Rostock. Math. Kolloq. **64**, 17–38 (2009)

Subject Classification (AMS)
35B55, 35B40, 65M06

Théodore K. Boni, Thibaut K. Kouakou

Numerical quenching for a semilinear parabolic equation with Dirichlet-Neumann boundary conditions and a potential

ABSTRACT. This paper concerns the study of the numerical approximation for a semilinear parabolic equation with Dirichlet-Neumann boundary conditions and a potential. Under some conditions, we show that the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time, and finally, we give some numerical experiments to illustrate our analysis.

KEY WORDS AND PHRASES. semidiscretization, semilinear parabolic equation, semidiscrete quenching time, convergence.

1 Introduction

Consider the following initial-boundary value problem

$$u_t(x,t) - u_{xx}(x,t) = -b(x)f(u(x,t)), \quad x \in (0,1), \quad t \in (0,T), \tag{1}$$

$$u_x(0,t) = 0, \quad u(1,t) = 1, \quad t \in (0,T), \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad x \in [0,1], \tag{3}$$

where $f : (0,\infty) \longrightarrow (0,\infty)$ is a C^1 convex, nonincreasing function, $\int_0^\alpha \frac{ds}{f(s)} < \infty$ for any positive real α, $\lim_{s\to 0^+} f(s) = \infty$, $b \in C^1([0,1])$, $b(x) > 0$, $x \in (0,1)$, $b'(0) = 0$, $b'(1) = 0$. The initial datum $u_0 \in C^2([0,1])$, $u_0(x) > 0$, $x \in [0,1]$,

$$u_0''(x) - b(x)f(u_0(x)) < 0, \quad x \in (0,1), \tag{4}$$

$$u_0'(x) > 0, \quad x \in (0,1), \tag{5}$$

$$u_0'(0) = 0, \quad u_0(1) = 1. \tag{6}$$

Here, $(0, T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, then we say that the solution u of (1)–(3) exists globally. When T is finite, then the solution u of (1)–(3) develops a singularity in a finite time, namely,

$$\lim_{t \to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{0 \leq x \leq 1} u(x, t)$. In this last case, we say that the solution u of (1)–(3) quenches in a finite time, and the time T is called the quenching time of the solution u. By virtue of the definition of the time T, we have

$$u(x, t) > 0, \quad (x, t) \in [0, 1] \times [0, T).$$

The theoretical study of solutions for semilinear parabolic equations which quench in a finite time, has been the subject of investigations of many authors (see, [2], [4], [7], [8], [10], [18], [21], [22], [29], and the references cited therein). In [7], Boni has proved the local in time existence and uniqueness of a classical solution under the hypotheses given in the introduction. The condition (4) allows the solution u to decrease with respect to the second variable, and the assumption (5) permits the solution to increase in space. Hence, the assumption (5) forces the solution u to attain its minimum at the first node. In the previous studies, with the help of the conditions (4) and (5), it is proved that the solution u of (1)–(3) quenches in a finite time at the first node. In addition, the quenching time is estimated (see, [10]). Let us notice that theoretically, it is not possible to determine the exact value of the quenching time.

In this paper, we are interested in the numerical study of the phenomenon of quenching. More precisely, we want to propose an algorithm which allows us to compute a good approximation of the real quenching time. We start by the construction of a semidiscrete scheme as follows. Let I be a positive integer, and define the grid $x_i = ih$, $0 \leq i \leq I$, where $h = 1/I$. Let $U_h(t) = (U_0(t), \cdots, U_I(t))^T$, and approximate the solution u of (1)-(3) by the solution $U_h(t)$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - \beta_i f(U_i(t)), \quad 0 \leq i \leq I-1, \quad t \in (0, T_q^h), \tag{7}$$

$$U_I(t) = 1, \quad t \in (0, T_q^h), \tag{8}$$

$$U_i(0) = \varphi_i, \quad 0 \leq i \leq I, \tag{9}$$

where $\varphi_h > 0$, and

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \tag{10}$$

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I-1,$$

β_i and φ_i are approximations of $b(x_i)$ and $u_0(x_i)$, respectively. One may also choose $\beta_i = b(x_i)$ and $\varphi_i = u_0(x_i)$, but sometimes, we are obliged to take approximations, especially when one does not know the exact value of either the potential or the initial datum. On the other hand, there is another motivation which has incited our choice. This motivation is that, we want to know the behavior of the quenching time when one perturbs slightly either the potential or the initial datum. This is very important in certain situations when for instance the exact values do not possess certain properties, which is not the case of the approximated values. One may list some remarks on the motivation of our study. Firstly, let us notice that the semidiscrete solution U_h of (7)–(9) is the solution of a differential system. Secondly, due to the fact that β_i and φ_i are approximations of $b(x_i)$ and $u_0(x_i)$, respectively, the scheme presented in (7)–(9) is not a standard scheme. Although this scheme is not standard, we shall see later that it allows us to obtain good approximations of the continuous quenching time. In addition, we shall observe that the solution of a semilinear parabolic equation and that of a differential system quench in finite times which are practically the same.

Here, $(0, T_q^h)$ is the maximal time interval on which $U_{hmin}(t) > 0$, where $U_{hmin}(t) = \min_{0\leq i\leq I} U_i(t)$. When T_q^h is finite, then we say that the solution $U_h(t)$ of (7)–(9) quenches in a finite time, and the time T_q^h is called the semidiscrete quenching time of the solution $U_h(t)$.

