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Comparison with ground state for solutions of non cooperative systems of Schrödinger operators on $\mathbb{R}^N$

ABSTRACT. We study the sign of solutions of a system $\mathcal{L}U = \lambda U + MU + F$, on the whole space $\mathbb{R}^N$, more precisely, we compare the components of U with the ground state solution. Here $\mathcal{L}$ is a diagonal matrix of Schrödinger operators of the form $Lu := -\Delta u + qu$, F is a vector of functions in $L^2(\mathbb{R}^N)$, and M is a matrix, not necessarily cooperative. When M is a constant matrix, we prove the existence of a real Λ playing the role of principal eigenvalue: if $|\lambda - \Lambda|$ is sufficiently small, U exists and the sign of each entry is fixed. The sign of each entry changes as λ grows and get over Λ. We study the case of a variable M for a 2×2 system.

1 Introduction

In this paper we study systems defined on the whole space $\mathbb{R}^N$ and acting on $(L^2(\mathbb{R}^N))^n$:

$$Lu_i := (-\Delta + q(x))u_i = \lambda u_i + \sum_{j=1}^{n} m_{ij}u_j + f_i,\ 1 \le i \le n \tag{1}$$

which we write:

$$\mathcal{L}U = \lambda U + MU + F, \tag{2}$$

with $U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} L & & 0 \\ & \ddots & \\ 0 & & L \end{pmatrix}$, and M is a $n \times n$ matrix with coefficients m_{ij}.

The potential $q(x)$ is assumed to be a continuous function $q\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$\inf_{\mathbb{R}^N} q > 0 \quad \text{and} \quad q(x) \to +\infty \text{ as } |x| \to \infty. \tag{3}$$

The potential is a "relatively small" perturbation of a radially symmetric potential which is assumed to be monotone increasing (in the radial variable) and growing somewhat faster than $|x|^2$ as $|x| \to \infty$.

For a unique equation

$$(-\Delta + q(x))\, u = \lambda u + f \text{ on } \mathbb{R}^N,$$

where q is a perturbation of a radially symmetric function, under the hypothesis $f \geq 0$, B. Alziary, J. Fleckinger, and P. Takáč consider the eigenvalue λ^*, associated to a function $\varphi^* > 0$. They show that for $|\lambda - \lambda^*|$ sufficiently small, if $\lambda < \lambda^*$ then $u > C\varphi^* > 0$ (fundamental positivity), and if $\lambda > \lambda^*$, and f comparable to φ^*, then $u < -C\varphi^* < 0$ (fundamental negativity).

First we are concerned with the anti-maximum principle for the system when M is a constant matrix. In the case of cooperative systems, there are several results related to the maximum principle. B. Alziary L. Cardoulis, and J. Fleckinger, obtained a maximum principle for cooperative systems, then B. Alziary, J. Fleckinger, and P. Takáč, proved a result of fundamental positivity. For the anti-maximum principle N. Besbas [10, Theorem 4.3.2, p. 40] gave a theorem on the fundamental negativity for a special cooperative problem involving a radial potential q. In the present work, we study general systems (in particular non cooperative systems are allowed) and we obtain a comparison with the ground state, for the spectral parameter λ close to the ground state energy level. In this part, we extend to a $n \times n$ system some results of fundamental positivity or negativity established by B. Alziary,J. Fleckinger and MH. Lécureux [3] for 2×2 systems.

In the second part, we tackle the case of a variable matrix M. Our result concerns 2×2 systems with M restricted to very specific forms.

Organization:

The paper is organized as follows. In Section 2, we introduce some notation. In Section 3 we recall some known results, in Section 4 we state our main results. Finally, in Section 5, we prove them.

2 Notations and hypotheses

2.1 Fundamental positivity, fundamental negativity, notation

It is established that the Schrödinger operator: $L_q \stackrel{\text{def}}{=} -\Delta + q(x)\bullet$ defined on $L^2(\mathbb{R}^N)$ with a positive continuous potential tending to $+\infty$ as $|x| \to \infty$ has a compact inverse and therefore a discrete spectrum. This holds since the variational space V_q is compactly embedded in $L^2(\mathbb{R}^N)$ (see D. E. Edmunds and W. D. Evans, [14], J. Fleckinger,[16]) where

$$V_q(\mathbb{R}^N) \stackrel{\text{def}}{=} \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q(x)|u|^2 < \infty \right\}. \tag{4}$$

The smallest eigenvalue is simple and is given by:

$$\lambda^*(q) = \inf_{u \in V_q(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q(x)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} \right\}. \tag{5}$$

Eigenfunctions associated to $\lambda^*(q)$ do not change sign and $\lambda^*(q)$ is referred to as the **"principal eigenvalue"**. Denote by φ^* (or $\varphi^*(q)$) the associated eigenfunction which is positive and normalized by $\|\varphi^*\|^2_{L^2(\mathbb{R}^N)} = 1$. The function φ^* is $C^1(\mathbb{R}^N)$, and exponentially decreasing near infinity. Usually, φ^* is called the **"ground state"** or **"principal eigenfunction"**.

As in the paper of B. Alziary and P. Takáč [8], we consider the operator $L_q \stackrel{\text{def}}{=} -\Delta + q(x)\bullet$ on a subspace X of $L^2(\mathbb{R}^N)$ defined, by

$$X \stackrel{\text{def}}{=} \{u \in L^2(\mathbb{R}^N) \colon u/\varphi^* \in L^\infty(\mathbb{R}^N)\}. \tag{6}$$

The space X equipped with the norm

$$\|u\|_X \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\mathbb{R}^N} \, (|u|/\varphi^*)$$

is a Banach space.

Notation: We note $u \stackrel{*}{\succ} 0$ and we say that $u \in X$ is fundamentally positive if there exists a real number $c > 0$ such that $u > c\varphi^*$.
Similarly we write $u \stackrel{*}{\prec} 0$ and we say that $u \in X$ is fundamentally negative if there exists a real number $c > 0$ such that $u < -c\varphi^*$.

