beta| \le 1 - \delta_{j,4}$.*

**Proof:** Obviously,

$$\begin{aligned}\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| &= \Big| \int_t^1 \frac{d}{d\tau} \partial_\xi^\beta G_{4,j}(x^* + \tau(x-x^*),\xi)\, d\tau \Big| \\ &\le r_k(x) \int_t^1 \big|(\nabla_x \partial_\xi^\beta G_{4,j})(x^*+\tau(x-x^*),\xi)\big|\, d\tau. \end{aligned} \tag{11}$$

Since $x^*$ is the nearest point to $x$ on the set $\bigcup\limits_{j\in I_1} M_j$ and the polyhedron $\Omega$ is convex, there exists a positive constant $c_0$ such that

$$c_0\,|x| < |x^*| < \big|x^* + \tau(x-x^*)\big| < |x| \quad \text{for } 0<\tau<1.$$

If $\xi \in \mathcal{V}_1$, $|\xi| < |x|/2$, $j \ne 4$ and $|\beta| \le 1$, then (11) together with Lemma 1 yields

$$\begin{aligned}&\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \\ &\quad \le c\, r_k(x)\, |x|^{-1-\Lambda_1-\mu'+2\varepsilon}\, |\xi|^{\Lambda_1-|\beta|-\varepsilon} \int_t^1 \big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\, d\tau \\ &\quad \le c'\, r_k(x)^{\mu'-1-\varepsilon}\, |x|^{-1-\mu'-|\beta|+\varepsilon}\, (1-t^{\mu'-1-\varepsilon}).\end{aligned}$$

and analogously

$$\big|G_{4,4}(x,\xi) - G_{4,4}(y,\xi)\big| \le c\, r_k(x)^{\mu'-1-\varepsilon}\, |x|^{-2-\mu'+\varepsilon}\, (1-t^{\mu'-1-\varepsilon}).$$

Suppose that $0 < \sigma \le \mu'-1-\varepsilon$. Then

$$\frac{1-t^{\mu'-1-\varepsilon}}{(1-t)^\sigma} \le \frac{1-t^{\mu'-1-\varepsilon}}{(1-t)^{\mu'-1-\varepsilon}} \le 1 \tag{12}$$

and, consequently,

$$\begin{aligned}&\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} = \frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|(1-t)r_k(x)|^\sigma} \\ &\quad \le c\, r_k(x)^{\mu'-1-\varepsilon-\sigma}\, |x|^{-1-\delta_{j,4}-\mu'-|\beta|+\varepsilon} \le c\, |x|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\, |x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}\end{aligned}$$

for $\xi \in \mathcal{V}_1$ and $|\xi| < |x|/2$, $j \ne 4$, $|\beta| \le 1-\delta_{j,4}$.

Suppose now that $\xi \in \mathcal{V}_1$ and $|\xi| > 2|x|$. Then (11) and Lemma 1 imply

$$\begin{aligned}&\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \\ &\quad \le c\, r_k(x)\, |\xi|^{-3-\delta_{j,4}-|\beta|} \Big(\frac{|x|}{|\xi|}\Big)^{\Lambda_1-2-\varepsilon} |x|^{2-\mu'+\varepsilon} \int_t^1 \big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\, d\tau \\ &\quad \le c'\, r_k(x)^{\mu'-1-\varepsilon}\, |x|^{\Lambda_1-\mu'}\, |\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}\, (1-t^{\mu'-1-\varepsilon})\end{aligned}$$

for $|\beta| \le 1-\delta_{j,4}$. If $0<\sigma<\min(\mu'-1-\varepsilon, \Lambda_1-1-\varepsilon)$, then it follows from the last inequality and (12) that

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} = \frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|(1-t)r_k(x)|^\sigma}$$
$$\le c\, r_k(x)^{\mu'-1-\varepsilon-\sigma}\, |x|^{\Lambda_1-\mu'} |\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}$$
$$\le c'\, |\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\, 2^{2+\sigma+\delta_{j,4}+|\beta|}\, |x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

for $|\beta| \le 1-\delta_{j,4}$.

