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Systems of Schrödinger Equations in the Whole Space

ABSTRACT. We present in this paper results for the sign of the weak solutions of some elliptic systems defined in $\mathbb{R}^N$ involving Schrödinger operators with indefinite weight functions and with potentials which tend to infinity at infinity.

KEY WORDS. Schrödinger operators, indefinite weight, principal eigenvalue, positivity and negativity, maximum and antimaximum principles, existence of solutions

1 Introduction

1.1 The problem settings

We study the elliptic system:

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + g_i(x, u_1, \ldots, u_n) \text{ in } \mathbb{R}^N, \quad i = 1, \ldots, n, \tag{1.1}$$

for $i = 1, \ldots, n$. We consider the following hypothesis for each $i = 1, \ldots, n$:

$(\mathbf{H_q^1})$ $q_i \in L^2_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N)$, $p > \frac{N}{2}$, such that $\lim_{|x|\to\infty} q_i(x) = \infty$ and $q_i \geq cst > 0$.

We will later specify the form and the hypotheses on each weight m_i and on each function g_i and we denote by μ_i real parameters for $i = 1, \ldots, n$. The variational space is denoted by $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$, where for each $i = 1, \ldots, n$, $V_{q_i}(\mathbb{R}^N)$ is the completion of $D(\mathbb{R}^N)$, the set of $\mathcal{C}^\infty$ functions with compact supports, with respect to the norm

$$\|u\|^2_{q_i} = \int_{\mathbb{R}^N} [|\nabla u|^2 + q_i u^2]. \tag{1.2}$$

We recall that the embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.

The aim of this paper is to study the sign of the solutions of (1.1). This extends earlier results already obtained for the Laplacian operator in a bounded domain (see [16, 18]), for equations or systems involving Schrödinger operators $-\Delta + q_i$ in $\mathbb{R}^N$ with positive weights (see [9–11]).

Our paper is organized as follows: In section 1.2 we recall some results for the scalar case, for the existence of principal eigenvalues in the case of indefinite weights. We also recall extensions of the maximum and antimaximum principles called ground state positivity and negativity (see [3, 4]). We study systems of the form (1.1) in Section 2. In Section 2.1 we give results for the maximum principle in the case of cooperative systems (2.1) by considering the positive principal eigenvalue and the negative principal eigenvalue of each operator $-\Delta + q_i$ associated with the indefinite weight m_i. Note that our results are more restrictive than those usually obtained when the weights m_i are positive (see [11, 16, 18]). In Section 2.2, first we give a result concerning the existence (and also Courant-Fischer formula) of a global positive eigenvalue $\Lambda_{1,M}$ for the cooperative system (2.8). Note that we can compare $\Lambda_{1,M}$ to each principal eigenvalue of $-\Delta + q_i$ associated with m_i. Then we obtain a maximum principal result for (2.8). Finally, in Section 2.3, for the two-by-two system (2.17), we present some results for the sign of the solutions. We decouple the system (2.17) in order to apply the results of the ground state positivity or negativity for each equation. Note that even if our conditions are restrictive, there are few results for the antimaximum principle for such systems (see [2]). Besides note that, to our knowledge, even the antimaximum principle, for the operator $-\Delta + q$ associated with an indefinite weight function m defined in the whole space, is not achieved yet (whereas it is well known for the Laplacian operator $-\Delta$ on a bounded domain in the case of an indefinite weight function, see [20], and for the Schrödinger operator $-\Delta + q$ in $\mathbb{R}^N$ but without any weight, see [3, 4]). In Appendix A, we give a brief recall of the proof of the antimaximum principle for the scalar case in the case of a positive and bounded weight m.

1.2 Review of results for the scalar case

1.2.1 The Schrödinger operator

We begin this section studying the Schrödinger operator $-\Delta + q$ associated with the weight m. We will assume throughout the paper that q is a potential which satisfies $(\mathbf{H_q^1})$. The weight m will assume one of the following hypotheses:

$(\mathbf{H_m^1})$ There exist two positive reals α and β such that $0 < \alpha \leq m \leq \beta$ in $\mathbb{R}^N$.

$(\mathbf{H_m^{*1}})$ $0 < m \leq cst$ in $\mathbb{R}^N$.

$(\mathbf{H_m^2})$ $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ $(N \geq 3)$, $m \geq 0$, $meas\{x \in \mathbb{R}^N, m(x) > 0\} \neq 0$.

$(\mathbf{H'^1_m})$ $m \in L^\infty(\mathbb{R}^N)$, m is positive in an open subset $\Omega_m^+ = \{x \in \mathbb{R}^N, m(x) > 0\}$ with non zero measure and m is negative in an open subset $\Omega_m^- = \{x \in \mathbb{R}^N, m(x) < 0\}$ with non zero measure.

$(\mathbf{H'^2_m})$ $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ $(N \geq 3)$, $meas(\Omega_m^+) > 0$, $meas(\Omega_m^-) > 0$.

For a positive weight m, we have:

Theorem 1.1 (cf. [12, Theorems 2.1,2.2]) *Assume that q satisfies $(\mathbf{H^1_q})$ and m satisfies $(\mathbf{H^1_m})$ or $(\mathbf{H^{*1}_m})$ or $(\mathbf{H^2_m})$. Then there exists a unique principal eigenvalue $\lambda_{1,q,m}$ which is simple and associated with a positive eigenfunction $\phi_{1,q,m}$ and:*

$$(-\Delta + q)\phi_{1,q,m} = \lambda_{1,q,m}\ m\ \phi_{1,q,m}\ \textit{in}\ \mathbb{R}^N; \quad \lambda_{1,q,m} > 0; \quad \phi_{1,q,m} > 0. \tag{1.3}$$

$$\lambda_{1,q,m} = \inf\{\frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q(\mathbb{R}^N)\ s.\ t.\ \int_{\mathbb{R}^N} m\phi^2 > 0\}. \tag{1.4}$$

For a weight m which changes sign in $\mathbb{R}^N$, we have:

Theorem 1.2 (cf. [12, Theorem 3.1]) *Assume that q satisfies $(\mathbf{H^1_q})$ and m satisfies $(\mathbf{H'^1_m})$ or $(\mathbf{H'^2_m})$. Then the operator $-\Delta + q$ associated with the weight m has a unique positive principal eigenvalue $\lambda_{1,q,m}$ associated with a positive eigenfunction $\phi_{1,q,m}$ and $(\lambda_{1,q,m}, \phi_{1,q,m})$ satisfy (1.3) and (1.4). Moreover the operator $-\Delta + q$ associated with the weight m has a unique negative principal eigenvalue $\tilde{\lambda}_{1,q,m}$ associated with a positive eigenfunction $\tilde{\phi}_{1,q,m}$ and $(\tilde{\lambda}_{1,q,m}, \tilde{\phi}_{1,q,m})$ satisfy*

$$(-\Delta + q)\tilde{\phi}_{1,q,m} = \tilde{\lambda}_{1,q,m}\ m\ \tilde{\phi}_{1,q,m}\ \textit{in}\ \mathbb{R}^N; \quad \tilde{\lambda}_{1,q,m} < 0; \quad \tilde{\phi}_{1,q,m} > 0. \tag{1.5}$$

$$\tilde{\lambda}_{1,q,m} = \sup\{\frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q(\mathbb{R}^N)\ s.\ t.\ \int_{\mathbb{R}^N} m\phi^2 < 0\}. \tag{1.6}$$

We have: $\tilde{\lambda}_{1,q,m} = -\lambda_{1,q,-m}$.

1.2.2 Maximum principle for the scalar case

We consider the following equation in a variational sense

$$(-\Delta + q)u = \mu m u + f \text{ in } \mathbb{R}^N \tag{1.7}$$

where μ is a real parameter and $f \in L^2(\mathbb{R}^N)$. First we recall the classical weak maximum principle for (1.7) in the case of a positive weight m.

Theorem 1.3 (cf. [12, Theorem 2.3]) *Assume that q satisfies $(\mathbf{H^1_q})$, m satisfies $(\mathbf{H^1_m})$ or $(\mathbf{H^{*1}_m})$ or $(\mathbf{H^2_m})$, $f \geq 0$ and u is a solution of the equation (1.7). If $\mu < \lambda_{1,q,m}$, then $u \geq 0$.*

Now we consider the equation (1.7) in the case of an indefinite weight m.