In this paper, under some assumptions, we show that the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size tends to zero. Recently, a similar study has been undertaken by Nabongo and Boni in [25] where they have considered the problem (1)–(3) for the case $b(x) = 1$ and $f(u) = u^{-p}$ with $p > 0$. It is worth noting that the potential and the nonlinearity of the current paper take into account those of Nabongo and Boni in [25]. One may also consult the papers of Nabongo and Boni in [26], [28] where semidiscrete and discrete schemes have been utilized to study the phenomenon of quenching for other parabolic problems. Let us notice that in these papers, the potential is equal one, and thus, the authors have not studied the effect of a perturbation of the potential on the semidiscrete quenching time. One of our source of motivation to undertake our study on numerical quenching comes from the study on numerical blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time) where the authors, for the treatment, have used semidiscrete and discrete schemes (see [1] and [24]). Our paper is organized as follows. In the next section, we give some results about the semidiscrete maximum principle and reveal certain properties of the semidiscrete solution. In the third section, under some conditions, we show that the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we also prove the convergence of the semidiscrete quenching time. Finally, in the last section, we give some numerical results to illustrate our analysis.

2 The semidiscrete scheme

In this section, we prove some results about the semidiscrete maximum principle and reveal certain properties concerning the operator δ^2 and the semidiscrete solution.

The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1 *Let $\alpha_h \in C^0([0,T),\mathbb{R}^{I+1})$ and let $V_h \in C^0([0,T),\mathbb{R}^{I+1})$ be such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + \alpha_i(t)V_i(t) \geq 0, \quad 0 \leq i \leq I-1, \quad t \in (0,T), \tag{11}$$

$$V_I(t) \geq 0, \quad t \in (0,T), \tag{12}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{13}$$

Then, the following estimates hold

$$V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0,T).$$

Proof: Let T_0 be any quantity satisfying $T_0 < T$, and introduce the vector $Z_h(t) = e^{\lambda t} V_h(t)$, where λ is such that $\alpha_i(t) - \lambda > 0$ for $t \in [0,T_0]$, $0 \leq i \leq I$. Let

$$m = \min_{t\in[0,T_0]} Z_{hmin}(t).$$

Since the vector $Z_i(t)$ is continuous on the compact $[0,T_0]$, then there exist $i_0 \in \{0,1,\cdots,I\}$ and $t_0 \in [0,T_0]$ such that $m = Z_{i_0}(t_0)$.

If $i_0 = I$, then according to (12), we have $m \geq 0$.

If $i_0 \in \{0,\cdots,I-1\}$, then we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \leq 0, \tag{14}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq \quad \text{if} \quad i_0 = 0, \tag{15}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1. \tag{16}$$

Due to (11), a straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (\alpha_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{17}$$

It follows from (14)–(16) that $(\alpha_{i_0}(t_0)-\lambda)Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$ because $\alpha_{i_0}(t_0) - \lambda > 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0, T_0]$, and the proof is complete. □

The lemma below shows a property of the semidiscrete solution.

Lemma 2.2 *Let U_h be the solution of (7)–(9). Assume that the initial datum satisfies $\varphi_i < 1$, $0 \leq i \leq I-1$. Then, we have*

$$U_i(t) < 1, \quad 0 \leq i \leq I-1, \quad t \in (0, T_q^h).$$

Proof: Let $t_0 \in (0, T_q^h)$ be the first time $t \in (0, T_q^h)$ such that $U_i(t) < 1$ for $0 \leq i \leq I-1$, $t \in (0, t_0)$, but $U_j(t_0) = 1$ for a certain $j \in \{0, \cdots, I-1\}$. We have

$$\frac{dU_j(t_0)}{dt} = \lim_{k \to 0} \frac{U_j(t_0) - U_j(t_0 - k)}{k} \geq 0, \tag{18}$$

$$\delta^2 U_j(t_0) = \frac{2U_1(t_0) - 2U_0(t_0)}{h^2} \leq \quad \text{if} \quad j = 0, \tag{19}$$

$$\delta^2 U_j(t_0) = \frac{U_{j+1}(t_0) - 2U_j(t_0) + U_{j-1}(t_0)}{h^2} \leq 0 \quad \text{if} \quad 1 \leq j \leq I-1, \tag{20}$$

which implies that

$$\frac{dU_j(t_0)}{dt} - \delta^2 U_j(t_0) + \beta_j f(U_j(t_0)) > 0.$$

But, this contradicts (7) and the proof is complete. □

Another version of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.3 *Let $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. If V_h, $W_h \in C^1([0,T), \mathbb{R}^{I+1})$ are such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + g(V_i(t), t) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + g(W_i(t), t), \tag{21}$$

$$0 \leq i \leq I-1, \quad t \in (0, T),$$

$$V_I(t) < W_I(t), \quad t \in (0, T), \tag{22}$$

$$V_i(0) < W_i(0), \quad 0 \leq i \leq I, \quad t \in (0, T), \tag{23}$$

then $V_i(t) < W_i(t)$, $0 \leq i \leq I$, $t \in (0, T)$.

Proof: Let $Z_h(t) = W_h(t) - V_h(t)$ and let t_0 be the first $t \in (0, T)$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \leq i \leq I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$. If $i_0 = I$, then we have a contradiction because of (22).

If $i_0 \in \{0, \cdots, I-1\}$, then we obtain

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq \quad \text{if} \quad i_0 = 0, \tag{24}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + g(W_{i_0}(t_0), t_0) - g(V_{i_0}(t_0), t_0) \leq 0.$$

But, this inequality contradicts (21), which ends the proof. $\square$

The following results show some properties of the semidiscrete solution.

Lemma 2.4 *Let U_h be the solution of (7)–(9) such that the initial datum satisfies*

$$\varphi_{i+1} > \varphi_i, \quad 0 \leq i \leq I-1. \tag{25}$$

Then, we have for $t \in (0, T_q^h)$,

$$U_{i+1}(t) > U_i(t), \quad 0 \leq i \leq I-1. \tag{26}$$

Proof: Invoking Lemma 2.2, we know that

$$U_i(t) < 1, \quad 0 \leq i \leq I-1, \quad t \in (0, T_q^h).$$

Let t_1 be the first $t \in (0, T_q^h)$ such that $U_{i+1}(t) > U_i(t)$ for $t \in (0, t_1)$, $0 \leq i \leq I-1$, but

$$U_{k+1}(t_1) = U_k(t_1) \quad \text{for a certain} \quad k \in \{0, \cdots, I-1\}. \tag{27}$$

Without loss of generally, we may suppose that k is the smallest integer which verifies (27).