2.2 Hypotheses on potential

Now we give the precise assumptions on the potential q, which guarantee the compactness of the resolvant $(\lambda I - L)^{-1}$. For a single equation, Alziary, Fleckinger, and Takáč obtain this compactness and so the fundamental positivity and negativity for different classes of potentials [6], [9]. We choose here hypotheses used in [9], but there is no problem for obtaining the same results with the class of potential used in [6].

More precisely, we introduce a class of growth for potentials:

$$\mathcal{C}_Q := \{Q \in \mathcal{C}(\mathbb{R}_+, (0,\infty)) \,/\, \exists r_0 > 0,\ Q' > 0\, a.e. \text{ on } [r_0,\infty),\ \int_{r_0}^{\infty} Q(r)^{-1/2}\, \mathrm{d}r < \infty\}. \tag{7}$$

We assume that the potential q satisfies Hypothesis (H_q):

Hypothesis (H_q) *The potential q is positive continuous and tends to $+\infty$ as $|x| \to \infty$. Moreover, there exist two functions Q_1 and Q_2 in $\mathcal{C}_Q$ and two positive constants $C_0, r_0 \in (0, \infty)$, such that*

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_0\, Q_1(|x|) \quad \text{for all } x \in \mathbb{R}^N\,, \tag{8}$$

$$\int_{r_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{r_0}^{s} \exp\Big(- \int_{r}^{s} [Q_1(t)^{1/2} + Q_2(t)^{1/2}]\, \mathrm{d}t \Big)\, \mathrm{d}r\, \mathrm{d}s < \infty\,. \tag{9}$$

In their paper, Alziary, and Takáč ([9] Corollary 3.3) show that the ground states $\varphi^*(q)$, $\varphi^*(Q_1)$ and $\varphi^*(Q_2)$ are comparable: there exist some constants $0 < \gamma_1 \leq \gamma_2 < \infty$ such that $\gamma_1 \varphi^*(q) \leq \varphi^*(Q_j) \leq \gamma_2 \varphi^*(q)$ with $j = 1, 2$. We have $X_q = X_{Q1} = X_{Q2}$.

Remark 2.1 The set X does not change if we change q into $q - \widetilde{q}$ where $\widetilde{q}$ is a bounded function such that $q - \widetilde{q} \geq 0$.

2.3 Hypotheses on matrix M and vector F

2.3.1 Case of constant matrix M

◇ **Hypothesis on** M

In this case, we suppose the whole spectrum of M real. More precisely:

Hypothesis (H_M): *The whole spectrum of Matrix M is in $\mathbb{R}$. We denote the p real eigenvalues $(\mu_i)_{1 \leq i \leq p}$ of matrix M, by*

$$\mu_1 > \mu_2 \geq \ldots \geq \mu_p.$$

We assume that the largest eigenvalue μ_1 of M is algebraically and geometrically simple.

Remark 2.2 We choose to write eigenvalues μ_i in decreasing order. The Jordan's canonical form allows us to write $M = PTP^{-1}$ with :

$$T = \left(\begin{array}{c|c|c|c} J_1 & & 0 & \\ \hline & J_2 & 0 & \\ \hline & 0 & \ddots & \\ \hline & & & J_p \end{array} \right)$$

where P is a change-of-basis matrix.

Every Jordan's block J_i is a square $k_i \times k_i$ matrix, in the form :

$$J_i = \begin{pmatrix} \mu_i & 1 & 0 & \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & \mu_i \end{pmatrix}$$

By Hypothesis (H_M), the first block is 1×1 : $J_1 = (\mu_1)$.

Notation: Let G be the eigenspace associated with μ_1 ($\dim G = 1$) and H the hyperplan spanned by other column vectors of Matrix P. By hypothesis (H_M), we have $\mathbb{R}^n = G \oplus H$. It is important to notice that, in matrix P, we can choose for the first column, every non null vector of G.

$\diamond$ **Hypothesis on** F

We recall that in the whole space, the anti-maximum principle could be violated for the equation

$$-\Delta u + q(x)u = \lambda u + f$$

if the function f is in $L^2(\mathbb{R}^N) \setminus X$ (cf. [5, Example 4.1, pp. 377–379]). So the fundamental negativity does not hold for $0 \leq f \not\equiv 0$. For results on systems presented in this article, of course we need to consider vector F with all the components f_k in X.

We can decompose $F(x)$ into $F(x) = F_G(x) + F_H(x)$ with $F_G(x) \in G$ and $F_H(x) \in H$.

Hypothesis (H_F): *All components f_i of vector F are in X and let us decompose $F(x) = F_G(x) + F_H(x)$ where $F_G(x) \in G$ and $F_H(x) \in H$. We assume there exists $\Psi \in G$ such that $F(x) = \widetilde{f}_1(x)\Psi + F_H(x)$ with $\widetilde{f}_1 \geq 0$ (a.e.), and $F_G = \widetilde{f}_1\Psi \not\equiv 0$.*

Vector Ψ is in G so we have : $M\Psi = \mu_1\Psi$. Its components ψ_i are constant real numbers. In Matrix $P = (p_{ij})$ we choose Ψ for the first column. So $\psi_i = p_{i1}$.

2.3.2 Case of variable M

In this case, M is a 2×2 matrix. We note $M = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$.

Assumptions on Matrix M allow us to diagonalise this matrix with the help of a change-of-basis matrix with real and constant coefficients. These very particular forms of matrix are studied by Cosner and Schaefer [13]. If $a \not\equiv d$, we need b and c proportional to $a - d$; if $a \equiv d$, b is proportional to c and have the same sign. In the first case, where $a \not\equiv d$, we need to have a constant sign for $a - d$. In the second case, we suppose $a \equiv d$.