Theorem 1.4 (cf. [12, Theorem 3.2] *Assume that q satisfies* $(\mathbf{H_q^1})$, *m satisfies* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$, *$\mu \in \mathbb{R}$, $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ and u is a solution of the equation* (1.7). *If $\tilde{\lambda}_{1,q,m} < \mu < \lambda_{1,q,m}$, then $u \geq 0$.*

1.2.3 Ground state positivity or negativity for the scalar case

We recall here a result of ground state positivity or negativity for the Schrödinger operator $-\Delta + q$ associated with a strictly positive and bounded weight m in $\mathbb{R}^N$ (see [1]). We will add in this section the following hypothesis upon the potentiel q.

$$(\mathbf{H_q^2}) \begin{cases} \text{(i)} & q \text{ is radially symmetric.} \\ \text{(ii)} & \text{There exists a constant } c_1 > 0 \text{ and a positive real } R_0 \text{ such that} \\ & c_1 Q(r) \leq q(r) \text{ for } R_0 \leq r \text{ with } Q \text{ an auxiliary function which satisfies} \\ & Q \text{ is positive and locally absolutely continuous }, Q'(r) \geq 0, \\ & \int_{R_0}^{+\infty} Q(r)^{-\beta} dr < +\infty \text{ with } 0 < \beta < \frac{1}{2}. \end{cases}$$

Definition 1.1 i) *A function $u \in L^2(\mathbb{R}^N)$ satisfies the ground state positivity if there exists a constant $c > 0$ such that $u \geq c\phi_{1q,m}$ almost everywhere in $\mathbb{R}^N$.*

ii) *A function $u \in L^2(\mathbb{R}^N)$ satisfies the ground state negativity if there exists a constant $c > 0$ such that $u \leq -c\phi_{1q,m}$ almost everywhere in $\mathbb{R}^N$.*

These notions are similar to the maximum and antimaximum principles in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, which have been established by [13], [22], [23] (for a function $f \in L^p(\Omega)$, $p > N$). But for the Schrödinger operator defined in the whole space, the hypothesis $f \in L^p(\Omega)$, $p > N$, is no longer sufficient and we need to take a smaller space for f, namely, a stronger ordered Banach space introduced in [4]

$$X_{q,m} = \{u \in L^2(\mathbb{R}^N), \frac{u}{\phi_{1,q,m}} \in L^\infty(\mathbb{R}^N)\}$$

endowed with the ordered norm $\|u\|_{X_{q,m}} = \inf\{C \in \mathbb{R}, |u| \leq C\phi_{1,q,m} \text{ a. e. in } \mathbb{R}^N\}$. We denote by S^{N-1} the unit sphere in $\mathbb{R}^N$ centered at the origin and by σ the surface measure on S^{N-1}. For any $s > 0$, we introduce the Banach space $X_{q,m}^{s,2}$ of all functions $f \in L^2_{loc}(\mathbb{R}^N)$ having the following properties:

$$[(-\Delta_S)^{s/2} f](r, .) \in L^2(S^{N-1}) \text{ for all } r > 0,$$

where Δ_S denotes the Laplace-Beltrami operator on the sphere S^{N-1}, and there is a constant $C \geq 0$ such that

$$\frac{1}{\sigma(S^{N-1})}\left(\int_{S^{N-1}} |f(r, x')|^2 d\sigma(x') + \int_{S^{N-1}} |[(-\Delta_S)^{s/2} f](r, x')|^2 d\sigma(x')\right) \leq [C\phi_{1,q,m}(r)]^2$$

for almost every $r > 0$. The smallest such constant C defined the norm $\|f\|_{X^{s,2}_{q,m}}$ in $X^{s,2}_{q,m}$. Notice that, for $f(x) = f(|x|)$, we have $f \in X^{s,2}_{q,m}$ if and only if $f \in X_{q,m}$ together with the norms $\|f\|_{X^{s,2}_{q,m}} = \|f\|_{X_{q,m}}$. We recall from [1] the following result (which extends, for a Schrödinger equation with weight, former results in [4]):

Theorem 1.5 (see [1] Theorem 2.1) *Assume that the potential q is radially symmetric and satisfies* $(\mathbf{H^1_q})$, $(\mathbf{H^2_q})$ *and the weight m satisfies* $(\mathbf{H^1_m})$. *Assume that $u \in D(-\Delta + q)$ is one solution of* (1.7), *$\mu \in \mathbb{R}$, $f \geq 0$ a.e. in $\mathbb{R}^N$ with $f > 0$ in some set of positive Lebesgue measure.*

(i) *For every $\mu \in (-\infty, \lambda_{1,q,m})$, there exists a constant $C(f, \mu) > 0$ such that: $u \geq C(f, \mu)\phi_{1,q,m}$ in $\mathbb{R}^N$. Moreover, if the weight m is radially symmetric and if $f \in X^{s,2}_{q,m}$, then there exists a positive number $\delta(f)$ (depending upon f) such that, for every $\mu \in (\lambda_{1,q,m} - \delta(f), \lambda_{1,q,m})$, $C(f, \mu) = \frac{\int_{\mathbb{R}^N} f\phi_{1,q,m}}{\lambda_{1,q,m}-\mu} + \Gamma(\mu, f)$ with $\lim_{\mu\to\lambda_{1,q,m}} \Gamma(\mu, f) = \Gamma < +\infty$. And furthermore, if $f \in X_{q,m}$, then there exists a constant $C'(\mu, f, m) > 0$ such that:*
$$C(f, \mu)\phi_{1,q,m} \leq u \leq \frac{C'(\mu, f, m)}{\lambda_{1,q,m} - \mu}\phi_{1,q,m} \text{ in } \mathbb{R}^N.$$

(ii) *Assume that the weight m is radially symmetric and that $f \in X^{s,2}_{q,m}$. Then there exists a positive number $\delta'(f)$ (depending upon f) such that, for every $\mu \in (\lambda_{1,q,m}, \lambda_{1,q,m} + \delta'(f))$, $u \leq -C''(f, \mu)\phi_{1,q,m}$ in $\mathbb{R}^N$ with $C''(f, \mu) = \frac{\int_{\mathbb{R}^N} f\phi_{1,q,m}}{\mu-\lambda_{1,q,m}} - \Gamma'(\mu, f)$ and with $\lim_{\mu\to\lambda_{1,q,m}} \Gamma'(\mu, f) = \Gamma' < +\infty$.*

For the proof, see Appendix A.

As for the case of a positive weight, we can obtain a result on ground state positivity but not on ground state negativity (because our proof for the antimaximum principle in Theorem 1.5 (ii) needs to consider a weight m such that $\|u\|_m = \sqrt{\int_{\mathbb{R}^N} mu^2}$ defines a norm in $L^2(\mathbb{R}^N)$ equivalent to the usual norm).

Theorem 1.6 *Assume that the potential q is radially symmetric and satisfies* $(\mathbf{H^1_q})$, $(\mathbf{H^2_q})$ *and the weight m satisfies* $(\mathbf{H'^1_m})$ *or* $(\mathbf{H'^2_m})$. *Furthermore if m satisfies* $(\mathbf{H'^2_m})$, *assume also that $m^+ \in L^\infty(\mathbb{R}^N)$ and that $|m(x)| \leq cstQ(|x|)^{1/2-\beta}$ for all $x \in \mathbb{R}^N$ (with Q the auxiliary function associated with q which satisfies* $(\mathbf{H^2_q})$*). Assume that $u \in D(-\Delta+q)$ is one solution of* (1.7), *$\mu \in \mathbb{R}$, $f \geq 0$ a.e. in $\mathbb{R}^N$ with $f > 0$ in some set of positive Lebesgue measure. Then for every μ such that $\tilde{\lambda}_{1,q,m} < \mu < \lambda_{1,q,m}$, there exists a constant $C(f, \mu) > 0$ such that: $u \geq C(f, \mu)\phi_{1,q,m}$ in $\mathbb{R}^N$.*

Proof: Assume that $\tilde{\lambda}_{1,q,m} < \mu < \lambda_{1,q,m}$ and $(-\Delta + q)u = \mu mu + f$ in $\mathbb{R}^N$. Note that $u \geq 0$ by the maximum principle (Theorem 1.4). Let $\alpha > 0$ be a positive real such that

$\alpha+(\mu-\lambda_{1,q,m})m > 0$ in $\mathbb{R}^N$ (which is possible for α sufficiently large since either m is bounded (case $(\mathbf{H_m'^1})$) or $m^+ \in L^\infty(\mathbb{R}^N)$ (case $(\mathbf{H_m'^2})$). Therefore u satisfies $(-\Delta + q - \lambda_{1,q,m}m)u = -\alpha u + g$ in $\mathbb{R}^N$ with $g = (\alpha + (\mu - \lambda_{1,q,m})m)u + f \geq 0$ in $\mathbb{R}^N$. Moreover 0 is the principal eigenvalue of the operator $-\Delta + q - \lambda_{1,q,m}m$ in $\mathbb{R}^N$. Thus, since $-\alpha < 0$ we can apply the Theorem 2.1 in [4] to obtain that $u \geq C\phi_{1,q,m}$ with C a positive constant which only depends of μ and f. $\square$

2 Results for systems

2.1 Results for linear systems

In this section, we consider (1.1) in the form:

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n \tag{2.1}$$

where each of the potentials q_i satisfy $(\mathbf{H_q^1})$ and each of the weights m_i satisfy one of the hypotheses among $(\mathbf{H_m^1})$, $(\mathbf{H_m^{*1}})$, $(\mathbf{H_m'^1})$, $(\mathbf{H_m^2})$, $(\mathbf{H_m'^2})$. We consider the hypotheses:

(H3) For all $i, j = 1, \cdots, n,\ a_{ij} \in L^\infty(\mathbb{R}^N)$ and $a_{ij} \geq 0$ if $i \neq j$.

