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Takagi's continuous nowhere differentiable function and binary digital sums

ABSTRACT. In this paper we derive functional relations and explicit representations at dyadic points for Takagi's continuous nowhere differentiable function T and also for functions which are connected with T. As consequence we get formulas for binary digital sums, namely the Trollope-Delange formula for the number of ones, a formula counting the zeros as well as a formula for the alternating sum of digits.

KEY WORDS. Takagi's continuous nowhere differentiable function, functional equations, Trollope-Delange formula for the sum-of-digits function, alternating sum of digits, Cantor sets.

1 Introduction

In 1903, T. Takagi [10] discovered an example of a continuous, nowhere differentiable function that was simpler than a well-known example of K. Weierstrass. Takagi's function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\Delta(2^n x)}{2^n} \qquad (x \in \mathbb{R}) \tag{1.1}$$

where $\Delta(x) = \operatorname{dist}(x, \mathbb{Z})$ is an 1-periodic function. T is given for $0 \le x \le 1$ by the following system of functional equations

$$T\left(\frac{x}{2}\right) = \frac{x}{2} + \frac{1}{2}T(x), \qquad T\left(\frac{1+x}{2}\right) = \frac{1-x}{2} + \frac{1}{2}T(x). \tag{1.2}$$

This function is connected with the well-known formula of Trollope-Delange for the sum of digits, cf. [11], [4]. Let $k \in \mathbb{N}$ have the binary expansion

$$k = \sum_{j=0}^{k'} a_j 2^j \qquad (a_j \in \{0, 1\}) \tag{1.3}$$

with $k' = [\log_2 k]$, and let $s_1(k) = a_0 + a_1 + \ldots + a_{k'}$ the number of ones then it holds the Trollope-Delange formula

$$\frac{1}{n}\sum_{k=0}^{n-1} s_1(k) = \frac{1}{2}\log_2 n + F_1(\log_2 n) \tag{1.4}$$

where $F_1(u)$ is a continuous, 1-periodic, nowhere differentiable function. In [4] was also determined the Fourier expansion of F_1. In [6] it was given a new proof of the Trollope-Delange formula by means of the Mellin transforms. We show that formula (1.4) is a consequence of functional relations for T and that the periodic function $F_1(u)$ for $u \le 0$ is representable by

$$F_1(u) = -\frac{u}{2} - \frac{1}{2^{u+1}}T(2^u) \qquad (u \le 0)$$

where T is Takagi's continuous nowhere differentiable function (Theorem 2.1). We shall verify the well-known bounds $\min F_1 = \frac{\log 3}{\log 4} - 1$, $\max F_1 = 0$ and determine the local maxima of $F_1(u)$ (Proposition 2.5).

If $s_0(k)$ denotes the number of zeros in the binary expansion (1.3) of k then it holds

$$\frac{1}{n}\sum_{k=1}^{n-1} s_0(k) = \frac{1}{2}\log_2 n + \frac{1}{n} + F_0(\log_2 n)$$

where F_0 is a continuous, 1-periodic, nowhere differentiable function which is given by

$$F_0(u) = \frac{1-u}{2} - 2^{1-u} + \frac{1}{2^u}T(2^{u-1}) \qquad (0 \le u < 1)$$

(Theorem 3.2). It holds $\min F_0 = -1$ and $\max F_0 = \frac{\log 3}{\log 4} - \frac{3}{2}$ (Proposition 3.3).

Moreover, for the alternating binary sum

$$\tilde{s}(k) = \sum_{j=0}^{k'} (-1)^j a_j \tag{1.5}$$

with k from (1.3) we show that

$$\frac{1}{n}\sum_{k=1}^{n-1} \tilde{s}(k) = \tilde{F}(\log_4 n)$$

where $\tilde{F}$ is a continuous, 1-periodic, nowhere differentiable function which is connected with Takagi's function as follows:

$$\tilde{F}(u) = \frac{1}{2^{2u+1}}\tilde{T}(4^u) \qquad (u \le 0)$$

where

$$\tilde{T}(x) = T(x) + \sum_{n=1}^{\infty} (-1)^n \frac{T(2^n x)}{2^{n-1}} \qquad (x \in \mathbb{R})$$

(Theorem 5.1). Finally, we investigate several properties of $\tilde{F}$. The bounds of $\tilde{F}$ are $\min \tilde{F} = 0$ and $\max \tilde{F} = \frac{1}{2}$. We show that the zero set of $\tilde{F}$ is a Cantor set of Lebesgue measure 0 and that $\tilde{F}$ satisfies the functional equation

$$\tilde{F}(u) + \tilde{F}\left(u + \frac{1}{2}\right) = \frac{1}{2} \qquad (u \in \mathbb{R})$$

(Proposition 5.5).

2 The binary sum-of-digit function

In [7] it was shown that for $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$ and $x \in [0, 1]$, the Takagi function T satisfies the functional equations

$$T\left(\frac{k+x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s_1(k)}{2^\ell}x + \frac{1}{2^\ell}T(x) \tag{2.1}$$

and that the representation

$$T\left(\frac{n}{2^\ell}\right) = \frac{n\ell}{2^\ell} - \frac{1}{2^{\ell-1}}\sum_{k=0}^{n-1} s_1(k) \tag{2.2}$$

with $n = 0, \ldots, 2^\ell$ is a consequence of (2.1).

Formula (2.2) implies that for $n \le 2^\ell$ the binary sum-of-digit function

$$S_1(n) = \sum_{k=0}^{n-1} s_1(k) \tag{2.3}$$

can be represented by

$$S_1(n) = \frac{n\ell}{2} - 2^{\ell-1}T\left(\frac{n}{2^\ell}\right) \tag{2.4}$$

where T is the Takagi function given by (1.1). In particular, for $n = 2^\ell$ we find from (2.4) in view of $T(1) = 0$ that $S_1(2^\ell) = \ell 2^{\ell-1}$. We show that the formula of Trollope-Delange is a consequence of (2.4).

