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Functional Equations for Knopp Functions and Digital Sums

ABSTRACT. The well-known Delange formula expressed the usual sum-of-digits function to a basis $q \geq 2$ by means of a continuous, nowhere differentiable function. The aim of this paper is to clarify the actually reason for this phenomenon. For this we show that specific Knopp functions satisfy functional equations which allow to calculate, for any positive integer n, the number of times of digits in the q-ary representation of n which are equal to a fixed $m \in \{1, \ldots, q-1\}$. By linear combination for arbitrary Knopp functions we get functional equations contained certain digital sums. These functional equations imply sum formulas for certain digital sums. Simple examples are the formula of Delange for the usual sum-of-digits function and a formula for the number of zeros

KEY WORDS. Knopp functions, functional equations, digital sums, Fourier expansion.

1 Introduction

Throughout in this paper let q be a fixed integer with $q \geq 2$. For an integer $k \in \mathbb{N}$ we introduce the q-ary representation

$$k = \sum_{j=0}^{\infty} a_j q^j \tag{1.1}$$

with $a_j \in \{0, 1, \ldots, q-1\}$ and $a_j = 0$ for $j > \log_q k$. It is known that the sum

$$S(n) = \sum_{k=1}^{n-1} s(k), \tag{1.2}$$

where $s(k) = a_0 + a_1 + \ldots$, can be represented by the Delange formula [3]

$$\frac{1}{n} S(n) = \frac{q-1}{2} \log_q n + F\left(\log_q n\right) \tag{1.3}$$

where $F(u)$ is a continuous, nowhere differentiable function with the period 1, cf. [9] for $q = 2$. In the case $q = 2$ this function can be expressed by

$$F(u) = -\frac{u}{2} - \frac{1}{2^{u+1}} T(2^u) \qquad (u \leq 0)$$

where T is Takagi's function, cf. [6]. Takagi's function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\Delta(2^n x)}{2^n} \qquad (0 \leq x \leq 1) \tag{1.4}$$

where $\Delta(x) = \operatorname{dist}(x, \mathbb{Z})$, and it was introduced in 1903 by T. Takagi [8] as an example of a continuous, nowhere differentiable function.

In this paper we investigate so-called Knopp functions ([7])

$$G(x) = \sum_{\nu=0}^{\infty} \frac{g(q^\nu x)}{q^\nu} \qquad (x \in \mathbb{R}) \tag{1.5}$$

where the function $g(x)$ is continuous, 1-periodic with $g(0) = g(1) = 0$ and linear in the intervals $[\frac{k}{q}, \frac{k+1}{q}]$, $(k \in \mathbb{Z})$. First we consider $q-1$ specific functions $g_m(x)$ $(m \in \{1, \ldots, q-1\})$ which form a basis for all such $g(x)$, i.e.

$$g(x) = \sum_{m=1}^{q-1} \lambda_m g_m(x) \qquad (x \in \mathbb{R}) \tag{1.6}$$

with suitable coefficients λ_m. By means of certain functional equations for the corresponding Knopp functions

$$G_m(x) = \sum_{\nu=0}^{\infty} \frac{g_m(q^\nu x)}{q^\nu} \qquad (x \in \mathbb{R}) \tag{1.7}$$

we are able to express the number $s_m(k)$ of exactly those digits of the integer k in the q-ary representation which equal m. We show that

$$\frac{1}{n} \sum_{k=1}^{n-1} s_m(k) = \frac{1}{q} \log_q n + F_m\left(\log_q n\right) \tag{1.8}$$

where $F_m(u)$ is a continuous nowhere differentiable function with period 1 which is connected with G_m by

$$F_m(u) = -\frac{u}{q} - \frac{1}{q^{u+1}} G_m(q^u) \qquad (u \leq 0).$$

The coefficients of the Fourier expansion of F_m can be expressed by means of the Hurwitz zeta function $\zeta(s, a)$ which for $\operatorname{Re} s > 1$ is defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{1.9}$$

where a is a fixed real number, $0 < a \leq 1$. When $a = 1$ this reduces to the Riemann zeta function, $\zeta(s) = \zeta(s, 1)$, cf. [1], p. 249.

Next for arbitrary numbers $\lambda_1, \ldots, \lambda_{p-1}$ we consider the Knopp function G from (1.5) with g from (1.6) and the function

$$s(k) = \sum_{m=1}^{q-1} \lambda_m s_m(k) \qquad (k \in \mathbb{N}_0). \tag{1.10}$$

For the sum (1.2) with $s(k)$ from (1.10) we show that it holds the formula

$$\frac{1}{n} S(n) = \frac{1}{q} S(q) \log_q n + F(\log_q n) \tag{1.11}$$

where F is a 1-periodic continuous nowhere differentiable function. Moreover, we can express the Fourier coefficients of F by means of the zeta function $\zeta(s,a)$. The connection between F in (1.11) and G from (1.5) with g from (1.6) is given by

$$F(u) = -\frac{1}{q} S(q) u - \frac{1}{q^{u+1}} G(q^u) \qquad (u \leq 0). \tag{1.12}$$

As application we get formulas for several digital sums. In particular, for $\lambda_m = m$ ($m = 1, \ldots, q-1$) we get the formula (1.3) of Delange for the sum-of-digits function and for $\lambda_m = 1$ a formula for the number of all digits which are different from zero. Finally, we also give a formula for the number of zeros.

