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A new strong convergence Theorem for non-Lipschitzian Mappings in a uniformly convex Banach Space

ABSTRACT. In the present paper, we establish new strong convergence theorems of the modified Mann and the modified Ishikawa iterative scheme with errors for a mapping which is asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space. Our theorems significantly extend and improve Kim and Kim's results. The results in the paper even in the case of asymptotically nonexpansive mappings are new.

KEY WORDS AND PHRASES. asymptotically nonexpansive in the intermediate sense, asymptotically nonexpansive, Mann-type iteration, Ishikawa-type iteration, uniformly convex Banach space.

1 Introduction

Let X be a real Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. If $k_n \equiv 1$, then T is known as a nonexpansive mapping. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. The mapping T is called *asymptotically nonexpansive in the intermediate sense* ([1]) provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x - y\|\big) \leq 0.$$

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From the above definitions, it follows that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian. Furthermore, if $T : C \to C$ is asymptotically nonexpansive and C is bounded, then T is asymptotically nonexpansive in the intermediate sense. There is a mapping which is asymptotically nonexpansive in the intermediate sense but is not Lipschitzian as the following example shows.

Example 1.(see [5]) Let $X = \mathbb{R}$, $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, we define

$$Tx = \begin{cases} k \sin \frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

The concept of asymptotic nonexpansiveness was introduced by Goebel and Kirk [4] in 1972. They proved that every asymptotically nonexpansive self-mapping of a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Several authors have studied methods for the iterative approximation of fixed points of mappings which are asymptotically nonexpansive and asymptotically nonexpansive in the intermediate sense (see for example [2, 3, 5, 7]). In [7], Schu introduced the modified Mann and the modified Ishikawa iterative schemes. Recently, Kim and Kim [5] considered the modified Mann and the modified Ishikawa iterative schemes with errors in the sense of Xu [11] of a mapping which is asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space. The scheme is defined as follows.

Let C be a nonempty convex subset of a Banach space X and $T : C \to C$ be a mapping.
Algorithm 1. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C. The iterative scheme (1.1) is called the *modified Ishikawa iterative scheme with errors* in the sense of Xu.

If $\beta'_n = \gamma'_n \equiv 0$ and $\alpha'_n \equiv 1$, then Algorithm 1 reduces to
Algorithm 2. For a given $x_1 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme

$$x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n u_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $\{u_n\}$ is a bounded sequence in C. The iterative scheme (1.2) is called the *modified Mann iterative scheme with errors* in the sense of Xu.

If $\gamma_n = \gamma'_n \equiv 0$, then Algorithm 1 reduces to modified Ishikawa iterative scheme, while setting $\beta'_n = \gamma_n = \gamma'_n \equiv 0$ and $\alpha'_n \equiv 1$, reduces to modified Mann iterative scheme.

The purpose of this paper is to establish several strong convergence theorems of the Ishikawa iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space.

2 Auxiliary Lemmas

For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$, and $\limsup_n \equiv \limsup_{n\to\infty}$. In the sequel, we shall need the following lemmas.

Lemma 2.1 ([9], Lemma 1) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \le a_n + b_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.*

Lemma 2.2 ([6], Lemma 2.2) *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of nonnegative numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n \mu_n < \infty$. Then $\liminf_n \mu_n = 0$.*

Lemma 2.3 *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative numbers such that $\sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\} = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$. Then there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_j b_{n_j} = 0$ and $\lim_j b_{n_j+1} = 0$.*

Proof: We first observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\}(b_n + b_{n+1}) &= \sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\} b_n + \sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\} b_{n+1} \\ &\le \sum_{n=1}^{\infty} a_n b_n + \sum_{n=1}^{\infty} a_{n+1} b_{n+1} < \infty.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\} = \infty$ and Lemma 2.2, we have $\liminf_n (b_n + b_{n+1}) = 0$. Then there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_j (b_{n_j} + b_{n_j+1}) = 0$. It follows from $b_n \ge 0$ that $\lim_j b_{n_j} = 0$ and $\lim_j b_{n_j+1} = 0$. $\square$

Lemma 2.4 *Let C be a nonempty convex subset of a Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set $F(T) := \{x \in C : x = Tx\}$ is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

$$\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \beta_n \gamma'_n < \infty, \ \textit{and} \ \sum_{n=1}^{\infty} \beta_n c_n < \infty,$$

where $c_n := \max\left\{0, \sup_{x,y\in C}\left(\|T^n x - T^n y\| - \|x - y\|\right)\right\}$ for each $n \ge 1$. Then we have the following conclusions.

