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Kriengsak Wattanawitoon, Usa Wannasingha Humphries, Poom Kumam[1]

Strong convergence by new hybrid methods of modified Ishikawa iterations for two asymptotically nonexpansive mappings and semigroups

ABSTRACT. In this paper, we introduce the iterative sequence for two asymptotically nonexpansive mappings and two asymptotically nonexpansive semigroups. Then we prove strong convergence theorems for a common fixed point of two asymptotically nonexpansive mappings and for a common fixed point of two asymptotically nonexpansive semigroups by using the new hybrid methods in a Hilbert space. Moreover, we discuss the problem of strong convergence and we also apply our results to generalizes extend and improve these announced by Plubtieng and Ungchittrakool's result [Strong convergence of modified Ishikawa iterations for two asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 67 (2007) 2306–2315.] and Takahashi et al. [Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276–286.].

KEY WORDS. strong convergence; nonexpansive mapping; Ishikawa iterations; asymptotically nonexpansive mappings; asymptotically nonexpansive semigroup

1 Introduction

Let E be a real Banach space, C be a nonempty closed convex subset of E, and $T : C \to C$ be a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for all} \quad x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : x = Tx\}$. A mapping T is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for all} \quad n \geq 1 \quad \text{and} \quad x, y \in C.$$

[1]Corresponding author

If S and T are two (asymptotically) nonexpansive mappings, then the point $x \in F(S) \cap F(T)$ is called the common fixed point of S and T.

Recall also that a one-parameter family $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitzian semigroup on C (see, e.g., [10]) if the following conditions are satisfied:
(a) $T(0)x = x, \quad x \in C$,
(b) $T(t + s)x = T(t)T(s)x, \quad$ for all $t, s \geq 0, \quad x \in C$,
(c) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,
(d) there exists a bounded measurable function $L : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$,
$\|T(t)x - T(t)y\| \leq L_t \|x - y\|, \quad$ for all $x, y \in C$.

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive if $\limsup_{t\to\infty} L_t \leq 1$. We denote by $F(\mathcal{T})$ the set of fixed points of the semigroup $\mathcal{T}$, that is $F(\mathcal{T}) = \{x \in C : T(s)x = x, \quad \forall s > 0\}$.

In 1953, Mann [5] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

The second iteration process is referred to as Ishikawa's iteration process [2], which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases} \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0, 1]$.

In 2003, Nakajo and Takahashi [6] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T. Moreover they introduced and studied an iteration process of a nonexpansive semigroup $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$

in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du, \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.4}$$

In 2006, Kim and Xu [3] adapted the iteration (1.3) to a asymptotically nonexpansive mapping in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.5}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$. They also proved that if $\alpha_n \le a$ for all n and for some $0 < a < 1$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T. Moreover, they modified an iterative method (1.4) to the case of an asymptotically nonexpansive semigroup $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.6}$$

where $\theta_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L_u du)^2 - 1](diamC)^2 \to 0$ as $n \to \infty$.

Recently, Takahashi et al.[9] introduced the modification Mann iteration method for a nonexpansive mapping and nonexpansive semigroup $J = \{T(t) : 0 \le t < \infty\}$. They proved strong convergence theorems in Hilbert spaces by a new hybrid method.

Very recently, Plubtieng and Ungchittrakool [8] modified Ishikawa iteration processes for two asymptotically nonexpansive mappings, and two asymptotically nonexpansive semigroups $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ and $\mathcal{S} = \{S(t) : 0 \le t < \infty\}$ with C be a closed convex bounded subset of Hilbert space H. They obtained strong convergence theorems.

In this paper, motivated by Plubtieng and Ungchittrakool's result [8] and Takahashi et al. [9], we prove strong convergence theorems for a common fixed point of two asymptotically nonexpansive mappings and two asymptotically nonexpansive semigroups in Hilbert spaces by using the new hybrid methods, which is introduced by Takahashi et al. [9]. Our results are extend and improve of some previous literature results.

2 Preliminaries

This section collects some lemma which will be used in the proofs for the main results in the next section.