We consider the case $\xi \in \mathcal{V}_1$, $\rho_\nu(x)/2 < \rho_\nu(\xi) < 2\rho_\nu(x)$. If $|x-\xi| > \min(r_k(x), r_k(\xi))$, then by (11) and Lemma 2,

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|$$
$$\le c\, r_k(x)\, |x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon} \int_t^1 \big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\, d\tau$$
$$\le c'\, r_k(x)^{\mu'-1-\varepsilon}\, |x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon}\, (1-t^{\mu'-1-\varepsilon})$$

for $|\beta| \le 1-\delta_{j,4}$. Thus for $0<\sigma\le \mu'-1-\varepsilon$, the estimate

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, r_k(x)^{\mu'-1-\varepsilon-\sigma}\, |x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon}$$
$$\le c'\, |x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

holds. If $\rho_\nu(x)/2 < \rho_\nu(\xi) < 2\rho_\nu(x)$ and $|x-\xi| < \min(r_k(x), r_k(\xi))$, then

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le |x-y|^{1-\sigma}\, \big|(\nabla_x \partial_\xi^\beta G_{4,j}(P,\xi)\big|,$$

where $P$ is a point on the line from $x$ to $y$. Therefore, by Lemma 2

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c|x-y|^{1-\sigma}\, |P-\xi|^{-3-\delta_{j,4}-|\beta|} \le c'\, |x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}.$$

Finally, we consider the case $\xi \notin \mathcal{V}_1$. In this case, we have $|x-\xi| > \varepsilon_0$, where $\varepsilon_0$ is the positive number introduced in Section 3. Furthermore, (11) and Lemma 3 imply

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big| \le c\, r_k(x)\, |x|^{\Lambda_1-\mu'} \int_t^1 \big(\tau r_k(x)\big)^{\mu'-2-\varepsilon}\, d\tau$$
$$\le c'\, r_k(x)^{\mu'-1-\varepsilon}\, |x|^{\Lambda_1-\mu'}\, (1-t^{\mu'-1-\varepsilon})$$

for $|\beta| \le 1-\delta_{j,4}$. If $\sigma \le \min(\mu'-1-\varepsilon, \Lambda_1-1-\varepsilon)$, we conclude that

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi)-\partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, r_k(x)^{\mu'-1-\varepsilon-\sigma}\, |x|^{\Lambda_1-\mu'} \le c'\, |x|^{\Lambda_1-1-\varepsilon-\sigma} \le C$$

for $|\beta| \le 1-\delta_{j,4}$. The proof of the lemma is complete. □

Next, we prove the estimate (9) for the case, where $x$ and $y$ lie in a plane perpendicular to the edge $M_k$ and have the same distance from $M_k$.

**Lemma 7** *Suppose that $x, y \in \Omega \cap \mathcal{U}_1$ and that $x^* \in M_k$ is the nearest point on the set $\bigcup_{j\in I_1} M_j$ both to $x$ and $y$. Furthermore, we assume that $r_k(x) = r_k(y)$ and $|x-y| < \delta|x-\xi|$, where $\delta$ is a sufficiently small positive number. Then the inequality (9) holds for $|\beta| \le 1-\delta_{j,4}$. Here $\sigma$ is an arbitrary positive number satisfying (8).*

**Proof:** 1) Suppose first that $\xi \in \mathcal{V}_1$ and $\rho_1(\xi) < \rho_1(x)/2$. Then by Lemma 1,

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le \big|\nabla_x \partial_\xi^\beta G_{4,j}(P,\xi)\big|\ |x-y|$$
$$\le c\,|x-y|\,\rho_1(P)^{-3-\Lambda_1+\varepsilon}\,\rho_1(\xi)^{\Lambda_1-|\beta|-\varepsilon}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-2-\varepsilon}$$

for $j \ne 4$, $|\beta| \le 1$, where $P$ is a point on the straight line between $x$ and $y$. From the inequality $|x-P| < |x-y| < \delta|x-\xi| < 3\delta\,\rho_1(x)/2$ it follows that

$$(2-3\delta)\,\rho_1(x) < 2\rho_1(P) < (2+3\delta)\,\rho_1(x).$$

Furthermore,

$$r_k(P) \ge (2\tan(\theta_k/2))^{-1}\,|x-y|, \tag{13}$$

where $\theta_k$ denotes the angle at the edge $M_k$. Thus,

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,\rho_1(x)^{-1-\Lambda_1-\mu'+2\varepsilon}\,\rho_1(\xi)^{\Lambda_1-|\beta|-\varepsilon}$$
$$\le c'\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,\rho_1(x)^{-1-\mu'-|\beta|+\varepsilon}$$

for $j \ne 4$, $|\beta| \le 1$. Setting $\varepsilon = \mu'-1-\sigma$ and using the inequality $3\rho_1(x) > 2|x-\xi|$, we obtain (9). Analogously, we obtain

$$\frac{\big|G_{4,4}(x,\xi) - G_{4,4}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,\rho_1(x)^{-2-\mu'+\varepsilon} \le c'\,|x-\xi|^{-3-\sigma}$$

for $\varepsilon = \mu'-1-\sigma$.