2 Functional equations for specific Knopp functions

Throughout in this paper let q be a fixed integer with $q \geq 2$. In this paper for $m \in \{1, \ldots, q-1\}$ we need the function G_m defined by (1.7) where the generated function g_m is given by

$$g_m(x) = \begin{cases} x & \text{for} \quad 0 \leq x \leq \frac{m}{q} \, , \\ m - (q-1)x & \text{for} \quad \frac{m}{q} \leq x \leq \frac{m+1}{q} \, , \\ x - 1 & \text{for} \quad \frac{m+1}{q} \leq x \leq 1 \, , \end{cases} \tag{2.1}$$

and by $g_m(x+1) = g_m(x)$ for $x \in \mathbb{R}$. This function can also be written as

$$g_m(x) = x - \left[\frac{qx}{m}\right](qx - m) + \left[\frac{qx}{m+1}\right](qx - m - 1) \qquad (0 \leq x \leq 1). \tag{2.2}$$

In particular, for $k \in \{0, 1, \ldots, q\}$ we have

$$g_m\left(\frac{k}{q}\right) = \begin{cases} \frac{k}{q} & \text{for} \quad 1 \leq k \leq m \, , \\ \frac{k-q}{q} & \text{for} \quad m < k \leq q-1 \, . \end{cases} \tag{2.3}$$

Obviously, the function G_m from (1.7) is continuous with $G_m(0) = 0$ and it holds $G_m(x+1) = G_m(x)$ for $x \in \mathbb{R}$. The function G_m satisfies the functional equation

$$G_m\left(\frac{x}{q}\right) = g_m\left(\frac{x}{q}\right) + \frac{1}{q}G_m(x) \qquad (x \in \mathbb{R}). \tag{2.4}$$

The function $s_m(k)$ which counts the digits m in (1.1) is given for $k \in \{0, 1, \ldots, q-1\}$ by

$$s_m(k) = \begin{cases} 1 & \text{for} \quad k = m\ , \\ 0 & \text{for} \quad k \neq m \end{cases} \tag{2.5}$$

and for arbitrary $k \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, q-1\}$ by

$$s_m(qk + r) = s_m(k) + s_m(r). \tag{2.6}$$

Proposition 2.1 *For $m \in \{1, \ldots, q-1\}$ the function G_m from (1.7) satisfies the functional equations*

$$G_m\left(\frac{k+x}{q^\ell}\right) = G_m\left(\frac{k}{q^\ell}\right) + \frac{\ell - qs_m(k)}{q^\ell}x + \frac{1}{q^\ell}G_m(x) \tag{2.7}$$

where $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, q^\ell - 1$, $x \in [0, 1]$. Moreover, for $n = 0, 1, \ldots, q^\ell$ we have

$$G_m\left(\frac{n}{q^\ell}\right) = \frac{n\ell}{q^\ell} - \frac{1}{q^{\ell-1}}\sum_{k=0}^{n-1} s_m(k). \tag{2.8}$$

Proof: Since $g_m(r) = 0$ for $r \in \mathbb{N}_0$ we get from (1.7) that

$$G_m\left(\frac{k}{q^\ell}\right) = \sum_{\nu=0}^{n-1} \frac{g_m(q^\nu \frac{k}{q^\ell})}{q^\nu}$$

and this implies

$$G_m\left(\frac{k+x}{q^\ell}\right) - G_m\left(\frac{k}{q^\ell}\right) = \sum_{\nu=0}^{\ell-1} \frac{g_m(q^\nu \frac{k+x}{q^\ell}) - g_m(q^\nu \frac{k}{q^\ell})}{q^\nu} + \sum_{\nu=\ell}^{\infty} \frac{g_m(q^\nu \frac{k+x}{q^\ell})}{q^\nu}.$$

For $\nu \geq \ell$ we find with $\mu = \nu - \ell \geq 0$ that $g_m(q^\nu \frac{k+x}{q^\ell}) = g_m(q^\mu k + q^\mu x) = g_m(q^\mu x)$ so that the last sum in the last equation is equal to $\frac{1}{q^\ell}G_m(x)$. For $\nu = 0, \ldots, \ell - 1$ there is no integer in the open interval $(q^\nu \frac{k}{q^\ell}, q^\nu \frac{k+1}{q^\ell})$, and hence the both numbers $q^\nu \frac{k+x}{q^\ell}$ and $q^\nu \frac{k}{q^\ell}$ belong to the same interval of the form $[r + \frac{s}{q}, r + \frac{s+1}{q}]$ with $r \in \mathbb{N}_0$ and $s \in \{0, 1, \ldots, q-1\}$. Since g_m is linear in each of these intervals we find that

$$\frac{g_m(q^\nu \frac{k+x}{q^\ell}) - g_m(q^\nu \frac{k}{q^\ell})}{q^\nu} = \varepsilon_\nu \frac{x}{q^\ell}$$

