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De Rham's singular function, its partial derivatives with respect to the parameter and binary digital sums

ABSTRACT. In this paper we show connections between sums related to the binary sum-of-digits function and the function of de Rham $R_a(x)$, and its partial derivatives with respect to the parameter. Starting point is a formula from [3] for the calculation of $R_a(x)$ for dyadic rational x. From this we derive an exact formula for exponential sums of digital sums, and by means of usual differentiations we find exact expressions for some digital sums. In particular, we get the well known result of Trollope-Delange concerning the sum-of-digits function and the formula of Coquet for power sums of digital sums.

KEY WORDS. De Rham's singular function, Takagi's nowhere differentiable function, digital sums, sum-of-digits function.

1 Introduction

In 1956 G. de Rham [13] proved that for a fixed parameter $a \in (0,1)$ the system of functional equations

$$\left.\begin{aligned} f\left(\tfrac{x}{2}\right) &= af(x), \\ f\left(\tfrac{x+1}{2}\right) &= a+(1-a)f(x) \end{aligned}\right\} \qquad (x \in [0,1]) \tag{1.1}$$

has a unique bounded solution $f = R_a(x)$ with $R_a(0) = 0$ and $R_a(1) = 1$. It is $R_{1/2}(x) = x$, but for $a \neq \frac{1}{2}$ de Rham's function $R_a(x)$ is a strictly singular function with the property

$$\int_0^1 R_a(x)dx = a. \tag{1.2}$$

In the literature this function is also called Lebesgue's singular function, cf. e.g. [16] or [1]. For $x \in [0,1]$ with the binary expansion

$$x = 0, d_1 d_2 \ldots \qquad (d_k \in \{0,1\}) \tag{1.3}$$

it holds

$$R_a(x) = \sum_{k=1}^{\infty} d_k a^{k-s_k}(1-a)^{s_k} \tag{1.4}$$

where $s_1 = 0$ and $s_k = d_1 + \ldots + d_{k-1}$ for $k \geq 1$, cf. [11]. In [2] it was shown that for $\ell \in \mathbb{N}$ and $n = 0, 1, \ldots, 2^\ell - 1$ de Rham's function satisfies the equations

$$R_a\left(\frac{n+x}{2^\ell}\right) = R_a\left(\frac{n}{2^\ell}\right) + a^\ell q^{s(n)} R_a(x) \qquad (x \in [0,1]) \tag{1.5}$$

where $q = \frac{1-a}{a}$ and where $s(n)$ denotes the number of ones in the binary representation of n. Let us mention that $s(n) = s_k$ for $n = [2^k x]$. Moreover, for $n = 0, 1, \ldots, 2^\ell$ it holds the formula

$$R_a\left(\frac{n}{2^\ell}\right) = a^\ell \sum_{k=0}^{n-1} q^{s(k)} \tag{1.6}$$

which is starting point of this paper.

In [3] it was considered de Rham's function in connection with two-scale-difference equations. In this paper we uncover connections between de Rham's function $R_a(x)$, its partial derivatives with respect to a and several binary digital sums. The fundamental result in the theory of digital sums is the well known Trollope-Delange formula. It expressed the sum-of-digits function

$$S(N) = \sum_{n=0}^{N-1} s(n) \tag{1.7}$$

by the exact formula ([15], [5])

$$S(N) = \frac{N \log_2 N}{2} + NF(\log_2 N) \tag{1.8}$$

where $F(u)$ is an 1-periodic function given by

$$F(u) = \frac{1-u}{2} - \frac{1}{2^u} T\left(\frac{1}{2^{1-u}}\right) \qquad (0 \leq u \leq 1)$$

with Takagi's function $T(x)$ defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \Delta(2^n x)$$

where $\Delta(x) = \operatorname{dist}(x, \mathbb{Z})$. The function F is continuous but nowhere differentiable. Formula (1.8) was proved in [7] by means of Mellin transforms and in [10] it was shown that

$$F(u) = -\frac{u}{2} - \frac{1}{2^{u+1}} T(2^u) \qquad (u \leq 0). \tag{1.9}$$

Digital sum problems are investigated by many authors, cf. e.g. [6], [4], [12]. For a historical survey see [14], [7].

In this paper we also investigate some digital sums. At first we show by means of formula (1.6) that the exponential sum

$$E_q(N) = \sum_{n=0}^{N-1} q^{s(n)} \tag{1.10}$$

with $q > 0$ can be expressed by the exact formula

$$E_q(N) = N^{\alpha} G_q(\log_2 N) \tag{1.11}$$

where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is a continuous, 1-periodic function which is connected with de Rham's function by

$$G_q(u) = a^u R_a(2^u) \qquad (u \le 0)$$

where $a = \frac{1}{1+q}$. The case $q = 1$ is trivial, namely $E_1(N) = N$. However for $q \neq 1$ the function $G_q(u)$ has the interesting property that it is differentiable almost everywhere and the local maxima and minima are dense in $\mathbb{R}$, Theorem 2.1. Let us mention that $E_2(N)$ is equal to the number of odd binomial coefficients in the first N rows of Pascal's triangle and that $E_2(N)$ was already investigated by many authors, cf. e.g. [14], [7].