Theorem 2.1 *It holds Trollope-Delange formula (1.4) where F_1 is a continuous, 1-periodic, nowhere differentiable function which is given by*

$$F_1(u) = -\frac{u}{2} - \frac{1}{2^{u+1}}T(2^u) \qquad (u \le 0). \tag{2.5}$$

Proof: According to the first equation in (1.2) the function

$$f_1(x) = -\frac{1}{2}\left\{\log_2 x + \frac{1}{x}T(x)\right\} \qquad (0 < x \le 1) \tag{2.6}$$

has for $0 < x \le \frac{1}{2}$ the property $f_1(2x) = f_1(x)$ so that it can be extended for all positive x by

$$f_1(2x) = f_1(x) \qquad (x > 0). \tag{2.7}$$

We show that for $n \in \mathbb{N}$ it holds

$$\frac{1}{n} S_1(n) = \frac{1}{2} \log_2 n + f_1(n). \tag{2.8}$$

For given n we choose ℓ so large that $n < 2^\ell$. From (2.4) we find in view of (2.6) and (2.7)

$$\begin{aligned} \frac{1}{n} S_1(n) &= \frac{\ell}{2} - \frac{2^{\ell-1}}{n} T\left(\frac{n}{2^\ell}\right) \\ &= \frac{1}{2} \log_2 n - \frac{1}{2}\left\{\log_2\left(\frac{n}{2^\ell}\right) + \frac{2^\ell}{n} T\left(\frac{n}{2^\ell}\right)\right\} \\ &= \frac{1}{2} \log_2 n + f_1\left(\frac{n}{2^\ell}\right) \\ &= \frac{1}{2} \log_2 n + f_1(n). \end{aligned}$$

If we put

$$F_1(u) = f_1(2^u) \qquad (u \in \mathbb{R}) \tag{2.9}$$

then (2.7) is equivalent to $F_1(u+1) = F_1(u)$. Moreover, (2.8) turns over into (1.4) and (2.6) yields (2.5). $\square$

According to (2.9) the functions F_1 and f_1 have the same bounds.

Proposition 2.2 *For the function $f_1 : (0, 1] \mapsto \mathbb{R}$ from (2.6) we have* $\max f_1 = 0$ *where $f_1(x) = 0$ if and only if $x = \frac{1}{2^\ell}$ with $\ell \in \mathbb{N}_0$. Furthermore,* $\min f_1 = \frac{\log 3}{\log 4} - 1$ *and we have $f_1(x) = \min f_1$ exactly for $x = \frac{2}{3}\frac{1}{2^\ell}$ with $\ell \in \mathbb{N}_0$.*

Proof: In view of (2.7) we only have to consider an interval of the form $(a, 2a]$ with any $a \in (0, \frac{1}{2}]$.

1. First we show that for $x \in (\frac{1}{2}, 1)$ we have $f_1(x) < 0 = f_1(1)$ which in view of (2.6) is equivalent to

$$T(x) > -x \log_2 x.$$

We consider the partial sum $T_2(x) = \Delta(x) + \frac{1}{2}\Delta(2x)$ of (1.1), which satisfies

$$T_2(x) = \begin{cases} \frac{1}{2} & \text{for } \frac{1}{2} \le x \le \frac{3}{4} \\ 2(1-x) & \text{for } \frac{3}{4} \le x \le 1, \end{cases}$$

and $T(x) > T_2(x)$ for $x \in (\frac{1}{2}, 1)$. For the function $f(x) = -x\log_2 x$ we have $f(\frac{1}{2}) = T_2(\frac{1}{2}) = \frac{1}{2}$ and $f(1) = T_2(1) = 0$. It follows in view of $f'(\frac{1}{2}) = 1 - \frac{1}{\log 2} < 0$ and $f'(1) = -\frac{1}{\log 2} > -2$ and the convexity of f that $T_2(x) > f(x)$ for $\frac{1}{2} < x < 1$. Consequently, $f_1(x) < 0$ for $\frac{1}{2} < x < 1$.

2. Let $c = f_1(\frac{2}{3}) = \frac{\log 3}{\log 4} - 1$. We show that $f_1(x) > c$ for $\frac{1}{3} < x < \frac{2}{3}$, i.e.

$$-\frac{1}{2}\left\{\log_2 x + \frac{1}{x}T(x)\right\} > c$$

which is equivalent to

$$x\log_2 x + T(x) + 2cx < 0 \qquad \left(\frac{1}{3} < x < \frac{2}{3}\right).$$

Since $\max T = \frac{2}{3}$ the inequality is true if the function

$$g(x) = x\log_2 x + \frac{2}{3} + 2cx$$

has the property $g(x) < 0$ for $\frac{1}{3} < x < \frac{2}{3}$. But this is valid since g is strictly convex and $g(\frac{1}{3}) = g(\frac{2}{3}) = 0$. □

In order to determine the local maxima of f_1 we shall show the

Lemma 2.3 *Let $a = \frac{k}{2^\ell}$ and $b = \frac{k+1}{2^\ell}$ with $\ell \in \mathbb{N}$ and $2^{\ell-1} \leq k < 2^\ell$. Then for $x = ta + (1-t)b$ with $0 \leq t \leq 1$ we have the inequality*

$$\frac{T(x)}{x} \geq t\frac{T(a)}{a} + (1-t)\frac{T(b)}{b}. \tag{2.10}$$

Proof: If this inequality is valid for $t = \frac{1}{2}$ then it follows by induction that it is valid for all dyadic $t = \frac{m}{2^n} \in (0,1)$ and hence for all $t \in (0,1)$ in view of continuity of T. So it is sufficient to prove (2.10) only for $t = \frac{1}{2}$.

In case $\ell = 1$ we have $k = 1$, i.e. $a = \frac{1}{2}$, $T(a) = \frac{1}{2}$, $b = 1$, $T(b) = 0$, $x = \frac{3}{4}$, $T(x) = \frac{1}{2}$, so that (2.10) is true for $t = \frac{1}{2}$.