where $\varepsilon_\nu = -(q-1)$ when $q^\nu \frac{k}{q^\ell} \in [r+\frac{m}{q}, r+\frac{m+1}{q}]$ and where $\varepsilon_\nu = +1$ elsewhere in view of (2.1). If k has the representation (1.1) then we write shortly $\frac{k}{q^\ell} = a_\ell, a_{\ell-1}\dots a_0$ with $a_\ell = 0$ since $k < q^\ell$ and then $q^\nu \frac{k}{q^\ell} = a_\ell \dots a_{\ell-\nu}, a_{\ell-\nu-1}\dots a_0$ for $0 \le \nu \le \ell-1$. Hence $\varepsilon_\nu = -(q-1)$ when $a_{\ell-\nu-1} = m$ which happens for $s_m(k)$ elements, and $\varepsilon_\nu = +1$ when $a_{\ell-\nu-1} \ne m$ which happens for $\ell - s_m(k)$ elements. This implies

$$\sum_{\nu=0}^{\ell-1} \varepsilon_\nu = -(q-1)s_m(k) + \ell - s_m(k) = \ell - q s_m(k)$$

and hence (2.7) is proved. Equation (2.8) follows from (2.7) with $x = 1$ and summation over k in view of $G_m(1) = 0$. □

3 The number of occurrences of a single digit

The equation (2.8) can be considered as sum formula for

$$S_m(n) = \sum_{k=1}^{n-1} s_m(k) \tag{3.1}$$

which is equal to the number of digits m in the q-ary representations of the integers $1, 2, \dots, n-1$. For this sum we have according to (2.8)

$$S_m(n) = \frac{n\ell}{q} - q^{\ell-1} G_m\left(\frac{n}{q^\ell}\right) \tag{3.2}$$

where $n \le q^\ell$ and G_m is given by (1.7). In particular, for $n = q^\ell$ we find from (3.2) in view of $G_m(1) = 0$ that the special sum $S_m(q^\ell) = \ell q^{\ell-1}$ is independent of m.

In order to obtain a representation of $S_m(k)$ $(m \in \{1, \dots, q-1\})$ which does not contain ℓ we introduce the function

$$f_m(x) = -\frac{1}{q}\left\{\frac{1}{x} G_m(x) + \log_q x\right\} \qquad (0 < x \le 1). \tag{3.3}$$

For $0 < x \le 1$ equation (2.4) simplifies to

$$G_m\left(\frac{x}{q}\right) = \frac{x}{q} + \frac{1}{q} G_m(x),$$

and therefore the function f_m has the property

$$f_m\left(\frac{x}{q}\right) = f_m(x) \qquad (0 < x \le 1).$$

Hence, we can extend the function $f_m(x)$ for all $x > 0$ by

$$f_m(qx) = f_m(x) \qquad (x > 0). \tag{3.4}$$

Theorem 3.1 *For the number of digits equal to m ($m \in \{1, \dots, q-1\}$) in the q-ary representation of the integers $1, 2, \dots, n-1$ we have*

$$\frac{1}{n} S_m(n) = \frac{1}{q} \log_q n + f_m(n) \tag{3.5}$$

where f_m is given by (3.3) and (3.4).

Proof: From (3.2) we get

$$\frac{1}{n} S_m(n) = \frac{1}{q} \left\{ \ell - \frac{q^\ell}{n} G_m\left(\frac{n}{q^\ell}\right) \right\}.$$

By means of (3.3) the term in brackets can be written as

$$\ell - \frac{q^\ell}{n} G_m\left(\frac{n}{q^\ell}\right) = \log_q n - \frac{q^\ell}{n} G_m\left(\frac{n}{q^\ell}\right) - \log_q \frac{n}{q^\ell} = \log_q n + q f_m\left(\frac{n}{q^\ell}\right).$$

In view of the property (3.4) we have

$$f_m\left(\frac{n}{q^\ell}\right) = f_m(n)$$

so that the representation (3.5) follows. □

4 Periodic functions and Fourier expansions

According to (3.4) the function

$$F_m(u) = f_m(q^u) \qquad (u \in \mathbb{R}) \tag{4.1}$$

is periodic with period 1 so that in view of (3.3) Theorem 3.1 implies the

Corollary 4.1 *Let m be a fixed integer with $1 \le m \le q-1$. Then for the sum (3.1) we have*

$$\frac{1}{n} S_m(n) = \frac{1}{q} \log_q n + F_m(\log_q n) \tag{4.2}$$

where F_m is a continuous function of period 1 which is given by

$$F_m(u) = -\frac{u}{q} - \frac{1}{q^{u+1}} G_m(q^u) \qquad (u \le 0) \tag{4.3}$$

with G_m from (1.7).

In order to determine the Fourier expansion of the periodic function $F_m(u)$ we need the zeta function $\zeta(s, a)$ defined by (1.9) for $\operatorname{Re} s > 1$ and $0 < a \le 1$. The only singularity of $\zeta(s, a)$ is at the point $s = 1$, cf. [10], p. 265.