Furthermore, for $m \in \mathbb{N}$ we investigate the partial derivative $\frac{\partial^m}{\partial a^m} R_a(x)$ which is a continuous function with respect to x, cf. [16], [1]. In particular we obtain the well known connection

$$\left.\frac{\partial R_a(x)}{\partial a}\right|_{a=1/2} = 2T(x) \tag{1.12}$$

between de Rham's function R_a and Takagi's function T, cf. [16]. Moreover, we calculate the functions $T_m(x) = \frac{\partial^m}{\partial a^m} R_a(x)|_{a=1/2}$ at all dyadic rational $x \in [0,1]$, Proposition 4.2. These functions are used for the representation of some digital sums.

For the binomial sum

$$B_m(N) = \sum_{n=0}^{N-1} \binom{s(n)}{m} \tag{1.13}$$

with $m \ge 1$ it holds the exact formula

$$B_m(N) = \frac{N}{m!}\left(\frac{\log_2 N}{2}\right)^m + N \sum_{\ell=0}^{m-1} (\log_2 N)^{\ell} F_{m,\ell}(\log_2 N)$$

where $F_{m,\ell}(u)$ are 1-periodic, continuous functions which can be expressed by the functions $T_1(x), \ldots, T_m(x)$, Theorem 5.3 and Proposition 5.1. In particular for $m = 1$ we get the Trollope-Delange formula (1.8).

Finally we investigate the digital power sum

$$S_m(N) = \sum_{n=0}^{N-1} s(n)^m \tag{1.14}$$

with $m \in \mathbb{N}$. Stolarsky [14] has proved the asymptotic formula

$$S_m(N) = N\left(\frac{\log_2 N}{2}\right)^m + \mathcal{O}\{N(\log N)^{m-1})\}$$

which is optimal in the sense that for the values

$$\alpha_m = \limsup \frac{S_m(N) - N\left(\frac{\log_2 N}{2}\right)^m}{N(\log N)^{m-1}} \tag{1.15}$$

and β_m, the corresponding lim inf, it holds: $-\infty < \beta_m < \alpha_m < \infty$. In particular $\alpha_1 = 0$, $\beta_1 = (\log 3/\log 4) - 1$. Coquet [4] obtained the precise formula

$$S_m(N) = N\left(\frac{\log_2 N}{2}\right)^m + N\sum_{\ell=0}^{m-1}(\log_2 N)^\ell G_{m,\ell}(\log_2 N) \tag{1.16}$$

where $G_{m,\ell}(u)$ are certain 1-periodic functions, and in [6] it was shown by means of binomial measures that these functions are continuous, cf. also [12].

We also prove the formula (1.16) of Coquet with explicit representations for the functions $G_{m,\ell}(u)$, Theorem 6.1, and determine the values α_m and β_m, Proposition 6.2.

2 Exponential sums of digital sums

For the exponential sum (1.10) with given $q > 0$ it follows from (1.6) that for $N \leq 2^\ell$ it holds

$$E_q(N) = \frac{1}{a^\ell} R_a\left(\frac{N}{2^\ell}\right) \tag{2.1}$$

where R_a is de Rham's function corresponding to $a = \frac{1}{1+q}$. In particular $E_q(2^\ell) = (q+1)^\ell$. In order to obtain a formula independent of ℓ we note that with

$$\alpha = -\log_2 a = \log_2(1+q) \tag{2.2}$$

the first equation of (1.1) with $f = R_a$ yields

$$\frac{R_a(\frac{x}{2})}{(\frac{x}{2})^\alpha} = \frac{R_a(x)}{x^\alpha} \qquad (0 < x \leq 1). \tag{2.3}$$

Hence the function

$$g_q(x) = \frac{R_a(x)}{x^\alpha} \qquad (0 < x \leq 1) \tag{2.4}$$

has for $0 < x < \frac{1}{2}$ the property $g_q(2x) = g_q(x)$ so that it can be extended for all $x > 0$ by

$$g_q(2x) = g_q(x). \tag{2.5}$$

According to (2.5) the function

$$G_q(u) = g_q(2^u) \qquad (u \in \mathbb{R}) \tag{2.6}$$

is periodic with period 1.

Theorem 2.1 *For $q > 0$ the exponential sum $E_q(N)$ from (1.10) can be expressed by the exact formula*

$$E_q(N) = N^\alpha G_q(\log_2 N) \tag{2.7}$$

where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is an 1-periodic function which is connected with de Rham's function R_a corresponding to $a = \frac{1}{1+q}$ by

$$G_q(u) = a^u R_a(2^u) \qquad (u \le 0). \tag{2.8}$$

G_q is continuous and differentiable almost everywhere. For $q \neq 1$ the maxima and minima of G_q are lying dense in $\mathbb{R}$.

Proof: For given $N \in \mathbb{N}$ we choose ℓ such that $2^\ell > N$. For the exponential sum (1.10) we get from (1.6)

$$E_q(N) = \frac{1}{a^\ell} R_a\left(\frac{N}{2^\ell}\right) = N^\alpha \left(\frac{2^\ell}{N}\right)^\alpha R_a\left(\frac{N}{2^\ell}\right)$$

where we have used that $a = \frac{1}{1+q}$ and $\frac{1}{a} = 2^\alpha$ according to (2.2). In view of (2.4) and (2.5) we get

$$E_q(N) = N^\alpha g_q(N) \tag{2.9}$$

and it follows (2.7) with the 1-periodic function $G_q(u)$ given by (2.6). For $u \le 0$ we have $0 < 2^u \le 1$ and from (2.6) and (2.4) with $x = 2^u$ we get in view of $\frac{1}{2^\alpha} = a$

$$G_q(u) = \frac{R_a(2^u)}{2^{u\alpha}} = a^u R_a(2^u)$$

i.e. the representation (2.8).