In the following let $\ell \geq 2$. If we put $A = T(a)$ and $B = T(b)$ then for $x = \frac{a+b}{2}$ we get from (2.1) that $T(x) = \frac{A+B}{2} + \frac{b-a}{2}$, and (2.10) with $t = \frac{1}{2}$ reads

$$\frac{\frac{A+B}{2} + \frac{b-a}{2}}{\frac{a+b}{2}} \geq \frac{A}{2a} + \frac{B}{2b}.$$

A simple calculation yields that this inequality is equivalent to

$$\frac{A}{a} \leq \frac{B}{b} + 2. \tag{2.11}$$

According to (2.1) we have

$$B = A + \frac{\ell - 2s_1(k)}{2^\ell}$$

and hence

$$\frac{A}{a} = \frac{B}{a} + \frac{2s_1(k) - \ell}{k}.$$

Because of $\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$ we have $\frac{1}{a} = \frac{1}{b} + \frac{1}{kb}$ and so inequality (2.11) is satisfied whenever

$$\frac{B}{kb} + \frac{2s_1(k) - \ell}{k} \leq 2$$

i.e. $\frac{1}{b}B + 2s_1(k) - \ell \leq 2k$. This is true for $\ell \geq 2$ since $b > \frac{1}{2}$, $B \leq \max T = \frac{2}{3}$, i.e. $\frac{1}{b}B < \frac{4}{3}$, and $s_1(k) \leq k$. □

Proposition 2.4 *The function f_1 from (2.6) has exactly at the dyadic points $\frac{k}{2^\ell}$ $(\ell \in \mathbb{N}, k \in \{1, \ldots, 2^\ell\})$ local maxima.*

Proof: In [7] it was shown that for dyadic points $x = \frac{k}{2^\ell}$ $(\ell \in \mathbb{N}, k \in \{1, \ldots, 2^\ell\})$ there exists the limit

$$\lim_{h \to 0} \frac{T(x+h) - T(x)}{|h| \log_2 \frac{1}{|h|}} = 1.$$

For the function f_1 from (2.6) by simple calculation it follows

$$\lim_{h \to 0} \frac{f_1(x+h) - f_1(x)}{|h| \log_2 \frac{1}{|h|}} = -\frac{1}{2x}.$$

Consequently, for dyadic $x = \frac{k}{2^\ell}$ it holds

$$\lim_{h \to 0} \frac{f_1(x+h) - f_1(x)}{|h|} = -\infty$$

which implies that f_1 has at x a local maximum (top).

Now let x be a nondyadic point where by (2.7) we can assume that $x \in (\frac{1}{2}, 1)$. Then for $\ell \in \mathbb{N}$ there is an integer $k = k(\ell)$ with $2^{\ell-1} \leq k < 2^\ell$ such that $\frac{k}{2^\ell} < x < \frac{k+1}{2^\ell}$, i.e. $x = ta + (1-t)b$ with $a = \frac{k}{2^\ell}$, $b = \frac{k+1}{2^\ell}$ and a certain $t \in (0, 1)$.

We show that $f_1(x) < tf_1(a) + (1-t)f_1(b)$ which by (2.6) is equivalent to

$$\log_2 x + \frac{T(x)}{x} > t\left\{\log_2 a + \frac{T(a)}{a}\right\} + (1-t)\left\{\log_2 b + \frac{T(b)}{b}\right\}. \tag{2.12}$$

By Lemma 2.3 it holds (2.10) and for the concave function $\log_2 x$ we have for $0 < t < 1$

$$\log_2 x > t \log_2 a + (1-t) \log_2 b.$$

Addition with (2.10) yields (2.12), so that indeed $f_1(x) < tf_1(a) + (1-t)f_1(b)$. It follows $f_1(x) < \max\{f_1(a), f_1(b)\}$ so that f_1 cannot have a local maximum at x. $\square$

It follows from (2.7), (2.9), Proposition 2.2 and Proposition 2.4

Proposition 2.5 *The continuous, 1-periodic function $F_1(u)$ in the formula (1.4) of Trollope-Delange has in $[0,1)$ its maximum exactly at $u_{\max} = 0$ with $F_1(0) = 0$, and its minimum exactly at $u_{\min} = 2 - \frac{\log 3}{\log 2} = 0,4150$ with $F_1(u_{\min}) = \frac{\log 3}{\log 4} - 1 = -0,2075$. The local maxima are exactly the numbers $\frac{\log k}{\log 2} + \ell$ $(k \in \mathbb{N}, \ell \in \mathbb{Z})$.*

As consequence of formula (1.4) we have the well-known inequality (cf. [5], [3], [8], [9]):

$$\frac{1}{2}\log_2 n - c_1 < \frac{1}{n}S_1(n) \le \frac{1}{2}\log_2 n \tag{2.13}$$

with the optimal constant $c_1 = 1 - \frac{\log 3}{\log 4}$.

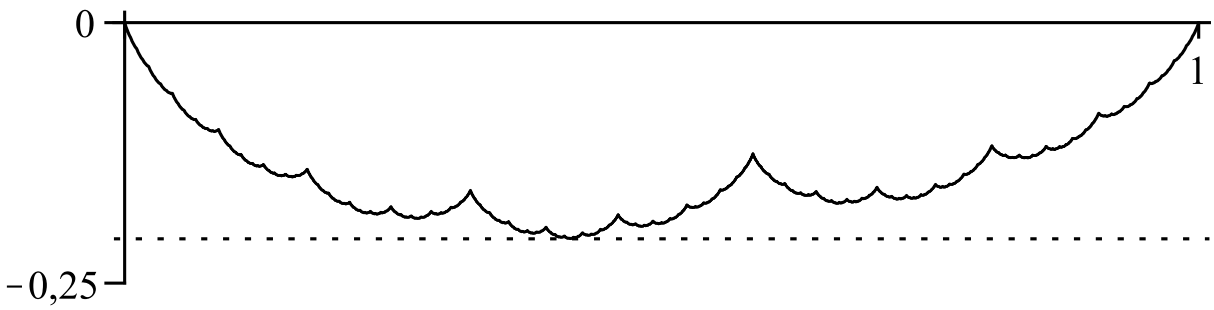

Figure 1: The graph of $F_1(u)$.