Lemma 2.1 *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.2 (*Opial* [7]). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

Lemma 2.3 (*Lin et al.* [4]). *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset of a bouded closed convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $Tx_n - x_n \to 0$, then $z \in F(T)$.*

Lemma 2.4 (*Nakajo and Takahashi* [6]). *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set $D := \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

Lemma 2.5 (*Kim and Xu* [3]). *Let C be a nonempty bounded closed convex subset of H and $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties:*
(a) $x_n \rightharpoonup z$; *and,*
(b) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$,
then $z \in F(\mathcal{T})$.

Lemma 2.6 (*Kim and Xu* [3]). *Let C be a nonempty bounded closed convex subset of H and $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Then it holds that*

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(u)x du - T(s)\left(\frac{1}{t}\int_0^t T(u)x du\right) \right\| = 0.$$

3 Convergence to a common fixed point of two asymptotically nonexpansive mappings

In this section, we prove a strong convergence theorem by the new hybrid method of modified Ishikawa iterations for two asymptotically nonexpansive mappings in Hilbert spaces.

Theorem 3.1 *Let C be a bounded closed convex subset of a Hilbert space H and let $S, T : C \to C$ be two asymptotically nonexpansive mappings with sequence $\{s_n\}$ and $\{t_n\}$ respectively such that $F := F(S) \cap F(T) \neq \emptyset$. Assume that $\alpha_n \leq a$ for all n and for some $0 < a < 1$ and $\beta_n \in [0, 1]$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T^n z_n, \\ z_n = \beta_n x_n + (1 - \beta_n) S^n x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x), \qquad n = 0, 1, 2 \dots, \end{cases} \tag{3.1}$$

where $\theta_n = (1 - \alpha_n)[(t_n^2 - 1) + (1 - \beta_n)t_n^2(s_n^2 - 1)](\operatorname{diam} C)^2 \to 0$ as $n \to \infty$. Then $\{x_n\}$ converges in norm to $P_F(x_0)$.

Proof: We first show that C_{n+1} is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. From the definition of C_{n+1} it is obvious that C_{n+1} is closed for each $n \geq 0$. By Lemma 2.4, we observe that C_{n+1} is convex.

Next, we show that $F \subset C_n$ for all $n \geq 0$. Indeed, let $p \in F$, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n x_n + (1 - \alpha_n) T^n z_n - p\|^2 \\ &= \|\alpha_n (x_n - p) + (1 - \alpha_n)(T^n z_n - p)\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|T^n z_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) t_n^2 \|z_n - p\|^2 \\ &= \|x_n - p\|^2 + (1 - \alpha_n)(t_n^2 \|z_n - p\|^2 - \|x_n - p\|^2). \end{aligned} \tag{3.2}$$

By Lemma 2.1, we get

$$\begin{aligned} \|z_n - p\|^2 &= \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|S^n x_n - p\|^2 - \beta_n (1 - \beta_n)\|S^n x_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 + (1 - \beta_n)(s_n^2 - 1)\|x_n - p\|^2 - \beta_n (1 - \beta_n)\|S^n x_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 + (1 - \beta_n)(s_n^2 - 1)\|x_n - p\|^2. \end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.2), we obtain

$$\begin{aligned} \|y_n - p\|^2 &\leq \|x_n - p\|^2 + (1 - \alpha_n)(t_n^2(\|x_n - p\|^2 + (1 - \beta_n)(s_n^2 - 1)\|x_n - p\|^2) - \|x_n - p\|^2) \\ &= \|x_n - p\|^2 + (1 - \alpha_n)(t_n^2(1 + (1 - \beta_n)(s_n^2 - 1)) - 1)\|x_n - p\|^2 \\ &= \|x_n - p\|^2 + ((1 - \alpha_n)(t_n^2 - 1) + (1 - \alpha_n)(1 - \beta_n)t_n^2(s_n^2 - 1))\|x_n - p\|^2 \\ &= \|x_n - p\|^2 + \theta_n \quad \text{with} \quad \theta_n \to 0. \end{aligned}$$

It follows that $p \in C_{n+1}$ and $F \subset C_{n+1}$ for all $n \geq 0$. Thus $\{x_n\}$ is well defined.
From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{Cn+1} x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0 \qquad \text{for all } p \in F \quad \text{and } n \in \mathbb{N} \tag{3.4}$$

