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On the improper derivatives of Takagi's continuous nowhere differentiable function

ABSTRACT. This note is a completion of [5] where it was investigated among other things the improper derivatives of Takagi's continuous nowhere differentiable function T. We determine all points x for which T has the one-sided improper derivatives $T'_+(x) = \infty$ and $T'_-(x) = \infty$.

KEY WORDS. Takagi's continuous nowhere differentiable function, improper derivatives

1 Introduction

In 1903, T. Takagi [6] discovered an example of a continuous, nowhere differentiable function that was simpler than a well-known example of K. Weierstrass. Takagi's function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\Delta\,(2^n x)}{2^n} \qquad (x \in \mathbb{R}) \tag{1.1}$$

where $\Delta(y) = \operatorname{dist}(y, \mathbb{Z})$ is a periodic function with period 1. The Takagi function was rediscovered independently by others, e.g. Knopp in 1918, Van der Waerden in 1930 and Hildebrandt in 1933, cf. [3].

It is known that T does not have a finite one-sided derivative anywhere. But at each dyadic rational point $x = \frac{m}{2^n}$ there exist the right-hand improper derivative

$$T'_+(x) = \lim_{h \to +0} \frac{T(x+h) - T(x)}{h} = +\infty$$

and left-hand improper derivative

$$T'_-(x) = \lim_{h \to -0} \frac{T(x+h) - T(x)}{h} = -\infty,$$

cf. [5]. Begle and Ayres [2] have investigated non-dyadic points $x \neq \frac{m}{2^n}$ for which the Takagi function (with the notation Hildebrandt function) does have an improper derivative $T'(x) = +\infty$ or $T'(x) = -\infty$. For given x let I_n and O_n represent the number of $1'$s and $0'$s among the first n terms in the dyadic expansion of x, and $D_n = O_n - I_n$. The claim of Begle and Ayres reads: If $\lim D_n = +\infty$ then $T'(x) = +\infty$ and if $\lim D_n = -\infty$ then $T'(x) = -\infty$. But this cannot be true since in [5] is a counterexample, cf. Example 7.2.

The purpose of this paper is to determine all non-dyadic points $x \neq \frac{m}{2^n}$ for which the improper derivatives do exist. We consider the right-hand and left-hand improper derivatives separately. In view of the symmetry $T(1-x) = T(x)$ it holds $T'_+(x) = \pm\infty$ if and only if $T'_-(1-x) = \mp\infty$. Therefore we only investigate the case $+\infty$. The main results of this note is that for non-dyadic x with the representation

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{a_n}} \tag{1.2}$$

where $1 \leq a_1 < a_2 < \ldots$ are integers, we have:

(i) $\quad T'_+(x) = \infty \quad \Longleftrightarrow \quad D_n \to \infty \quad (n \to \infty)$

and

(ii) $\quad T'_-(x) = \infty \quad \Longleftrightarrow \quad 2^{D_{a_n}} \frac{d_n}{2^{d_n}} \to \infty \quad (n \to \infty)$

where $d_n = a_{n+1} - a_n$, (Proposition 3.1, Proposition 4.5 and Remark 4.6).
Since $d_n 2^{-d_n}$ is bounded and $D_n \to \infty$ implies $D_{a_n} \to \infty$, from (i) and (ii) it follows

(iii) $\quad T'(x) = \infty \quad \Longleftrightarrow \quad 2^{D_{a_n}} \frac{d_n}{2^{d_n}} \to \infty \quad (n \to \infty)$.

Remark 1.1 It is remarkable that if $T'_-(x) = \infty$ then also $T'_+(x) = \infty$ but not conversely. In Example 7.2 from [5] it was considered a point (1.2) where $a_{n+1} \geq 4a_n$. Here $T'_+(x) = \infty$ since $D_n \to \infty$, but in [5] it was shown that $T'_-(x) = \infty$ does not be valid. Hence, the condition in (ii) cannot be satisfied.

Remark 1.2 The condition in (iii) is satisfied if and only if $D_{a_n} \to \infty$ and if e.g. d_n is bounded, but the condition also may be satisfied if $d_n \to \infty$.

Example 1.3 Take the point (1.2) with $a_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. Then $d_n = n + 1$, $D_{a_n} = a_n - 2n = \frac{n(n-3)}{2}$ and

$$2^{D_{a_n}} \frac{d_n}{2^{d_n}} = 2^{n(n-3)/2} \frac{n+1}{2^{n+1}} = 2^{(n^2-5n-2)/2}(n+1) \to \infty$$

as $n \to \infty$. So by (iii) we have $T'(x) = \infty$.