2) We consider the case $\xi \in \mathcal{V}_1$, $\rho_1(x)/2 < \rho_1(\xi) < 2\rho_1(x)$. There exists a point $P$ on the line between $x$ and $y$ such that

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le \big|\nabla_x \partial_\xi^\beta G_{4,j}(P,\xi)\big|\ |x-y|$$

If $|P-\xi| > \min(r_k(P), r_k(\xi))$, then Lemma 2 implies

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le c\,|x-y|\,|P-\xi|^{-3-\delta_{j,4}-|\beta|}\Big(\frac{r_k(P)}{|P-\xi|}\Big)^{\mu'-2-\varepsilon}$$

for $|\beta| \le 1-\delta_{j,4}$. Using the inequalities (13) and $|P-\xi| > |x-\xi| - |x-P| > (1-\delta)\,|x-\xi|$, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{\mu'-1-\varepsilon-\sigma}\,|x-\xi|^{-1-\mu'-\delta_{j,4}-|\beta|+\varepsilon}.$$

For $\varepsilon = \mu'-1-\sigma$, the inequality (9) holds. Analogously,

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\,|x-y|^{1-\sigma}\,|P-\xi|^{-3-\delta_{j,4}-|\beta|} \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

for $|P-\xi| < \min(r_k(P), r_k(\xi))$.

3) Suppose that $\xi \in \mathcal{V}_1$ and $\rho_1(\xi) > 2\rho_1(x)$. Then, for an arbitrary point $P$ on the line between $x$ and $y$, we have

$$\rho_1(P) < \rho_1(x) + |x-y| < (1+\delta)\rho_1(x) + \delta\rho_1(\xi) < \frac{1+3\delta}{2}\,\rho_1(\xi). \tag{14}$$

Thus by Lemma 1,

$$\begin{aligned}\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| &\le c\,|x-y|\,\rho_1(P)^{\Lambda_1-2-\varepsilon}\,\rho_1(\xi)^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-2-\varepsilon}\\ &= c\,|x-y|^\sigma\,\rho_1(P)^{\Lambda_1-1-\varepsilon-\sigma}\,\rho_1(\xi)^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}\Big(\frac{|x-y|}{r_k(P)}\Big)^{1-\sigma}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-1-\varepsilon-\sigma}\end{aligned}$$

for $|\beta| \le 1-\delta_{j,4}$. Using the inequalities (13), (14) and $r_k(P) < \rho_1(P)$, we obtain

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le c\,|x-y|^\sigma\,\rho_1(\xi)^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\,|x-y|^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

if $\varepsilon < \min(\Lambda_1-1-\sigma, \mu'-1-\varepsilon)$.

4) Finally, we consider the case $\xi \in \Omega\backslash\mathcal{V}_1$. Then by Lemma 3,

$$\begin{aligned}\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| &\le c\,|x-y|\,\rho_1(P)^{\Lambda_1-2-\varepsilon}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-2-\varepsilon}\\ &= c\,|x-y|^\sigma\,\rho_1(P)^{\Lambda_1-1-\varepsilon-\sigma}\Big(\frac{|x-y|}{r_k(P)}\Big)^{1-\sigma}\Big(\frac{r_k(P)}{\rho_1(P)}\Big)^{\mu'-1-\varepsilon-\sigma},\end{aligned}$$

where again $P$ is a point on the line from $x$ to $y$. Since all factors on the right-hand side have an upper bound independent of $x, y$ and $\xi$ if $\varepsilon < \min(\Lambda_1-1-\sigma, \mu'-1-\varepsilon)$, we get

$$\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big| \le c\,|x-y|^\sigma \le c'\,|x-y|^\sigma\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}.$$

The proof of the lemma is complete. □

In the next lemma, we assume that $x$ and $y$ lie on the same ray starting from the vertex $x^{(1)}$.