In order to obtain properties of G_q we investigate g_q. Obviously, g_q is continuous in $[\frac{1}{2}, 1]$ and hence at any $x > 0$. Moreover, g_q is differentiable almost everywhere in $(0, \infty)$ since R_a is increasing. For $q \neq 1$ we have $a = \frac{1}{1+q} \neq \frac{1}{2}$. Since for fixed $a \neq \frac{1}{2}$ the function R_a is singular there are points $x \in (0,1)$ where the derivative $R_a'(x) = +\infty$, and equation (1.5) implies that such points lie dense in $(0,1)$. This is valid also for the function g_q so that there is no interval where it can be decreasing. On the other side, since R_a is singular, it follows for such x with $R_a'(x) = 0$ that

$$\left(\frac{R_a(x)}{x^\alpha}\right)' = \frac{R_a'(x)x^\alpha - \alpha x^{\alpha-1} R_a(x)}{x^{2\alpha}} = -\frac{\alpha R_a(x)}{x^{\alpha+1}}.$$

Hence, the function g_q is differentiable almost everywhere in $(0,\infty)$ with $g_q'(x) = -\frac{\alpha}{x} g_q(x) < 0$ so that there is no interval where g_q can be increasing. By (2.6) it follow the assertions for G_q. □

3 Partial derivative of de Rham's function

It is known that $R_a(x)$ is differentiable with respect to a, cf. [16]. Here we show the differentiability by means of a method as in [2], Proposition 2.3.

Proposition 3.1 *For fixed $x \in [0,1]$ with the dyadic representation (1.3) de Rham's function R_a is differentiable with respect to a and the derivative $\frac{\partial}{\partial a}R_a(x)$ is a continuous function with respect to x which has the representation*

$$\frac{\partial}{\partial a}R_a(x) = \sum_{k=1}^{\infty} d_k(k(1-a)-s_k)a^{k-s_k-1}(1-a)^{s_k-1} \tag{3.1}$$

with $s_1 = 0$ and $s_k = d_1 + \ldots + d_{k-1}$ for $k \geq 1$.

Proof: We denote the formal derivative of (1.4) with respect to a by $\varphi(x,a)$ which reads

$$\varphi(x,a) = \sum_{k=1}^{\infty} d_k(k-s_k)a^{k-s_k-1}(1-a)^{s_k} - \sum_{k=1}^{\infty} d_k s_k a^{k-s_k}(1-a)^{s_k-1}. \tag{3.2}$$

We show that for fixed x this function is an analytic function with respect to a in the complex domain $D = \{a : |a| < 1, |1-a| < 1\}$. Namely, for fixed $\varepsilon > 0$ choosing $\max(|a|, |1-a|) \leq 1-\varepsilon$ we obtain for $x = \sum d_k 2^{-k}$ the estimate

$$|\varphi(x,a)| \leq 2\sum_{k=1}^{\infty} k(1-\varepsilon)^{k-1} = \frac{2}{\varepsilon^2}$$

in view of $0 \leq s_k \leq k-1$. This implies that the series (3.2) of polynomials in a is uniformly convergent in every compact subset of the domain $D = \{a : |a| < 1, |1-a| < 1\}$. Consequently, in this domain $\varphi(x,a)$ is a continuous function with respect to a and it is the derivative of R_a, cf. [9, p. 353]. Hence, from (3.2) we get the derivative (3.1) of R_a with respect to a. □

In the following we denote the first partial derivative of de Rham's function with respect to a by

$$D_1(x,a) = \frac{\partial}{\partial a}R_a(x). \tag{3.3}$$

Differentiation of (1.1) with respect to a yields the following system of functional equations:

$$\left.\begin{array}{rcl} f\left(\frac{x}{2}\right) &=& R_a(x) + af(x), \\ f\left(\frac{x+1}{2}\right) &=& 1 - R_a(x) + (1-a)f(x) \end{array}\right\} \quad (x \in [0,1]). \tag{3.4}$$

By a result of Girgensohn [8] this system has exactly one continuous solution f. Hence, we have $f(x) = D_1(x,a)$.

Proposition 3.2 *For fixed a the first partial derivative (3.3) satisfies $D_1(0,a) = D_1(1,a) = 0$ and $D_1(x,a) > 0$ for $0 < x < 1$ with $D_1(\frac{1}{2}, a) = 1$. Moreover, it holds*

$$D_1(1-x,a) = D_1(x,1-a) \qquad (x \in [0,1]) \tag{3.5}$$

and

$$\int_0^{1/2} D_1(x,a)dx = a, \qquad \int_{1/2}^1 D_1(x,a)dx = 1-a. \tag{3.6}$$

Proof: The property (3.5) follows from $R_a(1-x) = 1 - R_{1-a}(x)$, cf. [2, Proposition 2.3]. Equation (3.4) for $x = 0$ yields $D_1(0,a) = 0$ for each $a \in (0,1)$, and hence it holds $D_1(1,a) = 0$ in view of (3.5). The second equation (3.4) for $x = 0$ yields $D_1(\frac{1}{2}, a) = 1$. In order to show that $D_1(x,a) > 0$ for $0 < x < 1$ we remark that $D_1(x_0, a) > 0$ for a certain x_0 implies $D_1(\frac{x_0}{2}, a) > 0$, since (3.4), and also $D_1(1 - \frac{x_0}{2}, a) > 0$ in view of (3.5). Hence, $D_1(\frac{1}{2}, a) = 1$ implies $D_1(x,a) > 0$ for all dyadic rational $x \in (0,1)$, and in view of the continuity of $D_1(x,a)$ with respect to x it follows $D_1(x,a) \geq 0$ for all $0 < x < 1$. Equation (3.4) implies $D_1(x,a) > 0$ for $0 < x \leq \frac{1}{2}$, and therefore $D_1(x,a) > 0$ for $0 < x < 1$ according to (3.5).