If $k = I-1$, then $U_{I-1}(t_1) = U_I(t_1) = 1$, which contradicts the fact that $U_{I-1}(t_1) < 1$.

If $k = 1, \cdots, I-2$, then we observe that

$$\frac{d(U_{k+1} - U_k)(t_1)}{dt} = \lim_{\sigma \to 0} \frac{(U_{k+1} - U_k)(t_1) - (U_{k+1} - U_k)(t_1 - \sigma)}{\sigma} \leq 0,$$

and

$$\delta^2(U_{k+1}-U_k)(t_1)=\frac{(U_{k+2}-U_{k+1})(t_1)-2(U_{k+1}-U_k)(t_1)+(U_k-U_{k-1})(t_1)}{h^2}>0,$$

which implies that

$$\frac{d(U_{k+1}-U_k)(t_1)}{dt}-\delta^2(U_{k+1}-U_k)(t_1)+\beta_{k+1}f(U_{k+1}(t_1))-\beta_k f(U_k(t_1))<0.$$

But, this contradicts (7).

If $k=0$, then we get

$$\delta^2(U_{k+1}-U_k)(t_1)=\frac{(U_{k+2}-U_{k+1})(t_1)-3(U_{k+1}-U_k)(t_1)}{h^2}<0.$$

Thanks to the above inequality, it is easy to see that

$$\frac{d(U_{k+1}-U_k)(t_1)}{dt}-\delta^2(U_{k+1}-U_k)(t_1)+\beta_{k+1}f(U_{k+1}(t_1))-\beta_k f(U_k(t_1))<0,$$

which contradicts (7). This ends the proof. □

Remark 2.1 The above lemma says that, if the initial datum of the semidiscrete solution is increasing in space, then the semidiscrete solution also satisfies this property. This result will be used later to show that the semidiscrete solution attains its minimum at the first node.

To end this section, let us give some properties of the operator δ^2.

Lemma 2.5 *Let V_h and $U_h\in\mathbb{R}^{I+1}$. If $\delta^+(U_0)\delta^+(V_0)\geq 0$ and*

$$\delta^+(U_i)\delta^+(V_i)\geq 0,\quad \delta^-(U_i)\delta^-(V_i)\geq 0,\quad 1\leq i\leq I-1, \tag{28}$$

then

$$\delta^2(U_iV_i)\geq U_i\delta^2V_i+V_i\delta^2U_i,\quad 0\leq i\leq I-1,$$

where $\delta^+(U_i)=\frac{U_{i+1}-U_i}{h}$ and $\delta^-(U_i)=\frac{U_{i-1}-U_i}{h}$.

Proof: A straightforward computation reveals that

$$\delta^2(U_0V_0)=2\delta^+(U_0)\delta^+(V_0)+U_0\delta^2V_0+V_0\delta^2U_0,$$

$$\delta^2(U_iV_i)=\delta^+(U_i)\delta^+(V_i)+\delta^-(U_i)\delta^-(V_i)+U_i\delta^2V_i+V_i\delta^2U_i,\quad 1\leq i\leq I-1.$$

Taking into account the assumptions of the lemma, we obtain the desired result. □

Lemma 2.6 *Let $U_h \in \mathbb{R}^{I+1}$ be such that $U_h > 0$. Then, the following estimates hold*

$$\delta^2 f(U_i) \geq f'(U_i)\delta^2 U_i, \quad 0 \leq i \leq I-1.$$

Proof: Applying Taylor's expansion, we get

$$\delta^2 f(U_0) = f'(U_0)\delta^2 U_0 + f''(\theta_0)\frac{(U_1 - U_0)^2}{h^2},$$

$$\delta^2 f(U_i) = f'(U_i)\delta^2 U_i + f''(\theta_i)\frac{(U_{i+1} - U_i)^2}{2h^2} + f''(\eta_i)\frac{(U_{i-1} - U_i)^2}{2h^2},$$

$$1 \leq i \leq I-1,$$

where θ_i is an intermediate value between U_i and U_{i+1}, and η_i the one between U_{i-1} and U_i. Use the fact that $U_h > 0$ to complete the rest of the proof. □

3 Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_h of (7)–(9) quenches in a finite time and estimate its semidiscrete quenching time.

Our result is the following.

Theorem 3.1 *Let U_h be the solution of (7)–(9), and assume that there exists a constant $A \in (0,1]$ such that the initial datum satisfies*

$$\delta^2 \varphi_i - \beta_i f(\varphi_i) \leq -A\sin(ih\pi) f(\varphi_i), \quad 1 \leq i \leq I-1, \tag{29}$$

$$1 - \frac{2\pi^2}{A}\int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)} > 0. \tag{30}$$

Under the assumptions of Lemma 2.4, the solution U_h quenches in a finite time T_q^h, and the following estimation holds

$$T_q^h \leq -\frac{1}{\pi^2}\ln\left(1 - \frac{2\pi^2}{A}\int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)}\right).$$

Proof: Since $(0, T_q^h)$ is the maximal time interval of existence of the solution U_h our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} + C_i(t) f(U_i(t)), \quad 0 \leq i \leq I, \quad t \in [0, T_q^h),$$

where $C_i(t) = Ae^{-\lambda_h t}\cos(ih\frac{\pi}{2})$, $0 \le i \le I$, with $\lambda_h = \frac{2-2\cos(ih\frac{\pi}{2})}{h^2}$. A straightforward computation reveals that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) = \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)\right) + C_i(t)f'(U_i(t))\frac{dU_i(t)}{dt}$$

$$-\delta^2(C_i(t)f(U_i(t))) + \frac{dC_i(t)}{dt}f(U_i(t)), \quad 0 \le i \le I-1, \quad t \in (0, T_q^h).$$