So, for $x_{n+1} \in C_n$, we have,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle, \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle, \\ &= -\langle x_n - x_0, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle, \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \end{aligned}$$

for $n \in \mathbb{N}$. This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\| \|x_0 - x_{n+1}\|,$$

hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \qquad \text{for all } n \in \mathbb{N}.$$

Therefore $\{\|x_0 - x_n\|\}$ is nondecreasing. From $x_n = P_{C_n} x_0$, we also have $\langle x_0 - x_n, x_n - y \rangle \geq 0$, for all $y \in C_n$.
Since $F \subseteq C_n$, we get

$$\langle x_0 - x_n, x_n - p \rangle \geq 0 \quad \text{for all} \quad p \in F(T).$$

Thus, for $p \in F$, we obtain

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - p \rangle \\ &= \langle x_0 - x_n, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - p \rangle \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_n\| \|x_0 - p\|. \end{aligned}$$

for all $p \in F$ and $n \in \mathbb{N}$.
Hence $\{x_n\}$ is bounded. Thus $\lim_{n\to\infty} \|x_n - x_0\|$ exists.
Next, we show that $\|x_{n+1} - x_n\| \to 0$. From (3.4) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n \rangle - 2\langle x_0 - x_n, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, then $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.
Since $x_{n+1} \in C_{n+1} \subset C_n$, we have $\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n$, which implies that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\theta_n}$. We now claim that $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$. Indeed, by definition of y_n, we have

$$\begin{aligned} \|T^n z_n - x_n\| &= \frac{1}{1 - \alpha_n} \|y_n - x_n\| \\ &\leq \frac{1}{1 - \alpha_n} (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\leq \frac{1}{1 - \alpha_n} (\|x_n - x_{n+1}\| + \sqrt{\theta_n} + \|x_{n+1} - x_n\|) \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

By using the same argument as in the proof of [8, Theorem 3.1, pp. 2310], we have $\|S^n x_n - x_n\| \to 0$ as $n \to \infty$.
Putting $t_\infty = \sup\{t_n : n \geq 1\} < \infty$ and $s_\infty = \sup\{s_n : n \geq 1\} < \infty$, we deduce that

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\leq t_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1 + t_\infty)\|x_n - x_{n+1}\|.\end{aligned}$$

Since T is uniformly continuous, we have

$$\|Tx_n - x_n\| \to 0, \qquad \text{as} \quad n \to \infty. \tag{3.5}$$

Similarly, we have

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|Sx_n - S^{n+1}x_n\| + \|S^{n+1}x_n - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\leq t_\infty\|x_n - S^n x_n\| + \|S^{n+1}x_{n+1} - x_{n+1}\| + (1 + s_\infty)\|x_n - x_{n+1}\|.\end{aligned}$$

Since T is uniformly continuous, we have

$$\|Sx_n - x_n\| \to 0, \qquad \text{as} \quad n \to \infty. \tag{3.6}$$

Finally, we show that $x_n \to z_0$ where $z_0 = P_F x_0$. By (3.5), (3.6), Lemma 2.3 and boundedness of $\{x_n\}$ we obtain $\emptyset \neq \omega_w(x_n) \in P_F x_0$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_F(x_0)$ and with the weak lower semi-continuity of the norm, we have

$$\|x_0 - z_0\| \leq \|x_0 - w\| \leq \liminf_{n\to\infty} \|x_0 - x_n\| \leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,$$

for all $\omega_w(x_n)$. However, since $\omega_w(x_n) \subset F$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_F x_0$ by

$$\begin{aligned}\|x_n - z_0\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\ &\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

This completes the proof. □

The following corollary follows from Theorem 3.1 reduces (3.1) to the modified Mann's iteration for an aysmptotically nonexpansive mapping.

Corollary 3.2 *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $\alpha_n \leq a$ for all n and for some $0 < a < 1$. Then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_{n+1} = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x), \qquad n = 0, 1, 2 \ldots, \end{cases} \tag{3.7}$$

converges in norm to $P_{F(T)}x_0$, where $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$.