**Lemma 8** *Suppose that $x, y$ are points in $\Omega \cap \mathcal{U}_1$ satisfying the condition (10). If $x^{(1)}$ is the origin, $y = sx$ and $|x-y| < \delta|x-\xi|$, then (9) holds for $|\beta| \le 1-\delta_{j,4}$, where $c$ is independent of $x$ and $\xi$.*

**Proof:** Suppose first that $\xi \in \Omega \cap \mathcal{V}_1$. If $|\xi| < 2|x|$, then

$$|(s-1)x| = |x-y| < \delta\, |x-\xi| < 3\delta|x|$$

and, consequently, $|s-1| < 3\delta$. Let $\rho = |x|$. Using the equality $x \cdot \nabla_x = \rho\partial_\rho$, we obtain

$$\begin{aligned} &\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} = \frac{1}{|(s-1)x|^\sigma} \Big| \int_s^1 \frac{d}{d\tau}\, \partial_\xi^\beta G_{4,j}(\tau x,\xi)\, d\tau\Big| \\ &= \frac{1}{|(s-1)x|^\sigma} \Big| \int_s^1 x \cdot \big(\nabla_x\, \partial_\xi^\beta G_{4,j}\big)(\tau x,\xi)\, d\tau\Big| \\ &= \frac{1}{|(s-1)x|^\sigma} \Big| \int_s^1 \tau^{-1}\, (\rho\partial_\rho\, \partial_\xi^\beta G_{4,j})(\tau x,\xi)\, d\tau\Big|. \end{aligned} \tag{15}$$

We consider the case $|\xi| < |x|/2$. Then Lemma 4 yields

$$\begin{aligned} &\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} = |s-1|^{1-\sigma}\, |x|^{-\sigma}\, \tau^{-1}\, (\rho\partial_\rho\, \partial_\xi^\beta G_{4,j})(\tau x,\xi) \\ &\le c\, |s-1|^{1-\sigma}\, |x|^{-\sigma}\, \tau^{-1}\, |\tau x|^{-2-\Lambda_1+\varepsilon}\, |\xi|^{\Lambda_1-|\beta|-\varepsilon} \end{aligned}$$

for $j \ne 4$, $|\beta| \le 1$, where $\tau$ is a real number between $s$ and 1. Since $|\tau - 1| < |s-1| < 3\delta$ and $|x-\xi| < 3|x|/2$, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, |x|^{-2-\sigma-|\beta|} \le c'\, |x-\xi|^{-2-\sigma-|\beta|}$$

for $j \ne 4$, $|\beta| \le 1$. Furthermore,

$$\frac{\big|G_{4,4}(x,\xi) - G_{4,4}(y,\xi)\big|}{|x-y|^\sigma} \le c\, |s-1|^{1-\sigma}\, |x|^{-\sigma}\, \tau^{-1}\, |\tau x|^{-3} \le c'\, |x|^{-3-\sigma} \le c''\, |x-\xi|^{-3-\sigma}.$$

Analogously, in the case $\xi \in \mathcal{V}_1$, $|x|/2 < |\xi| < 2|x|$, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} \le c\, |(s-1)x|^{1-\sigma}\, |\tau x - \xi|^{-3-\delta_{j,4}-|\beta|},$$

where again $\tau$ is a number between $s$ and 1. Using the inequalities $|(s-1)x| = |x-y| < \delta|x-\xi|$ and

$$|\tau x - \xi| \ge |x-\xi| - |(\tau-1)x| \ge |x-\xi| - |(s-1)x| > (1-\delta)\, |x-\xi|,$$

we get (9). Now let $\xi \in \Omega \cap \mathcal{V}_1$, $|\xi| > 2|x|$. Then, by Lemma 4, (12) and (15), we have

$$\begin{aligned} \frac{\left|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\right|}{|x-y|^\sigma} &\leq \frac{c}{|(s-1)x|^\sigma} \left| \int_s^1 \tau^{-1} \, |\tau x|^{\Lambda_1 - 1 - \varepsilon} \, |\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon} \, d\tau \right| \\ &\leq c' \, \frac{|s^{\Lambda_1-1-\varepsilon} - 1|}{|s-1|^\sigma} \, |x|^{\Lambda_1-1-\varepsilon-\sigma} \, |\xi|^{-1-\Lambda_1-\delta_{j,4}-|\beta|+\varepsilon}. \end{aligned}$$