Hypothesis (H_{Mv1}) *(case $a \not\equiv d, a \geq d$): We assume*:

- $\diamond$ *Functions a and d are continuous, in $L^\infty(\mathbb{R}^N)$, and $a \geq d \geq 0$ with $a \not\equiv d$.*
- $\diamond$ *There exist two real numbers $\widehat{b}$ and $\widehat{c}$ such that $b = \widehat{b}(a-d)$ and $c = \widehat{c}(a-d)$, and $\widehat{D} = 1 + 4\widehat{b}\,\widehat{c} > 0$.*

Note that with hypotheses $a(x) \geq 0$ and $d(x) \geq 0$ we do not loose generality: we can add a positive number to each side to obtain these hypotheses.

In this case, we always use Hypothesis (H_F), but we can write it differently.

Hypothesis (H_{Fv1}) *(case $a \not\equiv d, a \geq d$): We assume $f_1,\ f_2 \in X$,*

$$\widetilde{f_1} = f_1 + \frac{2\widehat{b}}{1+\sqrt{\widehat{D}}} f_2 \geq 0 \quad \textit{and} \quad \widetilde{f_1} \not\equiv 0.$$

Hypothesis (H_{Mv2}) *(case $a \equiv d$): We assume*:

- ◇ *The equality $a = d$ and this function is in $L^\infty(\mathbb{R}^N)$. Moreover $\forall x \in \mathbb{R}^N$, $a(x) \geq 0$.*
- ◇ *There exist two positive real numbers $\widehat{b}$ and $\widehat{c}$ such that $b = \epsilon\widehat{b}r$ and $c = \epsilon\widehat{c}r$, where ϵ is ± 1 and $r \in L^\infty(\mathbb{R}^N)$ is a bounded, positive and continuous function.*

Hypothesis (H_F) can now be written:

Hypothesis (H_{Fv2}) *(case $a \equiv d$): We assume $f_1,\ f_2 \in X$,*

$$\sqrt{\widehat{c}} f_1 + \epsilon\sqrt{\widehat{b}} f_2 \geq 0 \textit{ and } \sqrt{\widehat{c}} f_1 + \epsilon\sqrt{\widehat{b}} f_2 \not\equiv 0.$$

Remark 2.3 Under Hypotheses (H_{Mv1}) or (H_{Mv2}), M has two real eigenvalues. We denote them by $\nu^+(x) \geq \nu^-(x)$. The two functions ν^+ and ν^- are in $L^\infty(\mathbb{R}^N)$.

3 Known Results

We recall here some results of fundamental positivity and fundamental negativity.

Our proof uses some results in Alziary, Takáč, ([8]) then Alziary, Fleckinger, Takáč,([5]) and Alziary, Takáč, ([9]) for fundamental positivity, in Besbas, ([10]) for fundamental negativity.

For q with superquadratical growth and for $f/\varphi^*(q) \in L^\infty$, they study

$$(-\Delta + q)u = \lambda u + f \tag{10}$$

and they show that there exist positive numbers c and δ (depending on q, f and λ) such that:

$$\lambda < \lambda^*(q) \;\Rightarrow\; u \overset{*}{\succ} 0, \text{ (fundamental positivity)}$$

$$\lambda^*(q) < \lambda < \lambda^*(q) + \delta \;\Rightarrow\; u \overset{*}{\prec} 0, \text{ (fundamental negativity).}$$

Fundamental Positivity

Theorem 3.1 ([8, Theorem 2.1, p. 284])([9, Theorem 3.1, p. 41])
Assume (H_q) *is satisfied and* $f \in L^2(\mathbb{R}^N)$*,* $f \geq 0$ *a.e. on* $\mathbb{R}^N$*,* $f \not\equiv 0$*. For* $\lambda < \lambda^*(q)$ *there exists a unique solution* u *to Equation* (10) *which is positive; and there exists a constant* $c > 0$ *such that*

$$u > c\varphi^*(q) > 0 \quad \textit{(fundamental positivity)}. \tag{11}$$

Moreover, if also $f \leq C\varphi^*(q)$*, with some constant* $C > 0$*, then we have*

$$u \leq c'\varphi^*(q) \quad \textit{everywhere, with } c' = \frac{C}{\lambda^*(q) - \lambda}. \tag{12}$$

Corollary 3.2 ([9]): *The constant* c *defined in* (11) *tends to* ∞ *as* $\lambda \to \lambda^*(q)$ *.*

This result plays an important role in the proof of our main Theorems:

Corollary 3.3 *Assume* $f \in X$ *(not necessarily* $f \geq 0$*), for* $\lambda < \lambda^*(q)$*,* u *exists and we have*

$$|u| \leq \frac{\|f\|_X}{\lambda^*(q) - \lambda}\varphi^*(q). \tag{13}$$

Indeed if we denote by $\mathcal{K}|_X$ the restriction of $\mathcal{K} = (L_q - \lambda I)^{-1}$ to the Banach space X, the operator $\mathcal{K}|_X$ is linear and bounded in X with norm $\leq \dfrac{1}{\lambda^*(q) - \lambda}$ ([9], p. 41).

Fundamental Negativity

It has been shown first in [1] for a radial potential and then in [9].

Theorem 3.4 ([9, Theorem 3.4, p. 42]) *Assume* (H_q) *is satisfied; let* $f \in X$ *be such that* $f \geq 0$ *a.e. on* $\mathbb{R}^N$*,* $f \not\equiv 0$*. Then there exists* $\delta(f) > 0$ *and* $c > 0$ *such that for all* $\lambda \in (\lambda^*(q); \lambda^*(q) + \delta)$*,*

$$u \leq -c\varphi^*(q) \quad \textit{(fundamental negativity)}. \tag{14}$$

Remark 3.5 The same holds if we assume only $\int_{\mathbb{R}^N} f\varphi^*(q)\, dx > 0$.

Corollary 3.6 ([10]): *The constant* c *defined in* (14) *tends to* ∞ *as* $\lambda \to \lambda^*(q)$ *.*

Remark 3.7 Besbas ([10]) uses a slightly different space $X^{1,2} \subset X$; it coincides with X for radially symmetric functions.