Lemma 4.2 *Let be $m \in \{1, \dots, q-1\}$ and $0 < \alpha \le \frac{m}{q}$. Then for the periodic function g_m from (2.1) we have for $\mathrm{Re}\, s > -1$, $s \neq 0, 1$*

$$\int_\alpha^\infty \frac{g_m(x)}{x^{s+2}} dx = \frac{1}{s\alpha^s} + q \frac{\zeta(s, \frac{m+1}{q}) - \zeta(s, \frac{m}{q})}{s(s+1)}. \tag{4.4}$$

Moreover, for the excluded values $s = 0$ and $s = 1$ we have

$$\int_\alpha^\infty \frac{g_m(x)}{x^2} dx = 1 - \log \alpha + q \log \frac{\Gamma(\frac{m+1}{q})}{\Gamma(\frac{m}{q})} \tag{4.5}$$

and

$$\int_\alpha^\infty \frac{g_m(x)}{x^3} dx = \frac{1}{\alpha} + \frac{q}{2} \left\{ \frac{\Gamma'(\frac{m}{q})}{\Gamma(\frac{m}{q})} - \frac{\Gamma'(\frac{m+1}{q})}{\Gamma(\frac{m+1}{q})} \right\}. \tag{4.6}$$

Proof: The integral (4.4), denoted by $I_m(s)$, converges absolutely for $\mathrm{Re}\, s > -1$. In view of (2.2) we have

$$I_m(s) = \int_\alpha^\infty \frac{x - [x]}{x^{s+2}} dx - J_m(s) + J_{m+1}(s)$$

where

$$J_m(s) = \int_\alpha^\infty \frac{1}{x^{s+2}} \left[\frac{(x - [x])q}{m} \right] ((x - [x])q - m)\, dx.$$

For $\mathrm{Re}\, s > 0$ the first integral can be computed by

$$\int_\alpha^\infty \frac{dx}{x^{s+1}} = \frac{1}{s\alpha^s}$$

and

$$\int_\alpha^\infty \frac{[x]}{x^{s+2}} dx = \int_1^\infty \frac{[x]}{x^{s+2}} dx = \frac{1}{s+1} \zeta(s+1),$$

cf. [3] (see also [1], p. 246). Moreover, for $\mathrm{Re}\, s > 1$ we have

$$\begin{aligned} J_m(s) &= \sum_{n=0}^\infty q \int_{n+m/q}^{n+1} \frac{dx}{x^{s+1}} - \sum_{n=0}^\infty (nq+m) \int_{n+m/q}^{n+1} \frac{dx}{x^{s+2}} \\ &= \frac{q}{s} \sum_{n=0}^\infty \left(\frac{1}{(n+\frac{m}{q})^s} - \frac{1}{(n+1)^s} \right) - \frac{1}{s+1} \sum_{n=0}^\infty \left(\frac{nq+m}{(n+\frac{m}{q})^{s+1}} - \frac{nq+m}{(n+1)^{s+1}} \right) \\ &= \frac{1}{s+1} \sum_{n=0}^\infty \frac{nq+m}{(n+1)^{s+1}} + \frac{q\zeta(s, \frac{m}{q})}{s(s+1)} - \frac{q}{s} \zeta(s) \end{aligned}$$

so that

$$J_{m+1}(s) - J_m(s) = \frac{1}{s+1} \zeta(s+1) + q \frac{\zeta(s, \frac{m+1}{q}) - \zeta(s, \frac{m}{q})}{s(s+1)}.$$

Hence,

$$I_m(s) = \frac{1}{s\alpha^s} + q \frac{\zeta(s, \frac{m+1}{q}) - \zeta(s, \frac{m}{q})}{s(s+1)}$$

which proves (4.4) for $\mathrm{Re}\, s > 1$. Since $\zeta(s,a)$ is analytic for $s \neq 1$ it follows that (4.4) is valid for $\mathrm{Re}\, s > -1$ excluded $s = 0$ and $s = 1$. In order to determine $I_m(0)$ we let s tend to zero and by means of the rule of de l' Hospital we get

$$\begin{aligned} I_m(0) &= \lim_{s\to 0} I_m(s) \\ &= \log \tfrac{1}{\alpha} + q\zeta'(0, \tfrac{m+1}{q}) - q\zeta'(0, \tfrac{m}{q}) - q\zeta(0, \tfrac{m+1}{q}) + q\zeta(0, \tfrac{m}{q}) \\ &= -\log \alpha + 1 + q \log \Gamma(\tfrac{m+1}{q}) - q \log \Gamma(\tfrac{m}{q}), \end{aligned}$$

since $\zeta(0,a) = \frac{1}{2} - a$ and $\zeta'(0,a) = \log \Gamma(a) - \frac{1}{2} \log(2\pi)$ (cf. [10], p. 271), and so we get (4.5). Finally, in view of

$$\lim_{s\to 1} \left(\zeta(s,a) - \frac{1}{s-1} \right) = -\frac{\Gamma'(a)}{\Gamma(a)}$$

(cf. [10], p. 271), we obtain

$$\lim_{s\to 1} (\zeta(s,a) - \zeta(s,b)) = \frac{\Gamma'(b)}{\Gamma(b)} - \frac{\Gamma'(a)}{\Gamma(a)}$$

and therefore (4.6). □

Proposition 4.3 *The continuous* 1*-periodic function* $F_m(u)$ *has the Fourier expansion*

$$F_m(u) = \sum_{k\in\mathbb{Z}} c_{mk} e^{2k\pi i u} \tag{4.7}$$

with

$$c_{m0} = \log_q \left(\frac{\Gamma(\frac{m}{q})}{\Gamma(\frac{m+1}{q})} \right) - \frac{1}{2q} - \frac{1}{q \log q}\,, \tag{4.8}$$

$$c_{mk} = \frac{\zeta(s_k, \frac{m}{q}) - \zeta(s_k, \frac{m+1}{q})}{s_k(s_k+1)\log q}\,, \qquad s_k = \frac{2\pi i k}{\log q}\,, \quad k \neq 0. \tag{4.9}$$