We observe that

$$\frac{dC_i(t)}{dt} - \delta^2 C_i(t) = 0, \quad C_{i+1}(t) < C_i(t), \quad 0 \le i \le I-1,$$

and due to Lemma 2.4, we find that $\delta^+(f(U_0))\delta^+(C_0) \ge 0$, and $\delta^+(f(U_i))\delta^+(C_i) \ge 0$, $\delta^-(f(U_i))\delta^-(C_i) \ge 0$, $0 \le i \le I-1$. It follows from Lemmas 2.5 and 2.6 that

$$\delta^2(C_i(t)f(U_i(t))) \ge C_i(t)f'(U_i(t))\delta^2 U_i(t) + f(U_i(t))\delta^2 C_i(t), \quad 0 \le i \le I-1.$$

Using the above estimates, we discover that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \le \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)\right) + C_i(t)f'(U_i(t))\left(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)\right)$$

$$+f(U_i(t))\left(\frac{dC_i(t)}{dt} - \delta^2 C_i(t)\right), \quad 0 \le i \le I-1, \quad t \in (0, T_q^h).$$

With the help of (7), we derive the following estimates

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \le -\beta_i(t)f'(U_i(t))\frac{dU_i(t)}{dt} - \beta_i(t)C_i(t)f'(U_i(t))f(U_i(t)), \quad 0 \le i \le I-1.$$

Taking into account the expression of $J_i(t)$, we arrive at

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \le -\beta_i(t)f'(U_i(t))J_i(t), \quad 0 \le i \le I-1.$$

Obviously, we note that

$$J_I(t) = \frac{dU_I(t)}{dt} + C_I(t)f(U_I(t)) = 0,$$

and due to inequalities (29), we get $J_h(0) \le 0$. It follows from Lemma 2.1 that $J_h(t) \le 0$ for $t \in [0, T_h)$. This estimate may be rewritten in the following manner

$$\frac{dU_i(t)}{dt} \le -Ae^{-\lambda_h t}\cos\left(\frac{i\pi h}{2}\right) f(U_i(t)), \quad 0 \le i \le I, \quad t \in (0, T_q^h). \tag{31}$$

At the first node, we have

$$\frac{dU_0(t)}{dt} \leq -Ae^{-\lambda_h t} f(U_0(t)), \quad t \in (0, T_q^h). \tag{32}$$

Apply Taylor's expansion to obtain

$$\cos\left(\frac{\pi h}{2}\right) = 1 - \frac{\pi^2 h^2}{4} + \frac{\pi^3 h^3}{48} \sin\left(\frac{\pi h}{2}\theta\right),$$

where $\theta \in [0, 1]$. This implies that $\lambda_h \leq \frac{\pi^2}{2}$. Therefore using (32), we discover that

$$\frac{dU_0(t)}{dt} \leq -Ae^{-\frac{\pi^2}{2}t} f(U_0(t)), \quad t \in (0, T_q^h).$$

After a little transformation, this inequality becomes

$$\frac{dU_0}{f(U_0)} \leq -Ae^{-\frac{\pi^2}{2}t} dt, \quad t \in (0, T_q^h). \tag{33}$$

Integrate the above estimate over $(0, T_q^h)$ to arrive at

$$T_q^h \leq -\frac{2}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A} \int_0^{U_0(0)} \frac{d\sigma}{f(\sigma)}\right). \tag{34}$$

We know from Lemma 2.4 that $U_0(t) = U_{hmin}(t)$ for $t \in (0, T_q^h)$, which implies that $U_0(0) = U_{hmin}(0) = \varphi_{hmin}$. Taking into account the above inequalities and (34), it is not difficult to check that

$$T_q^h \leq -\frac{2}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A} \int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)}\right). \tag{35}$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. $\square$

Remark 3.1 Let $t_0 \in (0, T_q^h)$. Integrate the estimate (33) over (t_0, T_q^h) and use the fact that $U_{hmin}(t_0) = U_0(t_0)$ to obtain

$$T_q^h - t_0 \leq -\frac{2}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A} e^{\frac{\pi^2}{2}t_0} \int_0^{U_{hmin}(t_0)} \frac{d\sigma}{f(\sigma)}\right). \tag{36}$$

The theorem below gives a lower bound of the semidiscrete quenching time.

Theorem 3.2 *Let U_h be the solution of (7)–(9). Assume that U_h quenches at the time T_q^h. Then, we have the following estimate*

$$T_q^h \geq \frac{1}{\|\beta_h\|_\infty} \int_0^{\varphi_{hmin}} \frac{ds}{f(s)}.$$

Proof: Let $\alpha(t)$ be the solution of the following differential equation

$$\alpha'(t) = -\|\beta_h\|_\infty f(\alpha(t)),\ t > 0,\ \alpha(0) = \varphi_{hmin},$$

and let $W_h(t)$ be the vector such that $W_i(t) = \alpha(t),\ 0 \le i \le I$. After a little transformation, it not hard to see that $\alpha(t)$ quenches in a finite time at the time $T_h = \frac{1}{\|\beta_h\|_\infty}\int_0^{\varphi_{hmin}} \frac{ds}{f(s)}$. Setting $Z_h(t) = W_h(t) - U_h(t)$, a straightforward computation reveals that