Remark 3.8 Fundamental negativity improves the antimaximum principle introduced in Clément-Pelletier ([12]).

4 Main Results

4.1 System $n \times n$

This result concerns System (2) where M is a constant matrix:

$$(2) \quad \mathcal{L}U := \begin{pmatrix} (-\Delta + q(x)) & & 0 \\ & \ddots & \\ 0 & & (-\Delta + q(x)) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \lambda U + MU + F,$$

Recall that, by hypothesis (H_q), (H_M) and (H_F), M has only real eigenvalues; its largest eigenvalue μ_1 is simple and there exists Ψ eigenvector of M associated with μ_1, such that $F(x) = \widetilde{f}_1(x)\Psi + F_H(x)$ with $\widetilde{f}_1 \geq 0$ (a.e.) Denote (ψ_i) the components of Ψ.

Theorem 4.1 *We assume Hypotheses (H_q), (H_M) and (H_F).*
Let $\Lambda := \lambda^*(q) - \mu_1$. Then there exist two real numbers $\delta > 0$ and $\delta' > 0$, depending on q, M, F, such that

- *If $\lambda \in (\Lambda - \delta; \Lambda)$ then System (2) admits a unique solution $U = (u_i)$. Moreover, for each integer $i \in [1, n]$, $u_i \in X$ and $\psi_i u_i \overset{*}{\succ} 0$.*
- *If $\lambda \in (\Lambda; \Lambda + \delta')$ then System (2) admits a unique solution $U = (u_i)$. Moreover, for each integer $i \in [1, n]$ $u_i \in X$ and $\psi_i u_i \overset{*}{\prec} 0$.*

Remark 4.2 If M is irreducible and cooperative, we know that there exists Ψ with all components strictly positive. We obtain the fundamental positivity below Λ and the fundamental negativity above Λ.

4.2 Variable Matrix M

Here M is a variable 2×2 matrix $M = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$.

The system is:

$$\begin{pmatrix} -\Delta + q(x) & 0 \\ 0 & -\Delta + q(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{15}$$

As we will see in the proof, the two real eigenvalues of M are $\nu^+(x) \geq \nu^-(x)$, and the functions ν^+ and ν^- are continuous, bounded. Let $\nu^+_{max} = \sup \{\nu^+(x),\, x \in \mathbb{R}\}$.
By Remark 2.1, we know that X is the same set for q, for $q^+ = q - \nu^+ + \nu^+_{max}$ and for $q^- = q - \nu^- + \nu^+_{max}$. We denote $\lambda^*(q^+)$ the principal eigenvalue of $-\Delta + q^+$ and $\lambda^*(q^-)$ the principal eigenvalue of $-\Delta + q^-$.

1. First case

Under Hypothesis 2.3.2 (H_{Mv1}), let us set $\widehat{b}$, $\widehat{c}$ the two real numbers such that $b = \widehat{b}(a-d)$ and $c = \widehat{c}(a-d)$.

Theorem 4.3 *(case $a \not\equiv d$) We assume Hypotheses (H_q), (H_{Mv1}) and (H_{Fv1}):*

$$f_1 + \frac{2\widehat{b}}{1+\sqrt{\widehat{D}}} f_2 \geq 0 \ \ a.e., \ \ f_1 + \frac{2\widehat{b}}{1+\sqrt{\widehat{D}}} f_2 \not\equiv 0$$

Let $\Lambda = \lambda^*(q^+) - \nu^+_{max}$. Then there exist two real numbers $\delta > 0$ and $\delta' > 0$, depending on q, M, F, such that

- *If $\Lambda - \delta < \lambda < \Lambda$, then System (15) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\succ} 0 \ \ and \ \ \ \widehat{c}u_2 \overset{*}{\succ} 0.$$

- *If $\Lambda < \lambda < \Lambda + \delta'$, then System (15) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\prec} 0 \ \ and \ \ \ \widehat{c}u_2 \overset{*}{\prec} 0.$$

Under Hypothesis 2.3.2 (H_{Mv2}), recall that functions $b = \epsilon\widehat{b}r$ and $c = \epsilon\widehat{c}r$ have the same sign, given by $\epsilon = \pm 1$.

Theorem 4.4 *(case $a \equiv d$) We assume Hypotheses (H_q), (H_{Mv2}) and (H_{Fv2}):*

$$\sqrt{\widehat{b}} f_1 + \epsilon\sqrt{\widehat{c}} f_2 \geq 0 \ \ a.e., \ \ \sqrt{\widehat{b}} f_1 + \epsilon\sqrt{\widehat{c}} f_2 \not\equiv 0.$$

Let $\Lambda = \lambda^*(q^+)$. Then there exist two real numbers $\delta > 0$ and $\delta' > 0$, depending on q, M, F, such that

- *If $\Lambda - \delta < \lambda < \Lambda$, then System (15) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\succ} 0 \ and \ \epsilon u_2 \overset{*}{\succ} 0.$$

- *If $\Lambda < \lambda < \Lambda + \delta'$, then System (15) admits a unique solution $U = (u_i)$. Moreover,*

$$u_1 \overset{*}{\prec} 0 \ and \ \epsilon u_2 \overset{*}{\prec} 0.$$

5 Proofs

5.1 Proof of Theorem 4.1

1/ First case: $\lambda < \Lambda = \lambda^*(q) - \mu_1$

First step: change of basis

We use the Jordan's block matrix $T = \left(\begin{array}{c|c|c|c} J_1 & & 0 & \\ \hline & J_2 & 0 & \\ \hline & 0 & \ddots & \\ \hline & & & J_p \end{array}\right)$ associated with matrix M in System (2):

$$\mathcal{L}U := \lambda U + MU + F.$$

There is a matrix P such that $T = P^{-1}MP$. More precisely, by Hypothesis (H_M) and Hypothesis (H_F) we can choose for the first column of change-of-basis matrix P : $\Psi \in G$ such that $F = \widetilde{f_1}\Psi + F_H$ with $\widetilde{f_1} \geq 0$ and $F_H(x) \in H$.
Now let us introduce the following notation:

$$U = P\widetilde{U} \quad \Leftrightarrow \quad \widetilde{U} = \begin{pmatrix} \widetilde{u_1} \\ \vdots \\ \widetilde{u_n} \end{pmatrix} = P^{-1}U \quad \text{and} \quad F = P\widetilde{F} \quad \Leftrightarrow \quad \widetilde{F} = \begin{pmatrix} \widetilde{f_1} \\ \vdots \\ \widetilde{f_n} \end{pmatrix} = P^{-1}F.$$