(i) $\lim_n \|x_n - p\|$ *exists for any* $p \in F(T)$.

(ii) $\lim_n d(x_n, F(T))$ *exists, where* $d(x, F(T))$ *denotes the distance from* x *to the fixed-point set* $F(T)$.

Proof: Let $p \in F(T)$. We note that $\{u_n - p\}$ and $\{v_n - p\}$ are two bounded sequences in C. Let

$$M := \sup\{\|u_n - p\|, \|v_n - p\| : n \geq 1\}.$$

By using (1.1), we have

$$\begin{aligned}
\|y_n - p\| &\leq \alpha'_n\|x_n - p\| + \beta'_n\|T^n x_n - p\| + \gamma'_n\|v_n - p\| \\
&\leq (\alpha'_n + \beta'_n)\|x_n - p\| + \beta'_n(\|T^n x_n - p\| - \|x_n - p\|) + \gamma'_n M \\
&\leq \|x_n - p\| + \beta'_n c_n + \gamma'_n M,
\end{aligned}$$

and so

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n\|T^n y_n - p\| + \gamma_n\|u_n - p\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n\|y_n - p\| + \beta_n(\|T^n y_n - p\| - \|y_n - p\|) + \gamma_n M \\
&\leq (\alpha_n + \beta_n)\|x_n - p\| + \beta_n\beta'_n c_n + \beta_n\gamma'_n M + \beta_n c_n + \gamma_n M \\
&\leq \|x_n - p\| + 2\beta_n c_n + \beta_n\gamma'_n M + \gamma_n M. \qquad (2.1)
\end{aligned}$$

Then

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + 2\beta_n c_n + \beta_n\gamma'_n M + \gamma_n M.$$

Consequently, the conclusions of the lemma follow from Lemma 2.1. This completes the proof. □

By Xu's inequality [10, Theorem 2], we have the following lemma.

Lemma 2.5 ([3], Lemma 1.4) *Let* X *be a uniformly convex Banach space and* $B_r := \{x \in X : \|x\| \leq r\}, r > 0$. *Then there exists a continuous strictly increasing convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + \mu y + \xi z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 - \lambda\mu g(\|x - y\|),$$

for all $x, y, z \in B_r$ *and* $\lambda, \mu, \xi \in [0, 1]$ *with* $\lambda + \mu + \xi = 1$.

The following lemmas are the important ingredients for proving our main results in the next section.

Lemma 2.6 *Let C be a nonempty convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set $F(T)$ is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

$$\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \beta_n \gamma_n' < \infty, \text{ and } \sum_{n=1}^{\infty} \beta_n c_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} \beta_n (1-\beta_n) g(\|T^n y_n - x_n\|) < \infty. \tag{2.2}$$

