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Iterative functional equations

ABSTRACT. Functional equations for iterates are solved or approximated by means of associated difference equations. Some examples and three Open Problems are pointed out.

KEY WORDS. Functional equations, difference equations, iterates, approximations.

1 Introduction

Let y be a strictly monotonous continuous function $\mathbb{R} \mapsto I \subset \mathbb{R}$ and $y^{[-1]}$ its inverse $I \mapsto \mathbb{R}$, then the functions

$$f^{[n]}(x) = y\left(n + y^{[-1]}(x)\right) \tag{1.1}$$

with integer n are the n-th iterates of the strictly increasing function $f = f^{[1]} : I \mapsto I$. Classical books concerning iterates are Aczél [1] and Kuczma [4], and recent surveys are Baron and Jarczyk [3], and Targonski [8]. Let us mention that the functions (1.1) are solutions of the well known translation equation, which also can be considered in the multidimensional case, cf. [2].

The connection (1.1) between y and the iterates of f enables us to solve the functional equation

$$f^{[k]} = F\left(x, f, \ldots, f^{[k-1]}\right) \tag{1.2}$$

with $k \in \mathbb{N}$ and $F : I^k \mapsto I$ by means of the associated difference equation

$$y(t + k) = F(y(t), y(t + 1), \ldots, y(t + k - 1)) \tag{1.3}$$

with the same F. Namely, any strictly monotonous continuous solution $y : \mathbb{R} \mapsto I$ of (1.3) yields a strictly increasing solution $f : I \mapsto I$ by means of (1.1).

In the case that F is a homogeneous function of degree 1, equation (1.2) has the so called characteristic solutions $f = rx$, so far as r is a real solution of the characteristic equation

$$r^k = F(1, r, \dots, r^{k-1}),$$

cf. Matkowski and Zhang [5] for linear homogeneous F.

In the following we mostly deal with the case $k = 2$. In the next section we give some examples for the method in the Introduction, and in the last two sections we deal with further methods for solving (1.2), at least approximatively. Three Open Problems are offered to the reader.

2 Linear associated equations

As a special case of (1.2) we consider the functional equation with constant coefficients

$$f^{[2]} = af + bx, \tag{2.1}$$

which was studied in detail by Matkowski and Zhang [5], see also the references therein. The associated difference equation (1.3) reads

$$y(t+2) = ay(t+1) + by(t) \tag{2.2}$$

and has the general solution

$$y(t) = \begin{cases} cp^t + dq^t & \text{for } p \neq q, \\ (ct+d)p^t & \text{for } p = q, \end{cases} \tag{2.3}$$

where c and d are arbitrary 1-periodic functions, and p, q the solutions of the corresponding characteristic equation

$$r^2 = ar + b. \tag{2.4}$$

In the following we restrict ourselves to the case that p, q are positive, and c, d some real constants.

The case of arbitrary real p, q different from zero can be reduced to the foregoing one by determining the function $g = f^{[2]}$ out of the equation

$$g^{[2]} + (2b - a^2)g + b^2 x = 0$$

with the squares of p, q as zeros of its characteristic equation. Afterwards, f is to determine as fractional iterate $f = g^{[1/2]}$, cf. [4, Theorems 15.7 and 15.9]. However, not all fractional iterates are solutions of (2.1), especially, in the case $pq < 0$ there exist at most the characteristic solutions of (2.1), disregarding exceptional cases, cf. [5, Theorem 6]. Let us mention that, if f is a solution of (2.1), then $g = -f(-x)$ is also a solution of it (what is trivial for odd f).

Applying formula (2.3) in the case $p \neq q$, we find that the iterates (1.1) of the solutions f of (2.1) can be written as

$$f^{[n]}(x) = cp^{n+y^{[-1]}(x)} + dq^{n+y^{[-1]}(x)} . \tag{2.5}$$

We introduce a real constant $s \neq 1$ such that

$$q = p^s , \tag{2.6}$$

and moreover the function

$$u(x) = cp^{y^{[-1]}(x)} .$$

Hence, (2.5) can be written as

$$f^{[n]}(x) = p^n u + Ap^{sn} u^s$$

with $A = dc^{-s}$. Since $f^{[0]}(x) = x$, we have to determine u by inversion of

$$x = u + Au^s , \tag{2.7}$$

and the foregoing functions turn into

$$f^{[n]}(x) = p^{sn} x + (p^n - p^{sn})u . \tag{2.8}$$

Since u must be strictly monotonous in x, we have to choose $A > 0$ for $s > 0$, and $A < 0$ for $s < 0$, whereas A remains arbitrary for $s = 0$.

Let us consider three examples for solutions of equation (2.1).