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + \beta_i f'(\xi_i(t))Z_i(t) \le 0, \quad 0 \le i \le I-1, \quad t \in (0, T_h^*),$$

$$Z_I(t) \le 0, \quad t \in (0, T_h^*),$$

$$Z_i(0) \le 0, \quad 0 \le i \le I,$$

where $T_h^* = \min\{T_h, T_q^h\}$, and $\xi_i(t)$ is an intermediate value between $U_i(t)$ and $W_i(t)$. Invoking Lemma 2.1, we derive the following estimate $W_h(t) \le U_h(t)$ for $t \in (0, T_h^*)$. Making use of the expression of W_h, we discover that

$$U_{hmin}(t) \ge \alpha(t) \quad \text{for} \quad t \in (0, T_h^*).$$

This implies that if $t < \frac{1}{\|\beta_h\|_\infty}\int_0^{\varphi_{hmin}} \frac{ds}{f(s)}$, then $U_{hmin}(t) > 0$. Therefore $T_q^h \ge \frac{1}{\|\beta_h\|_\infty}\int_0^{\varphi_{hmin}} \frac{ds}{f(s)}$, and the proof is complete. □

4 Convergence of the semidiscrete quenching time

In this section, under some assumptions, we prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero.

Firstly, we show that, in the interval $[0, T-\tau]$ with $\tau \in (0, T)$ where the continuous solution u obeys $u_{\min}(t) > 0$, the semidiscrete solution U_h approximates u when the mesh parameter h goes to zero. This result is stated in the following theorem.

Theorem 4.1 *Assume that (1)–(3) has a solution $u \in C^{4,1}([0,1] \times [0, T-\tau])$ such that $\min_{t\in[0,T]} u_{\min}(t) = \rho > 0$ with $\tau \in (0,T)$. Suppose that the potential and the initial datum of the problem (7)–(9) satisfy*

$$\|\beta_h - b_h\|_\infty = o(1) \quad as \quad h \to 0, \tag{37}$$

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0, \tag{38}$$

respectively, where $u_h(t) = (u(x_0,t), \cdots, u(x_I,t))^T$. Then, for h sufficiently small, the problem (7)–(9) has a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$ such that

$$\max_{0\le t\le T-\tau} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + h^2) \quad as\ h \to 0.$$

Proof: The problem (7)–(9) has for each h, a unique solution $U_h \in C^1([0,T_q^h),\mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t>0$ such that

$$\|U_h(t)-u_h(t)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad t\in(0,t(h)). \tag{39}$$

The relation (38) implies that $t(h)>0$ for h sufficiently small. Let $t^*(h)=\min\{t(h),T-\tau\}$. An application of the triangle inequality gives

$$U_{hmin}(t) \geq u_{\min}(t) - \|U_h(t)-u_h(t)\|_\infty \quad \text{for} \quad t\in(0,t^*(h)),$$

which implies that

$$U_{hmin}(t) \geq \rho - \frac{\rho}{2} = \frac{\rho}{2} \quad \text{for} \quad t\in(0,t^*(h)). \tag{40}$$

Apply Taylor's expansion to obtain

$$\delta^2 u(x_i,t) = u_{xx}(x_i,t) + \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t), \quad 0\leq i\leq I-1.$$

Exploiting the above equalities, we arrive at

$$\frac{du(x_i,t)}{dt} = \delta^2 u(x_i,t) - b(x_i)f(u(x_i,t)) - \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t), \quad 0\leq i\leq I-1.$$

Introduce the error of discretization

$$e_h(t) = U_h(t)-u_h(t), \quad t\in[0,t^*(h)).$$

Invoking the mean value theorem, we find that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = -\beta_i f'(\theta_i(t))e_i(t) - (\beta_i - b(x_i))f(u(x_i,t))$$

$$+\frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t), \quad 1\leq i\leq I-1, \quad t\in(0,t^*(h)), \tag{41}$$

where $\theta_i(t)$ is an intermediate value between $U_i(t)$ and $u(x_i,t)$. Let $M>0$ be such that

$$\frac{\|u_{xxxx}(\cdot,t)\|_\infty}{12} \leq M \text{ for } t\in[0,t^*(h)), \quad -\|\beta_h\|_\infty f'(\tfrac{\rho}{2}) \leq M, \quad f(\frac{\rho}{2}) \leq M.$$

Making use of the above inequalities, it is not hard to see that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq M|e_i(t)| + Mh^2 + M\|\beta_h - b_h\|_\infty,$$

$$0\leq i\leq I-1, \quad t\in(0,t^*(h)).$$

Introduce the vector z_h such that

$$z_i(t) = e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + Mh^2), \quad 0 \leq i \leq I, \quad t \in [0,T].$$

A straightforward computation yields

$$\frac{dz_i(t)}{dt} - \delta^2 z_i(t) > M|z_i(t)| + Mh^2 + M\|\beta_h - b_h\|_\infty,$$

$$0 \leq i \leq I-1, \quad t \in (0, t^*(h)),$$

$$z_I(t) > e_I(t), \quad t \in (0, t^*(h)),$$

$$z_i(0) > e_i(0), \quad 0 \leq i \leq I.$$

It follows from Lemma 2.3 that

$$z_i(t) > e_i(t), \quad t \in (0, t^*(h)), \quad 0 \leq i \leq I.$$

In the same way, we also prove that

$$z_i(t) > -e_i(t), \quad t \in (0, t^*(h)), \quad 0 \leq i \leq I,$$

which implies that

$$\|U_h(t) - u_h(t)\|_\infty \leq e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + Mh^2), \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T - \tau$. Suppose that $T - \tau > t(h)$. From (39), we obtain

$$\frac{\rho}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + Mh^2).$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $\frac{\rho}{2} \leq 0$, which is impossible. Consequently $t^*(h) = T - \tau$, and the proof is complete. □

Now, we are in a position to prove the main result of this section.