Remark 1.4 Let us mention that in (iii) the term D_{a_n} cannot be replaced by D_n. This shows the Example 1.3 since in view of $d_{a_n} = \frac{n(n+1)}{2} + 1 = \frac{n^2+n+2}{2}$ we have for $k = a_n$

$$2^{D_k} \frac{d_k}{2^{d_k}} = 2^{n(n-3)/2} \frac{\frac{n^2+n+2}{2}}{2^{(n^2+n+2)/2}} = \frac{n^2+n+2}{2^{2n+2}} \to 0$$

though $T'(x) = \infty$.

2 Relations for Takagi's function

In order to determine the improper derivatives we need some relations for the Takagi function. It is known that T satisfies for $0 \le x \le 1$ the following system of functional equations

$$T\left(\frac{x}{2}\right) = \frac{x}{2} + \frac{1}{2}T(x), \qquad T\left(\frac{1+x}{2}\right) = \frac{1-x}{2} + \frac{1}{2}T(x), \tag{2.1}$$

cf. e.g. [4], [5]. Moreover, for $\ell \in \mathbb{N}$, $k = 0, 1, \dots, 2^\ell - 1$ and $x \in [0,1]$, the Takagi function T satisfies the equations

$$T\left(\frac{k+x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell}x + \frac{1}{2^\ell}T(x) \tag{2.2}$$

and

$$T\left(\frac{k-x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{2s(k-1) - \ell}{2^\ell}x + \frac{1}{2^\ell}T(x) \tag{2.3}$$

where $s(k)$ denotes the binary sum-of-digit function which is the number of ones in the binary representation of k, cf. [5, Proposition 2.1].

Note that for given x with the dyadic expansion

$$x = 0, \xi_1, \xi_2 \dots \tag{2.4}$$

we have for the difference $D_n = O_n - I_n$ of the number of $0'$s and $1'$ in the first n terms of (2.4)

$$D_n = \sum_{\nu=1}^{n} (-1)^{\xi_\nu}.$$

Besides of (2.4) we consider $y = 0, \eta_1\eta_2 \dots$ with $\eta_n \in \{0, 1\}$. It is known that if x and y are different points in $[0,1]$ with $\xi_\nu = \eta_\nu$ for $\nu \le n \in \mathbb{N}$ then

$$\frac{T(x) - T(y)}{x - y} = D_n + \frac{T(x_n) - T(y_n)}{x_n - y_n}, \tag{2.5}$$

where $x_n = 0, \xi_{n+1}\xi_{n+2} \dots$ and $y_n = 0, \eta_{n+1}\eta_{n+2} \dots$, cf. [5, Formula (5.3)]. Let us mention that the index in Formula (5.3) is not correct.

The following estimate is already known for $0 < x \le \frac{1}{2}$ from [5, Lemma 3.1].

Lemma 2.1 *For $0 < x \le 1$ the Takagi function T satisfies the estimate*

$$\log_2 \frac{1}{x} \le \frac{1}{x}T(x) \le \log_2 \frac{1}{x} + c \tag{2.6}$$

with a positive constant $c < \frac{2}{3}$.

Proof: Since (2.6) is true for $0 < x \le \frac{1}{2}$ we can assume that $\frac{1}{2} < x \le 1$. By the first relation in (2.1) we have $T(x) = 2T(\frac{x}{2}) - x$ and hence $\frac{1}{x}T(x) = \frac{2}{x}T(\frac{x}{2}) - 1$. In view of $\log_2 \frac{2}{x} = 1 + \log_2 \frac{1}{x}$ and $\frac{x}{2} \le \frac{1}{2}$ it follows that (2.6) is also true for $\frac{1}{2} < x \le 1$. Thus, the lemma is proved. $\square$

3 Right-hand improper derivatives

First we investigate the existence of the right-hand improper derivative.

Proposition 3.1 *The Takagi function T has at the non-dyadic point x the right-hand improper derivative $T'_+(x) = \infty$ if and only if $D_n \to \infty$ as $n \to \infty$.*

Proof: Since x is a non-dyadic point the expansion (2.4) contains infinitely ones and zeros. Let y have the dyadic representation $y = \eta_0, \eta_1\eta_2\ldots$ where $\eta_\nu = \xi_\nu$ for $\nu \leq n$ and $\eta_{n+1} = 1$, $\xi_{n+1} = 0$ so that $x < y < x + 2^{1-n}$. We investigate the term

$$\frac{T(y) - T(x)}{y - x}$$

as $n \to \infty$.