Setting $\varepsilon = \Lambda_1 - 1 - \sigma$ and using the inequality $|s^\sigma - 1| \leq |s-1|^\sigma$ for $s > 0$, we obtain

$$\frac{\left|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\right|}{|x-y|^\sigma} \leq c \, |\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \leq c' \, |x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}.$$

It remains to consider the case $\xi \in \Omega \backslash \mathcal{V}_1$. Then Lemma 4, (12) and (15) imply

$$\begin{aligned} &\frac{\left|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\right|}{|x-y|^\sigma} \leq \frac{c}{|(s-1)x|^\sigma} \left| \int_s^1 \tau^{-1} \, |\tau x|^{\Lambda_1-1-\varepsilon} \, d\tau \right| \\ &\leq c' \, |x|^{\Lambda_1-1-\varepsilon-\sigma} \, \frac{|s^{\Lambda_1-1-\varepsilon} - 1|}{|s-1|^\sigma} \, . \end{aligned}$$

Setting $\varepsilon = \Lambda_1 - 1 - \sigma$ and using the inequality $|x - \xi| > \varepsilon_0$, we obtain (9). The proof of the lemma is complete. □

Now we can prove the main result of the paper

**Theorem 9** *Suppose that $\sigma$ is a positive number satisfying the condition* (8). *Then the elements $G_{4,j}(x,\xi)$ of Green's matrix satisfy the inequality* (7) *for $|\beta| \leq 1 - \delta_{j,4}$.*

**Proof:** For $|x-\xi| < m|x-y|$ the inequality (7) is already shown (see Lemma 5). We consider the case $|x-y| < \delta|x-\xi|$, where $\delta$ is a given sufficiently small positive number. Since then $|x-y| < \delta \, \mathrm{diam}(\Omega)$, we may assume in this case that $x$ and $y$ lie in the neighborhood $\mathcal{U}_1$ of the same vertex $x^{(1)}$ and that this vertex coincides with the origin. Let $I_1$ be the set of all indices $j$ such that $x^{(1)}$ is an endpoint of the edge $M_j$. Suppose first that that there exists an index $k \in I_1$ such that

$$r_k(x) = \min_{j \in I_1} r_j(x) \quad \text{and} \quad r_k(y) = \min_{j \in I_1} r_j(y). \tag{16}$$

By $x^*$ and $y^*$ we denote the nearest points to $x$ and $y$ on the edge $M_k$. Without loss of generality, we may assume that

$$\frac{r_k(x)}{|x|} > \frac{r_k(y)}{|y|}. \tag{17}$$

We define

$$s = \left( \frac{|x|^2 - r_k(x)^2}{|y|^2 - r_k(y)^2} \right)^{1/2}, \quad t = s \, \frac{r_k(y)}{r_k(x)} \quad \text{and} \quad z = x^* + t(x - x^*).$$

Then $x^*$ is also the nearest point to $sy$ on $M_k$. From (17) it follows that $t<1$. Furthermore, there exists a constant $c_0$ depending only on the domain $\Omega$ such that

$$|x-sy| < c_0\,|x-y|. \tag{18}$$

To see this, we consider the line $\ell_y$ through the origin and the point $y$. Then $|x-y| > \text{dist}(x,\ell_y)$, while $|x-sy|$ is the distance of $x$ from the intersection of $\ell_y$ with the plane perpendicular to $M_k$ through the point $x^*$. Since $M_k$ is the nearest edge to $y$ and $\Omega$ is convex, the angle between $\ell_y$ and the last plane is greater than a certain angle $\alpha_0>0$. Thus, $|x-sy| < (\sin\alpha_0)^{-1}\,\text{dist}(x,\ell_y)$ which proves (18). Since $sy$ and $z$ have the same distance $t\,r_k(x) = s\,r_k(y)$ from the point $x^*$ and $z$ lies on the straight line from $x$ to $x^*$, it follows that

$$|x-z| \le |x-sy| < c_0\,|x-y| \quad \text{and} \quad |z-sy| < 2c_0\,|x-y|.$$

Moreover,

$$|y-sy| \le |x-sy| + |x-y| < (c_0+1)\,|x-y|.$$

We assume in the following that $c_0\delta$ is sufficiently small. Applying Lemma 6, we obtain

$$\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(z,\xi)\big|}{|x-y|^\sigma} \le c\,\frac{\big|\partial_\xi^\beta G_{4,j}(x,\xi) - \partial_\xi^\beta G_{4,j}(z,\xi)\big|}{|x-z|^\sigma} \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}$$