(H4) For all $i = 1, \cdots, n,\ f_i \in L^2(\mathbb{R}^N)$.

(H5) For all $i, j = 1, \cdots, n,\ i \neq j$, there exists a positive constant K_{ij} such that $a_{ij} \leq K_{ij}\sqrt{|m_i m_j|}$.

Note that if each of the weights m_i satisfy $(\mathbf{H_m^1})$, then **(H5)** is automatically satisfied. Note also that in the particular case where $m_i = 1$ for each i, we can take $K_{ij} = \|a_{ij}\|_{L^\infty(\mathbb{R}^N)}$. We denote by

$$\lambda_i := \lambda_{1,q_i,m_i} \text{ and } \phi_i := \phi_{1,q_i,m_i} \tag{2.2}$$

the eigenpair for the operator $-\Delta + q_i$ associated with the weight m_i in $\mathbb{R}^N$. We denote by $L = (l_{ij})$ and $P = (p_{ij})$ the $n \times n$-matrices given as follows

$$l_{ii} := \lambda_i - \mu_i \text{ and } l_{ij} = -K_{ij}\ (i \neq j) \tag{2.3}$$

$$p_{ii} := 1 - |\mu_i|C_i\|m_i\| \text{ and } p_{ij} = -K_{ij}\sqrt{C_iC_j\|m_i\|\|m_j\|}\ (i \neq j) \tag{2.4}$$

where $\|m_i\|$ denotes either $\|m_i\|_{L^\infty(\mathbb{R}^N)}$ if m_i satisfies $(\mathbf{H_m'^1})$ or $\|m_i\|_{L^{N/2}(\mathbb{R}^N)}$ if m_i satisfies $(\mathbf{H_m'^2})$ and where $C_i = \max(1, \frac{1}{\inf q_i})\tilde{C}_0$ with either $\tilde{C}_0 = 1$ if m_i satisfies $(\mathbf{H_m'^1})$ or $\tilde{C}_0$ is the square of the Sobolev constant for the embedding of $H^1(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ if m_i satisfies $(\mathbf{H_m'^2})$. Note that (see (1.2))

$$\int_{\mathbb{R}^N} m_i u_i^2 \leq C_i \|m_i\| \, \|u_i\|_{q_i}^2 \text{ for all } u_i \in V_{q_i}(\mathbb{R}^N). \tag{2.5}$$

For positive weights m_i, we recall the maximum principle (see [11, Theorem 2.1] in the case of weights m_i which satisfy $(\mathbf{H_m^1})$).

Theorem 2.1 *Assume that each of the potentials q_i satisfy $(\mathbf{H_q^1})$ and each of the weights m_i satisfy $(\mathbf{H_m^{*1}})$ or $(\mathbf{H_m^2})$. Assume also that* **(H3)***-***(H5)** *are satisfied and that the matrix L, defined by* (2.3)*, is a non singular M-matrix.*

(i) *Then the cooperative system* (2.1) *satisfies the maximum principle (i. e. for any $f = (f_1, \cdots, f_n) \geq 0$, then $u_i \geq 0$ for all i, with $u = (u_1, \cdots, u_n)$ solution of* (2.1)*).*

(ii) *Assume here that each of the weights m_i satisfy $(\mathbf{H_m^{*1}})$. Then the cooperative system* (2.1) *satisfies the ground state positivity (i. e. for any $f = (f_1, \cdots, f_n) \geq 0$, $f_i \not\equiv 0$ then there exists a positive constant C such that $u_i \geq C\phi_i$ for all i, with ϕ_i defined by* (2.2)*).*

Proof:

(i) Assume that for all $i = 1, \cdots, n$, $f_i \geq 0$. Let $u = (u_1, \cdots, u_n)$ be a solution of the system (2.1) and define $u_i^- = \max(0, -u_i)$. Multiplying by u_i^- and integrating over $\mathbb{R}^N$, using **(H5)** we get:

$$0 \leq \|u_i^-\|_{q_i}^2 \leq \mu_i \int_{\mathbb{R}^N} m_i (u_i^-)^2 + \sum_{j=1; j\neq i}^{n} K_{ij} \left(\int_{\mathbb{R}^N} m_i (u_i^-)^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} m_j (u_j^-)^2 \right)^{1/2}. \tag{2.6}$$

Let X the vector be defined by ${}^tX = (x_1, \cdots, x_n)$ with $x_i = \left(\int_{\mathbb{R}^N} m_i (u_i^-)^2\right)^{1/2}$. From the characterization of λ_i and from (2.6), we have:

$$(\lambda_i - \mu_i) \int_{\mathbb{R}^N} m_i (u_i^-)^2 - \sum_{j=1; j\neq i}^{n} K_{ij} \left(\int_{\mathbb{R}^N} m_i (u_i^-)^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} m_j (u_j^-)^2 \right)^{1/2} \leq 0. \tag{2.7}$$

We denote by $(LX)_i = (\lambda_i - \mu_i) x_i - \sum_{j=1; j\neq i}^{n} K_{ij} x_j$. From (2.7) note that $(LX)_i \leq 0$ for each i and so $LX \leq 0$. Since L is a non singular M-matrix (see [6]), we can deduce that $X \leq 0$ and thus $X = 0$, i. e. $x_i = 0$ for each i. So from (2.6) we get for each i : $\|u_i^-\|_{q_i} = 0$ i. e. $u_i \geq 0$.

(ii) We combine the maximum principle for the system (2.1) with the ground sate positivity for an equation. Indeed, from (i) we know that $u_i \geq 0$ for all i and so $g_i := \sum_{j=1; j\neq i}^{n} a_{ij} u_j + f_i \geq 0$, $g_i > 0$ in a set of non zero measure. Therefore, since $\mu_i < \lambda_i$, we get that there exists a positive constant C_i such that $u_i \geq C_i \phi_i$.

□

Proceeding as for Theorem 2.1, we obtain the following maximum principle for indefinite weights.

Theorem 2.2 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and each of the weights m_i satisfy* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$*. Assume also that* $(\mathbf{H3})$-$(\mathbf{H5})$ *are satisfied.*

(i) *If the matrix P, defined by (2.4), is a non singular M-matrix, then the cooperative system (2.1) satisfies the maximum principle.*

(ii) *Assume also that, in the case of each of the weights m_i satisfy* $(\mathbf{H_m'^2})$*, $m_i^+ \in L^\infty(\mathbb{R}^N)$ and $|m_i(x)| \leq cstQ_i(|x|)^{1/2-\beta}$ for $x \in \mathbb{R}^N$ (with Q_i the auxiliary function associated with the potential q_i which satisfies* $(\mathbf{H_q^2})$*). If the matrix P is a non singular M-matrix, then the cooperative system (2.1) satisfies the ground state positivity.*

Note that, as for one equation, the condition "P is a non singular M-matrix" is a stronger hypothesis than the condition "L is a non singular M-matrix." Indeed, note that the hypothesis $1 - |\mu_i|C_i\|m_i\| > 0$ is stronger than the hypothesis $\tilde{\lambda}_{1,q_i,m_i} < \mu_i < \lambda_{1,q_i,m_i}$ (see (1.3)-(1.6),(2.5)).

For positive weights, we now recall the following result for the existence of solutions for the system (2.1) (see [11, Theorem 2.2 and Theorem 2.3] in the case of weights m_i which satisfy $(\mathbf{H_m^1})$).