Proof: By Theorem 3.1, if $S = I, \beta_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ then, (3.1) reduces to the modified Mann's iteration for an asymptotically nonexpansive mapping. □

Theorem 3.3 *Let C be a closed convex subset of a Hilbert space H and let $S, T : C \to C$ be two nonexpansive mappings such that $F = F(S) \cap F(T) \neq \emptyset$. Assume that $\alpha_n \leq 1 - \delta$ for all n and for some $\delta \in (0, 1]$ and $\beta_n \in [b, c]$ for all n and $0 < b < c < 1$, then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T z_n, \\ z_n = \beta_n x_n + (1 - \beta_n) S x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x), \qquad n = 0, 1, 2 \ldots, \end{cases} \tag{3.8}$$

converges in norm to $P_F x_0$.

Proof: Since every nonexpansive mapping is asymptotically nonexpansive mapping and using the same argument as in the proof of Theorem 3.1, we obtain $\|T x_n - x_n\| \to 0$ and $\|S x_n - x_n\| \to 0$ as $n \to \infty$. Hence, by Lemma 2.2, $w \subset F$ and therefore $\{x_n\}$ converges strongly to $P_F x_0$. □

By Theorem 3.3 reduces (3.8) to the modified Mann's iteration for a nonexpansive mapping, we obtain the following result:

Corollary 3.4 (*Takahashi et al.* [9], *Theorem 4.1*) *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\alpha_n \leq 1 - \delta$ for all n and for some $\delta \in (0, 1]$. Then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x), \qquad n = 0, 1, 2 \ldots, \end{cases} \tag{3.9}$$

converges in norm to $P_{F(T)} x_0$.

4 Convergence to a common fixed point of two asymptotically non-expansive semigroups

Theorem 4.1 *Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{T} = \{S(t) : 0 \leq t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$. Assume also that $0 < \alpha_n \leq a < 1$ and*

$0 < b \le \beta_n \le c < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\{t_n\}$ *is a positive real divergent sequence. Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u) z_n du, \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(u) x_n du, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \le \|x_n - v\|^2 + \tilde{\theta}_n\}, \\ x_{n+1} = P_{C_n}(x), \qquad n = 0, 1, 2 \ldots, \end{cases} \tag{4.1}$$

where $\tilde{\theta}_n = [(1-\alpha_n)\tilde{t}_n^2 + (1-\alpha_n)(1-\beta_n)\tilde{t}_n^2(\tilde{s}_n^2 - 1)](\text{diam } C)^2 \to 0$ as $n \to \infty$. *(hence* $\tilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_u^T du$ *and* $\tilde{s}_n = \frac{1}{s_n}\int_0^{s_n} L_u^S du$*). Then* $\{x_n\}$ *converges in norm to* $P_{F(\mathcal{T})\cap F(\mathcal{S})} x_0$.

Proof: First observe that $F(\mathcal{T}) \cap F(\mathcal{S}) \subset C_n$ for all n. Indeed, we have for all $p \in F(\mathcal{T}) \cap F(\mathcal{S})$

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u) z_n du - p\|^2 \\ &= \|\alpha_n(x_n - p) + (1-\alpha_n)(\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u) z_n du - p)\|^2 \\ &\le \|\alpha_n(x_n - p) + (1-\alpha_n)\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u) z_n du - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u) z_n du - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\tfrac{1}{t_n}\textstyle\int_0^{t_n} \|T(u) z_n - p\| du)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\tfrac{1}{t_n}\textstyle\int_0^{t_n} L_u^T du)\|z_n - p\|^2 \\ &\le \|x_n - p\|^2 + (1-\alpha_n)(\tilde{t}_n^2\|z_n - p\|^2). \end{aligned} \tag{4.2}$$