From (3.4) with $f(x) = D_1(x,a)$ we get

$$D_1\left(\frac{x}{2}, a\right) + D_1\left(\frac{x+1}{2}, a\right) = D_1(x,a) + 1,$$

and by integration we obtain

$$\int_0^1 D_1(x,a)dx = 1.$$

Hence and in view of (1.2) we get by integration of the first equation in (3.4) that

$$2\int_0^{1/2} D_1(x,a)dx = \int_0^1 R_a(x)dx + a\int_0^1 D_1(x,a)dx = 2a,$$

i.e. the first relation in (3.6), and the second relation follows in view of (3.5). □

We are especially interested in the case $a = \frac{1}{2}$ where the system (3.4) with $g = 2f$ attains the form

$$\left.\begin{array}{rcl} g\left(\frac{x}{2}\right) & = & \frac{x}{2} + \frac{1}{2}g(x), \\ g\left(\frac{x+1}{2}\right) & = & \frac{1-x}{2} + \frac{1}{2}g(x) \end{array}\right\} \qquad (x \in [0,1]) \tag{3.7}$$

which has the unique solution $g(x) = \frac{1}{2}D_1(x, \frac{1}{2})$, i.e.

$$g(x) = \frac{1}{2}\frac{\partial}{\partial a}R_a(x)\bigg|_{a=1/2}. \tag{3.8}$$

It is known that also Takagi's function T satisfies the system (3.7), cf. [10]. Hence, we have the interesting connection

$$\frac{\partial}{\partial a} R_a(x)\Big|_{a=1/2} = 2T(x) \tag{3.9}$$

which is proved in [16] by means of Schauder expansion of $R_a(x)$. According to Proposition 3.1 for the Takagi function T we find the representation

$$T(x) = \sum_{k=1}^{\infty} d_k \frac{k - 2s_k}{2^k} \tag{3.10}$$

with x from (1.3).

4 Partial derivatives of higher order

We investigate the derivatives of higher order with respect to a of de Rham's function $R_a(x)$, cf. [16]. The existence of

$$D_m(x,a) = \frac{\partial^m}{\partial a^m} R_a(x) \tag{4.1}$$

follows from the fact that R_a is holomorphic in a. From (1.1) we find for $m \geq 2$ by repeated differentiation with respect to a the functional equations

$$\left.\begin{array}{rcl} D_m\left(\frac{x}{2}, a\right) & = & mD_{m-1}(x,a) + aD_m(x,a) \\ D_m\left(\frac{x+1}{2}, a\right) & = & -mD_{m-1}(x,a) + (1-a)D_m(x,a) \end{array}\right\} \quad (x \in [0,1]) \tag{4.2}$$

and by a result of Girgensohn [8] the functions D_m are continuous with respect to x.

Proposition 4.1 *The function D_m $(m \in \mathbb{N})$ has the property*

$$D_m(1-x, a) = (-1)^{m+1} D_m(x, 1-a). \tag{4.3}$$

For $m \geq 2$ it holds

$$\int_0^1 D_m(x,a)dx = 0. \tag{4.4}$$

Proof: The symmetry property (4.3) follows from $R_a(1-x) = 1 - R_{1-a}(x)$, cf. [2, Proposition 2.3]. For $m \geq 2$ we get from (4.2)

$$D_m\left(\frac{x}{2}, a\right) + D_m\left(\frac{x+1}{2}, a\right) = D_m(x,a).$$

From this equation we find by induction on n that

$$\sum_{\nu=0}^{2^n-1} D_m\left(\frac{x+\nu}{2^n}, a\right) = D_m(x,a).$$

Dividing by 2^n we obtain as $n \to \infty$ equation (4.4). □

In particular we use the derivatives at $a = \frac{1}{2}$, i.e.

$$T_m(x) = \frac{\partial^m}{\partial a^m} R_a(x)\Big|_{a=1/2}. \tag{4.5}$$

Thus $T_0(x) = x$ and $T_1(x) = 2T(x)$ where T is Takagi's function. These functions were investigated already in [1] where it was shown that for $m \geq 1$ they are continuous but nowhere differentiable. In particular the extreme values of T_2 and T_3 were studied.

Proposition 4.1 implies that for $m \geq 1$ the function T_m has the symmetry property

$$T_m(1-x) = (-1)^{m+1} T_m(x) \tag{4.6}$$

and according to (4.2) for $m \geq 2$ it satisfies the functional equations

$$\left.\begin{array}{rcl} T_m\left(\frac{x}{2}\right) & = & mT_{m-1}(x) + \frac{1}{2}T_m(x) \\ T_m\left(\frac{x+1}{2}\right) & = & -mT_{m-1}(x) + \frac{1}{2}T_m(x) \end{array}\right\} \qquad (x \in [0,1]). \tag{4.7}$$

Hence, T_m is a so-called Knopp-function

$$T_m(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} g_m(2^n x)$$

with the generating function g_m given by $g_m(x) = mT_{m-1}(2x)$ for $0 \leq x \leq \frac{1}{2}$ as well as $g_m(1-x) = -g_m(x)$ and $g_m(1+x) = g_m(x)$ for $x \in \mathbb{R}$.