3 Counting zeros

In order to determine the number of zeros in binary expansion first we compute the number of all digits. Let $a(k)$ denote the number of all digits in the binary expansion of k, i.e. $a(k) = \ell$ if $2^\ell \le k < 2^{\ell+1}$. We state a formula for the sum

$$A(n) = \sum_{k=1}^{n-1} a(k). \tag{3.1}$$

Proposition 3.1 *For the number of all digits in the binary representations of the integers* $1, 2, \ldots, n-1$ *we have*

$$\frac{1}{n}A(n) = \log_2 n + \frac{1}{n} + F(\log_2 n) \tag{3.2}$$

where F *is a continuous,* 1*-periodic function which is given by*

$$F(u) = 1 - u - 2^{1-u} \qquad (0 \leq u < 1). \tag{3.3}$$

Proof: Obviously, $A(2^\ell) = 1 + 2 \cdot 2 + 3 \cdot 2^2 + \ldots + \ell \cdot 2^{\ell-1}$. In view of

$$1 + 2t + 3t^2 + \ldots + \ell t^{\ell-1} = \frac{(\ell+1)t^\ell(t-1) - (t^{\ell+1}-1)}{(t-1)^2} \qquad (t \neq 1)$$

we get

$$A(2^\ell) = (\ell+1)2^\ell - 2^{\ell+1} + 1 = (\ell-1)2^\ell + 1.$$

For $0 \leq m \leq 2^\ell$ we have $A(2^\ell + m) = A(2^\ell) + m(\ell+1)$, i.e.

$$A(2^\ell + m) = (\ell-1)2^\ell + 1 + m(\ell+1) = \ell(2^\ell + m) - 2^\ell + 1 + m.$$

Write $n = 2^\ell + m = 2^\ell(1+x)$ with $0 \leq x < 1$ we get in view of $\frac{2^\ell}{n} = \frac{1}{1+x}$ and $\frac{m}{n} = \frac{n-2^\ell}{n} = 1 - \frac{1}{1+x}$

$$\begin{aligned}
\frac{1}{n}A(n) &= \ell - \frac{2^\ell}{n} + \frac{m+1}{n} \\
&= \log_2 n + \log_2\left(\frac{2^\ell}{n}\right) - \frac{2^\ell}{n} + \frac{1}{n} + \frac{m}{n} \\
&= \log_2 n + \frac{1}{n} + \left\{1 - \log_2(1+x) - \frac{2}{1+x}\right\}.
\end{aligned}$$

This yields the assertion since in view of the periodicity of F we have for $n = 2^\ell(1+x)$

$$F(\log_2 n) = F(\log_2\{2^\ell(1+x)\}) = F(\log_2(1+x)) = F(u)$$

with $1 + x = 2^u$ $(0 \leq u < 1)$. □

Theorem 3.2 *For* $k \in \mathbb{N}_0$ *let* $s_0(k)$ *denote the number of zeros of* k *in the binary representation of* k*. Then it holds*

$$\frac{1}{n}\sum_{k=1}^{n-1} s_0(k) = \frac{1}{2}\log_2 n + \frac{1}{n} + F_0(\log_2 n) \tag{3.4}$$

where F_0 *is a continuous,* 1*-periodic, nowhere differentiable function which is given by*

$$F_0(u) = \frac{1-u}{2} - 2^{1-u} + \frac{1}{2^u}T(2^{u-1}) \qquad (0 \leq u < 1). \tag{3.5}$$

Proof: We have $s_0(n) = a(n) - s_1(n)$ where $a(n)$ counts the number of all digits of n in the binary expansion and $s_1(n)$ counts the number of ones. Formulas (1.4) and (3.2) imply (3.4) with $F_0(u) = F(u) - F_1(u)$. The representation (3.5) follows from (2.5) and (3.3). $\square$

Proposition 3.3 *The continuous, 1-periodic function $F_0(u)$ in formula (3.4) has in $[0,1)$ its maximum exactly at $u_{\max} = 2 - \frac{\log 3}{\log 2}$ with $F_0(u_{\max}) = \frac{\log 3}{\log 4} - \frac{3}{2} = -0,707519$, and its minimum exactly at $u_{\min} = 0$ with $F_0(0) = -1$.*

Proof: Put $2^{u-1} = x$ in (3.5) we see that $F_0(u)$ has in $[0,1)$ the same bounds as

$$f_0(x) = -\frac{1}{2}\log_2 x - \frac{1}{x} + \frac{1}{2x}T(x)$$

in $[\frac{1}{2}, 1)$. For $x = \frac{1+t}{2}$ with $0 \le t < 1$ we get in view of (1.1)

$$\begin{aligned} f_0\left(\frac{1+t}{2}\right) &= -\frac{1}{2}\log_2(1+t) + \frac{1}{2} - \frac{2}{1+t} + \frac{1}{1+t}\left\{\frac{1-t}{2} + \frac{1}{2}T(t)\right\} \\ &= -\frac{1}{2}\log_2(1+t) - \frac{1}{1+t} + \frac{1}{2(1+t)}T(t). \end{aligned}$$

1. Let $c_0 = f_0(\frac{2}{3}) = \frac{\log 3}{\log 4} - \frac{3}{2}$. We show that for $0 \le t < 1$ we have $f_0(\frac{1+t}{2}) \le f_0(\frac{2}{3}) = c_0$, i.e.

$$-\frac{1}{2}\log_2(1+t) - \frac{1}{1+t} + \frac{1}{2(1+t)}T(t) \le c_0$$

where we have equality if and only if $t = \frac{1}{3}$. The last inequality is equivalent to $T(t) \le g(t)$ where

$$g(t) = 2 + 2c_0(1+t) + (1+t)\log_2(1+t).$$

The derivative $g'(t) = 2c_0 + \frac{1+\log(1+t)}{\log 2}$ is strictly increasing with $g'(\frac{1}{3}) = \frac{1}{\log 2} - 1$. For $0 \le t < \frac{1}{3}$ we get by (1.2) with $x = 2t$ that $T(t) = t + \frac{1}{2}T(2t) \le t + \frac{1}{3} = T(\frac{1}{3}) - (\frac{1}{3} - t)$ where we have used that $\max T = \frac{2}{3} = T(\frac{1}{3})$. In view of $T(\frac{1}{3}) = g(\frac{1}{3})$ and $g'(t) < g'(\frac{1}{3}) < 1$ for $0 \le t < \frac{1}{3}$ it follows $T(t) < g(t)$ for these t. Moreover for $\frac{1}{3} < t < 1$ we have $g(\frac{1}{3}) < g(t)$ since $g'(t) > g'(\frac{1}{3}) > 0$. For these t it holds $T(t) \le T(\frac{1}{3})$ so that in view of $g(\frac{1}{3}) = T(\frac{1}{3})$ indeed we have $g(t) < T(t)$ for $\frac{1}{3} < t < 1$.