By Lemma 2.1, we have

$$\begin{aligned} \|z_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u) x_n du - p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u) x_n du\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\tfrac{1}{s_n}\textstyle\int_0^{s_n} \|S(u) x_n - p\| du)^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u) x_n du\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\tfrac{1}{s_n}\textstyle\int_0^{s_n} L_u^S du)^2\|x_n - p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u) x_n du\|^2 \\ &\le \|x_n - p\|^2 + (1-\beta_n)(\tilde{s}_n^2 - 1)\|x_n - p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u) x_n du\|^2 \\ &\le \|x_n - p\|^2 + (1-\beta_n)(\tilde{s}_n^2 - 1)\|x_n - p\|^2. \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2) yield,

$$\begin{aligned} \|y_n - p\|^2 &= \|x_n - p\|^2 + (1-\alpha_n)(\tilde{t}_n^2(\|x_n - p\|^2 + (1-\beta_n)(\tilde{s}_n^2 - 1)\|x_n - p\|^2)) \\ &= \|x_n - p\|^2 + ((1-\alpha_n)\tilde{t}_n^2 + (1-\alpha_n)(1-\beta_n)\tilde{t}_n^2(\tilde{s}_n^2 - 1))\|x_n - p\|^2 \\ &\le \|x_n - p\|^2 + \tilde{\theta}_n. \end{aligned}$$

So, $p \in C_{n+1}$. Hence $F(\mathcal{T}) \cap F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. By the same argument as in the proof of Theorem 3.1, C_n is closed and convex, $\{x_n\}$ is well defined. Also, similar to the proof of Theorem 3.1, we also have

$$\|x_n - x_{n+1}\| \to 0.$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we get

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \tilde{\theta}_n,$$

which in turn implies that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\tilde{\theta}_n}. \tag{4.4}$$

We can deduce that for all $0 \leq r < \infty$,

$$\begin{aligned} \|S(r)x_n - x_n\| &\leq \|S(r)x_n - S(r)(\tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u)x_n du)\| \\ &\quad + \|S(r)(\tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u)x_n du) - \tfrac{1}{s_n}\int_0^{s_n} S(u)x_n du\| \\ &\quad + \|\tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u)x_n du - x_n\| \\ &\leq (L_\infty + 1)\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u)x_n du - x_n\| \\ &\quad + \|S(r)(\tfrac{1}{s_n}\textstyle\int_0^{s_n} S(u)x_n du) - \tfrac{1}{s_n}\int_0^{s_n} S(u)x_n du\| \\ &:= (L_\infty + 1)A_n^S + B_n^S(r), \end{aligned} \tag{4.5}$$

where $A_n^S := \|\frac{1}{s_n}\int_0^{s_n} S(u)x_n du - x_n\|$ and $B_n^S(r) := \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(u)x_n du) - \frac{1}{s_n}\int_0^{s_n} S(u)x_n du\|$. We claim that $\lim_{n\to\infty} A_n^S = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty} B_n^S(r)$. By Lemma 2.6, we have $\limsup_{r\to\infty}\limsup_{n\to\infty} B_n^S(r) = 0$. By the proof of [8, Theorem 4.1, pp. 2312–2313, eqa. (4.4)], we obtain that $\lim_{n\to\infty} A_n^S = 0$. On the other hance, we can deduce that for all $0 \leq r < \infty$,

$$\begin{aligned} \|T(r)x_n - x_n\| &\leq \|T(r)x_n - T(r)(\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)x_n du)\| \\ &\quad + \|T(r)(\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)x_n du) - \tfrac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \\ &\quad + \|\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)x_n du - x_n\| \\ &\leq (L_\infty + 1)\|\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)x_n du - x_n\| \\ &\quad + \|T(r)(\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)x_n du) - \tfrac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \\ &:= (L_\infty + 1)A_n^T + B_n^T(r). \end{aligned} \tag{4.6}$$

where $A_n^T := \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\|$ and $B_n^T(r) := \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du) - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\|$. Moreover, we observe that

$$\begin{aligned} \|x_n - \tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)x_n du\| &\leq \|x_n - \tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)z_n du\| + \|\tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)z_n du - \tfrac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \\ &\leq \|x_n - \tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)z_n du\| + \tfrac{1}{t_n}\textstyle\int_0^{t_n} \|T(u)z_n - T(u)x_n\| du \\ &\leq \|x_n - \tfrac{1}{t_n}\textstyle\int_0^{t_n} T(u)z_n du\| + \tilde{t}_n\|z_n - x_n\|. \end{aligned}$$

Since $\|z_n - x_n\| = (1-\beta_n)\|\frac{1}{s_n}\int_0^{s_n} S(u)x_n du - x_n\| \to 0$ and $\|x_n - \frac{1}{t_n}\int_0^{t_n} T(u)z_n du\| \to 0$, we obtain

$$\lim_{n\to\infty}\|x_n - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| = 0 \tag{4.7}$$