1. Assume $T'_+(x) = \infty$. If we choose $\eta_\nu = 1 - \xi_\nu$ for $\nu > n+1$, then $y_n = 1 - x_n$ and by (2.5) we have

$$\frac{T(y) - T(x)}{y - x} = D_n. \tag{3.1}$$

Since $x < y \leq x + \frac{1}{2^n}$ it follows that $T'_+(x) = \infty$ implies $D_n \to \infty$.

2. Suppose $D_n \to \infty$. By (2.5) we have

$$\frac{T(y) - T(x)}{y - x} = D_n + \frac{T(y_n) - T(x_n)}{y_n - x_n}$$

where $x_n = 0, 0\xi_{n+2}\ldots$ and $y_n = 0, 1\eta_{n+2}\ldots$ so that $0 < x_n < \frac{1}{2}$ and $\frac{1}{2} \leq y_n \leq 1$. We consider two cases:

2.1 In case $\frac{7}{8} < y_n \leq 1$ we have $y_n - x_n > \frac{1}{8}$ and

$$\frac{T(y_n) - T(x_n)}{y_n - x_n} > \frac{-\frac{2}{3}}{\frac{1}{8}} = -\frac{16}{3}.$$

2.2 In case $\frac{1}{2} \leq y_n \leq \frac{7}{8}$ we put $y_n = \frac{1+t}{2}$ with $0 \leq t \leq \frac{1}{4}$. By (2.1) and $T(t) \geq 2t$ for $0 \leq t \leq \frac{1}{4}$

$$T(y_n) = T\left(\frac{1+t}{2}\right) = \frac{1-t}{2} + \frac{1}{2}T(t) \geq \frac{1+t}{2}$$

and

$$\frac{T(y_n) - T(x_n)}{y_n - x_n} \geq \frac{1 + t - 2T(x_n)}{1 + t - 2x_n}.$$

For the derivative of the function

$$f(t) = \frac{1 + t - 2T(x_n)}{1 + t - 2x_n}$$

we have

$$f'(t) = \frac{(1+t-2x_n) - (1+t-2T(x_n))}{(1+t-2x_n)^2} = \frac{2T(x_n) - 2x_n}{(1+t-2x_n)^2} \geq 0\,.$$

Hence, for $0 \leq t \leq \frac{1}{4}$ the function $f(t)$ is increasing and

$$\frac{T(y_n) - T(x_n)}{y_n - x_n} \geq f(0) = \frac{T(\frac{1}{2}) - T(x_n)}{\frac{1}{2} - x_n}.$$

With $h = 1 - 2x_n$, i.e. $x_n = \frac{1-h}{2}$, we find in view of the symmetry of T with respect to $\frac{1}{2}$ that

$$\frac{T(\frac{1}{2}) - T(x_n)}{\frac{1}{2} - x_n} = \frac{\frac{1}{2} - T(\frac{1+h}{2})}{h/2} = \frac{\frac{1}{2} - \frac{1-h}{2} - \frac{1}{2}T(h)}{h/2} = 1 - \frac{T(h)}{h}$$

where we have used the second equation in (2.1). By Lemma 2.1

$$\frac{T(h)}{h} \leq \log_2 \frac{1}{h} + c$$

with $c < \frac{2}{3}$. Note that $h = 1 - 2x_n = 0, \overline{\xi}_{n+2}\overline{\xi}_{n+3}\ldots$ with $\overline{\xi}_\nu = 1 - \xi_\nu$. If $\xi_{n+\nu} = 1$ for $\nu = 2, 3, \ldots, m$ and $\xi_{n+m+1} = 0$ then $m \geq 2$, $h \geq 1/2^m$ and $\log_2 \frac{1}{h} \leq m$. Note that $m = I_{n+m} - I_n$ and $O_{n+m} - O_n = 1$ since $\xi_{n+1} = 0$. Hence, $D_{n+m} - D_n = 1 - m$ and we get

$$D_n + \frac{T(y_n) - T(x_n)}{y_n - x_n} \geq D_n + 1 - m - c = D_{n+m} - c.$$

Both cases 2.1 and 2.2 together yield

$$\frac{T(y) - T(x)}{y - x} \geq \inf_{k \geq n} D_k + O(1)$$

which implies $T'_+(x) = \infty$ since $D_n \to \infty$. □

4 Left-hand improper derivatives

The determining of the conditions for the existence of the left-hand improper derivative $T'_-(x) = \infty$ is more complicated. We need some lemmas.