Proof: In view of the periodicity of $F_m(u)$ we have from (4.3)

$$F_m(u) = \frac{1}{q}(1-u) - \frac{1}{q^u} G_m(q^{u-1}) \qquad (0 \le u \le 1).$$

As in [3], p. 44, for the Fourier coefficients

$$c_{mk} = \int_0^1 F_m(u) e^{-2k\pi i u} du$$

we put $c_{mk} = a_{mk} + b_{mk}$ with

$$a_{mk} = \frac{1}{q} \int_0^1 (1-u) e^{-2k\pi i u} du,$$

i.e. $a_{m0} = \frac{1}{2q}$ and $a_{mk} = \frac{1}{2qk\pi i}$ for $k \neq 0$, and

$$b_{mk} = -\int_0^1 \frac{1}{q^u} G_m(q^{u-1}) e^{-2k\pi i u} du = -\sum_{\nu=0}^{\infty} \int_0^1 \frac{1}{q^{u+\nu}} g_m(q^{u+\nu-1}) e^{-2k\pi i u} du.$$

As in [3] we get by means of the substitution $u = 1 - \nu + \log_q x$ that

$$\int_0^1 \frac{1}{q^{u+\nu}} g_m(q^{u+\nu-1}) e^{-2k\pi i u} du = \frac{1}{q \log q} \int_{q^{\nu-1}}^{q^\nu} \frac{1}{x^2} g_m(x) e^{-2\pi i k \log_q x} dx$$

and hence

$$b_{mk} = -\frac{1}{q \log q} \int_{1/q}^{\infty} \frac{g_m(x)}{x^{2+2k\pi i/\log q}} dx.$$

By Lemma 4.2 with $\alpha = \frac{1}{q}$ we get the assertion. □

5 General Knopp functions

Now we consider the general Knopp function

$$G(x) = \sum_{\nu=0}^{\infty} \frac{g(q^\nu x)}{q^\nu} \qquad (x \in \mathbb{R}). \tag{5.1}$$

where the function g is continuous, 1-periodic with $g(0) = 0$, and linear in each interval $[\frac{k}{q}, \frac{k+1}{q}]$, $(k \in \mathbb{Z})$. Since the functions g_m from (2.1) form a basis for these functions, every g can be written as linear combination

$$g(x) = \sum_{m=1}^{q-1} \lambda_m g_m(x) \qquad (x \in \mathbb{R}) \tag{5.2}$$

with certain coefficients λ_m $(m \in \{1, \ldots, q-1\})$. From (2.1) we get

$$g\left(\frac{k}{q}\right) = \left(\frac{k}{q} - 1\right) \sum_{m=1}^{k-1} \lambda_m + \frac{k}{q} \sum_{m=k}^{q-1} \lambda_m \tag{5.3}$$

and it easy to see that

$$\lambda_m = g\left(\frac{1}{q}\right) + g\left(\frac{m}{q}\right) - g\left(\frac{m+1}{q}\right). \tag{5.4}$$

According to (5.2) the Knopp function G from (5.1) can be written as

$$G(x) = \sum_{m=1}^{q-1} \lambda_m G_m(x) \qquad (x \in \mathbb{R}) \tag{5.5}$$

with G_m from (1.7).

Now for $k \in \mathbb{N}_0$ we consider the function

$$s(k) = \sum_{m=1}^{q-1} \lambda_m s_m(k) \tag{5.6}$$

with $s_m(k)$ from (2.5) and (2.6). By (2.5) we have $s(0) = 0$ and $s(m) = \lambda_m$ for $m = 1, \ldots, q-1$, and (2.6) implies

$$s(kq + r) = s(k) + s(r) \tag{5.7}$$

for $k \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, q-1\}$.

Proposition 5.1 *Every function $s(k)$ with the property (5.7) can be written in the form (5.6) with*

$$\lambda_m = s(m) \qquad (m = 1, \ldots, q-1). \tag{5.8}$$

Proof: Assume that $s(k)$ is a given function satisfying (5.7) then for $k \in \mathbb{N}_0$ we put

$$s_0(k) = s(k) - \sum_{m=1}^{q-1} s(m) s_m(k). \tag{5.9}$$

In view of (2.5) it holds $s_0(k) = 0$ for $k = 0, 1, \ldots, q-1$. Moreover, according to (5.7) and (2.6) we have for $k \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, q-1\}$

$$s_0(qk + r) = s_0(k) + s_0(r).$$

It follows $s_0(k) = 0$ for all $k \in \mathbb{N}_0$ so that (5.9) implies the assertion. □

Let

$$S(n) = \sum_{k=1}^{n-1} s(k) \tag{5.10}$$

with $s(k)$ from (5.6), then (5.3) can be written as

$$g\left(\frac{k}{q}\right) = \frac{k}{q} S(q) - S(k) \qquad (k = 0, 1, \ldots, q). \tag{5.11}$$

In particular, $g(0) = g(1) = 0$ and $g(\frac{1}{q}) = \frac{1}{q} S(q)$.