Proposition 4.2 *For $m \geq 1$ the derivatives (4.5) of de Rham's function R_a satisfy the functional relations*

$$T_m\left(\frac{n+x}{2^\ell}\right) = T_m\left(\frac{n}{2^\ell}\right) + \sum_{\mu=0}^{m} a_\mu T_\mu(x) \tag{4.8}$$

where $\ell \in \mathbb{N}$, $n = 0, 1, \ldots, 2^\ell - 1$, $x \in [0,1]$, $T_0(x) = x$ and where a_μ are the constants

$$a_\mu = \binom{m}{\mu} \frac{\partial^{m-\mu}}{\partial a^{m-\mu}} a^{\ell - s(n)} (1-a)^{s(n)}\Big|_{a=1/2}. \tag{4.9}$$

Moreover, for $n = 0, 1, \ldots, 2^\ell$ it holds

$$T_m\left(\frac{n}{2^\ell}\right) = \frac{m!}{2^{\ell-m}} \sum_{j=0}^{n-1} \sum_{r=0}^{m} (-1)^r \binom{s(j)}{r} \binom{\ell - s(j)}{m-r}. \tag{4.10}$$

Proof: By m differentiations of (1.5) with respect to a we find

$$D_m\left(\frac{n+x}{2^\ell}\right) = D_m\left(\frac{n}{2^\ell}\right) + \sum_{\mu=0}^{m} \binom{m}{\mu} D_\mu(x) \frac{\partial^{m-\mu}}{\partial a^{m-\mu}} a^{\ell-s(n)}(1-a)^{s(n)}$$

so that for $a = \frac{1}{2}$ it follows (4.8) with the constants (4.9). Next we want to determine a_0. For this reason we compute

$$\frac{\partial^m}{\partial a^m}(1-a)^{s(n)}a^{\ell-s(n)} = \sum_{r=0}^{m}\binom{m}{r}(-1)^r \frac{s(n)!(1-a)^{s(n)-r}}{(s(n)-r)!}\frac{(\ell-s(n))!a^{\ell-s(n)-m+r}}{(\ell-s(n)-m+r)!}$$

so that for $a = \frac{1}{2}$ we obtain

$$a_0 = \frac{m!}{2^{\ell-m}}\sum_{r=0}^{m}(-1)^r\binom{s(n)}{r}\binom{\ell-s(n)}{m-r}.$$

In view of $T_0(1) = 1$ and $T_\mu(1) = 0$ for $\mu > 1$ equation (4.8) with j instead of n yields

$$T_m\left(\frac{j+1}{2^\ell}\right) = T_m\left(\frac{j}{2^\ell}\right) + \frac{m!}{2^{\ell-m}}\sum_{r=0}^{m}(-1)^r\binom{s(j)}{r}\binom{\ell-s(j)}{m-r}$$

and equation (4.10) follows by summation. □

In particular for $m = 1$ we obtain

$$T_1\left(\frac{n}{2^\ell}\right) = \frac{n\ell}{2^{\ell-1}} - \frac{1}{2^{\ell-2}}\sum_{j=0}^{n-1}s(j)$$

in accordance with $T_1(x) = 2T(x)$, cf. [10, formula (2.2)]. Equation (4.10) for $n = 1$ yields

$$T_m\left(\frac{1}{2^\ell}\right) = \frac{m!}{2^{\ell-m}}\binom{\ell}{m},$$

i.e. $T_m(\frac{1}{2^m}) = m!$ and $T_m(\frac{1}{2^\ell}) = 0$ for $\ell = 0, \ldots, m-1$.

5 Binomial sums of digital sums

Before we investigate the power sum (1.14) first we consider the binomial sum (1.13). By Theorem 2.1 it holds

$$B_m(N) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}N^\alpha G_q(\log_2 N)\bigg|_{q=1} \tag{5.1}$$

where $q = q(a) = \frac{a}{1-a}$. In order to obtain an explicit expression for the formula (5.1) we use (2.8). So for $u \le 0$ we get

$$\frac{\partial^m}{\partial q^m}N^\alpha G_q(u)\bigg|_{q=1} = \frac{\partial^m}{\partial q^m}N^\alpha a^u R_a(2^u)\bigg|_{q=1} \tag{5.2}$$

where $a = a(q) = \frac{1}{1+q}$. Since the function $u \mapsto a^u R_a(2^u)$ for $u \le 0$ is continuous and of period 1 also the functions $F_k(u)$ defined by

$$F_k(u) = \frac{\partial^k}{\partial q^k} a^u R_a(2^u)\bigg|_{q=1} \qquad (u \le 0) \tag{5.3}$$

are continuous and of period 1 so that they can be extended by $F_k(u+1) = F_k(u)$ for all $u \in \mathbb{R}$. In particular, in view of $a(1) = \frac{1}{2}$ and $R_{1/2}(x) = x$ we find

$$F_0(u) = a^u R_a(2^u)\big|_{q=1} = 1. \tag{5.4}$$

In order to express $F_k(u)$ for $k \ge 1$ we introduce polynomials

$$P_{k,\ell}(u) = (-1)^k \frac{k!}{\ell!} \binom{u+k-1}{k-\ell} \qquad (0 \le \ell \le k). \tag{5.5}$$

Obviously, $P_{k,\ell}(u)$ is a polynomial of degree $k - \ell$. In particular

$$P_{k,0}(u) = (-1)^k u(u+1)\cdots(u+k-1), \qquad P_{k,k}(u) = (-1)^k. \tag{5.6}$$

It is easy to see that for $1 \le \ell \le k$ we have

$$P_{k+1,\ell}(u) = -P_{k,\ell-1}(u) - (u+k+\ell)P_{k,\ell}(u). \tag{5.7}$$

Proposition 5.1 *For $k \ge 1$ the 1-periodic function $F_k(u)$ has the representation*

$$F_k(u) = (-1)^k \frac{u(u+1)\cdots(u+k-1)}{2^k} + \frac{1}{2^{u+k}} \sum_{\ell=1}^{k} \frac{P_{k,\ell}(u)}{2^\ell} T_\ell(2^u) \qquad (u \le 0) \tag{5.8}$$

with the polynomials $P_{k,\ell}(u)$ from (5.5) and the partial derivatives T_ℓ from (4.5).