Theorem 2.3 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and each of the weights m_i satisfy* $(\mathbf{H_m^{*1}})$ *or* $(\mathbf{H_m^2})$*. Assume also that* $(\mathbf{H3})$-$(\mathbf{H5})$ *are satisfied. If the matrix L is a non singular M-matrix, then the system (2.1) has a unique solution $u = (u_1, \cdots, u_n) \in V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$.*

For indefinite weights m_i, existence and uniqueness of a solution is stated as follows and is an application of the Lax-Milgram Theorem (see [11]).

Theorem 2.4 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and each of the weights m_i satisfy* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$*. Assume also that* $(\mathbf{H3})$-$(\mathbf{H5})$ *are satisfied. If the matrix P is a non singular M-matrix, then the system (2.1) has a unique solution $u = (u_1, \cdots, u_n) \in V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$.*

2.2 Existence of a global principal eigenvalue for a system

In this section, we consider the eigenvalue problem for the following system

$$(-\Delta + q_i)u_i = \lambda \left(m_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij} u_j \right) \text{ in } \mathbb{R}^N, \; i = 1, \cdots, n, \tag{2.8}$$

where each of the potentials q_i satisfy $(\mathbf{H_q^1})$ and each of the weights m_i satisfy one of the hypotheses among $(\mathbf{H_m^1})$, $(\mathbf{H_m^{*1}})$, $(\mathbf{H_m'^1})$. We denote by M is the $n \times n$-matrix given by $M = (m_{ij})$ with $m_{ii} := m_i$. We will consider the following hypotheses:

(H8) For all $i \neq j$, $m_{ij} \in L^\infty(\mathbb{R}^N)$ and $m_{ij} > 0$.

(H9) M is a symmetric matrix.

(H10) $\Omega := \cap_{i=1}^n \Omega_i^+$ is an open subset of $\mathbb{R}^N$ with non zero measure and with $\Omega_i^+ := \{x \in \mathbb{R}^N, m_i(x) > 0\}$.

We add another hypothesis upon the potentials q_i which assures that any weak solution $u_i \in V_{q_i}(\mathbb{R}^N)$ of the equation $(-\Delta + q_i)u_i = f_i$ in $\mathbb{R}^N$, with $f_i \in L^2(\mathbb{R}^N)$, belongs to the strong domain $D(-\Delta + q_i) \subset L^2(\mathbb{R}^N)$. It is the following hypothesis. For all $i = 1, \cdots, n$,

$(\mathbf{H_q^3})$ For all $x \in \mathbb{R}^N$ and all $h \in \mathbb{R}^N$, $h \neq 0$, $|\frac{q_i(x+h)-q_i(x)}{h}| \leq cst\ \sqrt{q_i(x)}$.

Note that for example, the potential $q(x) = 1 + |x|$ satisfies $(\mathbf{H_q^3})$.

Lemma 2.1 *Assume that the potential q satisfy $(\mathbf{H_q^1})$ and $(\mathbf{H_q^3})$. Let u be a weak solution of $(-\Delta + q)u = f$ in $\mathbb{R}^N$ with $f \in L^2(\mathbb{R}^N)$. Then $u \in H^2(\mathbb{R}^N)$, $qu \in L^2(\mathbb{R}^N)$ and therefore $u \in D(-\Delta + q)$.*

The proof of Lemma 2.1 is based on the methods of translations (see Appendix B). For strictly positive and bounded weights m_i, proceeding as for one equation (see [12, Theorem 2.1]), we can prove the existence of a positive eigenvalue associated with a positive eigenfunction for (2.8). Therefore, we extend here to Schrödinger operators defined in the whole space, some results of [21] and [8] for elliptic operators defined in a bounded domain.

Theorem 2.5 *Assume that each of the potentials q_i satisfy $(\mathbf{H_q^1})$ and each of the weights m_i satisfy $(\mathbf{H_m^{*1}})$. Assume also that* **(H8)** *is satisfied. Then there exists a unique principal eigenvalue $\Lambda_{1,M} > 0$ associated with a positive eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M}) \in V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ for the system (2.8). Moreover if* **(H9)** *and $(\mathbf{H_q^3})$ are satisfied then*

$$\Lambda_{1,M} = \inf \left\{ \frac{\sum_{i=1}^n \|u_i\|_{q_i}^2}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j},\ u = (u_1, \cdots, u_n) \in V \right.$$

$$\left. \textit{such that } \sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j > 0 \right\}. \tag{2.9}$$

Note that the condition $\sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j > 0$ is automatically satisfied if M is a definite positive matrix (i. e. for all $X \neq 0$, ${}^tXMX > 0$).

Proof: We denote by M the operator of multiplication by the matrix M in $(L^2(\mathbb{R}^N))^n$ and we consider the operator

$$L^{-1}M : ((L^2(\mathbb{R}^N))^n, \|.\|_{(L^2(\mathbb{R}^N))^n}) \to ((L^2(\mathbb{R}^N))^n, \|.\|_{(L^2(\mathbb{R}^N))^n}).$$

The operator $L^{-1}M$ is compact and strongly positive in the sense of quasi-interior points in $(L^2(\mathbb{R}^N))^n$, in the sense of Daners and Koch-Medina [15]. This implies that $L^{-1}M$ is irreducible and we apply the version of the Krein-Rutman Theorem given in [15, Theorem 12.3] to deduce that $r(L^{-1}M)$, the spectral radius of $L^{-1}M$, is a strictly positive and simple eigenvalue associated with an eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M})$ which is a quasi-interior point of $(L^2(\mathbb{R}^N))^n$, that is $\phi_{i,M} > 0$ in $\mathbb{R}^N$ for all i. Of course $\Lambda_{1,M} = \frac{1}{r(L^{-1}M)} > 0$ and $r(L^{-1}M)$ is the only one eigenvalue of $L^{-1}M$ associated with a positive eigenfunction.

We recall that $V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ and the inner product in V is defined by $< u, v >_V = \sum_{i=1}^n < u_i, v_i >_{q_i}$ for all $u = (u_1, \cdots, u_n) \in V$ and $v = (v_1, \cdots, v_n) \in V$. We set the bilinear form

$$\beta(u, v) = \sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i v_i + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_j v_i \text{ for all } u \in V \text{ and } v \in V.$$

From hypotheses **(H8)** and **(H9)**, β is a bilinear, symmetric and continous form. From the Riesz Theorem, we get the existence of a continuous operator $T : V \to V$, $T = (T_1, \cdots, T_n)$, such that $\beta(u, v) =< Tu, v >_V$ for all $u \in V$ and $v \in V$ (see [17] for the Lax-Milgram Theorem). We can easily prove that the operator T is compact.

Moreover, since the matrix M is assumed to be symmetric, the operator T is selfadjoint. So the largest eigenvalue of T is given by:

$$\mu_{1,M} = \sup_{u\in V, u\neq 0} \frac{< Tu, u >_V}{< u, u >_V} = \sup_{u\in V, u\neq 0} \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_j u_i}{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla u_i|^2 + q_i u_i^2]}.$$

Choosing $u = (u_1, \cdots, u_n) \in V$ such that *supp* $u_i \subset \{x \in \mathbb{R}^N, m_i(x) > 0\}$ for one i and $u_j = 0$ if $j \neq i$, we get that $\mu_{1,M} > 0$.

Now, we prove that $\Lambda_{1,M} = \frac{1}{\mu_{1,M}}$. We have $L^{-1}M\Phi_{1,M} = r(L^{-1}M)\Phi_{1,M}$ or equivalently $L\Phi_{1,M} = \Lambda_{1,M} M\Phi_{1,M}$. Therefore for all $i = 1, \cdots, n$:

$$(-\Delta + q_i)\phi_{i,M} = \Lambda_{1,M}(m_i\phi_{i,M} + \sum_{j=1;j\neq i} m_{ij}\phi_{j,M}) \text{ in } \mathbb{R}^N.$$

Thus for all $v = (v_1, \cdots, v_n) \in V$, we have:

$$\sum_{i=1}^n \int_{\mathbb{R}^N} [\nabla\phi_{i,M}.\nabla v_i + q_i\phi_{i,M} v_i] = \Lambda_{1,M} \sum_{i=1}^n \left(\int_{\mathbb{R}^N} m_i\phi_{i,M} v_i + \sum_{j=1;j\neq i}^n \int_{\mathbb{R}^n} m_{ij}\phi_{j,M} v_i \right).$$

For $v_i = \phi_{i,M}$, we get:

$$\frac{1}{\Lambda_{1,M}} = \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i \phi_{i,M}^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\phi_{j,M}\phi_{i,M}}{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla \phi_{i,M}|^2 + q_i \phi_{i,M}^2]} \leq \mu_{1,M}. \tag{2.10}$$