By (4.7) and Lemma 2.6, we have $\lim_{n\to\infty} A_n^T = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty} B_n^T(r)$. We thus conclude from (4.5) and (4.6) that

$$\limsup_{r\to\infty}\ \limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\ \limsup_{n\to\infty}\|S(r)x_n - x_n\|$$

An application of Lemma 2.5 implies that every weak limit point of $\{x_n\}$ is a member of $F(\mathcal{T}) \cap F(\mathcal{S})$. Repeating the last part of the proof of Theorem 3.1, we can prove that $P_{F(\mathcal{T})\cap F(\mathcal{S})}(x)$ is the only weak limit point of $\{x_n\}$, hence $\{x_n\}$ weakly convergence to $P_{F(\mathcal{T})\cap F(\mathcal{S})}(x)$, and therefore the convergence is strong. □

Corollary 4.2 *Let C be a bounded closed convex subset of a Hilbert space H and $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Assume also that $0 < \alpha_n \le a < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{t_n\}$ is a positive real divergent sequence. Then, the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \le \|x_n - v\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n}(x), \qquad n = 0, 1, 2\ldots, \end{cases} \tag{4.8}$$

converges in norm to $P_{F(\mathcal{T})}x_0$. where $\theta_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L_u du)^2 - 1](diam C)^2 \to 0$ as $n \to \infty$.

Proof: By Theorem 4.1, if the semigroup $\mathcal{S} = \{S(t) : 0 \le t < \infty\} = \mathcal{I} := \{I(t) : 0 \le t < \infty\}$ and $\beta_n = 1$, then $S(t)x_n = x_n$ for all n and for all $t > 0$. Hence $\frac{1}{s_n}\int_0^{s_n} S(u)x_n du = x_n$ for all n and $z_n = x_n$ then, (4.1) reduces to (4.8). □

Theorem 4.3 *Let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ and $\mathcal{S} = \{S(t) : 0 \le t < \infty\}$ be two nonexpansive semigroups on C. Assume also that $0 < \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{t_n\}$ and $\{s_n\}$ are two positive real divergent sequence. If $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \ne \emptyset$, then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 = x \in C, \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)z_n du, \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(u)x_n du, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_n}(x), \qquad n = 0, 1, 2\ldots, \end{cases} \tag{4.9}$$

converges in norm to $P_{F(\mathcal{F})}x_0$.

Proof: By Theorem 4.1 we have

$$\lim_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \lim_{n\to\infty}\|S(r)x_n - x_n\| \quad \text{for all} \quad 0 \le r < \infty.$$

Hence, by Lemma 2.5, $w \subset F(\mathcal{F})$ and therefore $\{x_n\}$ converges strongly to $P_F x_0$. □

The following corollary follows from Theorem 4.3

Corollary 4.4 (*Takahashi et al.* [9], *Theorem 4.4*) *Let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ be a nonexpansive semigroup on C. Assume that $0 < \alpha_n \le a < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{t_n\}$ is a positive real divergent sequence. If $F(\mathcal{T}) \neq \emptyset$, then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du, \\ C_{n+1} = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_n+1}(x_0), , \qquad n = 0, 1, 2 \ldots, \end{cases} \tag{4.10}$$

converges in norm to $P_{F(\mathcal{T})}x_0$.

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Authors:

Kriengsak Wattanawitoon
Department of Mathematics,
Faculty of Science,
King Mongkut's University of Technology
Thonburi Bang Mod,
Bangkok 10140,
Thailand

e-mail: s9510105@st.kmutt.ac.th

Poom Kumam
Department of Mathematics,
Faculty of Science,
King Mongkut's University of Technology
Thonburi Bang Mod,
Bangkok 10140,
Thailand

e-mail: poom.kum@kmutt.ac.th

Usa Wannasingha Humphries
Department of Mathematics,
Faculty of Science,
King Mongkut's University of Technology
Thonburi Bang Mod,
Bangkok 10140,
Thailand

e-mail: usa.wan@kmutt.ac.th