All potentials are equal, so System (2) becomes

$$\mathcal{L}\widetilde{U} = \lambda\widetilde{U} + T\widetilde{U} + \widetilde{F}. \tag{16}$$

By Hypothesis (H_M) the first equation in System (16) is

$$L\widetilde{u_1} = \lambda\widetilde{u_1} + \mu_1\widetilde{u_1} + \widetilde{f_1}, \tag{17}$$

where, by Hypothesis (H_F), $\widetilde{f_1} \geq 0$ and $\widetilde{f_1} \not\equiv 0$.

Look at the Jordan's block J_i with $2 \leq i \leq p$. The matrix J_i is $k_i \times k_i$. Set $s_i = \sum_{m=1}^{i-1} k_m$ with $k_1 = 1$.
From line $s_i + 1$ to line $s_i + k_i - 1$, we obtain $k_i - 1$ equations:

$$L\widetilde{u_j} = \lambda\widetilde{u_j} + \mu_i\widetilde{u_j} + \widetilde{u}_{j+1} + \widetilde{f_j} \qquad \text{if } s_i + 1 \leq j < s_i + k_i - 1, \tag{18}$$

and the last one:

$$L\widetilde{u_j} = \lambda\widetilde{u_j} + \mu_i\widetilde{u_j} + \widetilde{f_j} \qquad \text{for } j = s_i + k_i = s_{i+1}. \tag{19}$$

Second step: study of the triangular system (16)

In the first line
Using Theorem 3.1, we obtain that $L\widetilde{u_1} = \lambda\widetilde{u_1} + \mu_1\widetilde{u_1} + \widetilde{f_1}$ has a solution, $u_1 \overset{*}{\succ} 0$ (fundamental positivity), and since $\widetilde{f_1} \geq 0$ a.e. on $\mathbb{R}^N$,

$$c(\lambda)\varphi^* \leq \widetilde{u_1}.$$

If $\lambda \to \Lambda$, by Corollary (3.2) $c(\lambda) \to +\infty$.

In other lines we look at every Jordan's block.
In i^{th} block, with $2 \leq i \leq p$, from line $s_i + 1$ to line s_{i+1}.
• Line s_{i+1}: In Equation (19) $L\widetilde{u}_{s_{i+1}} = \lambda\widetilde{u}_{s_{i+1}} + \mu_i\widetilde{u}_{s_{i+1}} + \widetilde{f}_{s_{i+1}}$ by Corollary 3.3 the solution $\widetilde{u}_{s_{i+1}}$ exists and satisfies the inequality

$$|\widetilde{u}_{s_{i+1}}| \leq \frac{\|\widetilde{f}_{s_{i+1}}\|_X}{\lambda^*(q) - \mu_i - \lambda}\varphi^*. \tag{20}$$

By (H_M), $\lambda < \lambda^*(q) - \mu_1 < \lambda^*(q) - \mu_i$. So $|\widetilde{u}_{s_{i+1}}| \leq \dfrac{\|\widetilde{f}_{s_{i+1}}\|_X}{\mu_1 - \mu_i}\varphi^*$.
Hence, for $i > 1$, the function $\widetilde{u}_{s_{i+1}}$ is in X, and $\|\widetilde{u}_{s_{i+1}}\|_X \leq c_{s_{i+1}}$ where the constant $c_{s_{i+1}}$ depends only on F and M.
• From line $s_i + 1$ to line $s_{i+1} - 1$
For $j = s_{i+1} - 1$, we have $L\widetilde{u_j} = \lambda\widetilde{u_j} + \mu_i\widetilde{u_j} + \widetilde{u}_{s_{i+1}} + \widetilde{f_j}$.
Set $\widetilde{g_j} = \widetilde{u}_{s_{i+1}} + \widetilde{f}_{s_{i+1}}$. This function $\widetilde{g_j}$ is in X, and $\|\widetilde{g_j}\|_X \leq l_j$ where the constant l_j depends only on F and M.
Therefore, by Corollary 3.3 we obtain the existence of $\widetilde{u_j}$ and

$$|\widetilde{u_j}| \leq \frac{\|\widetilde{g_j}\|_X}{\lambda^*(q) - \mu_i - \lambda}\varphi^* \leq \frac{l_j}{\mu_1 - \mu_i}\varphi^*.$$

So, for $j = s_{i+1} - 1$, $\widetilde{u_j} \in X$, and $\|\widetilde{u_j}\|_X \leq c_j$ where c_j depends only on F and M.
Step by step, we can use the same argument from line $s_{i+1} - 1$ to line $s_i + 1$. Therefore we obtain, in each block, for each integer j with $s_i + 1 \leq j \leq s_{i+1} - 1$, the existence of the solution $\widetilde{u_j}$ which is in X. Moreover, $\|\widetilde{u_j}\|_X \leq c_j$ where the real c_j depends only on F and M.

To sum up, we have, for $2 \leq j \leq n$,

$$|\widetilde{u_j}| \leq c_j\varphi^*, \tag{21}$$

where the real c_j depends only on F and M,
and for $j = 1$,

$$c(\lambda)\varphi^* \le \widetilde{u_1}, \tag{22}$$

where $c(\lambda)$ depends on F, M, λ and $c(\lambda) \nearrow +\infty$ when $\lambda \nearrow \Lambda$.