2. We have to show that for $0 < t < 1$ we have $f_0(\frac{1+t}{2}) > f_0(\frac{1}{2}) = -1$, i.e.

$$-\frac{1}{2}\log_2(1+t) - \frac{1}{1+t} + \frac{1}{2(1+t)}T(t) > -1$$

which is equivalent to

$$T(t) - (1+t)\log_2(1+t) + 2t > 0 \qquad (0 < t < 1).$$

From (1.1) we get $T(t) \geq \Delta(t) + \frac{1}{2}\Delta(2t) \geq 2t(1-t)$ for $0 \leq t \leq 1$ so that the inequality is true if the function

$$h(t) = 2t(1-t) - (1+t)\log_2(1+t) + 2t$$

has the property $h(t) > 0$ for $0 < t < 1$. Since

$$h'(t) = 4 - 4t - \frac{\log(1+t)+1}{\log 2}, \qquad h''(t) = -4 - \frac{1}{(1+t)\log 2} < 0,$$

h is strictly concave in $[0,1]$ and by $h(0) = h(1) = 0$ it follows $h(t) > 0$ for $0 < t < 1$. $\square$

So we have

$$\frac{1}{2}\log_2 n + \frac{1}{n} - 1 \leq \frac{1}{n}S_0(n) < \frac{1}{2}\log_2 n + \frac{1}{n} + c_0$$

with the optimal constant $c_0 = \frac{\log 3}{\log 4} - \frac{3}{2}$.

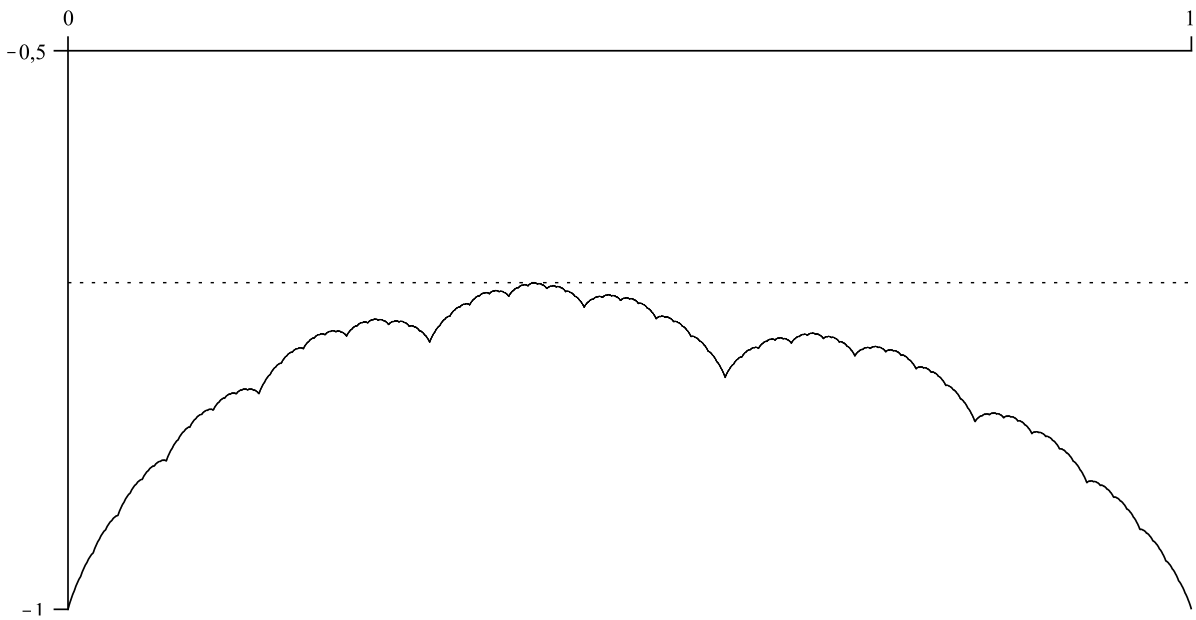

Figure 2: The graph of $F_0(u)$.

4 The alternating sum

Besides of (1.1) we also consider the alternating series

$$\tilde{T}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\Delta(2^n x)}{2^n} \qquad (x \in \mathbb{R}) \tag{4.1}$$

which can be written as

$$\tilde{T}(x) = T_+(x) - T_-(x) \qquad (x \in \mathbb{R}) \tag{4.2}$$

where

$$T_+(x) = \sum_{n=0}^{\infty} \frac{\Delta(2^{2n}x)}{2^{2n}}, \qquad T_-(x) = \sum_{n=0}^{\infty} \frac{\Delta(2^{2n+1}x)}{2^{2n+1}} = \frac{1}{2}T_+(2x). \tag{4.3}$$

Proposition 4.1 *The function $\tilde{T}$ from (4.1) is continuous, 1-periodic and nowhere differentiable. It can be expressed by the Takagi function T as follows:*

$$\tilde{T}(x) = T(x) + \sum_{n=1}^{\infty} (-1)^n \frac{T(2^n x)}{2^{n-1}} \qquad (x \in \mathbb{R}). \tag{4.4}$$

Proof: . Obviously, representation (4.1) implies that $\tilde{T}$ is continuous and 1-periodic. Further

$$\tilde{T}(x) = \sum_{\nu=0}^{\infty} a^\nu g(b^\nu x)$$

with $a = \frac{1}{4}$, $b = 4$ and $g(x) = \Delta(x) - \frac{1}{2}\Delta(2x)$ which is piecewise linear (polygonal) but not constant. Since $\tilde{T}$ is not polygonal it follows by Behrend [1], Theorem III on p. 477, that $\tilde{T}$ is nowhere differentiable.