Proof: At first we show by induction on k that for $u \le 0$

$$\frac{\partial^k}{\partial q^k} a^u R_a(2^u) = \sum_{\ell=0}^{k} P_{k,\ell}(u) a^{u+k+\ell} \frac{\partial^\ell}{\partial a^\ell} R_a(2^u).$$

This is valid for $k = 1$, then by $\frac{\partial a}{\partial q} = \frac{-1}{(1+q)^2} = -a^2$ we have for $u \le 0$

$$\begin{aligned} \frac{\partial}{\partial q} a^u R_a(2^u) &= \frac{\partial}{\partial a} a^u R_a(2^u) \frac{\partial a}{\partial q} \\ &= -u a^{u+1} R_a(2^u) - a^{u+2} \frac{\partial}{\partial a} R_a(2^u). \end{aligned}$$

If this is true for a fixed k then it follows in view of $\frac{da}{dq} = -a^2$

$$\begin{aligned} &\frac{\partial^{k+1}}{\partial q^{k+1}} a^u R_a(2^u) = -a^2 \frac{\partial}{\partial a} \left(\frac{\partial^k}{\partial q^k} a^u R_a(2^u) \right) \\ &= -\sum_{\ell=0}^{k} (u+k+\ell) P_{k,\ell}(u) a^{u+k+\ell+1} \frac{\partial^\ell}{\partial a^\ell} R_a(2^u) - \sum_{\ell=0}^{k} P_{k,\ell}(u) a^{u+k+\ell+2} \frac{\partial^{\ell+1}}{\partial a^{\ell+1}} R_a(2^u) \\ &= \sum_{\ell=0}^{k+1} P_{k+1,\ell}(u) a^{u+k+1+\ell} \frac{\partial^\ell}{\partial a^\ell} R_a(2^u). \end{aligned}$$

According to (5.3) we have to take $q = 1$, i.e. $a(1) = \frac{1}{2}$, and applying (4.5), $R_{1/2}(x) = x$ as well as (5.6) it follows the assertion. $\square$

Remark In particular, formula (5.8) for $k = 1$ yields

$$F_1(u) = -\frac{u}{2} - \frac{1}{2^{u+2}} T_1(2^u) \qquad (u \leq 0),$$

and it follows by (1.9) that

$$F_1(u) = F(u) \tag{5.9}$$

in the formula of Trollope-Delange.

Now for $k \in \mathbb{N}$ and $\ell = 0, \ldots, k$ we introduce numbers $a_{k,\ell}$ as follows:

$$\left.\begin{array}{lcll} a_{k,k} & = & 1 & \text{for} \quad k \geq 0 \\ a_{k,0} & = & 0 & \text{for} \quad k \geq 1 \\ a_{k+1,\ell} & = & a_{k,\ell-1} - k a_{k\ell} & \text{for} \quad k \geq 1, \;\; 1 \leq \ell \leq k \end{array}\right\}. \tag{5.10}$$

The numbers can be represented in a modified Pascal triangle, cf. Figure 1.

$$\begin{array}{ccccccccccccc} & & & & & 1 & & & & & & & k=0 \\ & & & & 0 & & 1 & & & & & & k=1 \\ & & & 0 & & -1 & & 1 & & & & & k=2 \\ & & 0 & & 2 & & -3 & & 1 & & & & k=3 \\ & 0 & & -6 & & 11 & & -6 & & 1 & & & k=4 \\ 0 & & 24 & & -50 & & 25 & & -10 & & 1 & & k=5 \end{array}$$

Figure 1: The first numbers $a_{k,\ell}$.

It is easy to show by induction that for $k \geq 1$ we have:

$$a_{k,1} = (-1)^{k-1}(k-1)!, \qquad a_{k,k-1} = -\binom{k}{2} \tag{5.11}$$

and for $k \geq 2$

$$a_{k,k-2} = \binom{k}{3} \frac{3k-1}{4}.$$

Lemma 5.2 *For integer $k \geq 1$ it holds*

$$\frac{\partial^k}{\partial q^k} N^\alpha = \frac{N^\alpha}{(1+q)^k} \sum_{\ell=1}^{k} (\log_2 N)^\ell a_{k,\ell} \tag{5.12}$$

with the coefficients $a_{k,\ell}$ from (5.10).