Third step: consequence for the initial system (2)

$U = P\widetilde{U}$ implies for each component $1 \le i \le n$:

$$u_i = p_{i1}\widetilde{u_1} + \sum_{j=2}^{n} p_{ij}\widetilde{u_j}.$$

As $\lambda \to \Lambda$, we have $\widetilde{u_1} \ge c(\lambda)\varphi^*(q)$, where $c(\lambda)$ tends to infinity; and by (21), $\sum_{j=2}^{n} p_{ij}\widetilde{u_j}$ is bounded by a constant times φ^*.

Therefore there exists $\delta_i > 0$ such that for $\lambda \in (\Lambda - \delta_i; \Lambda)$ the function

$$u_i = p_{i1}\widetilde{u_1} + \sum_{j=2}^{n} p_{ij}\widetilde{u_j}$$

has the same sign than p_{i1}. More precisely, if $p_{i1} > 0,\ u_i \overset{*}{\succ} 0$, and if $p_{i1} < 0\ u_i \overset{*}{\prec} 0$.

But the first eigenvector Ψ is the first column of the change-of-basis matrix $P : \psi_i = p_{i1}$ We obtain, in the case $\Lambda - \delta \le \lambda < \Lambda$, where $\delta = \min_i \delta_i$,

$$\psi_i u_i \overset{*}{\succ} 0 \text{ (fundamentally positive)}$$

2/ **Second case** $\lambda > \Lambda = \lambda^* - \mu_1$ and $|\lambda - \Lambda|$ small:
there is $\delta_0 > 0$ with $\Lambda < \lambda < \Lambda + \delta_0 < \lambda^* - \mu_2 \le \ldots \le \lambda^* - \mu_n$.

First step

We transform System (2) into System (16) exactly as above.

Second step: study of the triangular system (16)

In the first line (17) $\quad L\widetilde{u_1} = \lambda\widetilde{u_1} + \mu_1\widetilde{u_1} + \widetilde{f_1}$,
we can apply the fundamental negativity results (Theorem 3.4): there is $\delta_1(F) > 0$ such that if $\Lambda < \lambda < \Lambda + \delta_1 < \Lambda + \delta_0$, then $\widetilde{u_1} \le -c(\lambda)\varphi^*(q)$, and by Corollary 3.6: $c(\lambda)$ grows to $+\infty$ when $\lambda \to \Lambda$.

In the other equations, $L\widetilde{u_i} = \lambda\widetilde{u_i} + \mu_k\widetilde{u_i} + \widetilde{f_i}$ we have $\lambda < \lambda^* - \mu_i$. Hence by fundamental positivity and corollary 3.2, as in the case $\lambda < \Lambda$, we have by (21), $\sum_{j=2}^{n} p_{ij}\widetilde{u_j}$ bounded by a constant times φ^*.

Third step: consequence for the initial system (2)

In $u_i = p_{i1}\widetilde{u_1} + \sum_{j=2}^{n} p_{ij}\widetilde{u_j}$, we have $\sum_{j=2}^{n} p_{ij}\widetilde{u_j}$ bounded by a constant times φ^* and $\widetilde{u_1} < -c(\lambda)\varphi^*(q)$ tending to $-\infty$ when λ tends to Λ.
So there is $\delta' > 0$ such that : if $\Lambda < \lambda < \Lambda + \delta'$ we obtain $p_{j1}u_j = \psi_j u_j$ fundamentally negative: $\psi_j u_j \overset{*}{\prec} 0$. $\square$

5.2 Proof of Theorems 4.3 and 4.4

Here we study System (15):

$$\begin{pmatrix} -\Delta + q(x) & 0 \\ 0 & -\Delta + q(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

1/ Proof of Theorem 4.3

First step : study of eigenvalues

By Hypothesis (H_{Mv1}), there exist two real numbers $\widehat{b}$, $\widehat{c}$ such that $b = \widehat{b}(a-d)$ and $c = \widehat{c}(a-d)$. Since $a \geq d$, the two functions b and c never change sign. Moreover $\widehat{D} = 1 + 4\widehat{b}\,\widehat{c}$ is positive.

By calculation we obtain two eigenvalues : $\nu^+(x) = \frac{1}{2}\Big(a(x) + d(x) + (a(x) - d(x))\sqrt{\widehat{D}}\Big)$, and $\nu^-(x) = \frac{1}{2}\Big(a(x) + d(x) - (a(x) - d(x))\sqrt{\widehat{D}}\Big)$.

Since $a \geq d$, $a \not\equiv d$ and $\widehat{D} > 0$, we have $\nu^+ \geq \nu^-$, $\nu^+ \not\equiv \nu^-$. By (H_{Mv1}), the two functions a and d are continuous and bounded, so ν^+ and ν^- are continuous and bounded.
Set $\nu^+_{max} = \sup_x \nu^+(x)$.
By Remark 2.1, the set X is the same for the two potentials $q^+ = q + \nu^+_{max} - \nu^+$ and $q^- = q + \nu^+_{max} - \nu^-$. We have $q^- \geq q^+ > 0$, with $q^- \not\equiv q^+$.

