Rostock. Math. Kolloq. **63**, 25–35 (2008)

Subject Classification (AMS)
47H09, 47H10

Daruni Boonchari, Satit Saejung

Weak and strong convergence of a scheme with errors for three nonexpansive mappings

ABSTRACT. We establish weak and strong convergence theorems of modified Ishikawa iteration with errors with respect to three nonexpansive mappings. We improve and extend many results due to Khan and Fukhar-ud-din, Tamura and Takahashi and many authors. We also point out that an additional condition imposed in Rafiq's paper does not make sense.

KEY WORDS. nonexpansive mapping, Ishikawa iteration, uniformly convex space, Opial's condition, condition (A'')

1 Introduction

Nonexpansive mappings have been widely and extensively studied by many authors in many aspects. One is to approximate a common fixed point of nonexpansive mappings by means of an iteratively constructed sequence.

Let C be a nonempty convex subset of a normed space E and $R,\ S,\ T : C \to C$ be three mappings. Xu [13] introduced the following iterative scheme,

(a) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in [0,1] such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if $c_n \equiv 0$, i.e.,

$$\begin{cases} x_1 \in C \\ x_{n+1} = a_n x_n + (1 - a_n) T x_n, \quad n \geq 1, \end{cases} \tag{2}$$

where $\{a_n\}$ is a sequence in [0,1].

(b) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{3}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in [0,1] satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is called the Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme if $c_n \equiv 0 \equiv c'_n$, i.e.,

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + (1 - a'_n) T x_n \\ x_{n+1} = a_n x_n + (1 - a_n) T x_n, \quad n \geq 1, \end{cases} \tag{4}$$

where $\{a_n\}, \{a'_n\}$ are sequences in [0,1].

A generalization of Mann and Ishikawa iterative schemes was given by Das and Debata [3] and Takahashi and Tamura [11]. This scheme dealt with two mappings:

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + (1 - a'_n) T x_n \\ x_{n+1} = a_n x_n + (1 - a_n) S y_n, \quad n \geq 1, \end{cases} \tag{5}$$

(c) The sequence $\{x_n\}$, defined by

$$\begin{cases} x_1 \in C \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{6}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in [0,1] satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is studied by S. H. Khan and H. Fukhar-ud-din [4].

Inspired by [4] and [5], we generalize the scheme (6) to three nonexpansive mappings with errors as follows:

(d) The sequence $\{x_n\}$, defined by

$$\begin{cases} x_0 \in C \\ y_n = a'_n R x_n + b'_n T x_n + c'_n v_n \\ x_{n+1} = a_n R x_n + b_n S y_n + c_n u_n, \qquad n \geq 1, \end{cases} \tag{7}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in [0,1], $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

2 Preliminaries

Let E be a Banach space and let C be a nonempty closed convex subset of E. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A Banach space E is said to be satisfy Opial's condition [7] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta_E(\varepsilon)$ of convexity of E by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for every $\varepsilon > 0$.

A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ it follows that $x \in C$ and $Tx = y$.

Next we state the following useful lemmas.

Lemma 1 ([9]) *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2 ([12], Lemma 1) *Let $\{s_n\}, \{t_n\}$ be two nonnegative real sequences satisfying*

$$s_{n+1} \leq s_n + t_n \quad \textit{for all} \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

Lemma 3 ([1]) *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

3 Main results

In this section, we shall prove the weak and strong convergence theorems of the iteration scheme to a common fixed point of the nonexpansive mappings R, S and T. Let $F(T)$ denote the set of all fixed points of T.

Lemma 4 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $R,\ S,\ T : C \to C$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (7) with $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(R) \cap F(S) \cap F(T) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(R) \cap F(S) \cap F(T)$.*