Example 2.1 In the case $s = -1$ we find from (2.7)

$$u = \frac{1}{2}\left(x + \sqrt{x^2 - 4A}\right) ,$$

where also the negative sign of the root would be possible, and (2.8) with $k = 1$ yields the solutions

$$f(x) = \frac{1}{2}\left(p + \frac{1}{p}\right) x + \frac{1}{2}\left(p - \frac{1}{p}\right) \sqrt{x^2 - 4A} . \tag{2.9}$$

Example 2.2 In the case $s = 2$ we find from (2.7)

$$u = -B + \sqrt{B^2 + 2Bx}$$

with $B = 1/(2A)$ and $x \geq 0$, and therefore the solutions

$$f(x) = p^2 x + (p - p^2)\left(\sqrt{B^2 + 2Bx} - B\right) \tag{2.10}$$

with $B = 1/(2A)$ and $x \geq 0$.

Example 2.3 In the case $s = 3$ we get analogously by means of Cardano's formula the solutions

$$f(x) = p^3 x + \frac{3}{2}C(p - p^3)\left(\sqrt[3]{\sqrt{x^2 + C^3} + x} - \sqrt[3]{\sqrt{x^2 + C^3} - x}\right) \tag{2.11}$$

with $C^3 = 4/(27A)$.

Further examples are possible, e.g. in the cases $s = -2, 0$ or $\frac{3}{2}$.

3 Approximate solutions

In order to find an approximate solution of (1.2) with continuous F, we consider (1.3) for integer $t = n$ and write this equation as

$$y_{n+k}(z) = F(y_n(z), y_{n+1}(z), \ldots, y_{n+k-1}(z)), \tag{3.1}$$

where z is a certain parameter. Let the solution $y_n(z)$ be continuous and strongly increasing in z, so that $x = y_n(z)$ can be inverted by $z = y_n^{[-1]}(x)$, and put $f_n(x) = y_{n+1}\left(y_n^{[-1]}(x)\right)$. Replacing z in (3.1) by $y_n^{[-1]}(x)$ and considering that $y_{n+2}\left(y_n^{[-1]}(x)\right) = f_{n+1}(f_n(x))$, $y_{n+3}\left(y_n^{[-1]}(x)\right) = f_{n+2}(f_{n+1}(f_n(x)))$ etc., we see that the resulting equation converges to (1.3) in case that f_n converges to f.

We try this method for the example

$$f^{[2]} = x(1 + f) \tag{3.2}$$

with $x \geq 0$ and the nonlinear associated difference equation

$$y_{n+2} = y_n(1 + y_{n+1}) \tag{3.3}$$

from Stević [6], [7].

Proposition *Let y_n be the solution of (3.3) subject to the initial conditions $y_0 = y_1 = z$, then the corresponding functions $f_n = y_{n+1}\left(y_n^{[-1]}\right)$ satisfy*

$$f_{2n} < f_{2n+2} < f_{2n+3} < f_{2n+1} \tag{3.4}$$

for all $x > 0$ and all integers $n \geq 0$.

Proof: Obviously, all y_n and all f_n are continuous and strictly increasing and hence invertible. From the associated difference equation (3.3) it follows

$$f_{n+1}(f_n) = x(1 + f_n)\,, \tag{3.5}$$

and this equation implies

$$f_{n+1} = (1 + x) f_n^{[-1]}\,. \tag{3.6}$$

From $f_0 = x$, $f_1 = x + x^2$, $f_2(f_1) = x + x^2 + x^3$ and $f_3(f_2) = x + x^2 + x^3 + x^4$ it easily follows $f_0 < f_1$, $f_0(f_1) < f_2(f_1)$ and $f_3(f_2) < f_1(f_2)$, so that (3.4) is satisfied for $n = 0$, disregarding the inequality in the middle. From (3.6) it follows

$$\begin{aligned} f_{n+2} - f_{n+1} &= (1 + x)\left(f_{n+1}^{[-1]} - f_n^{[-1]}\right)\,, \\ f_{n+2} - f_n &= (1 + x)\left(f_{n+1}^{[-1]} - f_{n-1}^{[-1]}\right)\,, \end{aligned}$$

and since $f < g$ implies $f^{[-1]} > g^{[-1]}$, the inequalities (3.4) follow for all n by induction. □

Corollary *The functions f_{2n} converge to a function g, and the functions f_{2n+1} to a function h with $g \leq h$ and both*

$$g(h) = x(1 + h)\,, \qquad h(g) = x(1 + g)\,.$$

Open Problem 1 Prove that $g = h$.