Proof: Let $p \in F(T)$, it follows from Lemma 2.4 that $\{x_n - p\}$, $\{T^n x_n - p\}$, $\{y_n - p\}$, $\{T^n y_n - p\}$, $\{u_n - p\}$, and $\{v_n - p\}$ are all bounded. We may assume that such sequences belong to B_r where $r > 0$. By Lemma 2.5, we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\le \alpha_n' \|x_n - p\|^2 + \beta_n' \|T^n x_n - p\|^2 + \gamma_n' \|v_n - p\|^2 - \alpha_n' \beta_n' g(\|T^n x_n - x_n\|) \\
&\le (\alpha_n' + \beta_n') \|x_n - p\|^2 + \beta_n' (\|T^n x_n - p\|^2 - \|x_n - p\|^2) + \gamma_n' r^2 \\
&\le \|x_n - p\|^2 + \beta_n' (\|T^n x_n - p\| - \|x_n - p\|)(\|T^n x_n - p\| + \|x_n - p\|) + \gamma_n' r^2 \\
&\le \|x_n - p\|^2 + 2r\beta_n' c_n + \gamma_n' r^2 \\
&\le \|x_n - p\|^2 + (c_n + \gamma_n') M,
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|T^n y_n - p\|^2 + \gamma_n \|u_n - p\|^2 - \alpha_n \beta_n g(\|T^n y_n - x_n\|) \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|y_n - p\|^2 + \beta_n (\|T^n y_n - p\|^2 - \|y_n - p\|^2) + \gamma_n r^2 \\
&\quad + \gamma_n \beta_n g(\|T^n y_n - x_n\|) - (1-\beta_n) \beta_n g(\|T^n y_n - x_n\|) \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|y_n - p\|^2 + 2r\beta_n c_n + \gamma_n r^2 + \gamma_n \beta_n g(2r) \\
&\quad - (1-\beta_n) \beta_n g(\|T^n y_n - x_n\|) \\
&\le (\alpha_n + \beta_n) \|x_n - p\|^2 + \beta_n (c_n + \gamma_n') M + (\beta_n c_n + 2\gamma_n) M \\
&\quad - (1-\beta_n) \beta_n g(\|T^n y_n - x_n\|) \\
&\le \|x_n - p\|^2 + (2\beta_n c_n + \beta_n \gamma_n' + 2\gamma_n) M - (1-\beta_n) \beta_n g(\|T^n y_n - x_n\|),
\end{aligned}$$

where $M = \max\{2r, r^2, g(2r)\}$. This implies that

$$(1-\beta_n) \beta_n g(\|T^n y_n - x_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (2\beta_n c_n + \beta_n \gamma_n' + 2\gamma_n) M.$$

By Lemma 2.4(i), we obtain (2.2) and the proof is finished. □

Lemma 2.7 *Let C be a nonempty convex subset of a Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the restrictions that $\lim_n \gamma_n = \lim_n \gamma'_n = 0$. If there exists a subsequence $\{n_j\}$ of $\{n\}$ such that*

$$\lim_j \|T^{n_j} x_{n_j} - x_{n_j}\| = 0 = \lim_j \|T^{n_j+1} x_{n_j+1} - x_{n_j+1}\|, \tag{2.3}$$

then $\lim_j \|T x_{n_j} - x_{n_j}\| = 0$.

Proof: We note that

$$\lim_n c_n = \lim_n \max\left\{0, \sup_{x,y \in C} \left(\|T^n x - T^n y\| - \|x - y\|\right)\right\} = 0. \tag{2.4}$$

Using (1.1), we see that

$$\begin{aligned}
&\|T^{n_j} y_{n_j} - x_{n_j}\| \\
&\le (\|T^{n_j} y_{n_j} - T^{n_j} x_{n_j}\| - \|y_{n_j} - x_{n_j}\|) + \|y_{n_j} - x_{n_j}\| + \|T^{n_j} x_{n_j} - x_{n_j}\| \\
&\le c_{n_j} + \|y_{n_j} - x_{n_j}\| + \|T^{n_j} x_{n_j} - x_{n_j}\| \\
&\le c_{n_j} + \beta'_{n_j} \|T^{n_j} x_{n_j} - x_{n_j}\| + \gamma'_{n_j} \|v_{n_j} - x_{n_j}\| + \|T^{n_j} x_{n_j} - x_{n_j}\| \to 0,
\end{aligned}$$

and so,

$$\|x_{n_j+1} - x_{n_j}\| \le \beta_{n_j} \|T^{n_j} y_{n_j} - x_{n_j}\| + \gamma_{n_j} \|u_{n_j} - x_{n_j}\| \to 0. \tag{2.5}$$

Thus

$$\|T^{n_j+1} x_{n_j+1} - T^{n_j+1} x_{n_j}\| \le c_{n_j+1} + \|x_{n_j+1} - x_{n_j}\| \to 0. \tag{2.6}$$