Theorem 4.2 *Suppose that the solution u of (1)–(3) quenches in a finite time T such that $u \in C^{4,1}([0,1] \times [0,T))$. Assume that the potential and the initial datum of the problem (7)–(9) satisfy the conditions (34) and (35), respectively. Under the assumptions of Theorem 3.1, the problem (7)–(9) admits a unique solution U_h which quenches in a finite time T_q^h, and the following relation holds*

$$\lim_{h \to 0} T_q^h = T.$$

Proof: Let $0 < \varepsilon \leq T/2$. There exists a constant $R > 0$ such that

$$-\frac{1}{\pi^2}\ln\left(1-\frac{2\pi^2}{A}e^{\pi^2 T}\int_0^x \frac{ds}{f(s)}\right) < \frac{\varepsilon}{2} \text{ for } x \in [0, R]. \tag{42}$$

Due to the fact that the solution u quenches in a finite time T, there exists $T_1 \in (T-\frac{\varepsilon}{2}, T)$ such that $0 < u_{\min}(t) < \frac{R}{2}$ for $t \in [T_1, T)$. Let $T_2 = \frac{T_1+T}{2}$. Obviously $0 < u_{\min}(t) < \frac{R}{2}$ for $t \in [0, T_2]$. Exploiting Theorem 4.1, we know that the problem (7)–(9) admits a unique solution U_h which obeys the following estimate $\|U_h(t)-u_h(t)\|_\infty < \frac{R}{2}$ for $t \in [0, T_2]$, which implies that $\|U_h(T_2)-u_h(T_2)\|_\infty < \frac{R}{2}$. An application of the triangle inequality gives

$$U_{hmin}(T_2) \leq \|U_h(T_2)-u_h(T_2)\|_\infty + u_{\min}(T_2) \leq \frac{R}{2}+\frac{R}{2} = R.$$

Taking into account Theorem 3.1, we note that $U_h(t)$ quenches in a finite time T_q^h. We infer from Remark 3.1 that

$$|T_q^h - T_2| \leq -\frac{1}{\pi^2}\ln\left(1-\frac{2\pi^2}{A}e^{\pi^2 T_2}\int_0^{U_{hmin}(T_2)} \frac{ds}{f(s)}\right). \tag{43}$$

Since the function $s \longrightarrow -\ln(1-s)$ is an increasing function for positive values of s, it is not hard to see that the term on the right hand side of the above inequality is bounded from above by $-\frac{1}{\pi^2}\ln(1-\frac{2\pi^2}{A}e^{\pi^2 T}\int_0^{U_{hmin}(T_2)}\frac{ds}{f(s)})$. We deduce from (42) and (43) that $|T_q^h - T_2| \leq \frac{\varepsilon}{2}$, which implies that

$$|T_q^h - T| \leq |T_q^h - T_2| + |T_2 - T| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. $\square$

5 Numerical experiments

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the problem (1)–(3) in the case where $f(u) = u^{-p}$, $b(x) = 2+x^2$, $u_0(x) = \frac{1+x^2}{2}$ with $p > 0$. We start by the construction of an adaptive scheme as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$, and approximate the solution u of (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, \cdots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)}-2U_0^{(n)}}{h^2} - \beta_0(U_0^{(n)})^{-p},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2} - \beta_i(U_i^{(n)})^{-p}, \quad 1 \leq i \leq I-1,$$

$$U_I^{(n)} = 1,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I,$$

where $\varphi_i = \frac{1+(ih)^2}{2} - \varepsilon\frac{(\sin(\frac{i\pi h}{2})+1)}{5}$, $\beta_i = 1 + (ih)^2 - \epsilon \sin(i\pi h)$ with $\epsilon \in [0,1]$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\left\{\frac{(1-h^2)h^2}{2}, h^2(U_{hmin}^{(n)})^{p+1}\right\}.$$

Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution u of (1)–(3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - \beta_0 (U_0^{(n)})^{-p-1} U_0^{(n+1)},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} - \beta_i (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 1 \le i \le I-1,$$

$$U_I^{(n+1)} = 1,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I.$$

As in the case of the explicit scheme, here, we also choose $\Delta t_n = h^2(U_{hmin}^{(n)})^{p+1}$. Let us again remark that for the above implicit scheme, existence and positivity of the discrete solution are also guaranteed using standard methods (see, for instance [6]).

We need the following definition.

Definition 5.1 *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to\infty} U_{hmin}^{(n)} = 0$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.*

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \le 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $p = 1$

First case: $\epsilon = 0$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.0802018	1960	1.3	-
32	0.0800499	7550	5.3	-
64	0.0800131	28956	51	2.04
128	0.0800040	110721	1140	2.01

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.0808371	1975	2.8	-
32	0.0802084	7564	10.6	-
64	0.0800526	28969	164	2.01
128	0.0800014	110732	4620	1.61

Second case: $\epsilon = 1$

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.0176645	1856	1.3	-
32	0.0177477	7167	5.4	-
64	0.0177705	27512	45	1.86
128	0.0177765	105213	1097	1.92

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.0178058	1870	1.8	-
32	0.0177826	7179	11.4	-
64	0.0177791	27523	150	2.72
128	0.0177787	105223	3604	3.12

Third case: $\epsilon = 1/100$

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.0794108	1960	1.2	-
32	0.0792633	7551	5.4	-
64	0.0792276	28962	81	2.72
128	0.0792188	110749	1380	2.02

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.0800397	1975	1.8	-
32	0.0794202	7565	11.5	-
64	0.0792667	28974	193	2.01
128	0.0792286	110760	4260	2.01

Remark 5.1 Tables 1 and 2 provide us the results of the numerical quenching time when $\varepsilon = 0$. We observe that the numerical quenching time in this case is approximately equal to 0.08. It is worth noting that both explicit and implicit schemes give practically the same results, and we also observe that the variation of the different meshes has no important effects on the numerical quenching time. It is also important to point out that, when we look at Tables 5 and 6, we see that the numerical quenching times when $\epsilon \in (0, 1)$ is small enough, are slightly equal to that which corresponds to $\epsilon = 0$. On the other hand, when one examines Tables 1, 2, 3 and 4, one sees that an important perturbation on the potential and the initial datum has a meaningful impact on the numerical quenching time.