Moreover, since $\mu_{1,M}$ is an eigenvalue of the operator T defined above, let $\psi = (\psi_1, \cdots, \psi_n)$ be an eigenfunction associated with $\mu_{1,M}$. Since $T\psi = \mu_{1,M}\psi$, we have for all $v \in V$: $\mu_{1,M} < \psi, v >_V = < T\psi, v >_V = \beta(\psi, v)$ and so

$$\mu_{1,M} \sum_{i=1}^n \int_{\mathbb{R}^N} [\nabla \psi_i . \nabla v_i + q_i \psi_i v_i] = \sum_{i=1}^n \int_{\mathbb{R}^N} m_i \psi_i v_i + \sum_{i,j=1;j\neq i}^n \int_{\mathbb{R}^n} m_{ij}\psi_j v_i.$$

For $v = (0, \cdots, 0, v_i, 0, \cdots, 0) \in V$, we get:

$$\int_{\mathbb{R}^N} [\nabla \psi_i . \nabla v_i + q_i \psi_i v_i] = \frac{1}{\mu_{1,M}} \left(\int_{\mathbb{R}^N} m_i \psi_i v_i + \sum_{j=1;j\neq i}^n \int_{\mathbb{R}^n} m_{ij}\psi_j v_i \right).$$

Therefore, using Lemma 2.1, we have $L\psi = \frac{1}{\mu_{1,M}} M\psi$ or equivalently $L^{-1}M\psi = \mu_{1,M}\psi$. Thus $\mu_{1,M}$ is an eigenvalue of the operator $L^{-1}M$ and

$$0 < \mu_{1,M} \leq r(L^{-1}M) = \frac{1}{\Lambda_{1,M}}. \tag{2.11}$$

From (2.10) and (2.11), we deduce that $\mu_{1,M} = \frac{1}{\Lambda_{1,M}}$ and $\Lambda_{1,M}$ satisfies (2.9). $\square$

Now, for indefinite bounded weights m_i, proceeding as for one equation (see [12, Theorem 3.1]), we prove the existence and the uniqueness of a principal positive eigenvalue for (2.8). This is the following result.

Theorem 2.6 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and* $(\mathbf{H_q^3})$ *and each of the weights m_i satisfy* $(\mathbf{H_m'^1})$. *Assume also that* **(H8)**-**(H10)** *are satisfied. Then there exists a unique principal eigenvalue $\Lambda_{1,M} > 0$ associated with a positive eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M}) \in V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$, $\phi_{i,M} > 0$ and $\Lambda_{1,M}$ satisfies* (2.9).

Proof: We follow a method developed in [19] (for one equation in a bounded domain). Let $\Omega_i^+ = \{x \in \mathbb{R}^N, m_i(x) > 0\}$, *meas* $(\Omega_i^+) > 0$, $\Omega_i^- = \{x \in \mathbb{R}^N, m_i(x) < 0\}$, *meas* $(\Omega_i^-) > 0$, and $\Omega_i^0 = \{x \in \mathbb{R}^N, m_i(x) = 0\}$. Let $(u_1, \cdots, u_n)$ be a solution of (2.8). We have for all i :

$$(-\Delta + q_i)u_i + \lambda m_i^- u_i = \lambda(m_i^+ u_i + \sum_{j=1;j\neq i}^n m_{ij}u_j) \text{ in } \mathbb{R}^N. \tag{2.12}$$

For given $\lambda > 0$, we rewrite (2.12) as an eigenvalue problem with parameter $\sigma(\lambda)$. For all i,

$$(-\Delta + q_i)u_i + \lambda(m_i^- + 1_i)u_i = \sigma(\lambda) \left((m_i^+ + 1_i)u_i + \sum_{j=1;j\neq i}^n m_{ij}u_j \right) \text{ in } \mathbb{R}^N \tag{2.13}$$

where 1_i denotes the characteristic function of $\Omega_i^0 \cup \Omega_i^-$. We denote by $Q_i := q_i + \lambda(m_i^- + 1_i)$ and $\rho_i := m_i^+ + 1_i$. Then (2.13) is equivalent to

$$(-\Delta + Q_i)u_i = \sigma(\lambda)(\rho_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij}u_j) \text{ in } \mathbb{R}^N. \tag{2.14}$$

Note that the weight $\rho_i > 0$ in $\mathbb{R}^N$, $\rho_i \in L^\infty(\mathbb{R}^N)$ since $m_i \in L^\infty(\mathbb{R}^N)$ and Q_i satisfies $(\mathbf{H_q^1})$ since $\lambda > 0$. From Theorem 2.5, we deduce that the system (2.14) has a unique principal eigenvalue $\sigma(\lambda)$ associated with a principal eigenfunction $\Phi_\lambda = (\phi_{1,\lambda}, \cdots, \phi_{n,\lambda})$, $\phi_{i,\lambda} > 0$. Moreover, since $D(-\Delta + Q_i) = D(-\Delta) \cap D(Q_i)$, from (2.9), we get:

$$\sigma(\lambda) = \inf \left\{ \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\psi_i|^2 + q_i\psi_i^2] + \lambda \sum_{i=1}^n \int_{\mathbb{R}^N} (m_i^- + 1_i)\psi_i^2}{\sum_{i=1}^n \int_{\mathbb{R}^N} (m_i^+ + 1_i)\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j}, \psi = (\psi_1, \cdots, \psi_n) \in V \right.$$
$$\left. \text{such that } \sum_{i=1}^n \int_{\mathbb{R}^N} (m_i^+ + 1_i)\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j > 0 \right\}. \tag{2.15}$$

Therefore, $\sigma(\lambda) < \Lambda^+_{1,Q,N}$ the principal eigenvalue of the operator L_Q associated with the matrix $N = (n_{ij})$ where $L_Q = diag\ (-\Delta + Q_i)$, $n_{ii} = \rho_i$ and $n_{ij} = m_{ij}$ in $\Omega = \cap_{i=1}^n \Omega_i^+$ with Dirichlet boundary condition. Note that $\sigma : \lambda \mapsto \sigma(\lambda)$ is increasing and continuous and that $\sigma(0) > 0$. Therefore for all $\lambda > 0$, we have $0 < \sigma(0) < \sigma(\lambda) < \Lambda^+_{1,Q,N}$ and $\Lambda^+_{1,Q,N}$ is in fact independant of λ. Thus we deduce that there exists $0 < \tilde{\lambda} < \Lambda^+_{1,Q,N}$ such that $\sigma(\tilde{\lambda}) = \tilde{\lambda}$. Proceeding as in [19], we can show that $\tilde{\lambda}$ is unique.
Now, we verify that $\tilde{\lambda}$ satisfies (2.9). Let us denote by

$$\Lambda_{1,M} = \inf \left\{ \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\psi_i|^2 + q_i\psi_i^2]}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j}, \psi = (\psi_1, \cdots, \psi_n) \in V \right.$$
$$\left. \text{such that } \sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j > 0 \right\}.$$

Since

$$\tilde{\lambda} = \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\phi_{i,\tilde{\lambda}}|^2 + q_i\phi_{i,\tilde{\lambda}}^2] + \tilde{\lambda} \sum_{i=1}^n \int_{\mathbb{R}^N} (m_i^- + 1_i)\phi_{i,\tilde{\lambda}}^2}{\sum_{i=1}^n \int_{\mathbb{R}^N} (m_i^+ + 1_i)\phi_{i,\tilde{\lambda}}^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\phi_{i,\tilde{\lambda}}\phi_{j,\tilde{\lambda}}},$$

we have $\tilde{\lambda} \geq \Lambda_{1,M}$.
Moreover let $\psi = (\psi_1, \cdots, \psi_n) \in V$ be such that $\sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j > 0$. From (2.15), since $\tilde{\lambda} = \sigma(\tilde{\lambda})$, we get $\tilde{\lambda} \leq \frac{\sum_{i=1}^n \int_{\mathbb{R}^N} [|\nabla\psi_i|^2 + q_i\psi_i^2]}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i\psi_i^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}\psi_i\psi_j}$. Thus $\tilde{\lambda} \leq \Lambda_{1,M}$. □

Note that for all $i = 1, \cdots, n$, $\Lambda_{1,M} < \lambda_i$.
Indeed, from (1.4) and (2.9), we have $\Lambda_{1,M} \leq \lambda_i$. Suppose that $\Lambda_{1,M} = \lambda_i$. Then

$$(-\Delta + q_i)(\phi_{i,M} - \phi_i) = \lambda_i m_i(\phi_{i,M} - \phi_i) + \lambda_i \sum_{j=1;j\neq i}^{n} m_{ij}\phi_{j,M} \text{ in } \mathbb{R}^N,$$

where ϕ_i (resp. $\phi_{i,M}$) is defined by (2.2) (resp. Theorem 2.6). Multiplying by ϕ_i and integrating over $\mathbb{R}^N$, we obtain (since $\lambda_i > 0$), $\int_{\mathbb{R}^N} \sum_{j=1;j\neq i}^n m_{ij}\phi_{j,M}\phi_i = 0$. Since $m_{ij} > 0$, $\phi_{i,M} > 0$ and $\phi_i > 0$ we get a contradiction.