Proof: From $\alpha(q) = \log_2(1+q)$ and $\frac{\partial\alpha}{\partial q} = \frac{1}{(1+q)\log 2}$ we get

$$\frac{\partial}{\partial q} N^\alpha = N^\alpha \frac{\log_2 N}{1+q}$$

so that (5.12) is true for $k = 1$ since $a_{1,1} = 1$. Assume that (5.12) is valid for a fixed $k \geq 1$ then in view of $a_{k,0} = 0$ we get

$$\begin{aligned} \frac{\partial^{k+1}}{\partial q^{k+1}} N^\alpha &= \frac{N^\alpha}{(1+q)^{k+1}} \sum_{\ell=1}^{k} (\log_2 N)^{\ell+1} a_{k,\ell} - \frac{kN^\alpha}{(1+q)^{k+1}} \sum_{\ell=1}^{k} (\log_2 N)^\ell a_{k,\ell} \\ &= \frac{N^\alpha}{(1+q)^{k+1}} \sum_{\ell=1}^{k+1} (\log_2 N)^\ell a_{k+1,\ell} \end{aligned}$$

on account of (5.10) and $a_{k+1,k+1} = a_{k,k} = 1$. So (5.12) is proved by induction. □

Theorem 5.3 *For the binary binomial sum (1.14) with integer $m \geq 1$ we have the exact formula*

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \left(\frac{\log_2 N}{2}\right)^m + \frac{1}{m!} \sum_{\ell=0}^{m-1} (\log_2 N)^\ell F_{m,\ell}(\log_2 N) \tag{5.13}$$

where $F_{m,\ell}(u)$ are continuous functions of period 1 which have the representations

$$F_{m,\ell}(u) = \sum_{k=0}^{m-\ell} \binom{m}{k} \frac{a_{m-k,\ell}}{2^{m-k}} F_k(u) \tag{5.14}$$

with $a_{k,\ell}$ from (5.10) and $F_k(u)$ from (5.3).

Proof: We apply (5.1) with the 1-periodic function $G_q(u)$. In order to express the term

$$\left.\frac{\partial^m}{\partial q^m} N^\alpha G_q(u)\right|_{q=1}$$

as 1-periodic function we use (5.2) for $u \leq 0$. By Leibniz's formula

$$\frac{\partial^m}{\partial q^m} N^\alpha a^u R_a(2^u) = \sum_{k=0}^{m} \binom{m}{k} \frac{\partial^k}{\partial q^k} N^\alpha \frac{\partial^{m-k}}{\partial q^{m-k}} a^u R_a(2^u).$$

Now, $\alpha(q) = \log_2(1+q)$ yields $\alpha(1) = 1$ and by Lemma 5.2 we get

$$\left.\frac{\partial^k}{\partial q^k} N^\alpha\right|_{q=1} = \frac{N}{2^k} \sum_{\ell=0}^{k} (\log_2 N)^\ell a_{k,\ell}$$

with $a_{k,\ell}$ from (5.10). Using (5.3) we obtain for $u \leq 0$

$$\left.\frac{\partial^m}{\partial q^m} N^\alpha a^u R_a(2^u)\right|_{q=1} = N \sum_{k=0}^{m} \binom{m}{k} \frac{1}{2^k} \sum_{\ell=0}^{k} a_{k,\ell} (\log_2 N)^\ell F_{m-k}(u).$$

Since the right-hand side is 1-periodic for all $u \in \mathbb{R}$, it follows by (5.2)

$$\frac{\partial^m}{\partial q^m} N^\alpha G_q(u)\Big|_{q=1} = \sum_{\ell=0}^{m} \sum_{k=\ell}^{m} \binom{m}{k} \frac{a_{k,\ell}}{2^k} (\log_2 N)^\ell F_{m-k}(u)$$

and by (5.1) we get

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \sum_{\ell=0}^{m} \sum_{k=\ell}^{m} \binom{m}{k} \frac{a_{k,\ell}}{2^k} (\log_2 N)^\ell F_{m-k}(\log_2 N).$$

Replacing k by $m-k$ it follows

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \sum_{\ell=0}^{m} (\log_2 N)^\ell F_{m,\ell}(\log_2 N)$$

with the functions (5.14). In particular, $F_{m,m}(u) = \frac{a_{m,m}}{2^m} F_0(u) = \frac{1}{2^m}$ since $F_0(u) = 1$, cf. (5.4). This completes the proof. □

Remarks **1.** In case $m = 1$ we have $B_1(N) = S(N)$, cf. (1.7). Formula (5.14) yields $F_{1,0}(u) = F_1(u)$ and (5.13) simplifies to the Trollope-Delange formula (1.8) with $F(u) = F_1(u)$, cf. (5.9).

2. Note

$$\begin{aligned} F_{m,m-1}(u) &= \frac{a_{m,m-1}}{2^m} F_0(u) + \frac{m a_{m-1,m-1}}{2^{m-1}} F_1(u) \\ &= -\frac{1}{2^m} \binom{m}{2} + \frac{m}{2^{m-1}} F(u) \end{aligned}$$

where we have used (5.11), $F_0(u) = 1$ and (5.9). So (5.13) yields the asymptotic formula

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \left(\frac{L}{2}\right)^m + \frac{1}{m!} \left(\frac{L}{2}\right)^{m-1} \left\{ -\frac{1}{2} \binom{m}{2} + mF(L) \right\} + \mathcal{O}\left(L^{m-2}\right) \qquad (5.15)$$

with $L = \log_2 N$.

3. Further,

$$\begin{aligned} F_{m,m-2}(u) &= \frac{a_{m,m-2}}{2^m} F_0(u) + m \frac{a_{m-1,m-2}}{2^{m-1}} F_1(u) + \binom{m}{2} \frac{a_{m-2,m-2}}{2^{m-2}} F_2(u) \\ &= \frac{3m-1}{2^{m+2}} \binom{m}{3} - \frac{m}{2^{m-1}} \binom{m}{2} F_1(u) + \frac{1}{2^{m-2}} \binom{m}{2} F_2(u). \end{aligned}$$

Thus (5.13) yields for $m = 2$ the precise formula

$$\frac{1}{N} B_2(N) = \frac{1}{2} \left(\frac{L}{2}\right)^2 + \frac{L}{4} \left\{ -\frac{1}{2} + 2F_1(L) \right\} - \frac{1}{2} F_1(L) + \frac{1}{2} F_2(L)$$

again with $L = \log_2 N$ and $F_1(u) = F(u)$, cf. (5.9). Compare with (5.15) for $m = 2$.