In the following, we also give some plots to illustrate our analysis. In the figures below, we can see that the discrete solution quenches in a finite time at the first node. Here, all schemes are highly consistent.

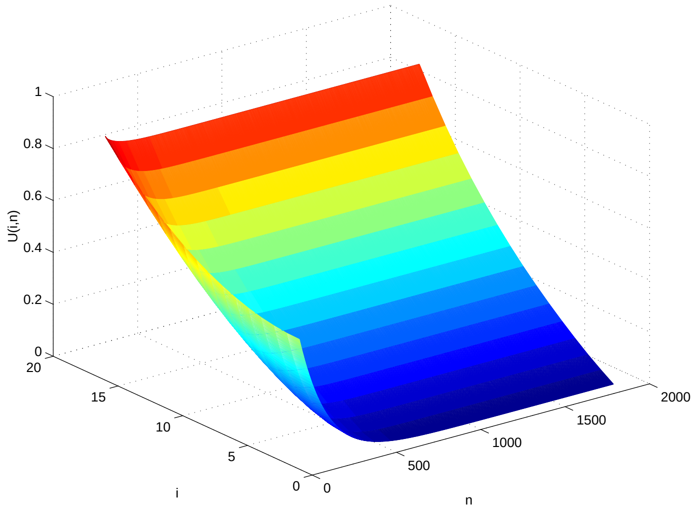

Figure 1: Evolution of the discrete solution $U_h^{(n)}$, $I = 16$, $\varepsilon = 0$, $f(s) = s^2$.

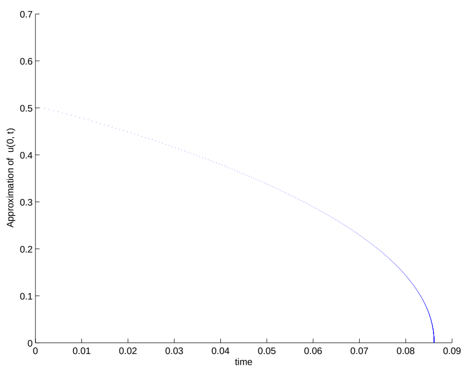

Figure 2: Approximation of $u(0,t)$, $I = 16$, $\varepsilon = 0$, $f(s) = s^2$.

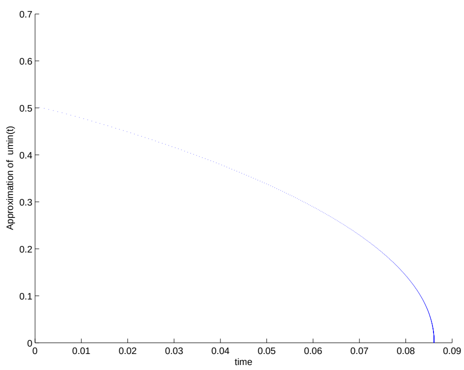

Figure 3: Approximation of $u_{min}(t)$, $I = 16$, $\varepsilon = 0$, $f(s) = s^2$.

6 Conclusion

In the present paper, we have considered a semilinear heat equation with a potential subject to Dirichlet-Neumann boundary conditions. We have constructed a semidiscrete scheme, and have shown that the solution of the semidiscrete scheme quenches in a finite time, and its semidiscrete quenching time converges to the continuous one when the mesh size tends to zero. We have studied in passing the continuity of the semidiscrete quenching time as a function of the potential and the initial datum. Our study can take into account some works where Dirichlet boundary conditions are considered. In fact, one knows that if a solution $u(x,t)$ is symmetric in $(-1,1)\times(0,T)$, then $u_x(0,t)=0$. This can permit to treat a problem where Dirichlet boundary condition is taken considering the problem developed in this paper. In the works to come, it will be better to consider the problem described in (1)–(3) using a full discrete scheme.

Acknowledgements

The authors would like to thank the referees for the throughout reading of the manuscript and several suggestions that helped us improve the presentation of the paper.

References

[1] **Abia, L. M., Lopez-Marcos, J. C.**, and **Martinez, J. :** *On the blow-up time convergence of semidiscretizations of reaction-diffusion equations.* Appl. Numer. Math. **26**, 399 – 414 (1998)

[2] **Acker, A.**, and **Walter, W. :** *The quenching problem for nonlinear parabolic differential equations.* Lecture Notes in Math., Springer-Verlag, New York, **564**, 1 – 12 (1976)

[3] **Acker, A.**, and **Kawohl, B. :** *Remarks on quenching.* Nonl. Anal. TMA **13**, 53 – 61 (1989)

[4] **Bandle, C.**, and **Braumer, C. M. :** *Singular perturbation method in a parabolic problem with free boundary.* in Proc. BAIL IVth Conference, Boole Press Conf. Ser. 8, Novosibirsk., 7 – 14 (1986)

[5] **Baras, P.**, and **Cohen, L. :** *Complete blow-up after T_{max} for the solution of a semilinear heat equation.* J. Funct. Anal., **71**, 142 – 174 (1987)

[6] **Boni, T. K. :** *Extinction for discretizations of some semilinear parabolic equations.* C. R. Acad. Sci, Paris, Sér. I **333**, 795 – 800 (2001)

[7] **Boni, T. K. :** *On quenching of solutions for some semilinear parabolic equations of second order.* Bull. Belg. Math. Soc. **7**, 73 – 95 (2000)