for $|\beta|\le 1-\delta_{j,4}$. Analogously, Lemmas 7 and 8 imply

$$\begin{aligned}\frac{\big|\partial_\xi^\beta G_{4,j}(z,\xi) - \partial_\xi^\beta G_{4,j}(sy,\xi)\big|}{|x-y|^\sigma} &\le c\,|z-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|},\\ \frac{\big|\partial_\xi^\beta G_{4,j}(sy,\xi) - \partial_\xi^\beta G_{4,j}(y,\xi)\big|}{|x-y|^\sigma} &\le c\,|y-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|} \le c'\,|x-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}\end{aligned}$$

for $|\beta|\le 1-\delta_{j,4}$. This proves the inequality (9) for the case where the nearest points to $x$ and $y$ on the set $\bigcup_{j\in I_1} M_j$ lie on the same edge $M_k$. If

$$r_k(x) = \min_{j\in I_1} r_j(x) \quad \text{and} \quad r_l(y) = \min_{j\in I_1} r_j(y), \tag{19}$$

where $k,l\in I_1$ and $k\ne l$, then one can find a set of points $z_1,\ldots,z_k$ on the straight line from $x=z_1$ to $y=z_k$, where for every pair $(i,i+1)$ there exits an index $n(i)\in I_1$ such that

$$r_{n(i)}(z_i) = \min_{j\in I_1} r_j(z_i) \quad \text{and} \quad r_{n(i)}(z_{i+1}) = \min_{j\in I_1} r_j(z_{i+1}) \quad \text{for } i=1,\ldots,k-1.$$

Obviously $(1-\delta)|x-\xi| < |z_i-\xi| < (1+\delta)|x-\xi|$ if $|x-y|<\delta|x-\xi|$. Thus, the inequalities

$$\frac{\big|\partial_\xi^\beta G_{4,j}(z_i,\xi) - \partial_\xi^\beta G_{4,j}(z_{i+1},\xi)\big|}{|z_i-z_{i+1}|^\sigma} \le c\,|z_i-\xi|^{-2-\sigma-\delta_{j,4}-|\beta|}, \quad i=1,\ldots,k-1,$$

imply (9). The proof of the theorem is complete. □

Using the analogous result for the elements $G_{i,j}(x,\xi)$, $i\ne 4$, in [8], we conclude that the estimate (4) is valid for $i,j=1,2,3,4$, $|\alpha|\le 1-\delta_{i,4}$, $|\beta|\le 1-\delta_{j,4}$. The estimate (5) can be deduced directly from (4) and (6).

## References

[1] **Fromm, S. J. :** *Potential space estimates for Green potentials in convex domains.* Proc. Amer. Math. Soc. **119**, 1, 225 – 233 (1993)

[2] **Grüter, M.**, and **Widman, K.-O. :** *The Green function for uniformly elliptic equations.* Manuscripta Math. **37**, 303 – 342 (1982)

[3] **Guzman, J., Leykekhman, D., Rossmann, J.**, and **Schatz, A. H. :** *Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods.* Numerische Mathematik **112**, 221 – 243 (2009)

[4] **Kozlov, V. A. :** *Behavior of solutions to the Dirichlet problem for elliptic systems in convex domains.* Comm. Partial Differential Equations **34**, 24 – 51 (2009)

[5] **Maz'ya, V. G.**, and **Plamenevskiĭ, B. A. :** *The first boundary value problem for classical equations of mathematical physics in domains with piecewise smooth boundaries.* Part 1: Z. Anal. Anwendungen **2**, 4, 335 – 359 (1983) Part 2: Z. Anal. Anwendungen **2**, 6, 523-551 (1983) (in Russian)

[6] **Maz'ya, V. G.**, and **Rossmann, J. :** *Pointwise estimates for Green's kernel of a mixed boundary value problem to the Stokes system in a polyhedral cone.* Math. Nachr. **278**, 15, 1766 – 1810 (2005)

[7] **Maz'ya, V. G**, and **Rossmann, J :** *Elliptic equations in polyhedral domains.* Amer. Math. Soc., Providence, Rhode Island, to appear

[8] **Roßmann, J. :** *Hölder estimates for Green's matrix of the Stokes system in convex polyhedra.* Around the Research of Vladimir Maz'ya II. Partial Differential Equations, pp. 315 – 336, Springer 2010

**received:** Datum 2010

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