Finally, we have

$$\begin{aligned}
\|x_{n_j} - T x_{n_j}\| &\le \|x_{n_j+1} - x_{n_j}\| + \|x_{n_j+1} - T^{n_j+1} x_{n_j+1}\| \\
&\quad + \|T^{n_j+1} x_{n_j+1} - T^{n_j+1} x_{n_j}\| + \|T^{n_j+1} x_{n_j} - T x_{n_j}\| \to 0,
\end{aligned}$$

since (2.5), (2.3), (2.6) and uniform continuity of T. We reach the desired conclusion. □

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set $F(T)$ is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$,

(ii) $\limsup_n \beta'_n < 1$,

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \beta_n \gamma'_n < \infty$, $\lim_n \gamma'_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n c_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Let $\{u_n\}$ be a given sequence in C. Recall that a mapping $T : C \to C$ with the nonempty fixed-point set $F(T)$ in C satisfies *Condition (A) with respect to the sequence* $\{u_n\}$ ([8]) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(u_n, F(T))) \leq \|Tu_n - u_n\|, \text{ for all } n \geq 1.$$

Proof: By (2.2) and Lemma 2.3, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_j g(\|T^{n_j} y_{n_j} - x_{n_j}\|) = 0 = \lim_j g(\|T^{n_j+1} y_{n_j+1} - x_{n_j+1}\|).$$

From g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_j \|T^{n_j} y_{n_j} - x_{n_j}\| = 0 = \lim_j \|T^{n_j+1} y_{n_j+1} - x_{n_j+1}\|. \tag{3.1}$$

By using (1.1), we have

$$\begin{aligned}
\|T^n x_n - x_n\| &\leq \|T^n y_n - T^n x_n\| + \|T^n y_n - x_n\| \\
&\leq (\|T^n y_n - T^n x_n\| - \|y_n - x_n\|) + \|y_n - x_n\| + \|T^n y_n - x_n\| \\
&\leq c_n + \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|v_n - x_n\| + \|T^n y_n - x_n\|.
\end{aligned}$$

This together with (2.4), (3.1) and $\lim_n \gamma'_n = 0$ gives

$$\lim_j (1 - \beta'_{n_j}) \|T^{n_j} x_{n_j} - x_{n_j}\| = 0 = \lim_j (1 - \beta'_{n_j+1}) \|T^{n_j+1} x_{n_j+1} - x_{n_j+1}\|.$$

As $\liminf_n (1 - \beta'_n) = 1 - \limsup_n \beta'_n > 0$, we have

$$\lim_j \|T^{n_j} x_{n_j} - x_{n_j}\| = 0 = \lim_j \|T^{n_j+1} x_{n_j+1} - x_{n_j+1}\|.$$

By Lemma 2.7, we have

$$\lim_j \|T x_{n_j} - x_{n_j}\| = 0. \tag{3.2}$$

Let f be a nondecreasing function corresponding to Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_{n_j}, F(T))) \leq \|T x_{n_j} - x_{n_j}\| \to 0.$$

Next, we prove that $\lim_n d(x_n, F(T)) = 0$. By Lemma 2.4(ii), suppose that $\lim_n d(x_n, F(T)) = b > 0$. Then there exists $K \in \mathbb{N}$ such that

$$0 < \frac{b}{2} \le d(x_{n_j}, F(T)) \quad \text{for all } j \ge K.$$

By the definition of f, we obtain

$$0 < f(\frac{b}{2}) \le f(d(x_{n_j}, F(T))) \quad \text{for all } j \ge K.$$

Therefore $\lim_j f(d(x_{n_j}, F(T))) \ge f(\frac{b}{2}) > 0$, this is a contradiction. Hence

$$\lim_n d(x_n, F(T)) = 0.$$

Let $a_n = 2\beta_n c_n + \beta_n \gamma_n' M + \gamma_n M$ for all $n \in \mathbb{N}$ and $p \in F(T)$. Then, by (2.1),

$$\|x_{n+1} - p\| \le \|x_n - p\| + a_n,$$

and hence

$$\|x_m - p\| \le \|x_n - p\| + \sum_{i=n}^{m-1} a_i, \tag{3.3}$$

for all $m \ge n$. We now prove that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$. Since

$$\lim_n d(x_n, F(T)) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} a_n < \infty,$$

there exists a positive integer N such that

$$d(x_N, F(T)) < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{i=N}^{\infty} a_i \le \frac{\varepsilon}{4}.$$