[8] **Deng, K.**, and **Levine, H. A. :** *On the blow-up of u_t at quenching.* Proc. Amer. Math. Soc. **106**, 1049 – 1056 (1989)

[9] **Fermanian Kammerer, C., Merle, F.**, and **Zaag, H. :** *Stability of the blow-up profile of nonlinear heat equations from the dynamical system point of view.* Math. Ann. **317**, 195 – 237 (2000)

[10] **Fila, M., Kawohl, B.**, and **Levine, H. A. :** *Quenching for quasilinear equations.* Comm. Part. Diff. Equat. **17**, 593 – 614 (1992)

[11] **Galaktionov, V.**, and **Vazquez, J. L. :** *Continuation of blow-up solutions of nonlinear heat equations in several space dimensions.* Comm. Pure Appl. Math. **50**, 1 – 67 (1997)

[12] **Galaktionov, V.**, and **Vazquez, J. L. :** *The problem of blow-up in nonlinear parabolic equation.* current developments in PDE (Temuco,1999), Discrete contin. Dyn. Syst. **8**, 399 – 433 (2002)

[13] **Galaktionov, V. A., Mikhailov, A. P.**, and **Samarskii, A. A. :** *Unbounded solutions of the Cauchy problem for the parabolic equation* $u_t = \nabla(u^\sigma \nabla u) + u^\beta$. Soviet Phys. Dokl. **25**, 458 – 459 (1980)

[14] **Guo, J. :** *On a quenching problem with Robin boundary condition.* Nonl. Anal. TMA **17**, 803 – 809 (1991)

[15] **Groisman, P.**, and **Rossi, J. D. :** *Dependance of the blow-up time with respect to parameters and numerical approximations for a parabolic problem.* Asympt. Anal. **37**, 79 – 91 (2004)

[16] **Groisman, P., Rossi, J. D.**, and **Zaag, H. :** *On the dependence of the blow-up time with respect to the initial data in a semilinear parabolic problem.* Comm. Part. Differential Equations **28**, 737 – 744 (2003)

[17] **Herreros, M. A.**, and **Vazquez, J. L. :** *Generic behaviour of one-dimensional blow up patterns.* Ann. Scuola. Norm. Sup. Pisa Cl. Sci. (4)**19**(3): 381 – 450, (1992)

[18] **Kawarada, H. :** *On solutions of initial-boundary problem for* $u_t = u_{xx} + 1/(1-u)$. Pul. Res. Inst. Math. Sci. **10**, 729 – 736 (1975)

[19] **Kirk, C. M.**, and **Roberts, C. A. :** *A review of quenching results in the context of nonlinear volterra equations.* Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **10**, 343 – 356 (2003)

[20] **Ladyzenskaya, A., Solonnikov, V. A.**, and **Ural'ceva, N. N. :** *Linear and quasililear equations parabolic type.* Trans. Math. Monogr. **23** AMS, Providence, RI, (1968)

[21] **Levine, H. A. :** *The phenomenon of quenching: a survey, in Trends.* In The Theory And Practice Of Nonlinear Analysis, North-Holland, Amsterdam, 275 – 286 (1985)

[22] **Levine, H. A. :** *Quenching, nonquenching and beyond quenching for solution of some parabolic equations.* Ann. Math. Pura Appl. **155**, 243 – 260 (1989)

[23] **Merle, F. :** *Solution of a nonlinear heat equation with arbitrarily given blow-up points.* Comm. Pure Appl. Math. **45**, 293 – 300 (1992)

[24] **Nakagawa, T. :** *Blowing up on the finite difference solution to* $u_t = u_{xx} + u^2$. Appl. Math. Optim. **2**, 337 – 350 (1976)

[25] **Nabongo, D.**, and **Boni, T. K. :** *Numerical quenching for a semilinear parabolic equation.* Math. Modelling and Anal. **13**, 521 – 538 (2008)

[26] **Nabongo, D.**, and **Boni, T. K. :** *Quenching for semidiscretization of a semilinear heat equation with Dirichlet and Neumann boundary condition.* Comment. Math. Univ. Carolinae **49**, 463 – 475 (2008)

[27] **Nabongo, D.**, and **Boni, T. K. :** *Quenching time of solutions for some nonlinear parabolic equations.* An. St. Univ. Ovidius Constanta **16**, 87 – 102 (2008)

[28] **Nabongo, D.**, and **Boni, T. K. :** *Quenching for semidiscretization of a heat equation with a singular boundary condition.* Asympt. Anal. **59**, 27 – 38 (2008)

[29] **Phillips, D. :** *Existence of solution of quenching problems.* Appl. Anal. **24**, 253 – 264 (1987)

[30] **Protter, M. H.**, and **Weinberger, H. F. :** *Maximum principles in differential equations.* Prentice Hall, Englewood Cliffs, NJ, (1967)

[31] **Quittner, P. :** *Continuity of the blow-up timeand a priori bounds for solutions in superlinear parabolic problems.* Houston J. Math. **29** (3): 757 – 799 (electronic), 2003

[32] **Shang, Q.**, and **Khaliq, A. Q. M. :** *A compound adaptive approach to degenerate nonlinear quenching problems.* Numer. Methods Partial Differential Equations **15**, 29 – 47 (1999)

[33] **Walter, W. :** *Differential- und Integral-Ungleichungen.* Springer, Berlin, (1964)

received: August 28, 2008

Authors:

Théodore K. Boni
Institut National Polytechnique
Houphouët-Boigny de Yamoussoukro,
BP 1093 Yamoussoukro,
(Côte d'Ivoire)

e-mail: theokboni@yahoo.fr

Thibaut K. Kouakou
Université d'Abobo-Adjamé,
UFR-SFA,
Département de Mathématiques
et Informatiques,
02 BP 801 Abidjan 02,
(Côte d'Ivoire)

e-mail: kkthibaut@yahoo.fr