From (4.3) and (1.1) we get $T(x) = T_+(x) + T_-(x)$. Hence

$$T_+(x) = T(x) - T_-(x) = T(x) - \frac{1}{2}T_+(2x)$$

Iteration gives

$$T_+(x) = (-1)^m \frac{1}{2^m} T_+(2^m x) + \sum_{n=0}^{m-1} (-1)^n \frac{T(2^n x)}{2^n}$$

for every $m \in \mathbb{N}$ and $x \in \mathbb{R}$. As T_+ is bounded we get

$$T_+(x) = \sum_{n=0}^{\infty} (-1)^n \frac{T(2^n x)}{2^n}.$$

Now from (4.2) and (4.3) it follows the assertion. □

Proposition 4.2 *For $\ell \in \mathbb{N}$ and $k \in \{0, 1, \dots, 4^\ell - 1\}$ the function $\tilde{T}$ from (4.1) satisfies the functional equations*

$$\tilde{T}\left(\frac{k+x}{4^\ell}\right) = \tilde{T}\left(\frac{k}{4^\ell}\right) + \frac{\tilde{s}(k)}{2^{2\ell-1}}x + \frac{1}{4^\ell}\tilde{T}(x) \qquad (0 \le x \le 1) \tag{4.5}$$

where $\tilde{s}(k)$, given by (1.5), denotes the alternating sum of digits of the number k in the binary representation. Moreover, for $n \in \{1, \dots, 4^\ell\}$ we have

$$\tilde{T}\left(\frac{n}{4^\ell}\right) = \frac{1}{2^{2\ell-1}} \sum_{k=0}^{n-1} \tilde{s}(k). \tag{4.6}$$

Proof: First we show that the function T_+ from (4.3) satisfies the functional equation

$$T_+\left(\frac{k+x}{4^\ell}\right) = T_+\left(\frac{k}{4^\ell}\right) + \frac{\ell - 2s_-(k)}{4\ell}x + \frac{1}{4^\ell}T_+(x) \qquad (0 \le x \le 1) \tag{4.7}$$

where $s_-(k) = a_1 + a_3 + \ldots$ denotes the sum of the digits a_{2j+1} with $2j+1 \le k'$ of the number k in the binary representation (1.3). Since $\Delta(m) = 0$ for $m \in \mathbb{N}_0$ we get from (4.3) that

$$T_+\left(\frac{k}{4^\ell}\right) = \sum_{n=0}^{\ell-1} \frac{\Delta(4^n \frac{k}{4^\ell})}{4^n}$$

and hence

$$T_+\left(\frac{k+x}{4^\ell}\right) - T_+\left(\frac{k}{4^\ell}\right) = \sum_{n=0}^{\ell-1} \frac{\Delta(4^n \frac{k+x}{4^\ell}) - \Delta(4^n \frac{k}{4^\ell})}{4^n} + \sum_{n=\ell}^{\infty} \frac{\Delta(4^n \frac{k+x}{4^\ell})}{4^n}.$$

For $n \ge \ell$ we find with $m = n - \ell \ge 0$ that $\Delta(4^n \frac{k+x}{4^\ell}) = \Delta(4^m k + 4^m x) = \Delta(4^m x)$ so that the last sum in the last equation is equal to $\frac{1}{4^\ell}T_+(x)$. For $n = 0, \ldots, \ell - 1$ there is no integer in the open interval $(4^n \frac{k}{4^\ell}, 4^n \frac{k+1}{4^\ell})$, and hence the both numbers $4^n \frac{k+x}{4^\ell}$ and $4^n \frac{k}{4^\ell}$ belong to the same interval $[m, m+\frac{1}{2}]$ or $[m+\frac{1}{2}, m+1]$ $(m \in \mathbb{N}_0)$ since $0 \le x \le 1$. Since $\Delta(\cdot)$ is linear in each of these intervals we find that

$$\frac{\Delta(4^n \frac{k+x}{4^\ell}) - \Delta(4^n \frac{k}{4^\ell})}{4^n} = \varepsilon_n \frac{x}{4^\ell}$$

where $\varepsilon_n = +1$ whenever $4^n \frac{k}{4^\ell} \in [m, m+\frac{1}{2})$ and where $\varepsilon_n = -1$ elsewhere. If k has the binary representation (1.3) then $k' < 2\ell$ since $k < 2^{2\ell}$ and

$$k = \sum_{j=0}^{2\ell} a_j 2^j$$

with $a_j = 0$ for $k' < j \le 2\ell$ for which we also write shortly $k = a_{2\ell}a_{2\ell-1}\ldots a_0$. Because of $4^n \frac{k}{4^\ell} = a_{2\ell}\ldots a_{2\ell-2n}, a_{2\ell-2n-1}\ldots a_0$ for $0 \le n \le \ell-1$ we have $\varepsilon_n = -1$ when $a_{2\ell-2n-1} = 1$ which happens $s_-(k)$ times, and $\varepsilon_n = +1$ when $a_{2\ell-2n-1} = 0$ which happens $\ell - s_-(k)$ times. This yields

$$\sum_{n=0}^{\ell-1} \varepsilon_n = -s_-(k) + \ell - s_-(k) = \ell - 2s_-(k)$$

and hence (4.7) is proved.

Analogously one can show for T_- from (4.3) the relation

$$T_-\left(\frac{k+x}{4^\ell}\right) = T_-\left(\frac{k}{4^\ell}\right) + \frac{\ell - 2s_+(k)}{4^\ell}x + \frac{1}{4^\ell}T_-(x) \qquad (0 \le x \le 1) \tag{4.8}$$

where $s_+(k) = a_0 + a_2 + \ldots$ denotes the sum of the digits a_{2j} of k in the representation (1.3). Obviously, the alternating sum (1.5) can be written as $\tilde{s}(k) = s_+(k) - s_-(k)$ so that (4.7) and (4.8) imply (4.5) in view of (4.2). Finally, equation (4.6) follows by summation from (4.5) in view of $\tilde{T}(0) = \tilde{T}(1) = 0$. $\square$

5 Alternating binary sums

Equation (4.6) yields for the alternating sum (1.5) the sum formula

$$\sum_{k=0}^{n-1} \tilde{s}(k) = 2^{2\ell-1}\tilde{T}\left(\frac{n}{4^\ell}\right) \tag{5.1}$$

provided that $n \leq 4^\ell$. We want to determine a formula which is independent of ℓ.

Theorem 5.1 *For the alternating sum (1.5) it holds the formula*

$$\frac{1}{n}\sum_{k=0}^{n-1} \tilde{s}(k) = \tilde{F}(\log_4 n) \tag{5.2}$$

where $\tilde{F}$ is a continuous, 1-periodic, nowhere differentiable function. This function is given by

$$\tilde{F}(u) = \frac{1}{2^{2u+1}}\tilde{T}(4^u) \qquad (u \leq 0) \tag{5.3}$$

where $\tilde{T}$ is given by (4.1) or (4.4) .