Now, we consider the following system

$$(-\Delta + q_i)u_i = \lambda \left(m_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij}u_j \right) + f_i \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n. \tag{2.16}$$

We give a maximum principle result.

Theorem 2.7 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and* $(\mathbf{H_q^3})$ *and each of the weights m_i satisfy* $(\mathbf{H_m^{*1}})$ *or* $(\mathbf{H_m^{\prime 1}})$. *Assume also that* **(H8)**-**(H9)** *are satisfied. Furthermore if the weights m_i satisfy* $(\mathbf{H_m^{\prime 1}})$, *assume also that* **(H10)** *is satisfied.Assume that $f_i \in L^2(\mathbb{R}^N)$ for all i. If $0 \leq \lambda < \Lambda_{1,M}$, then the system (2.16) satisfies the maximum principle: if $f = (f_1, \cdots, f_n) \geq 0$, then $u_i \geq 0$ for all i with $u = (u_1, \cdots, u_n)$ solution of (2.16).*

Note that we have the same condition $0 \leq \lambda < \Lambda_{1,M}$ as in [21, Proposition 2.2].

Proof: Multiplying (2.16) by u_i^-, integrating over $\mathbb{R}^N$, since $\lambda \geq 0$ and $f_i \geq 0$, we have:

$$0 \leq \sum_{i=1}^n \|u_i^-\|_{q_i}^2 \leq \lambda (\sum_{i=1}^n \int_{\mathbb{R}^N} m_i (u_i^-)^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij}u_i^- u_j^-) := \lambda C(u^-) = \lambda C(u_1^-, \cdots, u_n^-).$$

If $C(u^-) > 0$, then $\Lambda_{1,M} \leq \frac{\sum_{i=1}^n \|u_i^-\|_{q_i}^2}{C(u^-)} \leq \lambda$ and we get a contradiction with the hypothesis $\lambda < \Lambda_{1,M}$. Thus $C(u^-) = 0$. Then $\sum_{i=1}^n \|u_i^-\|_{q_i}^2 = 0$ and therefore $u_i \geq 0$ for all i. □

We can state a result for the existence of solutions for the system (2.16) as follows.

Theorem 2.8 *Assume that each of the potentials q_i satisfy* $(\mathbf{H_q^1})$ *and* $(\mathbf{H_q^3})$ *and each of the weights m_i satisfy* $(\mathbf{H_m^{*1}})$ *or* $(\mathbf{H_m^{\prime 1}})$. *Assume also that* **(H8)**-**(H9)** *are satisfied. Furthermore if the weights m_i satisfy* $(\mathbf{H_m^{\prime 1}})$, *assume also that* **(H10)** *is satisfied. Assume that $f_i \in L^2(\mathbb{R}^N)$ for all i. If $0 \leq \lambda < \Lambda_{1,M}$, then the system (2.16) has a unique solution $u = (u_1, \cdots, u_n) \in V$.*

Proof: We introduce a bilinear continuous form l and we apply the Lax-Milgram Theorem. Let $l : (V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N))^2 \to \mathbb{R}$ be defined by

$$l(u, v) = \sum_{i=1}^n \int_{\mathbb{R}^N} [\nabla u_i . \nabla v_i + q_i u_i v_i - \lambda m_i u_i v_i - \lambda \sum_{j=1;j\neq i}^n m_{ij} u_j v_i].$$

First note that from (2.9) we have: $\Lambda_{1,M} C(u) \leq \sum_{i=1}^n \|u_i\|_{q_i}^2$ for all $u = (u_1, \cdots, u_n) \in V$. Therefore, since $\lambda \geq 0$, we get: $l(u, u) \geq \frac{\Lambda_{1,M} - \lambda}{\Lambda_{1,M}} \sum_{i=1}^n \|u_i\|_{q_i}^2$ and so l is coercive. By the Lax-Milgram Theorem, we get the existence and the uniqueness of a weak solution for the system (2.16). □

2.3 Study of the signs of the solutions for a 2×2 system

We consider in this section the following system (for $N \geq 2$):

$$\begin{cases} (-\Delta + q)u = \lambda u + au + bv + f \text{ in } \mathbb{R}^N \\ (-\Delta + q)v = \lambda v + cu + dv + g \text{ in } \mathbb{R}^N. \end{cases} \tag{2.17}$$

The real λ is a real parameter and the potential q is radially symmetric and satisfies $(\mathbf{H_q^1})$ and $(\mathbf{H_q^2})$. The aim of this section is to present some results concerning positivity or negativity of the solutions of the system (2.17). We can find results for the antimaximum principle for a system of two equations with constant coefficients in [2]; the ideas, there, are to decouple the system, and then to apply the results of the antimaximum principle for each equation. We will follow this method in this section.
We denote by $M(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ the coupling matrix of the coefficients of the system (2.17). Following [14], we introduce S an invertible 2×2 matrix of constants such that S diagonalises $M(x)$ for all x. In [14], it is proved that such a choice is possible only if either (case I) $b(x)$ and $c(x)$ are both multiples of $a(x) - d(x)$ or (case II) $a(x) = d(x)$ for all x and $b(x)$ and $c(x)$ are positive multiples of each other. We define the functions u^* and v^* by

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = S^{-1} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} f^* \\ g^* \end{pmatrix} = S^{-1} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{2.18}$$

and since S is a constant matrix, we obtain from (2.17)

$$\begin{pmatrix} -\Delta + q & 0 \\ 0 & -\Delta + q \end{pmatrix} \begin{pmatrix} u^* \\ v^* \end{pmatrix} = \lambda \begin{pmatrix} u^* \\ v^* \end{pmatrix} + S^{-1} M(x) S \begin{pmatrix} u^* \\ v^* \end{pmatrix} + \begin{pmatrix} f^* \\ g^* \end{pmatrix}. \tag{2.19}$$

We suppose that the coefficients a, b, c, d of the system satisfy the following hypothesis:

$$\textbf{(H11)} \begin{cases} \text{(i)} & a, b, c, d \in L^\infty(\mathbb{R}^N). \\ \text{(ii)} & \text{either } b \text{ and } c \text{ are positive multiples of } a - d \text{ (case I)} \\ & \text{or } a = d \text{ and } b \text{ and } c \text{ are positive multiples of each other (case II)} \\ \text{(iii)} & a, b, c, d \text{ are radially symmetric functions.} \end{cases}$$

Note that the hypothesis **(H11)**(iii) upon the coefficients of the matrix M of the system (2.17) assures that the weights of each equation (after decoupling (2.17)) are radially symmetric. Here we consider the case I and we rewrite the matrix $M(x)$ under the following form:

$$M(x) = \begin{pmatrix} a(x) & b^*(a(x) - d(x)) \\ c^*(a(x) - d(x)) & d(x) \end{pmatrix} \quad \text{(case I)} \tag{2.20}$$

where $a \neq d$ and b^* and c^* are constants such that $1 + 4b^*c^* > 0$.

Moreover we assume that the following hypothesis is satisfied:

(H12) $f, g \in L^2(\mathbb{R}^N)$.