Lemma 4.1 *Assume that $x = \frac{k+r}{2^m}$ and $y = \frac{k-h}{2^m}$ where k is an odd integer and $0 < r < 1$, $0 \leq h \leq 1$. Then we have*

$$\frac{T(x) - T(y)}{x - y} = D_m + \frac{2h}{r+h} + \frac{T(r) - T(h)}{r+h}. \tag{4.1}$$

Proof: According to equation (2.2) we have

$$T(x) = T\left(\frac{k+r}{2^m}\right) = T\left(\frac{k}{2^m}\right) + \frac{m-2s(k)}{2^m}r + \frac{1}{2^m}T(r)$$

and by equation (2.3)

$$T(y) = T\left(\frac{k-h}{2^m}\right) = T\left(\frac{k}{2^m}\right) + \frac{2s(k-1)-m}{2^m}h + \frac{1}{2^m}T(h).$$

Since k is an odd integer, we have $s(k-1) = s(k) - 1$. It follows

$$T(x) - T(y) = \frac{m-2s(k)}{2^m}(r+h) + \frac{2h}{2^n} + \frac{T(r)-T(h)}{2^m}$$

and in view of $x - y = (r+h)/2^m$ and $D_m = m - 2s(k)$ it follows (4.1). $\square$

Assume that x is a non-dyadic point with the representation (1.2) so that

$$x = \frac{k_n + r_n}{2^{a_n}}, \qquad k_n = 2^{a_n}\sum_{\nu=1}^{n}\frac{1}{2^{a_\nu}}, \qquad r_n = 2^{a_n}\sum_{\nu=n+1}^{\infty}\frac{1}{2^{a_\nu}} \tag{4.2}$$

and that

$$y = \frac{k_n - h_n}{2^{a_n}}, \qquad 0 \le h_n \le 1. \tag{4.3}$$

Note that $r_n > 0$ since x is a non-dyadic point. Put $d_n = a_{n+1} - a_n$ then we have $d_n \ge 1$ and

$$r_n = \frac{1}{2^{d_n}}\sum_{\nu=n+1}^{\infty}\frac{1}{2^{a_\nu - a_{n+1}}} \le \frac{2}{2^{d_n}}$$

and therefore

$$d_n - 1 \le \log_2\frac{1}{r_n} < d_n. \tag{4.4}$$

Lemma 4.2 *If $h_n > 0$ then we put $h_n = 2^t r_n > 0$ and it holds*

$$\frac{T(x)-T(y)}{x-y} = D_{a_n} - d_n + \frac{t2^t + 2d_n}{1+2^t} + O(1) \tag{4.5}$$

for $t \le \log_2\frac{1}{r_n}$.

Proof: Because of $r_n > 0$ and $0 < h_n \le 1$, cf. (4.3), we can write $h_n = 2^t r_n$ with $t \le \log_2\frac{1}{r_n}$. By Lemma 4.1 with $m = a_n$

$$\frac{T(x)-T(y)}{x-y} = D_{a_n} - \frac{2h_n}{r_n+h_n} + \frac{T(r_n)-T(h_n)}{r_n+h_n}.$$

Moreover the term $2h_n/(r_n+h_n)$ is bounded and the last term can be written in the form

$$\frac{T(r_n)-T(h_n)}{r_n+h_n} = \frac{r_n}{r_n+h_n}\frac{T(r_n)}{r_n} - \frac{h_n}{r_n+h_n}\frac{T(h_n)}{h_n}.$$

By Lemma 2.1 and (4.4) we get

$$d_n - 1 \le \frac{1}{r_n} T(r_n) < d_n + c$$

with a constant $c < \frac{2}{3}$, i.e.

$$\frac{1}{r_n} T(r_n) = d_n + \varepsilon_n$$

with $|\varepsilon_n| \le 1$. For $h_n = 2^t r_n$ we have

$$\log_2 \frac{1}{h_n} = \log_2 \frac{1}{r_n} - t$$

and as before

$$\frac{1}{h_n} T(h_n) = d_n - t + \delta_n$$

with $|\delta_n| \le 2$. So with $h_n = 2^t r_n$ we get

$$\begin{aligned} \frac{T(r_n) - T(h_n)}{r_n + h_n} &= \frac{1}{1+2^t} \frac{T(r_n)}{r_n} - \frac{2^t}{1+2^t} \frac{T(h_n)}{h_n} \\ &= \frac{1}{1+2^t}(d_n + \varepsilon_n) - \frac{2^t}{1+2^t}(d_n - t + \delta_n) \\ &= -d_n + \frac{t2^t + 2d_n}{1+2^t} + \frac{1}{1+e^t}\varepsilon_n - \frac{2^t}{1+2^t}\delta_n \end{aligned}$$