Proof: By Proposition 4.1 the representations (4.1) and (4.4) are equivalent. Writing (4.1) in the form

$$\tilde{T}\left(\frac{x}{4}\right) = \Delta\left(\frac{x}{4}\right) - \frac{1}{2}\Delta\left(\frac{x}{2}\right) + \frac{1}{4}\tilde{T}(x) \qquad (x \in \mathbb{R})$$

we see that for $0 \leq x \leq 1$ it holds

$$\tilde{T}\left(\frac{x}{4}\right) = \frac{1}{4}\tilde{T}(x).$$

Hence, the function

$$\tilde{f}(x) = \frac{1}{2x}\tilde{T}(x) \qquad (0 < x \leq 1) \tag{5.4}$$

satisfies the equation

$$\tilde{f}\left(\frac{x}{4}\right) = \tilde{f}(x) \qquad (0 < x \leq 1),$$

and we can continue this function for all $x > 0$ such that

$$\tilde{f}(4x) = \tilde{f}(x) \qquad (x > 0). \tag{5.5}$$

It is easy to see that for $n \in \mathbb{N}$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1}\tilde{s}(k) = \tilde{f}(n). \tag{5.6}$$

For given n we choose ℓ so large that $n < 4^\ell$. From (5.1) divided by n we find

$$\frac{1}{n}\sum_{k=0}^{n-1}\tilde{s}(k) = \frac{2^{2\ell-1}}{n}\tilde{T}\left(\frac{n}{4^\ell}\right) = \tilde{f}\left(\frac{n}{4^\ell}\right) = \tilde{f}(n)$$

where we have used (5.4) and (5.5). If we put

$$\tilde{F}(u) = \tilde{f}(4^u) \qquad (u \in \mathbb{R}) \tag{5.7}$$

then (5.5) is equivalent to $\tilde{F}(u+1) = \tilde{F}(u)$, (5.4) turns over into (5.3) and (5.6) yields formula (5.2). Finally, from (5.3) we see that $\tilde{F}(u)$ is nowhere differentiable since $\tilde{T}$ has this property, cf. Proposition 4.1. □

In order to obtain more information on the functions $\tilde{f}$ and $\tilde{F}$, we need the following result of [2], p. 1005-1007 (cf. in particular formula (3.5) and the representations of x^-, x^+ on p. 1007).

Lemma 5.2 ([2]) *For $a > 2$ the set of numbers*

$$x = (a-1)\sum_{\nu=1}^{\infty}\frac{\xi_\nu}{a^\nu} \qquad (\xi_\nu \in \{0,1\}) \tag{5.8}$$

form a perfect Cantor set $\mathcal{F} \subset [0,1]$ of Lebesgue measure zero. The complement $\mathcal{G} = [0,1]\setminus\mathcal{F}$ is an open Cantor set of measure $|\mathcal{G}| = 1$. This set consists of all numbers of the form

$$x = (a-1)\sum_{\nu=0}^{n}\frac{\xi_\nu}{a^\nu} + \frac{t}{a^{n+1}} \qquad (1 < t < a-1).$$

Lemma 5.3 *Let x be a number in $[0,1]$. Then for all $k \in \mathbb{N}_0$ it holds the inequality*

$$\Delta(4^k x) \le \frac{1}{4}$$

if and only if x is representable in the form

$$x = \sum_{n=1}^{\infty}\frac{\eta_n}{4^n} \qquad (\eta_n \in \{0,3\}). \tag{5.9}$$

Proof: Assume that x has the form (5.9). We show that for $k \in \mathbb{N}_0$ we have

$$|4^k x - n_k| \leq \frac{1}{4}$$

where n_k is the integer

$$n_k = \eta_{k+1} + \sum_{n=1}^{k} 4^{k-n}\eta_n. \tag{5.10}$$

In the case $\eta_{k+1} = 0$ it is $n_k \leq 4^k x$, and in view of (5.9), (5.10) and $\eta_\nu \leq 3$ we have

$$4^k x - n_k = \sum_{n=k+2}^{\infty} \frac{\eta_n}{4^{n-k}} \leq \frac{3}{4^2} \sum_{m=0}^{\infty} \frac{1}{4^m} = \frac{1}{4}.$$

In the case $\eta_{k+1} = 3$ it is $n_k \geq 4^k x$, and in view of the (5.9), (5.10) and $\eta_\nu \geq 0$ we have the estimate

$$n_k - 4^k x = 1 - \sum_{n=k+1}^{\infty} \frac{\eta_n}{4^{n-k}} \leq 1 - \frac{3}{4} = \frac{1}{4}.$$

If x is not of the form (5.9) then according to Lemma 5.2 with $a = 4$ we have the representation

$$x = \sum_{n=1}^{k} \frac{\eta_n}{4^n} + \frac{t}{4^{k+1}} \qquad (1 < t < 3)$$

with a certain $k \in \mathbb{N}_0$. Therefore

$$4^k x = \sum_{n=1}^{k} 4^{k-n}\eta_n + \frac{t}{4}$$

and, in view of $1 < t < 3$, we find

$$\frac{1}{4} < 4^k x - [4^k x] < 1 - \frac{1}{4}.$$

Therefore in this case we have $\Delta(4^k x) > \frac{1}{4}$. □

Proposition 5.4 *The function $\tilde{f}$ from (5.4) satisfies the functional equation*

$$\tilde{f}(x) + \tilde{f}(2x) = \frac{1}{2} \qquad (x > 0). \tag{5.11}$$

We have $\min \tilde{f} = 0$ and $\tilde{f}(x) = 0$ if and only if $x > 0$ has the form

$$x = \sum_{n=-\infty}^{\infty} \zeta_n 4^n \qquad (\zeta_n \in \{0, 3\}) \tag{5.12}$$

where $\zeta_n = 0$ for $n > \log_4 x$.

Proof: Owing to (5.4) we investigate the function $\tilde{T}(x)$ for $0 < x \le 1$. From (4.1) we get

$$\tilde{T}(x) + \frac{1}{2}\tilde{T}(2x) = \Delta(x) \qquad (x \in \mathbb{R}).$$

By multiplication with $\frac{1}{2x}$ it follows (5.11) for $0 < x \le \frac{1}{2}$ in view of $\Delta(x) = x$ for these x and (5.4). Equation (5.5) implies the validity of (5.11) for all $x > 0$.