Then we define the two following constants $\rho_1 = \frac{1+\sqrt{1+4b^*c^*}}{2}$, $\rho_2 = \frac{1-\sqrt{1+4b^*c^*}}{2}$ and we choose $S = \begin{pmatrix} -b^* & -b^* \\ \rho_1 & \rho_2 \end{pmatrix}$. Thus we have $u = -b^*(u^* + v^*)$ and $v = \rho_1 u^* + \rho_2 v^*$. Now, if we define the functions

$$\mu_1(x) := \frac{1}{\rho_1 - \rho_2}[\rho_1 d(x) - \rho_2 a(x) + 2\rho_1\rho_2(a(x) - d(x))] \tag{2.21}$$

$$\mu_2(x) := \frac{1}{\rho_1 - \rho_2}[\rho_1 a(x) - \rho_2 d(x) - 2\rho_1\rho_2(a(x) - d(x))], \tag{2.22}$$

then we can write the decoupled system (see (2.17)-(2.22)) as

$$\begin{cases} (-\Delta + q)u^* = \lambda u^* + \mu_1 u^* + \frac{1}{b^*(\rho_1-\rho_2)}[\rho_2 f + b^* g] \text{ in } \mathbb{R}^N \\ (-\Delta + q)v^* = \lambda v^* + \mu_2 v^* - \frac{1}{b^*(\rho_1-\rho_2)}[\rho_1 f + b^* g] \text{ in } \mathbb{R}^N. \end{cases}$$

Theorem 2.9 *Assume that the potential q satisfies $(\mathbf{H_q^1})$-$(\mathbf{H_q^2})$ and that the hypotheses* **(H11)**-**(H12)** *are satisfied. Assume also that the matrix M has the form (2.20) with $b^*c^* < 0$ and $1 + 4b^*c^* > 0$. Let μ_1 and μ_2 functions be defined as in (2.21) and (2.22). Assume that μ_1 and μ_2 are functions such that $\lambda + \mu_1 \geq cst > 0$ and $\lambda + \mu_2 \geq cst > 0$. Define $f^* = \frac{1}{b^*(\rho_1-\rho_2)}[\rho_2 f + b^* g]$ and $g^* = -\frac{1}{b^*(\rho_1-\rho_2)}[\rho_1 f + b^* g]$.*

1. *Assume that $\lambda_{1,q,\lambda+\mu_1} - \delta(f^*) < 1 < \lambda_{1,q,\lambda+\mu_1}$, $\lambda_{1,q,\lambda+\mu_2} - \delta(g^*) < 1 < \lambda_{1,q,\lambda+\mu_2}$, $0 < f^* \in X^{s,2}_{q,\lambda+\mu_1}$ and $0 < g^* \in X^{s,2}_{q,\lambda+\mu_2}$, with $\delta(f^*)$, $\delta(g^*)$ defined in Theorem 1.5*
   Then u has the same sign as $-b^*$ and $v > 0$.

2. *Assume that $\lambda_{1,q,\lambda+\mu_1} < 1 < \lambda_{1,q,\lambda+\mu_1} + \delta'(f^*)$, $\lambda_{1,q,\lambda+\mu_2} < 1 < \lambda_{1,q,\lambda+\mu_2} + \delta'(g^*)$, $0 < f^* \in X^{s,2}_{q,\lambda+\mu_1}$ and $0 < g^* \in X^{s,2}_{q,\lambda+\mu_2}$.*
   Then $v < 0$ and u has the same sign as b^*.

Note that the above results are just consequences of the diagonalization of the coupling matrix M and applications of Theorem 1.5. We can also obtain similar results in the case II. Note that for λ sufficiently large, since each function μ_i is bounded, we have $\lambda + \mu_i \geq cst > 0$. Moreover if $b^* > 0$ e.g., choosing $g > 0$ and f such that $-\frac{b^* g}{\rho_2} < f < -\frac{b^* g}{\rho_1}$, we have $f^* > 0$ and $g^* > 0$.

A Appendix: Ground state positivity and negativity

We only give a sketch of the proof in $\mathbb{R}^2$. We recall that the space $X_{q,m}$ is defined by $X_{q,m} = \{u \in L^2(\mathbb{R}^2), \frac{u}{\phi_{1,q,m}} \in L^\infty(\mathbb{R}^2)\}$ and the space $X^{1,2}_{q,m}$ is defined by

$X^{1,2}_{q,m} = \{f : \mathbb{R}^2 \to \mathbb{R},\ \frac{\partial f}{\partial \theta}(r,.) \in L^2(-\pi,\pi)$ for all $r \geq 0$ and there exists a constant $C \geq 0$ such that $|f(r,\theta)| + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\frac{\partial f}{\partial \theta}(r,\theta)|^2\, d\theta\right)^{1/2} \leq C\phi_{1,q,m}(r)$ a.e.$\}$.

Note that the ground state positivity is a simple application of the weak maximum principle combined with [4, Theorem 2.1]. Note also that if $f \in X_{q,m}$ and $f \geq 0$ then there exists a positive constant $C(f)$ such that $0 \leq f \leq C(f)\phi_{1,q,m}$. Choosing $C'(f,m) = \frac{C(f)}{m_0(\lambda_{1,q,m}-\mu)}$ with $m_0 = \inf m > 0$, from the weak maximum principle for the scalar case, writing $(-\Delta + q)(C'(f,m)\phi_{1,q,m} - u) = \mu m(C'(f,m)\phi_{1,q,m} - u) + (\lambda_{1,q,m} - \mu)mC'(f,m)\phi_{1,q,m} - f$ in $\mathbb{R}^2$, we obtain that $u \leq C'(f,m)\phi_{1,q,m}$.

The proof of the ground state negativity is based upon ideas of [20] and [4]. We decompose it in several steps.

Step 1 : We denote by $L_q := -\Delta + q$ and by M the operator of multiplication by m. As in Hess [20] we consider the operator $L_q^{-1}M$ and the same decomposition of $L^2(\mathbb{R}^2) = span(\phi_{1,q,m}) \oplus R(I - \lambda_{1,q,m}L_q^{-1}M)$ where $R(I - \lambda_{1,q,m}L_q^{-1}M)$ is the range of the operator $I - \lambda_{1,q,m}L_q^{-1}M$. But because of the unboundedness of our domain, we cannot study $R(I - \lambda_{1,q,m}L_q^{-1}M)$ as it done in [20] and we adapt to our case an idea developed in [3] which is the following decomposition of $L^2(\mathbb{R}^2) = H_1 \oplus H_2 \oplus H_3$ with

$$\begin{aligned}
H_1 =& span(\phi_{1,q,m})\\
H_2 =& \{f \in L^2(\mathbb{R}^2)\colon f(x) \equiv f(|x|) \text{ with } \int_0^\infty m(r)f(r)\phi_{1,q,m}(r)r\,dr = 0\};\\
H_3 =& \{f \in L^2(\mathbb{R}^2)\colon \int_{-\pi}^{\pi} f(r,\theta)\,d\theta = 0 \text{ for almost all } r \geq 0\}.
\end{aligned}$$

Note that $\|.\|_m$ defined by (1.2) is a norm equivalent to the usual norm in $L^2(\mathbb{R}^2)$ since m satisfies $(\mathbf{H^1_m})$. It is obvious that $L^2(\mathbb{R}^2) = H_1 \oplus H_2 \oplus H_3$ is an orthogonal decomposition. The corresponding orthogonal projections P_1, P_2 and P_3, respectively, take the following forms, for each $f \in L^2(\mathbb{R}^2)$: $P_1 f = \frac{(f,\phi_{1,q,m})_m}{(\phi_{1,q,m},\phi_{1,q,m})_m}\,\phi_{1,q,m}$, $P_2 f = (I - P_1)f^*$ with $f^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)\,d\theta$, $P_3 f = f - f^*$.

Step 2 : Let u be a solution of (1.7), we decompose u and $L_q^{-1}f = g$ in $L^2(\mathbb{R}^2)$ under the following way: $u = \beta_\mu\phi_{1,q,m} + u_2 + u_3$ with $u_2 \in H_2$, $u_3 \in H_3$ and $g = g_1 + g_2 + g_3$. It is easy to check that: $g_1 = (I - \mu L_q^{-1}M)\beta_\mu\phi_{1,q,m}$, $g_2 = (I - \mu L_q^{-1}M)u_2$ and $g_3 = (I - \mu L_q^{-1}M)u_3$. The idea then is to show that the sign of u is given by β_μ and that u_2 and u_3 belong to $X_{q,m}$. For that we need the two following Propositions based on Propositions 3.5 and 3.6 in [3].