which yields (4.5). □

In view of (4.5) we want to estimate the minimum of the function

$$f_n(t) = \frac{t2^t + 2d_n}{1+2^t} \qquad (t \in \mathbb{R}). \tag{4.6}$$

Lemma 4.3 *For positive integer d the function $f(t) = (t2^t + 2d)/(1 + 2^t)$ attains its minimum exactly at one point $t_* = t_*(d)$ where $t_*(d) < d - 1$. It holds*

$$f(t_*) = \log_2 d + O(1). \tag{4.7}$$

Proof: 1. Note that $f(t) \to 2d$ as $t \to -\infty$ and $f(t) \to +\infty$ as $t \to +\infty$. Moreover, for the derivative

$$f'(t) = \frac{(2^t + t2^t \log 2)(1+2^t) - (t2^t + 2d)2^t \log 2}{(1+2^t)^2}$$

we have $f'(t) = 0$ if and only if

$$g(t) = 1 + 2^t + t \log 2 - 2d \log 2$$

vanishes. Now $g(t)$ is strictly increasing with $g(t) \to -\infty$ as $t \to -\infty$ and $g(t) \to +\infty$ as $t \to +\infty$ so that there is exactly one real number $t_* = t_*(d)$ with $g(t_*) = 0$.

In order to show that $t_* < d-1$ we prove the inequality

$$g(d-1) = 1 + 2^{d-1} + (d-1)\log 2 - 2d \log 2 > 0$$

which is true for $d = 1$. Moreover $g(d) - g(d-1) = 2^{d-1} - \log 2 \geq 1 - \log 2 > 0$ so that indeed $g(d-1) > 0$ for all $d \geq 1$. Consequently, $t_* < d-1$.

2. In order to show (4.7) we put $2^{t_*} = \tau_* d$ with suitable $\tau_* = \tau_*(d)$. Then we have

$$g(t_*) = 1 + \tau_* d + \log_2(\tau_* d) \log 2 - 2d \log 2 = 0$$

so that τ_* is a zero of the function

$$h(\tau, d) = 1 + \tau d + \log(\tau d) - 2d \log 2.$$

We show that $a < \tau_* < 2$ where $a = 2\log 2 - 1$. Note that $0 < a < 1$ and hence $h(a,1) = 1 + a + \log a - 2\log 2 = \log a < 0$. Moreover

$$h(a, d+1) - h(a,d) = a + \log(d+1) - \log d - 2\log 2 \leq a - 2\log 2 = -1$$

so that $h(a,d) < 0$ for all $d \geq 1$. On the other hand

$$h(2,d) = 1 + 2d + \log 2 + \log d - 4\log 2 \geq 3 - 3\log 2 > 0$$

and it follows $a < \tau_* < 2$ since $h(\tau, d)$ is strictly increasing with respect to τ.

Finally, with $t_* = \log_2(\tau_* d)$ we get

$$\begin{aligned} f(t_*) &= \frac{(\log_2 \tau_* + \log_2 d)\tau_* d + 2d}{1 + \tau_* d} \\ &= \log_2 d + \frac{\tau_* d(\log_2 \tau_* - 1)}{1 + \tau_* d} + \frac{2d}{1 + \tau_* d} \end{aligned}$$

where in view of $a < \tau_* < 2$ it holds

$$\frac{\tau_* d(\log_2 \tau_* - 1)}{1 + \tau_* d} \sim \log_2 \tau_* - 1, \qquad \frac{2d}{1 + \tau_* d} \sim \frac{2}{\tau_*}$$

as $d \to \infty$. This implies (4.7). □

Corollary 4.4 *The function (4.6) attains its minimum exactly at one point t_n where $t_n < d_n - 1$ and it holds $f_n(t_n) = \log_2 d_n + O(1)$, i.e.*

$$\frac{t2^t + 2d_n}{1 + 2^t} \geq \log_2 d_n + O(1).$$

Proposition 4.5 *The Takagi function has at the non-dyadic point x with the representation (1.2) the left-side improper derivative $T'_-(x) = \infty$ if and only if*

$$D_{a_n} - d_n + \log_2 d_n \to \infty \tag{4.8}$$

as $n \to \infty$.