Proof: Assume that $F(R)\cap F(S)\cap F(T) \neq \emptyset$. Let $p \in F(R)\cap F(S)\cap F(T)$. Since $S,\ T,\ R$ are nonexpansive mappings, we have

$$\begin{aligned}
\|y_n - p\| &= \|a'_n Rx_n + b'_n Tx_n + c'_n v_n - p\| \\
&\leq a'_n \|Rx_n - p\| + b'_n \|Tx_n - p\| + c'_n \|v_n - p\| \\
&\leq a'_n \|x_n - p\| + b'_n \|x_n - p\| + c'_n \|v_n - p\| \\
&= (a'_n + b'_n)\|x_n - p\| + c'_n \|v_n - p\| \\
&= (1 - c'_n)\|x_n - p\| + c'_n \|v_n - p\| \\
&\leq \|x_n - p\| + c'_n \|v_n - p\| \qquad (8)
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n Rx_n + b_n Sy_n + c_n u_n - p\| \\
&\leq a_n \|Rx_n - p\| + b_n \|Sy_n - p\| + c_n \|u_n - p\| \\
&\leq a_n \|y_n - p\| + b_n \|x_n - p\| + c_n \|u_n - p\| \\
&\leq a_n (\|x_n - p\| + c'_n \|v_n - p\|) + b_n \|x_n - p\| + c_n \|u_n - p\| \\
&= a_n \|x_n - p\| + a_n c'_n \|v_n - p\| + b_n \|x_n - p\| + c_n \|u_n - p\| \\
&= (a_n + b_n)\|x_n - p\| + a_n c'_n \|v_n - p\| + c_n \|u_n - p\| \\
&\leq \|x_n - p\| + c'_n \|v_n - p\| + c_n \|u_n - p\| \qquad (9)
\end{aligned}$$

By Lemma 2, $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 5 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S, T, R : C \to C$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (7) with $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$ and $0 < \delta \leq b_n, b'_n \leq 1 - \delta < 1$. If $F(R) \cap F(S) \cap F(T) \neq \emptyset$ and*

$$\|x - Sy\| \leq \|Rx - Sy\| \quad \text{for all } x, y \in C, \qquad (10)$$

then

$$\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Rx_n - x_n\| = 0$$

for all $p \in F(R) \cap F(S) \cap F(T)$.

Proof: From Lemma 4, we get $\lim_{n\to\infty}\|x_n-p\|$ exists. Let $\lim_{n\to\infty}\|x_n-p\|=c$. Then if $c=0$, we are done. Assume that $c>0$. Next, we want to show that $\lim_{n\to\infty}\|Sy_n-Rx_n\|=0$. We note that $\{u_n-Rx_n-p\}$ is a bounded sequence, so $\lim_{n\to\infty}c_n\|u_n-Rx_n-p\|=0$. Consider

$$\begin{aligned} c &= \lim_{n\to\infty}\|x_{n+1}-p\| \\ &= \lim_{n\to\infty}\|(1-b_n)Rx_n+b_nSy_n+c_nu_n-c_nRx_n-p\| \\ &= \lim_{n\to\infty}\|(1-b_n)(Rx_n-p)+b_n(Sy_n-p)+c_n(u_n-Rx_n-p)\| \\ &= \lim_{n\to\infty}\|(1-b_n)(Rx_n-p)+b_n(Sy_n-p)\| \end{aligned} \tag{11}$$

and from (8) we have

$$\limsup_{n\to\infty}\|Sy_n-p\|\le\limsup_{n\to\infty}\|y_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|+c'_n\|v_n-p\|=c \tag{12}$$

also,

$$\limsup_{n\to\infty}\|Rx_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|=c.$$

Using Lemma 1 and (11), we have

$$\lim_{n\to\infty}\|Sy_n-Rx_n\|=0. \tag{13}$$

It follows then that

$$\begin{aligned} \|Rx_n-x_n\| &\le \|Rx_n-Sy_n\|+\|Sy_n-x_n\| \\ &\le 2\|Rx_n-Sy_n\|\to 0, \end{aligned} \tag{14}$$

and hence

$$\|Sy_n-x_n\|\le\|Sy_n-Rx_n\|+\|Rx_n-x_n\|\to 0. \tag{15}$$

We are going to apply Lemma 1 again. To show that $\lim_{n\to\infty}\|y_n-p\|=c$, we observe that $\|x_n-p\|\le\|x_n-Sy_n\|+\|Sy_n-p\|\le\|x_n-Sy_n\|+\|y_n-p\|$ which implies that