Proposition A.1 (see [1, Proposition 3.1]) *Assume that q is a radially symmetric potential which satisfies* $(\mathbf{H^1_q})$-$(\mathbf{H^2_q})$ *and that m is a radially symmetric weight which satisfies*

($\mathbf{H^1_m}$). *Assume that $u_2, g_2 \in D(L_q)$, $L_q u_2 - \lambda_{1,q,m} M u_2 = L_q g_2 \in L^2(\mathbb{R}^2)$ with g_2 a radial symmetric function.*

(i) *If $\int_{\mathbb{R}^2} L_q g_2.\phi_{1,q,m} = 0$ and $\int_{\mathbb{R}^2} u_2 m\phi_{1,q,m} = 0$, then u_2 is radial and there exists a constant $\Gamma > 0$ (depending exclusively upon the potential q and the weight m) such that $|L_q g_2| \leq c\phi_{1,q,m} \Rightarrow |u_2| \leq \Gamma c\phi_{1,q,m}$.*

(ii) *If $\int_{\mathbb{R}^2} m.L_q g_2.\phi_{1,q,m} = 0$ and $\int_{\mathbb{R}^2} u_2 m\phi_{1,q,m} = 0$, then u_2 is radial and there exists a constant $\Gamma > 0$ (depending exclusively upon the potential q and the weight m) such that $|L_q g_2| \leq c\phi_{1,q,m} \Rightarrow |u_2| \leq \Gamma c\phi_{1,q,m}$.*

Proposition A.2 (see [1, Proposition 3.2]) *Assume that q is a radially symmetric potential which satisfies ($\mathbf{H^1_q}$)-($\mathbf{H^2_q}$) and that m is a radially symmetric weight which satisfies ($\mathbf{H^1_m}$). Assume that $u_3, g_3 \in D(L_q)$, $L_q u_3 - \lambda_{1,q,m} M u_3 = L_q g_3 \in L^2(\mathbb{R}^2)$ with $L_q g_3 \in H_3$ and $u_3 \in H_3$. If $L_q g_3 \in X^{1,2}_{q,m}$, then there exists a constant $\Gamma > 0$ (depending exclusively upon the potential q and the weight m) such that $\|u_3\|_{X^{1,2}_{q,m}} \leq \Gamma \|L_q g_3\|_{X^{1,2}_{q,m}}$.*

Step 3 : First note that if $f = L_q g = L_q g_1 + L_q g_2 + L_q g_3$ then $L_q g_1 + L_q g_2$ is obviously radially symmetric and so $L_q g_3 = P_3 f$. Note also that if $f \in X_{q,m}$ then $L_q g_1 \in X_{q,m}$, $L_q g_2 \in X_{q,m}$ and $L_q g_3 \in X_{q,m}$. Indeed $f^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)\, d\theta$ is in $X_{q,m}$ too and $L_q g_3 = P_3 f = f - f^*$ is in $X_{q,m}$. More $L_q g_1$ belongs to $X_{q,m}$ since m is bounded. Then we get $L_q g_2 \in X_{q,m}$.

Now, we study each component of the decomposition of u.
First, we calculate β_μ. Recall that $g_1 = \alpha\phi_{1,q,m}$ with the constant $\alpha = (L_q^{-1} f, \phi_{1,q,m})_m = \frac{1}{\lambda_{1,q,m}}\int_{\mathbb{R}^2} f\phi_{1,q,m}$. Since f is positive, $\alpha > 0$. Therefore, we get $\beta_\mu = \frac{\alpha\lambda_{1,q,m}}{\lambda_{1,q,m}-\mu}$.
Then, we prove that $u_2 \in X_{q,m}$. Writing down the Neumann series for the resolvant $(I - \mu L_q^{-1} M)^{-1}$:

$$u_2 = \sum_n (\mu - \lambda_{1,q,m})^n (M^{-1} L_q - \lambda_{1,q,m} I)^{-n} (I - \lambda_{1,q,m} L_q^{-1} M)^{-1} g_2.$$

Let call $g_2^0 = (I - \mu L_q^{-1} M)^{-1} g_2$ and apply Proposition A.1. Indeed $g_2 \in H_2$ and $L_q g_2$ satisfies:

$$\int_{\mathbb{R}^2} L_q g_2.\phi_{1,q,m} = \int_{\mathbb{R}^2} g_2.L_q\phi_{1,q,m} = \lambda_{1,q,m}\int_{\mathbb{R}^2} m.g_2.\phi_{1,q,m} = 0.$$

We obtain $g_2^0 \in H_2$ and $|g_2^0| \leq \Gamma c\phi_{1,q,m}$.
Then call $g_2^1 = (M^{-1} L_q - \lambda_{1,q,m} I)^{-1} g_2^0$; g_2^1 satisfies the following equation:

$$(I - \lambda_{1,q,m} L_q^{-1} M) g_2^1 = L_q^{-1} M g_2^0.$$

We check that

$$\int_{\mathbb{R}^2} m.L_q^{-1} M g_2^0.\phi_{1,q,m} = 0.$$

Applying again Proposition A.1, we get that $g_2^1 \in H_2$ and $|g_2^1| \leq \Gamma \|Mg_2^0\|_{X_{q,m}} \phi_{1,q,m}$. Using the same method at each step, we deduce that the following sequence:

$$g_2^{n+1} = (M^{-1}L_q - \lambda_{1,q,m}I)^{-1} g_2^n$$

satisfies $|g_2^{n+1}| \leq \Gamma \|Mg_2^n\|_{X_{q,m}} \phi_{1,q,m}$. Finally, we get that, if $|\mu - \lambda_{1,q,m}|$ is small enough,$u_2 \in X_{q,m}$. To conclude, we prove similarly that $u_3 \in X_{q,m}$.

We finish the proof, saying that there exists some λ_0 such that for $\lambda_{1,q,m} < \mu < \lambda_0$

$$u = \frac{\alpha\lambda_{1,q,m}}{\lambda_{1,q,m} - \mu}\phi_{1,q,m} + u_2 + u_3 \leq (\frac{\alpha\lambda_{1,q,m}}{\lambda_{1,q,m} - \mu} + C)\phi_{1,q,m}$$

where the constant C depends only on λ_0. Then the Theorem 1.5 follows immediately.

B Appendix: Proof of Lemma 2.1

We use the methods of translations (see [5], [7, p. 182]). Let u be a weak solution of $(-\Delta + q)u = f$ in $\mathbb{R}^N$.

Let $h \in \mathbb{R}^N$ and define

$$(D_h u)(x) = \frac{u(x+h) - u(x)}{|h|}.$$

Let $v = D_{-h}(D_h u)$, $v \in V_q(\mathbb{R}^N)$. From $\int_{\mathbb{R}^N} [\nabla u.\nabla v + quv] = \int_{\mathbb{R}^N} fv$, we get:

$$\int_{\mathbb{R}^N} |\nabla(D_h u)|^2 + \int_{\mathbb{R}^N} D_h(qu).(D_h u) = \int_{\mathbb{R}^N} f D_{-h}(D_h u).$$

Since $D_h(qu)(x) = q(x+h)D_h u(x) + u(x)D_h q(x)$, we get:

$$\int_{\mathbb{R}^N} |\nabla(D_h u)|^2 + \int_{\mathbb{R}^N} q(x+h)|D_h u(x)|^2\, dx + \int_{\mathbb{R}^N} u D_h q D_h u = \int_{\mathbb{R}^N} f D_{-h}(D_h u).$$

Using $q \geq cst > 0$, we deduce that there exists a positive constant $C = C(q)$ (depending upon q) such that:

$$\int_{\mathbb{R}^N} |\nabla(D_h u)|^2 + \int_{\mathbb{R}^N} |D_h u|^2 \leq C(q) \int_{\mathbb{R}^N} |f D_{-h}(D_h u)| + \int_{\mathbb{R}^N} |u D_h(q)||D_h u|.$$

Recall from [7, Proposition IX.3] that for all $w \in H^1(\mathbb{R}^N)$, $\|D_{-h} w\|_{L^2(\mathbb{R}^N)} \leq \|\nabla w\|_{L^2(\mathbb{R}^N)}$. Thus, since for all h, $|D_h(q)| \leq cst\ \sqrt{q}$, we have $|uD_h(q)| \leq cst\ |u|\sqrt{q}$ and there exists a positive constant C such that

$$\|D_h u\|_{H^1(\mathbb{R}^N)} \leq C[\|f\|_{L^2(\mathbb{R}^N)} + \|u\sqrt{q}\|_{L^2(\mathbb{R}^N)}] < \infty.$$

We conclude as in [7] by for all i and for all h,

$$\|D_h \frac{\partial u}{\partial x_i}\|_{L^2(\mathbb{R}^N)} \leq C[\|f\|_{L^2(\mathbb{R}^N)} + \|u\sqrt{q}\|_{L^2(\mathbb{R}^N)}].$$

Using [7, Proposition IX.3] we get that $\frac{\partial u}{\partial x_i} \in H^1(\mathbb{R}^N)$. Therefore $u \in H^2(\mathbb{R}^N)$ and $-\Delta u \in L^2(\mathbb{R}^N)$. Moreover, we have for all $\phi \in D(\mathbb{R}^N)$, $\int_{\mathbb{R}^N}(-\Delta u + qu - f)\phi = 0$ and $-\Delta u + qu - f \in L^1_{loc}(\mathbb{R}^N)$. From [7, Lemma IV.2], we get that $-\Delta u + qu = f$ a. e. in $\mathbb{R}^N$. Thus $qu \in L^2(\mathbb{R}^N)$ and in particular we deduce that $u \in D(-\Delta + q)$.

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