6 Power sums of digital sums

In order to obtain formulas for the power sums (1.14) we need the Stirling numbers of second kind $s_{m,n}$ defined by

$$x^m = \sum_{n=0}^{m} s_{m,n} n! \binom{x}{n} . \tag{6.1}$$

These numbers are nonnegative integers. In particular, $s_{m,m} = 1$ for $m \geq 0$ and $s_{m,0} = 0$ for $m \geq 1$. Now we prove the formula (1.16) of Coquet.

Theorem 6.1 *For the power sum (1.14) it holds the formula of Coquet*

$$\frac{1}{N} S_m(N) = \left(\frac{\log_2 N}{2}\right)^m + \sum_{\ell=0}^{m-1} (\log_2 N)^\mu G_{m,\ell}(\log_2 N) \tag{6.2}$$

where $G_{m,\ell}(u)$ are continuous, 1-peridic functions given by

$$G_{m,\ell}(u) = \sum_{k=0}^{m-\ell} \sum_{n=\ell+k}^{m} \binom{n}{k} \frac{a_{n-k,\ell}}{2^{n-k}} s_{m,n} F_k(u) \tag{6.3}$$

with the coefficients from (5.10) and (6.1) and the functions $F_k(u)$ from (5.3).

Proof: According to (6.1) we have

$$S_m(N) = \sum_{n=0}^{m} s_{m,n} n! B_n(N)$$

and by Theorem 5.3 we find with $L = \log_2 N$

$$\begin{aligned} \frac{1}{N} S_m(N) &= \sum_{n=0}^{m} s_{m,n} \sum_{\ell=0}^{n} L^\ell F_{n,\ell}(L) \\ &= \sum_{\ell=0}^{m} L^\ell \sum_{n=\ell}^{m} s_{m,n} F_{n,\ell}(L) \\ &= \sum_{\ell=0}^{m} L^\ell G_{m,\ell}(L) \end{aligned}$$

where in view of (5.14) with n instead of m

$$G_{m,\ell}(u) = \sum_{n=\ell}^{m} s_{m,n} \sum_{k=0}^{n-\ell} \binom{n}{k} \frac{a_{n-k,\ell}}{2^{n-k}} F_k(u)$$

which implies (6.3). In particular, we have

$$G_{m,m}(u) = \frac{a_{m,m}}{2^m} s_{m,m} F_0(u) = \frac{1}{2^m}$$

which completes the proof. $\square$

Proposition 6.2 *For α_m from (1.15) and β_m the correspond* $\liminf$ *it holds*

$$\alpha_m = \frac{m(m-1)}{2^{m+1}(\log 2)^{m-1}}, \qquad \beta_m = \alpha_m - \frac{m}{(2\log 2)^{m-1}}\left(1 - \frac{\log 3}{\log 4}\right). \tag{6.4}$$

For $m > 1$ it holds $0 < \beta_m < \alpha_m$ and $\alpha_m \to 0$ as $m \to \infty$.

Proof: First we compute α_m. For α_m from (1.15) we get by (6.2)

$$\begin{aligned}\alpha_m &= \limsup \frac{1}{(\log N)^{m-1}} \sum_{\mu=0}^{m-1} (\log_2 N)^\mu G_{m,\mu}(\log_2 N) \\ &= \frac{1}{(\log 2)^{m-1}} \max G_{m,m-1}(u).\end{aligned}$$

By (6.3) we get

$$\begin{aligned}G_{m,m-1}(u) &= \frac{a_{m-1,m-1}}{2^{m-1}} s_{m,m-1} F_0(u) + \frac{a_{m,m-1}}{2^m} s_{m,m} F_0(u) + m\frac{a_{m-1,m-1}}{2^{m-1}} s_{m,m} F_1(u) \\ &= \frac{1}{2^{m-1}}\binom{m}{2} - \frac{1}{2^m}\binom{m}{2} + \frac{m}{2^{m-1}} F(u) \\ &= \frac{1}{2^m}\binom{m}{2} + \frac{m}{2^{m-1}} F(u)\end{aligned}$$

where we have used that $s_{m,m-1} = \binom{m}{2}$, $F_0(u) = 1$ and $F_1(u) = F(u)$ in the formula of Trollope-Delange, cf. (5.9). We know $\max F(u) = 0$ and $\min F(u) = (\log 3/\log 4) - 1$, cf. e.g. [14]. Hence

$$\max G_{m,m-1}(u) = \frac{1}{2^m}\binom{m}{2}$$

which yields the value for α_m where $\alpha_m \to 0$ as $m \to \infty$. For β_m we get

$$\beta_m = \frac{1}{(\log 2)^{m-1}} \min G_{m,m-1}(u)$$

which yields

$$\beta_m = \frac{m}{(2\log 2)^{m-1}}\left(\frac{m-1}{4} - 1 + \frac{\log 3}{\log 4}\right)$$

and (6.4). Obviously $\beta_m < \alpha_m$. Now $\beta_2 > 0$ and hence $\beta_m > 0$ for all $m \geq 2$. □

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