According to (4.1) the function $\tilde{T}$ can be written as

$$\tilde{T}(x) = \sum_{k=0}^{\infty} \frac{g(4^k x)}{4^k} \qquad (x \in \mathbb{R}) \tag{5.13}$$

where $g(x) = \Delta(x) - \frac{1}{2}\Delta(2x)$ is a periodic function with period 1 which in $[0,1]$ is given by

$$g(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le \frac{1}{4} \\ 2x - \frac{1}{2} & \text{for} \quad \frac{1}{4} < x \le \frac{1}{2} \\ -2x + \frac{3}{2} & \text{for} \quad \frac{1}{2} < x \le \frac{3}{4} \\ 0 & \text{for} \quad \frac{3}{4} < x \le \frac{1}{4}. \end{cases} \tag{5.14}$$

Because of $g(x) \ge 0$ for $x \in \mathbb{R}$ we have $\tilde{T}(x) \ge 0$, too. For $0 < x \le 1$ we get from (5.4) that $\min \tilde{f} = \min \frac{1}{2x}\tilde{T}(x) = 0$ since $\tilde{T}(1) = 0$, and $\tilde{f}(x) = 0$ if and only if $\tilde{T}(x) = 0$. Equation (5.13) implies in view of $g(x) \ge 0$ that $\tilde{T}(x) = 0$ if and only if for all $k \in \mathbb{N}_0$ we have $g(4^k x) = 0$. According to (5.14) we have $g(x) = 0$ in [0,1] exactly for $0 \le x \le \frac{1}{4}$ and for $1 - \frac{1}{4} \le x \le 1$, i.e. $\Delta(x) \le \frac{1}{4}$. Consequently, for all $k \in \mathbb{N}_0$ it holds $g(4^k x) = 0$ if and only if $\Delta(4^k x) \le \frac{1}{4}$ so that by Lemma 5.3 we have $\tilde{T}(x) = 0$ for $0 < x \le 1$ if and only if x is of the form (5.9). It follows from (5.4) and (5.5) that $\tilde{f}(x) = 0$ for $x > 0$ if and only if x is of the form (5.12). □

Proposition 5.5 *The continuous, periodic function $\tilde{F}$ in formula (5.2), given by (5.3), satisfies the functional equation*

$$\tilde{F}(u) + \tilde{F}\left(u + \frac{1}{2}\right) = \frac{1}{2} \qquad (u \in \mathbb{R}). \tag{5.15}$$

The bounds of $\tilde{F}$ are $\min \tilde{F} = 0$ and $\max \tilde{F} = \frac{1}{2}$. It holds $\tilde{F}(u) = 0$ if and only if $u = \log_4 x$ where $x > 0$ has the form (5.12). The zeros of $\tilde{F}$ form a Cantor set of Lebesgue measure 0.

Proof: For the periodic function $\tilde{F}$ it holds (5.7). Proposition 5.4 implies that $\tilde{F}$ satisfies the functional equation (5.15) and that $\min \tilde{F} = 0$. It follows from (5.15) that $\max \tilde{F} = \frac{1}{2}$.

According to (5.7) Proposition 5.4 also implies the assertion on the zeros of $\tilde{F}$. By Lemma 5.2 the set of all x of the form (5.9) form a Cantor set of Lebesgue measure 0. This is true also for the zeros of $\tilde{F}$ according to (5.7). $\square$

Remark 5.6 **1.** Functional equation (5.15) implies

$$\int_0^1 \tilde{F}(u)du = \frac{1}{4}. \tag{5.16}$$

2. The map $4^u \mapsto x$ maps the interval $[0,1]$ onto $[1,4]$. In $(1,4]$ the number $x_0 = 3$ is the smallest number of the form (5.12), and $x_1 = 3 + \frac{3}{4} + \frac{3}{4^2} + \ldots = 4$ the largest such number. Hence, in $(0,1]$ the number $u_0 = \frac{\log 3}{\log 4}$ is the smallest zero of $\tilde{F}$ and $u_1 = 1$ the largest zero of $\tilde{F}$, cf. Figure 3.

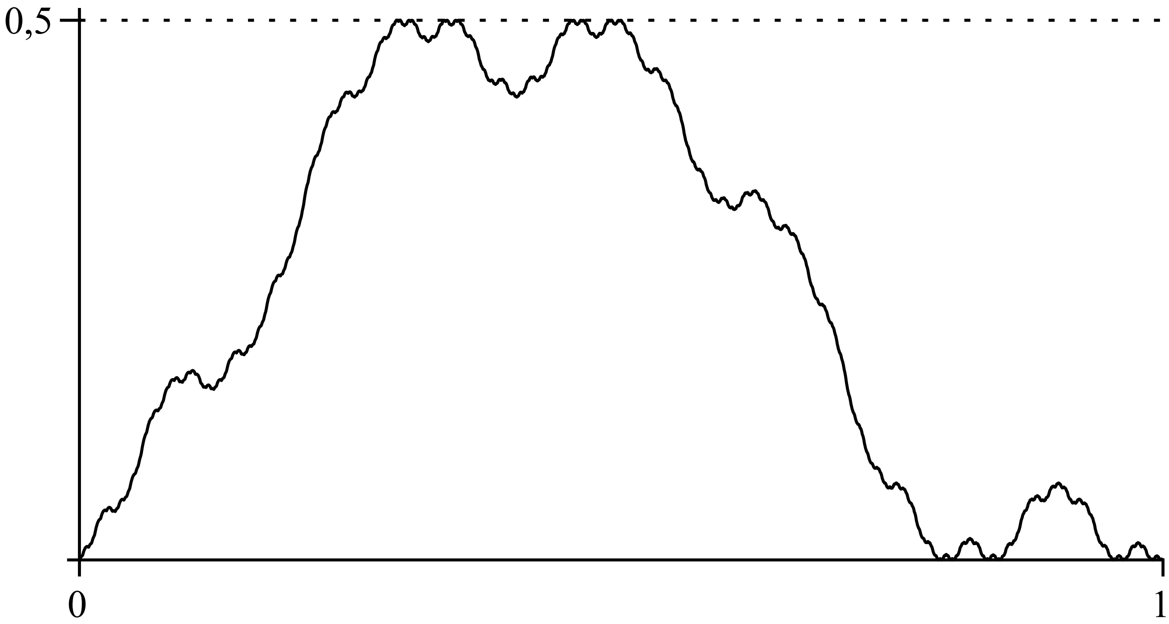

Figure 3: The graph of $\tilde{F}(u)$.

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