The principal eigenvalue of $L_{q^-} \overset{\text{def}}{=} -\Delta + q^-(x)\bullet$ is

$$\lambda^*(q^-) = \inf_{u \in V_{q^-}(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q^-(x)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} \right\}$$

and we know that

$$\lambda^*(q^-) = \int_{\mathbb{R}^N} |\nabla\varphi^*(q^-)|^2 dx + \int_{\mathbb{R}^N} q^- |\varphi^*(q^-)|^2 dx,$$

where $\varphi^*(q^-)$ is the ground state of $-\Delta + q^-(x)\bullet$, which is positive and normalized by $\|\varphi^*(q^-)\|^2_{L^2(\mathbb{R}^N)} = 1$.
By $\nu^-(x) \le \nu^+(x)$, $\nu^- \not\equiv \nu^+$, and by continuity we have

$$\int_{\mathbb{R}^N} \left(\nu^+_{max} - \nu^-(x)\right) |\varphi^*(q^-(x))|^2 dx > \int_{\mathbb{R}^N} \left(\nu^+_{max} - \nu^+(x)\right) |\varphi^*(q^-(x))|^2 dx,$$

so

$$\int_{\mathbb{R}^N} q^-(x)|\varphi^*(q^-)|^2 dx > \int_{\mathbb{R}^N} q^+(x)|\varphi^*(q^-)|^2 dx.$$

Therefore

$$\lambda^*(q^-) > \int_{\mathbb{R}^N} |\nabla\varphi^*(q^-)|^2 dx + \int_{\mathbb{R}^N} q^+(x)|\varphi^*(q^-)|^2 dx.$$

We obtain $\varphi^*(q^-) \in V_{q^+}$ and

$$\lambda^*(q^-) > \inf_{u \in V_{q^+}(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} q^+(x)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} \right\} = \lambda^*(q^+).$$

Second step: diagonalization of the system (15)

We choose the eigenvectors $v^+ = \begin{pmatrix} \dfrac{1+\sqrt{\widehat{D}}}{2} \\ \widehat{c} \end{pmatrix}$ associated with ν^+ and $v^- = \begin{pmatrix} -\widehat{b} \\ \dfrac{1+\sqrt{\widehat{D}}}{2} \end{pmatrix}$ associated with ν^-.

Let P the matrix with columns vectors v^+ and v^-. The inverse matrix is

$$P^{-1} = \frac{1}{\sqrt{\widehat{D}}} \begin{pmatrix} 1 & \dfrac{2\widehat{b}}{1+\sqrt{\widehat{D}}} \\ \dfrac{-\,2\widehat{c}}{1+\sqrt{\widehat{D}}} & 1 \end{pmatrix}.$$

As before, we note: $\widetilde{U} = \begin{pmatrix} \widetilde{u_1} \\ \widetilde{u_2} \end{pmatrix} = P^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\begin{pmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{pmatrix} = P^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.
The components of P and P^{-1} are constants. So, if $f_1,\ f_2 \in X$, then $\widetilde{f_1}$ and $\widetilde{f_2}$ are also in X.

By this change of basis, System (15)

$$\mathcal{L}U = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

is written in two equations:

$$-\Delta\widetilde{u_1} + q\widetilde{u_1} = \lambda\widetilde{u_1} + \nu^+\widetilde{u_1} + \widetilde{f_1},$$

$$-\Delta\widetilde{u_2} + q\widetilde{u_2} = \lambda\widetilde{u_2} + \nu^-\widetilde{u_2} + \widetilde{f_2}$$

where $\widetilde{f_1} \geq 0$, $\widetilde{f_1} \not\equiv 0$ by Hypothesis $(H_{Fv1}.)$

Set $q^+ = q + \nu^+_{max} - \nu^+$, and $q^- = q + \nu^+_{max} - \nu^-$, we derive

$$-\Delta\widetilde{u_1} + q^+\widetilde{u_1} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_1} + \widetilde{f_1}, \tag{23}$$

$$-\Delta\widetilde{u_2} + q^-\widetilde{u_2} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_2} + \widetilde{f_2}. \tag{24}$$

If $\lambda < \lambda^*(q^-) - \nu^+_{max}$, Equation (24) satisfies the Theorem of Fundamental Positivity, and by Corollary 3.3 we have

$$|\widetilde{u_2}| \leq \left(\lambda^*(q^-) - \nu^+_{max} - \lambda\right)^{-1} C_{\widetilde{f_2}}\varphi^*$$

• If $\lambda < \lambda^*(q^+) - \nu^+_{max} < \lambda^*(q^-) - \nu^+_{max}$,
we obtain

$$|\widetilde{u_2}| \leq \left(\lambda^*(q^-) - \lambda - \nu^+_{max}\right)^{-1} C_{\widetilde{f_2}}\varphi^* \leq \frac{C_{\widetilde{f_2}}}{\lambda^*(q^-) - \lambda^*(q^+)}\varphi^*.$$

Equation (23) satisfies the fundamental positivity result, so we have

$$\widetilde{u_1} \geq C(\lambda, \widetilde{f_1})\varphi^*$$

and $C(\lambda, \widetilde{f_1})$ tends to infinity, when λ tends to $\lambda^*(q^+) - \nu^+_{max}$. Consequently $\widetilde{u_2}$ is bounded, and $\widetilde{u_1}$ tends to infinity.
Now we can derive U from $U = P\widetilde{U}$; we have:

$$u_1 = \frac{1 + \sqrt{\widehat{D}}}{2}\widetilde{u_1} - \widehat{b}\widetilde{u_2}, \tag{25}$$

$$u_2 = \widehat{c}\widetilde{u_1} + \frac{1 + \sqrt{\widehat{D}}}{2}\widetilde{u_2}. \tag{26}$$

So there exists a real number $\delta > 0$, depending on F and M, such that for all
$\lambda^*(q^+) - \nu^+_{max} - \delta < \lambda < \lambda^*(q^+) - \nu^+_{max}$,

$$u_1 \overset{*}{\succ} 0 \text{ and } \widehat{c}u_2 \overset{*}{\succ} 0.$$

• If $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^-) - \nu^+_{max}$

By Theorem 3.4 in Equation (23) there exists δ_1 (depending on F) such that for all λ with $\lambda^*(q^+) < \lambda + \nu^+_{max} < \lambda^*(q^+) + \delta_1$, $\widetilde{u_1}$ exists and $\widetilde{u_1} \overset{*}{\prec} 0$. We can choose $\delta_1 < \lambda^*(q^-) - \lambda^*(q^+)$, and assume $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^+) - \nu^+_{max} + \delta_1 < \lambda^*(q^-) - \nu^+_{max}$.