Proof: First we assume that x has the expansion (2.4). For given positive integer m let be $y = 0, \eta_1\eta_2 \ldots$ a number with $\eta_\nu = \xi_\nu$ for $\nu < m$, $\eta_m = 0$, $\xi_m = 1$ so that $x - 2^{1-m} \leq y < x$. Again, we investigate the term

$$\frac{T(x) - T(y)}{x - y}$$

as $m \to \infty$. Note that

$$x = \frac{k + r}{2^m}, \qquad k = 2^m \sum_{\nu=1}^{m} \frac{\xi_\nu}{2^\nu}, \qquad r = 2^m \sum_{\nu=m+1}^{\infty} \frac{\xi_\nu}{2^\nu}$$

where $0 < r < 1$ since x is not dyadic rational. In view of

$$y \leq \sum_{\nu=1}^{m-1} \frac{\xi_\nu}{2^\nu} + \sum_{\nu=m+1}^{\infty} \frac{1}{2^\nu} = \frac{k-1}{2^m} + \frac{1}{2^m} = \frac{k}{2^m} \qquad y \geq \sum_{\nu=1}^{m-1} \frac{\xi_\nu}{2^\nu} = \frac{k-1}{2^m}$$

we have $y = (k-h)/2^m$ with $0 \leq h \leq 1$. Let $a_n \leq m < a_{n+1}$ then we get the representations (4.2) and (4.3) where $k_n = k/2^{m-a_n}$ is an odd integer, $r_n = r/2^{m-a_n}$, $h_n = h/2^{m-a_n}$, and $m \to \infty$ if and only if $n \to \infty$.
In case $h_n = 0$ we get by Lemma 4.1

$$\frac{T(x) - T(y)}{x - y} = D_{a_n} + \frac{T(r_n)}{r_n} > D_{a_n}. \tag{4.9}$$

In case $h_n > 0$ we put $h_n = 2^{t_n} r_n$ with t_n from Corollary 4.4 which is only possible if $2^{t_n} \leq 1$. But $t_n < d_n - 1$ and in view of $d_n - 1 < \log_2 \frac{1}{r_n}$, cf. (4.4), in fact $2^{t_n} r_n < 2^{d_n - 1} r_n < 1$. By Lemma 4.2 and Corollary 4.4 we have

$$\frac{T(x) - T(y)}{x - y} \geq D_{a_n} - d_n + \log_2 d_n + O(1) \tag{4.10}$$

where we have equality if we choose y such that $h_n = 2^{t_n} r_n$ in (4.3). From (4.9) we see that (4.10) is also valid in case $h_n = 0$ since $-d_n + \log_2 d_n < 0$.

Now it is easy to finish the proof. If (4.8) is satisfied then by (4.10) we obtain $T'_-(x) = \infty$. Conversely, if (4.8) fails then there is a strictly increasing sequence $\{n'\}$ of integers so that $D_{a_{n'}} - d_{n'} + \log_2 d_{n'} \to K < \infty$ as $n' \to \infty$. We use (4.2), (4.3) both with n' instead of n, where we put $h_{n'} = 2^{t_{n'}} r_{n'}$. Then by (4.10)

$$\frac{T(x) - T(y)}{x - y} = D_{n'} - d_{n'} + \log_2 d_{n'} + O(1)$$

so that

$$\liminf_{y \to x-} \frac{T(x) - T(y)}{x - y} < \infty.$$

Thus, the proposition is proved. □

Remark 4.6 The condition (4.8) can also be written as

$$2^{D_{a_n}} \frac{d_n}{2^{d_n}} \to \infty \tag{4.11}$$

as in (ii) of the Introduction.

Remark 4.7 Note that $D_{a_n} = a_n - 2n \to \infty$ is equivalent to $D_n \to \infty$. It is enough to show that $D_{a_n} \to \infty$ implies $D_n \to \infty$. We assume that $a_n \leq m < a_{n+1}$ then $O_m = m - n \geq a_n - n$, $I_m = n$ so that $D_m = O_m - I_m \geq a_n - 2n = D_{a_n} \to \infty$. So (4.11) is satisfied if $D_n \to \infty$ and d_n is bounded. It follows that $T'(x) = \infty$ if $D_n \to \infty$ and if the number of consecutive zeros in the dyadic representation of x is bounded, cf. [5, Proposition 5.3].

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Supplement. K. Kawamura and P. C. Allaart also have found the conditions for the existence of the improper derivatives of Takagi's function, cf. [1].

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