$$c\le\liminf_{n\to\infty}\|y_n-p\|.$$

This together with (12) gives

$$\lim_{n\to\infty}\|y_n-p\|=c. \tag{16}$$

Finally, from (16) and the boundedness of the sequence $\{v_n-Rx_n-p\}$, we have

$$\begin{aligned} c &= \lim_{n\to\infty}\|y_n-p\| \\ &= \lim_{n\to\infty}\|(1-b'_n)Rx_n+b'_nTx_n+c'_nv_n-c'_nRx_n-p\| \\ &= \lim_{n\to\infty}\|(1-b'_n)(Rx_n-p)+b'_n(Tx_n-p)+c'_n(v_n-Rx_n-p)\| \\ &= \lim_{n\to\infty}\|(1-b'_n)(Rx_n-p)+b'_n(Tx_n-p)\|. \end{aligned}$$

Moreover,

$$\limsup_{n\to\infty} \|Tx_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = c,$$

and

$$\limsup_{n\to\infty} \|Rx_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = c.$$

Applying Lemma 1, we get

$$\lim_{n\to\infty} \|Rx_n - Tx_n\| = 0. \tag{17}$$

Using (14) and (17), we get that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{18}$$

Consequently, using (13), (14), (18) and

$$\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\
&\le \|x_n - Sy_n\| + \|y_n - x_n\| \\
&\le \|x_n - Sy_n\| + a'_n\|Rx_n - x_n\| + b'_n\|Tx_n - x_n\| + c'_n\|v_n - x_n\| \\
&\le \|x_n - Sy_n\| + a'_n\|Rx_n - x_n\| + b'_n\|Tx_n - x_n\| + c'_n\|v_n - p\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{19}$$

This completes the proof. □

We first establish the weak convergence theorem of our iteration.

Theorem 6 *Let E be a uniformly convex Banach space satisfies the Opial's condition and C, S, T, R and $\{x_n\}$ be taken as in Lemma 5. If $F(R)\cap F(S)\cap F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of S, T and R.*

Proof: Let $p \in F(R)\cap F(S)\cap F(T)$, then as proved in Lemma 4, we get $\lim_{n\to\infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(R)\cap F(S)\cap F(T)$. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{m_j}\}$ of $\{x_n\}$, respectively. By Lemma 11, $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 3, therefore we obtain $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$ and $Rz_1 = z_1$. Again in the same way, we can prove that $z_2 \in F(R)\cap F(S)\cap F(T)$. Next, we prove the uniqueness.

For this we suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{i\to\infty} \|x_{n_i} - z_1\| \\ &< \lim_{i\to\infty} \|x_{n_i} - z_2\| \\ &= \lim_{n\to\infty} \|x_n - z_2\| \\ &= \lim_{j\to\infty} \|x_{m_j} - z_2\| \\ &< \lim_{j\to\infty} \|x_{m_j} - z_1\| \\ &= \lim_{n\to\infty} \|x_n - z_1\|.\end{aligned}$$

This is contradiction. Hence $\{x_n\}$ converges weakly to a point in $F(R) \cap F(S) \cap F(T)$. □

Our next goal is to prove a strong convergence theorem. Recall that a mapping $T : C \to C$ where C is a subset of E, is said to satisfy condition (A) ([10]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

Senter and Dotson [10] approximated fixed points of nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [6] and Tan and Xu [12] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same condition (A) which is weaker than the requirement that d is demicompact.

Three mappings $R, S, T : C \to C$ where C is a subset of E, are said to satisfy condition (A'') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{3}(\|x - Rx\| + \|x - Tx\| + \|x - Sx\|) \geq f(d(x, F))$$

for all $x \in C$ where $d(x, F) = \inf\{\|x - x^*\| : x^* \in F = F(R) \cap F(S) \cap F(T)\}$.

Note that condition (A'') reduces to condition (A) when $R = S = T$. We shall use condition (A'') instead of the compactness of C to study the strong convergence of $\{x_n\}$ defined in (7). It is noted that if $R = I$, then condition (A'') reduces to condition (A') of Khan and Fukhar-ud-din [4].