In Equation 24, by $\lambda < \lambda^*(q^-) - \nu^+_{max}$ we can apply the Fundamental Positivity Result. So $\widetilde{u_2}$ exists, and

$$|\widetilde{u_2}| \leq \left(\lambda^*(q^-) - \lambda - \nu^+_{max}\right)^{-1} C_{\widetilde{f_2}} \varphi^* \leq \frac{1}{\lambda^*(q^-) - \lambda^*(q^+) - \delta_1} \varphi^*.$$

We have $\widetilde{u_2}$ bounded by a constant times φ^*, and $\widetilde{u_1} \leq -C(\lambda, \widetilde{f_1})\varphi^*$, with $C(\lambda, \widetilde{f_1})$ tending to infinity when λ tends to $\lambda^*(q^+) - \nu^+_{max}$.

Relations (25) and (26) are always true. So there exists a real $0 < \delta \leq \delta_1$ such that: if $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^+) - \nu^+_{max} + \delta < \lambda^*(q^-) - \nu^+_{max}$, we have $u_1 \overset{*}{\prec} 0$ and $\widehat{c} u_2 \overset{*}{\prec} 0$. $\square$

2/ Proof of Theorem 4.4

By Hypothesis (H_{Mv2}), $a = d$ and there exist two real numbers $\widehat{b}$, $\widehat{c}$ such that $b = \epsilon \widehat{b} r$ and $c = \epsilon \widehat{c} r$, with $\epsilon = \pm 1$. The function $r \in L^\infty(\mathbb{R}^N)$ is continuous, positive and bounded.

The matrix $M(x)$ has two eigenvalues, $\nu^+(x) = a(x) + \sqrt{\widehat{b}\,\widehat{c}}\, r(x)$ and $\nu^-(x) = a(x) - \sqrt{\widehat{b}\,\widehat{c}}\, r(x)$. The function r is positive, bounded and continuous so the function $\nu^+ - \nu^- = 2\sqrt{\widehat{b}\,\widehat{c}}\, r(x)$ is positive, bounded and continuous. Let $q^+ = q + \nu^+_{max} - \nu^+$ and $q^- = q + \nu^+_{max} - \nu^-$. We have, as in the first step of the proof of Theorem 4.3, $\lambda(q^-) > \lambda(q^+)$.

Eigenvectors associated to ν^+ and ν^- are $v^+ = \begin{pmatrix} \sqrt{\widehat{b}} \\ \epsilon\sqrt{\widehat{c}} \end{pmatrix}$ and $v^- = \begin{pmatrix} -\epsilon\sqrt{\widehat{b}} \\ \sqrt{\widehat{c}} \end{pmatrix}$.

With these eigenvectors, we obtain

$$P^{-1} = \begin{pmatrix} \dfrac{1}{2\sqrt{\widehat{b}}} & \dfrac{\epsilon}{2\sqrt{\widehat{c}}} \\ \dfrac{-\epsilon}{2\sqrt{\widehat{b}}} & \dfrac{1}{2\sqrt{\widehat{c}}} \end{pmatrix}.$$

The components of P and P^{-1} are constants.

We always denote $\begin{pmatrix} \widetilde{u_1} \\ \widetilde{u_2} \end{pmatrix} = P^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\begin{pmatrix} \widetilde{f_1} \\ \widetilde{f_2} \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. Functions $\widetilde{f_1}$ and $\widetilde{f_2}$ are in X, and by Hypothesis (H_{Fv2}), $\widetilde{f_1} \geq 0$, and $\widetilde{f_1} \not\equiv 0$. We obtain the same equations as above:

$$(23) \quad -\Delta\widetilde{u_1} + q^+\widetilde{u_1} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_1} + \widetilde{f_1},$$

$$(24) \quad -\Delta\widetilde{u_2} + q^-\widetilde{u_2} = \left(\lambda + \nu^+_{max}\right)\widetilde{u_2} + \widetilde{f_2},$$

where $\widetilde{f_1} \geq 0$, $\widetilde{f_1} \not\equiv 0$ by Hypothesis (H_{Fv1}).

The study of the comparison with the ground state is the same as in Theorem 4.3. So $\widetilde{u_2}$ is still bounded in X. For $\widetilde{u_1}$:
• if $\lambda < \lambda^*(q^+) - \nu^+_{max}$, then $\widetilde{u_1} \geq C(\lambda, F)\varphi^*$, where $C(\lambda, F) \to \infty$ when $\lambda \to \lambda^*(q^+) - \nu^+_{max}$,
• if $\lambda > \lambda^*(q^+) - \nu^+_{max}$ and $|\lambda - (\lambda^*(q^+) - \nu^+_{max})|$ small, we have $\widetilde{u_1} \leq -C(\lambda, F)\varphi^*$, where $C(\lambda, F) \to \infty$ when $\lambda \to \lambda^*(q^+) - \nu^+_{max}$.

But now the change of basis gives:

$$u_1 = \sqrt{\widehat{b}}\,\widetilde{u_1} - \epsilon\sqrt{\widehat{b}}\,\widetilde{u_2}, \tag{27}$$

$$u_2 = \epsilon\sqrt{\widehat{c}}\,\widetilde{u_1} + \sqrt{\widehat{c}}\,\widetilde{u_2}. \tag{28}$$

By similar arguments, we obtain
- the existence of δ such that: if $\lambda^*(q^+) - \nu^+_{max} - \delta < \lambda < \lambda^*(q^+) - \nu^+_{max} < \lambda^*(q^-) - \nu^+_{max}$, then $u_1 \overset{*}{\succ} 0$ and $\epsilon u_2 \overset{*}{\succ} 0$,
- the existence of δ' such that: if $\lambda^*(q^+) - \nu^+_{max} < \lambda < \lambda^*(q^+) - \nu^+_{max} + \delta' < \lambda^*(q^-) - \nu^+_{max}$, then $u_1 \overset{*}{\prec} 0$ and $\epsilon u_2 \overset{*}{\prec} 0$. ■

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