By means of DERIVE we find that the solutions of (3.3) with $y_0 = y_1 = z$ are polynomials with the first terms

$$\begin{aligned} y_{2n} &= z + nz^2 + n(n-1)z^3 + n(n-1)^2 z^4 + \frac{n}{6}(n-1)(6n^2 - 13n + 5)z^5\,, \\ y_{2n+1} &= z + nz^2 + n^2 z^3 + \frac{n}{2}(n-1)(2n+1)z^4 + \frac{n}{6}(n-1)(6n^2 - n - 4)z^5\,, \end{aligned}$$

and that the corresponding functions f_n are power series with the first terms

$$\left.\begin{aligned} f_{2n} &= x + nx^3 - \frac{3}{2}n(n+1)x^4 + \frac{n}{2}(4n^2 + 11n + 3)x^5\,, \\ f_{2n+1} &= x + x^2 - nx^3 + \frac{n}{2}(3n+1)x^4 - n^2(2n+1)x^5\,. \end{aligned}\right\} \tag{3.7}$$

Obviously, it is not possible in these formulas to go to the limit $n \to \infty$ term by term. Nevertheless, since

$$nx^3 - \frac{3}{2}n^2x^4 + 2n^3x^5 + \frac{5}{2}n^4x^6 + \cdots = \frac{x^2}{2}\left(1 - \frac{1}{(1+nx)^2}\right)$$

for small nx, we expect the approximations

$$f_{2n} = x + \frac{x^2}{2} - \frac{x^2}{2(1+nx)^2}\,, \qquad f_{2n+1} = x + \frac{x^2}{2} + \frac{x^2}{2(1+nx)^2}\,.$$

Open Problem 2 Prove that the functions f_n with the first terms (3.7) converge to a solution f of (3.2) with the expansion

$$f = x + \frac{1}{2}x^2 - \frac{1}{16}x^4 + \frac{1}{16}x^5 - \frac{1}{64}x^6 + \ldots \tag{3.8}$$

for small x, and the asymptotic expansion

$$f = x^\lambda + \lambda - \frac{\lambda^2}{x} - \frac{\lambda^3}{2x^\lambda} + \frac{1}{x^{\lambda+1}} + \ldots \tag{3.9}$$

as $x \to \infty$, where $\lambda = \frac{1}{2}\left(\sqrt{5}+1\right)$ is the positive root of $\lambda^2 = \lambda + 1$.

The terms in (3.8) and (3.9) can be found by inserting suitable power series with indeterminate coefficients into (3.2) and comparing coefficients.

4 A new method

Let the parameter z in (3.1) be the initial value $x = y_0$. We deal with the question, whether it is possible that a solution $y_n(x)$ of (3.1) yields for $n = 1$ a solution $f(x) = y_1(x)$ of (1.2) in the case $k = 2$. Obviously, the answer is yes, if

$$y_1(y_1(x)) = y_2(x) \tag{4.1}$$

for all $x \in \mathbb{R}$. We discuss this answer in two cases.

4.1 Linear associated equations

As first example we consider once more equation (2.1) with the linear associated equation

$$y_{n+2} = ay_{n+1} + by_n \tag{4.2}$$

for $t = n$. According to $y_0 = x$ we can write the solution (2.3) of (4.2) as

$$y_n(x) = \begin{cases} cp^n + (x-c)q^n & \text{for } p \neq q\,, \\ (cn+x)p^n & \text{for } p = q\,. \end{cases}$$

In the case $p \neq q$ it follows

$$y_1(y_1(x)) = xq^2 + c(p-q)(1+q)\,,$$

and this is equal to $y_2(x)$ for $c(p-1) = 0$, i.e. for $c = 0$ resp. $p = 1$, cf. [5, Theorem 8], where the solution with $p = 1$ is excluded.

In the case $p = q$ it follows

$$y_1(y_1(x)) = xp^2 + cp(1+p)\,,$$

and this is equal to $y_2(x)$ also for $c(p-1) = 0$, i.e. for $c = 0$ resp. $p = 1$ cf. [5, Theorem 7]. Hence, we have found very few examples for the equation (4.1), i.e. for $f(x) = y_1(x)$, but we can consider a further one.

4.2 Nonlinear associated equation

Let us return to equation (3.2) with the associated equation (3.3). If we look for a solution of (3.3) being a power series in $x = y_0$, we find by means of DERIVE for the first terms

$$\begin{aligned} y_n(x) = x &+ \frac{n}{2}x^2 + \frac{n}{4}(n-1)x^3 + \frac{n}{16}(2n^2-5n+2)x^4 + \frac{n}{96}(6n^3-26n^2+27n-1)x^5 \\ &+ \frac{n}{384}(12n^4-77n^3+142n^2-49n-34)x^6 + \dots \end{aligned} \tag{4.3}$$

which for $n = 1$ turn into (3.8).

Open Problem 3 Show that $f(x) = y_1(x)$ is indeed a solution of (3.2).

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