Theorem 7 *Let E be a uniformly convex Banach space and C, $\{x_n\}$ be taken as in Lemma 5. Let R, S, $T : C \to C$ be three mappings satisfying condition (A''). If $F(R) \cap F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of R, S and T.*

Proof: By Lemma 4, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(R) \cap F(S) \cap F(T)$. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \geq 0$. If $c = 0$, we are done. Suppose that $c > 0$. By

Lemma 5, $\lim_{n\to\infty}\|Sx_n - x_n\| = \lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|Rx_n - x_n\| = 0$. Let $M = \sup\{\|v_n - x_n\|, \|u_n - x_n\| : n \in \mathbb{N}\}$. Moreover, by (9),

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq \|x_n - p\| + c_n'\|v_n - p\| + c_n\|u_n - p\| \\
&\leq \|x_n - p\| + c_n'\|v_n - x_n\| + c_n'\|x_n - p\| + c_n\|u_n - x_n\| + c_n\|x_n - p\| \\
&\leq (1 + c_n' + c_n)\|x_n - p\| + c_n'\|v_n - x_n\| + c_n\|u_n - x_n\| \\
&\leq (1 + c_n' + c_n)\|x_n - p\| + (c_n' + c_n)M.
\end{aligned} \tag{20}$$

This implies that $d(x_{n+1}, F) \leq (1 + c_n' + c_n)d(x_n, F) + (c_n' + c_n)M$ and hence $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of Lemma 2. By condition (A''),

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in E.

Let $\epsilon > 0$. We choose a positive integer N_1 such that

$$d(x_{N_1}, F) < \frac{\varepsilon}{4}. \tag{21}$$

We next choose $q \in F$ such that

$$\|x_{N_1} - q\| < \frac{\varepsilon}{4}. \tag{22}$$

By $\lim_{n\to\infty}\|x_n - q\|$ exists, the sequence $\{\|x_n - p\|\}$ is bounded. Let $K = \sup_{n\in\mathbb{N}}\{\|x_n - q\|, M\}$. Then from (20), we have

$$\|x_{n+1} - q\| \leq \|x_n - q\| + (c_n' + c_n)K. \tag{23}$$

Since $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c_n' < \infty$, there exists N_2 such that

$$\sum_{i=N_2}^{\infty} Q_i < \frac{\varepsilon}{4}, \tag{24}$$

where $Q_i = (c_i + c_i')K$. We take $N = \max\{N_1, N_2\}$. Let $n \geq N$ and $m \geq 1$. It follows from (22), (23) and (24) that

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|p - x_n\| \\
&\leq \|x_n - p\| + \|p - x_n\| + \sum_{i=n}^{n+m-1} Q_i \\
&= 2\|x_n - p\| + \sum_{i=n}^{n+m-1} Q_i \\
&\leq 2\|x_N - p\| + 2\sum_{i=N}^{n-1} Q_i + \sum_{i=n}^{n+m-1} Q_i \\
&\leq 2\|x_N - p\| + 2\sum_{i=N}^{n+m-1} Q_i \\
&\leq 2\|x_N - p\| + 2\sum_{i=N}^{\infty} Q_i \\
&\leq 2(\frac{\varepsilon}{4}) + 2(\frac{\varepsilon}{4}) = \varepsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in E. Since C is closed, $x_n \to x \in C$. By the continuities of S, R, T and (14), (18), (19), we get $Sx = Rx = Tx = x$. So $x \in F(R) \cap F(S) \cap F(T)$. This completes the proof. □

If R is the identity mapping, then (10) is automatically satisfied and we have the following.

Corollary 8 ([4],Theorem 1, Theorem 2) *Let E be a uniformly convex Banach space and C, S, T and $\{x_n\}$ be taken as in Theorem 7. Suppose that $F(S) \cap F(T) \neq \emptyset$. Then*

1. *If E has the Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of S and T,*
2. *If the mappings S and T satisfy condition (A'), then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Remark 1 Theorem 6 and Theorem 7 extend and improve Theorem 1 and Theorem 2 of [4] in the following ways:

1. the iteration methods in [4] are included as a special case of ours. Indeed, the identity mapping is replaced by the more general nonexpansive mapping,
2. the boundedness of C is not assumed as was the case in [4].

Remark 2 The following example [5, see Example 3.1] shows that our results extend substantially results in [4].