In view of (5.5), (5.6) and (5.10) we get from Proposition 2.1 the

Theorem 5.2 *For $\ell \in \mathbb{N}$, $k = 0, 1, \dots, q^\ell - 1$, $x \in [0, 1]$ the Knopp function G from (5.1) with g from (5.2) satisfies the functional equations*

$$G\left(\frac{k+x}{q^\ell}\right) = G\left(\frac{k}{q^\ell}\right) + \frac{S(q)\ell - qs(k)}{q^\ell}x + \frac{1}{q^\ell}G(x). \tag{5.12}$$

Moreover, for $n = 0, 1, \dots, q^\ell$ we have

$$G\left(\frac{n}{q^\ell}\right) = \frac{S(q)n\ell - qS(n)}{q^\ell} \tag{5.13}$$

with $S(n)$ from (5.10).

It is known that in case $g(x) \not\equiv 0$ the Knopp function G from (5.1) is nowhere differentiable, cf. [2] and [5]. In [5] it was shown even that in the case $g(x) \not\equiv 0$ the function G from (5.1) does not have anywhere a finite one-sided derivative. We show that this property is a consequence of (5.12) where we need the following simple lemma, cf. [4].

Lemma 5.3 *Let $f : [0, 1] \mapsto \mathbb{R}$ have a finite right-hand derivative $f'_+(x_0)$ at the point $x_0 \in [0, 1)$. Let further (u_ℓ) and (v_ℓ) be sequences in $[0, 1]$ with $x_0 < u_\ell < v_\ell$ for all $\ell \in \mathbb{N}$ and $v_\ell \to x_0$ as $\ell \to \infty$. If there exists a $p > 0$ with $u_\ell - x_0 \le p(v_\ell - u_\ell)$ for all $\ell \in \mathbb{N}$ then*

$$\frac{f(v_\ell) - f(u_\ell)}{v_\ell - u_\ell} \to f'_+(x_0) \qquad (\ell \to \infty).$$

Proposition 5.4 *If $g(x) \not\equiv 0$ then the Knopp function G from (5.1) has nowhere a finite one-sided derivative.*

Proof: Assume, at $x_0 \in [0, 1)$ there exists the finite right-hand derivative $G'_+(x_0)$. For $\ell \in \mathbb{N}$ and $k = 0, 1, \dots, q^\ell - 1$ we put $x_{k,\ell} = k/q^\ell$ and $N_{a,b} = \{k \in \mathbb{N} : a \le k \le b\}$. If $x_{k',\ell} \le x_0 < x_{k'+1,\ell}$ then for every $k \in N_{k'+1,k'+2q-1}$ we put $u_{k,\ell} = x_{k,\ell}$ and $v_{k,\ell} = x_{k+1,\ell}$ so that $x_0 < u_{k,\ell} < v_{k,\ell}$ and $u_{k,\ell} - x_0 \le p(v_{k,\ell} - u_{k,\ell})$ with $p = 2q$. Applying (5.12) with $x = 1$ we get

$$\frac{G(v_{k,\ell}) - G(u_{k,\ell})}{v_{k,\ell} - u_{k,\ell}} - \frac{G(v_{k+1,\ell}) - G(u_{k+1,\ell})}{v_{k+1,\ell} - u_{k+1,\ell}} = \{S(q)\ell - qs(k)\} - \{S(q)\ell - qs(k+1)\}$$

and Lemma 5.3 implies that for $k \in N_{k'+1,k'+2q-1}$ we have

$$s(k+1) - s(k) \to 0 \qquad (\ell \to \infty).$$

The set $N_{k'+1,k'+2q-1}$ contains a section of the form $N_{d,d+q-2}$ with $d = qk_0 \le k' + q$. For $k \in N_{d,d+q-2}$, i.e. $k = qk_0 + r$ with $r = 0, 1, \dots, q-2$, we have in view of (5.7) and (5.8) that $s(k) = s(qk_0 + r) = s(k_0) + s(r) = s(k_0) + \lambda_r$ with $\lambda_0 = 0$ and hence

$$s(k+1) - s(k) = \lambda_{r+1} - \lambda_r \to 0 \qquad (\ell \to \infty).$$

This implies $\lambda_r = 0$ for all $r = 1, \dots, q-1$ since $\lambda_0 = 0$. □

6 Digital sums

From Corollary 4.1, Proposition 4.3 and Proposition 5.4 we get for the sum $S(n)$ from (5.10) in view of $\lambda_m = s(m)$ for $m = 1, \ldots, q-1$ and $\lambda_1 + \ldots + \lambda_{q-1} = S(q)$ the main result concerning digital sums.

Theorem 6.1 *For $S(n)$ from (5.10) with $s(k)$ from (5.6) we have the formula*

$$\frac{1}{n}S(n) = \frac{S(q)}{q}\log_q n + F(\log_q n) \tag{6.1}$$

where $F(u) = \lambda_1 F_1(u) + \ldots + \lambda_{q-1}F_{q-1}(u)$ is a continuous, nowhere differentiable function of period 1 which is given by

$$F(u) = -\frac{S(q)u}{q} - \frac{1}{q^{u+1}}G(q^u) \qquad (u \le 0) \tag{6.2}$$

with G from (5.1). The Fourier coefficients of F read

$$c_k = \sum_{m=1}^{q-1} \lambda_m c_{mk} \tag{6.3}$$

with c_{mk} from (4.8), (4.9).