There must exist $q \in F(T)$ such that

$$\|x_N - q\| = d(x_N, q) < \frac{\varepsilon}{4}.$$

From (3.3), it follows that, for all $m, n \ge N$,

$$\begin{aligned}
\|x_m - x_n\| &\le \|x_m - q\| + \|x_n - q\| \\
&\le 2\|x_N - q\| + \sum_{i=N}^{n-1} a_i + \sum_{i=N}^{m-1} a_i \\
&\le 2\|x_N - q\| + 2\sum_{i=N}^{\infty} a_i \\
&< 2 \cdot \frac{\varepsilon}{4} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C. In virtue of the completeness of C, we may assume that $x_n \to q'$ as $n \to \infty$ where $q' \in C$. By the continuity of T and (3.2), we have $Tq' = q'$, so q' is a fixed point of T. This completes the proof. □

Letting $\beta'_n = \gamma'_n \equiv 0$ and $\alpha'_n \equiv 1$ in Theorem 3.1, we obtain the following Mann-type convergence.

Theorem 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$be a sequence in C defined by Algorithm 2 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} \beta_n c_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 3.3 ([5], Theorem 1) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \varepsilon \le \beta_n \le 1-\varepsilon < 1$,

(ii) $0 < \varepsilon' \le \alpha_n$,

(iii) $\limsup_n \beta'_n < 1$,

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

If T is completely continuous, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof: By (i), we have $\varepsilon^2 \le \beta_n(1-\beta_n)$. Then

$$0 < \varepsilon^2 \le \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\}.$$

Therefore $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$. Moreover, since T is completely continuous, T satisfies Condition (A) with respect to the sequence $\{x_n\}$ (see [3, Corollary 2.5]). The proof is finished, by using Theorem 3.1. □

Remark 3.4 Condition (ii) is superfluous because (ii) is exactly implied by (i).

Corollary 3.5 **([5], Theorem 2)** *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 2 with the following restrictions:*

(i) $0 < \varepsilon \le \beta_n \le 1 - \varepsilon < 1$,

(ii) $0 < \varepsilon' \le \alpha_n$,

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

If T is completely continuous, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.6 Theorem 3.1 extends and improves Theorem 1 of [5] in the following ways:

(i) The condition $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$ is *strictly* weaker than the condition $0 < \varepsilon \le \beta_n \le 1 - \varepsilon < 1$. In fact, our result is applicable to the case of $\beta_n = 1/n$ while such chosen parameters are not satisfied the requirement of [5, Theorem 1].

(ii) The condition $0 < \varepsilon' \le \alpha_n$ is removed.

(iii) The restrictions $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$ are weakened and replaced by $\sum_{n=1}^{\infty} \beta_n \gamma'_n < \infty$, $\lim_n \gamma'_n = 0$, and $\sum_{n=1}^{\infty} \beta_n c_n < \infty$.

(iv) The complete continuity imposed on T is replaced by the more general Condition (A) with respect to $\{x_n\}$.

Since every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense whenever C is bounded, we have the following theorems.

Theorem 3.7 *Let C be a nonempty closed convex and bounded subset of a uniformly convex Banach space X, $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$,

(ii) $\limsup_n \beta'_n < 1$,

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \beta_n \gamma'_n < \infty$, $\lim_n \gamma'_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n(k_n - 1) < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 3.8 *Let C be a nonempty closed convex and bounded subset of a uniformly convex Banach space X, $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers, and $\{x_n\}$ be a sequence in C defined by Algorithm 2 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} \beta_n(k_n - 1) < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.9 Theorem 3.7 and 3.8 extend and improve Theorem 1.1 and 1.2 of Chang [2], respectively.

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