Example 9 Let E be the real line with the usual norm and let $C = [-1,1]$. Define $R, S, T : C \to C$ by

$$Rx = \begin{cases} x, & x \in [0,1] \\ -x, & x \in [-1,0) \end{cases}$$

$$Sx = \begin{cases} -\sin x, & x \in [0,1] \\ \sin x, & x \in [-1,0) \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{2}x, & x \in [0,1] \\ -\frac{1}{2}x, & x \in [-1,0). \end{cases}$$

Obviously, $F(R) \cap F(S) \cap F(T) = \{0\}$. Moreover, it is not hard to see that nonexpansive mappings R, S and T satisfy condition (A'').

Remark 3 Recently, Rafiq [8] introduced the following condition: two mappings $S, T : C \to C$ are said to satisfy (AU-N) if

$$\|Sx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

It the clear that if $S = T$, then (AU-N) is the definition of nonexpansive mappings. Unfortunately, if S and T satisfy (AU-N), then

$$\|Sx - Tx\| \leq \|x - x\| = 0 \quad \text{for all } x \in C,$$

from which $S = T$. This means (AU-N) is meaningless. Consequently, all results in [8] are just dealing with only one mapping.

Acknowledgements

The second author supported by the Thailand Research Fund (Grant MRG4980022). The authors would like to thank W. Nilsrakoo for drawing our attention to [8].

References

[1] **Browder, F. E. :** *Nonlinear operators and nonlinear equations of evolution in Banach spaces.* Nonlinear functional analysis (Proc. Sympos. Pure Math., Vol. 18, Part 2, Chicago, Ill., 1968). 1–308. Amer. Math. Soc., Providence, R. I., (1976)

[2] **Chidume, C. E.**, and **Moore, C. :** *Fixed point iteration for pseudocontractive maps.* Proc. Amer. Math. Soc. **127(4)**, 1163–1170, (1999)

[3] **Das, G.**, and **Debata, J. P. :** *Fixed points of quasinonexpansive mappings.* Indian J. Pure Appl. Math. **17(11)**, 1263–1269, (1986)

[4] **Khan, S. H.**, and **Fukhar-ud-din, H. :** *Weak and strong convergence of a scheme with errors for two nonexpansive mappings.* Nonlinear Anal. **61(8)**, 1295–1301, (2005)

[5] **Liu, Z., Agarwal, R. P., Feng, C.**, and **Kang, S. M. :** *Weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings.* Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **44** , 83–96, (2005)

[6] **Maiti, M.**, and **Ghosh, M. K. :** *Approximating fixed points by Ishikawa iterates.* Bull. Austral. Math. Soc. **40(1)**, 113–117, (1989)

[7] **Opial, Z. :** *Weak convergence of the sequence of successive approximations for nonexpansive mappings.* Bull. Amer. Math. Soc. **73**, 591–597, (1967)

[8] **Rafiq, A. :** *Convergence of an iterative scheme due to Agarwal et al.* Rostock. Math. Kolloq. **61**, 95–105, (2006)

[9] **Schu, J. :** *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings.* Bull. Austral. Math. Soc. **43(1)**, 153–159, (1991)

[10] **Senter, H. F.**, and **Dotson, W. G. :** *Approximating fixed points of nonexpansive mappings.* Proc. Amer. Math. Soc. **44**, 375–380, (1974)

[11] **Takahashi, W.**, and **Tamura, T. :** *Convergence theorems for a pair of nonexpansive mappings.* J. Convex Anal. **5(1)**, 45–56, (1998)

[12] **Tan, K. K.**, and **Xu, H. K. :** *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process.* J. Math. Anal. Appl. **178(2)**, 301–308, (1993)

[13] **Xu, Y. :** *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations.* J. Math. Anal. Appl. **224(1)**, 91–101, (1998)

received: August 29, 2007

Authors:

Daruni Boonchari
Department of Mathematics,
Faculty of Science,
Mahasarakham University,
Maha Sarakham 44150,
Thailand

e-mail: boonchari@hotmail.com

Satit Saejung
Department of Mathematics,
Faculty of Science,
Khon Kaen University,
Khon Kaen 40002,
Thailand

e-mail: saejung@kku.ac.th