We want to point out this for two examples.

1. The sum-of-digits function. For the sum of digits in the q-ary expansion of the integer k we have $\lambda_m = s(m) = m$ for $m \in \{1, \ldots, q-1\}$. Theorem 6.1 for $\lambda_m = m$ yields the well-known formula (1.3) of Delange where F is a continuous nowhere differentiable function which is given by

$$F(u) = -\frac{q-1}{2}u - \frac{1}{q^{u+1}}G(q^u) \qquad (u \le 0) \tag{6.4}$$

where G is given by (5.1) with g from (5.2). The Fourier coefficients of $F(u)$ are

$$\begin{aligned} c_0 &= \frac{q-1}{2}\log_q(2\pi) - \frac{q+1}{4} - \frac{q-1}{2\log q}\,, \\ c_k &= -\frac{q-1}{\log q}\frac{\zeta(s_k)}{s_k(s_k+1)}\,, \qquad s_k = \frac{2k\pi i}{\log q}\,, \quad k \neq 0 \end{aligned}$$

which follow from (6.3) with $\lambda_m = m$ in view of the relations

$$\prod_{m=1}^{q-1}\left(\frac{\Gamma(\frac{m}{q})}{\Gamma(\frac{m+1}{q})}\right)^m = \prod_{m=1}^{q-1}\Gamma\left(\frac{m}{q}\right) = \frac{(2\pi)^{\frac{q-1}{2}}}{\sqrt{q}}$$

and

$$\begin{aligned}\sum_{m=1}^{q-1} m\left(\zeta(s,\tfrac{m}{q})-\zeta(s,\tfrac{m+1}{q})\right) &= \sum_{m=1}^{q-1}\zeta(s,\tfrac{m}{q})-(q-1)\zeta(s)\\ &= (q^s-q)\zeta(s).\end{aligned}$$

2. The number of digits different from zero. For the number of digits which are different from zero in the q-ary representation of the integer k we have (5.6) with $\lambda_m = 1$ for $m \in \{1,\ldots,q-1\}$ and the function (5.2) for $0 \le x \le 1$ reads

$$g(x) = \sum_{m=1}^{q-1} g_m(x) = \begin{cases} (q-1)x & \text{for } \ 0 \le x \le \frac{1}{q}\ , \\ 1-x & \text{for } \ \frac{1}{q} < x \le 1\ . \end{cases} \tag{6.5}$$

Theorem 6.1 for $\lambda_m = 1$ yields:

Corollary 6.2 *Let $S(n)$ denote the numbers of digits different from zero in the q-ary representations of the integers $1, 2, \ldots, n-1$. Then it holds*

$$\frac{1}{n}S(n) = \frac{q-1}{q}\log_q n + F(\log_q n) \tag{6.6}$$

where $F(u)$ is a continuous nowhere differentiable function of period 1 which is given by

$$F(u) = -\frac{(q-1)u}{q} - \frac{1}{q^{u+1}}G(q^u) \qquad (u \le 0) \tag{6.7}$$

where G is given by (5.1) with g from (5.2). The Fourier expansion of the periodic function $F(u)$ has the coefficients

$$\begin{aligned} c_0 &= \log_q \Gamma\left(\frac{1}{q}\right) - \frac{q-1}{2q} - \frac{q-1}{q\log q}\ ,\\ c_k &= \frac{\zeta(s_k,\frac{1}{q})-\zeta(s_k)}{s_k(s_k+1)\log q}\ , \qquad s_k = \frac{2k\pi i}{\log q}\ , \qquad k \ne 0. \end{aligned}$$

7 The number of zeros

In Corollary 4.1 we have given a formula for the number of a fixed digit $m \in \{1,\ldots,q-1\}$. Now, we consider the digit $m=0$. In order to determine the number of zeros in the q-ary expansion first we compute the number of all digits. Let $a(k)$ denote the number of all digits in the q-ary expansion of k, i.e. $a(k) = \ell+1$ if $q^\ell \le k < q^{\ell+1}$. We state a formula for the sum

$$A(n) = \sum_{k=1}^{n-1} a(k). \tag{7.1}$$

Proposition 7.1 *For the number of all digits in the q-ary representations of the integers $1, 2, \ldots, n-1$ we have*

$$\frac{1}{n}A(n) = \log_q n + \frac{1}{(q-1)n} + H(\log_q n) \tag{7.2}$$

where H is a continuous function of period 1 which is given by

$$H(u) = 1 - u - \frac{1}{q-1}q^{1-u} \qquad (0 \le u < 1). \tag{7.3}$$

Proof: We have $a(k) = 1$ for $k = 1, \ldots, q-1$, $a(k) = 2$ for $k = q, \ldots, q^2-1$ and so on. Since for $k \ge 1$ the first digit may be $1, \ldots, q-1$ and the following digits may be $0, 1, \ldots, q-1$ we get for the sum (7.1) the special values $A(q) = q-1$, $A(q^2) = q-1+2q(q-1)$, $A(q^3) = q-1+2q(q-1)+3q^2(q-1)$ and in general

$$A(q^\ell) = (q-1)(1+2q+3q^2+\ldots+\ell q^{\ell-1}).$$

In view of

$$1 + 2t + 3t^2 + \ldots + \ell t^{\ell-1} = \frac{(\ell+1)t^\ell(t-1) - (t^{\ell+1}-1)}{(t-1)^2} \qquad (t \ne 1)$$

we get

$$A(q^\ell) = \ell q^\ell - \frac{q^\ell - 1}{q-1}.$$

It follows for $0 \le k \le q^{\ell+1} - q^\ell$ that

$$A(q^\ell + k) = \ell q^\ell - \frac{q^\ell - 1}{q-1} + (\ell+1)k$$

i.e.

$$A(q^\ell + k) = \ell(q^\ell + k) - \frac{q^\ell - 1}{q-1} + k.$$

Write $n = q^\ell + k = q^\ell(1+x)$ with $0 \le x < q-1$ we get in view of $\frac{q^\ell}{n} = \frac{1}{1+x}$, $\frac{k}{n} = 1 - \frac{1}{1+x}$ and $\ell = \log_q n + \log_q(\frac{q^\ell}{n}) = \log_q n - \log_q(1+x)$

$$\begin{aligned}
\frac{1}{n}A(n) &= \ell - \frac{q^\ell - 1}{n(q-1)} + \frac{k}{n} \\
&= \log_q n + \frac{1}{n(q-1)} + \left\{ -\log_q(1+x) - \frac{1}{(q-1)(1+x)} + 1 - \frac{1}{1+x} \right\} \\
&= \log_q n + \frac{1}{n(q-1)} + \left\{ 1 - \log_q(1+x) - \frac{q}{(q-1)(1+x)} \right\}.
\end{aligned}$$

This yields the assertion since in view of the periodicity of H we have for $n = q^\ell(1+x)$

$$H(\log_q n) = H(\log_q[q^\ell(1+x)]) = H(\log_q(1+x)) = H(u)$$

with $1 + x = q^u$ $(0 \le u < 1)$. □

The following result is a generalization of Theorem 3.2 in [6].

Proposition 7.2 *Let $s_0(k)$ be the number of zeros of k in the q-ary representation of k. Then it holds*

$$\frac{1}{n}\sum_{k=1}^{n-1} s_0(k) = \frac{\log_q n}{q} + \frac{1}{(q-1)n} + F_0(\log_q n) \tag{7.4}$$

where F_0 is a continuous nowhere differentiable function of period 1 which is given by

$$F_0(u) = \frac{1-u}{q} + \frac{1}{q^u}G(q^{u-1}) - \frac{q^{1-u}}{q-1} \qquad (0 \le u < 1) \tag{7.5}$$

where G is given by (5.1) with the 1-periodic function g given by (6.5). The continuous periodic function $F_0(u)$ has the Fourier expansion

$$F_0(u) = \sum_{k\in\mathbb{Z}} c_{0k}e^{2k\pi iu} \tag{7.6}$$

with

$$\begin{aligned} c_{00} &= \frac{2q-1}{2q} - \frac{1}{q\log q} - \log_q \Gamma\left(\frac{1}{q}\right), \\ c_{0k} &= \frac{1-\zeta(s_k)+\zeta(s_k,\frac{1}{q})}{s_k(s_k+1)\log q}, \qquad s_k = \frac{2\pi ik}{\log q}, \quad k \neq 0. \end{aligned}$$

Proof: We have $s_0(n) = a(n) - s(n)$ where $a(n)$ counts the number of all digits of n in the q-ary expansion and $s(n)$ counts the number of all digits different from zero. Hence Proposition 7.1 and Corollary 6.2 imply the assertion. Formulas (6.6) and (7.2) imply (7.4) with $F_0(u) = H(u) - F(u)$. Since the Fourier coefficients c_k of F are known, we have to compute the Fourier coefficients h_k of

$$H(u) = \sum_{k\in\mathbb{Z}} h_k e^{2k\pi iu}.$$

We put $h_k = a_k + b_k$ with

$$a_k = \int_0^1 (1-u)e^{-2k\pi iu}du,$$

i.e. $a_0 = \frac{1}{2}$ and $a_k = \frac{1}{2k\pi i}$ for $k \neq 0$, and

$$b_k = \frac{-q}{q-1}\int_0^1 q^{-u}e^{-2k\pi iu}du.$$

Substitution $x = q^u$ yields that

$$\int_0^1 q^{-u}e^{-2k\pi iu}du = \frac{1}{\log q}\int_1^q \frac{1}{x^2}e^{-2\pi ik\log_q x}dx$$

and hence

$$b_k = -\frac{q}{(q-1)\log q}\int_1^q \frac{dx}{x^{2+2k\pi i/\log q}}.$$

So $b_k = \frac{-1}{\log q + 2k\pi i}$ and by $h_k = a_k + b_k$ we get $h_0 = \frac{1}{2} - \frac{1}{\log q}$ and

$$h_k = \frac{1}{s_k(1+s_k)\log q}\,, \qquad s_k = \frac{2k\pi i}{\log q}, \quad k \neq 0.$$

This completes the proof. □

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